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Three-region inequalities for the second order elliptic equation with discontinuous coefficients and size estimate

E. Francini*  C.-L. Lin†  S. Vessella‡  J.-N. Wang§

Abstract

In this paper, we would like to derive a quantitative uniqueness estimate, the three-region inequality, for the second order elliptic equation with jump discontinuous coefficients. The derivation of the inequality relies on the Carleman estimate proved in our previous work [5]. We then apply the three-region inequality to study the size estimate problem with one boundary measurement.

1 Introduction

In this work we aim to study the size estimate problem with one measurement when the background conductivity has jump interfaces. A typical application of this study is to estimate the size of a cancerous tumor inside an organ by the electric impedance tomography (EIT). In this case, considering discontinuous medium is typical, for instance, the conductivities of heart, liver, intestines are 0.70 (S/m), 0.10 (S/m), 0.03 (S/m), respectively. Previous works on this problem assumed that the conductivity of the studied body is Lipschitz continuous, see, for example, [3, 4]. The first result on the size estimate problem with a discontinuous background conductivity was given in [18], where only the two dimensional case was considered. In this paper, we will study the problem in dimension $n \geq 2$.

The main ingredients of our method are quantitative uniqueness estimates for

$$\text{div}(A \nabla u) = 0 \quad \Omega \subset \mathbb{R}^n.$$  

Those estimates are well-known when $A$ is Lipschitz continuous. The derivation of the estimates is based on the Carleman estimate or the frequency function method. For $n = 2$ and $A \in L^\infty$, quantitative uniqueness estimates are obtained via the connection

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between (1.1) and quasiregular mappings. This is the method used in [18]. For \( n \geq 3 \), the connection with quasiregular mappings is not true. Hence we return to the old method – the Carleman estimate, to derive quantitative uniqueness estimates when \( A \) is discontinuous. Precisely, when \( A \) has a \( C^{1,1} \) interface and is Lipschitz away from the interface, a Carleman estimate was obtained in [5] (see [11, 12, 13] for related results). Here we will derive three-region inequalities using this Carleman estimate. The three-region inequality provides us a way to propagate “smallness” across the interface (see also [12] for similar estimates). Relying on the three-region inequality, we then derive bounds of the size of an inclusion with one boundary measurement. For other results on the size estimate, we mention [1] for the isotropic elasticity, [15, 16, 17] for the isotropic/anisotropic thin plate, [7, 6] for the shallow shell.

2 The Carleman estimate

In this section, we would like to describe the Carleman estimate derived in [5]. We first denote \( H_\pm = \chi_{R^n_\pm} \) where \( R^n_\pm = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y \geq 0 \} \) and \( \chi_{R^n_\pm} \) is the characteristic function of \( R^n_\pm \). Let \( u_\pm \in C^\infty(\mathbb{R}^n) \) and define

\[
  u = H_+ u_+ + H_- u_- = \sum_\pm H_\pm u_\pm,
\]

hereafter, \( \sum_\pm a_\pm = a_+ + a_- \), and

\[
  \mathcal{L}(x, y, \partial) u := \sum_\pm H_\pm \text{div}_{x,y}(A_\pm(x,y) \nabla_{x,y} u_\pm),
\]

where

\[
  A_\pm(x,y) = \{a_{ij}^\pm(x,y)\}_{i,j=1}^n, \quad x \in \mathbb{R}^{n-1}, y \in \mathbb{R}
\]

is a Lipschitz symmetric matrix-valued function satisfying, for given constants \( \lambda_0 \in (0, 1], M_0 > 0 \),

\[
  \lambda_0 |z|^2 \leq A_\pm(x,y)z \cdot z \leq \lambda_0^{-1} |z|^2, \quad \forall (x,y) \in \mathbb{R}^n, \forall z \in \mathbb{R}^n
\]

and

\[
  |A_\pm(x',y') - A_\pm(x,y)| \leq M_0(|x' - x| + |y' - y|).
\]

We write

\[
  h_0(x) := u_+(x, 0) - u_-(x, 0), \quad \forall x \in \mathbb{R}^{n-1},
\]

\[
  h_1(x) := A_+(x, 0) \nabla_{x,y} u_+(x, 0) \cdot \nu - A_-(x, 0) \nabla_{x,y} u_-(x, 0) \cdot \nu, \quad \forall x \in \mathbb{R}^{n-1},
\]

where \( \nu = -e_n \).

For a function \( h \in L^2(\mathbb{R}^n) \), we define

\[
  \hat{h}(\xi, y) = \int_{\mathbb{R}^{n-1}} h(x, y) e^{-ix \cdot \xi} \, dx, \quad \xi \in \mathbb{R}^{n-1}.
\]
As usual $H^{1/2}(\mathbb{R}^{n-1})$ denotes the space of the functions $f \in L^2(\mathbb{R}^{n-1})$ satisfying
\[ \int_{\mathbb{R}^{n-1}} |\xi| |\hat{f}(\xi)|^2 d\xi < \infty, \]
with the norm
\[ ||f||_{H^{1/2}(\mathbb{R}^{n-1})}^2 = \int_{\mathbb{R}^{n-1}} (1 + |\xi|^2)^{1/2} |\hat{f}(\xi)|^2 d\xi. \tag{2.7} \]
Moreover we define
\[ [f]_{1/2,\mathbb{R}^{n-1}} = \left[ \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|f(x) - f(y)|^2}{|x-y|^n} dydx \right]^{1/2}, \]
and recall that there is a positive constant $C$, depending only on $n$, such that
\[ C^{-1} \int_{\mathbb{R}^{n-1}} |\xi||\hat{f}(\xi)|^2 d\xi \leq [f]_{1/2,\mathbb{R}^{n-1}}^2 \leq C \int_{\mathbb{R}^{n-1}} |\xi||\hat{f}(\xi)|^2 d\xi, \]
so that the norm (2.7) is equivalent to the norm $||f||_{L^2(\mathbb{R}^{n-1})} + [f]_{1/2,\mathbb{R}^{n-1}}$. From now on, we use the letters $C, C_0, C_1, \cdots$ to denote constants (depending on $\lambda_0, M_0, n$). The value of the constants may change from line to line, but it is always greater than 1. We will denote by $B_r(x)$ the $(n-1)$-ball centered at $x \in \mathbb{R}^{n-1}$ with radius $r > 0$. Whenever $x = 0$ we denote $B_r = B_r(0)$.

**Theorem 2.1** Let $u$ and $A_{\pm}(x,y)$ satisfy (2.3)-(2.4). There exist $L, \beta, \delta_0, r_0, \tau_0$ positive constants, with $r_0 \leq 1$, depending on $\lambda_0, M_0, n$, such that if $\alpha_+ > L \alpha_-, \delta \leq \delta_0$ and $\tau \geq \tau_0$, then
\begin{align*}
&\sum_{\pm} \sum_{|k|=0}^{2} \tau^{-2|k|} \int_{\mathbb{R}^{n \pm}} |D^k u_{\pm}|^2 e^{2\tau \phi_{\pm}(x,y)} dxdy + \sum_{\pm} \sum_{|k|=0}^{1} \tau^{-2|k|} \int_{\mathbb{R}^{n-1}} |D^k u_{\pm}(x,0)|^2 e^{2\tau \phi_{\pm}(x,0)} dx \\
&+ \sum_{\pm} \tau^2 |e^{\tau \phi_{\pm}(\cdot,0)} u_{\pm}(\cdot,0)|_{1/2,\mathbb{R}^{n-1}}^2 + \sum_{\pm} [D(e^{\tau \phi_{\pm}} u_{\pm})(\cdot,0)]_{1/2,\mathbb{R}^{n-1}}^2 \\
\leq C \left( \sum_{\pm} \int_{\mathbb{R}^{n \pm}} |\mathcal{L}(x,y,\theta)(u_{\pm})|^2 e^{2\tau \phi_{\pm}(x,y)} dxdy + [e^{\tau \phi_{\pm}} h_1]_{1/2,\mathbb{R}^{n-1}}^2 \\
+ [\nabla_x(e^{\tau \phi_{\pm}} h_0)(\cdot,0)]_{1/2,\mathbb{R}^{n-1}}^2 + \tau^3 \int_{\mathbb{R}^{n-1}} |h_0|^2 e^{2\tau \phi_{\pm}(x,0)} dx + \tau \int_{\mathbb{R}^{n-1}} |h_1|^2 e^{2\tau \phi_{\pm}(x,0)} dx \right), \tag{2.8}
\end{align*}
where $u = H_+ u_+ + H_- u_-, u_{\pm} \in C^\infty(\mathbb{R}^n)$ and supp $u \subset B_{r/2} \times [-\delta r_0, \delta r_0]$, and $\phi_{\pm}(x,y)$ is given by
\begin{align*}
\phi_{\pm}(x,y) &= \begin{cases} 
\frac{\alpha_+ y}{\delta} + \frac{\beta y^2}{2\delta^2} - \frac{|x|^2}{2\delta}, & y \geq 0, \\
\frac{\alpha_- y}{\delta} + \frac{\beta y^2}{2\delta^2} - \frac{|x|^2}{2\delta}, & y < 0,
\end{cases} \tag{2.9}
\end{align*}
and $\phi_{\pm}(x,0) = \phi_{\pm}(x,0) = \phi_{\mp}(x,0)$. 

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Remark 2.2 It is clear that (2.8) remains valid if we can add lower order terms
\[ \sum_{\pm} H_{\pm} (W \cdot \nabla_{x,y} u_{\pm} + V u_{\pm}), \]
where \( W, V \) are bounded functions, to the operator \( \mathcal{L} \). That is, one can substitute
\[ \mathcal{L}(x, y, \partial) u = \sum_{\pm} H_{\pm} \text{div}_{x,y}(A_{\pm}(x, y) \nabla_{x,y} u_{\pm}) + \sum_{\pm} H_{\pm} (W \cdot \nabla_{x,y} u_{\pm} + V u_{\pm}) \] (2.10)
in (2.8).

3 Three-region inequalities

Based on the Carleman estimate given in Theorem 2.1, we will derive three-region inequalities across the interface \( y = 0 \). Here we consider \( u = H_+ u_+ + H_- u_- \) satisfying
\[ \mathcal{L}(x, y, \partial) u = 0 \quad \text{in} \quad \mathbb{R}^n, \]
where \( \mathcal{L} \) is given in (2.10) and
\[ \|W\|_{L^\infty(\mathbb{R}^n)} + \|V\|_{L^\infty(\mathbb{R}^n)} \leq \lambda_0^{-1}. \]
Fix any \( \delta \leq \delta_0 \), where \( \delta_0 \) is given in Theorem 2.1.

**Theorem 3.1** Let \( u \) and \( A_{\pm}(x, y) \) satisfy (2.3)-(2.4) with \( h_0 = h_1 = 0 \). Then there exist \( C \) and \( R \), depending only on \( \lambda_0, M_0, n \), such that if \( 0 < R_1, R_2 \leq R \), then
\[ \int_{U_2} |u|^2 \, dx \leq (e^{\tau_0 R_2} + CR_1^{-4}) \left( \int_{U_1} |u|^2 \, dxdy \right)^{\frac{R_2}{2R_1 + 3\delta^2}} \left( \int_{U_3} |u|^2 \, dxdy \right)^{\frac{2R_2}{2R_1 + 3\delta^2}}, \] (3.1)
where \( \tau_0 \) is the constant derived in Theorem 2.1,
\[ U_1 = \{ z \geq -4R_2, \frac{R_1}{8a} < y < \frac{R_1}{a} \}, \]
\[ U_2 = \{ -R_2 \leq z \leq \frac{R_1}{2a}, y < \frac{R_1}{8a} \}, \]
\[ U_3 = \{ z \geq -4R_2, y < \frac{R_1}{a} \}, \]
a = \( \alpha_+ / \delta \), and
\[ z(x, y) = \frac{\alpha_- y}{\delta} + \frac{\beta y^2}{2\delta^2} - \frac{|x|^2}{2\delta}. \] (3.2)

**Proof.** To apply the estimate (2.8), \( u \) needs to satisfy the support condition. Also, we can choose \( \alpha_+ \) and \( \alpha_- \) in Theorem 2.1 such that \( \alpha_+ > \alpha_- \). We can choose \( r \leq r_0 \) satisfying
\[ r \leq \min \left\{ \frac{13\alpha_-}{8\beta}, \frac{2\delta}{19\alpha_- + 8\beta} \right\}. \] (3.3)
Figure 1: $U_1$ and $U_2$ are shown in green and red, respectively. $U_3$ is the region enclosed by blue boundaries.

Note that the choices of $\delta, r$ also depend on $\lambda_0, M_0, n$. We then set

$$R = \frac{\alpha - r}{16}.$$  \hfill (3.3)

It follows from (3.3) that

$$R \leq \frac{13\alpha^2}{128\beta}. \hfill (3.4)$$

Given $0 < R_1 < R_2 \leq R$. Let $\vartheta_1(t) \in C_0^\infty(\mathbb{R})$ satisfy $0 \leq \vartheta_1(t) \leq 1$ and

$$\vartheta_1(t) = \begin{cases} 
1, & t > -2R_2, \\
0, & t \leq -3R_2. 
\end{cases}$$

Also, define $\vartheta_2(y) \in C_0^\infty(\mathbb{R})$ satisfying $0 \leq \vartheta_2(y) \leq 1$ and

$$\vartheta_2(y) = \begin{cases} 
0, & y \geq \frac{R_1}{2a}, \\
1, & y < \frac{R_1}{4a}. 
\end{cases}$$

Finally, we define $\vartheta(x, y) = \vartheta_1(z(x, y))\vartheta_2(y)$, where $z$ is defined by (3.2).

We now check the support condition for $\vartheta$. From its definition, we can see that $\text{supp} \vartheta$ is contained in

$$\begin{cases} 
z(x, y) = \frac{\alpha - y}{\delta} + \frac{\beta y^2}{2\delta^2} - \frac{|x|^2}{2\delta} > -3R_2, \\
y < \frac{R_1}{2a}.
\end{cases} \hfill (3.5)$$
In view of the relation 
\[ \alpha_+ > \alpha_- \quad \text{and} \quad a = \frac{\alpha_+}{\delta}, \]
we have that 
\[ \frac{R_1}{2a} < \frac{\delta}{2\alpha_-} \cdot R_1 < \frac{\delta}{\alpha_-} \cdot \frac{\alpha_r}{16} < \delta, \]
i.e., \( y < \delta r \leq \delta r_0 \). Next, we observe that
\[ -3R_2 > -3R = -\frac{3\alpha - r}{16} > \frac{\alpha_-}{\delta}(-\delta r) + \frac{\beta}{2\delta^2}(-\delta r)^2, \]
which gives \(-\delta r < y\) due to (3.3). Consequently, we verify that \(|y| < \delta r\). One the other hand, from the first condition of (3.5) and (3.3), we see that
\[ \frac{|x|^2}{2\delta} < 3R_2 + \frac{\alpha_- y}{\delta} + \frac{\beta y^2}{2\delta^2} \leq 3\frac{\alpha_- r}{16} + \frac{\alpha_-}{\delta} \cdot \delta r + \frac{\beta}{2\delta^2} \cdot \delta^2 r^2 \leq \frac{\delta}{8}, \]
which gives \(|x| < \delta/2\).

Since \( h_0 = 0 \), we have that
\[ \vartheta(x,0)u_+(x,0) - \vartheta(x,0)u_-(x,0) = 0, \quad \forall \ x \in \mathbb{R}^{n-1}. \] (3.6)

Applying (2.8) to \( \vartheta u \) and using (3.6) yields
\[
\sum_{\pm} \sum_{|k|=0}^{2} \tau^{3-2|k|} \int_{\mathbb{R}^n_{\pm}} |D^k(\vartheta u_\pm)|^2 e^{2\tau \phi_{\delta,\pm}(x,y)} dxdy \\
\leq C \sum_{\pm} \int_{\mathbb{R}^n_{\pm}} |L(x,y,\vartheta)(\vartheta u_\pm)|^2 e^{2\tau \phi_{\delta,\pm}(x,y)} dxdy \\
+ C \tau \int_{\mathbb{R}^{n-1}} |A_+(x,0) \nabla_{x,y}(\vartheta u_+)(x,0) \cdot \nu - A_-(x,0) \nabla_{x,y}(\vartheta u_-)(x,0) \cdot \nu|^2 e^{2\tau \phi_{\delta}(x,0)} dx \\
+ C [e^{2\tau \phi_{\delta}(x,0)}(A_+(x,0) \nabla_{x,y}(\vartheta u_+)(x,0) \cdot \nu - A_-(x,0) \nabla_{x,y}(\vartheta u_-)(x,0) \cdot \nu)]^2_{1/2,\mathbb{R}^{n-1}}. \]
(3.7)

We now observe that \( \nabla_{x,y} \vartheta_1(z) = \vartheta'_1(z) \nabla_{x,y} z = \vartheta'_1(z)(-\frac{z}{\delta}, \frac{\alpha_-}{\delta} + \frac{\beta y}{\delta^2}) \) and it is nonzero only when
\[-3R_2 < z < -2R_2. \]

Therefore, when \( y = 0 \), we have
\[ 2R_2 < \frac{|x|^2}{2\delta} < 3R_2. \]

Thus, we can see that
\[ |\nabla_{x,y} \vartheta(x,0)|^2 \leq CR_2^2 \left( \frac{6R_2}{\delta} + \frac{\alpha^2}{\delta^2} \right) \leq CR_2^{-2}. \] (3.8)
By $h_0(x) = h_1(x) = 0$, (3.8), and the easy estimate of [5, Proposition 4.2], it is not hard to estimate

$$
\begin{align*}
\tau \int_{\mathbb{R}^n} & |A_+(x, 0) \nabla_{x,y} (\partial u_+(x, 0)) \cdot \nu - A_-(x, 0) \nabla_{x,y} (\partial u_-(x, 0)) \cdot \nu|^2 e^{2r \phi_1(x, 0)} dx \\
+ & \left[ e^{r \phi_1(\cdot, 0)} (A_+(x, 0) \nabla_{x,y} (\partial u_+(x, 0)) \cdot \nu - A_-(x, 0) \nabla_{x,y} (\partial u_-(x, 0)) \cdot \nu) \right]^2_{1/2, \mathbb{R}^{n-1}} \\
\leq & CR_2^2 e^{-4rR_2} \left( \tau \int_{\{\sqrt{4R^2} \leq |x| \leq \delta R^2\}} |u_+(x, 0)|^2 dx + [u_+(x, 0)]^2_{1/2, \{\sqrt{4R^2} \leq |x| \leq \delta R^2\}} \right) \\
+ & CR_2^{-3} e^{-4rR_2} \int_{\{\sqrt{4R^2} \leq |x| \leq \delta R^2\}} |u_+(x, 0)|^2 dx \\
\leq & CR_2^{-3} e^{-4rR_2} E, \\
\end{align*}
\tag{3.9}
$$

where

$$E = \int \left\{\sqrt{4R^2} \leq |x| \leq \delta R^2\right\} |u_+(x, 0)|^2 dx + [u_+(x, 0)]^2_{1/2, \{\sqrt{4R^2} \leq |x| \leq \delta R^2\}}.$$

Expanding $\mathcal{L}(x, y, \partial)(\partial u_\pm)$ and considering the set where $D\partial \neq 0$, we can estimate

\begin{align*}
\sum_{|k|=0}^{1} \tau^{3-2|k|} \int_{\{-2R_2 \leq z \leq \frac{R_1}{2}, y < \frac{R_1}{a}\}} |D^k u_\pm|^2 e^{2r \phi_{\pm}(x, y)} dx dy \\
\leq & C \sum_{|k|=0}^{1} \sum_{|k|=0}^{1} R_2^{2(|k|-2)} \int_{\{-3R_2 \leq z \leq -2R_2, y < \frac{R_1}{2a}\}} |D^k u_\pm|^2 e^{2r \phi_{\pm}(x, y)} dx dy \\
+ & C \sum_{|k|=0}^{1} R_1^{2(|k|-2)} \int_{\{-3R_2 \leq z, \frac{R_1}{2a} < y < \frac{R_1}{a}\}} |D^k u_+|^2 e^{2r \phi_{+}(x, y)} dx dy \\
+ & CR_2^{-3} e^{-4rR_2} E, \\
\tag{3.10}
\end{align*}

\begin{align*}
\leq & C \sum_{|k|=0}^{1} \sum_{|k|=0}^{1} R_2^{2(|k|-2)} e^{-4rR_2} e^{2r (\alpha_+ - \alpha_-) \frac{R_1}{a}} \int_{\{-3R_2 \leq z \leq -2R_2, y < \frac{R_1}{a}\}} |D^k u_\pm|^2 dx dy \\
+ & C \sum_{|k|=0}^{1} R_1^{2(|k|-2)} e^{2r \alpha_+ \frac{R_1}{2a}} e^{2r \alpha_- \frac{R_1}{2a}} \int_{\{z \geq -3R_2, \frac{R_1}{2a} < y < \frac{R_1}{a}\}} |D^k u_+|^2 dx dy \\
+ & C r_2^{-3} e^{-4rR_2} E.
\end{align*}

Let us denote $U_1 = \{ z \geq -4R_2, \frac{R_1}{8a} < y < \frac{R_1}{a} \}$, $U_2 = \{ -R_2 \leq z \leq \frac{R_1}{2a}, y < \frac{R_1}{8a}\}$. 

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From (3.10) and interior estimates (Caccioppoli’s type inequality), we can derive that

\[
\tau^3 e^{-2\tau R_2} \int_{U_2} |u|^2 \, dx \, dy \\
\leq \tau^3 e^{-2\tau R_2} \int_{\{-R_2 \leq z \leq \frac{R_1}{2a}, y < \frac{R_1}{4a}\}} |u|^2 \, dx \, dy \\
\leq \sum_{\pm} \tau^3 \int_{\{-2R_2 \leq z \leq \frac{R_1}{2a}, y < \frac{R_1}{4a}\}} |u_\pm|^2 e^{2\tau \phi_{\pm}(x,y)} \, dx \, dy \\
\leq C \sum_{|k|=0} \left[ R_2^{2(|k|-2)} e^{-4\tau R_2} e^{2\tau \frac{(a_+ - a_-)}{a} \frac{R_1}{4a}} \int_{\{-3R_2 \leq z \leq -2R_2, y < \frac{R_1}{4a}\}} |D^k u_\pm|^2 \, dx \, dy \\
+ C \tau^2 R_2^{-3} e^{-4\tau R_2} E \right] \\
\leq CR_1^{-4} e^{-3\tau R_2} \int_{\{-4R_2 \leq z \leq -R_2, y < \frac{R_1}{4a}\}} |u|^2 \, dx \, dy + CR_2^{-4} e^{-3\tau R_2} e^{-4\tau R_2} E \\
+ CR_1^{-4} e^{(1 + \frac{2R_2}{4\alpha})^\tau R_1} \int_{\{z \geq -4R_2, \frac{R_1}{4a} < y < \frac{R_1}{2a}\}} |u|^2 \, dx \, dy \\
\leq CR_1^{-4} \left[ e^{2\tau R_1} \int_{U_1} |u|^2 \, dx \, dy + \tau^2 e^{-3\tau R_2} F \right],
\]

where

\[
F = \int_{\{z \geq -4R_2, y < \frac{R_1}{4a}\}} |u|^2 \, dx \, dy
\]

and we used the inequality \(\frac{\beta R_1}{4\alpha} < 1\) due to (3.4).

Dividing \(\tau^3 e^{-2\tau R_2}\) on both sides of (3.11) implies that

\[
\int_{U_2} |u|^2 \, dx \, dy \leq CR_1^{-4} \left( e^{2\tau (R_1 + R_2)} \int_{U_1} |u|^2 \, dx \, dy + e^{-\tau R_2 F} \right). 
\]

Now, we consider two cases. If \(\int_{U_1} |u|^2 \, dx \, dy \neq 0\) and

\[
e^{2\tau_0 (R_1 + R_2)} \int_{U_1} |u|^2 \, dx \, dy < e^{-\tau_0 R_2 F},
\]

then we can pick a \(\tau > \tau_0\) such that

\[
e^{2\tau (R_1 + R_2)} \int_{U_1} |u|^2 \, dx \, dy = e^{-\tau R_2 F}.
\]
Using such $\tau$, we obtain from (3.12) that
\[
\int_{U_2} |u|^2 dxdy \leq C R_1^{-4} e^{2\tau(R_1+R_2)} \int_{U_1} |u|^2 dxdy
\]
\[
= C R_1^{-4} \left( \int_{U_1} |u|^2 dxdy \right)^{\frac{R_2}{2\theta_1 + 3\theta_2}} (F)^{\frac{2R_1 + 2R_2}{2\theta_1 + 3\theta_2}}. \tag{3.13}
\]

If $\int_{U_1} |u|^2 dxdy = 0$, then letting $\tau \to \infty$ in (3.12) we have $\int_{U_2} |u|^2 dxdy = 0$ as well. The three-regions inequality (3.1) obviously holds.

On the other hand, if
\[
e^{-\tau_0 R_2} F \leq e^{2\tau_0(R_1+R_2)} \int_{U_1} |u|^2 dxdy,
\]
then we have
\[
\int_{U_2} |u|^2 dx \leq (F)^{\frac{R_2}{2\theta_1 + 3\theta_2}} (F)^{\frac{2R_1 + 2R_2}{2\theta_1 + 3\theta_2}}
\]
\[
\leq \exp (\tau_0 R_2) \left( \int_{U_1} |u|^2 dxdy \right)^{\frac{R_2}{2\theta_1 + 3\theta_2}} (F)^{\frac{2R_1 + 2R_2}{2\theta_1 + 3\theta_2}}. \tag{3.14}
\]

Putting together (3.13), (3.14), we arrive at
\[
\int_{U_2} |u|^2 dx \leq (\exp (\tau_0 R_2) + C R_1^{-4}) \left( \int_{U_1} |u|^2 dxdy \right)^{\frac{R_2}{2\theta_1 + 3\theta_2}} (F)^{\frac{2R_1 + 2R_2}{2\theta_1 + 3\theta_2}}. \tag{3.15}
\]

\[\square\]

4 \quad \text{Size estimate}

We will apply the three-region inequality (3.1) to estimate the size of embedded inclusion in this section. Here we denote $\Omega$ a bounded open set in $\mathbb{R}^n$ with $C^{1, \alpha}$ boundary $\partial \Omega$ with constants $s_0, L_0$, where $0 < \alpha \leq 1$. Assume that $\Sigma$ is a $C^2$ closed hypersurface with constants $r_0, K_0$ satisfying
\[
\text{dist}(\Sigma, \partial \Omega) \geq d_0 \tag{4.1}
\]
for some $d_0 > 0$. We divide $\Omega$ into three sets, namely,
\[
\Omega = \Omega_+ \cup \Sigma \cup \Omega_-
\]
where $\Omega_\pm$ are open subsets. Note that $\partial \Omega_- = \partial \Omega \cup \Sigma$ and $\partial \Omega_+ = \Sigma$. We also define
\[
\Omega_h = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > h \}.
\]
Definition 4.1 [C^{1,\alpha} regularity] We say that $\Sigma$ is $C^2$ with constants $r_0, K_0$ if for any $P \in \Sigma$ there exists a rigid transformation of coordinates under which $P = 0$ and
\[
\Omega_\pm \cap B(0, r_0) = \{(x, y) \in B(0, r_0) \subset \mathbb{R}^n : y \geq \psi(x)\},
\]
where $\psi$ is a $C^2$ function on $B_{r_0}(0)$ satisfying $\psi(0) = 0$ and
\[
\|\psi\|_{C^2(B_{r_0}(0))} \leq K_0.
\]
The definition of $C^{1,\alpha}$ boundary is similar. Note that $B(a, r)$ stands for the $n$-ball centered at $a$ with radius $r > 0$. We remind the reader that $B_r(a)$ denotes the $(n-1)$-ball centered at $a$ with radius $r > 0$.

Assume that $A_{\pm} = \{a_{ij}^\pm(x, y)\}_{i,j=1}^n$ satisfy (2.3) and (2.4). Let us define $H_{\pm}^{(\Omega)} = \chi_{\Omega_{\pm}}, A = H_{\pm}^{(\Omega)} A_++H_{\pm}^{(\Omega)} A_-,$ $u = H_{\pm}^{(\Omega)} u_++H_{\pm}^{(\Omega)} u_-$. We now consider the conductivity equation
\[
\text{div}(A \nabla u) = 0 \quad \text{in} \quad \Omega. \tag{4.2}
\]
It is not hard to check that $u$ satisfies $h_0 = h_1 = 0$, where $h_0$ and $h_1$ are defined by (2.5), (2.6), where in this case $\nu$ is the outer normal of $\Sigma$. For $\phi \in H^{1/2}(\partial \Omega)$, let $u$ solve (4.2) and satisfy the boundary value $u = \phi$ on $\partial \Omega$.

Next we assume that $D$ is a measurable subset of $\Omega$. Suppose that $\hat{A}$ is a symmetric $n \times n$ matrix with $L^\infty(\Omega)$ entries. In addition, we assume that there exist $\eta > 0, \zeta > 1$ such that
\[
(1 + \eta)A \leq \hat{A} \leq \zeta A \quad \text{a.e. in} \quad \Omega \tag{4.3}
\]
or $\eta > 0, 0 < \zeta < 1$ such that
\[
\zeta A \leq \hat{A} \leq (1 - \eta)A \quad \text{a.e. in} \quad \Omega. \tag{4.4}
\]
Now let $v = H_+^{(\Omega)} v_++H_-^{(\Omega)} v_-$ be the solution of
\[
\begin{aligned}
\text{div}(A \chi_{\Omega \setminus D} + \hat{A} \chi_D) \nabla v) = 0 & \quad \text{in} \quad \Omega, \\
v = \phi & \quad \text{on} \quad \partial \Omega. \tag{4.5}
\end{aligned}
\]
The inverse problem considered here is to estimate $|D|$ by the knowledge of $\{\phi, A \nabla v \cdot \nu|_{\partial \Omega}\}$. In this work we would like to consider the most interesting case where
\[
\bar{D} \subseteq \Omega_+. \tag{4.6}
\]
In practice, one could think of $\Omega_+$ being an organ and $D$ being a tumor. The aim is to estimate the size of $D$ by measuring one pair of voltage and current on the surface of the body.
We denote $W_0$ and $W$ the powers required to maintain the voltage $\phi$ on $\partial \Omega$ when the inclusion $D$ is absent or present. It is easy to see that

$$W_0 = \int_{\partial \Omega} \phi A \nabla u \cdot \nu = \int_{\Omega} A \nabla u \cdot \nabla u$$

and

$$W = \int_{\partial \Omega} \phi A \nabla v \cdot \nu = \int_{\Omega} (A \chi_{\Omega \setminus D} + \hat{A} \chi_D) \nabla v \cdot \nabla v.$$

The size of $D$ will be estimate by the power gap $W - W_0$. To begin, we derive the following energy inequalities which are similar to those proved in [4] for the Neumann boundary value problem.

**Lemma 4.1** Assume that $A$ satisfies the ellipticity condition (2.3). If either (4.3) or (4.4) holds, then

$$C_1 \int_D |\nabla u|^2 \leq |W_0 - W| \leq C_2 \int_D |\nabla u|^2,$$

where $C_1, C_2$ are constants depending only on $\lambda, \eta,$ and $\zeta$.

**Proof.** We prove the lemma by adopting methods from [4] (and [10]). For simplicity, we denote $g = A \nabla u \cdot \nu|_{\partial \Omega}$ and $\tilde{g} = A \nabla v \cdot \nu|_{\partial \Omega}$. Note that $v$ and $u$ have the same Dirichlet data. Also, we have

$$\int_{\Omega} (A - A \chi_{\Omega \setminus D} - \hat{A} \chi_D) \nabla v \cdot \nabla u = \int_{\partial \Omega} \phi (g - \tilde{g}) = W_0 - W.$$  \hspace{1cm} (4.8)

By (4.8) and Green’s identity, we can derive

$$\int_{\Omega} (A \chi_{\Omega \setminus D} + \hat{A} \chi_D) \nabla (v - u) \cdot \nabla (v - u)$$

$$= \int_{\Omega} (A \chi_{\Omega \setminus D} + \hat{A} \chi_D) \nabla (v - u) \cdot \nabla v - \int_{\Omega} (A \chi_{\Omega \setminus D} + \hat{A} \chi_D) \nabla (v - u) \cdot \nabla u$$

$$= - \int_{\Omega} (A \chi_{\Omega \setminus D} + \hat{A} \chi_D) \nabla (v - u) \cdot \nabla u + \int_{\Omega} A \nabla (v - u) \cdot \nabla u$$

$$\quad = \int_D \hat{A} \nabla u \cdot \nabla u + \int_{\Omega} (A - A \chi_{\Omega \setminus D} - \hat{A} \chi_D) \nabla v \cdot \nabla u$$

$$\quad = \int_D \hat{A} \nabla u \cdot \nabla u + W_0 - W.$$  \hspace{1cm} (4.9)

In the same way, we can obtain

$$\int_{\Omega} A \nabla (v - u) \cdot \nabla (v - u) = - \int_D \hat{A} \nabla v \cdot \nabla v - (W_0 - W).$$  \hspace{1cm} (4.10)

Formulae (4.9), (4.10) are exactly (2.9), (2.10) in [4, page 58]. The rest of arguments then follow those of [4, Lemma 2.1]. \hfill \Box

The derivation of bounds on $|D|$ will be based on (4.7) and the following Lipschitz propagation of smallness for $u$. 

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Proposition 4.1 (Lipschitz propagation of smallness) Let $u \in H^1(\Omega)$ be the solution of (4.2) with Dirichlet data $\phi$. For any $B(x, \rho) \subset \Omega_+$, we have that

$$\int_{B(x, \rho)} |\nabla u|^2 \geq C \int_{\Omega} |\nabla u|^2,$$

(4.11)

where $C$ depends on $\Omega_+$, $d_0$, $\lambda_0$, $M_0$, $r_0$, $K_0$, $s_0$, $L_0$, $\alpha$, $\alpha'$, $\rho$, and

$$\frac{\|\phi - \phi_0\|_{C^{1, \alpha'}(\partial\Omega)}}{\|\phi - \phi_0\|_{H^{1/2}(\partial\Omega)},}$$

for $\phi_0 = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \phi$. Here $\alpha'$ satisfies $0 < \alpha' < \frac{\alpha}{(\alpha + 1)n}$.

Before proving Proposition 4.1, we need to adjust the three-region inequality (3.1) for the $C^2$ interface $\Sigma$. Let $0 \in \Sigma$ and the coordinate transform $(x', y') = T(x, y) = (x, y - \psi(x))$ for $x \in B_{s_0}(0)$. Denote $\tilde{U} = T(B(0, s_0))$ and $\tilde{A}_\pm = \{\tilde{a}_{i,j}^\pm\}_{i,j=1}^n$ the coefficients of $A_\pm$ in the new coordinates $(x', y')$. It is easy to see that $\tilde{A}_\pm$ satisfies (2.3) and (2.4) with possible different constants $\tilde{\lambda}_0$, $\tilde{M}_0$, depending on $\lambda_0$, $M_0$, $r_0$, $K_0$. Then there exist $C$ and $\tilde{R}$, depending on $\tilde{\lambda}_0$, $\tilde{M}_0$, $n$, such that for

$$0 < R_1 < R_2 < \tilde{R}$$

(4.12)

and $U_1, U_2, U_3$ defined as in Theorem 3.1, we have that $U_3 \subset \tilde{U}$ (so $U_1, U_2$ are contained in $\tilde{U}$ as well) and (3.1) holds. Now let $\tilde{U}_j = T^{-1}(U_j)$, $j = 1, 2, 3$, then (3.1) becomes

$$\int_{\tilde{U}_2} |u|^2 dxdy \leq C \left( \int_{\tilde{U}_1} |u|^2 dxdy \right)^{\frac{R_2}{2R_1 + 3R_2}} \left( \int_{\tilde{U}_3} |u|^2 dxdy \right)^{\frac{2R_1 + 2R_2}{2R_1 + 3R_2}}$$

(4.13)

where $C$ depends on $\lambda_0$, $M_0$, $r_0$, $K_0$, $n$, $R_1$, $R_2$. Furthermore, by Caccioppoli’s inequality and generalized Poincaré’s inequality (see (3.8) in [2]), we obtain from (4.13) that

$$\int_{\tilde{U}_2} |\nabla u|^2 dxdy \leq C \left( \int_{\tilde{U}_1} |\nabla u|^2 dxdy \right)^{\frac{R_2}{2R_1 + 3R_2}} \left( \int_{\tilde{U}_3} |\nabla u|^2 dxdy \right)^{\frac{2R_1 + 2R_2}{2R_1 + 3R_2}}$$

(4.14)

with a possibly different constant $C$.

Since $A_+ \ (\text{respectively } A_-)$ is Lipschitz in $\Omega_+ \ (\text{respectively } \Omega_-)$, the following three-sphere inequality is well-known. Let $u_\pm$ be a solution to $\text{div}(A_\pm \nabla u_\pm) = 0$ in $\Omega_\pm$. Then for $B(x_0, r) \subset \Omega_+$ (or $B(x_0, \tilde{r}) \subset \Omega_-$) and $0 < r_1 < r_2 < r_3 < \tilde{r}$, we have that

$$\int_{B(x_0, r_2)} |\nabla u_\pm|^2 dxdy \leq C \left( \int_{B(x_0, r_1)} |\nabla u_\pm|^2 dxdy \right)^\theta \left( \int_{B(x_0, r_3)} |\nabla u_\pm|^2 dxdy \right)^{1-\theta},$$

(4.15)
where $0 < \theta < 1$ and $C$ depend on $\lambda_0, M_0, n, r_1/r_3, r_2/r_3$.

Now we are ready to prove Proposition 4.1.

**Proof of Proposition 4.1.** It suffices to study the case where $\rho$ is small. Since $\Sigma \in C^2$, it satisfies both the uniform interior and exterior sphere properties, i.e., there exists $a_0 > 0$ such that for all $z \in \Sigma$, there exist balls $B \subset \Omega_+$ and $B' \subset \Omega_-$ of radius $a_0$ such that $B \cap \Sigma = B' \cap \Sigma = \{z\}$. Next let $\nu_z$ be the unit normal at $z \in \Sigma$ pointing into $\Omega_+$ (inwards) and $L = \{z + t\nu_z \subset \mathbb{R}^n : t \in [\rho_0, -3\rho_0]\}$. We then fix $R_1, R_2$ satisfying (4.12) and choose $\rho_0 > 0$ so that

$$S_z = \cup_{y \in L} B(y, \rho_0) \subset \tilde{U}_2.$$

Denote $\kappa = R_2/(2R_1 + 3R_2)$. Note that we move the construction of the three-region inequality from 0 to $z$.

Let $x \in \Omega_+$ and consider $B(x, \rho) \subset \Omega_+$, where $\rho \leq \min\{a_0, \rho_0\}$. For any $y \in \Omega_{2\rho}$, we discuss three cases.

(i) Let $y \in \Omega_{+, \rho}$, then by (4.15) and the chain of balls argument, we have that

$$\frac{\int_{B(y, \rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \leq C \left( \frac{\int_{B(x, \rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \right)^{\theta N_1},$$

where $N_1$ depends on $\Omega_+$ and $\rho$.

(ii) Let $y \in \{y \in \bar{\Omega}_+ : \text{dist}(y, \Sigma) \leq \rho\} \cup \{y \in \Omega_+ : \text{dist}(y, \Sigma) \leq 3\rho\}$, then $B(y, \rho) \subset S_z$ for some $z \in \Sigma$. Note that $\tilde{U}_1 \subset \Omega_{+, \rho}$ (taking $\rho$ even smaller if necessary). We then apply (4.16) iteratively to estimate

$$\frac{\int_{\tilde{U}_1} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \leq C \left( \frac{\int_{B(x, \rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \right)^{\theta N_1},$$

where $C$ depends on $\tilde{U}_1$ and $\rho$. Combining estimates (4.17) and (4.14) yields

$$\frac{\int_{B(y, \rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \leq C \left( \frac{\int_{B(x, \rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \right)^{\kappa \theta N_1}.$$  

(iii) Finally, we consider the case where $y \in \Omega_- \cap \Omega_{2\rho}$ and $\text{dist}(y, \Sigma) > 3\rho$. We observe that if $y_* = z + (3\rho)\nu_z$, then (4.18) implies

$$\frac{\int_{B(y_*, \rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \leq C \left( \frac{\int_{B(x, \rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \right)^{\kappa \theta N_1}.$$  

Again using (4.15) and the chain of balls argument (starting with (4.19)), we obtain that

$$\frac{\int_{B(y, \rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \leq C \left( \frac{\int_{B(x, \rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \right)^{\kappa \theta N_1 \theta N_2}.$$  

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Putting together (4.16), (4.18), and (4.20) gives
\[ \frac{\int_{B(y,\rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \leq C \left( \frac{\int_{B(x,\rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \right)^s \]  
(4.21)
for all \( y \in \Omega_{2\rho} \), where \( 0 < s < 1 \) and \( C \) depends on \( \lambda_0, M_0, n, r_0, K_0, \rho, \Omega_\pm \).

In view of (4.21) and covering \( \Omega_{3\rho} \) with balls of radius \( \rho \), we have that
\[ \frac{\int_{\Omega_{3\rho}} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \leq C \left( \frac{\int_{B(x,\rho)} |\nabla u|^2}{\int_{\Omega} |\nabla u|^2} \right)^s. \]  
(4.22)

Note that \( u - \phi_0 \) is the solution to (4.2) with Dirichlet boundary value \( \phi - \phi_0 \). By Corollary 1.3 in [14], we have that
\[ \|\nabla u\|_{L^\infty(\Omega)} = \|\nabla (u - \phi_0)\|_{L^\infty(\Omega)} \leq C \|\phi - \phi_0\|_{C^{1,\alpha'}(\partial \Omega)} \]
with \( 0 < \alpha' < \frac{\alpha}{(\alpha+1) n} \), which implies
\[ \int_{\Omega \setminus \Omega_{5\rho}} |\nabla u|^2 \leq C |\Omega \setminus \Omega_{5\rho}| \|\phi - \phi_0\|_{C^{1,\alpha'}(\partial \Omega)} \leq C \rho \|\phi - \phi_0\|_{C^{1,\alpha'}(\partial \Omega)}. \]  
(4.23)

Here we have used \( |\Omega \setminus \Omega_{5\rho}| \lesssim \rho \) proved in [3]. Using the Poincaré inequality, we have
\[ \|\phi - \phi_0\|_{H^{1/2}(\partial \Omega)} \leq C \|u - \phi_0\|_{H^1(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}. \]
Combining this and (4.23), we see that if \( \rho \) is small enough depending on \( \Omega_\pm, d_0, \lambda_0, M_0, r_0, K_0, s_0, L_0, \alpha, \alpha', \rho, \) and \( \|\phi - \phi_0\|_{C^{1,\alpha'}(\partial \Omega)}/\|\phi - \phi_0\|_{H^{1/2}(\partial \Omega)} \), then
\[ \frac{\|\nabla u\|_{L^2(\Omega_{3\rho})}}{\|\nabla u\|_{L^2(\Omega)}} \geq \frac{1}{2}. \]

The proposition follows from this and (4.22).

We now have enough tools to derive bounds on \( |D| \).

**Theorem 4.2** Suppose that the assumptions of this section hold.

(i) If, moreover, there exists \( h > 0 \) such that
\[ |D_h| \geq \frac{1}{2} |D| \]  
(fatness condition).

Then there exist constants \( K_1, K_2 > 0 \) depending only on \( \Omega_\pm, d_0, h, \lambda_0, M_0, r_0, K_0, s_0, L_0, \alpha, \alpha' \), and \( \|\phi - \phi_0\|_{C^{1,\alpha'}(\partial \Omega)}/\|\phi - \phi_0\|_{H^{1/2}(\partial \Omega)} \), such that
\[ K_1 \left| \frac{W_0 - W}{W_0} \right| \leq |D| \leq K_2 \left| \frac{W_0 - W}{W_0} \right|. \]
Remark 4.3 We point out that part (i) of Theorem 4.2 still holds if the assumption (4.6) is replaced by
\[ \text{dist}(D, \partial \Omega) \geq d_2 > 0. \]
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