



UNIVERSITÀ  
DEGLI STUDI  
FIRENZE

# FLORE

## Repository istituzionale dell'Università degli Studi di Firenze

### Dirichlet problems for fully anisotropic elliptic equations

Questa è la Versione finale referata (Post print/Accepted manuscript) della seguente pubblicazione:

*Original Citation:*

Dirichlet problems for fully anisotropic elliptic equations / Barletta, Giuseppina; Cianchi, Andrea. - In: PROCEEDINGS OF THE ROYAL SOCIETY OF EDINBURGH. SECTION A. MATHEMATICS. - ISSN 0308-2105. - STAMPA. - 147:(2017), pp. 25-60. [10.1017/S0308210516000020]

*Availability:*

The webpage <https://hdl.handle.net/2158/1040587> of the repository was last updated on 2021-03-11T18:54:53Z

*Published version:*

DOI: 10.1017/S0308210516000020

*Terms of use:*

Open Access

La pubblicazione è resa disponibile sotto le norme e i termini della licenza di deposito, secondo quanto stabilito dalla Policy per l'accesso aperto dell'Università degli Studi di Firenze (<https://www.sba.unifi.it/upload/policy-oa-2016-1.pdf>)

*Publisher copyright claim:*

La data sopra indicata si riferisce all'ultimo aggiornamento della scheda del Repository FloRe - The above-mentioned date refers to the last update of the record in the Institutional Repository FloRe

(Article begins on next page)

# Dirichlet problems for fully anisotropic elliptic equations

Giuseppina Barletta

*Dipartimento di Ingegneria Civile, dell'Energia, dell'Ambiente e dei Materiali*

*Università Mediterranea di Reggio Calabria*

*Via Graziella - Loc. Feo di Vito, 89122, Reggio Calabria (Italy)*

*e-mail: giuseppina.barletta@unirc.it*

Andrea Cianchi

*Dipartimento di Matematica e Informatica " U. Dini "*

*Università di Firenze*

*Viale Morgagni 67/A, 50134 Firenze (Italy)*

*e-mail: cianchi@unifi.it*

## Abstract

The existence of a nontrivial bounded solution to the Dirichlet problem, for a class of nonlinear elliptic equations involving a fully anisotropic partial differential operator, is established. The relevant operator depends on the gradient of the unknown through the differential of a general convex function. This function need not be radial, nor have a polynomial type growth. Besides providing genuinely new conclusions, our result recovers and embraces, in a unified framework, several contributions in the existing literature, and augments them in various special instances.

## 1 Introduction

We are concerned with Dirichlet problems for elliptic equations of the form

$$(1.1) \quad \begin{cases} -\operatorname{div}(\Phi_\xi(\nabla u)) = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega$  is an open set in  $\mathbb{R}^n$ ,  $n \geq 2$ , with finite Lebesgue measure  $|\Omega|$ , the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and vanishes at 0, and  $\Phi : \mathbb{R}^n \rightarrow [0, \infty)$  is an even, strictly convex function, vanishing at 0. The notation  $\Phi_\xi$  stands for the gradient of  $\Phi$ . Let us emphasize that  $\Phi(\xi)$  neither necessarily depends on  $\xi$  through its length  $|\xi|$ , nor necessarily has a power type behavior.

The equation in (1.1) is the Euler equation of the functional

$$(1.2) \quad J_\Phi(u) = \int_\Omega (\Phi(\nabla u) - F(u)) \, dx,$$

---

*This research was partly supported by the Research Project of Italian Ministry of University and Research (MIUR) "Elliptic and parabolic partial differential equations: geometric aspects, related inequalities, and applications" 2012, and by the GNAMPA Research Project of INdAM (National Institute of High Mathematics) "Variational methods for quasi-linear differential problems in non-standard settings".*

where  $F : \mathbb{R} \rightarrow \mathbb{R}$  is given by

$$(1.3) \quad F(t) = \int_0^t f(s) ds \quad \text{for } t \in \mathbb{R}.$$

Clearly, the function  $u = 0$  is a trivial solution to (1.1). The aim of the present paper is to show that, under suitable assumptions on  $\Phi$  and  $f$ , problem (1.1) also admits a nontrivial solution, which is a critical point of the functional (1.2).

Reference contributions on critical point methods for nonlinear elliptic boundary value problems with lack of coercivity include the paper [AR] and the monographs [MW, Ra, St, Wi]. The vast literature on these topics consists of a huge amount of papers. We do not attempt an even partial list of them.

The existence of nontrivial solutions to elliptic equations associated with non-coercive functionals is well known to depend on a balance between the nonlinearity in the trial functions, and the nonlinearity in their gradient. In particular, the behavior near infinity of the functions governing these nonlinearities is dictated by a Sobolev type inequality. Clarifying this issue with regard to problem (1.1) is one of the main focuses of our research.

Prototypal results in this line of investigations deal with semilinear equations of the form

$$(1.4) \quad -\Delta u = f(u),$$

or with the more general  $p$ -Laplacian type equations

$$(1.5) \quad -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = f(u).$$

Equation (1.5) is the Euler equation of (1.2) with  $\Phi(\xi) = \frac{1}{p}|\xi|^p$ ; equation (1.4) corresponds to the special case when  $p = 2$ . Besides other assumptions, the existence of nontrivial solutions to Dirichlet problems associated with (1.5) is guaranteed if either  $p > n$ , or  $1 < p < n$  and

$$(1.6) \quad \lim_{|t| \rightarrow \infty} \frac{tf(t)}{|t|^q} = 0 \quad \text{for some } q < p^*,$$

where  $p^* = \frac{np}{n-p}$ , the Sobolev conjugate of  $p$ . The threshold  $p^*$  for  $q$  in (1.6) is known to be sharp, as a consequence of the Pohozaev identity. In the borderline case when  $p = n$ , growths of  $tf(t)$  slower than  $e^{tn'}$  are allowed, where  $n' = \frac{n}{n-1}$ . The function  $e^{tn'}$  appears in an embedding theorem by [Po, Tr, Yu], which replaces the standard Sobolev embedding in this critical situation.

Equations associated with non-coercive functionals with non-necessarily polynomial growth in the gradient have been investigated e.g. in [CGMS] in an Orlicz-Sobolev space setting (see also [FOR] for related problems). The relevant equations read

$$(1.7) \quad -\operatorname{div} \left( \frac{A'(|\nabla u|)}{|\nabla u|} \nabla u \right) = f(u),$$

where  $A : [0, \infty) \rightarrow [0, \infty)$  is a continuously differentiable, strictly convex function vanishing at 0, and  $A'$  denotes its derivative. These equations are still isotropic, in the sense that the coefficient of the differential operator, and the associated functional, just depend on the length of the gradient. Indeed,  $\Phi(\xi) = A(|\xi|)$  in this case. The result of [CGMS] requires that

$$(1.8) \quad \lim_{|t| \rightarrow \infty} \frac{tf(t)}{B(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0,$$

where  $B$  is a Young function introduced in a (non-sharp, in general) embedding theorem for Orlicz-Sobolev spaces of [DT].

Genuinely anisotropic equations of the form

$$(1.9) \quad -\sum_{i=1}^n (|u_{x_i}|^{p_i-2} u_{x_i})_{x_i} = f(u),$$

where  $p_i > 1$  for  $i = 1, \dots, n$ , and the subscript  $x_i$  denotes partial derivative with respect to  $x_i$ , are the subject of [FGK]. They are the Euler equations of functionals whose integrand is endowed with a peculiar structure, and agrees with the sum of multiples of the powers  $p_i$  of the partial derivatives of trial functions. Precisely,  $\Phi(\xi) = \sum_{i=1}^n \frac{1}{p_i} |\xi_i|^{p_i}$ . A basic role in discussing anisotropic equations of the form (1.9) is played by the harmonic average  $\bar{p}$  of the exponents  $p_i$ , defined by

$$(1.10) \quad \frac{1}{\bar{p}} = \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}.$$

In [FGK], it is assumed that either  $\bar{p} > n$ , or  $1 < \bar{p} < n$  and

$$(1.11) \quad \lim_{|t| \rightarrow \infty} \frac{tf(t)}{|t|^q} = 0 \quad \text{for some } q < \bar{p}^*,$$

where  $\bar{p}^*$  stands for the Sobolev conjugate of  $\bar{p}$ .

The novelty of our contribution is twofold. On one hand, it deals with general problems as in (1.1) involving a function  $\Phi$  without any additional special structure. In particular, Dirichlet problems associated with equations of the form (1.4), (1.7) and (1.9) are encompassed as special instances. On the other hand, even in these special instances, our result enhances the available results in the literature under some respect.

The underlying functional framework of the present paper is quite unconventional, due to the general structure of the equations in question. This calls for the development of some new aspects of the theory of anisotropic Orlicz-Sobolev spaces, which provide a natural function space setting for the problems under consideration. A key role is played by a notion of subcritical growth for  $f$  near infinity, which depends on a sharp embedding theorem for anisotropic Orlicz-Sobolev spaces. Such an embedding involves a Young function  $\Phi_n$ , which enters as an optimal Sobolev conjugate of  $\Phi$ . A precise statement of our main result can be found in the next section. Here, we limit ourselves to mentioning that our requirement on  $f$  near infinity amounts to

$$(1.12) \quad \lim_{|t| \rightarrow \infty} \frac{tf(t)}{\Phi_n(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0,$$

unless (a suitable average of)  $\Phi$  grows so fast for every admissible function  $u$  to be automatically bounded, in which case no assumption on  $f$  near infinity is needed. Let us emphasize that, not only conditions (1.6), (1.8) and (1.11) are included in (1.12), but they are also weakened by (1.12) in certain situations. For instance, even when  $\Phi$  depends on  $\xi$  just through its length  $|\xi|$ , the function  $\Phi_n$  may actually grow faster than the function  $B$  appearing in (1.8).

## 2 Main result

A formulation of our existence theorem requires a few notations. Given any function  $\Phi \in C^1(\mathbb{R}^n)$  as above, define the quantities

$$i_\Phi = \liminf_{|\xi| \rightarrow \infty} \frac{\xi \cdot \Phi_\xi(\xi)}{\Phi(\xi)}, \quad s_\Phi = \limsup_{|\xi| \rightarrow \infty} \frac{\xi \cdot \Phi_\xi(\xi)}{\Phi(\xi)},$$

where the dot “ $\cdot$ ” denotes scalar product in  $\mathbb{R}^n$ . Note that, owing to our assumptions on  $\Phi$ , one has that  $1 \leq i_\Phi \leq s_\Phi \leq \infty$ .

By  $\Phi_* : [0, \infty) \rightarrow [0, \infty)$  we denote the (convex) function obeying

$$(2.1) \quad |\{\xi \in \mathbb{R}^n : \Phi(\xi) \leq t\}| = |\{\xi \in \mathbb{R}^n : \Phi_*(|\xi|) \leq t\}| \quad \text{for } t \geq 0.$$

Observe that the function  $\xi \mapsto \Phi_*(|\xi|)$  agrees with the spherically increasing symmetral of  $\Phi$ , and can be regarded as a kind of “average in measure” of  $\Phi$ .

Next, we call  $\Phi_n : [0, \infty) \rightarrow [0, \infty]$  the optimal Sobolev conjugate of  $\Phi$ , defined as

$$(2.2) \quad \Phi_n(t) = \Phi_*(H^{-1}(t)) \quad \text{for } t \geq 0,$$

where  $H : [0, \infty) \rightarrow [0, \infty)$  is given by

$$H(t) = \left( \int_0^t \left( \frac{\tau}{\Phi_*(\tau)} \right)^{\frac{1}{n-1}} d\tau \right)^{\frac{n-1}{n}} \quad \text{for } t \geq 0,$$

provided that the integral is convergent. Here,  $H^{-1}$  denotes the generalized left-continuous inverse of  $H$ . The function  $\Phi_n$  was introduced in [Ci3], where a sharp embedding theorem for anisotropic Orlicz-Sobolev spaces is presented (see also [Ci1, Ci2] for the isotropic case).

We are now ready to state and briefly comment on the assumptions of our main result, contained in Theorem 2.1 below. To begin with, we require that

$$(2.3) \quad 1 < i_\Phi \quad \text{and} \quad s_\Phi < \infty.$$

Condition (2.3) ensures, in particular, the reflexivity of the anisotropic Orlicz-Sobolev space associated with the function  $\Phi$  – see Proposition 3.1, Section 3.

In the present setting, an Ambrosetti-Rabinowitz type condition takes the form

$$(2.4) \quad \liminf_{t \rightarrow \pm\infty} F(t) > 0,$$

and

$$(2.5) \quad \liminf_{t \rightarrow \pm\infty} \frac{t f(t)}{F(t)} > s_\Phi.$$

A decay assumption for  $f$  at 0 depends on  $\Phi$  only through  $\Phi_*$ , and reads

$$(2.6) \quad \lim_{t \rightarrow 0} \frac{t f(t)}{\Phi_*(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0.$$

Finally, the subcritical growth condition on  $f$  at infinity to which we alluded above comes into play. The growth condition in question is only needed when

$$(2.7) \quad \int_0^\infty \left( \frac{\tau}{\Phi_*(\tau)} \right)^{\frac{1}{n-1}} d\tau = \infty,$$

and amounts to requiring that

$$(2.8) \quad \int_0^\infty \left( \frac{\tau}{\Phi_*(\tau)} \right)^{\frac{1}{n-1}} d\tau < \infty,$$

and

$$(2.9) \quad \lim_{t \rightarrow \pm\infty} \frac{tf(t)}{\Phi_n(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0 .$$

If, on the contrary,

$$(2.10) \quad \int^\infty \left( \frac{\tau}{\Phi_*(\tau)} \right)^{\frac{1}{n-1}} d\tau < \infty ,$$

a condition which ensures that any function from the Orlicz-Sobolev space associated with  $\Phi$  is bounded, then no further condition on  $f$  near infinity has to be imposed.

**Theorem 2.1** *Let  $\Omega$  be an open set in  $\mathbb{R}^n$ , with  $n \geq 2$ , such that  $|\Omega| < \infty$ . Assume that  $\Phi \in C^1(\mathbb{R}^n)$  is an even, strictly convex function, vanishing at 0. Let  $f$  be a continuous function. Assume that conditions (2.3)–(2.6) are fulfilled, and that either (2.7) holds and (2.8)–(2.9) are in force, or (2.10) holds. Then the Dirichlet problem (1.1) admits a non trivial, bounded, weak solution  $u$ .*

Theorem 2.1 is specialized to a few special instances, including (1.4), (1.7) and (1.9), in Section 5. Section 3 is devoted to some preliminary results on anisotropic Orlicz-Sobolev spaces built upon  $n$ -dimensional Young functions. The proof of Theorem 2.1 is then accomplished in Section 4.

### 3 Anisotropic Orlicz-Sobolev spaces and the Nemytskii operator

A function  $A : [0, \infty) \rightarrow [0, \infty]$  is called a Young function if it is convex, vanishes at 0, and is neither identically equal to 0, nor to infinity. Definitions and properties concerning Young functions, as well as  $n$ -dimensional Young functions, to be used in what follows, are collected in the Appendix.

Let  $G$  be a measurable set in  $\mathbb{R}^N$ , with  $N \geq 1$ . The Orlicz space  $L^A(G)$  is the set of all measurable functions  $u : G \rightarrow \mathbb{R}$  such that the Luxemburg norm

$$\|u\|_{L^A(G)} = \inf \left\{ \lambda > 0 : \int_G A\left(\frac{1}{\lambda}|u|\right) dx \leq 1 \right\}$$

is finite. The functional  $\|\cdot\|_{L^A(G)}$  is a norm on  $L^A(G)$ , which makes the latter a Banach space. If  $|G| < \infty$  and  $A \in \Delta_2$  near infinity, then  $\int_G A(|u|)dx < \infty$  for every  $u \in L^A(G)$ .

The Hölder type inequality

$$(3.1) \quad \int_G |uv| dx \leq 2\|u\|_{L^A(G)}\|v\|_{L^{\tilde{A}}(G)}$$

holds for every  $u \in L^A(G)$  and  $v \in L^{\tilde{A}}(G)$ . Here,  $\tilde{A}$  denotes the Young conjugate of  $A$ .

Assume that  $|G| < \infty$ . Let  $A$  and  $B$  be Young functions such that  $A$  dominates  $B$  near infinity. Then

$$L^A(G) \rightarrow L^B(G),$$

where the arrow “ $\rightarrow$ ” stands for continuous embedding. In particular,

$$L^A(G) \rightarrow L^1(G)$$

for any Young function  $A$ .

Orlicz spaces of  $\mathbb{R}^n$ -valued measurable functions are built upon  $n$ -dimensional Young functions. Let  $n \geq 1$ . A function  $\Phi : \mathbb{R}^n \rightarrow [0, \infty]$  is called an  $n$ -dimensional Young function if it is convex,  $\Phi(0) = 0$ ,  $\Phi(\xi) = \Phi(-\xi)$  for  $\xi \in \mathbb{R}^n$ , and for every  $t > 0$ , the set  $\{\xi \in \mathbb{R}^n : \Phi(\xi) < t\}$  is bounded and contains an open neighborhood of 0.

The Orlicz space  $L^\Phi(G, \mathbb{R}^n)$  is the set of all measurable functions  $U : G \rightarrow \mathbb{R}^n$  such that the norm

$$\|U\|_{L^\Phi(G, \mathbb{R}^n)} = \inf \left\{ \lambda > 0 : \int_G \Phi\left(\frac{1}{\lambda}U\right) dx \leq 1 \right\}$$

is finite. The space  $L^\Phi(G, \mathbb{R}^n)$ , equipped with this norm, is a Banach space.

The Hölder type inequality

$$(3.2) \quad \int_G |U \cdot V| dx \leq 2 \|U\|_{L^\Phi(G, \mathbb{R}^n)} \|V\|_{L^{\tilde{\Phi}}(G, \mathbb{R}^n)}$$

holds for every  $U \in L^\Phi(G, \mathbb{R}^n)$  and  $V \in L^{\tilde{\Phi}}(G, \mathbb{R}^n)$ , where  $\tilde{\Phi}$  denotes the Young conjugate of  $\Phi$ . Assume that  $|G| < \infty$ . If  $\Phi \in \Delta_2$  near infinity, then  $\int_G \Phi(U) dx < \infty$  for every  $U \in L^\Phi(G, \mathbb{R}^n)$ . By [Sch, Corollary 7.2],

$$(3.3) \quad L^\Phi(G, \mathbb{R}^n) \text{ is reflexive if and only if } \Phi \in \Delta_2 \cap \nabla_2 \text{ near infinity.}$$

If  $\Phi$  and  $\Psi$  are  $n$ -dimensional Young functions such that  $\Phi$  dominates  $\Psi$  near infinity, then

$$L^\Phi(G, \mathbb{R}^n) \rightarrow L^\Psi(G, \mathbb{R}^n).$$

In particular,

$$(3.4) \quad L^\Phi(G, \mathbb{R}^n) \rightarrow L^1(G, \mathbb{R}^n)$$

for any  $n$ -dimensional Young function  $\Phi$ .

Now, let  $\Omega$  be an open set in  $\mathbb{R}^n$ ,  $n \geq 2$ , such that  $|\Omega| < \infty$ . Given an  $n$ -dimensional Young function  $\Phi$ , the anisotropic Orlicz-Sobolev space  $W_0^{1,\Phi}(\Omega)$  is defined as

$$W_0^{1,\Phi}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} : \text{the continuation of } u \text{ by 0 outside } \Omega \\ \text{is weakly differentiable in } \mathbb{R}^n, \text{ and } \nabla u \in L^\Phi(\Omega, \mathbb{R}^n)\}.$$

The isotropic Orlicz-Sobolev space  $W_0^{1,A}(\Omega)$  associated with a Young function  $A$  is defined analogously, on requiring that  $|\nabla u| \in L^A(\Omega)$ .

One has that  $W_0^{1,\Phi}(\Omega)$ , equipped with the norm

$$\|u\|_{W_0^{1,\Phi}(\Omega)} = \|\nabla u\|_{L^\Phi(\Omega, \mathbb{R}^n)},$$

is a Banach space. A proof of this fact relies upon standard properties of weak derivatives, and of  $n$ -dimensional Young functions.

**Proposition 3.1** *Let  $\Phi$  be an  $n$ -dimensional Young function such that  $\Phi \in \Delta_2 \cap \nabla_2$  near infinity. Then the Orlicz-Sobolev space  $W_0^{1,\Phi}(\Omega)$  is reflexive.*

**Proof.** This is a consequence of property (3.3), and of the fact that  $W_0^{1,\Phi}(\Omega)$  is isometrically isomorphic to a closed subspace of the Orlicz spaces  $L^\Phi(\Omega, \mathbb{R}^n)$  via the map

$$W_0^{1,\Phi}(\Omega) \ni u \mapsto (u_{x_1}, \dots, u_{x_n}) \in L^\Phi(\Omega, \mathbb{R}^n).$$

□

An anisotropic Poincaré type inequality for functions in  $W_0^{1,\Phi}(\Omega)$  is stated in the next proposition.

**Proposition 3.2** *Let  $\Phi$  be an  $n$ -dimensional Young function, and let  $\Phi_*$  be the Young function given by (2.1). Then*

$$(3.5) \quad \int_{\Omega} \Phi_*(|u|) dx \leq \int_{\Omega} \Phi(\omega_n^{-\frac{1}{n}} |\Omega|^{\frac{1}{n}} \nabla u) dx,$$

and

$$(3.6) \quad \|u\|_{L^{\Phi_*}(\Omega)} \leq \omega_n^{-\frac{1}{n}} |\Omega|^{\frac{1}{n}} \|\nabla u\|_{L^{\Phi}(\Omega, \mathbb{R}^n)}$$

for every  $u \in W_0^{1,\Phi}(\Omega)$ . Here,  $\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$ , the Lebesgue measure of the unit ball in  $\mathbb{R}^n$ .

**Proof.** Let us call  $\Omega^\star$  the open ball, centered at 0, with the same measure as  $\Omega$ . Given any function  $u \in W_0^{1,\Phi}(\Omega)$ , denote by  $u^\star : \Omega^\star \rightarrow [0, \infty)$  the spherical symmetral of  $u$ , namely the radially decreasing function equimeasurable with  $u$ . An anisotropic version of the Polyá-Szegő principle tells us that  $u^\star \in W_0^{1,\Phi_*}(\Omega^\star)$ , and

$$(3.7) \quad \int_{\Omega} \Phi(\nabla u) dx \geq \int_{\Omega^\star} \Phi_*(|\nabla u^\star|) dx.$$

Inequality (3.7) is stated in [K1]; a full proof can be found in [Ci4, Theorem 3.5]. On the other hand, an isotropic Poincaré type inequality ensures that

$$(3.8) \quad \int_{\Omega^\star} \Phi_*(|\nabla u^\star|) dx \geq \int_{\Omega^\star} \Phi_*(\omega_n^{\frac{1}{n}} |\Omega|^{-\frac{1}{n}} u^\star) dx,$$

see [Ta, Lemma 3]. Finally, since  $u$  and  $u^\star$  are equimeasurable,

$$(3.9) \quad \int_{\Omega^\star} \Phi_*(\omega_n^{\frac{1}{n}} |\Omega|^{-\frac{1}{n}} u^\star) dx = \int_{\Omega} \Phi_*(\omega_n^{\frac{1}{n}} |\Omega|^{-\frac{1}{n}} |u|) dx.$$

Inequality (3.5) is a consequence of (3.7)–(3.9). Inequality (3.6) follows on applying (3.5) with  $u$  replaced with  $\frac{u}{\|\nabla u\|_{L^{\Phi}(\Omega, \mathbb{R}^n)}}$ , via the very definition of Luxemburg norm.  $\square$

A Sobolev-Poincaré inequality, with optimal Orlicz target norm, reads as follows. Assume that  $\Phi$  is an  $n$ -dimensional Young function fulfilling (2.8), and let  $\Phi_n$  be the Sobolev conjugate of  $\Phi$  defined as in (2.2). By [Ci3, Theorem 1 and Remark 1], there exists a constant  $C = C(n)$  such that

$$(3.10) \quad \int_{\Omega} \Phi_n \left( \frac{|u|}{C(\int_{\Omega} \Phi(\nabla u) dy)^{\frac{1}{n}}} \right) dx \leq \int_{\Omega} \Phi(\nabla u) dx,$$

and

$$(3.11) \quad \|u\|_{L^{\Phi_n}(\Omega)} \leq C \|u\|_{W_0^{1,\Phi}(\Omega)}$$

for every  $u \in W_0^{1,\Phi}(\Omega)$ . Moreover,  $L^{\Phi_n}(\Omega)$  is the optimal, i.e. smallest possible, Orlicz space which renders (3.11) true for all  $n$ -dimensional Young functions  $\Phi$  with prescribed  $\Phi_*$ .

In particular, if (2.10) holds, then  $\Phi_n(t) = \infty$  for large  $t$ , and (3.11) yields

$$(3.12) \quad \|u\|_{L^\infty(\Omega)} \leq C \|u\|_{W_0^{1,\Phi}(\Omega)}$$

for every  $u \in W_0^{1,\Phi}(\Omega)$ .



**Remark 3.3** Since we are assuming that  $|\Omega| < \infty$ , inequalities (3.11) and (3.12) continue to hold even if (2.8) fails, provided that  $\Phi_n$  is defined with  $\Phi_*$  replaced by another Young function equivalent near infinity, which renders (2.8) true. We shall adopt the convention that  $\Phi_n$  is defined according to this procedure in what follows, whenever needed.

Let us notice that, under assumption (2.7),

$$(3.13) \quad \int_{\Omega} \Phi_n(c|u|)dx < \infty$$

for every  $u \in W_0^{1,\Phi}(\Omega)$  and every  $c \geq 0$ . This fact can be shown to follow from (3.10).

The embedding

$$W_0^{1,\Phi}(\Omega) \rightarrow L^1(\Omega)$$

is compact for any  $n$ -dimensional Young function  $\Phi$ . Indeed, by (3.4),  $W_0^{1,\Phi}(\Omega) \rightarrow W_0^{1,1}(\Omega)$ , and the embedding  $W_0^{1,1}(\Omega) \rightarrow L^1(\Omega)$  is compact.

We denote by  $(W_0^{1,\Phi}(\Omega))^*$  the topological dual of  $W_0^{1,\Phi}(\Omega)$ , and by  $\langle \cdot, \cdot \rangle$  the duality brackets for the pair  $((W_0^{1,\Phi}(\Omega))^*, W_0^{1,\Phi}(\Omega))$ .

In Proposition 3.6 below we analyze properties of the Nemytskii operator, associated with a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , in the anisotropic Orlicz-Sobolev space  $W_0^{1,\Phi}(\Omega)$ . In preparation for this, we need a few technical results contained in the following Lemmas 3.4 and 3.5.

Let  $F$  be defined as in (1.3). We introduce the auxiliary functions  $\bar{f} : \mathbb{R} \rightarrow [0, \infty)$ , defined as

$$(3.14) \quad \bar{f}(t) = \max_{s \in [-|t|, |t|]} |f(s)| \quad \text{for } t \in \mathbb{R},$$

and  $\bar{F} : [0, \infty) \rightarrow [0, \infty)$ , defined as

$$(3.15) \quad \bar{F}(t) = \int_0^t \bar{f}(\tau) d\tau \quad \text{for } t \in [0, \infty).$$

Note that  $\bar{f}$  is even and non-decreasing in  $[0, \infty)$ , and hence  $\bar{F}$  is a Young function.

**Lemma 3.4** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Then*

$$(3.16) \quad |f(t)| \leq \frac{\bar{F}(2|t|)}{|t|} \quad \text{for } t \neq 0,$$

and

$$(3.17) \quad |f(t)| \leq 2\bar{F}^{-1}(\bar{F}(2|t|)) \quad \text{for } t \in \mathbb{R}.$$

**Proof.** We have that

$$\bar{F}(2|t|) = \int_0^{2|t|} \bar{f}(\tau) d\tau \geq \int_{|t|}^{2|t|} \bar{f}(\tau) d\tau \geq \bar{f}(|t|)|t| \geq |f(t)||t| \quad \text{for } t \in \mathbb{R},$$

namely (3.16). Inequality (3.17) follows from (3.16), via (6.2). □

**Lemma 3.5** Assume that  $|G| < \infty$ . Let  $B$  and  $E$  be Young functions such that  $E$  increases essentially more slowly than  $B$  near infinity. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function such that

$$(3.18) \quad |f(t)| \leq c(1 + \tilde{E}^{-1}(E(c|t|))) \quad \text{for } t \in \mathbb{R},$$

and some constant  $c > 0$ .

(i) Let  $u \in L^E(G)$  and let  $\{u_k\}$  be a bounded sequence in  $L^B(G)$  such that  $u_k \rightarrow u$  in  $L^E(G)$ . Then

$$(3.19) \quad \lim_{k \rightarrow \infty} \int_G f(u_k)(u_k - u) dx = 0.$$

(ii) Assume that  $u \in L^B(G)$  and  $\{u_k\} \subset L^B(G)$ . If  $u_k \rightarrow u$  in  $L^B(G)$ , then

$$\lim_{k \rightarrow \infty} \|f(u_k) - f(u)\|_{L^{\tilde{B}}(G)} = 0.$$

**Proof.** (i) By (3.18) and (3.2),

$$(3.20) \quad \left| \int_G f(u_k)(u_k - u) dx \right| \leq c \left( \|u_k - u\|_{L^1(G)} + \int_G \tilde{E}^{-1}(E(c|u_k|))(u_k - u) dx \right) \\ \leq c \left( \|u_k - u\|_{L^1(G)} + 2 \|\tilde{E}^{-1}(E(c|u_k|))\|_{L^{\tilde{E}}(G)} \|u_k - u\|_{L^E(G)} \right).$$

Since the sequence  $\{u_k\}$  is bounded in  $L^B(G)$ , and  $E$  increases essentially more slowly than  $B$  near infinity, there exists a constant  $c' > 0$  such that

$$\int_G \tilde{E}(\tilde{E}^{-1}(E(c|u_k|))) dx = \int_G E(c|u_k|) dx \leq c',$$

for  $k \in \mathbb{N}$ . Hence, by property (6.1), applied with  $A = \tilde{E}$ ,

$$1 \geq \frac{1}{\max\{1, c'\}} \int_G \tilde{E}(\tilde{E}^{-1}(E(c|u_k|))) dx \geq \int_G \tilde{E} \left( \frac{\tilde{E}^{-1}(E(c|u_k|))}{\max\{1, c'\}} \right) dx,$$

namely,

$$(3.21) \quad \|\tilde{E}^{-1}(E(c|u_k|))\|_{L^{\tilde{E}}(G)} \leq \max\{1, c'\}.$$

Since  $u_k \rightarrow u$  in  $L^E(G)$ , and hence also in  $L^1(G)$ , equation (3.19) follows from (3.20) and (3.21).

(ii) By the definition of Luxemburg norm it suffices to show that

$$(3.22) \quad \lim_{k \rightarrow \infty} \int_G \tilde{B} \left( \frac{|f(u_k) - f(u)|}{\lambda} \right) dx = 0 \quad \text{for every } \lambda > 0.$$

Since  $u_k \rightarrow u$  in  $L^B(G)$ , there exists a subsequence, still denoted by  $\{u_k\}$ , and a function  $v \in L^B(G)$  such that  $u_k \rightarrow u$  a.e. in  $G$ , and  $|u_k(x)| \leq v(x)$  for a.e.  $x \in G$ , for every  $k \in \mathbb{N}$ . A proof of this fact follows along the same lines as in the classical special case when  $L^B(G)$  is a Lebesgue space. The Fatou type property of the norm  $\|\cdot\|_{L^B(G)}$ , which tells us that  $\|w_k\|_{L^B(G)} \nearrow \|w\|_{L^B(G)}$  if  $\{w_k\}$  is any sequence such that  $0 \leq w_k \nearrow w$ , plays a role here.

Hence,

$$\lim_{k \rightarrow \infty} f(u_k(x)) = f(u(x)) \quad \text{for a.e. } x \in G.$$

Equation (3.22) will thus follow if we prove that, for every  $\lambda > 0$ , there exists a function  $v_\lambda \in L^1(G)$  such that

$$(3.23) \quad \tilde{B} \left( \frac{|f(u_k(x)) - f(u(x))|}{\lambda} \right) \leq v_\lambda(x) \quad \text{for a.e. } x \in G,$$

for  $k \in \mathbb{N}$ . By the convexity of  $\tilde{B}$ , and (3.18)

$$(3.24) \quad \begin{aligned} \tilde{B} \left( \frac{|f(u_k(x)) - f(u(x))|}{\lambda} \right) &\leq \tilde{B} \left( \frac{2|f(u_k(x))|}{\lambda} \right) + \tilde{B} \left( \frac{2|f(u(x))|}{\lambda} \right) \\ &\leq \tilde{B} \left( \frac{2c}{\lambda} + \frac{2c}{\lambda} \tilde{E}^{-1}(E(c|u_k(x)|)) \right) + \tilde{B} \left( \frac{2c}{\lambda} + \frac{2c}{\lambda} \tilde{E}^{-1}(E(c|u(x)|)) \right) \\ &\leq \tilde{B} \left( \frac{4c}{\lambda} \right) + \frac{1}{2} \left[ \tilde{B} \left( \frac{4c}{\lambda} \tilde{E}^{-1}(E(c|u_k(x)|)) \right) + \tilde{B} \left( \frac{4c}{\lambda} \tilde{E}^{-1}(E(c|u(x)|)) \right) \right] \end{aligned}$$

for a.e.  $x \in G$ , and for  $k \in \mathbb{N}$ . Thanks to (6.6), there exists  $t_0 \geq 0$  such that

$$\tilde{E}^{-1}(E(ct)) \leq \frac{\lambda}{4c} \tilde{B}^{-1}(E(ct)) \quad \text{if } t > t_0.$$

Hence

$$\begin{aligned} \tilde{B} \left( \frac{4c}{\lambda} \tilde{E}^{-1}(E(c|u_k(x)|)) \right) &\leq \tilde{B} \left( \frac{4c}{\lambda} \tilde{E}^{-1}(E(ct_0)) \right) + \tilde{B} \left( \frac{4c}{\lambda} \frac{\lambda}{4c} \tilde{B}^{-1}(E(c|u_k(x)|)) \right) \\ &= \tilde{B} \left( \frac{4c}{\lambda} \tilde{E}^{-1}(E(ct_0)) \right) + E(c|u_k(x)|) \quad \text{for a.e. } x \in G. \end{aligned}$$

An analogous estimate holds with  $u_k$  replaced by  $u$ . Altogether, from (3.24) we infer that

$$(3.25) \quad \begin{aligned} \tilde{B} \left( \frac{|f(u_k(x)) - f(u(x))|}{\lambda} \right) &\leq \tilde{B} \left( \frac{4c}{\lambda} \right) + \tilde{B} \left( \frac{4c}{\lambda} \tilde{E}^{-1}(E(ct_0)) \right) \\ &\quad + \frac{1}{2} [E(c|u_k(x)|) + E(c|u(x)|)] \quad \text{for a.e. } x \in G. \end{aligned}$$

for  $k \in \mathbb{N}$ . Since  $u, v \in L^B(G)$ , and  $E$  increases essentially more slowly than  $B$  near infinity, the right-hand side of (3.25) is an integrable function in  $G$ . Inequality (3.23) follows. The proof is complete.  $\square$

**Proposition 3.6** *Let  $\Phi$  be an  $n$ -dimensional Young function, and let  $\Phi_n$  be its Sobolev conjugate defined by (2.2) (according to the convention of Remark 3.3). Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function.*

(i) *Assume that  $\Phi$  fulfils (2.7), and*

$$(3.26) \quad \limsup_{t \rightarrow \pm\infty} \frac{|tf(t)|}{\Phi_n(\lambda|t|)} < \infty \quad \text{for some } \lambda > 0.$$

*Then the operator  $\mathcal{N}_f : W_0^{1,\Phi}(\Omega) \rightarrow (W_0^{1,\Phi}(\Omega))^*$ , given by*

$$\langle \mathcal{N}_f(u), v \rangle = \int_{\Omega} f(u)v \, dx$$

*for  $u, v \in W_0^{1,\Phi}(\Omega)$ , is well defined.*

*Moreover, if (3.26) is strengthened by assuming (2.9), then the operator  $\mathcal{N}_f$  is continuous.*

(ii) *Assume that  $\Phi$  fulfils (2.10). Then the operator  $\mathcal{N}_f$  is well defined and continuous.*

**Proof.** (i) Assumption (3.26) implies that (and is in fact equivalent to)

$$(3.27) \quad |f(t)| \leq c \left( 1 + \frac{\Phi_n(c|t|)}{|t|} \right) \quad \text{for } t \neq 0,$$

for some constant  $c > 0$ . We begin by proving that  $f(u) \in L^{\tilde{\Phi}_n}(\Omega)$  for every  $u \in W_0^{1,\Phi}(\Omega)$ . Here,  $\tilde{\Phi}_n$  stands for the Young conjugate of  $\Phi_n$ . By (3.27), the convexity of  $\tilde{\Phi}_n$ , and (6.2),

$$(3.28) \quad \begin{aligned} \tilde{\Phi}_n \left( \frac{|f(u)|}{2c^2} \right) &\leq \tilde{\Phi}_n \left( \frac{1}{2c} \left( 1 + \frac{\Phi_n(c|u|)}{|u|} \right) \right) \leq \frac{1}{2} \tilde{\Phi}_n \left( \frac{1}{c} \right) + \frac{1}{2} \tilde{\Phi}_n \left( \frac{\Phi_n(c|u|)}{c|u|} \right) \\ &\leq \frac{1}{2} \tilde{\Phi}_n \left( \frac{1}{c} \right) + \frac{1}{2} \tilde{\Phi}_n \left( \tilde{\Phi}_n^{-1}(\Phi_n(c|u|)) \right) = \frac{1}{2} \tilde{\Phi}_n \left( \frac{1}{c} \right) + \frac{1}{2} \Phi_n(c|u|) \quad \text{a.e. in } \Omega. \end{aligned}$$

From (3.28) and (3.13), we deduce that

$$\int_{\Omega} \tilde{\Phi}_n \left( \frac{f(u)}{2c^2} \right) dx < \infty,$$

whence  $f(u) \in L^{\tilde{\Phi}_n}(\Omega)$ . Therefore, owing to (3.1) and (3.11), there exists a constant  $C$  such that

$$\int_{\Omega} |f(u)v| dx \leq 2 \|f(u)\|_{L^{\tilde{\Phi}_n}(\Omega)} \|v\|_{L^{\Phi_n}(\Omega)} \leq 2C \|f(u)\|_{L^{\tilde{\Phi}_n}(\Omega)} \|v\|_{W_0^{1,\Phi}(\Omega)}$$

for every  $u, v \in W_0^{1,\Phi}(\Omega)$ . This shows that  $\mathcal{N}_f : W_0^{1,\Phi}(\Omega) \rightarrow (W_0^{1,\Phi}(\Omega))^*$  is well defined.

Assume now that (2.9) is fulfilled. In order to prove the continuity of  $\mathcal{N}_f$ , consider any function  $u \in W_0^{1,\Phi}(\Omega)$  and any sequence  $\{u_k\} \subset W_0^{1,\Phi}(\Omega)$  such that  $u_k \rightarrow u$  in  $W_0^{1,\Phi}(\Omega)$ . By (3.1) and the Sobolev inequality (3.11), there exists a constant  $C$  such that

$$\begin{aligned} \left| \int_{\Omega} (f(u_k) - f(u))v dx \right| &\leq 2 \|f(u_k) - f(u)\|_{L^{\tilde{\Phi}_n}(\Omega)} \|v\|_{L^{\Phi_n}(\Omega)} \\ &\leq C \|f(u_k) - f(u)\|_{L^{\tilde{\Phi}_n}(\Omega)} \|v\|_{W_0^{1,\Phi}(\Omega)} \end{aligned}$$

for every  $v \in W_0^{1,\Phi}(\Omega)$ , and every  $k \in \mathbb{N}$ . Hence,

$$(3.29) \quad \begin{aligned} \|\mathcal{N}_f(u_k) - \mathcal{N}_f(u)\|_{(W_0^{1,\Phi}(\Omega))^*} &= \sup_{\|v\|_{W_0^{1,\Phi}(\Omega)} \leq 1} |\langle \mathcal{N}_f(u_k), v \rangle - \langle \mathcal{N}_f(u), v \rangle| \\ &\leq C \|f(u_k) - f(u)\|_{L^{\tilde{\Phi}_n}(\Omega)}. \end{aligned}$$

On the other hand, the Sobolev inequality (3.11) again implies that  $u_k \rightarrow u$  in  $L^{\Phi_n}(\Omega)$ . Assumption (2.9), via inequality (3.17), allows us to apply Lemma 3.5, Part (ii), with  $c = 2$ ,  $E = \bar{F}$ ,  $B = \Phi_n$ , and deduce that

$$(3.30) \quad \lim_{k \rightarrow \infty} \|f(u_k) - f(u)\|_{L^{\tilde{\Phi}_n}(\Omega)} = 0.$$

The conclusion follows from (3.29) and (3.30).

(ii) The Sobolev inequality (3.12) holds. Thus, given any  $u, v \in W_0^{1,\Phi}(\Omega)$ , we have that  $u, v \in L^\infty(\Omega)$ , and, in particular,  $f(u) \in L^1(\Omega)$ . Consequently, by (3.12) again, there exists a constant  $C$  such that

$$\int_{\Omega} |f(u)v| dx \leq \|f(u)\|_{L^1(\Omega)} \|v\|_{L^\infty(\Omega)} \leq C \|f(u)\|_{L^1(\Omega)} \|v\|_{W_0^{1,\Phi}(\Omega)}$$

for every  $u, v \in W_0^{1,\Phi}(\Omega)$ . Hence,  $\mathcal{N}_f : W_0^{1,\Phi}(\Omega) \rightarrow (W_0^{1,\Phi}(\Omega))^*$  is well defined.

As for the continuity of  $\mathcal{N}_f$ , if  $u \in W_0^{1,\Phi}(\Omega)$  and  $\{u_k\} \subset W_0^{1,\Phi}(\Omega)$  are such that  $u_k \rightarrow u$  in  $W_0^{1,\Phi}(\Omega)$ , then, by (3.12),  $u_k \rightarrow u$  in  $L^\infty(\Omega)$ , and hence  $f(u_k) \rightarrow f(u)$  in  $L^1(\Omega)$ . Therefore

$$\|\mathcal{N}_f(u_k) - \mathcal{N}_f(u)\|_{(W_0^{1,\Phi}(\Omega))^*} = \sup_{\|v\|_{W_0^{1,\Phi}(\Omega)} \leq 1} |\langle \mathcal{N}_f(u_k), v \rangle - \langle \mathcal{N}_f(u), v \rangle| \leq C \|f(u_k) - f(u)\|_{L^1(\Omega)},$$

whence the conclusion follows.  $\square$

## 4 Proof of Theorem 2.1

Assume throughout that  $\Omega$  is an open set in  $\mathbb{R}^n$ , with  $n \geq 2$ , such that  $|\Omega| < \infty$ . Let  $\Phi$  be an  $n$ -dimensional Young function, and let  $f$  be any continuous function such that  $f(u)\varphi \in L^1(\Omega)$  for every  $u, \varphi \in W_0^{1,\Phi}(\Omega)$ . A function  $u \in W_0^{1,\Phi}(\Omega)$  will be called a weak solution to problem (1.1) if

$$(4.1) \quad \int_{\Omega} \Phi_{\xi}(\nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} f(u) \varphi \, dx$$

for every test function  $\varphi \in W_0^{1,\Phi}(\Omega)$ .

The energy functional associated with problem (1.1) is the functional  $J_{\Phi} : W_0^{1,\Phi}(\Omega) \rightarrow \mathbb{R}$  defined by (1.2). Any critical point of  $J_{\Phi}$  satisfies (4.1), and is hence a solution to (1.1). In order to establish Theorem 2.1 it will thus suffice to show that  $J_{\Phi}$  has a nontrivial critical point. To this purpose, we shall make use of a version of the Mountain Pass Theorem, stated below, for functionals defined on a Banach space  $X$ , and satisfying the Palais-Smale condition. Recall that a functional  $I : X \rightarrow \mathbb{R}$  is said to satisfy the Palais-Smale condition if

$$(4.2) \quad \begin{aligned} &\text{any sequence } \{u_k\} \subset X \text{ such that } \{I(u_k)\} \text{ is bounded,} \\ &\text{and } \lim_{k \rightarrow \infty} \|I'(u_k)\|_{X^*} = 0, \text{ has a convergent subsequence in } X. \end{aligned}$$

A sequence  $\{u_k\}$  as in (4.2) will be called a Palais-Smale sequence for the functional  $I$ .

**Mountain Pass Theorem [AR]** *Let  $X$  be a real Banach space. Assume that the functional  $I : X \rightarrow \mathbb{R}$  is of class  $C^1$ , satisfies the Palais-Smale condition (4.2), and fulfills the following properties:*

$$(4.3) \quad I(0) = 0,$$

$$(4.4) \quad \text{there exist } \rho, \sigma > 0 \text{ such that } \inf_{\|u\|_X = \rho} I(u) \geq \sigma,$$

$$(4.5) \quad \text{there exists } \bar{u} \in X \text{ such that } \|\bar{u}\|_X > \rho \text{ and } I(\bar{u}) \leq 0.$$

*Then  $I$  has a critical point  $u$ , such that  $I(u) = c \geq \sigma$ , where*

$$c = \inf_{\gamma \in \mathcal{G}} \max_{s \in [0,1]} I(\gamma(s)),$$

*and*

$$\mathcal{G} = \{\gamma \in C^0([0,1], X) : \gamma(0) = 0, \gamma(1) = \bar{u}\}.$$

The continuous differentiability of the functional  $J_{\Phi}$  is the object of the following result.

**Proposition 4.1** *Let  $\Phi \in C^1(\mathbb{R}^n)$  be a strictly convex  $n$ -dimensional Young function satisfying (2.3). Let  $\Phi_n$  be its Sobolev conjugate defined by (2.2) (according to the convention of Remark 3.3). Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Assume that either (2.7) and (2.9) hold, or (2.10) holds. Then the functional  $J_\Phi$ , defined by (1.2), is of class  $C^1$ .*

Proposition 4.1 is a consequence of the next two propositions.

**Proposition 4.2** *Assume that  $\Phi \in C^1(\mathbb{R}^n)$  is a strictly convex  $n$ -dimensional Young function satisfying (2.3). Then the functional  $I_\Phi : W_0^{1,\Phi}(\Omega) \rightarrow \mathbb{R}$ , defined as*

$$I_\Phi(u) = \int_{\Omega} \Phi(\nabla u) dx$$

*for  $u \in W_0^{1,\Phi}(\Omega)$ , is of class  $C^1$ .*

**Proof.** It suffices to show that  $I_\Phi$  is Gâteaux differentiable, and that its Gâteaux derivative  $(I_\Phi)'_G$  is continuous. Let  $u, \varphi \in W_0^{1,\Phi}(\Omega)$ , and let  $\mu \in (0, 1)$ . Since  $\Phi \in C^1(\mathbb{R}^n)$ ,

$$(4.6) \quad \lim_{\mu \rightarrow 0^+} \Phi_\xi(\nabla(u(x) + \mu \nabla \varphi(x)) \cdot \nabla \varphi(x) = \Phi_\xi(\nabla u(x)) \cdot \nabla \varphi(x) \quad \text{for a.e. } x \in \Omega.$$

Moreover, for a.e.  $x \in \Omega$  there exists  $\rho_{\mu,x} \in (0, 1)$  such that

$$(4.7) \quad \frac{\Phi(\nabla u(x) + \mu \nabla \varphi(x)) - \Phi(\nabla u(x))}{\mu} = \Phi_\xi(\nabla u(x) + \mu \rho_{\mu,x} \nabla \varphi(x)) \cdot \nabla \varphi(x).$$

On the other hand, by (6.14), (6.23), the convexity of  $\Phi$  and the fact that  $\mu \rho_{\mu,x} \in (0, 1)$  we deduce that

$$(4.8) \quad \begin{aligned} |\Phi_\xi(\nabla u(x) + \mu \rho_{\mu,x} \nabla \varphi(x)) \cdot \nabla \varphi(x)| &\leq \tilde{\Phi}(\Phi_\xi(\nabla u(x) + \mu \rho_{\mu,x} \nabla \varphi(x))) + \Phi(\nabla \varphi(x)) \\ &\leq \Phi(2\nabla u(x) + 2\mu \rho_{\mu,x} \nabla \varphi(x)) + \Phi(\nabla \varphi(x)) \\ &\leq \frac{1}{2}\Phi(4\nabla u(x)) + \frac{1}{2}\Phi(4\nabla \varphi(x)) + \Phi(\nabla \varphi(x)) \quad \text{for a.e. } x \in \Omega. \end{aligned}$$

Since  $\Phi \in \Delta_2$  near infinity, by (4.8) the right-hand side of (4.7) belongs to  $L^1(\Omega)$ . From (4.6) and (4.7), via the dominated convergence theorem, we obtain that

$$\langle (I_\Phi)'_G(u), \varphi \rangle = \lim_{\mu \rightarrow 0^+} \int_{\Omega} \frac{\Phi(\nabla u + \mu \nabla \varphi) - \Phi(\nabla u)}{\mu} dx = \int_{\Omega} \Phi_\xi(\nabla u) \cdot \nabla \varphi dx$$

for  $u, \varphi \in W_0^{1,\Phi}(\Omega)$ .

We next show that the operator  $(I_\Phi)'_G : W_0^{1,\Phi}(\Omega) \rightarrow (W_0^{1,\Phi}(\Omega))^*$  is continuous. Let  $\{u_k\}$  be any sequence in  $W_0^{1,\Phi}(\Omega)$ , converging to some function  $u \in W_0^{1,\Phi}(\Omega)$ . Then,  $\|\nabla u_k - \nabla u\|_{L^\Phi(\Omega)} \rightarrow 0$  as  $k \rightarrow \infty$ , and hence

$$(4.9) \quad \lim_{k \rightarrow \infty} \int_{\Omega} \Phi(\lambda(\nabla u_k - \nabla u)) dx = 0 \quad \text{for every } \lambda > 0.$$

Moreover, on passing, if necessary, to a subsequence, still denoted by  $\{u_k\}$ , we have that  $\nabla u_k \rightarrow \nabla u$  a.e. in  $\Omega$ . Hence,

$$(4.10) \quad \Phi(\nabla u_k) \rightarrow \Phi(\nabla u) \quad \text{and} \quad \Phi_\xi(\nabla u_k) \rightarrow \Phi_\xi(\nabla u) \quad \text{a.e. in } \Omega.$$

We have that

$$(4.11) \quad 0 \leq \Phi(2\nabla u_k(x)) \leq \frac{1}{2}\Phi(4(\nabla u_k(x) - \nabla u(x))) + \frac{1}{2}\Phi(4\nabla u(x)) \quad \text{for a.e. } x \text{ in } \Omega,$$

for  $k \in \mathbb{N}$ . Equation (4.9) ensures that there exists  $w \in L^1(\Omega)$  such that  $\Phi(4(\nabla u_k - \nabla u)) \leq w$  a.e. in  $\Omega$ , for  $k \in \mathbb{N}$ . Thus, inequality (4.11) implies that

$$(4.12) \quad \Phi(2\nabla u_k(x)) \leq \frac{w(x) + \Phi(4\nabla u(x))}{2} \quad \text{for a.e. } x \text{ in } \Omega,$$

the right-hand side of (4.12) being a function in  $L^1(\Omega)$ . By (2.3), and Proposition 6.5, Part (ii), one has that  $\Phi \in \nabla_2$  near infinity. This property is easily seen to imply that  $\lim_{|\xi| \rightarrow \infty} \frac{\Phi(\xi)}{|\xi|} = \infty$ . Proposition 6.6 then yields  $\tilde{\Phi} \in \Delta_2$  near infinity. Consequently, there exist constants  $C > 2$  and  $M \geq 0$  such that

$$(4.13) \quad \tilde{\Phi}(2\eta) \leq \tilde{C}\Phi(\eta) \quad \text{if } |\eta| > M.$$

Finally, on making use of the fact that  $\tilde{\Phi}$  is an even convex function, and of (4.13), (6.23) and (4.12), one obtains that

$$(4.14) \quad \begin{aligned} \tilde{\Phi}(\Phi_\xi(\nabla u_k(x)) - \Phi_\xi(\nabla u(x))) &\leq \frac{\tilde{\Phi}(2\Phi_\xi(\nabla u_k(x)))}{2} + \frac{\tilde{\Phi}(-2\Phi_\xi(\nabla u(x)))}{2} \\ &\leq \max_{|\eta| \leq M} \tilde{\Phi}(2\eta) + \frac{C}{2} \left[ \tilde{\Phi}(\Phi_\xi(\nabla u_k(x))) + \tilde{\Phi}(\Phi_\xi(\nabla u(x))) \right] \\ &\leq \max_{|\eta| \leq M} \tilde{\Phi}(2\eta) + \frac{C}{2} [\Phi(2\nabla u_k(x)) + \Phi(2\nabla u(x))] \\ &\leq \max_{|\eta| \leq M} \tilde{\Phi}(2\eta) + \frac{C}{2} \left[ \frac{w(x) + \Phi(4\nabla u(x))}{2} + \Phi(2\nabla u(x)) \right] \quad \text{for a.e. } x \in \Omega. \end{aligned}$$

Owing to (4.10) and (4.14), via the dominated convergence theorem one deduces that

$$\lim_{k \rightarrow \infty} \int_{\Omega} \tilde{\Phi}(\Phi_\xi(\nabla u_k) - \Phi_\xi(\nabla u)) \, dx = 0.$$

Since  $\tilde{\Phi} \in \Delta_2$  near infinity, this also implies that

$$(4.15) \quad \lim_{k \rightarrow \infty} \|\Phi_\xi(\nabla u_k) - \Phi_\xi(\nabla u)\|_{L^{\tilde{\Phi}}(\Omega, \mathbb{R}^n)} = 0.$$

Clearly, the above argument applies to any subsequence of  $\{u_k\}$ . This ensures that equation (4.15) holds, in fact, for the whole sequence  $\{u_k\}$ .

Now, let  $\varphi \in W_0^{1,\Phi}(\Omega)$ . Thanks to (3.2),

$$\begin{aligned} | \langle (I_\Phi)'_G(u_k) - (I_\Phi)'_G(u), \varphi \rangle | &= \left| \int_{\Omega} (\Phi_\xi(\nabla u_k) - \Phi_\xi(\nabla u)) \cdot \nabla \varphi \, dx \right| \\ &\leq 2 \|\Phi_\xi(\nabla u_k) - \Phi_\xi(\nabla u)\|_{L^{\tilde{\Phi}}(\Omega, \mathbb{R}^n)} \|\nabla \varphi\|_{L^\Phi(\Omega, \mathbb{R}^n)}. \end{aligned}$$

Thereby, from (4.15) we infer that

$$\lim_{k \rightarrow \infty} \|(I_\Phi)'_G(u_k) - (I_\Phi)'_G(u)\|_{(W_0^{1,\Phi}(\Omega))^*} \leq 2 \lim_{k \rightarrow \infty} \|\Phi_\xi(\nabla u_k) - \Phi_\xi(\nabla u)\|_{L^{\tilde{\Phi}}(\Omega, \mathbb{R}^n)} = 0$$

The continuity of  $(I_\Phi)'_G$  is thus established.  $\square$

**Proposition 4.3** *Let  $\Phi$  be a  $n$ -dimensional Young function, and let  $\Phi_n$  be its Sobolev conjugate defined by (2.2) (according to the convention of Remark 3.3). Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Assume that either (2.7) and (2.9) hold, or (2.10) holds. Then the functional  $L_f : W_0^{1,\Phi}(\Omega) \rightarrow \mathbb{R}$ , defined by*

$$L_f(u) = \int_{\Omega} F(u) dx$$

*for  $u \in W_0^{1,\Phi}(\Omega)$ , is of class  $C^1$ .*

The following Lemma is needed in the proof of Proposition 4.3. Its proof makes use of calculus arguments, and will be omitted for brevity.

**Lemma 4.4** *Let  $B$  be a Young function, and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Let  $\bar{f}$ ,  $F$  and  $\bar{F}$  be the functions associated with  $f$  as in (1.3), (3.14) and (3.15), respectively.*

*(i) If*

$$(4.16) \quad \lim_{t \rightarrow 0} \frac{tf(t)}{B(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0,$$

*then,*

$$(4.17) \quad \lim_{t \rightarrow 0} \frac{t\bar{f}(t)}{B(\lambda|t|)} = 0, \quad \text{and} \quad \lim_{t \rightarrow 0} \frac{F(t)}{B(\lambda|t|)} = \lim_{t \rightarrow 0} \frac{\bar{F}(|t|)}{B(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0.$$

*(ii) If*

$$(4.18) \quad \lim_{t \rightarrow \infty} \frac{B(t)}{t} = \infty,$$

*and*

$$(4.19) \quad \lim_{t \rightarrow \pm\infty} \frac{tf(t)}{B(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0,$$

*then,*

$$(4.20) \quad \lim_{t \rightarrow \pm\infty} \frac{t\bar{f}(t)}{B(\lambda|t|)} = \lim_{t \rightarrow \pm\infty} \frac{F(t)}{B(\lambda|t|)} = \lim_{t \rightarrow \pm\infty} \frac{\bar{F}(|t|)}{B(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0.$$

*In particular, the function  $\bar{F}$  increases essentially more slowly than  $B$  near infinity.*

**Proof of Proposition 4.3.** We shall show that  $L_f$  is Gâteaux differentiable in  $W_0^{1,\Phi}(\Omega)$ , and that its Gâteaux derivative  $(L_f)'_G$  is continuous. To this purpose, fix any  $u, \varphi \in W_0^{1,\Phi}(\Omega)$  and let  $\mu \in (0, 1)$ . By the continuity of  $f$ ,

$$(4.21) \quad \lim_{\mu \rightarrow 0} \frac{F(u(x) + \mu\varphi(x)) - F(u(x))}{\mu} = f(u(x))\varphi(x) \quad \text{for a.e. } x \in \Omega.$$

Moreover, for a.e.  $x \in \Omega$ , there exists  $\theta_{\mu,x} \in (0, 1)$  such that

$$(4.22) \quad \frac{F(u(x) + \mu\varphi(x)) - F(u(x))}{\mu} = f(u(x) + \mu\theta_{\mu,x}\varphi(x))\varphi(x).$$



By (3.16), and the fact that the function  $(0, \infty) \mapsto \frac{\bar{F}(2s)}{s}$  is non-decreasing, we obtain that

$$(4.23) \quad \begin{aligned} |f(u(x) + \mu\theta_{\mu,x}\varphi(x))\varphi(x)| &\leq \frac{\bar{F}(2(|u(x) + \mu\theta_{\mu,x}\varphi(x)|))}{|u(x) + \mu\theta_{\mu,x}\varphi(x)|} |\varphi(x)| \\ &\leq \frac{\bar{F}(2(|u(x)| + |\varphi(x)|))}{|u(x)| + |\varphi(x)|} |\varphi(x)| \leq \bar{F}(2(|u(x)| + |\varphi(x)|)) \quad \text{for a.e. } x \in \Omega. \end{aligned}$$

Assume first that conditions (2.7) and (2.9) are in force. One can verify that the function  $\Phi_n(t)$  dominates  $t^{\frac{n}{n-1}}$  near infinity, whatever  $\Phi$  is, and hence  $\lim_{t \rightarrow \infty} \frac{\Phi_n(t)}{t} = \infty$ . Thus, by Lemma 4.4 applied with  $B = \Phi_n$ , the Young function  $\bar{F}$  increases essentially more slowly than  $\Phi_n$  near infinity. Owing to (3.13), the right-hand side of (4.23) belongs to  $L^1(\Omega)$ . If, instead, condition (2.10) holds, then the same assertion is true, owing to embedding (3.12). In any case, from (4.21)–(4.23) we obtain, via the dominated convergence theorem, that

$$\langle (L_f)'_G(u), \varphi \rangle = \int_{\Omega} f(u) \varphi \, dx$$

for every  $u, \varphi \in W_0^{1,\Phi}(\Omega)$ .

The continuity of  $(L_f)'_G$  is a straightforward consequence of Proposition 3.6.  $\square$

Our next task consists in showing that the functional  $J_{\Phi}$  satisfies the Palais-Smale condition. This is accomplished in the next proposition.

**Proposition 4.5** *Let  $\Phi \in C^1(\mathbb{R}^n)$  be an  $n$ -dimensional Young function satisfying (2.3). Let  $\Phi_n$  be its Sobolev conjugate defined by (2.2) (according to the convention of Remark 3.3). Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function satisfying (2.5). Assume that either (2.7) and (2.9) hold, or (2.10) holds. Then the functional  $J_{\Phi}$  satisfies the Palais-Smale condition (4.2).*

The proof of Proposition 4.5 makes use of the next lemma. In what follows, the arrow “ $\rightharpoonup$ ” denotes weak convergence.

**Lemma 4.6** *Assume that  $\Phi \in C^1(\mathbb{R}^n)$  is an even, strictly convex, nonnegative function, vanishing at 0, and satisfying (2.3). Then the operator  $T : W_0^{1,\Phi}(\Omega) \rightarrow (W_0^{1,\Phi}(\Omega))^*$ , defined as*

$$\langle Tu, v \rangle = \int_{\Omega} \Phi_{\xi}(\nabla u) \cdot \nabla v \, dx$$

*for  $u, v \in W_0^{1,\Phi}(\Omega)$ , is well defined. Moreover, if  $u \in W_0^{1,\Phi}(\Omega)$ , and  $\{u_k\} \subset W_0^{1,\Phi}(\Omega)$  is a sequence such that*

$$u_k \rightharpoonup u \quad \text{in } W_0^{1,\Phi}(\Omega),$$

*and*

$$(4.24) \quad \limsup_{k \rightarrow \infty} \langle T(u_k), u_k - u \rangle \leq 0,$$

*then  $u_k \rightarrow u$  in  $W_0^{1,\Phi}(\Omega)$ .*

**Proof.** Let us begin by showing that  $T$  is well defined. Owing to (6.23) and (3.2), if  $u \in W_0^{1,\Phi}(\Omega)$ , then  $\Phi_{\xi}(\nabla u) \in L^{\tilde{\Phi}}(\Omega, \mathbb{R}^n)$ . Moreover, if also  $v \in W_0^{1,\Phi}(\Omega)$ , then, by (3.2),

$$\int_{\Omega} |\Phi_{\xi}(\nabla u) \cdot \nabla v| \, dx \leq 2 \|\Phi_{\xi}(\nabla u)\|_{L^{\tilde{\Phi}}(\Omega, \mathbb{R}^n)} \|\nabla v\|_{L^{\Phi}(\Omega, \mathbb{R}^n)}.$$

This guarantees that  $T$  is well defined, and that

$$\|T(u)\|_{(W_0^{1,\Phi}(\Omega))^*} \leq 2\|\Phi_\xi(\nabla u)\|_{L^{\tilde{\Phi}}(\Omega, \mathbb{R}^n)}.$$

Now, let  $\{u_k\}$  be a sequence as in the statement. Observe that

$$(4.25) \quad \lim_{k \rightarrow \infty} \int_{\Omega} \Phi_\xi(\nabla u) \cdot (\nabla u_k - \nabla u) dx = 0.$$

By (4.24) and (4.25), for every  $\sigma > 0$ , there exists  $k_\sigma \in \mathbb{N}$  such that

$$(4.26) \quad \int_{\Omega} (\Phi_\xi(\nabla u_k) - \Phi_\xi(\nabla u)) \cdot (\nabla u_k - \nabla u) dx < \sigma$$

if  $k > k_\sigma$ . Given  $t, \tau > 0$ , set

$$l = \inf\{(\Phi_\xi(\xi) - \Phi_\xi(\eta)) \cdot (\xi - \eta) : |\xi| \leq \tau, |\eta| \leq \tau, |\xi - \eta| > t\}.$$

Inasmuch as  $\Phi \in C^1(\mathbb{R}^n)$  and is strictly convex,

$$l > 0 \quad \text{for } t, \tau > 0.$$

Set

$$\Omega_k = \{x \in \Omega : |\nabla u_k(x)| \leq \tau, |\nabla u(x)| \leq \tau, |\nabla u_k(x) - \nabla u(x)| > t\}.$$

Owing to (4.26),

$$l|\Omega_k| = \int_{\Omega_k} l dx \leq \int_{\Omega} (\Phi_\xi(\nabla u_k) - \Phi_\xi(\nabla u)) \cdot (\nabla u_k - \nabla u) dx < \sigma$$

if  $k > k_\sigma$ . Thus,

$$|\Omega_k| < \frac{\sigma}{l} \quad \text{if } k > k_\sigma.$$

By the strict convexity of  $\Phi$ , and property (6.7), there exists a constant  $c > 0$  such that

$$\Phi(\xi) \geq c|\xi| \quad \text{if } |\xi| \geq 1.$$

Since  $u_k \rightharpoonup u$ , there exists  $M > 0$  such that  $\|u_k\|_{W_0^{1,\Phi}(\Omega)} \leq M$  for  $k \in \mathbb{N}$ , and  $\|u\|_{W_0^{1,\Phi}(\Omega)} \leq M$ . Fix  $\tau > M$ . Then,

$$\frac{c\tau}{M} |\{x \in \Omega : |\nabla u_k(x)| \geq \tau\}| \leq c \int_{\{|\nabla u_k| \geq \tau\}} \frac{|\nabla u_k|}{M} dx \leq \int_{\{|\nabla u_k| \geq \tau\}} \Phi\left(\frac{\nabla u_k}{M}\right) dx \leq 1$$

for  $k \in \mathbb{N}$ . Analogously,

$$\frac{c\tau}{M} |\{x \in \Omega : |\nabla u(x)| \geq \tau\}| \leq 1.$$

Hence,

$$|\{x \in \Omega : |\nabla u_k(x)| \geq \tau\}| \leq \frac{M}{c\tau} \quad \text{and} \quad |\{x \in \Omega : |\nabla u(x)| \geq \tau\}| \leq \frac{M}{c\tau} \quad \text{for } \tau > M.$$

Fix  $\varepsilon > 0$ , and choose  $\sigma = \frac{l\varepsilon}{3}$  and  $\tau > \max\{M, \frac{3M}{c\varepsilon}\}$ . Then,

$$\begin{aligned} |\{x \in \Omega : |\nabla u_k(x) - \nabla u(x)| > t\}| &\leq |G_k| + |\{x \in \Omega : |\nabla u_k(x)| \geq \tau\}| \\ &\quad + |\{x \in \Omega : |\nabla u(x)| \geq \tau\}| < \varepsilon \quad \text{if } k > k_\sigma. \end{aligned}$$

Thus,  $\nabla u_k \rightarrow \nabla u$  in measure and, up to a subsequence still denoted by  $\{u_k\}$ , a.e. in  $\Omega$ . Consequently,  $\Phi(\nabla u_k) \rightarrow \Phi(\nabla u)$  a.e. in  $\Omega$ , and by Fatou's theorem,

$$(4.27) \quad \liminf_{k \rightarrow \infty} \int_{\Omega} \Phi(\nabla u_k) dx \geq \int_{\Omega} \Phi(\nabla u) dx.$$

On the other hand, the convexity of  $\Phi$  implies that

$$\Phi(\eta) \geq \Phi(\xi) + \Phi_{\xi}(\xi)(\eta - \xi) \quad \text{for } \xi, \eta \in \mathbb{R}^n.$$

Hence,

$$\int_{\Omega} \Phi_{\xi}(\nabla u_k) \cdot (\nabla u_k - \nabla u) dx \geq \int_{\Omega} \Phi(\nabla u_k) dx - \int_{\Omega} \Phi(\nabla u) dx$$

for  $k \in \mathbb{N}$ , and, by (4.24),

$$(4.28) \quad \limsup_{k \rightarrow \infty} \int_{\Omega} \Phi(\nabla u_k) dx \leq \int_{\Omega} \Phi(\nabla u) dx.$$

Coupling (4.27) with (4.28) tells us that

$$(4.29) \quad \lim_{k \rightarrow \infty} \int_{\Omega} \Phi(\nabla u_k) dx = \int_{\Omega} \Phi(\nabla u) dx.$$

Since  $\Phi$  is  $\Delta_2$  near infinity, the convergence of  $\nabla u_k$  to  $\nabla u$  a.e. and equation (4.29) imply that  $\|\nabla u_k - \nabla u\|_{L^{\Phi}(\Omega)} \rightarrow 0$  as  $k \rightarrow \infty$ . This follows along the same lines as in the case when standard isotropic Orlicz norms are involved – see e.g. [RR1, Chapter 3, Theorem 12]. Inasmuch as the whole argument clearly applies to any subsequence of  $\{u_k\}$ , the conclusion follows.  $\square$

**Proof of Proposition 4.5.** Let  $\{u_k\} \subset W_0^{1,\Phi}(\Omega)$  be a Palais-Smale sequence for  $J_{\Phi}$ . Since the sequence  $\{J_{\Phi}(u_k)\}$  is bounded, there exists a subsequence, still denoted by  $\{u_k\}$ , and a number  $c \in \mathbb{R}$  such that  $\lim_{k \rightarrow \infty} J_{\Phi}(u_k) = c$ . Thus, for every  $\varepsilon > 0$ , there exists  $k_{\varepsilon} \in \mathbb{N}$  such that

$$(4.30) \quad c - \varepsilon < J_{\Phi}(u_k) < c + \varepsilon \quad \text{if } k > k_{\varepsilon}.$$

On the other hand, since  $\lim_{k \rightarrow \infty} \|J'_{\Phi}(u_k)\|_{(W_0^{1,\Phi}(\Omega))^*} = 0$ , there exists a sequence  $\{\varepsilon_k\}$  such that  $\varepsilon_k \rightarrow 0^+$ , and

$$(4.31) \quad -\varepsilon_k \|\varphi\|_{W_0^{1,\Phi}(\Omega)} \leq \int_{\Omega} \Phi_{\xi}(\nabla u_k) \cdot \nabla \varphi \, dx - \int_{\Omega} f(u_k) \varphi \, dx \leq \varepsilon_k \|\varphi\|_{W_0^{1,\Phi}(\Omega)}$$

for every  $\varphi \in W_0^{1,\Phi}(\Omega)$ .

Given any  $\sigma > 0$ , there exists  $M \geq 0$  such that

$$(s_{\Phi} + \sigma)\Phi(\xi) - \Phi_{\xi}(\xi) \cdot \xi \geq 0 \quad \text{if } |\xi| \geq M.$$

On setting  $\alpha = s_{\Phi} + 2\sigma$ , the last inequality can be rewritten as

$$(4.32) \quad \sigma\Phi(\xi) \leq \alpha\Phi(\xi) - \Phi_{\xi}(\xi) \cdot \xi \quad \text{if } |\xi| \geq M.$$

By (2.5), if  $\sigma$  is sufficiently small, then

$$(4.33) \quad \alpha F(t) - f(t)t < 0 \quad \text{if } |t| \geq M,$$

provided that  $M$  is sufficiently large.

Now, choose  $\varphi = u_k$  in (4.31), multiply through (4.30) by  $\alpha$ , and add the resulting equations to obtain that

$$(4.34) \quad \int_{\Omega} (\alpha \Phi(\nabla u_k) - \Phi_{\xi}(\nabla u_k) \cdot \nabla u_k) dx - \int_{\Omega} (\alpha F(u_k) - f(u_k)u_k) dx \\ \leq \alpha(c + \varepsilon) + \varepsilon_k \|u_k\|_{W_0^{1,\Phi}(\Omega)}$$

for  $k > k_{\varepsilon}$ . From (4.32), (4.34), (4.33) we deduce that

$$(4.35) \quad \sigma \int_{\Omega} \Phi(\nabla u_k) dx \leq \int_{\{|\nabla u_k| \geq M\}} (\alpha \Phi(\nabla u_k) - \Phi_{\xi}(\nabla u_k) \cdot \nabla u_k) dx + \sigma \int_{\{|\nabla u_k| < M\}} \Phi(\nabla u_k) dx \\ = \int_{\Omega} (\alpha \Phi(\nabla u_k) - \Phi_{\xi}(\nabla u_k) \cdot \nabla u_k) dx \\ + \int_{\{|\nabla u_k| < M\}} (\sigma \Phi(\nabla u_k) - \alpha \Phi(\nabla u_k) + \Phi_{\xi}(\nabla u_k) \cdot \nabla u_k) dx \\ \leq \alpha(c + \varepsilon) + \varepsilon_k \|u_k\|_{W_0^{1,\Phi}(\Omega)} + \int_{|u_k| \leq M} (\alpha F(u_k) - f(u_k)u_k) dx + C \\ \leq C' + \varepsilon_k \|u_k\|_{W_0^{1,\Phi}(\Omega)}$$

for  $k > k_{\varepsilon}$ , for some constants  $C = C(M, \Phi, \alpha)$  and  $C' = C'(M, \Phi, \alpha)$ .

We claim that  $\{u_k\}$  is bounded in  $W_0^{1,\Phi}(\Omega)$ . To verify this claim, suppose, by contradiction, that  $\{u_k\}$  is unbounded. In particular, on passing, if necessary, to a subsequence, we may assume that

$$\|u_k\|_{W_0^{1,\Phi}(\Omega)} - \varepsilon > 1 \quad \text{for } k \in \mathbb{N}.$$

By the very definition of Luxemburg norm, and property (6.7)

$$1 < \int_{\Omega} \Phi \left( \frac{\nabla u_k}{\|u_k\|_{W_0^{1,\Phi}(\Omega)} - \varepsilon} \right) dx \leq \int_{\Omega} \frac{\Phi(\nabla u_k)}{\|u_k\|_{W_0^{1,\Phi}(\Omega)} - \varepsilon} dx$$

for  $k \in \mathbb{N}$ . Hence,

$$(4.36) \quad \|u_k\|_{W_0^{1,\Phi}(\Omega)} - \varepsilon < \int_{\Omega} \Phi(\nabla u_k) dx$$

for  $k \in \mathbb{N}$ . Coupling (4.35) with (4.36) yields

$$(4.37) \quad 1 - \frac{\varepsilon}{\|u_k\|_{W_0^{1,\Phi}(\Omega)}} \leq \frac{C'}{\sigma \|u_k\|_{W_0^{1,\Phi}(\Omega)}} + \frac{\varepsilon_k}{\sigma}$$

for  $k > k_{\varepsilon}$ . Passing to the limit as  $k \rightarrow \infty$  in (4.37) leads to a contradiction. Our claim is thus proved. Assumption (2.3) and Lemma 6.5 ensure, owing to Proposition 3.1, that the space  $W_0^{1,\Phi}(\Omega)$  is reflexive. Thus, there exists a subsequence of  $\{u_k\}$ , still denoted by  $\{u_k\}$ , that weakly converges to some function  $u \in W_0^{1,\Phi}(\Omega)$ . Now, if (2.7) and (2.9) hold, then by (3.17) and (4.20), we may apply Lemma 3.5 with  $E = \overline{F}$  and  $B = \Phi_n$ . Therefore, on choosing  $\varphi = u - u_k$  in (4.31), and exploiting (3.19), we deduce that

$$(4.38) \quad \limsup_{k \rightarrow \infty} \int_{\Omega} \Phi_{\xi}(\nabla u_k) \cdot (\nabla u - \nabla u_k) dx = 0.$$

Equation (4.38) continues to hold even if, instead, (2.10) is in force, since (3.19) trivially holds thanks to embedding (3.12). Equation (4.38), via Lemma 4.6, implies that  $u_k \rightarrow u$  in  $W_0^{1,\Phi}(\Omega)$ .  $\square$

**Lemma 4.7** *Let  $\Phi$  be an  $n$ -dimensional Young function and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Assume that condition (2.6) is fulfilled, and that either (2.7) holds and (2.8)–(2.9) are in force, or (2.10) holds. Then,*

$$(4.39) \quad \lim_{\|u\|_{W_0^{1,\Phi}(\Omega)} \rightarrow 0} \frac{\int_{\Omega} F(u) dx}{\int_{\Omega} \Phi(\nabla u) dx} = 0.$$

**Proof.** By Lemma 4.4, for every  $\varepsilon > 0$  there exists  $t_{\varepsilon} \geq 0$  such that

$$(4.40) \quad |F(t)| < \varepsilon \Phi_* \left( \omega_n^{\frac{1}{n}} |\Omega|^{-\frac{1}{n}} |t| \right) \quad \text{if } |t| \leq t_{\varepsilon}.$$

From (4.40) and (3.5) we deduce that

$$(4.41) \quad \frac{|\int_{|u| \leq t_{\varepsilon}} F(u) dx|}{\int_{\Omega} \Phi(\nabla u) dx} < \frac{\varepsilon \int_{|u| \leq t_{\varepsilon}} \Phi_* \left( \omega_n^{\frac{1}{n}} |\Omega|^{-\frac{1}{n}} |u| \right) dx}{\int_{\Omega} \Phi(\nabla u) dx} < \frac{\varepsilon \int_{\Omega} \Phi(\nabla u) dx}{\int_{\Omega} \Phi(\nabla u) dx} = \varepsilon.$$

Suppose first that (2.7), (2.8) and (2.9) hold. Then we can apply Lemma 4.4 with  $B = \Phi_n$ . In particular,  $\bar{F}$  increases essentially more slowly than  $\Phi_n$ . Thus, one can show that there exists  $\lambda > 0$  such that

$$(4.42) \quad \bar{F}(|t|) \leq \varepsilon \Phi_n(\lambda |t|) \quad \text{if } |t| \geq t_{\varepsilon}.$$

Moreover, we can choose  $\lambda$  such that  $C\lambda > 1$ , where  $C$  denotes the constant appearing in (3.10). Fix  $\delta < (C\lambda)^{-n}$ . Since, in particular,  $\delta < 1$ , if  $\|u\|_{W_0^{1,\Phi}(\Omega)} < \delta$  then  $\int_{\Omega} \Phi(\nabla u) dx < \delta$  as well. Thus,

$$\lambda < \frac{1}{C\delta^{\frac{1}{n}}} < \frac{1}{C(\int_{\Omega} \Phi(\nabla u) dx)^{\frac{1}{n}}}.$$

Hence, by (4.42) and (3.10),

$$(4.43) \quad \frac{|\int_{|u| > t_{\varepsilon}} F(u) dx|}{\int_{\Omega} \Phi(\nabla u) dx} < \frac{\varepsilon \int_{|u| > t_{\varepsilon}} \Phi_n(\lambda |u|) dx}{\int_{\Omega} \Phi(\nabla u) dx} < \frac{\varepsilon \int_{\Omega} \Phi_n \left( \frac{|u|}{C(\int_{\Omega} \Phi(\nabla u) dy)^{\frac{1}{n}}} \right) dx}{\int_{\Omega} \Phi(\nabla u) dx} \leq \varepsilon.$$

Coupling (4.41) with (4.43) tells us that  $\frac{|\int_{\Omega} F(u) dx|}{\int_{\Omega} \Phi(\nabla u) dx} < 2\varepsilon$ , if  $\|u\|_{W_0^{1,\Phi}(\Omega)} < \delta$ . Thus, equation (4.39) follows.

Assume next that (2.10) holds. In this case, by (3.12), there exists  $\delta > 0$  such that  $\|u\|_{L^{\infty}(\Omega)} < t_{\varepsilon}$  if  $\|u\|_{W_0^{1,\Phi}(\Omega)} < \delta$ . Equation (4.41) then yields  $\frac{|\int_{\Omega} F(u) dx|}{\int_{\Omega} \Phi(\nabla u) dx} < \varepsilon$ , and equation (4.39) follows also in this case.  $\square$

Our last preparatory result in view of the proof of Theorem 2.1 is contained in the following lemma.

**Lemma 4.8** *Let  $\Phi \in C^1(\mathbb{R}^n)$  be an  $n$ -dimensional Young function, such that  $s_{\Phi} < \infty$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function fulfilling (2.4) and (2.5). If  $u \in C_0^1(\Omega)$  and  $u$  does not vanish identically, then*

$$(4.44) \quad \lim_{t \rightarrow \infty} \int_{\Omega} (\Phi(t \nabla u) - F(tu)) dx = -\infty.$$

**Proof.** Owing to (6.21), for every  $\varepsilon$ , there exists  $M \geq 0$  such that

$$(4.45) \quad \Phi(t\xi) \leq \Phi(\xi)t^{s_\Phi+\varepsilon} \quad \text{if } t \geq 1 \text{ and } |\xi| \geq M.$$

By (2.5), if  $\varepsilon$  and  $\alpha > 0$  are chosen in such a way that  $\liminf_{t \rightarrow \pm\infty} \frac{tf(t)}{F(t)} > \alpha > s_\Phi + \varepsilon$ , then

$$\frac{tf(t)}{F(t)} \geq \alpha \quad \text{if } |t| \text{ is sufficiently large.}$$

Owing to assumption (2.4) and to the last inequality, there exist  $a, b > 0$  such that

$$(4.46) \quad F(t) \geq a|t|^\alpha - b \quad \text{for } t \in \mathbb{R}.$$

Now, let  $u$  be as in the statement and let  $t \geq 1$ . Owing to (4.45) and (4.46),

$$\begin{aligned} J_\Phi(tu) &\leq \int_{\{|\nabla u| \leq M\}} \Phi(t\nabla u) dx + \int_{\{|\nabla u| > M\}} \Phi(t\nabla u) dx - \int_\Omega a|tu|^\alpha dx + b|\Omega| \\ &\leq \int_{\{|\nabla u| \leq M\}} \Phi\left(tM \frac{\nabla u}{|\nabla u|}\right) dx + t^{s_\Phi+\varepsilon} \int_{\{|\nabla u| > M\}} \Phi(\nabla u) dx - at^\alpha \|u\|_{L^\alpha(\Omega)} + b|\Omega| \\ &\leq t^{s_\Phi+\varepsilon} \left[ \int_{\{|\nabla u| \leq M\}} \Phi\left(M \frac{\nabla u}{|\nabla u|}\right) dx + \int_{\{|\nabla u| > M\}} \Phi(\nabla u) dx \right] - at^\alpha \|u\|_{L^\alpha(\Omega)} + b|\Omega|, \end{aligned}$$

where  $\frac{\nabla u}{|\nabla u|}$  is taken to be 0 if  $\nabla u = 0$ . Equation (4.44) follows, inasmuch as  $s_\Phi + \varepsilon < \alpha$ .  $\square$

We are now in a position to accomplish the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Corollary 4.1 and Proposition 4.5 ensure that the functional  $J_\Phi$ , defined by (1.2), is of class  $C^1$  and satisfies the Palais-Smale condition (4.2). Lemmas 4.7 and 4.8 tell us that conditions (4.4) and (4.5), respectively, are fulfilled. Thus,  $J_\Phi$  satisfies the assumptions of the Mountain Pass Theorem stated above, and hence  $J_\Phi$  has a nontrivial critical point  $u \in W_0^{1,\Phi}(\Omega)$ , which is a solution to (1.1).

The boundedness of  $u$  follows from an application of [Al, Theorem 4.1].  $\square$

## 5 Special instances

In this section, we specialize Theorem 2.1 to some classes of functions  $\Phi$ , which govern the differential operator in the equation in (1.1), with a distinctive structure, including those corresponding to equations (1.4), (1.7) and (1.9). In particular, the novelties of our conclusions in comparison with the existing literature are pointed out.

### 5.1 Isotropic growth

Consider first the isotropic case when  $\Phi$  is radial, namely

$$(5.1) \quad \Phi(\xi) = A(|\xi|) \quad \text{for } \xi \in \mathbb{R}^n,$$

where  $A$  is as in (1.7). Problem (1.1) then reads

$$(5.2) \quad \begin{cases} -\operatorname{div} \left( \frac{A'(|\nabla u|)}{|\nabla u|} \nabla u \right) = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Owing to (5.1), we have that  $\Phi_*(t) = A(t)$ , and  $\Phi_n(t) = A_n(t)$ , where

$$A_n(t) = A(H_A^{-1}(t)) \quad \text{for } t \geq 0,$$

and

$$H_A(t) = \left( \int_0^t \left( \frac{\tau}{A(\tau)} \right)^{\frac{1}{n-1}} d\tau \right)^{\frac{n-1}{n}} \quad \text{for } t \geq 0.$$

Moreover,  $i_\Phi = i_A$ , and  $s_\Phi = s_A$ , where we have set

$$i_A = \liminf_{t \rightarrow \infty} \frac{tA'(t)}{A(t)}, \quad s_A = \limsup_{t \rightarrow \infty} \frac{tA'(t)}{A(t)}.$$

By Theorem 2.1, problem (5.2) has a nontrivial solution, provided that

$$1 < i_A, \quad s_A < \infty,$$

$$\liminf_{t \rightarrow \pm\infty} \frac{tf(t)}{F(t)} > s_A,$$

$$\lim_{t \rightarrow 0} \frac{tf(t)}{A(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0,$$

and either

$$\int_0^\infty \left( \frac{\tau}{A(\tau)} \right)^{\frac{1}{n-1}} d\tau < \infty,$$

or

$$\int_0^\infty \left( \frac{\tau}{A(\tau)} \right)^{\frac{1}{n-1}} d\tau = \infty, \quad \int_0^\infty \left( \frac{\tau}{A(\tau)} \right)^{\frac{1}{n-1}} d\tau < \infty,$$

and

$$\lim_{t \rightarrow \pm\infty} \frac{tf(t)}{A_n(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0.$$

This result strengthens [CGMS, Theorem 1.1], where an analogous conclusion is derived with the function  $A_n$  replaced with another function, which, in general, can grow more slowly near infinity (see, for instance, the next example). Furthermore, in [CGMS] the assumption  $s_A < \infty$  is replaced with the more stringent assumption that  $\sup_{t \geq 0} \frac{tA'(t)}{A(t)} < \infty$ .

## 5.2 Isotropic power type growth

Let us further specialize problem (5.2) to functions  $A$  having an explicit asymptotic behavior near infinity. Assume first that

$$A(t) = \frac{1}{p} t^p \quad \text{for } t \geq 0,$$

for some  $p \in (1, n)$ . With this choice of  $A$ , problem (5.2) agrees with the classical Dirichlet problem for the  $p$ -Laplace equation

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We then obtain that a nontrivial solution exists, provided that  $f$  fulfills the following conditions:

$$\liminf_{t \rightarrow \pm\infty} \frac{tf(t)}{F(t)} > p,$$

$$\lim_{t \rightarrow 0} \frac{f(t)}{|t|^{p-1}} = 0,$$

$$(5.3) \quad \lim_{t \rightarrow \pm\infty} \frac{f(t)}{|t|^{p^*-1}} = 0,$$

where  $p^* = \frac{np}{n-p}$ , the Sobolev exponent associated with  $p$ . Note that this conclusion somewhat augments standard results for the  $p$ -Laplace equation, which require that  $\lim_{t \rightarrow \pm\infty} \frac{f(t)}{|t|^{q-1}} = 0$  for some  $q < p^*$ .

If  $p > n$ , the same result holds, without assumption (5.3).

In the borderline case when  $A(t) = t^n$  for every  $t \geq 0$ , Theorem 2.1 does not apply, since both  $\int_0^\infty (\frac{\tau}{A(\tau)})^{\frac{1}{n-1}} d\tau = \infty$  and  $\int_0 (\frac{\tau}{A(\tau)})^{\frac{1}{n-1}} d\tau = \infty$ . However, if  $A(t) = t^n$  for large  $t$ , but the latter integral converges, then Theorem 2.1 entails that a nontrivial solution to problem (5.2) exists, provided that

$$\liminf_{t \rightarrow \pm\infty} \frac{tf(t)}{F(t)} > n,$$

$$\lim_{t \rightarrow 0} \frac{tf(t)}{A(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0,$$

and

$$\lim_{t \rightarrow \pm\infty} \frac{tf(t)}{e^{\lambda|t|^{n'}}} = 0 \quad \text{for every } \lambda > 0.$$

In [CGMS], the stronger assumption  $\lim_{t \rightarrow \pm\infty} \frac{tf(t)}{e^{\lambda|t|}} = 0$  for every  $\lambda > 0$  was instead required, as well as a more stringent condition at 0 than  $\int_0 (\frac{\tau}{A(\tau)})^{\frac{1}{n-1}} d\tau < \infty$ .

### 5.3 Anisotropic growth in split form

Here, we deal with anisotropic functions  $\Phi$  with a split structure, namely functions given by

$$\Phi(\xi) = \sum_{i=1}^n A_i(|\xi_i|) \quad \text{for } \xi \in \mathbb{R}^n,$$

where  $A_i : [0, \infty) \rightarrow [0, \infty)$ ,  $i = 1, \dots, n$ , are strictly convex, continuously differentiable functions vanishing at 0. In this case, problem (1.1) takes the form

$$(5.4) \quad \begin{cases} -\sum_{i=1}^n (A'_i(u_{x_i}))_{x_i} = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The function  $\Phi_*$  is equivalent to the convex function  $\hat{A} : [0, \infty) \rightarrow [0, \infty)$ , whose inverse is given by

$$\hat{A}^{-1}(r) = \left( \prod_{i=1}^n A_i^{-1}(r) \right)^{\frac{1}{n}} \quad \text{for } r \geq 0,$$

in the sense that there exist positive constants  $c_1$  and  $c_2$  such that

$$(5.5) \quad \hat{A}(c_1 t) \leq \Phi_*(t) \leq \hat{A}(c_2 t) \quad \text{for } t \geq 0,$$

see [Ci3, Equation (1.9)]. Hence, the Sobolev conjugate  $\Phi_n$  is in turn equivalent to the function  $\hat{A}_n$ , given by

$$\hat{A}_n(t) = \hat{A}(H_{\hat{A}}^{-1}(t)) \quad \text{for } t \geq 0,$$



where now

$$H_{\widehat{A}}(t) = \left( \int_0^t \left( \frac{\tau}{\widehat{A}(\tau)} \right)^{\frac{1}{n-1}} d\tau \right)^{\frac{n-1}{n}} \text{ for } t \geq 0.$$

Furthermore, one can show that

$$(5.6) \quad i_{\Phi} = \min_{1 \leq i \leq n} i_{A_i}, \quad s_{\Phi} = \max_{1 \leq i \leq n} s_{A_i}.$$

Hence, an application of Theorem 2.1 tells us that problem (5.4) has a nontrivial solution, provided that

$$\begin{aligned} 1 &< \min_{1 \leq i \leq n} i_{A_i}, \quad \max_{1 \leq i \leq n} s_{A_i} < \infty, \\ \liminf_{t \rightarrow \pm\infty} \frac{tf(t)}{F(t)} &> \max_{1 \leq i \leq n} s_{A_i}, \\ \lim_{t \rightarrow 0} \frac{tf(t)}{\widehat{A}(\lambda|t|)} &= 0 \quad \text{for every } \lambda > 0, \end{aligned}$$

and either

$$\int^{\infty} \left( \frac{\tau}{\widehat{A}(\tau)} \right)^{\frac{1}{n-1}} d\tau < \infty,$$

or

$$\int^{\infty} \left( \frac{\tau}{\widehat{A}(\tau)} \right)^{\frac{1}{n-1}} d\tau = \infty, \quad \int_0^{\infty} \left( \frac{\tau}{\widehat{A}(\tau)} \right)^{\frac{1}{n-1}} d\tau < \infty$$

and

$$\lim_{t \rightarrow \infty} \frac{tf(t)}{\widehat{A}_n(\lambda t)} = 0 \quad \text{for every } \lambda > 0.$$

#### 5.4 Anisotropic power type growth

Consider the standard instance of (5.4), when

$$A_i(t) = \frac{1}{p_i} t^{p_i} \quad \text{for } t \geq 0,$$

for some powers  $p_i > 1$ ,  $i = 1, \dots, n$ . Namely,

$$\Phi(\xi) = \sum_{i=1}^n \frac{1}{p_i} |\xi_i|^{p_i} \quad \text{for } \xi \in \mathbb{R}^n.$$

Thus, problem (1.1) agrees with

$$(5.7) \quad \begin{cases} -\sum_{i=1}^n (|u_{x_i}|^{p_i-2} u_{x_i})_{x_i} = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that here

$$i_{A_i} = s_{A_i} = p_i \quad \text{for } i = 1, \dots, n.$$

Owing to (5.5)

$$\Phi_*(t) \approx t^{\bar{p}} \quad \text{for } t \geq 0,$$

where the relation “ $\approx$ ” means that the two sides are bounded by each other up to multiplicative constants independent of  $t$ , and  $\bar{p}$  is the harmonic average of the powers  $p_i$ , defined via (1.10). In particular, when  $\bar{p} < n$ , one has that

$$\Phi_n(t) \approx t^{\bar{p}^*} \quad \text{for } t \geq 0,$$

where  $\bar{p}^* = \frac{n\bar{p}}{n-\bar{p}}$ , the Sobolev conjugate of  $\bar{p}$ .

By the result of Subsection 5.3, we can thus conclude that problem (5.7) has a nontrivial solution, provided that

$$\liminf_{t \rightarrow \pm\infty} \frac{tf(t)}{F(t)} > \max_{1 \leq i \leq n} p_i,$$

$$\lim_{t \rightarrow 0} \frac{f(t)}{|t|^{\bar{p}-1}} = 0,$$

and either  $\bar{p} > n$ , or  $\bar{p} < n$  and

$$(5.8) \quad \lim_{t \rightarrow \pm\infty} \frac{f(t)}{|t|^{\bar{p}^*-1}} = 0.$$

This recovers [FGK, Theorem 4], and extends it, in that, unlike [FGK], here we are not assuming that  $f$  is just a power. Let us point out that in [FGK] the sharpness of assumption (5.8) is also shown. This is accomplished by proving, via suitable anisotropic Pohozaev type identities, the non-existence of nontrivial solutions to (5.7), in suitable classes of domains, in case of nonlinearities  $f$  of the form  $f(t) = t^{q-1}$  with  $q > \bar{p}^*$ .

### 5.5 Anisotropic power-logarithmic type growth

We deal here with a somewhat more general case than that of Subsection 5.4, corresponding to (5.4) with the choice:

$$(5.9) \quad A_i(t) = \frac{1}{p_i} t^{p_i} \log^{\alpha_i}(c+t) \quad \text{for } t \geq 0,$$

where  $p_i > 1$ ,  $\alpha_i \in \mathbb{R}$ ,  $i = 1, \dots, n$ , and  $c$  is a positive constant, sufficiently large (depending on the powers  $p_i$  and  $\alpha_i$ ) for all functions  $A_i$  to be convex. Thus,

$$\Phi(\xi) = \sum_{i=1}^n \frac{1}{p_i} |\xi_i|^{p_i} \log^{\alpha_i}(c + |\xi_i|) \quad \text{for } \xi \in \mathbb{R}^n.$$

Note that

$$i_{A_i} = s_{A_i} = p_i \quad \text{for } i = 1, \dots, n$$

if the functions  $A_i$  are given by (5.9). Let  $\bar{p}$  be defined as in (1.10), and let  $\bar{\alpha}$  be defined as

$$\bar{\alpha} = \frac{\bar{p}}{n} \sum_{i=1}^n \frac{\alpha_i}{p_i}.$$

One can verify, via (5.5), that

$$\Phi_*(t) \approx \begin{cases} t^{\bar{p}} & \text{near } 0, \\ t^{\bar{p}} \log^{\bar{\alpha}}(c+t) & \text{near infinity,} \end{cases}$$

up to multiplicative constants independent of  $t$ . Moreover, if  $\bar{p} < n$ , then conditions (2.7) and (2.8) are fulfilled, and

$$\Phi_n(t) \approx t^{\bar{p}^*} \log^{\frac{\bar{\alpha}n}{n-\bar{p}}}(c+t) \quad \text{near infinity,}$$

where  $\bar{p}^*$  denotes the Sobolev conjugate of  $\bar{p}$ .

The result of Subsection 5.3 ensures that problem (5.4) has a nontrivial solution, provided that

$$\liminf_{t \rightarrow \pm\infty} \frac{tf(t)}{F(t)} > \max_{1 \leq i \leq n} p_i,$$

$$\lim_{t \rightarrow 0} \frac{f(t)}{|t|^{\bar{p}-1}} = 0,$$

and either  $\bar{p} > n$ , or  $\bar{p} < n$  and

$$\lim_{t \rightarrow \pm\infty} \frac{f(t)}{|t|^{\bar{p}^*-1} \log^{\frac{\bar{\alpha}n}{n-\bar{p}}}(|t|)} = 0.$$

## 6 Appendix: Young functions and $n$ -dimensional Young functions

Standard Young functions have been extensively treated in the literature. Notations and properties involving Young functions, which are exploited in this paper, are recalled in the first part of this appendix. For a comprehensive treatment of this matter we refer the reader to the monographs [KR, RR1, RR2].

The Young conjugate of a Young function  $A$  is the Young function  $\tilde{A}$  defined as

$$\tilde{A}(s) = \sup\{st - A(t) : t \geq 0\} \quad \text{for } s \geq 0.$$

One has that  $\tilde{\tilde{A}} = A$  for any Young function  $A$ .

On denoting by  $A^{-1}$  the (generalized) left-continuous inverse of  $A$ , one has that

$$(6.1) \quad t \leq \tilde{A}^{-1}(t)A^{-1}(t) \leq 2t \quad \text{for } t \geq 0.$$

Hence,

$$(6.2) \quad \frac{A(t)}{t} \leq \tilde{A}^{-1}(A(t)) \leq 2\frac{A(t)}{t} \quad \text{for } t > 0.$$

If  $A$  is a Young function, then

$$\lambda A(t) \leq A(\lambda t) \quad \text{for } \lambda \geq 1 \text{ and } t \geq 0.$$

A Young function  $A$  is said to satisfy the  $\Delta_2$ -condition near infinity if there exist constants  $C \geq 2$  and  $M \geq 0$  such that

$$(6.3) \quad A(2t) \leq CA(t) \quad \text{if } t \geq M.$$

A Young function  $A$  is said to dominate another Young function  $B$  near infinity, if there exist constants  $c > 0$  and  $M \geq 0$  such that

$$(6.4) \quad B(t) \leq A(ct) \quad \text{if } t \geq M.$$

If (6.4) holds with  $M = 0$ , then we say that  $A$  dominates  $B$  globally. Two Young functions  $A$  and  $B$  are called equivalent near infinity [globally] if they dominate each other near infinity [globally]. The function  $B$  is said to increase essentially more slowly than  $A$  near infinity, if

$$(6.5) \quad \lim_{t \rightarrow \infty} \frac{B(\lambda t)}{A(t)} = 0 \quad \text{for every } \lambda > 0.$$

Condition (6.5) is equivalent to

$$(6.6) \quad \lim_{s \rightarrow \infty} \frac{A^{-1}(s)}{B^{-1}(s)} = 0.$$

The theory of  $n$ -dimensional Young functions seems to be much less developed than that of standard Young functions. Contributions to this topic can be found in [Ro, Sch, Sk, Sk, Tr]. The remaining part of this appendix is devoted to definitions and proofs of some results on this subject, which are not straightforward consequences of parallel results for usual Young functions.

For technical reasons, we distinguish between Young functions and 1-dimensional Young functions. However, extending a Young function to an even function to the whole of  $\mathbb{R}$  results in a 1-dimensional Young function; conversely, the restriction of a 1-dimensional Young function to  $[0, \infty)$  is a Young function. Thus, any definition or result concerning Young functions translates into a corresponding definition or result for 1-dimensional Young functions, and viceversa.

Given a Young function  $A$ , the function  $\mathbb{R}^n \ni \xi \mapsto A(|\xi|)$  is an (isotropic)  $n$ -dimensional Young function. Moreover, given an  $n$ -dimensional Young function  $\Phi$ , and a point  $\xi \in \mathbb{R}^n$ , the function  $[0, \infty) \ni t \mapsto \Phi(t\xi)$  is a Young function.

If  $\Phi$  is an  $n$ -dimensional Young function, then

$$(6.7) \quad \lambda\Phi(\xi) \leq \Phi(\lambda\xi) \quad \text{for } \lambda \geq 1 \text{ and } \xi \in \mathbb{R}^n.$$

An  $n$ -dimensional Young function  $\Phi$  is said to satisfy the  $\Delta_2$ -condition near infinity if there exist constants  $C \geq 2$  and  $M \geq 0$  such that

$$(6.8) \quad \Phi(2\xi) \leq C\Phi(\xi) \quad \text{if } |\xi| \geq M.$$

The function  $\Phi$  is said to satisfy the  $\nabla_2$ -condition near infinity if there exist constants  $C > 2$  and  $M \geq 0$  such that

$$(6.9) \quad \Phi(2\xi) \geq C\Phi(\xi) \quad \text{if } |\xi| \geq M.$$

The global  $\Delta_2$ -condition and the global  $\nabla_2$ -condition are defined accordingly, with  $M = 0$ .

Our applications mainly require properties of functions satisfying these conditions near infinity, and we thus focus this case in what follows. The relevant properties have, however, global counterparts, which are usually simpler to prove.

**Proposition 6.1** *Let  $\Phi$  be an  $n$ -dimensional Young function.*

(i)  $\Phi \in \Delta_2$  near infinity if and only if there exist constants  $M \geq 0$  and  $k > 1$  such that

$$(6.10) \quad \Phi(k\xi) \leq 2\Phi(\xi) \quad \text{if } |\xi| \geq M.$$

(ii)  $\Phi \in \nabla_2$  near infinity if and only if there exist constants  $M \geq 0$  and  $k > 1$  such that

$$(6.11) \quad \Phi(k\xi) \geq 2k\Phi(\xi) \quad \text{if } |\xi| \geq M.$$

**Proof.** (i) Assume that (6.8) holds. Set  $\delta = \frac{1}{C-1} \leq 1$ , and  $k = 1 + \delta$ . By the convexity of  $\Phi$  and (6.8), we have that

$$\begin{aligned}\Phi(k\xi) &= \Phi((k - 2\delta + 2\delta)\xi) = \Phi((1 - \delta)\xi + \delta(2\xi)) \\ &\leq (1 - \delta)\Phi(\xi) + \delta\Phi(2\xi) \leq (1 - \delta)\Phi(\xi) + \delta C\Phi(\xi) = 2\Phi(\xi) \quad \text{if } |\xi| \geq M,\end{aligned}$$

namely (6.10).

Conversely, assume that (6.10) is in force. Fix  $m \in \mathbb{N}$  such that  $k^m \geq 2$ , and choose  $C = 2^m$ . Iterating (6.10) tells us that

$$\Phi(2\xi) \leq \Phi(k^m\xi) \leq 2^m\Phi(\xi) = C\Phi(\xi) \quad \text{if } |\xi| \geq M.$$

Hence, (6.8) follows.

(ii) Assume that (6.9) holds. Fix  $m \geq 1$  such that  $(\frac{C}{2})^m > 2$ , namely  $C^m > 2^{m+1}$ , and choose  $k = 2^m$ . Thus, by iteration of (6.9), we deduce that

$$\Phi(\xi) \leq \frac{1}{C^m}\Phi(2^m\xi) < \frac{1}{2^{m+1}}\Phi(2^m\xi) = \frac{1}{2k}\Phi(k\xi) \quad \text{if } |\xi| \geq M.$$

namely (6.11).

Finally, suppose that (6.11) holds. Thus,

$$(6.12) \quad \Phi\left(\frac{2\xi}{k}\right) \leq \frac{1}{2k}\Phi(2\xi) \quad \text{if } |\xi| \geq \frac{Mk}{2}.$$

Set  $C = \frac{4k}{2k-1} > 2$ , and  $M_1 = \frac{Mk}{2}$ . Owing to the convexity of  $\Phi$ , to inequality (6.12), and to (6.7),

$$\Phi(\xi) \leq \frac{1}{4k}\Phi(2\xi) + \frac{1}{2} \cdot \frac{k-1}{k}\Phi(2\xi) = \frac{1}{C}\Phi(2\xi) \quad \text{if } |\xi| \geq M_1.$$

This establishes property (6.9). □

An  $n$ -dimensional Young function  $\Psi$  is said to dominate another  $n$ -dimensional Young function  $\Phi$  near infinity if there exist constants  $c > 0$  and  $M \geq 0$  such that

$$(6.13) \quad \Phi(\xi) \leq \Psi(c\xi) \quad \text{if } |\xi| \geq M.$$

If (6.13) holds with  $M = 0$ , then we say that  $\Psi$  dominates  $\Phi$  globally. Two  $n$ -dimensional Young functions  $\Psi$  and  $\Phi$  are called equivalent near infinity [globally] if they dominate each other near infinity [globally].

The Young conjugate of  $\Phi$  is the  $n$ -dimensional Young function  $\tilde{\Phi}$  given by

$$(6.14) \quad \tilde{\Phi}(\eta) = \sup\{\eta \cdot \xi - \Phi(\xi) : \xi \in \mathbb{R}^n\} \quad \text{for } \eta \in \mathbb{R}^n.$$

One has that

$$(6.15) \quad \tilde{\tilde{\Phi}} = \Phi.$$

**Proposition 6.2** *Let  $\Phi$  be a  $n$ -dimensional Young function. Then  $\Phi$  is finite-valued if and only if*

$$(6.16) \quad \lim_{|\eta| \rightarrow \infty} \frac{\tilde{\Phi}(\eta)}{|\eta|} = \infty.$$

**Proof.** Let  $\Phi$  be finite-valued. Assume, by contradiction, that (6.16) fails. Then, there exist a constant  $c > 0$  and a sequence  $\{\eta_k\} \subset \mathbb{R}^n$  such that  $\lim_{k \rightarrow \infty} |\eta_k| = \infty$ , and  $\tilde{\Phi}(\eta_k) \leq c|\eta_k|$  for  $k \in \mathbb{N}$ . Since  $\frac{\eta_k}{|\eta_k|} = 1$ , there exists subsequence (still denoted by  $\{\eta_k\}$ ), and  $\theta \in \mathbb{R}^n$ , with  $|\theta| = 1$ , such that  $\lim_{k \rightarrow \infty} \frac{\eta_k}{|\eta_k|} = \theta$ . In particular, this limit implies that  $\lim_{k \rightarrow \infty} \frac{\theta \cdot \eta_k}{|\eta_k|} = |\theta|^2 = 1$ . Now, fix any  $t > c$ . Therefore,

$$\Phi(t\theta) = \tilde{\Phi}(t\theta) = \sup_{\eta \in \mathbb{R}^n} [t\theta \cdot \eta - \tilde{\Phi}(\eta)] \geq \sup_{k \in \mathbb{N}} [t\theta \cdot \eta_k - \tilde{\Phi}(\eta_k)] \geq \sup_{k \in \mathbb{N}} \left[ \left( t \frac{\theta \cdot \eta_k}{|\eta_k|} - c \right) |\eta_k| \right] = \infty.$$

This contradicts the fact that  $\Phi$  is finite-valued.

Conversely, assume that (6.16) holds. Then,

$$(6.17) \quad \Phi(\xi) = \tilde{\Phi}(\xi) = \sup_{\eta \in \mathbb{R}^n} [\xi \cdot \eta - \tilde{\Phi}(\eta)] \leq \sup_{\eta \in \mathbb{R}^n} \left[ |\eta| |\xi| - \tilde{\Phi}(\eta) \right] = \sup_{\eta \in \mathbb{R}^n} \left[ |\eta| \left( |\xi| - \frac{\tilde{\Phi}(\eta)}{|\eta|} \right) \right].$$

Fix any  $\xi \in \mathbb{R}^n$ . By (6.16), there exists  $K > 0$  such that  $\frac{\tilde{\Phi}(\eta)}{|\eta|} \geq 2|\xi|$  if  $|\eta| > K$ , whence

$$(6.18) \quad |\eta| \left( |\xi| - \frac{\tilde{\Phi}(\eta)}{|\eta|} \right) \leq 0 \quad \text{if } |\eta| > K.$$

From (6.17) and (6.18) we deduce that

$$\Phi(\xi) \leq \sup_{|\eta| \leq K} \left[ |\eta| \left( |\xi| - \frac{\tilde{\Phi}(\eta)}{|\eta|} \right) \right] \leq K|\xi| < \infty,$$

and hence  $\Phi$  is finite-valued. □

The following corollary is a straightforward consequence of Proposition 6.2 and of equation (6.15).

**Corollary 6.3** *Let  $\Phi$  be a  $n$ -dimensional Young function. Then*

$$\Phi \text{ is finite-valued and } \lim_{|\xi| \rightarrow \infty} \frac{\Phi(\xi)}{|\xi|} = \infty$$

*if and only if*

$$\tilde{\Phi} \text{ is finite-valued and } \lim_{|\eta| \rightarrow \infty} \frac{\tilde{\Phi}(\eta)}{|\eta|} = \infty.$$

**Proposition 6.4** *Let  $\Phi$  and  $\Psi$  be  $n$ -dimensional Young functions. Assume that  $\Psi$  is finite-valued and there exists  $M_0 \geq 0$  such that*

$$(6.19) \quad \Phi(\xi) \geq \Psi(\xi) \quad \text{if } |\xi| \geq M_0.$$

*Then there exists  $M_1 \geq 0$  such that*

$$(6.20) \quad \tilde{\Phi}(\eta) \leq \tilde{\Psi}(\eta) \quad \text{if } |\eta| \geq M_1.$$

*Conversely, assume that  $\lim_{|\xi| \rightarrow \infty} \frac{\Phi(\xi)}{|\xi|} = \infty$  and (6.20) holds for some  $M_1 \geq 0$ . Then (6.19) holds for some  $M_0 \geq 0$ .*

**Proof.** Assume that  $\Psi$  is finite-valued, and (6.19) is in force. By Proposition 6.2, there exists  $M_1 \geq 0$  such that  $\tilde{\Psi}(\eta) > M_0|\eta|$  if  $|\eta| \geq M_1$ . Thus, for any such  $\eta$ ,

$$\begin{aligned}\tilde{\Phi}(\eta) &= \sup_{\xi \in \mathbb{R}^n} [\xi \cdot \eta - \Phi(\xi)] = \max \left\{ \sup_{|\xi| \leq M_0} [\xi \cdot \eta - \Phi(\xi)], \sup_{|\xi| > M_0} [\xi \cdot \eta - \Phi(\xi)] \right\} \\ &\leq \max \left\{ M_0|\eta|, \sup_{|\xi| > M_0} [\xi \cdot \eta - \Psi(\xi)] \right\} \leq \max \{ M_0|\eta|, \tilde{\Psi}(\eta) \} = \tilde{\Psi}(\eta).\end{aligned}$$

Inequality (6.20) is thus established.

Conversely, assume that  $\lim_{|\xi| \rightarrow \infty} \frac{\Phi(\xi)}{|\xi|} = \infty$ , and (6.20) holds. Then there exists  $M_0 \geq 0$  such that  $\Phi(\xi) > M_1|\xi|$  whenever  $|\xi| \geq M_0$ . By (6.15),

$$\begin{aligned}\Psi(\xi) &= \tilde{\tilde{\Psi}}(\xi) = \max \left\{ \sup_{|\eta| \leq M_1} [\xi \cdot \eta - \tilde{\Psi}(\eta)], \sup_{|\eta| > M_1} [\xi \cdot \eta - \tilde{\Psi}(\eta)] \right\} \\ &\leq \max \left\{ M_1|\xi|, \sup_{|\eta| > M_1} [\xi \cdot \eta - \tilde{\Phi}(\eta)] \right\} \leq \max \{ M_1|\xi|, \Phi(\xi) \} = \Phi(\xi) \quad \text{if } |\xi| \geq M_0,\end{aligned}$$

namely (6.19). □

**Proposition 6.5** *Let  $\Phi \in C^1(\mathbb{R}^n)$  be an  $n$ -dimensional Young function.*

- (i)  $\Phi \in \Delta_2$  near infinity if and only if  $s_\Phi < \infty$ .
- (ii)  $\Phi \in \nabla_2$  near infinity if and only if  $i_\Phi > 1$ .

**Proof.** Given any  $\xi \in \mathbb{R}^n$ , define the continuously differentiable Young function  $A$  by

$$A(t) = \Phi(t\xi) \quad \text{for } t \geq 0.$$

Note that  $A'(t) = \Phi_\xi(t\xi) \cdot \xi$  for all  $t \geq 0$ .

- (i) Assume that  $s_\Phi < \infty$ . Then, for every  $\varepsilon > 0$  there exists  $M \geq 0$  such that  $\frac{\Phi_\xi(\xi) \cdot \xi}{\Phi(\xi)} < s_\Phi + \varepsilon$  if  $|\xi| \geq M$ . Given any  $\xi \in \mathbb{R}^n$  such that  $|\xi| \geq M$ , we have that  $\frac{A'(t)}{A(t)} \leq \frac{s_\Phi + \varepsilon}{t}$  if  $t \neq 0$ , whence

$$(6.21) \quad A(t) \leq t^{s_\Phi + \varepsilon} A(1) \quad \text{if } t \geq 1.$$

The choice  $t = 2$  in the last inequality yields

$$\Phi(2\xi) \leq 2^{s_\Phi + \varepsilon} \Phi(\xi) \quad \text{if } |\xi| \geq M.$$

This tells us that  $\Phi \in \Delta_2$  near infinity.

Conversely, assume that  $\Phi \in \Delta_2$  near infinity. Since  $A'(t)$  is nonnegative and non-decreasing,

$$\Phi(2\xi) = A(2) = \int_0^2 A'(t) dt \geq \int_1^2 A'(t) dt \geq A'(1) = \Phi_\xi(\xi) \cdot \xi \quad \text{for } \xi \in \mathbb{R}^n.$$

Thus, inasmuch as  $\Phi \in \Delta_2$  near infinity, there exist  $C \geq 2$  and  $M \geq 0$  such that

$$C\Phi(\xi) \geq \Phi(2\xi) \geq \Phi_\xi(\xi) \cdot \xi \quad \text{if } |\xi| \geq M.$$

This shows that  $s_\Phi \leq C$ .

- (ii) Assume that  $i_\Phi > 1$ . Then, for every  $\varepsilon \in (0, i_\Phi - 1)$  there exists  $M > 0$  such that  $\frac{\Phi_\xi(\xi) \cdot \xi}{\Phi(\xi)} >$

$i_\Phi - \varepsilon > 1$  if  $|\xi| \geq M$ . Hence, given any  $\xi \in \mathbb{R}^n$  such that  $|\xi| \geq M$ ,  $\frac{A'(t)}{A(t)} \geq \frac{i_\Phi - \varepsilon}{t}$  if  $t \geq 1$ . Therefore,  $A(t) \geq t^{i_\Phi - \varepsilon} A(1)$  if  $t \geq 1$ . Since  $i_\Phi - \varepsilon > 1$ , we may choose  $t = 2^{\frac{1}{i_\Phi - \varepsilon - 1}}$  in the last inequality. So doing, we obtain that

$$\Phi\left(2^{\frac{1}{i_\Phi - \varepsilon - 1}} \xi\right) \geq 2^{\frac{i_\Phi - \varepsilon}{i_\Phi - \varepsilon - 1}} \Phi(\xi) \quad \text{if } |\xi| \geq M.$$

Thus  $\Phi \in \nabla_2$  near infinity, since equation (6.11) holds with  $k = 2^{\frac{1}{i_\Phi - \varepsilon - 1}}$ .

Conversely, assume that  $\Phi \in \nabla_2$  near infinity. By (6.9), there exist  $C > 2$  and  $M \geq 0$  such that

$$C \int_0^1 A'(t) dt = C\Phi(\xi) \leq \Phi(2\xi) = \int_0^2 A'(t) dt \quad \text{if } |\xi| \geq M.$$

Consequently,

$$C \int_0^2 A'(t) dt \leq \int_0^2 A'(t) dt + C \int_1^2 A'(t) dt \quad \text{if } |\xi| \geq M,$$

and hence

$$(C - 1)\Phi(2\xi) \leq C \int_1^2 A'(t) dt \leq CA'(2) = C\Phi_\xi(2\xi) \cdot \xi \quad \text{if } |\xi| \geq M.$$

Altogether,

$$1 < \frac{2(C - 1)}{C} \leq \frac{\Phi_\xi(2\xi) \cdot 2\xi}{\Phi(2\xi)} \quad \text{if } |\xi| \geq M,$$

whence  $i_\Phi > 1$ . □

**Proposition 6.6** *Let  $\Phi$  be a finite-valued  $n$ -dimensional Young function such that  $\lim_{|\xi| \rightarrow \infty} \frac{\Phi(\xi)}{|\xi|} = \infty$ .*

- (i)  $\Phi \in \Delta_2$  near infinity if and only if  $\tilde{\Phi} \in \nabla_2$  near infinity.
- (ii)  $\Phi \in \nabla_2$  near infinity if and only if  $\tilde{\Phi} \in \Delta_2$  near infinity.

**Proof.** By Corollary 6.3 and equation (6.15), it suffices to prove part (ii). Assume that  $\Phi \in \nabla_2$  near infinity. Let  $k$  and  $M$  be as in (6.11). Define the  $n$ -dimensional Young function  $\Phi_1$  as  $\Phi_1(\xi) = \frac{\Phi(k\xi)}{2k}$  for  $\xi \in \mathbb{R}^n$ . Then

$$(6.22) \quad \Phi_1(\xi) \geq \Phi(\xi) \quad \text{if } |\xi| \geq M,$$

and

$$\tilde{\Phi}_1(\eta) = \frac{\tilde{\Phi}(2\eta)}{2k} \quad \text{for } \eta \in \mathbb{R}^n.$$

Owing to (6.22) and Proposition 6.4, there exists  $M_1 > 0$  such that  $\tilde{\Phi}_1(\eta) \leq \tilde{\Phi}(\eta)$  if  $|\eta| \geq M_1$ . This implies that  $\tilde{\Phi} \in \Delta_2$  near infinity.

Conversely, assume that  $\tilde{\Phi} \in \Delta_2$  near infinity, and let  $C > 2$  and  $M \geq 0$  as in (6.8). Define the  $n$ -dimensional Young function  $\Psi$  by  $\Psi(\eta) = \frac{\tilde{\Phi}(2\eta)}{C}$  for  $\eta \in \mathbb{R}^n$ . By Corollary (6.3),  $\lim_{|\xi| \rightarrow \infty} \frac{\Psi(\xi)}{|\xi|} = \infty$ . We can thus argue as above, and find  $M_1 > 0$  such that

$$\frac{1}{C} \Phi\left(\frac{C\xi}{2}\right) = \tilde{\Psi}(\xi) \geq \tilde{\Phi}(\xi) = \Phi(\xi) \quad \text{if } |\xi| \geq M_1.$$

Hence,  $\Phi \in \nabla_2$  near infinity. □



**Proposition 6.7** *Let  $\Phi \in C^1(\mathbb{R}^n)$  be an  $n$ -dimensional Young function. If  $\lim_{|\xi| \rightarrow \infty} \frac{\Phi(\xi)}{|\xi|} = \infty$ , then*

$$(6.23) \quad \tilde{\Phi}(\Phi_\xi(\xi)) \leq \Phi(2\xi) \quad \text{for } \xi \in \mathbb{R}^n.$$

**Proof.** If  $\xi = 0$ , then equation (6.23) holds trivially. Assume now that  $\xi \neq 0$ . Set

$$(6.24) \quad \eta = \Phi_\xi(\xi).$$

The function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$ , defined as

$$g(\zeta) = \eta \cdot \zeta - \Phi(\zeta) \quad \text{for } \zeta \in \mathbb{R}^n,$$

is concave. Moreover, owing to our assumptions,  $\lim_{|\zeta| \rightarrow \infty} g(\zeta) = -\infty$ . Thus, it attains its maximum at every point  $\zeta$  where its gradient vanishes, namely such that  $\eta = \Phi_\xi(\zeta)$ . In particular, by (6.24),  $g$  attains its maximum at  $\xi$ . Therefore,

$$\tilde{\Phi}(\Phi_\xi(\xi)) = \tilde{\Phi}(\eta) = \max g = g(\xi) = \Phi_\xi(\xi) \cdot \xi - \Phi(\xi) \leq \Phi_\xi(\xi) \cdot \xi \leq \Phi(2\xi).$$

Hence, inequality (6.23) follows.  $\square$

**Acknowledgement.** The authors wish to thank the referee for his careful reading of the manuscript.

## References

- [Al] A.Alberico, Boundedness of solutions to anisotropic variational problems, *Comm. Part. Diff. Equat.* **36** (2011), 470–486.
- [AR] A.Ambrosetti and P.H.Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381.
- [Ci1] A.Cianchi, A sharp embedding theorem for Orlicz-Sobolev spaces, *Indiana Univ. Math. J.* **45** (1996), 39–65.
- [Ci2] A.Cianchi, Boundedness of solutions to variational problems under general growth conditions, *Comm. Part. Diff. Eq.* **22** (1997), 1629–1646.
- [Ci3] A.Cianchi, A fully anisotropic Sobolev inequality, *Pacific J. Math.* **196** (2000), 283–295.
- [Ci4] A.Cianchi, Symmetrization in anisotropic elliptic equations, *Comm. Part. Diff. Equat.* **32** (2007), 693–717.
- [CGMS] Ph.Clément, M.Garcia-Huidobro, R.Manásevich & K.Schmitt, Mountain pass type solutions for quasilinear elliptic equations, *Calc. Var.* **11** (2000), 33–62.
- [DT] D.T.Donaldson & N.S.Trudinger, Orlicz-Sobolev spaces and embedding theorems, *J. Funct. Anal.* **8** (1971), 52–75.
- [FOR] D.G. de Figueiredo, J M. do Ó & B.Ruf, An Orlicz-space approach to superlinear elliptic systems, *J. Funct. Anal.* **224** (2005), 471–496.
- [FGK] I.Fragalà, F.Gazzola & B.Kawohl, Existence and nonexistence results for anisotropic elliptic equations, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **21** (2004), 715–734.

- [Kl] V.S.Klimov, Isoperimetric inequalities and embedding theorems, *Dokl. Akad. Nauk SSSR* **217** (1974), 272–275 (Russian); english translation: *Soviet Math. Dokl.* **15**.
- [KR] M.A.Krasnosel'skii & Ja.B.Rutickii, "*Convex functions and Orlicz spaces*", P. Noordhoff Ltd., Groningen 1961.
- [MW] J.Mawhin, M.Willem, "*Critical point theory and Hamiltonian systems*", Springer-Verlag, New York, 1989.
- [Po] S.I.Pohozaev, On the imbedding Sobolev theorem for  $pl = n$ , *Doklady Conference, Section Math. Moscow Power Inst.* (1965), 158–170
- [Ra] P.H.Rabinowitz, "*Minimax methods in critical point theory with applications to differential equations*", CBMS Reg. Conf. Ser. in Math. **65**, Amer. Math. Soc., Providence, 1986.
- [RR1] M.M.Rao & Z.D.Ren, "*Theory of Orlicz spaces*", Marcel Dekker, New York, 1991.
- [RR2] M.M.Rao & Z.D.Ren, "*Applications of Orlicz spaces*", Marcel Dekker, New York, 2002.
- [Ro] R.T.Rockafellar, "*Convex analysis*", Princeton University Press, Princeton, 1970.
- [Sch] G. Schappacher, A notion of Orlicz spaces for vector valued functions, *Appl. Math.* **50** (2005), 355–386.
- [Sk] M.S.Skaff, Vector valued Orlicz spaces generalized N-functions. I., *Pacific J. Math.* **28** (1969), 193–206.
- [Sk] M.S.Skaff, Vector valued Orlicz spaces generalized N-functions. II., *Pacific J. Math.* **28** (1969), 413–430.
- [St] M.Struwe, "*Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*", Second Edition, Springer-Verlag, Berlin, 1996.
- [Ta] G.Talenti, Boundedness of minimizers, *Hokkaido Math. J.* **19** (1990), 259–279.
- [Tr] N.S.Trudinger, An imbedding theorem for  $H^0(G, \Omega)$  spaces, *Studia Math.* **50** (1974), 17–30.
- [Wi] M.Willem, "*Minimax theorems*", Birkhäuser, Boston (MA), 1996.
- [Yu] V.I.Yudovich, Some estimates connected with integral operators and with solutions of elliptic equations, *Soviet Math. Doklady* **2** (1961), 746–749.