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# A Dirichlet problem on the half-line for nonlinear equations with indefinite weight

Zuzana Došlá, Mauro Marini and Serena Matucci

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**Abstract.** We study the existence of positive solutions on the half-line  $[0, \infty)$  for the nonlinear second order differential equation

$$(a(t)x')' + b(t)F(x) = 0, \quad t \geq 0,$$

satisfying Dirichlet type conditions, say  $x(0) = 0$ ,  $\lim_{t \rightarrow \infty} x(t) = 0$ . The function  $b$  is allowed to change sign and the nonlinearity  $F$  is assumed to be asymptotically linear in a neighborhood of zero and infinity. Our results cover also the cases in which  $b$  is a periodic function for large  $t$  or it is unbounded from below.

*Keywords.* Second order nonlinear differential equation, boundary value problem on the half line, Dirichlet conditions, globally positive solution, disconjugacy, principal solution.

*MSC 2010:* Primary 34B40, Secondary 34B18.

## 1 Introduction

Consider the boundary value problem (BVP) on the half-line  $[0, \infty)$

$$(a(t)x')' + b(t)F(x) = 0, \tag{1}$$

$$x(0) = 0, \quad x(t) > 0 \text{ on } (0, \infty), \quad \lim_{t \rightarrow \infty} x(t) = 0, \tag{2}$$

where we assume the following:

(i) The function  $a$  is continuous on  $[0, \infty)$ ,  $a(t) > 0$ , and

$$\int_0^\infty \frac{1}{a(t)} dt < \infty. \quad (3)$$

(ii) The function  $b$  is continuous on  $[0, \infty)$ , nonnegative and not identically zero on  $[0, 1]$ , and is allowed to change sign for  $t > 1$ . Moreover,  $b$  is bounded from above, that is, there exists a positive constant  $B$  such that

$$b(t) \leq B \text{ on } [1, \infty). \quad (4)$$

(iii) The function  $F$  is continuous on  $\mathbb{R}$ ,  $F(u)u > 0$  for  $u \neq 0$ ,  $F$  is differentiable on  $[0, \infty)$  with bounded nonnegative derivative:

$$0 \leq \frac{dF(u)}{du} \leq 1 \text{ for } u \geq 0, \quad (5)$$

and satisfies

$$\lim_{u \rightarrow 0^+} \frac{F(u)}{u} = k_0, \quad \lim_{u \rightarrow \infty} \frac{F(u)}{u} = k_\infty, \quad (6)$$

where

$$0 \leq k_0 \neq k_\infty.$$

Observe that (5) implies that  $k_0, k_\infty \leq 1$ .

The BVP (1)-(2) is a Dirichlet-type BVP on an unbounded domain. Recently, there has been a growing interest in studying infinite interval problems associated to second order nonlinear differential equations, under various points of view. For a wide bibliography, we refer the reader to [1, 2, 21] and the references therein. When the weight  $b$  is of fixed sign or it is sign-indefinite, we refer to [7, 14, 26] or [15, 16, 23, 24], respectively. The BVP (1)-(2) arises in the investigation of positive radial solutions for elliptic equations, when the nonlinearity is asymptotically linear, see, e.g., [3].

Our main aim is to continue this study when the function  $b$  is allowed to change its sign and the nonlinearity  $F$  can be, roughly speaking, close to a linear function. The investigated problem can be viewed as an extension to the half-line of recent results on nonlinear BVPs with a sign-indefinite weight on a compact interval, see, e.g., [4, 5], and reference therein for a brief survey on this topic.

Denote by  $|\cdot|_L$  the norm in  $L^1[0, 1]$  and set

$$A(t) = \int_0^t \frac{1}{a(s)} ds. \quad (7)$$

Our main result is the following, in which the disconjugacy of a suitable auxiliary differential linear equation plays a key role, see Section 3 below.

**Theorem 1.** *Assume that the linear differential equation*

$$v'' + \frac{B}{a(t)}v = 0 \tag{8}$$

*is disconjugate on  $[1, \infty)$ , where the constant  $B$  is defined in (4).*

*If there exist  $t_1, t_2 \in (0, 1)$ ,  $t_1 < t_2$  such that  $\int_{t_1}^{t_2} b(t) dt > 0$ , and*

$$0 \leq \min\{k_0, k_\infty\}A(1) |b|_L < 1, \tag{9}$$

$$\max\{k_0, k_\infty\} \int_{t_1}^{t_2} b(t) dt > \frac{A(1)}{A(t_1)(A(1) - A(t_2))}, \tag{10}$$

*then the BVP (1)-(2) has a solution.*

*Moreover, the solution  $x$  has a local maximum in the interval  $(0, 1]$ , is decreasing in  $[1, \infty)$  and satisfies*

$$\int_1^\infty \frac{1}{a(t)x^2(t)} dt = \infty. \tag{11}$$

Theorem 1 covers also the cases in which the weight  $b$  is a periodic function for large  $t$  or it is unbounded from below.

Our approach is based on a shooting method and a continuity result. More precisely, Theorem 1 is proved by considering two auxiliary BVPs, the first one on the compact interval  $[0, 1]$ , where  $b$  is nonnegative, and the second one on the half-line  $[1, \infty)$ , where  $b$  is allowed to change its sign. The problem of the existence of solutions for (1), emanating from zero, positive in the interval  $(0, 1)$ , and satisfying additional assumptions at  $t = 1$ , is considered in Section 2 and is solved by using some results from [22], with minor changes. The BVP on  $[1, \infty)$  is examined in Section 4. It deals with positive decreasing solutions on  $[1, \infty)$  for (1) which tend to zero as  $t \rightarrow \infty$ . This second problem is solved by using a fixed point theorem for operators defined in a Fréchet space by a Schauder's linearization device, see [11, Theorem 1.3]. This method does not require the explicit form of the fixed point operator, but only some *a-priori* bounds. These estimations are obtained using some properties of principal solutions of disconjugate second order linear

equations, see [20, Chapter 11]. Finally, roughly speaking, the solvability of (1)-(2) is obtained by using a shooting method on  $[0, 1]$  and, by means some continuity arguments, pasting a solution of (1) on  $[0, 1]$  with a solution of the BVP on  $[1, \infty)$ . This last argument can be viewed as a generalization to non compact intervals of some ideas in [19].

Notice that our approach allows us to obtain also an estimation of the decay to zero of solutions of (1)-(2). Some examples complete the paper.

## 2 Two auxiliary BVPs on $[0, 1]$

In this section, we recall some results about the existence of solutions of (1) on  $[0, 1]$ , which belong either to  $\Delta_1$  or  $\Delta_2$ , where

$$\begin{aligned}\Delta_1 &= \{u \in C[0, 1] : u(0) = u(1) = 0, u(t) > 0 \text{ on } (0, 1)\} \\ \Delta_2 &= \{u \in C[0, 1] : u(0) = u'(1) = 0, u(t) > 0 \text{ on } (0, 1)\}.\end{aligned}$$

These results can be obtained from [22], with minor changes.

BVPs on a compact interval, associated to equations of the form

$$z'' + g(t)F(z) = 0, \tag{12}$$

where  $g$  is a continuous nonnegative function on  $[0, 1]$ , have been widely investigated in the literature, under many different points of view. We refer to [6, Introduction] and references therein for a brief survey.

In particular, the existence of solutions of (12), which satisfy either  $z \in \Delta_1$  or  $z \in \Delta_2$ , has been considered in [18], where the key conditions on the nonlinearity are either that  $F$  is superlinear, that is,  $k_0 = 0, k_\infty = \infty$  or  $F$  is sublinear, that is,  $k_0 = \infty, k_\infty = 0$ . When the nonlinearity  $F$  is not necessarily superlinear nor sublinear, these results have been extended in several ways in [22].

Using [22, Corollaries 3.1 and 3.5] and the continuity of  $g$ , we obtain the following result.

**Lemma 1.** *Assume that there exist  $t_1, t_2 \in (0, 1)$ ,  $t_1 < t_2$ , such that  $\int_{t_1}^{t_2} g(t) dt > 0$  and*

$$0 \leq \min\{k_0, k_\infty\} |g|_L < 1, \quad \max\{k_0, k_\infty\} \int_{t_1}^{t_2} g(t) dt > \frac{1}{t_1(1-t_2)}. \tag{13}$$

*Then (12) has both solutions  $z_1 \in \Delta_1$  and  $z_2 \in \Delta_2$ .*

*Proof.* In virtue of the continuity of  $g$ , every nonnegative solution  $z$  of (12),  $z \not\equiv 0$ , satisfies  $z(t) > 0$  on  $(0, 1)$ , since  $z'$  is nonincreasing. Hence, the assertion follows from [22, Corollaries 3.1 and 3.5].  $\square$

When  $g$  does not have zeros on  $[0, 1]$ , from Lemma 1 we obtain the following.

**Lemma 2.** *Let  $g$  be positive on  $[0, 1]$ . If*

$$0 \leq \min\{k_0, k_\infty\}|g|_L < 1, \quad \max\{k_0, k_\infty\} \min_{t \in [0,1]} g(t) > 27, \quad (14)$$

then (12) has both solutions  $z_1 \in \Delta_1$  and  $z_2 \in \Delta_2$ .

*Proof.* Fixed  $t_1, t_2 \in (0, 1)$ ,  $t_1 < t_2$ , we have

$$\int_{t_1}^{t_2} g(\tau) d\tau \geq (t_2 - t_1) \min_{t \in [0,1]} g(t).$$

Thus, the second condition in (13) is satisfied if

$$\max\{k_0, k_\infty\} \min_{t \in [0,1]} g(t) \geq \frac{1}{t_1(1-t_2)(t_2-t_1)}$$

for a suitable choice of  $t_1, t_2$ . Put  $\rho = \varrho(t_1, t_2) = t_1(1-t_2)(t_2-t_1)$ , it is easily checked that  $\varrho$  takes its maximum  $1/27$  on the region  $0 \leq t_1 < t_2 \leq 1$  when  $t_1 = 1/3, t_2 = 2/3$ . Therefore, the second inequality in (14) follows.  $\square$

Define for  $t \in [0, 1]$

$$\tau(t) = \frac{A(t)}{A(1)}, \quad (15)$$

where  $A$  is given in (7). Thus,  $\tau$  maps the interval  $[0, 1]$  into itself. Let  $x$  be a solution of (1) on  $[0, 1]$  and put  $z(\tau) = x(t(\tau))$ , where  $t(\tau)$  is the inverse function of  $\tau(t)$ . Then,  $z$  is a solution on  $[0, 1]$  of

$$\frac{d^2 z}{d\tau^2} + \tilde{b}(\tau)F(z) = 0, \quad (16)$$

where  $\tilde{b}(\tau) = A^2(1)a(t(\tau))b(t(\tau))$ . Vice versa, if  $z$  is a solution of (16) on  $[0, 1]$ , then  $x(t) = z(\tau(t))$  is a solution of (1) on the same interval. Moreover, it is easy to show that  $x$  belongs to  $\Delta_i$  if and only if  $z \in \Delta_i, i = 1, 2$ . Hence, Lemmas 1 and 2 read for (1) as follow.

**Proposition 1.** *Assume that one of the following conditions is satisfied.*

(i) *There exist  $t_1, t_2 \in (0, 1)$ ,  $t_1 < t_2$  such that  $\int_{t_1}^{t_2} b(t) dt > 0$ , and*

$$0 \leq \min\{k_0, k_\infty\} A(1) |b|_L < 1,$$

$$\max\{k_0, k_\infty\} \int_{t_1}^{t_2} b(t) dt > \frac{A(1)}{A(t_1)(A(1) - A(t_2))}.$$

(ii)  *$b(t) > 0$  on  $[0, 1]$  and*

$$0 \leq \min\{k_0, k_\infty\} A(1) |b|_L < 1,$$

$$27 < \max\{k_0, k_\infty\} A(1) \min_{t \in [0, 1]} b(t).$$

*Then (1) has both solutions  $x_1 \in \Delta_1$  and  $x_2 \in \Delta_2$ .*

*Proof.* Since

$$\int_0^1 \tilde{b}(\tau) d\tau = A^2(1) \int_0^1 b(t(\tau))a(t(\tau)) d\tau = A(1) \int_0^1 b(t) dt = A(1) |b|_L$$

and

$$\int_{t_1}^{t_2} b(t) dt = \frac{1}{A(1)} \int_{\tau_1}^{\tau_2} \tilde{b}(\tau) d\tau,$$

where  $\tau_i = \tau(t_i) = A(t_i)/A(1)$ ,  $i = 1, 2$ , the assertion follows from Lemmas 1 and 2. □

Other sufficient conditions for the existence of solutions of (1) in the sets  $\Delta_1$  and  $\Delta_2$ , can be obtained in a similar way from other results in [22].

### 3 Principal solutions and disconjugacy

Consider the linear equation

$$(a(t)y')' + \beta(t)y = 0, \tag{17}$$

where  $\beta$  is a continuous function for  $t \geq T \geq 0$ . In our study, an important role is played by the disconjugacy property and the notion of principal solutions for (17).

We recall that (17) is said to be *disconjugate* on an interval  $I \subset [T, \infty)$  if any nontrivial solution of (17) has at most one zero on  $I$ . We refer to [13, 20] and references therein for basic properties of disconjugacy. In particular, the following results will be useful in the sequel.

**Lemma 3.** *Let  $T_1 \geq T$ . The following statements are equivalent.*

- (i<sub>1</sub>) *Equation (17) is disconjugate on  $[T_1, \infty)$ ;*
- (i<sub>2</sub>) *Equation (17) is disconjugate on  $(T_1, \infty)$ ;*
- (i<sub>3</sub>) *Equation (17) has a solution without zeros on  $(T_1, \infty)$ .*

*Proof.* (i<sub>1</sub>)  $\iff$  (i<sub>2</sub>). If (17) is disconjugate on  $[T_1, \infty)$ , then it is disconjugate on  $(T_1, \infty)$ . The vice versa follows from [13, Theorem 2, Chapt.1], with minor changes. Finally, (i<sub>2</sub>)  $\iff$  (i<sub>3</sub>) follows from [20, Corollary 6.1].  $\square$

The concept of principal solution was introduced in 1936 by W. Leighton and M. Morse and, later on, analyzed by P. Hartman and A. Wintner, see, e.g., [20, Chapter 11]. If (17) is nonoscillatory, then there exists a solution  $u_0$  of (17), which is uniquely determined up to a constant factor by one of the following conditions (in which  $u$  denotes an arbitrary solution of (17), linearly independent of  $u_0$ ):

$$\lim_{t \rightarrow \infty} \frac{u_0(t)}{u(t)} = 0, \tag{18}$$

$$\frac{u'_0(t)}{u_0(t)} < \frac{u'(t)}{u(t)} \quad \text{for large } t,$$

$$\int_{t_u}^{\infty} \frac{dt}{a(t)u_0^2(t)} = \infty, \tag{19}$$

where  $t_u \geq T$  is such that  $u_0(t) \neq 0$  on  $[t_u, \infty)$ . The solution  $u_0$  is called *principal solution* of (17) and any solution  $u$  of (17), which is linearly independent of  $u_0$ , is called a *nonprincipal solution* of (17). Property (18) is the simplest and most typical property characterizing principal solutions, because, roughly speaking, it means that the principal solution is the smallest one in a neighborhood of infinity.

**Remark 1.** If (17) is disconjugate on  $[T_1, \infty)$ ,  $T_1 \geq T$ , then any principal solution of (17) does not have zeros on  $(T_1, \infty)$ , see [20, Chapter XI, Exercise 6.6]. Thus, a necessary condition for positiveness of the principal solution on the open interval  $(T, \infty)$ , is the disconjugacy of the equation. Nevertheless,

disconjugacy cannot be sufficient for the positiveness of principal solution on the close half-line  $[T, \infty)$ , as the following example shows.

**Example 1.** Consider the equation

$$(a(t)y')' + y = 0, \quad t \geq 0, \quad (20)$$

where  $a(1) = 1$  and

$$a(t) = \frac{1 + t - 2e^{t-1}}{1 - t} \text{ if } t \neq 1.$$

Hence,  $a$  is a positive continuous function on  $[0, \infty)$  and (3) holds for  $a$ . Using (19), we get that  $y_0(t) = te^{-t}$  is the principal solution of (20). Moreover, in view of Lemma 3, equation (20) is disconjugate on  $[0, \infty)$ .

Consider now the special case  $\beta(t) \equiv M > 0$  in (17), i.e. the equation

$$(a(t)y')' + My = 0. \quad (L)$$

In view of Example 1, the disconjugacy of (L) on  $[T, \infty)$  does not guarantee the positiveness of principal solution at the initial point  $t = T$ . To obtain this additional property, consider the so-called *dual equation to (L)*, that is the equation

$$v'' + \frac{M}{a(t)}v = 0, \quad (D)$$

which is obtained from (L) by the change of variable  $v(t) = a(t)y'(t)$ . The dual equation has been often used in the literature for studying oscillatory properties of second order self-adjoint linear equations, see, e.g., [8, 9, 25], and, for the half-linear case, [10, 17].

The following necessary and sufficient condition for the disconjugacy of (D) holds, see also [20, page 352].

**Lemma 4.** *Equation (D) is disconjugate on  $[T, \infty)$  if and only if (D) has a solution  $v_0$  such that  $v_0(t) > 0$  on  $(T, \infty)$  and  $v'_0(t) > 0$  on  $[T, \infty)$ .*

*Proof.* Assume that (D) is disconjugate on  $[T, \infty)$ . From Lemma 3 there exists a solution  $v_0$  of (D) such that  $v_0(t) > 0$  for  $t > T$ . Thus,  $v'_0$  is decreasing for  $t > T$ . We claim that  $v'_0(t) > 0$  on the whole interval  $[T, \infty)$ . By contradiction, if  $v'_0$  has a zero on  $[T, \infty)$ , then there exists  $t_1 > T$  such that  $v'_0(t) \leq v'_0(t_1) < 0$  for  $t \geq t_1$ . Integrating this inequality we get

$v_0(t) \leq v_0(t_1) + v_0'(t_1)(t - t_1)$ , which gives a contradiction with the positive-ness of  $v_0$  when  $t$  tends to infinity. The opposite statement follows again in virtue of Lemma 3.  $\square$

From Lemma 4 we obtain the following.

**Lemma 5.** *If (D) is disconjugate on  $[T, \infty)$ , then (L) has a principal solution  $y_0$  such that  $y_0(t) > 0$  on  $[T, \infty)$  and  $y_0'(t) < 0$  on  $(T, \infty)$ .*

*Proof.* In view of Lemma 4 and the change of variable  $y(t) = v'(t)$ , equation (L) has a solution  $y_0$  which satisfies  $y_0(t) > 0$  on  $[T, \infty)$  and  $y_0'(t) < 0$  on  $(T, \infty)$ . Hence, the disconjugacy of (L) follows from Lemma 3. If  $y_0$  is not principal solution, from [20, Corollary 6.3] the solution  $\bar{y}$  given by

$$\bar{y}(t) = y_0(t) \int_t^\infty \frac{ds}{a(s)y_0^2(s)},$$

is the desired principal solution of (L).  $\square$

**Remark 2.** Example 1 shows that the assumption on disconjugacy of (D) in Lemma 5 cannot be replaced by the disconjugacy of (L). Moreover, observe that the dual equation of (20) is

$$v'' + a^{-1}(t)v = 0, \tag{21}$$

where  $a$  is defined in Example 1. It is easy to verify that the function  $v_0(t) = 2e^{-1} - (1+t)e^{-t}$  is a principal solution of (21). Since  $v_0(1) = 0$ , any principal solution of (21) has a zero at  $t = 1$ . Consequently, (21) is not disconjugate on  $[0, \infty)$ .

## 4 An auxiliary BVP on $[1, \infty)$

For any  $c > 0$ , consider for  $t \geq 1$  the existence of solutions  $x$  of (1) which satisfy the boundary conditions

$$x(1) = c, \quad x'(1) \leq 0, \quad x(t) > 0 \text{ on } [1, \infty), \quad \lim_{t \rightarrow \infty} x(t) = 0. \tag{22}$$

The solvability of this BVP is based on a general fixed point theorem for operators defined in a Fréchet space, see [11, Theorem 1.3]. In particular, this result reduces the existence of solutions of a BVP for differential equations on noncompact intervals to the existence of suitable *a-priori* bounds and it is mainly useful when the associated fixed point operator is not known in an explicit form. We recall this result in the form that will be used.

**Theorem 2.** Consider the BVP on  $[T, \infty)$ ,  $T \geq 0$ ,

$$(a(t)x')' + b(t)F(x) = 0, \quad x \in S, \quad (23)$$

where  $S$  is a nonempty subset of the Fréchet space  $C[T, \infty)$ . Let  $G$  be a continuous function on  $\mathbb{R}^2$ , such that  $F(d) = G(d, d)$  for any  $d \in \mathbb{R}$  and assume that there exist a nonempty, closed, convex and bounded subset  $\Omega \subset C[T, \infty)$  such that for any  $u \in \Omega$  the BVP on  $[T, \infty)$

$$(a(t)x')' + b(t)G(u(t), x(t)) = 0, \quad x \in S$$

admits a unique solution  $x_u$ . Let  $\Psi$  be the operator  $\Omega \rightarrow C[T, \infty)$ , such that  $\Psi(u) = x_u$ . Assume

- (i<sub>1</sub>)  $\Psi(\Omega) \subset \Omega$ ;
- (i<sub>2</sub>) if  $\{u_n\} \subset \Omega$  is a sequence converging in  $\Omega$  and  $\Psi(u_n) \rightarrow x$ , then  $x \in S$ .

Then  $\Psi$  has a fixed point in  $\Omega$ , which is a solution of the BVP (23).

Let  $\tilde{F}$  be the function

$$\tilde{F}(v) = \frac{F(v)}{v} \text{ if } v > 0, \quad \tilde{F}(0) = k_0, \quad (24)$$

where  $k_0$  is defined in (6) and set  $b_+(t) = \max\{b(t), 0\}$ ,  $b_-(t) = -\min\{b(t), 0\}$ . Thus  $b(t) = b_+(t) - b_-(t)$ . The following holds.

**Theorem 3.** Assume that equation (8) is disconjugate on  $[1, \infty)$ . Then, for any  $c > 0$ , equation (1) has a unique globally positive decreasing solution  $x$  on  $[1, \infty)$  satisfying (22) and (11).

*Proof.* Fixed  $c > 0$ , consider the equations

$$(a(t)y')' + By = 0, \quad (25)$$

$$(a(t)w')' - b_-(t)w = 0. \quad (26)$$

From Lemma 5, equation (25) is disconjugate on  $[1, \infty)$  and has a principal solution  $y_0$  such that  $y_0(1) = c$ ,  $y_0(t) > 0$  on  $[1, \infty)$ ,  $y_0'(t) < 0$  on  $(1, \infty)$ . Moreover, from [9, Theorem 1] we obtain  $\lim_{t \rightarrow \infty} y_0(t) = 0$ .

Since  $-b_-(t) \leq 0$ , equation (25) is a Sturm majorant for (26). Thus (26) has a positive principal solution  $w_0$  such that  $w_0(1) = c$ ,  $w_0'(t) \leq 0$  for  $t \geq 1$ ,

see, e.g., [20, Corollary 6.4]. Using the comparison result for the principal solutions, see e.g. [20, Corollary 6.5], we get on  $(1, \infty)$

$$\frac{w'_0(t)}{w_0(t)} \leq \frac{y'_0(t)}{y_0(t)}$$

and so  $0 < w_0(t) \leq y_0(t)$  for  $t \geq 1$ .

Let  $\Omega$  and  $S$  be the subsets of the Fréchet space  $C[1, \infty)$  given by

$$\Omega = \left\{ u \in C[1, \infty), \frac{1}{2}w_0(t) \leq u(t) \leq y_0(t) \right\},$$

$$S = \left\{ x \in C[1, \infty), x(1) = c, x(t) > 0, \int_1^\infty \frac{1}{a(t)x^2(t)} dt = \infty \right\},$$

respectively.

For any  $u \in \Omega$  consider the linear equation

$$(a(t)x')' + b(t)\tilde{F}(u(t))x(t) = 0, \quad (27)$$

where  $\tilde{F}$  is given in (24). In view of (5), we have  $\sup_{v \geq 0} \tilde{F}(v) \leq 1$ . Hence, (25) is a majorant for (27). Thus, using again the comparison result [20, Corollary 6.5], equation (27) has a unique positive principal solution  $x_u$ , such that  $x_u(1) = c$ , and for  $t > 1$

$$\frac{x'_u(t)}{x_u(t)} \leq \frac{y'_0(t)}{y_0(t)}.$$

Hence, taking into account that  $y_0$  is decreasing to zero as  $t$  tends to infinity, we get

$$\begin{aligned} 0 < x_u(t) &\leq y_0(t) \quad \text{on } [1, \infty), \\ \lim_{t \rightarrow \infty} x_u(t) &= 0, \quad x'_u(t) < 0 \quad \text{on } (1, \infty). \end{aligned} \quad (28)$$

Thus, for any  $u \in \Omega$ , equation (27) has a solution  $x_u \in S$ , which is unique in view of (19).

Denote by  $\Psi : \Omega \rightarrow C[1, \infty)$  the operator

$$\Psi(u) = x_u.$$

Using again the comparison result [20, Corollary 6.5] for equations (27) and (26), we obtain for any  $u \in \Omega$  and  $t \geq 1$

$$\frac{w'_0(t)}{w_0(t)} \leq \frac{x'_u(t)}{x_u(t)}. \quad (29)$$

Then, in view of (28) we get for any  $u \in \Omega$  and  $t \geq 1$

$$w_0(t) \leq x_u(t) \leq y_0(t),$$

i.e., the operator  $\Psi$  maps  $\Omega$  into itself.

Now, let  $\{u_n\} \subset \Omega$  be a sequence converging in  $\Omega$  and  $x_{u_n} = \Psi(u_n) \rightarrow x$ . Clearly  $x(1) = c$ . Since  $\overline{\Psi(\Omega)} \subset \Omega$ , we get  $x(t) > 0$ . Moreover, since  $y_0$  is a principal solution of (25), from (28) we obtain

$$\int_1^\infty \frac{1}{a(t)x^2(t)} dt \geq \int_1^\infty \frac{1}{a(t)y_0^2(t)} dt = \infty.$$

Thus,  $x \in S$  and, by Theorem 2, there exists a fixed point  $\bar{x}$  of  $\Psi$  in  $\Omega$ . Clearly,  $\bar{x}$  is a solution of (1) on  $[1, \infty)$  and  $\bar{x}(1) = c$ . Since  $\bar{x}$  is also a principal solution of (27) with  $u = \bar{x}$ , from (28) we get  $\bar{x}(t) > 0$ ,  $\bar{x}'(t) < 0$  for  $t > 1$ ,  $\bar{x}'(1) \leq 0$  and  $\lim_{t \rightarrow \infty} \bar{x}(t) = 0$ . Thus  $\bar{x}$  is positive decreasing on  $(1, \infty)$  and satisfies (22) and (11).

Finally, it remains to verify that (1) has a unique solution which satisfies (22). Let  $x, v$  be two positive solutions of (1) defined on  $[1, \infty)$  and satisfying (22). In view of the first part of the proof, we can suppose also that

$$\int_1^\infty \frac{dt}{a(t)x^2(t)} = \infty. \quad (30)$$

Denote by  $\Phi(u, v)$  the function ( $u \geq 0, v \geq 0$ )

$$\Phi(u, v) = \begin{cases} (F(u) - F(v))/(u - v) & \text{if } u \neq v \\ dF(u)/du & \text{if } u = v \end{cases}$$

and set  $z(t) = x(t) - v(t)$ . Thus,  $z$  is a solution of the equation

$$(a(t)z')' + b(t)\bar{\Phi}(t)z = 0, \quad (31)$$

where  $\bar{\Phi}(t) = \Phi(x(t), v(t))$ . In virtue of (5), we have

$$b(t)\bar{\Phi}(t) \leq B.$$

Since, from Lemma 5, equation (25) is disconjugate on  $[1, \infty)$ , the equation (31) is disconjugate on  $[1, \infty)$  too. Since  $z(1) = 0$ , the solution  $z$  does not have zeros for  $t > 1$  and so, without loss of generality, we can suppose  $z(t) > 0$

for  $t > 1$ . Because  $\lim_{t \rightarrow \infty} z(t) = 0$ , there exists  $t_1 > 1$  such that  $z'(t_1) = 0$ . Moreover, taking into account that  $x$  satisfies (30) and  $z(t) < x(t)$ , we get that  $z$  is a principal solution of (31). Using again the comparison result [20, Corollary 6.5] for equations (25) and (31), we obtain for  $t > 1$

$$\frac{z'(t)}{z(t)} \leq \frac{y_0'(t)}{y_0(t)}, \quad (32)$$

where  $y_0$  is the positive decreasing principal solution of (25) defined in the first part of the proof. Thus, the inequality (32) gives a contradiction at  $t = t_1$ , because

$$\frac{y_0'(t_1)}{y_0(t_1)} < 0.$$

□

We conclude this section with the following continuity result for starting points of solutions of (1) which satisfy (22).

**Theorem 4.** *Assume that equation (8) is disconjugate on  $[1, \infty)$ . Let  $\{c_n\}$  be a positive sequence converging to zero and denote by  $x_n$  the unique solution of (1) which satisfies (22) with  $c_n = c$ . Then the sequence  $\{x_n'(1)\}$  converges to zero.*

*Proof.* In virtue of Theorem 3, for any  $c_n > 0$ , equation (1) has a unique solution  $x_n$  which satisfies (22) with  $c_n = c$ . Denote by  $w_n$  the principal solution of (26) such that  $w_n(1) = c_n$ . From (29) and (22) we get

$$w_n'(1) \leq x_n'(1) \leq 0. \quad (33)$$

Since principal solutions are determined up to a constant factor, we have

$$w_n(t) = \frac{c_n}{c_1} w_1(t).$$

Hence  $w_n'(1) = c_n w_1'(1)/c_1$  and from (33) the assertion follows. □

## 5 Proof of the main result

In this section we prove Theorem 1 and we show some its consequences. To this aim, the following generalization of the well known Kneser's theorem (see for instance [12, Section 1.3]), plays a key role.

**Proposition 2.** *Consider the system*

$$z' = F(t, z), \quad (t, z) \in [T_1, T_2] \times \mathbb{R}^n$$

where  $F$  is continuous and bounded, and let  $K_0$  be a continuum (i.e., a compact and connected subset) of  $\{(T_1, w) : w \in \mathbb{R}^n\}$ . Let  $\mathcal{Z}(K_0)$  be the family of all the solutions emanating from  $K_0$ . If any solution  $z \in \mathcal{Z}(K_0)$  is defined on the whole interval  $[T_1, T_2]$ , then the cross-section  $\mathcal{Z}(T_2; K_0) = \{z(T_2) : z \in \mathcal{Z}(K_0)\}$  is a continuum in  $\mathbb{R}^n$ .

*Proof of Theorem 1.* Let  $x_1 \in \Delta_1$  and  $x_2 \in \Delta_2$  be the solutions on  $[0, 1]$  of (1), whose existence is guaranteed by Proposition 1, and let  $\alpha = \max\{x'_1(0), x'_2(0)\} > 0$ ,  $\beta = \min\{x'_1(0), x'_2(0)\} > 0$ . Put

$$L = \alpha a(0)A(1) \tag{34}$$

and let  $\widehat{F}$  be a Lipschitz function on  $\mathbb{R}$  such that

$$\widehat{F}(u) = \begin{cases} 0, & u < 0 \\ F(u), & 0 \leq u \leq L \\ F(L), & u > L \end{cases} .$$

For  $\ell \in (0, \alpha]$ , consider the Cauchy problem

$$\begin{cases} (a(t)x')' + b(t)\widehat{F}(x) = 0, & t \in [0, 1] \\ x(0) = 0, x'(0) = \ell, \end{cases} \tag{35}$$

and denote by  $x_\ell$  the unique solution of (35). Let us show that  $x_\ell$  is defined on the whole interval  $[0, 1]$ . For any solution  $x$  of the equation in (35), the function  $a(\cdot)x'(\cdot)$  is nonincreasing, so  $a(t)x'_\ell(t) \leq a(0)x'_\ell(0) = a(0)\ell$ . Integrating this inequality, in view of (34) we get for  $t \in [0, 1]$

$$x_\ell(t) \leq a(0) \ell \int_0^t \frac{1}{a(s)} ds \leq a(0) \ell A(1) \leq L.$$

Assume now that  $x_\ell(t) > 0$  on  $(0, t_1)$ ,  $0 < t_1 \leq 1$ , and  $x_\ell(t_1) = 0$ . Then, in virtue of the uniqueness of the Cauchy problem (35), we obtain  $x'_\ell(t_1) < 0$ . If  $t_1 < 1$ , then  $x_\ell(t) < 0$  in a right neighborhood of  $t_1$  and satisfies  $(a(t)x'_\ell)' = 0$ ,

which gives  $x_\ell(t) < 0$  for every  $t \geq t_1$  for which this solution exists. Since  $x'_\ell(t_1) < 0$ , by integration we obtain for  $t > t_1$

$$x_\ell(t) = a(t_1)x'_\ell(t_1) \int_{t_1}^t \frac{1}{a(s)} ds > a(t_1)x'_\ell(t_1)A(1),$$

that is,  $x_\ell$  is bounded from below. Therefore the solution  $x_\ell$  of (35) is defined on the whole interval  $[0, 1]$ .

Let  $x$  be any solution of (1), nonnegative on  $[0, 1]$  and satisfying  $x(0) = 0$ ,  $x'(0) = \ell \in (0, \alpha]$ . Then  $x$  is also a solution of (35) for  $0 \leq t \leq 1$ , and vice versa. Indeed, reasoning as above, we obtain  $x(t) \leq L$  on  $[0, 1]$  and therefore  $F(x(t)) = \widehat{F}(x(t))$  for all  $t \in [0, 1]$ .

Put  $K_0 = \{(x(1), x'(1)) : x \text{ is solution of (35) with } \ell \in [\beta, \alpha]\}$ . Since any solution of (35) is defined on the whole  $[0, 1]$ , by Proposition 2 the set  $K_0$  is a continuum in  $\mathbb{R}^2$ , containing the points  $(0, x'_1(1))$ ,  $(x_2(1), 0)$ , with  $x'_1(1) < 0$ ,  $x_2(1) > 0$ . Further,  $K_0$  does not contain any point  $(0, c)$  with  $c \geq 0$ . Therefore a continuum  $K_1 \subseteq K_0$  exists,  $K_1 \subseteq \bar{\pi} = \{(u, v) : u \geq 0, v \leq 0\}$ ,  $(0, 0) \notin K_1$ , and there exist two points  $P, Q \in K_1$ ,  $P = (p, 0)$ ,  $Q = (0, -q)$ ,  $p > 0, q > 0$ .

In order to complete the proof, we use a similar argument to the one given in [23, Theorem 1.1], with minor changes. Consider equation (1) for  $t \geq 1$ . By Theorem 3, for every  $c > 0$ , (1) has a unique positive decreasing solution  $x$  satisfying (22) and (11). Then, the set  $S_1$  of the initial data of the solutions of (1) on  $[1, \infty)$  satisfying (11) and (22) is connected,  $S_1 \subset \bar{\pi}$ , and its projection on the first component is the half-line  $(0, \infty)$ . Further, from Theorem 4,  $(0, 0) \in \bar{S}_1$ . Therefore we have

$$K_1 \cap S_1 \neq \emptyset.$$

Let us show that each point  $(c, d) \in K_1 \cap S_1$  corresponds to a solution of the BVP (1)-(2). Let  $(c, d) \in K_1 \cap S_1$ . Then  $c > 0, d \leq 0$ . Since  $(c, d) \in K_1$ , there exists a solution  $u$  of (35), for a suitable  $\ell \in [\beta, \alpha]$ , such that  $u(1) = c > 0$  and  $u'(1) = d \leq 0$ . Since  $u(1) > 0$  we have  $u(t) > 0$  on  $(0, 1]$ . Therefore  $u$  is also a solution of (1) in  $[0, 1]$ , with  $u(0) = 0$ ,  $u(t) > 0$  for  $t \in (0, 1]$ . As  $(c, d) \in S_1$ , a positive decreasing solution  $v$  of (1) exists on  $[1, \infty)$ , which satisfies (22) and  $v(1) = c = u(1)$ ,  $v'(1) = d = u'(1)$ . Hence, the function

$$x(t) = \begin{cases} u(t), & t \in [0, 1], \\ v(t), & t > 1. \end{cases}$$

is a solution of the BVP (1)-(2) and the proof is complete.  $\square$

From Theorem 1 and Proposition 1, we get the following.

**Corollary 1.** *Let assumptions of Proposition 1-(ii) be satisfied and equation (8) be disconjugate on  $[1, \infty)$ . Then the BVP (1)-(2) has a solution.*

We close this section with the solvability of our BVP for the perturbed equation

$$(a(t)z')' + (b(t) + b_1(t))F(z) = 0, \quad (36)$$

where  $b_1$  is a continuous function for  $t \geq 0$  such that  $b_1(t) \equiv 0$  on  $[0, 1]$  and  $b_1(t) \leq 0$  for  $t > 1$ .

**Corollary 2.** *If assumptions of Theorem 1 are satisfied, then equation (36) has a solution  $z$  satisfying boundary conditions (2).*

## 6 Examples and concluding remarks

Theorem 1 is illustrated by the following example.

**Example 2.** Consider the equation

$$(a(t)x')' + b(t) F(x) = 0, \quad (37)$$

where

$$a(t) = (1+t)^2, \quad b(t) = \frac{1}{5e} \exp\left(\frac{16}{1+16t^4}\right) \cos\left(\frac{\pi t}{2}\right) \quad \text{for } t \geq 0. \quad (38)$$

and  $F$  satisfies (5) and (6) with

$$k_0 = \frac{9}{e^{15}}, \quad k_\infty = 1.$$

Since  $b$  is decreasing on  $[0, 1]$ , we get

$$\int_{1/3}^{1/2} b(\tau) d\tau \geq \frac{1}{6} b(1/2) = \frac{\sqrt{2}}{60} e^7.$$

For equation (37), the function  $A$  in (7) becomes

$$A(t) = \frac{t}{1+t},$$

so assumptions (9), (10) are verified for  $t_1 = 1/3$  and  $t_2 = 1/2$ , because

$$A(1)|b|_L \geq \frac{1}{2}b(0) = \frac{e^{15}}{10},$$

$$\frac{A(1)}{A(t_1)(A(1) - A(t_2))} = 12 < \frac{\sqrt{2}}{60}e^7.$$

Finally, for  $t \in [1, \infty)$  we have

$$b(t) \leq \frac{1}{5e} < \frac{1}{4}$$

and the equation (8) becomes the Euler equation

$$v'' + \frac{1}{4(1+t)^2}v = 0,$$

which is disconjugate on  $[1, \infty)$ , see, e.g., [25, Chapter 2.1]. Hence, in view of Theorem 1, equation (37) has solutions  $x$  which satisfy the boundary conditions (2) and

$$\int_1^\infty \frac{1}{(1+t)^2 x^2(t)} dt = \infty.$$

**Remark 3.** Example 2 can be slightly modified for the nonlinearity

$$F(u) = \frac{u^2}{1+u}$$

or the nonlinearity

$$F(u) = \frac{u}{1+\sqrt{u}}.$$

**Remark 4.** Consider the equation

$$(a(t)x')' + (b(t) + b_1(t))F(x) = 0, \quad (39)$$

where the functions  $a, b$  are given in (38),  $b_1$  is the function

$$b_1(t) = (e - e^t)(|\cos t| - \cos t) \text{ for } t \geq 1, \quad b_1(t) \equiv 0 \text{ for } t \in [0, 1),$$

and  $F$  is as in Example 2. Since  $b_1(t) \leq 0$ , in view of Corollary 2, equation (39) has solutions  $x$  which satisfy the boundary conditions (2).

**Remark 5.** Theorem 1 and Corollaries 1, 2 continue to hold if the assumption (5) is replaced by the more general condition

$$\exists K > 0 : 0 \leq \frac{dF(u)}{du} \leq K \quad \text{for } u \geq 0$$

and the disconjugacy of (8) is substituted by the disconjugacy on  $[1, \infty)$  of the linear equation

$$v'' + \frac{BK}{a(t)}v = 0.$$

**Remark 6.** The assumption  $k_0 \neq k_\infty$  implies that  $F$  cannot be a linear function on  $[0, \infty)$ . If the linear equation

$$(a(t)x')' + b(t)x = 0 \tag{40}$$

has a solution  $x$  satisfying (2), then in virtue of Lemma 3, (40) is disconjugate on  $[0, \infty)$ . However,  $x$  is not necessarily the principal solution of (40). The following example illustrates this case.

**Example 3.** Consider the equation

$$(e^{2t}x')' + e^{2t}x = 0, \quad t \geq 0 \tag{41}$$

A standard calculation shows that  $x_0(t) = e^{-t}$ ,  $x_1(t) = te^{-t}$  are solutions of (41). Obviously,  $x_1$  satisfies (2). Observe that  $x_1$  is a nonprincipal solution and  $x_0$  is the principal solution.

In a forthcoming paper we will consider this kind of BVPs for nonlinear equations for which  $k_0 = k_\infty$ .

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