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# INTEGRAL ESTIMATES FOR TRANSPORT DENSITIES

L. DE PASCALE, L. C. EVANS, AND A. PRATELLI

ABSTRACT. We introduce some integration-by-parts methods that improve upon the  $L^p$  estimates on transport densities from the recent paper by L. De Pascale and A. Pratelli, *Calculus of Variations and Partial Differential Equations* 14 (2002), 249-274.

## 1. INTRODUCTION

This paper provides some PDE methods that improve upon the  $L^p$  estimates on the “transport densities” in certain Monge–Kantorovich mass transfer problems, as derived in the earlier paper [4] by the first and third authors and in some case also in [7]. Our main estimate provides the bound

$$\|\sigma_k\|_{L^q} \leq C (\|f\|_{L^q} + 1) \quad (1)$$

for each  $2 \leq q < \infty$ , when  $u$  solves the quasilinear elliptic equation

$$-\operatorname{div}(\sigma_k Du_k) = f \quad (2)$$

for

$$\sigma_k := e^{\frac{k}{2}(|Du_k|^2 - 1)} \quad (3)$$

and  $k$  sufficiently large. The constant  $C$  in (1) depends on  $q$ , but not on the parameter  $k$ .

This problem arises as an approximation of the fundamental *transport (or continuity) equation* for the Monge–Kantorovich mass transfer problem, as explained for instance in [6]. In this interpretation, we seek an optimal rearrangement of the measure  $\mu^+ := f^+ dx$  into  $\mu^- := f^- dy$ . In the limit  $k \rightarrow \infty$ , we have  $u_k \rightarrow u$ ,  $\sigma_k \rightarrow a$  and the potential  $u$  solves

$$\begin{cases} -\operatorname{div}(aDu) = f, \\ |Du| \leq 1, \\ |Du| = 1 \text{ where } a > 0. \end{cases} \quad (4)$$

We call  $a$  the *transport density*. It turns out that an optimal mass reallocation plan can be constructed using  $u$  and  $a$ .

The paper [4] by De Pascale and Pratelli studied how the integrability properties of  $f = f^+ - f^-$  affect those of the transport density. They showed that

- (i)  $a \in L^\infty$  if  $f \in L^\infty$ , and
- (ii)  $a \in L^{q-\epsilon}$  if  $f \in L^q$ , for  $1 \leq q < \infty$  and each  $\epsilon > 0$ .

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We introduce in this paper some PDE integration-by-parts methods to improve assertion (ii), by demonstrating

$$a \in L^q \quad \text{if } f \in L^q, \quad \text{for } 2 \leq q < \infty.$$

We have tried, and failed, to extend our methods to include  $q = \infty$ .

A PDE like (4) comes up also in the general formulation of Bouchitté and Buttazzo [1] for finding a distribution of a given amount of conductor to best dissipate heat. Then  $f$  represents a heat source and  $u$  the temperature of the system. The survey [6] describes several more applications.

## 2. APPROXIMATION

We will for simplicity take  $U = B^0(0, R)$ , the open ball with center 0 and radius  $R > 0$ . Hereafter we always suppose that  $f \in L^1(U)$ , with  $\int_U f \, dx = 0$ . Denote by  $u_k$  the solution of the nonlinear boundary-value problem

$$\begin{cases} -\operatorname{div}(\sigma_k Du_k) = f & \text{in } U \\ u_k = 0 & \text{on } \partial U, \end{cases} \quad (5)$$

where we write

$$\sigma_k := e^{\frac{k}{2}(|Du_k|^2 - 1)}. \quad (6)$$

Observe that  $u_k$  is the unique minimizer of the functional

$$F_k[v] := \int_U \frac{1}{k} e^{\frac{k}{2}(|Dv|^2 - 1)} - f v \, dx$$

in  $W_0^{1,k}$ . This approximation is suggested by the recent paper [5]. Regularity theory (Cf. Marcellini [9]) implies that  $u_k$  is smooth, provided  $f$  is.

We want to study the limits of  $u_k$  and  $\sigma_k$  as  $k \rightarrow \infty$ , and begin with some uniform bounds.

**Lemma 2.1.** *Suppose that  $f \in L^1(U)$ . Then the sequence  $\{u_k\}_{k=1}^\infty$  is bounded in  $W_0^{1,q}(U)$ , for each  $1 \leq q < \infty$ .*

*Proof.* Observe first that  $x \leq e^{\frac{x^2-1}{2}}$  for  $x \geq 0$ , and therefore that  $|Du_k| \leq \sigma_k^{\frac{1}{k}}$ . Recalling then (5), (6), we deduce for  $k > n$  that

$$\int_U |Du_k|^{k+2} \, dx \leq \int_U |Du_k|^2 \sigma_k \, dx = \int_U f u_k \, dx \leq C \|u_k\|_{L^\infty} \leq C \|Du_k\|_{L^k}.$$

Note that  $\|Du_k\|_{L^k}^k \leq \|Du_k\|_{L^{k+2}}^{k+2} + C$ . Hence  $\|Du_k\|_{L^k}^k \leq C + C \|Du_k\|_{L^k}$ , and so  $\|Du_k\|_{L^k} \leq C$ . We deduce for each  $k > q$  that

$$\|Du_k\|_{L^q} \leq \|Du_k\|_{L^k} \|1\|_{L^{\frac{kq}{k-q}}} \leq C.$$

□

We next identify the  $\Gamma$ -limit of problem (5), (6) as  $k \rightarrow \infty$ . For this, define

$$F[v] := \begin{cases} -\int_U f v \, dx & \text{if } v \in C_0^{0,1}(U), |Dv| \leq 1 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases} \quad (7)$$

**Theorem 2.2.** *As  $k$  goes to infinity, we have*

$$F_k \xrightarrow{\Gamma} F.$$

*with respect to the uniform convergence of functions.*

*Proof.* **1.** Since the mapping  $u \mapsto \langle f, u \rangle = \int_U f u \, dx$  is linear, it is enough to prove

$$E_k[v] := \frac{1}{k} \int_U e^{\frac{k}{2}(|Dv|^{2-1})} \, dx \xrightarrow{\Gamma} E[v], \quad (8)$$

for

$$E[v] := \begin{cases} 0 & \text{if } v \in C_0^{0,1}(U), \, |Dv| \leq 1 \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases} \quad (9)$$

**2.** If  $E[v] < \infty$ , we clearly have

$$E[v] = 0 = \lim_{k \rightarrow \infty} E_k[v].$$

Suppose now that  $v_k \rightarrow v$  uniformly, and  $\limsup_{k \rightarrow \infty} E_k[v_k] \leq C < \infty$ . Fix an integer  $m$  and let  $k > m$ . Since  $e^{\frac{x^2-1}{2}} \geq x$ , we have for each open set  $V \subseteq U$  that

$$\begin{aligned} \left( \int_V |Dv_k|^m \, dx \right)^{1/m} &\leq |V|^{1/m-1/k} \left( \int_V |Dv_k|^k \, dx \right)^{1/k} \\ &\leq |V|^{1/m-1/k} k^{1/k} E_k(v_k)^{1/k} \leq |V|^{1/m-1/k} k^{1/k} C^{1/k}. \end{aligned}$$

Passing to limits in  $k$  and recalling the lower semicontinuity of the  $L^m$  norm of the gradient, we discover

$$\left( \int_V |Dv|^m \, dx \right)^{1/m} \leq |V|^{1/m}.$$

This inequality, valid for all  $V$  as above, implies that  $Dv$  is in  $L^\infty$ , with  $|Dv| \leq 1$  almost everywhere. Consequently,

$$E[v] = 0 \leq \liminf_{k \rightarrow \infty} E_k[v_k].$$

□

Introduce next the vector fields

$$\mathbf{G}_k := \sigma_k Du_k \quad (k = 1, \dots).$$

**Theorem 2.3.** *Suppose that for some  $1 < q < \infty$  we have the uniform bounds*

$$\sup_k \|\mathbf{G}_k\|_{L^q(U; \mathbb{R}^n)} < \infty.$$

Define

$$f_k := -\operatorname{div}(\mathbf{G}_k),$$

and assume

$$\begin{cases} f_k \rightharpoonup f & \text{weakly in } L^q(U) \\ \mathbf{G}_k \rightharpoonup \mathbf{G} & \text{weakly in } L^q(U; \mathbb{R}^n), \\ u_k \rightarrow u & \text{uniformly.} \end{cases}$$

Then there exists a positive function  $a \in L^q$  such that

$$\begin{cases} \mathbf{G} = aDu, \\ |Du| = 1 \text{ a.e. on } \{a > 0\}, \text{ and} \\ \sigma_k \rightharpoonup a & \text{weakly in } L^q(U). \end{cases}$$

In particular,  $a = |\mathbf{G}|$ .

*Proof.* **1.** First of all, note that  $-\operatorname{div} \mathbf{G} = f$ ; that is,

$$\int_U \mathbf{G} \cdot D\psi \, dx = \int_U f\psi \, dx$$

for all  $\psi \in C^1$ ,  $\psi = 0$  on  $\partial U$ .

Let us now fix  $0 < \lambda < 1$  and calculate:

$$\begin{aligned} \int_U |\mathbf{G}| \, dx &\leq \liminf_{k \rightarrow \infty} \int_U |\mathbf{G}_k| \, dx = \liminf_{k \rightarrow \infty} \left( \int_{\{|Du_k|^2 > 1-\lambda\}} |\mathbf{G}_k| \, dx + \int_{\{|Du_k|^2 \leq 1-\lambda\}} |\mathbf{G}_k| \, dx \right) \\ &\leq \liminf_{k \rightarrow \infty} \left( \frac{1}{\sqrt{1-\lambda}} \int_U |\mathbf{G}_k| |Du_k| \, dx + \int_U e^{-\frac{k}{2}\lambda} \sqrt{1-\lambda} \, dx \right). \end{aligned}$$

When  $k$  goes to infinity, the last integral goes to 0. Notice also that

$$\int_U |\mathbf{G}_k| |Du_k| \, dx = \int_U \sigma_k |Du_k|^2 \, dx = \int_U f_k u_k \, dx.$$

Therefore

$$\sqrt{1-\lambda} \int_U |\mathbf{G}| \, dx \leq \liminf_{k \rightarrow \infty} \int_U f_k u_k \, dx = \int_U f u \, dx = \int_U \mathbf{G} \cdot Du \, dx$$

for each  $0 < \lambda < 1$ , and consequently

$$\int_U |\mathbf{G}| \, dx \leq \int_U \mathbf{G} \cdot Du \, dx. \quad (10)$$

**2.** Reasoning now as in the proof of Theorem 2.2, we fix an integer  $m$  and let  $k > m$ . Then for each open set  $V \subseteq U$

$$\begin{aligned} \left( \int_V |Du_k|^m \, dx \right)^{1/m} &\leq |V|^{1/m-1/k+1} \left( \int_V |Du_k|^{k+1} \, dx \right)^{1/k+1} \\ &\leq |V|^{1/m-1/k+1} \|\mathbf{G}_k\|_{L^1}^{1/k+1} \leq |V|^{1/m-1/k+1} C^{1/k+1}. \end{aligned}$$

Pass to limits in  $k$  to find

$$\left( \int_V |Du|^m \, dx \right)^{1/m} \leq |V|^{1/m},$$

and therefore  $|Du| \leq 1$  almost everywhere. The first two assertions of the Theorem now follow from (10).

**3.** To show also that  $\sigma_k \rightarrow a$ , let us fix  $\psi \in C_0^\infty$  and prove

$$\int_U \sigma_k \psi \, dx \rightarrow \int_U a \psi \, dx.$$

We write

$$\int_U \sigma_k \psi \, dx = \int_U \sigma_k |Du_k|^2 \psi \, dx + \int_U \sigma_k (1 - |Du_k|^2) \psi \, dx =: A_1 + A_2.$$

Notice now that

$$\begin{aligned} A_1 &= \int_U \psi \mathbf{G}_k \cdot Du_k \, dx = \int_U \mathbf{G}_k \cdot D(u_k \psi) \, dx - \int_U u_k \mathbf{G}_k \cdot D\psi \, dx \\ &= \int_U f_k u_k \psi \, dx - \int_U u_k \mathbf{G}_k \cdot D\psi \, dx. \end{aligned}$$

This expression converges as  $k \rightarrow \infty$  to

$$\begin{aligned} \int_U f u \psi \, dx - \int_U u \mathbf{G} \cdot D\psi \, dx &= \int_U \mathbf{G} \cdot D(u\psi) \, dx - \int_U u \mathbf{G} \cdot D\psi \, dx \\ &= \int_U \psi \mathbf{G} \cdot Du \, dx = \int_U \psi a |Du|^2 \, dx = \int_U a \psi \, dx. \end{aligned}$$

4. It remains to show that  $A_2 \rightarrow 0$ . If we write  $\varphi_k := |Du_k|^2 - 1$ , then

$$|A_2| \leq \|\psi\|_{L^\infty} \int_U e^{\frac{k}{2}\varphi_k} |\varphi_k| \, dx.$$

Since  $x e^{-\frac{x}{2}} \leq 1$  for each  $x > 0$ , we have

$$\int_{\{\varphi_k < 0\}} e^{\frac{k}{2}\varphi_k} |\varphi_k| \, dx \leq \frac{1}{k} \int_U k |\varphi_k| e^{-\frac{k|\varphi_k|}{2}} \, dx \leq \frac{|U|}{k} \rightarrow 0.$$

Finally, since  $q > 1$  there exists a constant  $c_q > 0$  such that

$$\frac{e^{x(q-1)}}{x} \geq c_q > 0$$

for all  $x > 0$ . Consequently,

$$\begin{aligned} \int_{\{\varphi_k > 0\}} e^{\frac{k}{2}\varphi_k} |\varphi_k| \, dx &= \frac{2}{k} \int_{\{\varphi_k > 0\}} e^{\frac{k}{2}\varphi_k} \frac{k}{2} \varphi_k \, dx \\ &\leq \frac{2}{c_q k} \int_{\{\varphi_k > 0\}} e^{\frac{qk}{2}\varphi_k} \, dx = \frac{2}{c_q k} \int_U \sigma_k^q \, dx \leq \frac{2C^q}{c_q k} \rightarrow 0. \end{aligned}$$

This completes the proof that  $A_2 \rightarrow 0$ .  $\square$

### 3. ESTIMATES I

The full calculations for our main estimate in §4 are fairly involved, and so for the reader's convenience we provide in this section a simpler computation illustrating the main ideas. Suppose  $2 \leq q < \infty$ .

**Theorem 3.1.** *There exists a constant  $C$ , depending on  $q$ , but independent of  $k$ , such that*

$$\int_U \sigma_k^q \, dx \leq C \left( \int_U |f|^q \, dx + 1 \right). \quad (11)$$

*Proof. 1.* To simplify notation, we hereafter in the proof do not write the subscripts  $k$ . Observe that since  $Du$  is bounded in each space  $L^q$  and  $u = 0$  on  $\partial U$ , we have the bound

$$|u| \leq C$$

for some constant  $C$ .

2. Multiply (5) by  $\sigma^{q-1}u$  and integrate by parts:

$$\begin{aligned} \int_U \sigma^q |Du|^2 + (q-1)\sigma^{q-1} Du \cdot D\sigma u \, dx &= \int_U \sigma u_i (\sigma^{q-1}u)_i \, dx = \int_U f \sigma^{q-1} u \, dx \\ &\leq C \left( \int_U |f|^q \, dx \right)^{\frac{1}{q}} \left( \int_U \sigma^q \, dx \right)^{1-\frac{1}{q}}. \end{aligned} \quad (12)$$

Here and afterwards we write the subscript  $i$  to denote the partial derivative with respect to the variable  $x_i$ .

Notice that  $|Du|^2 \geq 1$  if  $\sigma \geq 1$ . Therefore

$$\int_U \sigma^q dx \leq C \left( \int_U |f|^q dx + \int_U \sigma^{q-1} |Du \cdot D\sigma| dx + 1 \right). \quad (13)$$

**3.** Next, multiply (5) by  $-(\sigma^{q-1}u_j)_j$ :

$$\begin{aligned} \int_U (\sigma u_i)_i (\sigma^{q-1}u_j)_j dx &= - \int_U f (\sigma^{q-1}u_j)_j dx \\ &= \int_U f \sigma^{q-2} (-\sigma u_j)_j dx - \int_U f (q-2) \sigma^{q-2} \sigma_j u_j dx \\ &\leq C \int_U f^2 \sigma^{q-2} + |f| \sigma^{q-2} |Du \cdot D\sigma| dx. \end{aligned} \quad (14)$$

The term on the left is

$$\begin{aligned} A &:= - \int_U \sigma u_i (\sigma^{q-1}u_j)_{ij} dx + \int_{\partial U} \sigma u_i \nu^i (\sigma^{q-1}u_j)_j d\mathcal{H}^{n-1} \\ &= \int_U (\sigma u_i)_j (\sigma^{q-1}u_j)_i dx + \int_{\partial U} \sigma u_i \nu^i (\sigma^{q-1}u_j)_j - \sigma u_i \nu^j (\sigma^{q-1}u_j)_i d\mathcal{H}^{n-1}, \end{aligned} \quad (15)$$

where  $\nu = (\nu^1, \dots, \nu^n)$  is the unit outer normal to  $\partial U$ . The boundary integral is

$$\begin{aligned} B &:= \int_{\partial U} \sigma^q (u_i \nu^i u_{jj} - u_i \nu^j u_{ij}) d\mathcal{H}^{n-1} \\ &\quad + \int_{\partial U} (q-1) \sigma^{q-1} (u_i \nu^i u_j \sigma_j - u_i \nu^j \sigma_i u_j) d\mathcal{H}^{n-1}. \end{aligned} \quad (16)$$

The integrand of the last term equals 0, since  $\sigma = e^{\frac{k}{2}(|Du|^2-1)}$  and so  $\sigma_j = k u_i u_{ij} \sigma$ .

Consider a point  $x_0 \in \partial U$ ; without loss, we can take  $x_0 = (0, \dots, R)$ . Then  $\nu = (0, \dots, 1)$  and  $Du = (0, \dots, u_n)$ , since  $u = 0$  on  $\partial U$ . The integrand of the first term on the right hand side of (16) at  $x_0$  therefore equals

$$\sigma^q (\Delta u - u_{nn}) u_n. \quad (17)$$

Lastly, write  $x' = (x_1, \dots, x_{n-1})$  and observe that  $u(x', \sqrt{R^2 - |x'|^2}) \equiv 0$  for small  $x'$ . We differentiate this identity twice and set  $x' = 0$ , to compute  $\Delta u - u_{nn} = \frac{n-1}{R} u_n$  at  $x_0$ . Hence

$$B = \frac{n-1}{R} \int_{\partial U} \sigma^q |Du|^2 d\mathcal{H}^{n-1} \geq 0.$$

**4.** Therefore

$$\begin{aligned} A &= \int_U (\sigma u_i)_i (\sigma^{q-1}u_j)_j dx \geq \int_U (\sigma u_i)_j (\sigma^{q-1}u_j)_i dx \\ &= \int_U (\sigma u_{ij} + \sigma_j u_i) (\sigma^{q-1}u_{ij} + (q-1) \sigma^{q-2} \sigma_i u_j) dx \\ &= \int_U \sigma^q |D^2 u|^2 + (q-1) \sigma^{q-2} |Du \cdot D\sigma|^2 + q \sigma^{q-1} \sigma_j u_i u_{ij} dx. \end{aligned} \quad (18)$$

Recall that  $\sigma_j = ku_i u_{ij} \sigma$ . Hence (14) and (18) imply

$$\begin{aligned} \int_U \sigma^q |D^2 u|^2 + (q-1) \sigma^{q-2} |Du \cdot D\sigma|^2 + \frac{q}{k} \sigma^{q-2} |D\sigma|^2 dx \\ \leq C \int_U f^2 \sigma^{q-2} + |f| \sigma^{q-2} |Du \cdot D\sigma| dx \\ \leq \frac{q-1}{2} \int_U \sigma^{q-2} |Du \cdot D\sigma|^2 + C \int_U |f|^2 \sigma^{q-2} dx; \end{aligned} \quad (19)$$

and consequently

$$\int_U \sigma^{q-2} |Du \cdot D\sigma|^2 dx \leq C \int_U |f|^2 \sigma^{q-2} dx. \quad (20)$$

5. Combining now (13) and (20) one obtains

$$\begin{aligned} \int_U \sigma^q dx &\leq C \int_U |f|^q dx + C \int_U \sigma^{q-1} |Du \cdot D\sigma| dx + C \\ &\leq C \int_U |f|^q dx + \frac{1}{3} \int_U \sigma^q dx + C \int_U \sigma^{q-2} |Du \cdot D\sigma|^2 dx + C \\ &\leq C \int_U |f|^q dx + \frac{1}{3} \int_U \sigma^q dx + C \int_U |f|^2 \sigma^{q-2} dx + C \\ &\leq C \int_U |f|^q dx + \frac{2}{3} \int_U \sigma^q dx + C; \end{aligned} \quad (21)$$

and this gives (11).  $\square$

**Remark.** The boundary integral term  $B$  is in fact nonnegative for any convex, smooth domain replacing  $U = B(0, R)$ : see for instance the similar calculations in §1.5 of Ladyzhenskaja [8].  $\square$

#### 4. ESTIMATES II

In this section we derive our main integral estimate.

**Theorem 4.1.** *Assume that  $2 \leq q < \infty$  and that  $f \in C^\infty(\bar{U})$ . Then there exist a constant  $C$ , depending only on  $q$ , and a constant  $K$ , depending only on  $\|f\|_{L^\infty}$ , such that*

$$\int_U \sigma_k^q |Du_k|^q dx \leq C \left( \int_U |f|^q dx + 1 \right) \quad (22)$$

for all  $k \geq K$ .

The proof is similar to that of Theorem 3.1, except that we must handle the additional term  $|Du_k|^q$  on the left. This makes our multipliers and estimates more intricate.

*Proof.* 1. For notational simplicity we hereafter write  $\sigma$  and  $u$  in place of  $\sigma_k$  and  $u_k$ .

Since  $f$  is smooth, the same is true for  $u$  and  $\sigma$ . Observe also the bound

$$|u| \leq C.$$

We record for later reference these consequences of (6):

$$|Du|_i = \frac{\sigma_i}{k\sigma |Du|}, \quad u_i u_{ij} = \frac{\sigma_j}{k\sigma}. \quad (23)$$



2. We multiply the PDE (5) by  $\sigma^{q-1}|Du|^{q+1}u$  and integrate by parts, to find

$$\int_U \sigma Du \cdot D(\sigma^{q-1}|Du|^{q+1}u) dx = \int_U \sigma^{q-1}|Du|^{q+1}uf dx. \quad (24)$$

The right hand term in (24) is less than or equal to

$$C \int_U \sigma^{q-1}|Du|^{q+1}|f| dx \leq \frac{1}{2} \int_{\{|f| \leq \frac{\sigma|Du|}{2C}\}} \sigma^q |Du|^{q+2} dx + 2^{q-1} C^q \int_{\{|f| > \frac{\sigma|Du|}{2C}\}} |Du|^2 |f|^q dx.$$

But if  $\sigma|Du| < 2C|f|$ , then obviously  $\sigma|Du| \leq 2C\|f\|_{L^\infty}$ . Recalling (6), we see that this implies  $|Du| \leq 2$  provided  $k \geq K$ , for some constant  $K$  depending only upon  $\|f\|_{L^\infty}$ . Therefore

$$\int_U \sigma^{q-1}|Du|^{q+1}uf dx \leq \frac{1}{2} \int_U \sigma^q |Du|^{q+2} dx + C \int_U |f|^q dx. \quad (25)$$

3. We use (23) to evaluate the left hand term in (24):

$$\begin{aligned} \int_U \sigma Du \cdot D(\sigma^{q-1}|Du|^{q+1}u) dx &= \int_U \sigma^q |Du|^{q+3} dx \\ &+ (q-1) \int_U \sigma^{q-1}|Du|^{q+1}u D\sigma \cdot Du dx + (q+1) \int_U \sigma^q u |Du|^q Du \cdot (D|Du|) dx \\ &= \int_U \sigma^q |Du|^{q+3} dx \\ &+ (q-1) \int_U \sigma^{q-1}|Du|^{q+1}u D\sigma \cdot Du dx + \frac{q+1}{k} \int_U \sigma^{q-1}u |Du|^{q-1} Du \cdot D\sigma dx. \end{aligned} \quad (26)$$

But  $\sigma \geq 1$  only if  $|Du| \geq 1$ ; and hence

$$\int_U \sigma^q |Du|^{q+2} dx \leq \int_U \sigma^q |Du|^{q+3} dx + C, \quad (27)$$

since  $U$  is bounded.

Combining (27), (26), (24) and (25), we deduce the inequality

$$\begin{aligned} \int_U \sigma^q |Du|^{q+2} dx &\leq C + \frac{1}{2} \int_U \sigma^q |Du|^{q+2} dx + C \int_U |f|^q dx \\ &+ C \int_U \sigma^{q-1}|Du|^{q+1}|D\sigma \cdot Du| dx + \frac{C}{k} \int_U \sigma^{q-1}|Du|^{q-1}|D\sigma \cdot Du| dx. \end{aligned}$$

Arguing as before (this means dividing the integrals in the set where  $|D\sigma \cdot Du| \leq \epsilon \sigma |Du|/C$  and in the rest of  $U$ ), we see that therefore

$$\begin{aligned} \int_U \sigma^q |Du|^{q+2} dx &\leq C + C \int_U |f|^q dx + \epsilon \int_U \sigma^q |Du|^{q+2} dx + \frac{\epsilon}{k} \int_U \sigma^q |Du|^q dx \\ &+ \frac{C^2}{\epsilon} \int_U \sigma^{q-2}|Du|^q |D\sigma \cdot Du|^2 dx + \frac{C^2}{k\epsilon} \int_U \sigma^{q-2}|Du|^{q-2}|D\sigma \cdot Du|^2 dx \end{aligned} \quad (28)$$

for any  $\epsilon > 0$ . Since  $\int_U \sigma^q |Du|^q dx \leq \int_U \sigma^q |Du|^{q+2} dx + C$ , this implies our first main estimate:

$$\begin{aligned} \int_U \sigma^q |Du|^{q+2} dx &\leq C + C \int_U |f|^q dx + C \int_U \sigma^{q-2}|Du|^q |D\sigma \cdot Du|^2 dx \\ &+ \frac{C}{k} \int_U \sigma^{q-2}|Du|^{q-2}|D\sigma \cdot Du|^2 dx. \end{aligned} \quad (29)$$

4. The last two terms in (29) involving  $D\sigma \cdot Du$  are dangerous, since  $D\sigma$  is of order  $k$ : we need another estimate to control them.

Let us therefore continue by multiplying the PDE (5) by  $-\operatorname{div}(\sigma^{q-1}|Du|^q Du)$  and thereby deriving the identity

$$\int_U \operatorname{div}(\sigma Du) \operatorname{div}(\sigma^{q-1}|Du|^q Du) dx = - \int_U f \operatorname{div}(\sigma^{q-1}|Du|^q Du) dx. \quad (30)$$

The term on the right equals

$$\begin{aligned} \int_U f \sigma^{q-2}|Du|^q (-\operatorname{div}(\sigma Du)) dx - \int_U f \sigma Du \cdot D(\sigma^{q-2}|Du|^q) dx &= \int_U |f|^2 \sigma^{q-2}|Du|^q dx \\ - (q-2) \int_U f \sigma^{q-2}|Du|^q Du \cdot D\sigma dx - q \int_U f \sigma^{q-1}|Du|^{q-1} Du \cdot (D|Du|) dx. \end{aligned}$$

We again recall (23) and deduce

$$\begin{aligned} - \int_U f \operatorname{div}(\sigma^{q-1}|Du|^q Du) dx &\leq \int_U |f|^2 \sigma^{q-2}|Du|^q dx \\ + (q-2) \int_U |f| \sigma^{q-2}|Du|^q |Du \cdot D\sigma| dx + \frac{q}{k} \int_U |f| \sigma^{q-2}|Du|^{q-2} |Du \cdot D\sigma| dx. \end{aligned} \quad (31)$$

The left hand term of (30) is

$$\begin{aligned} A &:= - \int_U \sigma u_i (\sigma^{q-1}|Du|^q u_j)_{ij} dx + \int_{\partial U} \sigma u_i \nu^i (\sigma^{q-1}|Du|^q u_j)_j d\mathcal{H}^{n-1} \\ &= \int_U (\sigma u_i)_j (\sigma^{q-1}|Du|^q u_j)_i dx \\ &\quad + \int_{\partial U} \sigma u_i \nu^i (\sigma^{q-1}|Du|^q u_j)_j - \sigma u_i \nu^j (\sigma^{q-1}|Du|^q u_j)_i d\mathcal{H}^{n-1}. \end{aligned} \quad (32)$$

Call the boundary term  $B$ . Then, almost exactly as in step 3 of the previous proof, we can show that

$$B = \frac{n-1}{R} \int_{\partial U} \sigma^q |Du|^{q+2} d\mathcal{H}^{n-1} \geq 0.$$

Consequently,

$$\begin{aligned} A &= \int_U (\sigma u_i)_i (\sigma^{q-1}|Du|^q u_j)_j dx \geq \int_U (\sigma u_i)_j (\sigma^{q-1}|Du|^q u_j)_i dx \\ &= \int_U (\sigma u_{ij} + \sigma_j u_i) \left( \sigma^{q-1}|Du|^q u_{ij} + (q-1)\sigma^{q-2}|Du|^q \sigma_i u_j + \frac{q}{k}\sigma^{q-2}|Du|^{q-2}\sigma_i u_j \right) dx \\ &= \int_U \sigma^q |Du|^q |D^2 u|^2 + \frac{q}{k}\sigma^{q-2}|Du|^q |D\sigma|^2 + (q-1)\sigma^{q-2}|Du|^q |D\sigma \cdot Du|^2 + \\ &\quad + \frac{q}{k^2}\sigma^{q-2}|Du|^{q-2}|D\sigma|^2 + \frac{q}{k}\sigma^{q-2}|Du|^{q-2}|D\sigma \cdot Du|^2 dx. \end{aligned}$$

The first, the second and the fourth terms in the last expression are positive, and so we deduce

$$\begin{aligned} (q-1) \int_U \sigma^{q-2}|Du|^q |D\sigma \cdot Du|^2 dx + \frac{q}{k} \int_U \sigma^{q-2}|Du|^{q-2}|D\sigma \cdot Du|^2 dx \\ \leq \int_U \operatorname{div}(\sigma Du) \operatorname{div}(\sigma^{q-1}|Du|^q Du) dx. \end{aligned} \quad (33)$$

Collecting (33), (30) and (31), we find

$$\begin{aligned} & \int_U \sigma^{q-2} |Du|^q |D\sigma \cdot Du|^2 dx + \frac{1}{k} \int_U \sigma^{q-2} |Du|^{q-2} |D\sigma \cdot Du|^2 dx \\ & \leq C \int_U |f|^2 \sigma^{q-2} |Du|^q dx + C \int_U |f| \sigma^{q-2} |Du|^q |Du \cdot D\sigma| dx \\ & \quad + \frac{C}{k} \int_U |f| \sigma^{q-2} |Du|^{q-2} |Du \cdot D\sigma| dx. \end{aligned} \quad (34)$$

Take  $\epsilon > 0$  to be a small constant, which will be fixed later on. Then

$$\begin{aligned} \int_U f^2 \sigma^{q-2} |Du|^q dx & \leq \epsilon \int_U \sigma^q |Du|^{q+2} dx + C \int_{\{|f| > \sigma |Du| \sqrt{\epsilon}\}} |f|^q |Du|^2 dx \\ & \leq \epsilon \int_U \sigma^q |Du|^{q+2} dx + C \int_U |f|^q dx, \end{aligned} \quad (35)$$

since  $|Du| \leq 2$  wherever  $|f| > \sigma |Du| \sqrt{\epsilon}$ , provided  $k \geq K$  and  $K$  is large.

Recalling (35), we can likewise estimate for each  $\delta > 0$  that

$$\begin{aligned} \int_U |f| \sigma^{q-2} |Du|^q |D\sigma \cdot Du| dx & \leq \delta \int_U \sigma^{q-2} |Du|^q |D\sigma \cdot Du|^2 dx + C \int_U f^2 \sigma^{q-2} |Du|^q dx \\ & \leq \delta \int_U \sigma^{q-2} |Du|^q |D\sigma \cdot Du|^2 dx + C \left( \epsilon \int_U \sigma^q |Du|^{q+2} dx + \int_U |f|^q dx \right). \end{aligned} \quad (36)$$

Similarly,

$$\begin{aligned} \int_U |f| \sigma^{q-2} |Du|^{q-2} |Du \cdot D\sigma| dx & \leq \delta \int_U \sigma^{q-2} |Du|^{q-2} |D\sigma \cdot Du|^2 dx + C \int_U f^2 \sigma^{q-2} |Du|^{q-2} dx \\ & \leq \delta \int_U \sigma^{q-2} |Du|^{q-2} |D\sigma \cdot Du|^2 dx + C \left( \epsilon \int_U \sigma^q |Du|^q dx + C \int_U |f|^q dx \right). \end{aligned} \quad (37)$$

Since  $\sigma \geq 1$  only if  $|Du| \geq 1$ , we have

$$\int_U \sigma^q |Du|^q dx \leq \int_U \sigma^q |Du|^{q+2} dx + C,$$

and therefore

$$\begin{aligned} \int_U |f| \sigma^{q-2} |Du|^{q-2} |Du \cdot D\sigma| dx & \leq \delta \int_U \sigma^{q-2} |Du|^{q-2} |D\sigma \cdot Du|^2 dx \\ & \quad + C \left( \epsilon \int_U \sigma^q |Du|^{q+2} dx + \int_U |f|^q dx + 1 \right). \end{aligned} \quad (38)$$

Taking  $\delta > 0$  small, we then derive from (34), (35), (36) and (38) our second main inequality

$$\begin{aligned} & \int_U \sigma^{q-2} |Du|^q |D\sigma \cdot Du|^2 dx + \frac{1}{k} \int_U \sigma^{q-2} |Du|^{q-2} |D\sigma \cdot Du|^2 dx \\ & \leq \epsilon \int_U \sigma^q |Du|^{q+2} dx + C \int_U |f|^q + C. \end{aligned} \quad (39)$$

5. Putting together inequalities (29) and (39) and fixing  $\epsilon > 0$  small, we finally discover

$$\int_U \sigma^q |Du|^{q+2} dx \leq C + C \int_U |f|^q dx.$$

As  $\sigma \geq 1$  only if  $|Du| \geq 1$ , estimate (22) follows.  $\square$

Theorem 4.1 concerns only smooth functions  $f$ . However, since the bound for the  $L^q$  norm of the transport density depends only upon the  $L^q$  norm of  $f$ , we can approximate:

**Theorem 4.2.** *For each  $2 \leq q < \infty$  and  $f \in L^q(U)$  the associated transport density  $a$  belongs to  $L^q(U)$ . Furthermore, there is a constant  $C$ , depending only upon  $n$  and  $U$ , such that*

$$\|a\|_{L^q} \leq C (\|f\|_{L^q} + 1). \quad (40)$$

*Proof.* **1.** Let us first define

$$f_j := f * \eta_{1/j},$$

the convolution of  $f$  with a standard mollifier. For each integer  $j$ , we then solve

$$\begin{cases} -\operatorname{div}(\sigma_{k,j} Du_{k,j}) = f_j & \text{in } U \\ u_{k,j} = 0 & \text{on } \partial U, \end{cases} \quad (41)$$

for

$$\sigma_{k,j} := e^{\frac{k}{2}(|Du_{k,j}|^2 - 1)}. \quad (42)$$

**2.** According to (22), we have the estimate

$$\int_U \sigma_{k,j}^q |Du_{k,j}|^q dx \leq C \left( \int_U |f_j|^q dx + 1 \right) \leq C \left( \int_U |f|^q dx + 1 \right) \quad (43)$$

for all  $k$  greater than or equal to some constant  $K = k(j)$ , depending only on the  $L^\infty$  norm of  $f_j$ . Now define

$$\sigma_j := \sigma_{k(j),j}, \quad u_j := u_{k(j),j}, \quad \mathbf{G}_j := \sigma_j Du_j.$$

Clearly  $f_j \rightarrow f$  in  $L^q$ . Furthermore, (43) implies that  $\mathbf{G}_j$  is bounded in  $L^q$ . We may therefore assume upon reindexing that

$$\mathbf{G}_j \rightharpoonup \mathbf{G} \quad \text{weakly in } L^q(U; \mathbb{R}^n).$$

Finally we may pass as necessary to a further subsequence to ensure  $u_j$  converges uniformly to a limit  $u$ . Apply Theorem 2.3.  $\square$

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