The Structural analysis of polygonal masonry domes. The case of Brunelleschi's dome in Florence

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The paper deals with the structural analysis of polygonal masonry domes by taking into account the thickness. A numerical procedure is presented through which the actual behaviour of the material is considered in the analysis. The structure is modelled as a discrete system of rigid blocks linked through elastic mortar layers. No-tension behaviour of the material is assumed to be totally concentrated in the mortar joints located between the adjacent blocks. Such a joint can therefore be assumed as a unilateral elastic contact constraint. The solution is achieved by a step by step algorithm in which the starting solution, relative to the standard material (linear elastic and bilateral), is subsequently corrected according to the actual material behaviour. The numerical procedure can be applied to the analysis of any type of polygonal masonry domes, with a spherical or pointed shape, subject to self weight loads, complete or with hole and lantern. As a particular case of analysis, some interesting remarks dealing with the case of Brunelleschi's Dome are presented.

Keywords: *masonry dome, no-tension analysis, numerical method*

Intoduction

The study of masonry domes has been the object of several research projects starting from some eighteenth century memoirs of the Academiae Royale des Sciences in Paris. One of the first aspects of the problem to be examined, was the search for the optimal shape to be given to a masonry dome. Pierre Bougure (Bouguer 1734), while answering the question of what shape a masonry dome, subjected toself-weight, should have, proposed for the first time the equation of the funicular meridian. He stated that in order to have equilibrium, the meridian must have the same shape of the curve that represents the funicular of the loads which are relative to a slice of dome. Subsequently, a similar solution to the same problem was found by Charles Bossut[2], Lorenzo Mascheroni[3], and Giuseppe Venturoli[4]. All the solutions suggested by these authors have a common characteristic: the dome is considered, *de facto*, as a one-dimensional behaviour structure, composed of a series of distinct segments or slices or "lunes", wider at the base and tapering to zero at the crown, placed in mutual contact with each other but without any interactions among them. In conclusion, since the equilibrium of each slice is investigated separately, if it can be shown that each element of the sliced structure is stable, then it is argued that the original structure must be stable. Under these assumptions the analysis of masonry domes does not present any difference compared with the analysis of masonry arches. It is interesting to note that the fact of not considering any action among the slices of the dome, corresponding to the hypothesis of zero hoop stresses, means that the problem lies, actually, in the field of modern limit analysis, as applied by J.Heyman (Heyman 1967) in his fundamental studies on masonry structures. Moreover, in his introductory assumptions, Heyman himself makes explicit reference to the eighteen-century model, considering that it was still perfectly suitable to deal with the general solution of the problem. The model of limit analysis proposed by Heyman in any way differs from the analysis performed by Poleni (Poleni 1734) or the *Three Mathematicians* (Le Seur et al.1742) for evaluating the stability of the dome of St.Peter. The similarity between the "*settore solido*" of Poleni and the "*orange slice*" of Heyman is very clear.

Masonry dome. An appropriate three-dimensional finite element modelling

Following a typically eighteenth century idea, let us consider the general problem of a masonry structure consisting of rigid blocks linked through elastic mortar layers. In such a model the no-tension behaviour of the material is totally supposed to be concentrated in the mortar joint located between two adjacent blocks. Such a joint, can therefore, be assumed as a unilateral elastic contact constraint.

In particular, the mortar joint can be idealized, in a Drucker's way, through an interface device consisting of a set of elastic links, orthogonal to the contact surface, capable of transmitting only compressive forces between the blocks, and additional links, parallel to the interface, through which the shear forces can be transmitted. The behaviour of the orthogonal links is assumed to be unilateral and linear elastic, whereas for the parallel ones further hypotheses can be added in order to specify either the shear strength and to calibrate, for instance, the influence of the friction between the blocks, or a bilateral rigid behaviour totally capable of preventing sliding (Fig. 1). In practice a reasonably low number of orthogonal bars is sufficient to describe, with significant expressiveness, the behaviour of the joint and to clearly appraise the location and depth of possible cracks.

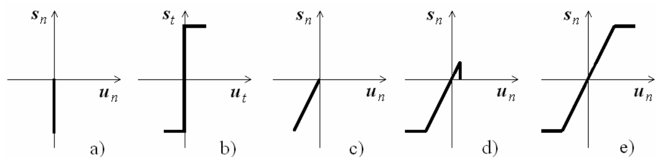


Fig. 1. Behaviour of the joints: rigid-cracking behaviour of the links orthogonal to the interface (a) and tangential (b); elastic-cracking behaviour of the links orthogonal to the interface (c) or (d) and tangential (e).

In the case of a masonry dome the structure is modelled by a set of discrete three-dimensional rigid elements which represent single or multiple blocks of stone. The element of the shell, considering the actual thickness, is cut out by two meridian planes and two sloping planes perpendicular to the generating curve of the middle surface of the shell.

Let us consider, therefore, the general problem of a masonry dome consisting of n three dimensional shell rigid elements linked through m unilateral elastic contact interfaces. (Fig. 2).

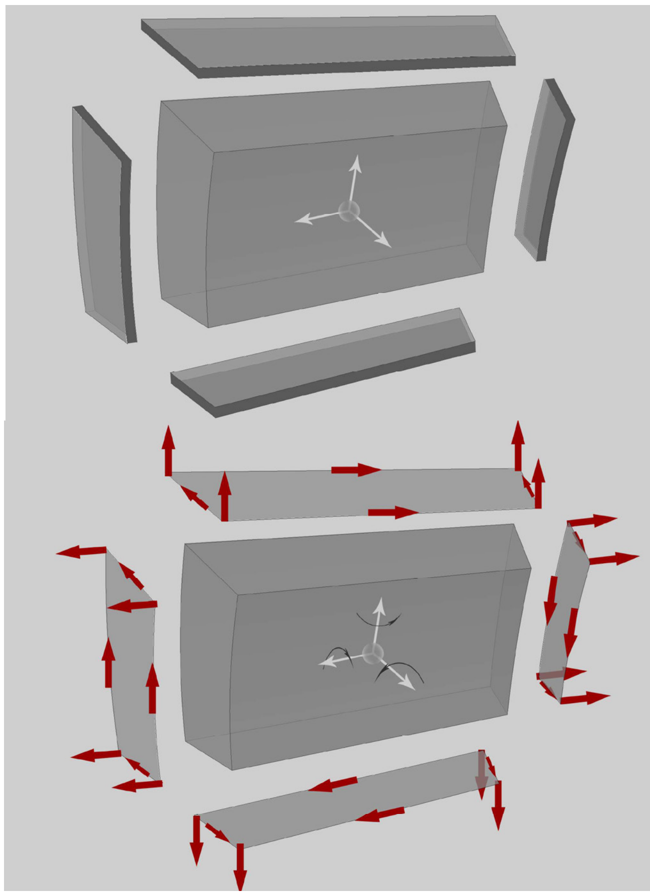


Fig. 2. Internal forces and loads acting on the joint for each element

Assuming the structure subjected to the action of external loads represented by the vectors $F \in \Re^{6n}$, the problem can be expressed through a system of equilibrium and elastic-kinematical equations, which variables, those that correspond to the unilateral links in the interface model, are subject to inequalities:

$$\begin{cases} AX + F = 0 \\ A^T x + KX = \bar{\delta} \end{cases} \quad \text{sub} \begin{cases} X_n \leq 0 \\ X_t \leq f \cdot \sum_{j=1}^{k} X_{nj} \\ \bar{\delta} \geq 0 \end{cases} \qquad (1)$$

In the previous form (1) $X \in \Re^{km}$ indicates the unknown vector of internal forces located on the interface joints, where X_n includes the components orthogonal to the contact surface of the joint and X_t includes the shear components; $x \in \Re$ represents the unknown vector of components of the displacements related to the centroids of the elements ; $K \in \Re^{km \times km}$; is the diagonal stiffness matrix of the contact constraints.; $\delta \in \Re^{km}$ indicates the unknown vector whose components are internal distortions which need for obtaining a solution capable of satisfying both the equilibrium equations, while respecting the sign conditions, and the elastic-kinematical compatibility of the actual reacting structure. On this subject, it is convenient to distinguish, within the vector δ , two types of entities, assuming for the former, related to the equilibrium aspects, the notation δ_1 and for the latter, related to the compatibility ones, the notation δ_2 . Notice that the value *k* depends on the number of contact constraints chosen to characterize the interface device and states the degree of statically indeterminacy of the structure.

Of course the system of equations (1) could have no solution under the sign conditions expressed in the first inequality; in such a case it means that the structure cannot be equilibrated under the given system of the external actions. In this case there is no vector $X \in \Re^{km}$ which simultaneously satisfies, the $6n$ equations and the km inequalities.

However let us suppose that the system (1) is consistent. In such a case, the general solution $X = X_0 + X_N$, that is able to satisfy the equilibrium problem and the first of the two inequalities, can be obtained assuming, as the initial solution X_0, which is relative to the bilateral linear elastic behaviour of the contact constraints:

$$X_0 = K^{-1} A^T (AK^{-1} A^T)^{-1} F \qquad (2)$$

Such an initial solution is then modified through the vector:

$$X_N = (I - K^{-1} A^T (AK^{-1} A^T)^{-1} A) \cdot \overline{\delta_1} = C \cdot \overline{\delta_1} \qquad (3)$$

which, added to X_0, satisfies the first equation of the system (1) while respecting the sign conditions. Computing the Moore-Penrose generalized inverse of C_i, it is easily possible to evaluate the vector $\delta_{1i} = -C_i X_{0i}$, where X_{0i} are the components of the interactions that, in the joint, do not respect the inequalities of (1). If the solution of the unilateral problem exists, the vector solution, which satisfies simultaneously the equilibrium equations and the first and the second inequality of (1), assumes the form:

$$X = \begin{bmatrix} 0 \\ X_t \\ X_c \end{bmatrix}$$

where for each joint:

$$X_c < 0 \text{ and } X_t \leq f \cdot \sum_{j=1}^{k} X_{nj} \qquad (4)$$

Since the final vector X is different from the first elastic vector solution X_0, it cannot satisfy the kinematical compatibility expressed through the second set of equations in the system (1). A very easy way to again build up such a compatibility is to consider the second set of equations in system (1) in the form:

$$\bar{x} = -(A_c A_c^T)^{-1} A_c K_c X_c \qquad (5)$$

which represents the vector of the displacements of the centroids of the elements only relative to the actual reacting structure. Finally, one can determine the vector δ_2, in such a way that the compatibility of the second of system (1) is again obtained : $\delta_2 = A_i^T x$.

The components of the vector $\delta_2 \neq 0$ give the position and width of the cracks located in the mortar joints.

First numerical applications

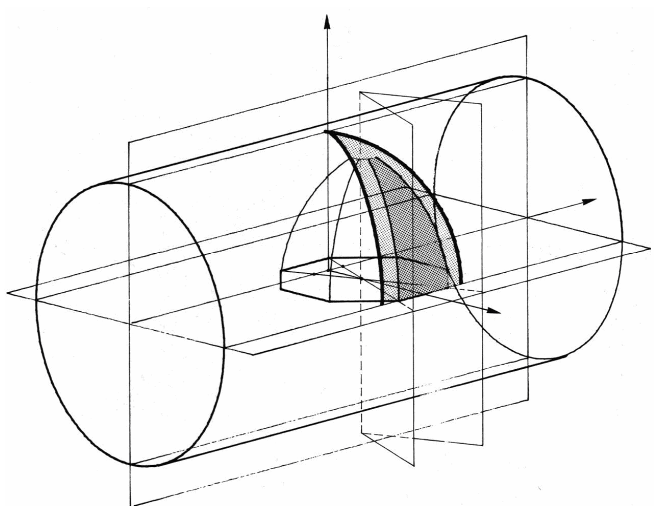
The numerical procedure described in the previous paragraph can be applied to the analysis of two types of axial-symmetrical masonry domes, respectively hemispherical and ogival, subject to self weight loads, with three different conditions at the crown: complete, with hole and lantern. In any case, with assigned radius R, the thickness *s* has been considered as a constant. According to the axial symmetry of the structure the internal shear forces acting along the rings and on the planes of the meridians become zero. The stiffness in the contact devices, which simulates the behaviour of the mortar layers, has been considered uniform. The procedure allows one to define the limit configuration of equilibrium and, correspondently, the collapse mechanism that is due to the formation of cracks along the meridian and parallel interface sections. With reference to the limit configuration of equilibrium the minimum thickness of the dome is identified and the corresponding ratio *s / R* is performed.

Afterwards an interesting comparison is presented between the numerical method described here and two different solutions of the same cases (Table 1).

The first solution concerns Heyman's hypothesis, also used by Hoppenheim, based on the assumption that the resultant hoop stress between contiguous lunes of the dome is zero. This means that one considers the dome as a one-dimensional behaviour structure, to be seen as composed of a series of distinct segments or slices or "lunes", wider at the base and tapering to zero at the crown, placed in mutual contact with each other but without any interactions among them: of course, such a solution must be considered as limit solution.

Brunelleschi's dome in Florence

The numerical procedure described above has been applied to the analysis of the Dome of Santa Maria del Fiore. The first aspect to tackle was to define the overall geometrical model of the structure. The Dome, formed by two shells divided by a space of approximately 1.2 m, is made up of eight lunes that represent segments of an elliptical cylinder. To better understand the spatial location of the volume of the building see figure 3.



		Heyman's hypothesis ▪ limit thickness	Method presented here ▪ limit thickness ▪ last compressive ring
Hemispherical dome	complete	▪ 0.044 R	▪ 0.043 R ▪ φ = 29.59°
	hole and lantern	▪ 0.050 R	▪ 0.044 R ▪ φ = 24.00°
Ogival dome	complete	▪ 0.085 R	▪ 0.050 R ▪ φ = 48.00°
	hole and lantern	▪ 0.032 R	▪ 0.022 R ▪ φ = 40.00°

Tab. 1: Comparison between different approaches.

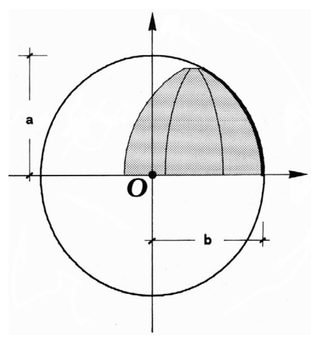
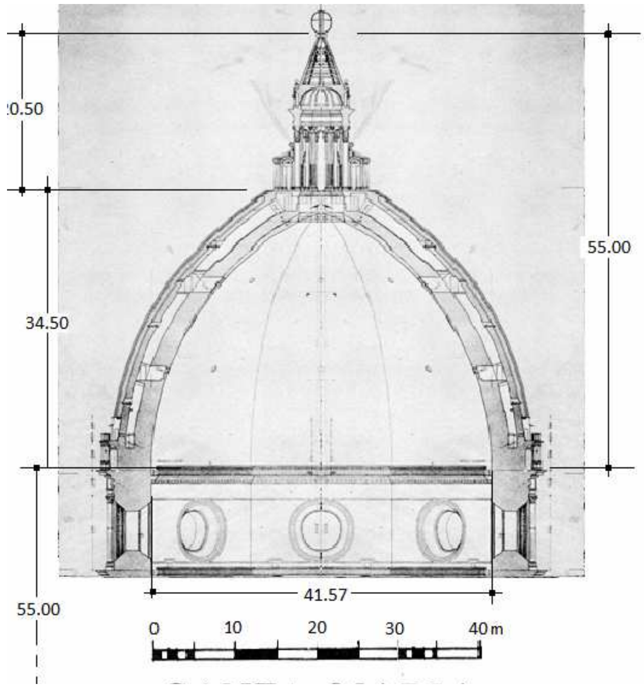
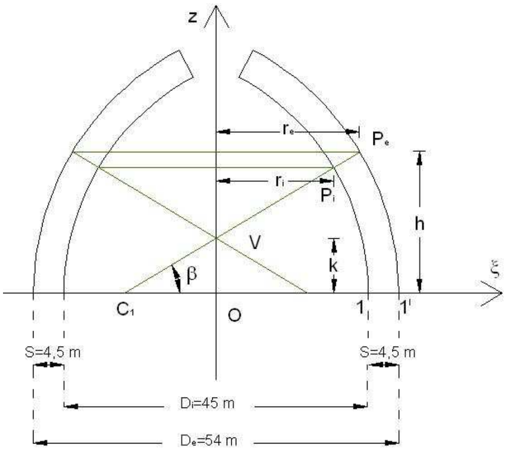
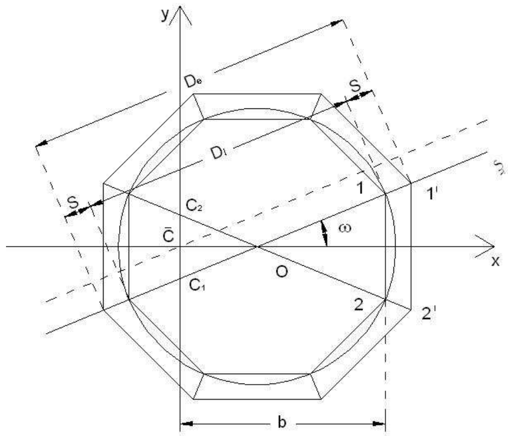


Fig. 3 Axonometric view of the general geometry of the segment of dome relating to the elliptical cylinder.

The geometrical model adopted for the analysis is based on the studies, published in 1977, by Salvatore Di Pasquale; for the contents refer to the bibliography. The principal aspect is that every laying bed of the bricks is lying on a conic surface, and is easily indentified by the intersection between the cylindrical segment of the dome, with elliptical directrix, and the surface of ideal cones, defined by rotating a generatrix line with a variable angle and a vertex, whose level moves progressively upwards, located on the axis of the dome (fig. 4 - 5).







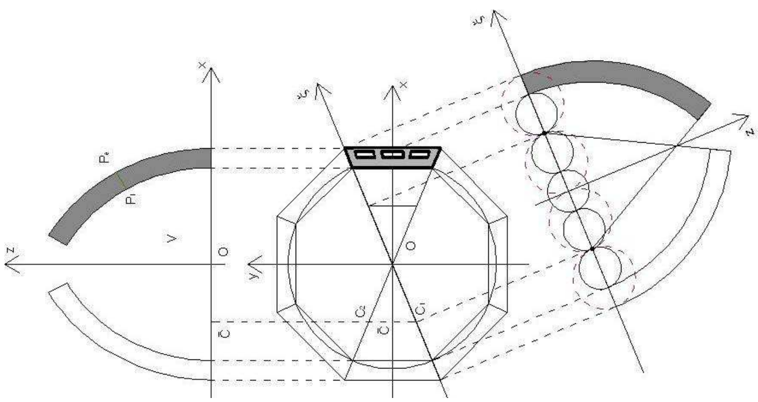


Fig. 4 Brunelleschi's Dome. Geometry and structure.

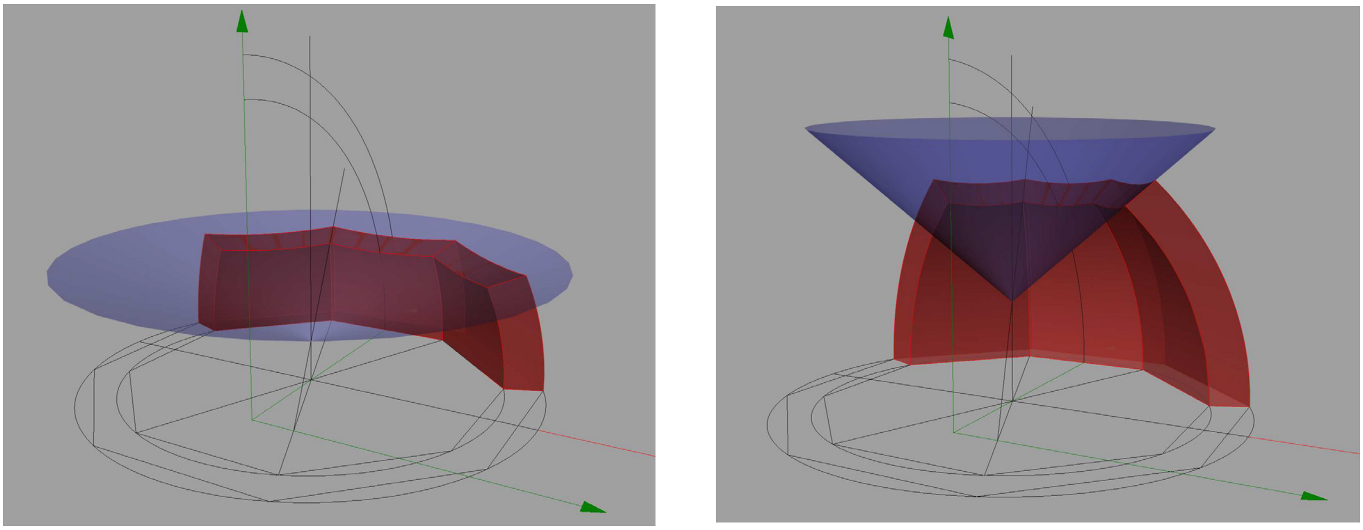


Fig. 5. Axonometric view of geometrical model of conic laying bed of the bricks.

Below are some mathematical notations for the analytic reading of the problem:

$a = \overline{C_1 1}$ = radius of curvature of the big corner rib ($a = \frac{4}{5}D = 36\ m$)

$b = a \cdot \cos\varpi$ (where $\varpi = \frac{\pi}{8}$)

$h = a \cdot sen\beta$ (z of P on the big corner rib – with $0 \le \beta \le 62°$)

$k = \frac{3}{8}a \cdot tg\beta$ (z of the vertex V of the cone)

$r = a(\cos\beta - \frac{3}{8})$ (radius of the circle at the base of the generic cone)

Equation of the cone (x, y, z) :

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} - \frac{(z-k)^2}{(h-k)^2} = 0 \tag{6}$$

Let us define a orthogonal plane (x, y) through the axis z as a function of φ ($-\frac{\pi}{8} \le \beta \le \frac{\pi}{8}$)

$$y = x \cdot tg\varphi \tag{7}$$

Substituting in (6) the equation above becomes:

$$\frac{x^2}{2} + \frac{x^2 \cdot tg^2\varphi}{rr^2} - \frac{(z-k)^2}{(h-k)^2} = 0$$

The equation of the elliptical cylinder is:

$$z^2 + \frac{\left(x - \frac{3}{8}ac\right)^2}{c^2} = a^2 \qquad \text{with} \quad c = \cos\varpi \tag{8}$$

Numerical remarks

The structure of the Dome has been analyzed using the numerical model presented above. The numerical tests have focused on the general stability of the Monument despite the presence of inevitable and manifest cracks along the meridians in the middle of the lune and on the big corner ribs (Figures 7 and 8). The numerical results, obtained under the no-tension hypothesis for the material, confirmed in every case, the actual behaviour of the structure (Table 2).
In Figure 6 a series of synthesized images of the various models tested is shown. In the construction of the geometrical shape the conical laying beds of the bricks has been taken into account: this has influenced the mesh of the finite element model.

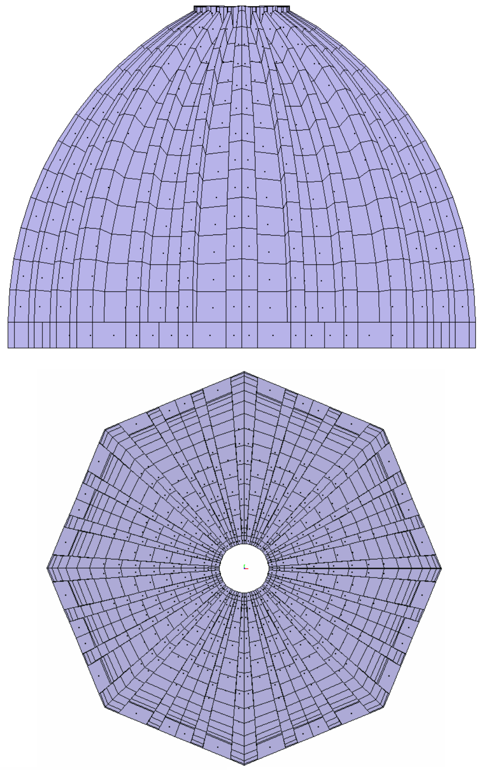


Fig. 6 . Views of the geometrical model and mesh.

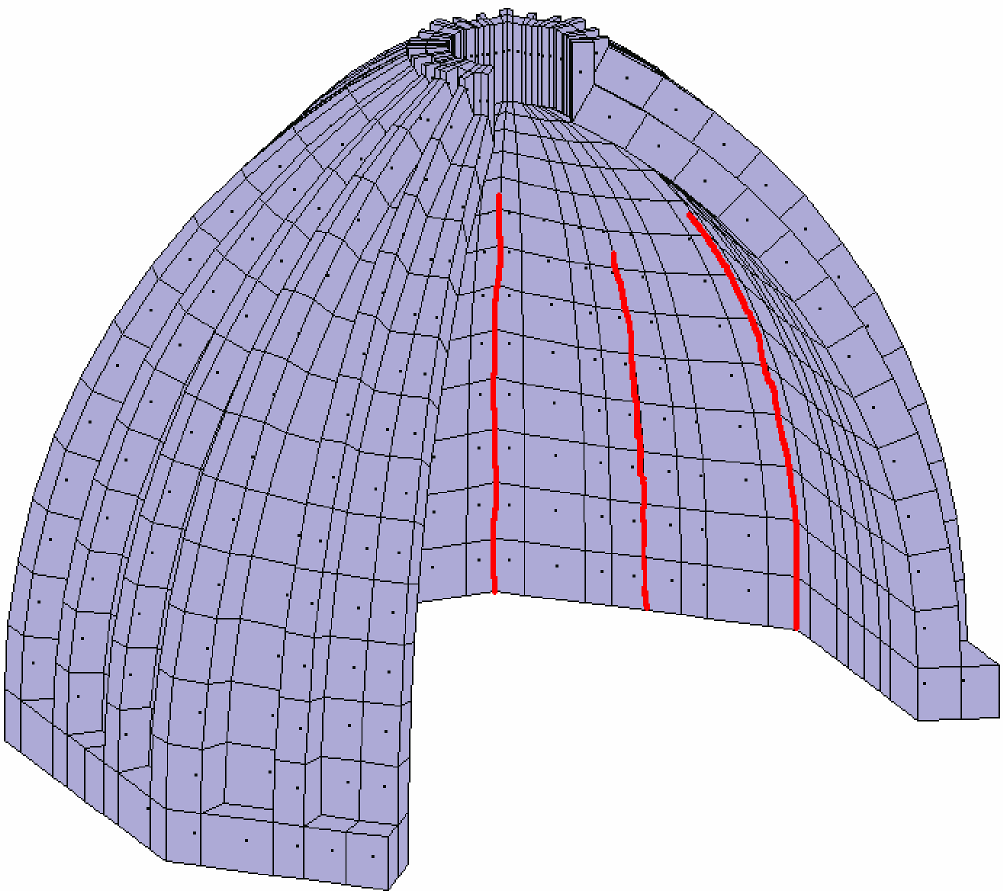


Fig.7. Axonometric view. Location and spreading of cracks.

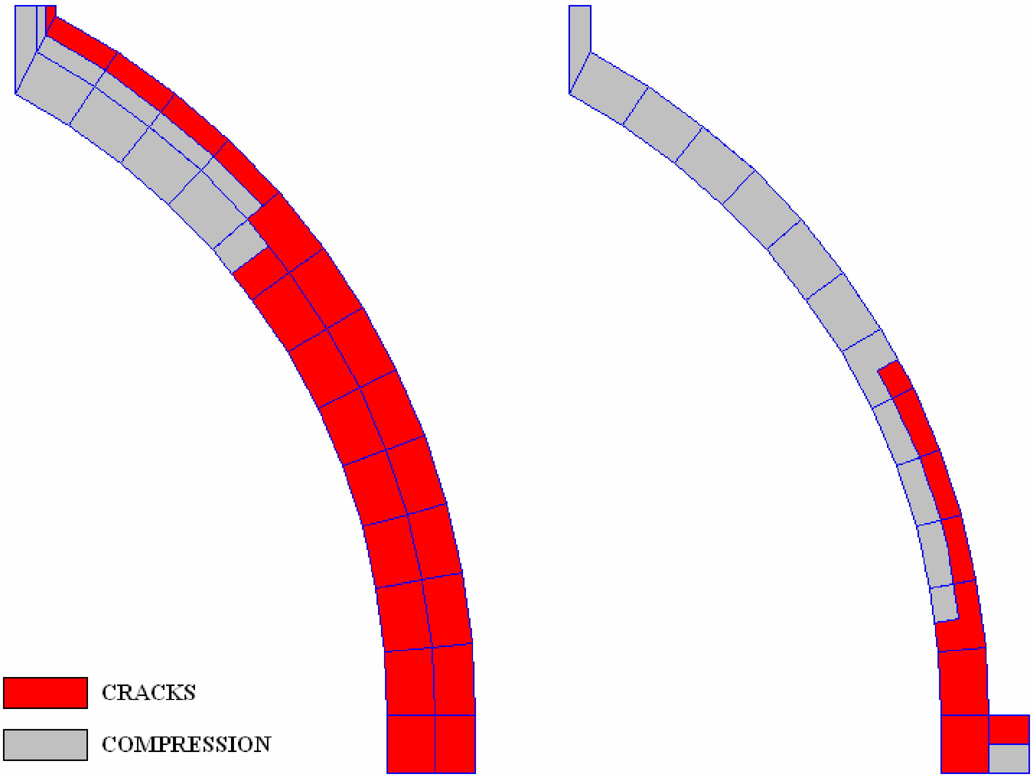


Fig. 8. Location and spreading of cracks. Cross section on corner big rib (left) and on the middle meridian of the lune (right).

Maximum compressive stress (MPa) at the base of the Dome		
Corner big rib	Medium rib	Middle of the lune
0.79	0.78	0.47

Table 2: Values of compressive stress at the base of the Dome.

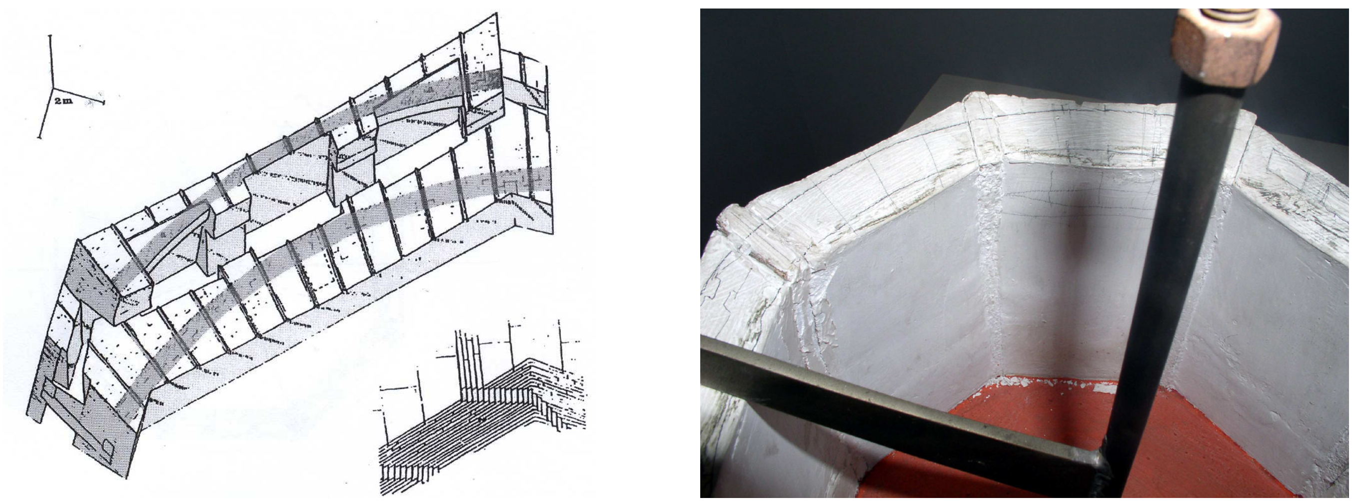


Fig. 9. Ideal circular dome within the actual thickness.

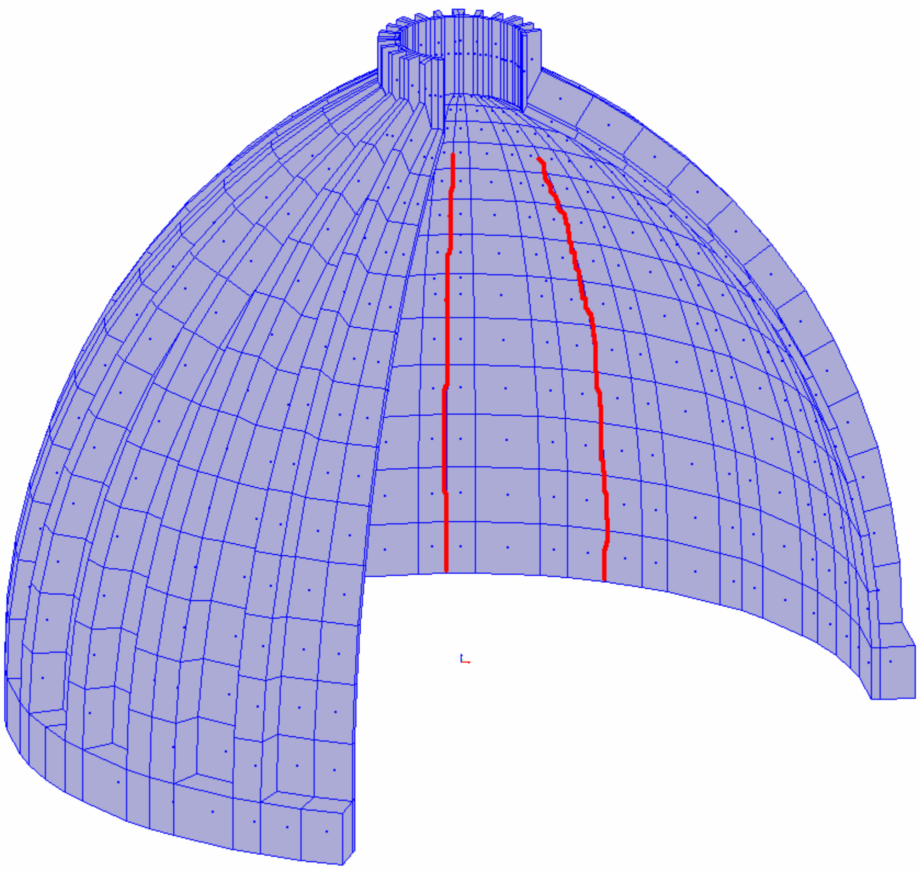


Fig. 10. Axonometric view.
Location and spreading of cracks.

Following the idea of Di Pasquale, a significant test was performed by modeling the structure in order to obtain, within the actual structure, a circular dome that, while keeping the ribs, has a possible thickness of only 48 cm (fig. 9). The Dome is still stable also in this case (fig. 10)

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