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Schröder Partitions and Schröder Tableaux

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Abstract. We introduce the notions of *Schröder shape* and of *Schröder tableau*, which provide some kind of analogs of the classical notions of Young shape and Young tableau. We investigate some properties of the partial order given by containment of Schröder shapes. Then we propose an algorithm which is the natural analog of the well known RS correspondence for Young tableaux, and we characterize those permutations whose insertion tableaux have some special shapes. We end our paper with a few suggestions for possible further work.

1 Introduction

Given a positive integer n , a *partition* of n is a finite sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$ and $n = \lambda_1 + \lambda_2 + \dots + \lambda_r$. When λ is a partition of n we also write $\lambda \vdash n$. A graphical way of representing partitions is given by Young shapes. The *Young shape* of the above partition $\lambda \vdash n$ consists of r left-justified rows having $\lambda_1, \dots, \lambda_r$ boxes (also called cells) stacked in decreasing order of length. The set of all Young shapes can be endowed with a poset structure by containment (of top-left justified shapes). Such a poset turns out to be in fact a lattice, called the *Young lattice*. A *standard Young tableau* with n cells is a Young shape whose cells are filled in with positive integers from 1 to n in such a way that entries in each row and each column are (strictly) increasing.

Young tableaux are among the most investigated combinatorial objects. The widespread interest in Young tableaux is certainly due both to their intrinsic combinatorial beauty (which is witnessed by several surprising facts concerning, for instance, their enumeration, such as the hook length formula and the RSK algorithm) and to their usefulness in several algebraic contexts, typically in the representation theory of groups and related matters (such as Schur functions and the Littlewood-Richardson rule).

Apart from their classical definition, there are several alternative ways to introduce Young tableaux. In the present paper we are interested in the possibility

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of defining standard Young tableaux in terms of a certain lattice structure on Dyck paths. The main advantage of this point of view lies in the possibility of giving an analogous definition in a modified setting, in which Dyck paths are replaced by some other class of lattice paths. Here we will try to see what happens if we replace Dyck paths with Schröder paths, just scratching the surface of a theory that, in our opinion, deserves to be better studied.

Given a Cartesian coordinate system, a *Dyck path* is a lattice path starting from the origin, ending on the x -axis, never falling below the x -axis and using only two kinds of steps, $u(p) = (1, 1)$ and $d(own) = (1, -1)$. A Dyck path can be encoded by a word w on the alphabet $\{u, d\}$ such that in every prefix of w the number of u 's is greater than or equal to the number of d 's and the total number of u and d in w is the same (the resulting language is called *Dyck language* and its words *Dyck words*). The *length* of a Dyck path is the length of the associated Dyck word (which is necessarily an even number).

Consider the set \mathbf{D}_n of all Dyck paths of length $2n$; it can be endowed with a very natural poset structure, by declaring $P \leq Q$ whenever P lies weakly below Q in the usual two-dimensional drawing of Dyck paths (for any $P, Q \in \mathbf{D}_n$). This partial order actually induces a distributive lattice structure on \mathbf{D}_n , to be denoted \mathcal{D}_n and called *Dyck lattice of order n* . This can be shown both in a direct way, using the combinatorics of lattice paths (see [FP]), and as a consequence of the fact that \mathcal{D}_n is order-isomorphic to (the dual of) the Young lattice of the staircase partition $(n-1, n-2, \dots, 2, 1)$ (that is the principal down-set generated by such a staircase partition in the Young lattice). Referring to the latter approach, any $P \in \mathbf{D}_n$ uniquely determines a Young shape, which can be obtained by taking the region included between P and the maximum path of \mathcal{D}_n , then slicing it into square cells using diagonal lines of slope 1 and -1 passing through all points having integer coordinates, and finally rotating the sheet of paper by 45° anticlockwise (see Figure 1).

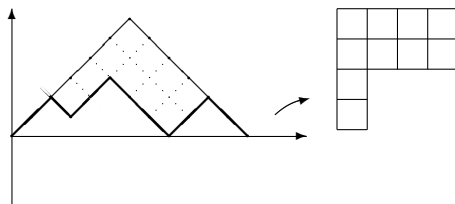


Fig. 1. A Dyck path and the associated Young shape.

It is well known that there is a bijection between standard Young tableaux of a given shape and saturated chains in the Young lattice starting from the empty shape and ending with that shape. Translating this fact on Dyck lattices, we can thus state that standard Young tableaux of a given shape are in bijection with saturated chains (inside a Dyck lattice of suitable order) starting from the Dyck path associated with that shape and ending with the maximum of the lattice.

This suggests us to try to find an analog of this fact in which Dyck paths are replaced by other types of paths. As already mentioned, the case treated in the present paper is that of Schröder paths.

In section 2 we introduce the notion of Schröder shape and study some properties of the poset of Schröder shapes (in some sense analogous to those of the Young lattice). In section 3 we introduce the notion of Schröder tableau and we define an algorithm which, given a permutation and a Schröder shape, produces a pair of Schröder tableaux having that shape; this is made in analogy with the classical RS algorithm. In particular, we will address the problem of determining which permutations are mapped into the same Schröder insertion tableau, and we solve it for a few special shapes. Finally, we devote Section 4 to the presentation of some directions of further research.

2 The Poset of Schröder Partitions

A *Schröder shape* is a set of triangular cells in the plane obtained from a Young shape by drawing the NE-SW diagonal of each of its (square) cells, and possibly adding at the end of some rows one more triangular cell, provided that, in a group of rows having equal length, only the first (topmost) one can have an added triangle. The number of cells of a Schröder shape is called the *order* of that shape. An example of a Schröder shape is illustrated in Figure 2.

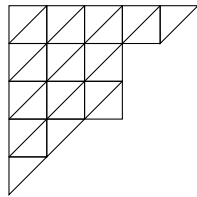


Fig. 2. A Schröder shape of order 25.

A Schröder shape has triangular cells of two distinct types, which will be referred to as *lower triangular cells* and *upper triangular cells*. In particular, rows having an odd number of cells necessarily terminate with an upper triangular cell. A Schröder shape determines a unique integer partition, whose parts are the number of cells in the rows of the shape. For instance, the partition associated with the shape in Figure 2 is $(9, 6, 6, 3, 1)$. As a consequence of the definition of a Schröder shape, it is clear that not every partition can be represented using a Schröder shape. More precisely, we have the following result, whose proof is completely trivial and so it is left to the reader.

Proposition 1. *An integer partition can be represented with a Schröder shape if and only if its odd parts are simple (i.e. have multiplicity 1).*

Those integer partitions which can be represented with a suitable Schröder shape will be called *Schröder partitions*. The set of all Schröder partitions will be denoted **Sch**, and the set of Schröder partitions of order n with **Sch** $_n$. From now on we will frequently refer to Schröder shapes and to Schröder partitions interchangeably, when no confusion is likely to arise.

From the enumerative point of view, the number of Schröder partitions is known, and is recorded in [Sl] as sequence A006950. In particular, the generating function of Schröder partitions is given by

$$\prod_{k>0} \frac{1+x^{2k-1}}{1-x^{2k}}.$$

There are several combinatorial interpretations for the resulting sequence, however an appropriate reference for the present one (in terms of Schröder partitions) appears to be [D]. In that paper the author proves a far more general result, concerning partitions such that the multiplicity of each odd part is in a prescribed set and the multiplicity of each even part is unrestricted.

It is interesting to notice that this sequence is also relevant from an algebraic point of view. Indeed it coincides with the sequence of numbers of nilpotent conjugacy classes in the Lie algebras $o(n)$ of skew-symmetric $n \times n$ matrices. This suggests that Schröder partitions have a role in representation theory that certainly deserves to be better investigated.

Here we propose a refined enumerative result, namely we describe a simple recurrence for the number of Schröder partitions of n into k parts.

Proposition 2. *Denote with $s_{n,k}$ the number of Schröder partitions of n into k parts and with $s'_{n,k}$ the number of Schröder partitions of n into k parts having smallest part different from 1. Then, for all $n \geq k \geq 1$:*

- (i) $s_{n,k} = s'_{n,k} + s'_{n-1,k-1}$;
- (ii) $s'_{n,k} = s'_{n-2,k-1} + s'_{n-2k-1,k-1} + s'_{n-2k,k}$.

Proof. We immediately observe that the set of Schröder partitions of n into k parts whose smallest part is equal to 1 is in bijection with the set of Schröder partitions of n into $k-1$ parts whose smallest part is different from 1. This gives at once the formula in (i).

Concerning (ii), given a Schröder partition λ of n into k parts with no part equal to 1, we distinguish two cases. If λ has at least one part equal to 2, then removing it leaves us with a Schröder partition of $n-2$ into $k-1$ parts, still having no part equal to 1. Otherwise, removing the first two columns of λ returns a Schröder partition of $n-2k$ into k parts, possibly having one part equal to 1. From here, using (i), we immediately obtain (ii). ■

Though the formalism of Schröder shapes seems not to add relevant information on the enumerative combinatorics of Schröder partitions, it suggests at least an interesting family of maps on integer partitions, which turns out to define a

family of involutions if suitably restricted. Consider the family of maps $(c_n)_{n \in \mathbf{N}}$ defined on the set of all integer partitions as follows: given a partition λ and a positive integer n , $c_n(\lambda)$ is the integer partition $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ (of the same size as λ) whose i -th part μ_i is given by the sum of the n columns of (the Young shape of) λ from the $((i-1)n+1)$ -th one to the (in) -th one. So, for instance, $c_3((7, 6, 6, 6, 4, 3, 3, 1)) = (22, 13, 1)$. Since each of the above maps preserves the size of a partition, it is clearly an endofunction when restricted to the set of all integer partitions of size n . Notice that c_1 is the well-known conjugation map (which exchanges rows with columns in a Young shape). In spite of the fact that c_1 is an involution (on the set of all partitions), it is easy to see that all the other c_n 's are not involutions. However, it is possible to characterize the set of those partitions for which c_n^2 acts as the identity map.

Proposition 3. *Given $n \in \mathbf{N}$ and an integer partition λ (whose i -th part will be denoted λ_i , as usual), we have that $c_n^2(\lambda) = \lambda$ if and only if the following two conditions hold:*

- if $\lambda_i \not\equiv 0 \pmod{n}$, then λ_i is simple;
- there is at most one part λ_i of λ such that $\lambda_i < n$.

Proof. For any given λ , suppose that there exists one part of $c_n(\lambda)$ which is $\not\equiv 0 \pmod{n}$; denoting with μ' the first of them, this means that λ has a set of n consecutive columns whose sum is equal to μ' . Since $\mu' \not\equiv 0 \pmod{n}$, this implies that such n columns are not all equal. Now, since in a Young shape columns are in decreasing order of length, it is impossible that the successive n columns of λ sum up to μ' , hence μ' is simple. We have thus proved that the first of the two conditions in the above statement holds for every partition in the image of c_n . This is enough to conclude that, if $c_n^2(\lambda) = \lambda$, then necessarily the same condition holds for λ (which lies indeed in the image of c_n). Moreover, if λ has at least two parts $< n$, then certainly $c_n^2(\lambda) \neq \lambda$, since each part of a partition in the image of c_n has length at least n , except at most for its smallest part.

Conversely, observe that we can represent every partition λ by means of a Young-like shape, which is obtained from the usual Young shape of λ by simply grouping together the cells of each row n by n . In this way we obtain a shape (call it $\tilde{\lambda}$) in which each cell is a horizontal rectangle made of n cells of the original Young shape, except at most the last cell of each row, which is a horizontal rectangle having *at most* n cells. Now observe that $c_n(\lambda)$ can be obtained by exchanging the rows and the columns of $\tilde{\lambda}$ and then breaking the horizontal rectangles of the resulting shape into n square cells. This construction is illustrated below for the partition $\lambda = (9, 7, 6, 6, 6, 4, 3, 3, 2)$ and $n = 3$: cells with the same label have to be grouped together, and the resulting partition $c_3(\lambda) = (26, 16, 4)$ is depicted on the right.

Given two Schröder partitions λ and μ , their join in the Young lattice is the partition $\lambda \vee \mu$ whose i -th part is the maximum between λ_i and μ_i , for all i . We will now show that $\lambda \vee \mu$ is a Schröder partition.

Suppose that $(\lambda \vee \mu)_i$ is an odd part of $\lambda \vee \mu$. Moreover, suppose w.l.o.g. that $(\lambda \vee \mu)_i = \lambda_i$ (which means that $\lambda_i \geq \mu_i$). Now $\lambda_i > \lambda_{i+1}$ (since we are supposing that λ_i is odd). Moreover, if μ_i is even, then $\lambda_i > \mu_i \geq \mu_{i+1}$ (once again since λ_i is odd), whereas if μ_i is odd, then $\lambda_i \geq \mu_i > \mu_{i+1}$ (since μ is a Schröder partition). Thus, in all cases $(\lambda \vee \mu)_i = \lambda_i > \lambda_{i+1}, \mu_{i+1}$, hence $(\lambda \vee \mu)_i > (\lambda \vee \mu)_{i+1}$. Using a similar argument it is possible to show that $(\lambda \vee \mu)_i < (\lambda \vee \mu)_{i-1}$. We have thus shown that $(\lambda \vee \mu)$ has simple odd parts, i.e. it is a Schröder partition.

Using a completely similar argument one can also show that the meet of two Schröder partitions in the Young lattice is again a Schröder partition, thus completing the proof. ■

Remark. Notice that the above theorem can be generalized as follows. For a given function $f : \mathbf{N} \rightarrow \mathbf{N} \cup \{\infty\}$, consider the set of integer partitions in which part i appears at most $f(i)$ times. Such set is a sublattice of Young lattice (with partwise join and meet). This generalization, though interesting, will play no role in the present paper.

3 An RSK-like Algorithm for Schröder Tableaux

From the algorithmic point of view, the main application of Young tableaux is in the context of the RSK algorithm. This algorithm, named after Robinson, Schensted and Knuth, takes as input a word (on the alphabet of positive integers) of length n and produces in output two semistandard Young tableaux with n cells having the same shape. For what concerns us, we will deal with a special case of the RSK algorithm, often referred to as *Robinson-Schensted correspondence* (briefly, RS correspondence), in which the input is a permutation of length n and the output is given by a pair of standard Young tableaux. A brief description of such an algorithm is given below (Algorithm 1, where $\pi = \pi_1 \pi_2 \cdots \pi_n$ is a generic permutation of length n).

The RSK algorithm is extensively described in the literature. For instance, the interested reader can find a modern and elegant presentation of it in [Be]. Among other things, one of the most beautiful properties of the RS correspondence is that it establishes a bijection between permutations of length n and pairs of standard Young tableaux with n cells having the same shape. This fact bears important enumerative consequences, as well as strictly algebraic ones. For a given permutation π , the tableaux of the pair (P, Q) returned by the RS algorithm are usually referred to as the *insertion tableau* (the tableau P) and the *recording tableau* (the tableau Q). As a consequence, we have the following nice result, which can again be found in [Be].

Theorem 2. *Denote with f^λ the number of standard Young tableaux of shape λ . Then we have:*

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2.$$

Algorithm 1: RS(π)

```

 $P := \boxed{\pi_1};$ 
 $Q := \boxed{1};$ 
for  $k$  from 2 to  $n$  do
     $\alpha := \pi_k;$ 
    for  $i \geq 1$  do
        if  $\alpha$  is bigger than all elements in the  $i$ -th row of  $P$  then
            append a cell with  $\pi_k$  inside at the end of the  $i$ -th row of  $P$ ;
            append the cell  $\boxed{k}$  at the end of the  $i$ -th row of  $Q$ ;
            break;
        else
            write  $\alpha$  in the cell of the  $i$ -th row containing the smallest element  $\beta$ 
            bigger than  $\alpha$ ;
             $\alpha := \beta$ ;
        end
    end
end

```

A *standard Schröder tableau* (from now on, simply *Schröder tableau*) with n cells is a Schröder shape whose cells are filled in with positive integers from 1 to n in such a way that entries in each row and each column are (strictly) increasing.

We propose here a natural analog of the RS algorithm for Schröder tableaux. The main difference (which is due to the specific underlying shape of a Schröder tableaux) lies in the fact that there are two distinct ways of managing the insertion of a new element in the tableau, depending on whether the cell it should be inserted in is an upper triangle or a lower triangle. As a consequence, our algorithm does not establish a bijection between permutations and pairs of Schröder tableaux; nevertheless, due to the strict analogy with the RS correspondence, we believe that it is very likely to have interesting combinatorial properties. A description of our algorithm is given below (Algorithm 2, where π is as in Algorithm 1).

Example. Consider the permutation $\pi = 465193287$. The pair (P, Q) of Schröder tableaux produced by applying the algorithm Sch to π is:

1	2	7	8
3	4	9	
5	6		

1	2	5	8
3	4	9	
6	7		

Algorithm 2: $\text{Sch}(\pi)$

```

 $P :=$  the 1-cell Schröder tableau with  $\pi_1$  written in the cell;
 $Q :=$  the 1-cell Schröder tableau with 1 written in the cell;
for  $k$  from 2 to  $n$  do
     $\alpha := \pi_k$ ;
    for  $i \geq 1$  do
        if  $\alpha$  is bigger than all elements in the  $i$ -th row of  $P$  then
            append a cell (either an upper or a lower triangle) with  $\pi_k$  inside at
            the end of the  $i$ -th row of  $P$ ;
            append a cell (either an upper or a lower triangle) with  $k$  inside at
            the end of the  $i$ -th row of  $Q$ ;
            break;
        else
            let  $A$  be the cell of the  $i$ -th row containing the smallest element
            bigger than  $\alpha$ ;
            if  $A$  is an upper triangle then
                 $\beta :=$  content of the lower triangle immediately below  $A$ ;
                move the content of  $A$  to the lower triangle immediately below
                 $A$ ;
                write  $\alpha$  in  $A$ ;
                 $\alpha := \beta$ ;
            else
                 $\beta :=$  content of  $A$ ;
                write  $\alpha$  in  $A$ ;
                 $\alpha := \beta$ ;
            end
        end
    end
end

```

In this section we aim at starting the investigation of the combinatorial properties of this RS-analog. More specifically, we will address the following problem: given a Schröder shape P , can we characterize those permutations having a Schröder tableau of shape P as their insertion tableau? How many of them are there? This problem seems to be quite difficult in its full generality; here we will deal with very few simple cases, for which we can provide complete answers.

3.1 Permutations with Given Schröder Insertion Shape: Some Cases

The first case we investigate is that of a Schröder shape consisting of a single row (which can terminate either with an upper or a lower triangle). To state our result we first need to recall a classical definition.

Given a permutation $\pi = \pi_1 \cdots \pi_n$, we say that π_i is a *left-to-right maximum* (or, briefly, *LR maximum*) whenever $\pi_i = \max(\pi_1, \dots, \pi_i)$.

Proposition 4. *Let $\pi = \pi_1 \cdots \pi_n$ be a permutation of length n . The Schröder insertion tableau of π has a single row if and only if, for all $i \geq n$:*

1. *if i is odd, then π_i is a LR maximum of π ;*
2. *if i is even, then π_i is a LR maximum of the permutation obtained from π by removing π_{i-1} (and suitably renaming the remaining elements).*

Proof. Suppose we are inserting π_i in the insertion tableau P , which is assumed to consist of a single row. If i is odd, then the last cell of P is a lower triangle; in order not to create new rows, π_i has necessarily to be a LR maximum. On the other hand, if i is even, then the last cell of P is an upper triangle; in this case, π_i can be inserted in P in two ways: either π_i is a LR maximum, and so it is simply appended at the end of the unique row of P , or π_i is greater than all previous elements of π but π_{i-1} , hence π_i is inserted in the cell containing π_{i-1} (which is the last cell of the unique row of P , and so it is an upper triangle) and a new cell (a lower triangle) containing π_{i-1} is added at the end of the unique row of P .

Conversely, it is easy (and so left to the reader) to check that a permutation satisfying conditions 1 and 2 in the statement of the present proposition must have a Schröder insertion tableau consisting of a single row. ■

The permutations π of length n whose Schröder insertion tableau have a single row can therefore be simply characterized as follows: for all i , $\{\pi_{2i+1}, \pi_{2i+2}\} = \{2i+1, 2i+2\}$. As a consequence of this fact, a formula for the number of such permutations follows immediately.

Proposition 5. *The set of permutations of length n whose Schröder insertion tableau consists of a single row has cardinality $2^{\lfloor \frac{n}{2} \rfloor}$.*

The second case we consider is the natural counterpart of the previous one, that is Schröder shapes having a single column. Despite the similarities with the previous case, it turns out that the set of permutations having Schröder insertion tableau of this form can be nicely described in terms of *pattern avoidance*.

Given two permutations σ and $\tau = \tau_1 \cdots \tau_n$ (of length k and n respectively, with $k \leq n$), we say that there is an *occurrence* of σ in τ when there exists indices $i_1 < i_2 < \cdots < i_k$ such that $\tau_{i_1} \tau_{i_2} \cdots \tau_{i_k}$ is order isomorphic to σ . When there is an occurrence of σ in τ , we also say that τ *contains the pattern* σ . When τ does not contain σ , we say that τ *avoids the pattern* σ . The set of all permutations of length n avoiding a given pattern σ is denoted with $Av_n(\sigma)$. Some useful references for the combinatorics of patterns in permutations are [Bo] and [K], whereas similar notions of patterns in set partitions and in compositions and words are studied in [M] and [HM], respectively.

Proposition 6. *Let $\pi = \pi_1 \cdots \pi_n$ be a permutation of length n . The Schröder insertion tableau of π has a single column if and only if $\pi \in Av_n(123, 213)$.*

Proof. An argument similar to that of the preceding proposition shows that the Schröder insertion tableau of π has a single column if and only if, for all

$i \leq n$, $\pi_i < \min(\{\pi_1, \dots, \pi_{i-1}\} \setminus \min\{\pi_1, \dots, \pi_{i-1}\})$ (i.e., π_i is smaller than the second minimum of set of all previous elements). Thus π can be factored into subpermutations (made of consecutive elements of π), say $\pi = \tilde{\pi}_1 \cdots \tilde{\pi}_r$, in such a way that each factor $\tilde{\pi}_i$ is isomorphic to a permutation of the form $1t(t-1) \cdots 32$ (for some t) and each element of $\tilde{\pi}_i$ is greater than each element of $\tilde{\pi}_{i+1}$ (for all i). In the language of permutation patterns, this is usually expressed by saying that π is a *skew sum* of permutations of the form $1t(t-1) \cdots 32$. It is now a known fact (see, for instance, [AA]) that such permutations are precisely those avoiding the two patterns 123 and 213. ■

Many classes of permutations avoiding a given set of patterns have been enumerated. The above one is among them, see [SiSc].

Proposition 7. *The set of permutations of length n whose Schröder insertion tableau consists of a single column has cardinality 2^{n-1} .*

We close this section by simply stating (without proof) one more case, which is, in some sense, a generalization of both the cases described above. Namely, we consider the case of what can be called *Schröder hooks*, that is Schröder shapes having at most one row and one column with more than one cell.

Again, we need to recall a classical definition, and also to give a new one. A *shuffle* of two permutations σ and τ (having length n and m , respectively) is a permutation of length $n+m$ having two disjoint subpermutations (not made in general by adjacent elements of π) isomorphic to σ and τ . Moreover, if the subpermutations of σ and τ formed by the first k elements are isomorphic, a *k -rooted shuffle* of σ and τ is a permutation obtained by concatenating the permutation formed by the first k elements of σ (or τ) (with elements suitably renamed) with a shuffle of the subpermutations formed by the remaining elements of σ and τ . For instance, a shuffle of 25143 and $\overline{4132}$ is given by 479218536, and a 3-rooted shuffle of 253461 and 25413 is given by 37561824.

Proposition 8. *Let $\pi = \pi_1 \cdots \pi_n$ be a permutation of length n . The Schröder insertion tableau of π is a Schröder hook if and only if π is a 2-rooted shuffle of two permutations having a single row Schröder insertion tableau and a single column Schröder insertion tableau, respectively.*

4 Further Work

The algebraic and combinatorial properties of the distributive lattice \mathcal{Sch} of Schröder shapes needs to be further investigated. In particular, the analogies with differential posets should be much deepened.

We have just started the characterization and enumeration of permutations having a given Schröder insertion tableau. Many more shapes should be investigated. Moreover, we still have to understand the role of the recording tableau.

Can we find a nice closed formula for the number of Schröder tableaux of a given shape? In the case of Young tableaux there is a famous *hook formula*, which however seems to be unlikely in our case, since we have numerical evidence that, for certain shapes, this number has large prime factors.

An alternative presentation of Schröder tableaux is as Young shapes whose cells are filled in with pairs of distinct integers. This description shows that Schröder tableaux could be somehow related to *interval orders*.

The analogies between Young tableaux and Schröder tableaux should be investigated more, especially from a purely algebraic point of view. Combinatorial objects related to Young tableaux, such as Schur functions and the plactic monoid, as well as algorithmic and algebraic constructions, such as Schützenberger’s *jeu de taquin*, the Littlewood-Richardson rule and the Schubert calculus on Grassmannians and flag varieties, could have some interesting counterparts in the context of Schröder tableaux.

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