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ON THE ORDERS OF REAL ELEMENTS OF SOLVABLE GROUPS

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ABSTRACT

We study the finite solvable groups in which the orders of the real elements are either 2-powers or not divisible by 2. Equivalently, we describe the finite solvable groups in which the centralizer of every involution is 2-closed.

1. Introduction

The prime graph $\Gamma(G)$ of a finite group G provides a convenient framework for many questions about the orders of the elements of G. The vertices of $\Gamma(G)$ are the prime divisors of $|G|$ and vertices p and q are connected by an edge if G contains an element of order pq.

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Reality is of considerable importance in finite group theory. We define the **real prime graph** $\Gamma_{\mathbb{R}}(G)$ as follows. The vertices of $\Gamma_{\mathbb{R}}(G)$ are the primes p such that G contains a real element of order p, and vertices p and q are connected by an edge if G contains a real element of order pq. In [DMN], the vertices of $\Gamma_{\mathbb{R}}(G)$ were determined; if p a prime divisor of $|G|$ is not a vertex of $\Gamma_{\mathbb{R}}(G)$ and $G = \mathbf{O}^{2'}(G)$, then G is non-solvable with abelian p-Sylow subgroups.

Here we investigate the groups G for which 2 is an isolated vertex of $\Gamma_{\mathbb{R}}(G)$, that is, the finite groups all of whose real elements have 2-power or odd order. The analogous question for $\Gamma(G)$ was studied by M. Suzuki in the landmark paper [S2]. If $|G|$ is even, then 2 is an isolated vertex of $\Gamma(G)$ if and only if the centralizer of every involution in G is a 2-group. Suzuki called such groups *CIT groups*. He showed that the nonabelian simple CIT groups are the Zassenhaus groups in characteristic 2, together with $L_2(q)$ for certain q, and $L_3(4)$, and he also determined the solvable CIT groups [S1, p. 435-436].

We thank H. Tong-Viet for pointing out a second major connection between our topic and Suzuki's work. Tong-Viet shows in Proposition 2.7 below that 2 is an isolated vertex of $\Gamma_{\mathbb{R}}(G)$ if and only if the centralizer of every involution in G is 2-closed (i.e., it has a normal Sylow 2-subgroup). Suzuki [S2] called groups satisfying the latter property *C-groups*. He showed that the simple C-groups are the simple CIT groups, together with the families $\mathrm{L}_3(2^n)$ and $\mathrm{U}_3(2^n)$. Suzuki also had something to say about the solvable C-groups [S2, Theorem 5], but our results are far more detailed.

We recall now that since the real elements of G are the real elements of $\mathbf{O}^{2'}(G)$, the smallest normal subgroup of G with odd index, when studying the real elements of a finite group, it is no loss to assume that $G = \mathbf{O}^{2'}(G)$.

THEOREM A: Suppose that G is a finite solvable group with $\mathbf{O}^{2'}(G) = G$. Assume that every real element of G is either a 2-element or a $2'$-element. Let $N = \mathbf{O}_2(G)$ and $Q \in \mathrm{Syl}_2(G)$, and assume that G is not a 2-group. Then:

(1) G/N has a normal 2-complement K/N and Q/N is cyclic or quaternion. If zN is the unique involution of Q/N, then $\mathbf{C}_{K/N}(Q/N) = \mathbf{C}_{K/N}(zN)$.

(2) Suppose that $N > 1$. Then $N = \mathbf{F}(G)$, Q/N is cyclic and G splits over N. If $|Q/N| > 2$, then K/N is cyclic and G is a CIT group. In any case, K/F_2 is metabelian and F_2/N is abelian, where $F_2/N = \mathbf{F}(G/N)$. If $|G|$ is coprime to 3, then K/F_2 is abelian.

Both parts of Theorem A are best possible; part (1) is best possible by Lemma 2.2 and Theorem 2.4, and part (2) is best possible by Theorem 3.1 and Example 3.10. In the situation of part (2), Theorem 3.9, Proposition 3.5 and Corollary 3.8 provide additional information about G/N. While we have much to say about the structure of G/N, little can be said, in general, about the structure of N. Indeed it is not hard to show that the derived length of N can be arbitrarily large. Character-theoretic methods, in particular Isaacs π-theory, are used to prove part (1) of Theorem A. To prove part (2), we show that G/N acts on N so that the centralizers in G/N of the nonidentity elements of N have normal Sylow 2-subgroups. After extensive analysis, we deduce strong restrictions on the structure of G/N.

2. The Case $\mathbf{O}_2(G) = 1$.

For the sake of brevity, let us say that a finite group G **satisfies R** if every real element of G has 2-power order or $2'$-order. If $H \leq G$ and G satisfies **R**, then notice that H satisfies **R**.

Our first lemma gives a character theoretical characterization of solvable groups satisfying **R**, which we shall heavily use later on. In order to do this, we use the so called *Isaacs π-theory*. Recall if π is a set of primes, in every π-separable group G, there exists a canonical subset $B_\pi(G)$ of the set of irreducible complex characters $\mathrm{Irr}(G)$ of G, containing the trivial character 1_G, such that their restrictions form a basis of the complex space of class functions defined on the π-elements of G. In particular, $|B_\pi(G)|$ is the number of conjugacy classes of π-elements of G, and the square matrix $(\psi_i(x_j))$ where $\psi \in B_\pi(G)$ and x_j is complete set of G-representatives of π-elements, is invertible. (See Theorem A and Corollary (10.2) of [I2].) From the very definition, one can check that if $N \lhd G$ is contained in $\ker(\chi)$ for some $\chi \in \mathrm{Irr}(G)$, then $\chi \in B_\pi(G)$ if and only if $\bar{\chi} \in B_\pi(G/N)$, where $\bar{\chi} \in \mathrm{Irr}(G/N)$ is the character defined by $\bar{\chi}(Ng) = \chi(g)$.

Lemma 2.1: *Suppose that G is solvable. Then every real element of G has 2-power or $2'$-order if and only if every real $\chi \in \mathrm{Irr}(G)$ belongs to $B_2(G)$ or to $B_{2'}(G)$.*

Proof. First of all notice that for every set of primes π, the number of real characters in $B_\pi(G)$ coincides with the number of real conjugacy classes of G consisting of π-elements, using Brauer's Lemma on character tables with complex conjugation (Theorem (6.32) of [I]). Second, we claim that if π and σ are disjoint sets of primes, then $B_\pi(G) \cap B_\sigma(G)$ consists just of the trivial character. In order to show this, work by induction on $|G|$. Let $\chi \in B_\pi(G) \cap B_\sigma(G)$ and let N be a minimal normal subgroup of G. Then N is a p-group for some prime p. Since π and σ are disjoint, let us say that p is not in π. Then N is contained in the kernel of χ, because $\chi \in B_\pi(G)$ (Corollary (5.3) of [I2]). Then $\chi \in \mathrm{Irr}(G/N)$ and we apply induction.

Now, let $\mathrm{cl}_{\mathbb{R}}(G)$ be the set of the real conjugacy classes of G, let $\mathrm{cl}_{2,\mathbb{R}}(G)$ be the set of real classes consisting of 2-elements, and let $\mathrm{cl}_{2',\mathbb{R}}(G)$ the set of real classes consisting of $2'$-elements. Also, let $\mathrm{Irr}_{\mathbb{R}}(G)$ be the set of the irreducible real characters of G, let $B_{2,\mathbb{R}}(G)$ be the set of the real B_2-characters, and let $B_{2',\mathbb{R}}(G)$ be the set of the real $B_{2'}$-characters. Then

$$\mathrm{cl}_{\mathbb{R}}(G) - \{1\} = (\mathrm{cl}_{2,\mathbb{R}}(G) - \{1\}) \cup (\mathrm{cl}_{2',\mathbb{R}}(G) - \{1\})$$

if and only if

$$|\mathrm{cl}_{\mathbb{R}}(G)| = |\mathrm{cl}_{2,\mathbb{R}}(G)| + |\mathrm{cl}_{2',\mathbb{R}}(G)| - 1$$

if and only if

$$|\mathrm{Irr}_{\mathbb{R}}(G)| = |B_{2,\mathbb{R}}(G)| + |B_{2',\mathbb{R}}(G)| - 1$$

if and only if

$$\mathrm{Irr}_{\mathbb{R}}(G) - \{1_G\} = (B_{2,\mathbb{R}}(G) - \{1_G\}) \cup (B_{2',\mathbb{R}}(G) - \{1_G\}) .$$

∎

Lemma 2.2: *Suppose that G is a finite group and let $N \lhd G$.*

(a) If G satisfies **R**, *then G/N satisfies* **R**.

(b) Assume that G is solvable and that $N \leq \mathbf{Z}(G)$ has odd order. Then G satisfies **R** *if and only if G/N satisfies* **R**.

Proof. (a) By induction on $|G|$. If $Q \in \mathrm{Syl}_p(N)$, then $\mathbf{N}_G(Q)$ satisfies **R**, and if Q is not normal in G, by induction $\mathbf{N}_G(Q)/\mathbf{N}_N(Q) \cong G/N$ satisfies **R**, so we may assume that N is nilpotent.

Now, suppose that Nx is a real element of G/N of even, non-2-power order. Then $Nx^{-1} = Nx^t$ for some $t \in G$. If $H = \langle x, t\rangle$, then we may assume that $G = HN$, by induction. Since G/N is solvable, then we have that G is solvable. Finally, since G satisfies **R**, then every real character of G lies in $B_2(G)$ or

$B_{2'}(G)$, and the same happens for every real character of G/N. Hence, G/N satisfies $\mathbf{R}$, but this is a contradiction.

(b) Suppose that G/N satisfies $\mathbf{R}$, and let $\chi \in \mathrm{Irr}(G)$ be real valued. Let $\lambda \in \mathrm{Irr}(N)$ be under χ. Since $\chi_N = \chi(1)\lambda$, then λ is real valued and we conclude that $\lambda = 1$. Hence $\chi \in \mathrm{Irr}(G/N)$. Therefore $\chi \in B_2(G/N)$ or $\chi \in B_{2'}(G/N)$, and $\chi \in B_2(G)$ or $\chi \in B_{2'}(G)$. Now, we apply Lemma 2.1. ∎

In order to prove the following key technical lemma, we shall need to use the Gadjendragadkar special characters. We refer the reader to [Ga] for the definition and main properties.

Lemma 2.3: *Suppose that G is a finite solvable group such that all real elements of G have 2-power order or $2'$-order. If M is a $2'$-subgroup of G normalized by a 2-subgroup D of G, and $1 \neq \alpha \in \mathrm{Irr}(M)$ is such that $\alpha^r = \bar{\alpha}$ for some $r \in D$, then $\alpha^{MD} \in \mathrm{Irr}(MD)$.*

Proof. It is no loss to assume that $MD = G$. Let $T = I_G(\alpha)$ be the stabilizer in G of α. Since $I_G(\alpha) = I_G(\bar{\alpha})$, it follows that r normalizes T. We want to prove that $T = M$. Suppose that $T > M$. Then let T/S be a chief factor of $T\langle r\rangle$, where $M \leq S$, so that T/S has order 2 and S is r-invariant. Let $1 \neq \lambda \in \mathrm{Irr}(T/S)$, and notice that λ is real of order 2 and r-invariant. Now, by Corollary (6.28) of [Is], α has a unique extension $\beta \in \mathrm{Irr}(T)$ such that the determinantal order $o(\beta)$ is odd. Since $\bar{\beta}^r$ is an extension of α with odd determinantal order, it follows that $\beta^r = \bar{\beta}$. Now, $\lambda\beta \in \mathrm{Irr}(T|\alpha)$, and by the Clifford correspondence, we have that $\chi = (\lambda\beta)^G \in \mathrm{Irr}(G)$. Now,

$$\chi = (\lambda\beta)^G = ((\lambda\beta)^r)^G = (\lambda\bar{\beta})^G = (\bar{\lambda}\bar{\beta})^G = \bar{\chi}\,,$$

and we conclude that χ is real. Therefore $\chi \in B_2(G)$ or $\chi \in B_{2'}(G)$ by Lemma 2.1. Since $T \lhd\lhd G$, we conclude that the irreducible constituents of χ_T all lie in $B_2(T)$ or in $B_{2'}(T)$, by Corollary (7.5) of [I2]. Now, β is $2'$-special and λ is 2-special (see, for instance, Proposition (2.3) of [Ga]). Hence we conclude that $\lambda\beta$ is both 2 and $2'$-factorable (see the definition after Theorem (2.5) of [I2]) and also lies in $B_2(T)$ or in $B_{2'}(T)$. By Theorem (2.5) of [I2] and Lemma (5.4) of [I2], this is impossible. ∎

Theorem 2.4: *Let $G > 1$ be a solvable group such that $\mathbf{O}^{2'}(G) = G$ and $\mathbf{O}_2(G) = 1$. Let Q be a Sylow 2-subgroup of G. Then the following are equivalent:*

(a) *all real elements of G are either 2-elements or $2'$-elements;*
(b) *G has a normal 2-complement K, Q is either cyclic or generalized quaternion and $\mathbf{C}_K(Q) = \mathbf{C}_K(z)$, where z is the unique involution of Q.*

Proof. Assume first that the real elements of G are either 2-elements or $2'$-elements. This hypothesis is inherited by the subgroups of G and their quotients. Also $\mathbf{O}_2(M) = 1$ for every $M \lhd G$. We show that G satisfies (b) by induction on $|G|$. Let $K = \mathbf{O}_{2'}(G)$ and $L/K = \mathbf{O}_2(G/K)$. Assume, working by contradiction, that K is not a 2-complement of G. So, $L < G$ and by induction a Sylow 2-subgroup Q_0 of $\mathbf{O}^{2'}(L)$ is either cyclic or generalized quaternion. As $L/K \cong Q_0$, then $G/\mathbf{C}_G(L/K)$ is either a 2-group or it is isomorphic to a subgroup of S_4. Since $\mathbf{C}_G(L/K) \leq L$ (by the Hall-Higman Lemma 1.2.3) and $L/K = \mathbf{O}_2(G/K)$, then $L/K \cong Q_8$. As G has no nontrivial factor group of odd order, it follows that G/K is either $GL(2,3)$ or the double cover of S_4 with Q_{16} Sylow 2-subgroups. These two groups have real elements of order 6, and this cannot happen by hypothesis and Lemma 2.2. So we conclude that $G = L$. Hence, K is the normal 2-complement of G.

Let Q be a Sylow 2-subgroup of G. We wish to show next that Q is cyclic or generalized quaternion. As $\mathbf{O}_2(G) = 1$, $F = \mathbf{F}(K) = \mathbf{F}(G)$ and Q acts faithfully on F. Also, FQ satisfies $\mathbf{R}$. By coprime action, we also know that Q acts faithfully on $F/\Phi(F)$ and $FQ/\Phi(F)$ satisfies $\mathbf{R}$, so in order to show that Q is cyclic or generalized quaternion, we may assume in this paragraph that K is abelian and KQ satisfies $\mathbf{R}$. Let $1 \neq z \in \mathbf{Z}(Q)$ of order 2. Then $D = \mathbf{C}_K(z) < K$ and $K = D \times [K, z]$. Notice that z inverts $[K, z]$ and therefore z inverts K/D. Now $D \lhd KQ$ and KQ/D satisfies $\mathbf{R}$. If $1 \neq \lambda \in \mathrm{Irr}(K/D)$, then $\lambda^z = \bar{\lambda}$, and by Lemma 2.3, it follows that $\lambda^{KQ} \in \mathrm{Irr}(KQ)$. Thus we have that Q acts Frobeniusly on K/D, and we conclude that Q is either cyclic or generalized quaternion. Also z is the unique involution of Q.

Now that we have that Q is cyclic or generalized quaternion, let us come back to our original notation, where $G = KQ$, and K is a normal 2-complement. Assume finally that $\mathbf{C}_K(Q) < \mathbf{C}_K(z) = D$. Then Q acts non-trivially on $\mathbf{C}_K(z)$. Then there exists $1 \neq x \in D$ such that $x^u = x^{-1}$ for some $u \in Q$, by Lemma (3.1.d) of [DMN], for instance. Then there exists $1 \neq \eta \in \mathrm{Irr}(D)$ such that $\eta^u = \bar{\eta}$. (Consider the action of $Q \times \langle\sigma\rangle$ on the classes of D and of the characters of D, and apply Brauer's lemma on character tables, where σ is complex-conjugation: see Lemma (6.1) of [NT].) Now, by the z-Glauberman correspondence, there exists $1 \neq \theta \in \mathrm{Irr}(K)$ such that θ is z-invariant and $\theta^u = \bar{\theta}$. (The z-Glauberman correspondent of θ is η, and it follows that $\theta^u = \bar{\theta}$ because the Glauberman correspondence commutes with Galois and group automorphisms.) Now, by Lemma 2.3, we conclude that $\theta^G \in \mathrm{Irr}(G)$. However, this is impossible, since θ is z-invariant. We conclude that $\mathbf{C}_K(Q) = \mathbf{C}_K(z)$.

Suppose, conversely, that Q is cyclic or generalized quaternion, that G has a normal 2-complement K and that $\mathbf{C}_K(Q) = \mathbf{C}_K(z)$, where z is the unique involution of Q. Let g be a real element of G and denote by a and b the $2'$-part and the 2-part of g, respectively. Up to conjugation, we can assume $b \in Q$. Note that both a and b are real elements of G, using the definition. Assume that g is not a $2'$-element. Then $b \neq 1$, so z is a power of b and hence $a \in \mathbf{C}_K(z) = \mathbf{C}_K(Q)$. Hence $\mathbf{N}_G(\langle a\rangle)/\mathbf{C}_G(\langle a\rangle)$ has odd order and then, as a is real, it follows that $a = 1$. ∎

Recalling that if Q is either a cyclic 2-group or if Q is generalized quaternion group, $Q \neq Q_8$, then $\mathrm{Aut}(Q)$ is a 2-group, one immediately gets the following:

Corollary 2.5: *Let G be a solvable group such that all real elements of G are either 2-elements or $2'$-elements. Assume that $\mathbf{O}_2(G) = 1$. Then there exists a normal subgroup N of G, with $|G : N| \in \{1, 3\}$, such that N has a normal 2-complement.*

A group G is said to be p-closed, for a prime p, if it has a normal Sylow p-subgroup. The following result describes the structure of finite groups which have no nontrivial real element of odd order.

Lemma 2.6: *A group G has no nontrivial real element of odd order if and only if G is 2-closed.*

Proof. This follows from Proposition 6.4 in [DNT]. ∎

We recall that a group G is called a C-group if the centralizer of every involution is 2-closed (see [S2]). We thank H. Tong-Viet for pointing out the following:

Proposition 2.7 (H. Tong-Viet): *A group G is a C-group if and only if G satisfies $\mathbf{R}$.*

Proof. Clearly, we can assume that G has even order.

Suppose first that G satisfies $\mathbf{R}$. Let x be an involution of G and $H = \mathbf{C}_G(x)$. In order to show that G is a C-group, by Lemma 2.6 we have to show that H has no nontrivial real element of odd order. By way of contradiction, suppose that $y \in H$ is a nontrivial real element of odd order $m > 1$. Then $y^g = y^{-1}$ for some $g \in H$. Consider $z = xy \in H$. As $x \in \mathbf{Z}(H)$ we have

$$z^g = (xy)^g = x^g y^g = xy^{-1} = y^{-1}x^{-1} = z^{-1} .$$

Therefore, z is a real element of H, and hence of G, of order $2m$, a contradiction. Since x is an arbitrary involution, we deduce that G is a C-group.

Conversely, assume that G is a C-group. Working by contradiction, assume that G has a real element z whose order is even but not a 2-power. As any power of a real element is also real, we can assume that z has order $2m$, where $m > 1$ is odd. Then $x = z^m$ is an involution and $y = z^2 \in \mathbf{C}_G(x)$ has order m. As z is a real element of G, there exists a $g \in G$ such that $z^g = z^{-1}$. It follows that $x^g = x^{-1} = x$, so $g \in \mathbf{C}_G(x)$, and $y^g = y^{-1}$. Hence, y is a nontrivial real element of $\mathbf{C}_G(x)$, against Lemma 2.6. ∎

3. The Case $\mathbf{O}_2(G) > 1$.

In order to prove the second part of Theorem A, it is convenient to state the following:

Standard Hypotheses. Let $G = KQ$, where $K > 1$ is normal of odd order, $Q \in \mathrm{Syl}_2(K)$ is cyclic or quaternion and $\mathbf{C}_K(Q) = \mathbf{C}_K(z)$, where $\langle z \rangle = \Omega_1(Q)$. Suppose also that $\mathbf{O}^{2'}(G) = G$. Assume that G acts on a 2-group V and that $\mathbf{C}_G(v)$ has a normal Sylow 2-subgroup (equivalently, $\mathbf{C}_G(v) = \mathbf{O}_2(\mathbf{C}_G(v)) \times \mathbf{O}_{2'}(\mathbf{C}_G(v)))$ for all $1 \neq v \in V$. In this case, we say that G satisfies the Standard Hypotheses with respect to V.

Our aim in this section is to pin down the structure of the groups G satisfying the Standard Hypotheses. Before we go deep into analyzing these groups, it might be convenient to show how the hypotheses of Theorem A have naturally led us to the Standard Hypotheses.

THEOREM 3.1: *Suppose that G is a finite solvable group with $\mathbf{O}^{2'}(G) = G$. Assume that G satisfies $\mathbf{R}$ and that $G > \mathbf{O}_2(G) = N > 1$. Then there exists a* *subgroup H of G such that $G = NH$, with $N \cap H = 1$, such that H satisfies the Standard Hypotheses with respect to N. Moreover, $\mathbf{O}_{2'}(G) \leq \mathbf{Z}(G)$.*

Conversely, if $H = KQ$ satisfies the Standard Hypotheses with respect to V, then the semidirect product $G = VH$ satisfies $\mathbf{R}$.

Proof. Assume that G statisfies $\mathbf{R}$ and that $N = \mathbf{O}_2(G)$ is a proper nontrivial subgroup of G. Let $Q_0 \in \mathrm{Syl}_2(G)$. Since $\mathbf{O}_2(G/N) = 1$, by Lemma 2.2(a) we have that G/N satisfies the hypothesis of Theorem 2.4. Hence, G/N has a normal 2-complement L/N and Q_0/N is either cyclic or quaternion, with $\mathbf{C}_{L/N}(zN) = \mathbf{C}_{L/N}(Q_0/N)$, where zN is the unique involution of Q_0/N. Let K be a 2-complement in G and $H = \mathbf{N}_G(K)$. By the Frattini Argument, we have that $G = NH$. If $N \cap H = \mathbf{C}_N(K) > 1$, then H would contain a central involution x. Now, a Sylow 2-subgroup of H acts nontrivially on K, as $K > 1$ and $\mathbf{O}^{2'}(H) = H$, so there exists a nontrivial element $y \in K$ such that y is real in H (see [DMN, Lemma 3.1], for example). Thus xy is a real element of H, and so of G, against $\mathbf{R}$. Hence, H is a complement of N in G. Write $H = KQ$, where $Q \cong Q_0/N$ is either cyclic or quaternion, and observe that $\mathbf{C}_K(Q) = \mathbf{C}_K(z)$, where z is the involution of Q. Let $X \leq N$ be a non-trivial cyclic subgroup and let x be the involution in X. Let $C = \mathbf{C}_H(x)$, M the normal 2-complement of C and $T \in \mathrm{Syl}_2(C)$. If T does not act trivially on M, then we know that there is $1 \neq y \in M$ and $t \in T$ such that $y^t = y^{-1}$. Thus, $(xy)^t = xy^{-1} = (xy)^{-1}$ and xy is a real element of G whose order is divisible by 2 and is not a 2-power, a contradiction. Therefore T acts trivially on M and C has a normal Sylow 2-subgroup. Since $\mathbf{C}_H(X) \leq C$, the same is true for $\mathbf{C}_H(X)$. Hence, H satisfies the Standard Hypotheses with respect to N.

As $N > 1$, there exists a non-trivial element $x_0 \in N \cap \mathbf{Z}(Q_0)$. Now, $\mathbf{O}_{2'}(G) \leq \mathbf{C}_H(N) \leq \mathbf{C}_H(x_0)$ and $Q_0 \lhd \mathbf{C}_H(x_0)$ by the previous paragraph. So $Q_0 \leq \mathbf{C}_G(\mathbf{O}_{2'}(G)) \lhd G$. Since $\mathbf{O}^{2'}(G) = G$, we have that $\mathbf{O}_{2'}(G) \leq \mathbf{Z}(G)$.

Conversely, assume that H satisfies the Standard Hypotheses with respect to V. Assume, working by contradiction, that there exists a real element g of $G = VH$ whose order is even and not a 2-power. By taking a suitable power, we can assume that the order of g is 2 times an odd number. We can write $g = xt$, where $x \in H$ has odd order, $t \in G$ is an involution and $[x, t] = 1$. So, there is a 2-element $z \in G$ such that $(xt)^z = (xt)^{-1} = x^{-1}t$ and hence z centralizes t and inverts x. Since H satisfies $\mathbf{R}$ by Theorem 2.4, and xtV is a real element of G/V, we have $t \in V$. Thus $\langle x, z\rangle \le \mathbf{C}_H(t)$, contradicting the Standard Hypotheses. ∎

Let now introduce another set of assumptions that lends itself more conveniently to induction. Recall that, given a group action of Q on K, A/B is a *Q-invariant p-section of K* (p a prime) if A, B are Q-invariant subgroups of K, $B \lhd A$ and A/B is a p-group.

Hypotheses H2. Let G be a group with a cyclic Sylow 2-subgroup $Q > 1$ and a normal 2-complement K. Assume that $\mathbf{O}^{2'}(G) = G$ and $\mathbf{C}_K(Q) = \mathbf{C}_K(z)$, where $\langle z\rangle = \Omega_1(Q)$. Assume also that, for every (odd) prime p and for every Q-invariant p-section A/B of K, $[A/B, Q]$ is cyclic,

Lemma 3.2: *Assume the Hypotheses H2 for G and let N be a normal subgroup of G, $N \le K$. Then also G/N satisfies the Hypotheses H2.*

Proof. Write $\bar{G} = G/N$. Clearly, $\mathbf{O}^{2'}(\bar{G}) = \bar{G}$, $\bar{Q} \ne 1$ is a cyclic Sylow 2-subgroup of $\bar{G}$, $\bar{K}$ the normal 2-complement and $\bar{z}$ the only involution of $\bar{Q}$.

Every $\bar{Q}$-invariant p-section of $\bar{G}$ is of the form $\bar{A}/\bar{B}$, where A and B are subgroups of G containing N. So A/B is a Q-invariant p-section of G and $[\bar{A}/\bar{B}, \bar{Q}] = \overline{[A/B, Q]}$ is cyclic. Finally, by coprimality, $\mathbf{C}_{\bar{K}}(\bar{Q}) = \overline{\mathbf{C}_K(Q)} = \overline{\mathbf{C}_K(z)} = \mathbf{C}_{\bar{K}}(\bar{z})$. So, $\bar{G}$ satisfies the Hypotheses H2. ∎

In the following, we denote by $\pi(G)$ the set of the prime divisors of the order of a group G.

Lemma 3.3: *Let $G = K\langle z\rangle$, where $K = \mathbf{O}_{2'}(G)$ and z is an involution. Suppose that G acts on a 2-group V and $\mathbf{C}_G(v)$ has a normal Sylow 2-subgroup for all nonidentity $v \in V$.*

Then

- **(a):** *If A is a z-invariant abelian subgroup of K, then $[A, z]$ is cyclic and $[A, z]$ acts Frobeniusly on V.*
- **(b):** *Let $p \in \pi(K)$ and let P be a z-invariant p-subgroup of K. Then $[P, z]$ is cyclic.*

Proof. We note that the hypotheses are inherited by every subgroup of G that contains z. To prove (a), let A be a z-invariant abelian subgroup of K and let $B = [A, z]$. Since B is abelian, we have $\mathbf{C}_B(z) = 1$ and so z inverts every element of B. If $1 \ne b \in B$, then the dihedral group $\langle z, b\rangle$ acts on $\mathbf{C}_V(b)$. If

$\mathbf{C}_V(b) > 1$, then z fixes a nonidentity element v of $\mathbf{C}_V(b)$, and so $\langle z, b\rangle$ fixes v, contrary to the hypothesis. Hence B acts Frobeniusly on V and then B is cyclic.

To prove (b), we proceed by induction on $|P|$. We may assume that $P = [P, z]$. Let $B < P$ be a maximal z-invariant subgroup. Since $P\langle z\rangle$ is supersolvable, $|P : B| = p$. Let $y \in P \setminus B$. Then $[y, z] \notin B$, since z does not centralize P/B. Let $x = [y, z]$. Then z inverts x and $P = B\langle x\rangle$. Now, $B = [B, z]\mathbf{C}_B(z)$, so $P = \langle x\rangle[B, z]\mathbf{C}_B(z)$. Since $P = [P, z]$, it follows that z inverts every element of $P/\Phi(P)$. Hence $\mathbf{C}_B(z) \le \Phi(P)$, and so $P = \langle x\rangle[B, z]$. Since z inverts $\langle x\rangle$, we have $x^p \in [B, z]$. Hence $|P| = |\langle x\rangle||[B, z]|/|\langle x^p\rangle|$ and so $|P : [B, z]| = p$. Thus $B = [B, z]$.

By induction, B is cyclic. Since P has a cyclic subgroup of index p, [Go, 5.4.4] yields that P is either abelian or a modular group $M_m(p)$ of order p^m for some $m \ge 3$. Suppose that $P = M_m(p)$. Then $W = \Omega_1(P)$ is elementary abelian of order p^2 by [Go, 5.4.3]. But z inverts both $W \cap B$ and $W/W \cap B \cong P/B$. Hence $[W, z] = W$, contradicting part (a). We conclude that P is abelian, and so $P = [P, z]$ is cyclic by part (a). ∎

Proposition 3.4: *Suppose $G = KQ$ satisfies the Standard Hypotheses. Then G satisfies Hypotheses $H2$. Also $\mathbf{F}(K) = \mathbf{F}(G)$.*

Proof. Note that $\mathbf{F}(G)$ has odd order; otherwise $z \in \mathbf{O}_2(G)$ and $K = \mathbf{C}_K(z) = \mathbf{C}_K(Q)$ would be a direct factor of G, against the assumptions that $K > 1$ and $\mathbf{O}^{2'}(G) = G$.

Thus $\mathbf{F}(G) = \mathbf{F}(K)$. There exists a Sylow subgroup P of $\mathbf{F}(K)$ such that $[P, z] \neq 1$. Since z is the only involution in Q, we see that Q acts faithfully on $[P, z]$. Since $[P, z]$ is cyclic by Lemma 3.3, we conclude that Q is abelian and hence cyclic.

Now let A be a Q-invariant subgroup of K. Then $[A, Q] = [A, z]$ as follows. Indeed, since $[A, z] \le [A, Q]$ and $A = [A, z]\mathbf{C}_A(z)$, we have

$$[A, Q] = [A, Q, Q] = [[A, z]([A, Q] \cap \mathbf{C}_A(z)), Q] \, .$$

Since Q centralizes $\mathbf{C}_A(z)$ by assumption, and since Q normalizes both A and $\langle z\rangle$, we have $[A, Q] = [[A, z], Q] \le [A, z]$, as desired.

Next, if B is a Q-invariant normal subgroup of A, then $[A/B, Q] = [A, Q]B/B$. In fact, Q acts trivially on $(A/B)/([A, Q]B/B) \cong A/[A, Q]B$, and so $[A/B, Q] \leq [A, Q]B/B$; the other inclusion is straightforward. Similarly, $[A/B, z] = [A, z]B/B$.

Now let A/B be a Q-invariant p-section of G. Then, by coprime action, $A = BP$, where P is a Q-invariant Sylow p-subgroup of A. Using the natural isomorphism between the Q-invariant sections A/B and $P/(P \cap B)$, one sees that $[A/B, Q] \cong [P/(P \cap B), Q]$, and similarly for z. By the two paragraphs above, we conclude that $[A/B, Q] \cong [P/(P \cap B), z] \cong [P, z](P \cap B)/(P \cap B)$ is a homomorphic image of $[P, z]$, and hence is cyclic by Lemma 3.3. Thus G satisfies (H2). ∎

Note that if G satisfies (H2), then G is solvable. Thus every chief factor X of G is an elementary abelian p-group for some prime p; in this case, we say X is a p-chief factor. As usual, if X is elementary abelian of order p^n, we call n the *rank* of X. We recall that an A-group is a solvable group whose Sylow p-subgroups are abelian, for all primes p. An A-group has p-length 1 for every prime p.

Proposition 3.5: *Suppose G satisfies (H2). Let $p \in \pi(K)$ and let X be a p-chief factor of G. Let $\bar{G} = G/\mathbf{C}_G(X)$. Then*

- **(a):** *$rank(X) \leq 3$; if $|Q| > 2$, then $rank(X) = 1$ and $K \leq \mathbf{C}_G(X)$.*
- **(b):** *$\bar{G}$ is an A-group of p'-order.*
- **(c):** *$\bar{K}$ has derived length at most 2. If $(3, |\bar{K}|) = 1$, then $\bar{K}$ is cyclic.*

Proof. We observe first that $\mathbf{C}_G(X) \leq K$; otherwise $\mathbf{C}_X(z) = X$ and so $\mathbf{C}_X(Q) = X$ by coprime action and (H2). Since $\bar{G} = \mathbf{O}^{2'}(\bar{G})$, X would then be central in G, which is not the case.

So by Lemma 3.2, $\bar{G}$ satisfies (H2). If $|X| = p$, then $\bar{G}$ is abelian. Since $\bar{G} = \mathbf{O}^{2'}(\bar{G})$, it follows that $\bar{G}$ is a 2-group. Hence $K \leq \mathbf{C}_G(X)$.

Assume from now on that $rank(X) > 1$. We will show that $|\bar{Q}| = 2$ and that $\bar{G}$ is either a dihedral group of order $2d$, where d is an odd divisor of $p+1$ or $p-1$, or $\bar{G}$ is an extension of S_3 by an abelian group $\bar{C}$, where $|\bar{C}|$ divides $(p-1)^2$ and $(|\bar{C}|, |S_3|) = 1$.

As Q maps injectively into $\bar{G}$, we often identify Q with its image in $\bar{G}$. We observe also that $|\mathbf{F}(\bar{G})|$ is odd. Indeed if $\mathbf{O}_2(\bar{G}) > 1$, then $z \in \mathbf{Z}(\bar{G})$ and so $[X, z] = X$. Since $|[X, z]| = p$ by (H2), this contradicts our assumption that $rank(X) > 1$.

First suppose that G acts primitively on X and $\mathbf{F}(\bar{G})$ is abelian. By primitivity, $\mathbf{F}(\bar{G})$ is cyclic and, since X is a faithful irreducible $\bar{G}$-module, $\mathbf{F}(\bar{G})$ acts Frobeniusly on X. Let ϕ be the Brauer character of $\bar{G}$ afforded by $X \otimes \overline{\mathrm{GF}(p)}$. Since $\mathbf{F}(\bar{G})$ is a normal self-centralizing cyclic subgroup of $\bar{G}$, we have $\phi = \alpha^{\bar{G}}$ for a faithful character α of $\mathbf{F}(\bar{G})$. It follows that $\phi(z) = 0$, so $\mathbf{C}_X(z)$ has rank $[\phi_{\langle z\rangle}, 1_{\langle z\rangle}] = rank(X)/2$. Similarly, if $y \in Q$ has order 4, then $\mathbf{C}_X(y)$ has rank equal to $rank(X)/4$. Since $[X,z]$ has rank 1, however, we have $rank(\mathbf{C}_X(z)) = rank(X) - 1$. It follows that $rank(X) = 2$ and $Q = \langle z\rangle$. Now [MW, Theorem 2.11] yields that $\bar{G}$ is a subgroup of the semilinear group $\Gamma(p^2)$. Since $\mathbf{O}^{2'}(\bar{G}) = \bar{G}$, it follows that $\bar{G}$ is dihedral of order $2d$, where $d = |\bar{K}|$ divides $p+1$.

Next suppose G acts primitively on X and $\mathbf{F}(\bar{G})$ is nonabelian. By [MW, 1.9], $\mathbf{F}(\bar{G}) = \bar{E}\bar{Z}$, where each Sylow subgroup of $\bar{E}$ is extraspecial of prime exponent, $\bar{Z} = \mathbf{Z}(\mathbf{F}(\bar{G}))$ is cyclic, and $\bar{E} \lhd \bar{G}$. Choose $\bar{D}$ to be a Sylow subgroup of $\bar{E}$ of maximal order. Let $|\bar{D}| = r^{2m+1}$ for a prime $r \neq p$ and $m \geq 1$. Faithful absolutely irreducible $\bar{D}$-modules have dimension r^m, so if z does not centralize $\mathbf{Z}(\bar{D})$, then every irreducible $\langle z\rangle\bar{D}$-submodule W of $X \otimes \overline{\mathrm{GF}(p)}$ has dimension $2r^m$. Since z interchanges the two irreducible $\overline{\mathrm{GF}(p)}[\bar{D}]$-summands of W, we have $\dim(\mathbf{C}_W(z)) = \dim([W,z]) = r^m$. Hence $[X,z]$ has rank at least r^m, contradicting the fact that $[X,z]$ has rank 1. We conclude that z centralizes $\mathbf{Z}(\bar{D})$.

Now z preserves the nondegenerate symplectic commutator form on $\bar{D}/\mathbf{Z}(\bar{D})$. It follows that $[\bar{D}/\mathbf{Z}(\bar{D}), z]$ has even rank; see [D, Lemma 2.2] for example. Since $[\bar{D}/\mathbf{Z}(\bar{D}), z]$ is cyclic by (H2), we conclude that z centralizes $\bar{D}/\mathbf{Z}(\bar{D})$. Since z also centralizes $\mathbf{Z}(\bar{D})$, we have $[\bar{D}, \bar{z}] = 1$. Let $D \lhd G$ with $D/\mathbf{C}_G(X) = \bar{D}$. Then $D = \mathbf{C}_D(z)\mathbf{C}_G(X) = \mathbf{C}_D(Q)\mathbf{C}_G(X)$ by coprime action and (H2). Hence $[\bar{D}, Q] = 1$. Since $\bar{G} = \mathbf{O}^{2'}(\bar{G})$, we have $\bar{D} \leq \mathbf{Z}(\bar{G})$, which is absurd.

We may therefore assume that G acts imprimitively on X. Let $X = X_1 \oplus \cdots \oplus X_k$ be an imprimitivity decomposition for the action of G (so $k > 1$). Let

$$C = \bigcap_{i=1}^{k} \mathbf{N}_G(X_i)\,.$$

We claim that $z \notin C$. Suppose the contrary. Since the X_i are G-conjugate and all the involutions in G are conjugate, it follows that $|[X_i, z]| = |[X_j, z]|$ for all i, j. Since $[X,z] \neq 1$, all $[X_i, z]$ are nontrivial, and so $|[X,z]| \geq p^k > p$. This contradicts (H2), proving the claim.

Thus Q maps injectively into G/C. If z interchanges X_i and X_j, then $|[X_i \oplus X_j, z]| = |X_i|$. It follows that each $|X_i| = p$ and that z acts as a transposition on the set $\{X_1, \dots, X_k\}$. Since this holds for every imprimitivity decomposition of X, it follows that $k = \log_p|X|$ for all such decompositions, and so G/C acts faithfully and primitively on $\{X_1, \dots, X_k\}$. Thus G acts monomially on X. Since G/C is a primitive permutation group containing a transposition, a theorem of Jordan (see [H, II.4.5] for example) yields that $G/C \cong S_k$. By solvability, $k \leq 4$. Since G is 2-nilpotent, $k \leq 3$. Hence $|X| \leq p^3$, as desired. Since $|Q|$ divides $|G/C| = |S_k|$, we have $|Q| = 2$, as desired.

We may view $\bar{G}$ as a subgroup of the wreath product $W = C_{p-1} \wr S_k$. Let B be the base group of W. If $k = 2$, then B has a cyclic direct summand of order $p-1$ that is central in W. Since $\bar{G} = \mathbf{O}^{2'}(\bar{G})$, it follows that $\bar{C} = \bar{K}$ is cyclic of order dividing $p-1$, and hence $\bar{G}$ is dihedral of order $2|\bar{C}|$. Thus all assertions of Proposition 3.5 hold when $k = 2$.

Suppose that $k = 3$. We show first that $(3, |\bar{C}|) = 1$. Let $\bar{M} = \Omega_1(\mathbf{O}_3(\bar{C}))$. Then $\bar{M}$ is a GF(3)[S_3]-submodule of $\bar{P}$, the natural GF(3)[S_3]-permutation module. Now $\bar{P}$ is uniserial, with only two proper nontrivial submodules, namely $\bar{S}$ and $\bar{T}$ of ranks 1 and 2, respectively. We have $[\bar{S}, z] = 1$ and $[\bar{T}/\bar{S}, z] = \bar{T}/\bar{S}$. Let $\bar{x} \in \bar{G} \setminus \bar{C}$ be a 3-element inverted by $\bar{z}$. If $|\bar{x}| > 3$, then $\langle \bar{x} \rangle \cap \bar{M}$ would be a rank 1 submodule of $\bar{P}$ which is inverted by $\bar{z}$, which is impossible. Hence $\bar{x}$ has order 3.

Suppose first that $\bar{M} = \bar{S}$. Then $\mathbf{O}_3(\bar{C})$ is a cyclic group centralized by $\langle \bar{x}, \bar{z} \rangle$. It follows that $\langle \bar{x}, \bar{z} \rangle \times \mathbf{O}_3(\bar{C})$ is a factor group of $\bar{G}$, contradicting $\bar{G} = \mathbf{O}^{2'}(\bar{G})$. Suppose next that $\bar{M}$ contains $\bar{T}$. Then $\langle \bar{x} \rangle \bar{T}/\bar{S}$ is a noncyclic 3-section of $\bar{G}$ inverted by $\langle z \rangle = Q$, contradicting (H2). We conclude that $\bar{M} = 1$, and so $(3, |\bar{C}|) = 1$. In particular, $\bar{G}$ is an A-group. As in the case $k = 2$, B now has a cyclic direct summand of order $p-1$ that is central in W, and so $|\bar{C}|$ divides $(p-1)^2$. Also $\bar{K}'' = 1$.

Finally suppose, to get a contradiction, that $p = 3$. Then $|\bar{C}|$ divides $(p-1)^2$ and $|\bar{C}|$ is odd, so $\bar{C} = 1$. Hence $\bar{G} \cong S_3$ and so X is not an irreducible $\bar{G}$-module, a contradiction. Thus (b) holds when $k = 3$, completing the proof. ∎

Lemma 3.6: *Let G satisfy (H2) and let $F = \mathbf{F}(G)$. If $K > 1$, then $F = \mathbf{F}(K)$. In any event G/F has p-length 1 for every prime $p \in \pi(G/F)$.*

Proof. As in the proof of Proposition 3.4, if $z \in \mathbf{O}_2(G)$, then $G = Q \times K$. This contradicts $\mathbf{O}^{2'}(G) = G$ if $K > 1$. The other assertion follows from

Proposition 3.5 and the fact that F is the intersection of the centralizers of all the chief factors of G. ∎

PROPOSITION 3.7: *Let G satisfy (H2). Let $p \in \pi(K)$ and let $P = \mathbf{O}_p(G)$, $R = \Phi(G) \cap P$ and $X = P/R$. If $X \neq 1$, then X is a noncentral chief factor of G, $P \in \mathrm{Syl}_p(G)$, and $R = \Phi(P)$.*

Proof. First we remark that if $M \leq K$ is a nontrivial normal subgroup of G and M is complemented in G, then $[M, z] > 1$. Otherwise $M \leq \mathbf{C}_K(z) = \mathbf{C}_K(Q)$, and so $G/\mathbf{C}_G(M)$ has odd order. Since $\mathbf{O}^{2'}(G) = G$, we have $M \leq \mathbf{Z}(G)$. Since M is complemented in G, we see that G has a factor group isomorphic to M, contradicting $G = \mathbf{O}^{2'}(G)$.

We may assume that $X \neq 1$. By [H, III.3.4(b)], we have $\Phi(G/R) = \Phi(G)/R$, and so $\Phi(G/R) \cap (P/R) = 1$. Now [H, III.4.4] yields that P/R is complemented in G/R. We have $X = P/R = \mathbf{O}_p(G/R)$. Hence to prove that X is a chief factor of G and $P \in \mathrm{Syl}_p(G)$, we may assume that $R = 1$. Note that G/R satisfies (H2) by Lemma 3.2. Write $G = XH$ with $X = \mathbf{O}_p(G)$, $X \cap H = 1$, and $Q \leq H$. Clearly, $X \not\leq \mathbf{Z}(G)$.

Since X is isomorphic to a subgroup of $\mathbf{F}(G)/\Phi(G)$, X is a completely reducible G-module by [H, III.4.5]. Suppose $X = X_1 \times X_2$, with X_1 and X_2 nontrivial normal subgroups of G. Then, by the first paragraph, $[X_i, z] > 1$ for $i = 1, 2$. But (H2) implies that $[X, z]$ is cyclic, a contradiction. We conclude that X is a noncentral chief factor of G.

Let $F = \mathbf{F}(G)$. By Lemma 3.6, $H/H \cap F \cong G/F$ has p-length 1. Since $X = \mathbf{O}_p(G)$, it follows that $H \cap F$ is a p'-group, and so H has p-length 1. By Proposition 3.5, $C = \mathbf{C}_G(X)$ contains a Sylow p-subgroup P_0 of H. Let $N = \mathbf{O}_{p'}(C) \lhd G$. Now C/XN has p-length 1 and $\mathbf{O}_{p'}(C/XN) = 1$. Hence XP_0N/XN is a normal Sylow p-subgroup of C/XN and of G/XN, and so XP_0N/N is a normal Sylow p-subgroup of G/N. Let $\bar{G} = G/N$. Then $[\bar{P}_0 \times \bar{X}, z]$ is cyclic by (H2). Since $[\bar{X}, z]$ is nontrivial, it follows that $\mathbf{C}_{\bar{P}_0}(z) = \mathbf{C}_{\bar{P}_0}(Q) = \bar{P}_0$. Since $\bar{G} = \mathbf{O}^{2'}(\bar{G})$, we conclude that $\bar{P}_0 \leq \mathbf{Z}(\bar{H})$. Since $\bar{P}_0$ is a Sylow subgroup of $\bar{H}$ and $\bar{H} = \mathbf{O}^{2'}(\bar{H})$, we have $\bar{P}_0 = 1$. Thus $P_0 = 1$ and X is a Sylow subgroup of G.

Returning to the original notation, we have shown that $P \in \mathrm{Syl}_p(G)$ and X is a noncentral chief factor of G. Let $T = \Phi(P)$; observe that $T \leq R$. By coprimeness, P/T is a direct product $X_0 \times X_1$ of G-submodules, with $X \cong X_0$. As above, (H2) implies that $[X_1, z] = [X_1, Q] = 1$, whence $X_1 \leq \mathbf{Z}(G/T)$. Since

X_1 is complemented in G/T and since $\mathbf{O}^{2'}(G/T) = G/T$, we have $X_1 = 1$ and so $T = R$, as desired. ∎

COROLLARY 3.8: *Suppose G satisfies (H2), $p \in \pi(K)$, and $P \in \mathrm{Syl}_p(G)$. Then G has p-length 1 and P has class at most 2. Moreover G contains normal Hall subgroups H_i $(1 \leq i \leq |\pi(K)|)$ such that $1 < H_1 < \ldots < H_{|\pi(K)|} = K$.*

Proof. If $\mathbf{O}_p(G) \not\leq \Phi(G)$, then $P \lhd G$ by Proposition 3.7. Suppose $\mathbf{O}_p(G) \leq \Phi(G)$. Lemma 3.6 implies that $G/\mathbf{O}_p(G)$ has p-length 1. Hence $G/\Phi(G)$ has p-length 1. By [H, VI.6.4(e)], G has p-length 1. Thus G has p-length 1 in all cases.

To prove that $P' \leq \mathbf{Z}(P)$, we can work in $G/\mathbf{O}_{p'}(G)$ by Lemma 3.2. Then $P = \mathbf{F}(G)$ and so the p'-group G/P acts faithfully on $P/\Phi(P)$. In particular, $[P/\Phi(P), z] \neq 1$. By Proposition 3.7, $P/\Phi(P)$ is a noncentral chief factor of G. Let $P_0 = \Pi[P,t]$, where t ranges over all involutions of G. Since $P_0 \not\leq \Phi(P)$, we have $\Phi(P)P_0 = P$ and so $P_0 = P$. Since each $[P,t]$ is cyclic and normal in P, we have $[P', [P,t]] = 1$. Thus $[P', P] = 1$ and so $P' \leq \mathbf{Z}(P)$.

To prove the final assertion, note that, since $\Phi(G) < \mathbf{F}(G)$, Proposition 3.7 implies that G has a nontrivial normal Sylow subgroup H_1. Since G/H_1 satisfies (H2), the final assertion follows by induction. ∎

THEOREM 3.9: *Let G satisfy (H2) with $K > 1$. Let $p \in \pi(K)$ and $P \in \mathrm{Syl}_p(G)$. Then P is homocyclic abelian of rank at most 3. We have $\mathbf{Z}(G) = 1$. If p divides $|K/K'|$, then P is cyclic.*

Proof. We show first that P is abelian. Since G has p-length 1 and $G/\mathbf{O}_{p'}(G)$ satisfies (H2), we may assume that $\mathbf{O}_{p'}(G) = 1$ and $P \lhd G$. By Proposition 3.7, $X = P/\Phi(P)$ is a chief factor of G. Working by induction on $|P|$, we may assume that P' is the unique minimal normal subgroup of G.

By Proposition 3.5, $|X| = p^k$ with $k \leq 3$. If $k = 1$, then $|P/\Phi(P)| = p$ and so P is cyclic. Observe also that if P has exponent p, then P is abelian. Indeed the proof of Corollary 3.8 shows that $P = \Pi[P,t]$, where t ranges over the involutions of G. Since each subgroup $[P,t]$ is cyclic by (H2), each such subgroup has order p and therefore lies, by normality, in $\mathbf{Z}(P)$. Thus $P = \mathbf{Z}(P)$ is abelian. We therefore assume that $\exp(P) = p^n > p$. We will show that $|P| = p^{kn}$.

Let $A = \mho_1(P) = \langle x^p : x \in P \rangle$. Since $\exp(P) > p$, we have $A > 1$ and hence $P' \leq A$.

Since p is odd and $P' \leq \mathbf{Z}(P)$, we have $\exp(\Omega_1(P)) = p$. This follows from the identity $(xy)^p = x^p y^p [x,y]^{p(p-1)/2}$ for $x, y \in P$. Since P' has exponent p, the same identity shows that the pth power map from P to A is a surjective homomorphism. Hence $A = \{x^p : x \in P\}$.

Let P_a be the additive abelian group whose elements coincide with those of P, under the operation $x + y = xy[x,y]^{-\frac{1}{2}}$. This is the "Baer trick"; see [I1, Lemma 4.37]. If $x \in P$, then x has the same order in P_a as it does in P. Hence $\Omega_1(P) = \Omega_1(P_a) = \{x \in P : x^p = 1\}$ and $\mho_1(P) = \mho_1(P_a) = \{x^p : x \in P\}$. Now $P' \leq \Phi(P)$ and $P' \leq A = \mho_1(P) = \mho_1(P_a) = \Phi(P_a)$. If M is a subset of P containing P', then the addition formula shows that M is a subgroup of P if and only if M is a subgroup of P_a. It then follows from the definition of the Frattini subgroup that $\Phi(P) = \Phi(P_a)$.

If $g \in G$, then the automorphism of P induced by g is also an automorphism of P_a, by [I1, Lemma 4.37]. It follows that $P/\Phi(P)$ and $P_a/\Phi(P_a)$ are isomorphic G-modules. Let H be a p-complement in G. If P_a were a nontrivial direct product $P_1 \times P_2$ of H-invariant subgroups, then $P_a/\Phi(P_a) = P_1/\Phi(P_1) \times P_2/\Phi(P_2)$ would be a reducible G-module, contradicting $P_a/\Phi(P_a) \cong X$. We conclude that P_a is indecomposable under the action of H. By [Go, 5.2.2], P_a is homocyclic, and so $|P_a| = |P_a/\Phi(P_a)|^m = |X|^m$, where $p^m = \exp(P_a)$. But $\exp(P_a) = \exp(P) = p^n$. Hence $|P| = |X|^n = p^{kn}$, as desired.

Suppose that $k = 3$. We claim that $X = [X, z_1] \times [X, z_2] \times [X, z_3]$ for suitable involutions $z_1, z_2, z_3 \in G$. To see this, let $Y = \Pi[X, t]$, as t ranges over the involutions of G. Since Y is a nontrivial G-submodule of X, we have $Y = X$. Hence there certainly exist involutions z_1 and z_2 such that $[X, z_1] \neq [X, z_2]$. By (H2), each $|[X, z_i]| = p$, and so $[X, z_1][X, z_2]$ is an hyperplane in X. Since $Y = X > [X, z_1][X, z_2]$ the claim follows.

Let $Y_i = [P, z_i]$ for $1 \leq i \leq 3$. By (H2), each Y_i is a (normal) cyclic subgroup of P. We have $P = Y_1 Y_2 Y_3 \Phi(P) = Y_1 Y_2 Y_3$. Since $\exp(P) = p^n$, we have $|Y_i| \leq p^n$ for each i. Now $p^{3n} = |Y_1 Y_2 Y_3| \leq |Y_1 Y_2||Y_3| \leq |Y_1||Y_2||Y_3| \leq p^{3n}$. Hence $|Y_i| = p^n$ for each i and $|Y_1 Y_2| = p^{2n}$. Thus $Y_1 \cap Y_2 = 1$ and so $[Y_1, Y_2] \leq Y_1 \cap Y_2 = 1$. Similarly, $[Y_1, Y_3] = [Y_2, Y_3] = 1$, and so $P = Y_1 \times Y_2 \times Y_3$ is abelian.

When $k = 2$, a similar argument shows that P is a direct product of two cyclic subgroups of order p^n. Thus P is abelian in all cases, as desired.

We claim next that $P \cap \mathbf{Z}(G) = 1$. To see this, we may still assume that $\mathbf{O}_{p'}(G) = 1$ and $P \lhd G$. Since P is abelian and indecomposable under the action of a p'-group, it follows as above that P is homocyclic. Moreover the only G-invariant subgroups of P are of the form $\Omega_i(P)$, with $i \leq n$, and all p-chief factors of G are isomorphic to X (as G-modules); see [Ha]. Since $G = \mathbf{O}^{2'}(G)$, we have $P \not\leq \mathbf{Z}(G)$. Hence X is not central in G and so $P \cap \mathbf{Z}(G) = 1$ for all $p \in \pi(K)$, as claimed. Since $\mathbf{O}_2(G) = 1$ by Lemma 3.6, we have $\mathbf{Z}(G) = 1$, as desired.

Finally, suppose p divides $|K/K'|$. To prove that P is cyclic, we may assume once more that $\mathbf{O}_{p'}(G) = 1$ and $P \lhd G$. Since a nontrivial quotient of $X = P/\Phi(P)$ is central in K, it follows that X is central in K, and that $X = [X, z]$ has rank 1. Hence P is cyclic, as desired. ∎

Proof of Theorem A. The first part of Theorem A follows by Lemma 2.2 and Theorem 2.4. Suppose then that $N = \mathbf{O}_2(G) > 1$. By Theorem 3.1, $G = NH$, where $N \cap H = 1$ and $H = KQ$ satisfies the Standard Hypotheses with respect to N. By Proposition 3.4, H satisfies (H2). In particular, Q is cyclic. By Theorem 3.9, $\mathbf{Z}(H) = 1$. By Theorem 3.1, $\mathbf{O}_{2'}(G) \leq \mathbf{Z}(G)$. Hence $\mathbf{O}_{2'}(G)N/N \leq \mathbf{Z}(G/N) \cong \mathbf{Z}(H) = 1$. Thus $\mathbf{O}_{2'}(G) = 1$ and $N = \mathbf{F}(G)$.

Suppose $|Q| > 2$. Then Proposition 3.5 implies that every chief factor X of H is centralized by K. Hence $K \leq \mathbf{F}(H) = \mathbf{F}(K)$ and so K is nilpotent. Hence $K = [K, Q] = \Pi_{p \in \pi(K)} [\mathbf{O}_p(H), Q]$ is cyclic by (H2). Thus $\mathbf{C}_K(Q) = 1$ and $\mathbf{C}_K(z) = \mathbf{C}_K(Q) = 1$ by (H2). Then $H = KQ$ is a Frobenius group. By Lemma 3.3(a), NK is also a Frobenius group, and so G is a CIT group.

We may now assume that $|Q| = 2$. Since $F = \mathbf{F}(K) = \mathbf{F}(H)$ is the intersection of the centralizers $\mathbf{C}_H(X)$, where X ranges over all the chief factors of H, we see that K/F is a subdirect product of the corresponding factor groups $K/\mathbf{C}_K(X)$. By Proposition 3.5, $(K/F)'' = 1$ and $(K/F)' = 1$ when $(3, |K|) = 1$. By Theorem 3.9, all Sylow subgroups of K are abelian. In particular, F is abelian.

Thus all assertions of Theorem A hold. ∎

The following example shows that K/F can be nonabelian when $G = KQ$ satisfies the Standard Hypotheses.

Example 3.10: There is a group G satisfying the Standard Hypotheses such that K has derived length 3.

In the wreath product $\mathbb{Z}_{25} \operatorname{wr} S_3$, let $U = [(\mathbb{Z}_{25})^3, S_3]$, so that U is isomorphic to $\mathbb{Z}_{25} \times \mathbb{Z}_{25}$. Let D be dihedral of order 18. Then D maps onto S_3 with kernel D_0 of order 3, and so D acts on U with kernel D_0. Let $U_0 = \Omega_1(U)$ and let $U^* = U/U_0$. Then U^* acts faithfully and diagonally on $A = \mathrm{GF}(101)^3$, and $U^* : S_3$ acts faithfully, monomially, and irreducibly on A. Thus DU acts irreducibly and monomially on A, with kernel of order 75.

Let z be a fixed involution in D. Let $T = D'$. Then $[z, T] = T, [z, TU] = TU$, and $[z, TUA] = TUA$. Let $K = TUA$. Then $[z, K] = K$ and K has derived length 3. Let $G = \langle z \rangle K$.

Let δ be a fixed non principal linear character of D_0. Choose $\mu \in \mathrm{Irr}(U_0)$ so that no involution in DU inverts $\ker(\mu)$. This is possible because DU acts on U_0 as S_3. Thus only 3 of the 6 one-dimensional subspaces of U_0 are inverted by involutions in DU. Next choose $\lambda \in \mathrm{Irr}(A)$ so that $\ker(\lambda)$ does not contain $[A, t]$, for any involution t in DU. This is possible because $DU/\mathbf{C}_{DU}(A)$ contains 15 involutions, and for each such involution t, 102 hyperplanes in A contain $[A, t]$. Thus at most $15 \cdot 102$ of the $101^2 + 101 + 1$ hyperplanes in A contain some $[A, t]$.

Let $\chi \in \mathrm{Irr}(G)$ lie over $\delta \times \mu \times \lambda$. Since no involution in G fixes δ, Clifford's theorem implies that $\chi = \theta^G$ for some $\theta \in \mathrm{Irr}(K)$. Now $\theta \in \mathrm{IBr}(K)$, and $(\theta^G)^0 = \chi^0$ is the Brauer character of an irreducible G-module V.

Let $0 \neq v \in V$. Since G has a cyclic Sylow 3-subgroup T and the restriction of χ to D_0 is a multiple of $\delta + \delta^{-1}$, it follows that $(3, |\mathbf{C}_G(v)|) = 1$. We claim that v is centralized by no dihedral group of order 10. Suppose the contrary. Since $\langle z \rangle U$ is a Hall $\{2, 5\}$-subgroup of G, we may assume, after replacing v by a conjugate vector, that v is centralized by a D_{10} subgroup $\langle t \rangle \langle u \rangle$, with $u \in U_0$ and t an involution in DU. Since every constituent of χ_{U_0} is a DU-conjugate of μ, there must exist $x \in DU$ such that $\langle u \rangle = \ker(\mu^x)$. Then t inverts $\ker(\mu^x)$, and so some involution in DU inverts $\ker(\mu)$, contrary to the definition of μ. This proves the claim.

We claim next that v is centralized by no dihedral group of order 202. Suppose the contrary. Since $\langle z \rangle$ is a Sylow 2-subgroup of G, we may assume that v is centralized by by a D_{202}-subgroup $\langle z \rangle \langle a \rangle$, with $a \in A$. Since every constituent of χ_A is a DU-conjugate of λ, there must exist $x \in DU$ such that $\langle a \rangle \leq \ker(\lambda^x)$. Then $\ker(\lambda^x)$ contains $\langle a \rangle = [A, z]$ and so $\ker(\lambda)$ contains $[A, xzx^{-1}]$, contrary to the definition of λ. This proves the claim. Thus $\mathbf{C}_G(v)$ is the direct product of a 2-group and a $\{5, 101\}$-group.

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