Variants of theorems of Baer and Hall on finite-by-hypercentral groups

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dedicated to the memory of Guido Zappa

Abstract We show that if a group $G$ has a finite normal subgroup $L$ such that $G/L$ is hypercentral, then the index of the hypercenter of $G$ is bounded by a function of the order of $L$. This completes recent results generalizing classical theorems by R. Baer and P. Hall. Then we apply our results to groups of automorphisms of a group $G$ acting in a restricted way on an ascending normal series of $G$. 

1 Introduction

A classical theorem by R. Baer states that, if the $m$-th term $Z_m(G)$ of the upper central series a group $G$ has finite index $t$ in $G$ for some positive integer $m$, then there is a finite normal subgroup $L$ of $G$ such that $G/L$ is nilpotent of class at most $m$, that is $G/L = Z_m(G/L)$ (see 14.5.1 in [7], which shall be the reference for undefined notation). Recently, in [6] it has been shown that there is such an $L$ with finite order $d$ bounded by a function of $t$ and $m$.

In the opposite direction, P. Hall showed that, if there is a normal subgroup $L$ with finite order $d$ such that $G/L$ is nilpotent of class at most $m$, then $G/Z_{2m}(G)$ has finite order bounded by a function of $d$ and $m$ (see [7], page 118).

Recently, in [2] it has been shown that the hypercenter of $G$ has finite index $t$ if and only if there is a finite normal subgroup $L$ with order $d$ such that $G/L$ is hypercentral, that is coincides with its hypercenter. Recall that the hypercenter of a group $G$ is the last term of the upper central series of $G$ (see details below). Then in [5] it has been shown that $d$ may be bounded by a function of $t$, namely $t^{(1+\log_2 t)/2}$. Here we complete the picture by showing that $t$ in turn may be bounded by a function of $d$.

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Theorem 1 If a group $G$ has a finite normal subgroup $L$ such that $G/L$ is hypercentral, then the hypercenter of $G$ has index bounded by $|\text{Aut}(L)| \cdot |Z(L)|$.

Corollary 1 If a group $G$ has a finite normal subgroup $L$ such that $G/L$ is nilpotent of class $m$, then $|G/Z_{2^m}(G)|$ is bounded by a function of $d := |L|$.

There are many generalizations and variants of Baer and Hall theorems. By applying Theorem 1 above, we improve the results in [3] which are concerned with possibly non-inner automorphisms.

Before stating our Theorem 2 we recall some definition. As usual, we say that the group $A$ acts on a group $G$ if and only if there is a homomorphism $\varphi: A \to \text{Aut}(G)$ (called action). We will regard both $G$ and $A$ as subgroups of the holomorph group $G \rtimes \text{Aut}(G)$ of $G$. In particular, we will denote by a bar $\bar{}$ the action of a group $G$ on itself by conjugation, that is the natural $\text{Aut}(G)$-homomorphism $G \to \bar{G} \leq \text{Aut}(G)$. If an action is such that its image $\bar{A}$ is normalized by $\bar{G} = \text{Inn}(G)$, we define by recursion an ascending $G$-series $Z_\alpha(G, A)$ (with $\alpha$ ordinal number) by $Z_0(G, A) := 1$, $Z_{\alpha+1}(G, A) := C_{G/Z_\alpha(G, A)}(A)$ and $Z_{\lambda}(G, A) := \cup_{\alpha<\lambda}Z_\alpha(G, A)$ when $\lambda$ is a limit ordinal. We call $Z_\alpha(G, A)$ the $\alpha$th $A$-center of $G$. Recall that an ascending $G$-series is a well ordered (by inclusion) set of normal subgroups of $G$. Clearly the series $Z_\alpha(G, A)$ is stabilized by $A$, in the sense that $A$ acts trivially on the factors between consecutive terms. The last term $Z_\infty(G, A)$ of this series is called $A$-hypercenter of $G$.

We say that $G$ is $A$-hypercentral with (ordinal) type at most $\alpha$ if and only if $G = Z_\alpha(G, A)$. Clearly $Z_\alpha(G) := Z_\alpha(G, \bar{G})$ is the usual $\alpha$th center of $G$ and if $G = Z_\alpha(G)$, then $G$ is hypercentral of type at most $\alpha$.

Now we are in a position to state our second result, which consists in two parts that refer to theorems of Baer and Hall, respectively. In fact, if $A = \text{Inn}(G)$, then part (B) reduces to Theorem B in [5] and part (H) to Theorem 1 above.

**Theorem 2** Let $G$ be a group and $A$ be a subgroup of $\text{Aut}(G)$ such that $A^{\text{Inn}(G)} = A$ and the hypercenter of $A/(A \cap \text{Inn}(G))$ has finite index $k$.

(B) If the $A$-hypercenter of $G$ has finite index $t$, then there is a finite normal $A$-subgroup $L$ with order bounded by a function of $(t, k)$ such that $G/L$ is $A$-hypercentral.

(H) If there is a finite normal $A$-subgroup $L$ with order $d$ such that $G/L$ is $A$-hypercentral, then the $A$-hypercenter of $G$ has finite index bounded by a function of $(d, k)$.
Remark that this theorem generalizes Theorems 4 and 3 of [3] where the same picture is considered, but with more restrictive conditions, that is $A$ contains $\text{Inn}(G)$, the factor $A/\text{Inn}(G)$ is finite and the involved series which are stabilized by $A$ are finite. Clearly, our bounding functions do not depend on the length of the considered series.

Finally note that the hypothesis that $A$ is normalized by $\text{Inn}(G)$ is necessary, as shown by Example in Sect. 2 below.

2 Proof of Theorem

To prove Theorem we use a key lemma. Recall that we denote the hypercenter of a group $G$ by $Z_{\infty}(G)$.

Lemma 1 Let $A \leq H$ be normal subgroups of a group $G$ with $A$ finite and $A \leq Z(H)$. If $G/C_G(H)$ is locally nilpotent and $H/A \leq Z_{\infty}(G/A)$, then $H \leq Z_{\infty}(G)A$.

Proof. Arguing by induction on the order of $A$, we may assume that $A$ is minimal normal in $G$. Then $A$ is an elementary abelian $p$-group for some prime $p$. If $A \cap Z(G) \neq 1$, then $A \leq Z(G)$ by minimality of $A$ and so we have $H \leq Z_{\infty}(G)A$.

Suppose then $A \cap Z(G) = 1$ (and so $A \cap Z_{\infty}(G) = 1$) and let $N := Z_{\infty}(G) \cap H$. Note that the hypotheses hold for the subgroups $\bar{A} := AN/N$, $\bar{H} := H/N$ of the group $\bar{G} := G/N$. Since from $\bar{H} \leq Z_{\infty}(\bar{G})\bar{A}$ it follows $H \leq Z_{\infty}(G)A$, we may assume $Z_{\infty}(G) \cap H = 1$.

We claim that $H = A$ (note that $H \leq Z_{\infty}(G)A$ if and only if $H = H \cap Z_{\infty}(G)A = (H \cap Z_{\infty}(G))A = A$). Suppose, by contradiction, $H > A$ and let $X/A \neq 1$ be either infinite cyclic or of prime order $r$ and contained in $(H/A) \cap Z(G/A)$. Since by hypotheses $A \leq Z(H)$, then $X$ is abelian and $X \triangleleft G$, clearly.

Let us show now that $X$ is a $p$-group. If, by contradiction, $X/A$ is infinite or $r \neq p$, then $X^p \neq 1$ and $X^p \cap A = 1$. Thus $X^p$ is $G$-isomorphic to $X^p A/A \leq Z_{\infty}(G/A)$. Hence $X^p \leq H \cap Z_{\infty}(G) = 1$, a contradiction. So $X/A$ has order $p$.

Assume, again by contradiction, $X^p \neq 1$. By minimality of $A$, we have $X^p = A = [G, X]$ and so $[G, A] = [G, X^p] = [G, X]^p = A^p = 1$, a contradiction.

Then $X$ is a finite elementary abelian $p$-group. Since $[G, A] = A \leq X$, the subgroup $X \rtimes (G/C_G(X))$ of the holomorph of $X$ is not nilpotent, and so
$G/C_G(X)$ is not a $p$-group. Hence there are a prime $q \neq p$ and a normal non-trivial $q$-subgroup $Q/C_G(X)$ of $G/C_G(X)$. Since $Q \nsubseteq C_G(X)$, then $[X, Q] \neq 1$. Thus $[X, Q] = A$, as $[X, Q] \leq A$ and by minimality of $A$.

By a standard argument on coprime actions (see for example Exercise 4.1 in [1]), we have

$$X = [X, Q] \times C_X(Q) = A \times C_X(Q),$$

therefore $C_X(Q) \neq 1$. On the other hand, $C_X(Q)$ is a normal subgroup of $G$ and so $C_X(Q) \leq Z_\infty(G) \cap H = 1$, a contradiction which gives the claim $H = A$. □

Proof of Theorem 1. Let us apply Lemma 1 with $A := Z(L)$ and $H := C_G(L)$. In fact on one hand $H/A = H/(H \cap L) \cong G \ L/H/L$, then $H/A \leq Z_\infty(G/A)$. On the other hand $L \leq C_G(H)$ and so $G/C_G(H)$ is hypercentral, since it is an image of $G/L$. Therefore $H \leq Z_\infty(G)A$. Hence

$$|H/(H \cap Z_\infty(G))| = |A(Z_\infty(G) \cap H)/(Z_\infty(G) \cap H)| \leq |A| = |Z(L)|.$$

Since $H = C_G(L)$, then $|G/H| \leq |\text{Aut}(L)|$. Thus

$$|G/Z_\infty(G)| \leq |G/H| \cdot |H/(H \cap Z_\infty(G))| \leq |\text{Aut}(L)| \cdot |Z(L)|.$$

Proof of Corollary 1. Note that $Z_{d+m}(G) = Z_\infty(G)$ has finite index. Thus if $d \leq m$, the statement follows directly from Theorem 1. Otherwise, $|G/Z_{2m}(G)|$ is bounded by the maximum of the $h(d, i)$ with $i = 1, \ldots, d$, where $h(d, m)$ is the bounding function in Hall Theorem. □

From Theorem 1 and the above quoted result from [5] we deduce a corollary which gives a rather complete picture of finite-by-hypercentral groups.

Corollary 2 If $G$ is a group with a (finite) normal series

$$G = G_0 \geq F_1 \geq G_1 \geq \ldots \geq F_n \geq G_n = 1$$

where

- each factor $F_i/G_i$ is finite with order $t_i > 1$,
- each factor $G_{i-1}/F_i$ is contained in the hypercenter of $G/F_i$,

then there is a normal subgroup $L$ with finite order bounded by a function of $t = t_1 \cdot \ldots \cdot t_n$ such that $G/L$ is hypercentral.

Moreover the hypercenter of $G$ has finite index bounded by a function of $t$.
**Proof.** Define recursively a function $f : \mathbb{N} \to \mathbb{N}$ by means of $f(1) = 1$ and $f(t + 1) = (t + 1)g(g(f(t)))$ for each $t \in \mathbb{N}$, where $g(t) := t^{1 + \log_2 t}$.

We show that there is $L \triangleleft G$ such that $|L| \leq f(t)$ and $G/L = Z_{\alpha}(G/L)$ for $\alpha := \alpha_n + \ldots + \alpha_1 + m'$, where the $\alpha_i$'s are ordinal numbers such that $G_{i-1}/F_i \leq Z_{\alpha_i}(G/F_i)$ for each $i$ and $m' \in \mathbb{N}$ may be bounded by a function of $t$ and of the $\alpha_i$'s which are finite. Since $f(t) \geq t$ for each $t$, the statement is trivial if $n = 1$.

Assume then by induction on $n$ that there is a normal series

$$G \geq F_{n-1} \geq G_{n-1} \geq F_n \geq G_n = 1$$

such that $G/F_{n-1}$ is hypercentral of type $\alpha' = \alpha_{n-1} + \ldots + \alpha_1 + m''$, with $m'' \in \mathbb{N}$ and $|F_{n-1}/G_{n-1}| \leq f(t_s)$ with $t_s = t_1 \cdot \ldots \cdot t_{n-1}$. Applying Theorem \[1\] to $G/G_{n-1}$, if $Z/G_{n-1} := Z_{(\log_2 g(f(t_s)) + \alpha')} (G/G_{n-1})$, then $|G/Z| \leq g(f(t_s))$. Thus, applying Theorem B of [KOS] to $G/F_n$, we have that there is a normal subgroup $L$ such that $G/L$ is hypercentral with ordinal type at most $\alpha + \left\lceil \log_2 g(f(t_s)) \right\rceil + \alpha' + \left\lceil \log_2 g(f(t_s)) \right\rceil$ and $|L/F_n| \leq g(g(f(t_s)))$. We have: $|L| \leq t_1 g(g(f(t_s))) \leq tg(g(f(t - 1))) = f(t)$, as wished. \[\square\]

**Remark:** In the above proof, if $\alpha$ is infinite, then clearly $G/Z_\alpha(G)$ is finite. Otherwise, if $G_{i-1}/F_i \leq Z_{m_i}(G/F_i)$ for each $i$ with $m_i \in \mathbb{N}$, then there is a finite normal subgroup $L$ such that $G/L = Z_m(G/L)$ with $m := m_1 + m_2 + \ldots + m_n$, by Theorem B in [1]. Hence, in this case, $G/Z_2m(G)$ is finite.

## 3 Proof of Theorem 2

**Proof of Theorem 2.** Let $\alpha'$ such that $B/(A \cap \text{Inn}(G)) := Z_{\alpha'}(A/(A \cap \text{Inn}(G))$ has finite index in $A/(A \cap \text{Inn}(G))$. Consider the subgroup $S := G \times A$ of the holomorph group of $G$.

Assume first $A \geq \text{Inn}(G)$. Let $G_\delta := Z_\delta(G, A)$ for any ordinal $\delta$. We claim:

$$(*), \forall \delta ~ S_\delta := G_\delta \bar{G}_\delta \leq Z_\delta(S).$$

By induction, suppose true for $\delta$. Note that $\bar{G} \leq A$ acts by conjugation on $G$ the same way as $G$. We have $[S_{\delta+1}, S] = [G_{\delta+1} \bar{G}_{\delta+1}, GA]$. On one hand, we have $[G_{\delta+1}, GA] \leq [G_{\delta+1}, A] \cdot [G_{\delta+1}, G]^A \leq G_\delta$. On the other hand, $[G_{\delta+1}, GA] \leq [G_{\delta+1}, A] \cdot [G_{\delta+1}, G]^A \leq G_\delta G_\delta = S_\delta$. It follows $S_{\delta+1} \leq Z_{\delta+1}(S)$ and the claim is proved since the limit ordinal step is trivial.
To prove (B) in the case $A \geq \text{Inn}(G)$, let $\alpha$ be such that $Z_{\alpha}(G, A)$ has finite index in $G$ and note that in the normal series

$$S = GA \geq GB \geq G\bar{G} \geq G_\alpha \bar{G}_\alpha \geq 1$$

the factors $GA/GB$ and $G\bar{G}/G_\alpha \bar{G}_\alpha$ are finite with order $k$ and $t^2$, respectively. Moreover, by $(\ast)$, factors $GB/GG$ and $G_\alpha \bar{G}_\alpha$ are contained in the $\alpha$'th and $\alpha$th center of $S/G\bar{G}$ and $S$, respectively. Thus we apply Corollary 2 to the group $S = GA$. Then the statement (for the group $G$) follows easily.

Concerning part (H) in the case $A \geq \text{Inn}(G)$, consider the normal series

$$S = GA \geq GB \geq G\bar{G} \geq LL \geq 1.$$  

Note that $GA/GB$ and $LL$ are finite with order $k$ and $d^2$, respectively. Moreover, if $\alpha_1$ is such that $Z_{\alpha_1}(G/L, A)$ has finite index in $G/L$, then by $(\ast)$ we have that $GB/LL$ is contained in the $(\alpha_1 + \alpha')$th $A$-center of $S/LL$. We may apply Corollary 2 and get the statement.

To deal with the more general case, let $\bar{N} := A \cap \text{Inn}(G)$ such that $Z(G) \leq N \leq G$. Note that $[G, A] \leq N$, as $[\overline{g}_1, \overline{g}_2] = [\overline{g}, \overline{g}] \in A \cap \text{Inn}(G)$ for all $\overline{g} \in A$ since $A \cap \text{Inn}(G) = A$. Thus $A$ acts trivially on $G/N$. Moreover the group $\bar{A} := A/C_A(N)$ may be considered as a group of automorphisms on $N$ containing $\text{Inn}(N)$. Thus, to prove (H), one may apply the above case to $N$ and $\bar{A} := A/C_A(N)$.

To prove (B) in the general case note that, by the above, the subgroup $Z := Z_{\infty}(N, A)$ has finite index in $N$, bounded by a function of $|L \cap N| \leq |L|$. Let $K/Z$ be the $A$-hypercenter of $G/Z$. Clearly, $K \cap N = Z$. Moreover $K/Z = Z(G/Z, A)$. Consider then $C/Z := C_{G/Z}([G, A]Z/Z)$ and note that $C$ has finite index in $G$, since $[G, A] \leq N$. By applying the Three Subgroup Lemma to $A, C/Z, C/Z$, we have that $A$ acts trivially on the derived subgroup of $C/Z$. Thus $C'/Z \leq C_{G/Z}(A) \leq K/Z$. Therefore $CK/K$ is abelian. We consider the series

$$G \geq CK \geq K \geq Z \geq 1.$$  

The index of $CK$ in $G$ is finite and bounded by a function of $d = |L|$, as $|N/Z|$ is. Then consider the action of $A$ on the abelian group $\hat{G} := CK/K$. Since $K \cap N = Z$, we have that $|NK/K|$ is bounded by a function of $d$. Thus the image of $A \cap \text{Inn}(G)$ in $\hat{A} := A/C_A(\hat{G})$ is finite with order bounded by a function of $d$. By Corollary 2, $Z_{\alpha'}(\hat{A})$ has finite index $q$ in $\hat{A}$, bounded by a function of $d$ and $k$. Recall that $\hat{G}$ is abelian and $[\hat{G}, \hat{A}]$ is finite, as $[G, A]$ is finite modulo $K$. Let $\hat{S} := \hat{G} \rtimes \hat{A}$. Then $Z_{1+\alpha'}(\hat{S}/[\hat{G}, \hat{A}])$ has finite index at
most \( q \). By Theorem 1, the index of \( Z_{1+\alpha'(\hat{S})} \) in \( \hat{S} \) is finite and bounded by a function of \( d \) and \( q \). Thus the \( A \)-hypercenter of \( \hat{G} := CK/K \) has finite index and bounded by a function of \( d \) and \( k \), as wished. \( \square \)

**Remark:** in the case \( A \geq \text{Inn}(G) \) of the above proof, if \( \alpha, \alpha_1 \) and \( \alpha' \) are finite, we have that:

- in case (B), the \( 2(\alpha + \alpha') \)th \( A \)-center has finite index in \( G \), by the above quoted result in [4]. In particular, for \( \alpha' = 0 \) we have Theorem 3 of [3].
- in case (H), there is a boundedly finite normal \( A \)-subgroup \( L \) such that \( G/L \) coincides with its \((\alpha_1 + \alpha')\)th \( A \)-center. This follows by applying the remarks after Corollary 2 to the group \( S \). In particular, for \( \alpha' = 0 \) we have Theorem 2 and 4 of [3].

Let us see that the condition that \( A \) is normalized by \( \text{Inn}(G) \) is necessary.

**Example** There is an elementary abelian group \( G \) and a bounded abelian group \( A \leq \text{Aut}(G) \) such that \( G/Z_\omega(G, A) \) is finite (of prime order), while \( G/L \) is not \( A \)-hypercentral, for any finite \( A \)-subgroup \( L \leq G \).

**Proof.** Let \( G := Dr_{i<\omega}\langle a_i \rangle \) be an elementary abelian \( p \)-group, where \( p \) is an odd prime and let \( Z := Dr_{0<i<\omega}\langle a_i \rangle \). For any \( i > 0 \), consider \( \gamma_i \in \text{Aut}(G) \) centralizing \( Z \), and such that \( a_0^\gamma := a_0 a_i \). Let \( \tau \in \text{Aut}(G) \) centralizing \( Z \) and such that \( a_0^\tau := a_0^2 \). Let \( A \) be the subgroup of \( \text{Aut}(G) \) generated by \( \tau \) and all the \( \gamma_i \)'s. Then \( Z = Z_1(G, A) \) has index \( p \) in \( G \), while if \( K \) is a proper \( A \)-subgroup of \( G \), then \( a_0 \notin K \), as \( a_0^A = G \). Clearly \( \tau \) does not centralizes \( a_0 \) mod \( K \). Thus \( G/K \) is not \( A \)-hypercentral, for any proper \( A \)-subgroup \( K \) of \( G \) and in particular for any finite \( A \)-subgroup \( L \leq G \). \( \square \)

We finish by noticing that Theorem 2 may be formulated in a different way. Recall that the factor of two consecutive terms of a series is called just factor.

**Corollary 3** Let \( A \) be a finite-by-hypercentral group of automorphisms of a group \( G \) such that \( A^{\text{Inn}(G)} = A \).

If there is an ascending normal series in \( G \) with a finite number of finite factors and such that \( A \) acts trivially on all other factors, then:

i) there is a finite index normal \( A \)-subgroup \( G_0 \) of \( G \) such that \( A \) stabilizes an ascending \( G \)-series of \( G_0 \);

ii) there is a finite normal \( A \)-subgroup \( L \) such that \( A \) stabilizes an ascending \( G \)-series of \( G/L \).
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