Integral estimates for transport densities

INTEGRAL ESTIMATES FOR TRANSPORT DENSITIES

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ABSTRACT. We introduce some integration-by-parts methods that improve upon the L^p estimates on transport densitites from the recent paper by L. De Pascale and A. Pratelli, Calculus of Variations and Partial Differential Equations 14 (2002), 249-274.

1. INTRODUCTION

This paper provides some PDE methods that improve upon the L^p estimates on the "transport densities" in certain Monge–Kantorovich mass transfer problems, as derived in the earlier paper [4] by the first and third authors and in some case also in [7]. Our main estimate provides the bound

$$\|\sigma_k\|_{L^q} \leq C\left(\|f\|_{L^q} + 1\right) \tag{1}$$

for each $2 \leq q < \infty$, when u solves the quasilinear elliptic equation

$$-\operatorname{div}(\sigma_k Du_k) = f \tag{2}$$

for

$$\sigma_k := e^{\frac{k}{2}\left(|Du_k|^2 - 1\right)} \tag{3}$$

and k sufficiently large. The constant C in (1) depends on q, but not on the parameter k.

This problem arises as an approximation of the fundamental *transport (*or *continuity) equation* for the Monge–Kantorovich mass transfer problem, as explained for instance in [6]. In this interpretation, we seek an optimal rearrangement of the measure $\mu^+ := f^+dx$ into $\mu^- := f^-dy$. In the limit $k \to \infty$, we have $u_k \to u, \sigma_k \to a$ and the potential u solves

$$\begin{cases} -\operatorname{div}(aDu) = f, \\ |Du| \leq 1, \\ |Du| = 1 \text{ where } a > 0. \end{cases} \tag{4}$$

We call a the *transport density.* It turns out that an optimal mass reallocation plan can be constructed using u and a.

The paper [4] by De Pascale and Pratelli studied how the integrability properties of $f = f^+ - f^-$ affect those of the transport density. They showed that

(i) $a \in L^\infty$ if $f \in L^\infty$, and
(ii) $a \in L^{q-\epsilon}$ if $f \in L^q$, for $1 \leq q < \infty$ and each $\epsilon > 0$.

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We introduce in this paper some PDE integration–by–parts methods to improve assertion (ii), by demonstrating

$$a \in L^q \quad \text{if } f \in L^q, \quad \text{for} \quad 2 \le q < \infty.$$

We have tried, and failed, to extend our methods to include $q = \infty$.

A PDE like (4) comes up also in the general formulation of Bouchitté and Buttazzo [1] for finding a distribution of a given amount of conductor to best dissipate heat. Then f represents a heat source and u the temperature of the system. The survey [6] describes several more applications.

2. Approximation

We will for simplicity take $U = B^0(0, R)$, the open ball with center 0 and radius $R > 0$. Hereafter we always suppose that $f \in L^1(U)$, with $\int_U f\, dx = 0$. Denote by u_k the solution of the nonlinear boundary–value problem

$$\begin{cases} -\operatorname{div}(\sigma_k Du_k) = f & \text{in } U \\ \qquad\qquad u_k = 0 & \text{on } \partial U, \end{cases} \tag{5}$$

where we write

$$\sigma_k := e^{\frac{k}{2}\left(|Du_k|^2 - 1\right)}. \tag{6}$$

Observe that u_k is the unique minimizer of the functional

$$F_k[v] := \int_U \frac{1}{k} e^{\frac{k}{2}(|Dv|^2-1)} - fv\, dx$$

in $W_0^{1,k}$. This approximation is suggested by the recent paper [5]. Regularity theory (Cf. Marcellini [9]) implies that u_k is smooth, provided f is.

We want to study the limits of u_k and σ_k as $k \to \infty$, and begin with some uniform bounds.

Lemma 2.1. *Suppose that $f \in L^1(U)$. Then the sequence $\{u_k\}_{k=1}^{\infty}$ is bounded in $W_0^{1,q}(U)$, for each $1 \le q < \infty$.*

Proof. Observe first that $x \le e^{\frac{x^2-1}{2}}$ for $x \ge 0$, and therefore that $|Du_k| \le \sigma_k^{\frac{1}{k}}$. Recalling then (5), (6), we deduce for $k > n$ that

$$\int_U |Du_k|^{k+2}\ dx \le \int_U |Du_k|^2 \sigma_k\ dx = \int_U f u_k\, dx \le C\|u_k\|_{L^\infty} \le C\|Du_k\|_{L^k}.$$

Note that $\|Du_k\|_{L^k}^k \le \|Du_k\|_{L^{k+2}}^{k+2} + C$. Hence $\|Du_k\|_{L^k}^k \le C + C\|Du_k\|_{L^k}$, and so $\|Du_k\|_{L^k} \le C$. We deduce for each $k > q$ that

$$\|Du_k\|_{L^q} \le \|Du_k\|_{L^k}\ \|1\|_{L^{\frac{kq}{k-q}}} \le C.$$

$\square$

We next indentify the Γ–limit of problem (5), (6) as $k \to \infty$. For this, define

$$F[v] := \begin{cases} -\int_U fv\, dx & \text{if } v \in C_0^{0,1}(U),\ |Dv| \le 1 \ a.e. \\ +\infty & \text{otherwise.} \end{cases} \tag{7}$$

Theorem 2.2. *As k goes to infinity, we have*

$$F_k \xrightarrow{\Gamma} F.$$

with respect to the uniform convergence of functions.

Proof. **1.** Since the mapping $u \mapsto \langle f, u\rangle = \int_U fu\,dx$ is linear, it is enough to prove

$$E_k[v] := \frac{1}{k}\int_U e^{\frac{k}{2}(|Dv|^2-1)}dx \xrightarrow{\Gamma} E[v], \tag{8}$$

for

$$E[v] := \begin{cases} 0 & \text{if } v \in C_0^{0,1}(U),\ |Dv| \le 1 \ a.e. \\ +\infty & \text{otherwise.} \end{cases} \tag{9}$$

2. If $E[v] < \infty$, we clearly have

$$E[v] = 0 = \lim_{k\to\infty} E_k[v].$$

Suppose now that $v_k \to v$ uniformly, and $\limsup_{k\to\infty} E_k[v_k] \le C < \infty$. Fix an integer m and let $k > m$. Since $e^{\frac{x^2-1}{2}} \ge x$, we have for each open set $V \subseteq U$ that

$$\begin{aligned}\left(\int_V |Dv_k|^m\ dx\right)^{1/m} &\le |V|^{1/m-1/k}\left(\int_V |Dv_k|^k dx\right)^{1/k} \\ &\le |V|^{1/m-1/k}\,k^{1/k}\,E_k(v_k)^{1/k} \le |V|^{1/m-1/k}\,k^{1/k}\,C^{1/k}.\end{aligned}$$

Passing to limits in k and recalling the lower semicontinuity of the L^m norm of the gradient, we discover

$$\left(\int_V |Dv|^m dx\right)^{1/m} \le |V|^{1/m}.$$

This inequality, valid for all V as above, implies that Dv is in L^∞, with $|Dv| \le 1$ almost everywhere. Consequently,

$$E[v] = 0 \le \liminf_{k\to\infty} E_k[v_k].$$

□

Introduce next the vector fields

$$\mathbf{G}_k := \sigma_k Du_k \qquad (k = 1, \dots).$$

Theorem 2.3. *Suppose that for some $1 < q < \infty$ we have the uniform bounds*

$$\sup_k \|\mathbf{G}_k\|_{L^q(U;\mathbb{R}^n)} < \infty.$$

Define

$$f_k := -\operatorname{div}(\mathbf{G}_k),$$

and assume

$$\begin{cases} f_k \rightharpoonup f & \text{weakly in } L^q(U) \\ \mathbf{G}_k \rightharpoonup \mathbf{G} & \text{weakly in } L^q(U;\mathbb{R}^n), \\ u_k \to u & \text{uniformly.} \end{cases}$$

Then there exists a positive function $a \in L^q$ such that

$$\begin{cases} \mathbf{G} = aDu, \\ |Du| = 1 \ a.e. \text{ on } \{a > 0\}, \text{ and} \\ \sigma_k \rightharpoonup a \quad \text{weakly in } L^q(U). \end{cases}$$

In particular, $a = |\mathbf{G}|$.

Proof. **1.** First of all, note that $-\operatorname{div}\mathbf{G} = f$; that is,

$$\int_U \mathbf{G}\cdot D\psi\,dx = \int_U f\psi\,dx$$

for all $\psi \in C^1$, $\psi = 0$ on ∂U.

Let us now fix $0 < \lambda < 1$ and calculate:

$$\begin{aligned}\int_U |\mathbf{G}|\,dx &\le \liminf_{k\to\infty}\int_U |\mathbf{G}_k|\,dx = \liminf_{k\to\infty}\left(\int_{\{|Du_k|^2>1-\lambda\}}|\mathbf{G}_k|\,dx + \int_{\{|Du_k|^2\le 1-\lambda\}}|\mathbf{G}_k|\,dx\right)\\ &\le \liminf_{k\to\infty}\left(\frac{1}{\sqrt{1-\lambda}}\int_U |\mathbf{G}_k||Du_k|\,dx + \int_U e^{-\frac{k}{2}\lambda}\sqrt{1-\lambda}\,dx\right).\end{aligned}$$

When k goes to infinity, the last integral goes to 0. Notice also that

$$\int_U |\mathbf{G}_k||Du_k|\,dx = \int_U \sigma_k|Du_k|^2\,dx = \int_U f_k u_k\,dx.$$

Therefore

$$\sqrt{1-\lambda}\int_U |\mathbf{G}|\,dx \le \liminf_{k\to\infty}\int_U f_k u_k\,dx = \int_U fu\,dx = \int_U \mathbf{G}\cdot Du\,dx$$

for each $0<\lambda<1$, and consequently

$$\int_U |\mathbf{G}|\,dx \le \int_U \mathbf{G}\cdot Du\,dx. \tag{10}$$

2. Reasoning now as in the proof of Theorem 2.2, we fix an integer m and let $k > m$. Then for each open set $V \subseteq U$

$$\begin{aligned}\left(\int_V |Du_k|^m\,dx\right)^{1/m} &\le |V|^{1/m-1/k+1}\left(\int_V |Du_k|^{k+1}dx\right)^{1/k+1}\\ &\le |V|^{1/m-1/k+1}||\mathbf{G}_k||_{L^1}^{1/k+1} \le |V|^{1/m-1/k+1}C^{1/k+1}.\end{aligned}$$

Pass to limits in k to find

$$\left(\int_V |Du|^m dx\right)^{1/m} \le |V|^{1/m},$$

and therefore $|Du| \le 1$ almost everywhere. The first two assertions of the Theorem now follow from (10).

3. To show also that $\sigma_k \rightharpoonup a$, let us fix $\psi \in C_0^\infty$ and prove

$$\int_U \sigma_k\psi\,dx \to \int_U a\psi\,dx.$$

We write

$$\int_U \sigma_k\psi\,dx = \int_U \sigma_k|Du_k|^2\psi\,dx + \int_U \sigma_k(1-|Du_k|^2)\psi\,dx =: A_1 + A_2.$$

Notice now that

$$\begin{aligned}A_1 &= \int_U \psi\,\mathbf{G}_k\cdot Du_k\,dx = \int_U \mathbf{G}_k\cdot D(u_k\psi)\,dx - \int_U u_k\,\mathbf{G}_k\cdot D\psi\,dx\\ &= \int_U f_k u_k\psi\,dx - \int_U u_k\mathbf{G}_k\cdot D\psi\,dx.\end{aligned}$$

This expression converges as $k \to \infty$ to

$$\int_U fu\psi\,dx - \int_U u\,\mathbf{G}\cdot D\psi\,dx = \int_U \mathbf{G}\cdot D(u\psi)\,dx - \int_U u\,\mathbf{G}\cdot D\psi\,dx$$
$$= \int_U \psi\,\mathbf{G}\cdot Du\,dx = \int_U \psi a|Du|^2\,dx = \int_U a\psi\,dx.$$

4. It remains to show that $A_2 \to 0$. If we write $\varphi_k := |Du_k|^2 - 1$, then

$$|A_2| \le \|\psi\|_{L^\infty} \int_U e^{\frac{k}{2}\varphi_k}|\varphi_k|\,dx.$$

Since $xe^{-\frac{x}{2}} \le 1$ for each $x > 0$, we have

$$\int_{\{\varphi_k<0\}} e^{\frac{k}{2}\varphi_k}|\varphi_k|\,dx \le \frac{1}{k}\int_U k|\varphi_k|e^{-\frac{k|\varphi_k|}{2}}\,dx \le \frac{|U|}{k} \to 0.$$

Finally, since $q > 1$ there exists a constant $c_q > 0$ such that

$$\frac{e^{x(q-1)}}{x} \ge c_q > 0$$

for all $x > 0$. Consequently,

$$\int_{\{\varphi_k>0\}} e^{\frac{k}{2}\varphi_k}|\varphi_k|\,dx = \frac{2}{k}\int_{\{\varphi_k>0\}} e^{\frac{k}{2}\varphi_k}\,\frac{k}{2}\varphi_k\,dx$$
$$\le \frac{2}{c_q k}\int_{\{\varphi_k>0\}} e^{\frac{qk}{2}\varphi_k}\,dx = \frac{2}{c_q k}\int_U \sigma_k^q\,dx \le \frac{2C^q}{c_q k} \to 0.$$

This completes the proof that $A_2 \to 0$. □

3. Estimates I

The full calculations for our main estimate in §4 are fairly involved, and so for the reader's convenience we provide in this section a simpler computation illustrating the main ideas. Suppose $2 \le q < \infty$.

Theorem 3.1. *There exists a constant C, depending on q, but independent of k, such that*

$$\int_U \sigma_k^q\,dx \le C\left(\int_U |f|^q\,dx + 1\right). \tag{11}$$

Proof. **1.** To simplify notation, we hereafter in the proof do not write the subscripts k. Observe that since Du is bounded in each space L^q and $u = 0$ on ∂U, we have the bound

$$|u| \le C$$

for some constant C.

2. Multiply (5) by $\sigma^{q-1}u$ and integrate by parts:

$$\begin{aligned}\int_U \sigma^q|Du|^2 + (q-1)\sigma^{q-1}Du\cdot D\sigma\,u\,dx &= \int_U \sigma u_i(\sigma^{q-1}u)_i\,dx = \int_U f\sigma^{q-1}u\,dx \\ &\le C\left(\int_U |f|^q\,dx\right)^{\frac{1}{q}}\left(\int_U \sigma^q\,dx\right)^{1-\frac{1}{q}}.\end{aligned} \tag{12}$$

Here and afterwards we write the subscript i to denote the partial derivative with respect to the variable x_i.

Notice that $|Du|^2 \geq 1$ if $\sigma \geq 1$. Therefore

$$\int_U \sigma^q \, dx \leq C \left(\int_U |f|^q \, dx + \int_U \sigma^{q-1} |Du \cdot D\sigma| \, dx + 1 \right). \tag{13}$$

3. Next, multiply (5) by $-(\sigma^{q-1} u_j)_j$:

$$\begin{aligned} \int_U (\sigma u_i)_i (\sigma^{q-1} u_j)_j \, dx &= -\int_U f (\sigma^{q-1} u_j)_j \, dx \\ &= \int_U f \sigma^{q-2} (-(\sigma u_j)_j) \, dx - \int_U f(q-2) \sigma^{q-2} \sigma_j u_j \, dx \\ &\leq C \int_U f^2 \sigma^{q-2} + |f| \sigma^{q-2} |Du \cdot D\sigma| \, dx. \end{aligned} \tag{14}$$

The term on the left is

$$\begin{aligned} A : &= -\int_U \sigma u_i (\sigma^{q-1} u_j)_{ij} \, dx + \int_{\partial U} \sigma u_i \nu^i (\sigma^{q-1} u_j)_j \, d\mathcal{H}^{n-1} \\ &= \int_U (\sigma u_i)_j (\sigma^{q-1} u_j)_i \, dx + \int_{\partial U} \sigma u_i \nu^i (\sigma^{q-1} u_j)_j - \sigma u_i \nu^j (\sigma^{q-1} u_j)_i \, d\mathcal{H}^{n-1}, \end{aligned} \tag{15}$$

where $\nu = (\nu^1, \ldots, \nu^n)$ is the unit outer normal to ∂U. The boundary integral is

$$\begin{aligned} B := &\int_{\partial U} \sigma^q (u_i \nu^i u_{jj} - u_i \nu^j u_{ij}) \, d\mathcal{H}^{n-1} \\ &+ \int_{\partial U} (q-1) \sigma^{q-1} (u_i \nu^i u_j \sigma_j - u_i \nu^j \sigma_i u_j) \, d\mathcal{H}^{n-1}. \end{aligned} \tag{16}$$

The integrand of the last term equals 0, since $\sigma = e^{\frac{k}{2}(|Du|^2 - 1)}$ and so $\sigma_j = k u_l u_{lj} \sigma$.

Consider a point $x_0 \in \partial U$; without loss, we can take $x_0 = (0, \ldots, R)$. Then $\nu = (0, \ldots, 1)$ and $Du = (0, \ldots, u_n)$, since $u = 0$ on ∂U. The integrand of the first term on the right hand side of (16) at x_0 therefore equals

$$\sigma^q (\Delta u - u_{nn}) u_n. \tag{17}$$

Lastly, write $x' = (x_1, \ldots, x_{n-1})$ and observe that $u(x', \sqrt{R^2 - |x'|^2}) \equiv 0$ for small x'. We differentiate this identity twice and set $x' = 0$, to compute $\Delta u - u_{nn} = \frac{n-1}{R} u_n$ at x_0. Hence

$$B = \frac{n-1}{R} \int_{\partial U} \sigma^q |Du|^2 \, d\mathcal{H}^{n-1} \geq 0.$$

4. Therefore

$$\begin{aligned} A &= \int_U (\sigma u_i)_i \left(\sigma^{q-1} u_j \right)_j \, dx \geq \int_U (\sigma u_i)_j (\sigma^{q-1} u_j)_i \, dx \\ &= \int_U (\sigma u_{ij} + \sigma_j u_i)(\sigma^{q-1} u_{ij} + (q-1) \sigma^{q-2} \sigma_i u_j) \, dx \\ &= \int_U \sigma^q |D^2 u|^2 + (q-1) \sigma^{q-2} |Du \cdot D\sigma|^2 + q \sigma^{q-1} \sigma_j u_i u_{ij} \, dx. \end{aligned} \tag{18}$$

Recall that $\sigma_j = k u_l u_{lj} \sigma$. Hence (14) and (18) imply

$$\begin{aligned}
\int_U \sigma^q |D^2 u|^2 + (q-1)\sigma^{q-2}|Du \cdot D\sigma|^2 + \frac{q}{k}\sigma^{q-2}|D\sigma|^2 dx \\
\leq C \int_U f^2 \sigma^{q-2} + |f| \sigma^{q-2} |Du \cdot D\sigma| \, dx \\
\leq \frac{q-1}{2} \int_U \sigma^{q-2} |Du \cdot D\sigma|^2 + C \int_U |f|^2 \sigma^{q-2} \, dx;
\end{aligned} \tag{19}$$

and consequently

$$\int_U \sigma^{q-2} |Du \cdot D\sigma|^2 \, dx \leq C \int_U |f|^2 \sigma^{q-2} \, dx. \tag{20}$$

5. Combining now (13) and (20) one obtains

$$\begin{aligned}
\int_U \sigma^q \, dx &\leq C \int_U |f|^q \, dx + C \int_U \sigma^{q-1} |Du \cdot D\sigma| \, dx + C \\
&\leq C \int_U |f|^q \, dx + \frac{1}{3} \int_U \sigma^q \, dx + C \int_U \sigma^{q-2} |Du \cdot D\sigma|^2 \, dx + C \\
&\leq C \int_U |f|^q \, dx + \frac{1}{3} \int_U \sigma^q \, dx + C \int_U |f|^2 \sigma^{q-2} \, dx + C \\
&\leq C \int_U |f|^q \, dx + \frac{2}{3} \int_U \sigma^q \, dx + C;
\end{aligned} \tag{21}$$

and this gives (11). □

Remark. The boundary integral term B is in fact nonnegative for any convex, smooth domain replacing $U = B(0, R)$: see for instance the similar calculations in §1.5 of Ladyzhenskaja [8]. □

4. Estimates II

In this section we derive our main integral estimate.

Theorem 4.1. *Assume that $2 \leq q < \infty$ and that $f \in C^\infty(\bar{U})$. Then there exist a constant C, depending only on q, and a constant K, depending only on $\|f\|_{L^\infty}$, such that*

$$\int_U \sigma_k^q |Du_k|^q \, dx \leq C \left(\int_U |f|^q \, dx + 1 \right) \tag{22}$$

for all $k \geq K$.

The proof is similar to that of Theorem 3.1, except that we must handle the additional term $|Du_k|^q$ on the left. This makes our multipliers and estimates more intricate.

Proof. **1.** For notational simplicity we hereafter write σ and u in place of σ_k and u_k.
Since f is smooth, the same is true for u and σ. Observe also the bound

$$|u| \leq C.$$

We record for later reference these consequences of (6):

$$|Du|_i = \frac{\sigma_i}{k\sigma|Du|}, \; u_i u_{ij} = \frac{\sigma_j}{k\sigma}. \tag{23}$$

2. We multiply the PDE (5) by $\sigma^{q-1}|Du|^{q+1}u$ and integrate by parts, to find

$$\int_U \sigma Du \cdot D\left(\sigma^{q-1}|Du|^{q+1}u\right)\,dx = \int_U \sigma^{q-1}|Du|^{q+1}uf\,dx. \tag{24}$$

The right hand term in (24) is less than or equal to

$$C\int_U \sigma^{q-1}|Du|^{q+1}|f|\,dx \le \frac{1}{2}\int_{\{|f|\le\frac{\sigma|Du|}{2C}\}} \sigma^q|Du|^{q+2}\,dx + 2^{q-1}\,C^q \int_{\{|f|>\frac{\sigma|Du|}{2C}\}} |Du|^2|f|^q\,dx.$$

But if $\sigma|Du| < 2C|f|$, then obviously $\sigma|Du| \le 2C\|f\|_{L^\infty}$. Recalling (6), we see that this implies $|Du| \le 2$ provided $k \ge K$, for some constant K depending only upon $\|f\|_{L^\infty}$. Therefore

$$\int_U \sigma^{q-1}|Du|^{q+1}uf\,dx \le \frac{1}{2}\int_U \sigma^q|Du|^{q+2}\,dx + C\int_U |f|^q\,dx. \tag{25}$$

3. We use (23) to evaluate the left hand term in (24):

$$\begin{aligned}
\int_U \sigma Du \cdot D\left(\sigma^{q-1}|Du|^{q+1}u\right)\,dx &= \int_U \sigma^q|Du|^{q+3}\,dx \\
+(q-1)\int_U \sigma^{q-1}|Du|^{q+1}u\,D\sigma\cdot Du\,dx &+ (q+1)\int_U \sigma^q u|Du|^q Du\cdot(D|Du|)\,dx \\
&= \int_U \sigma^q|Du|^{q+3}\,dx \\
+(q-1)\int_U \sigma^{q-1}|Du|^{q+1}u\,D\sigma\cdot Du\,dx &+ \frac{q+1}{k}\int_U \sigma^{q-1}u|Du|^{q-1}Du\cdot D\sigma\,dx.
\end{aligned} \tag{26}$$

But $\sigma \ge 1$ only if $|Du| \ge 1$; and hence

$$\int_U \sigma^q|Du|^{q+2}\,dx \le \int_U \sigma^q|Du|^{q+3}\,dx + C, \tag{27}$$

since U is bounded.

Combining (27), (26), (24) and (25), we deduce the inequality

$$\begin{aligned}
\int_U \sigma^q|Du|^{q+2}\,dx \le C &+ \frac{1}{2}\int_U \sigma^q|Du|^{q+2}\,dx + C\int_U |f|^q\,dx \\
&+ C\int_U \sigma^{q-1}|Du|^{q+1}|D\sigma\cdot Du|\,dx + \frac{C}{k}\int_U \sigma^{q-1}|Du|^{q-1}|D\sigma\cdot Du|\,dx.
\end{aligned}$$

Arguing as before (this means dividing the integrals in the set where $|D\sigma\cdot Du| \le \epsilon\,\sigma|Du|/C$ and in the rest of U), we see that therefore

$$\begin{aligned}
\int_U \sigma^q|Du|^{q+2}\,dx \le C &+ C\int_U |f|^q\,dx + \epsilon\int_U \sigma^q|Du|^{q+2}\,dx + \frac{\epsilon}{k}\int_U \sigma^q|Du|^q\,dx \\
&+ \frac{C^2}{\epsilon}\int_U \sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2\,dx + \frac{C^2}{k\epsilon}\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\,dx
\end{aligned} \tag{28}$$

for any $\epsilon > 0$. Since $\int_U \sigma^q|Du|^q\,dx \le \int_U \sigma^q|Du|^{q+2}\,dx + C$, this implies our first main estimate:

$$\begin{aligned}
\int_U \sigma^q|Du|^{q+2}\,dx \le C + C\int_U |f|^q\,dx &+ C\int_U \sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2\,dx \\
&+ \frac{C}{k}\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\,dx.
\end{aligned} \tag{29}$$

4. The last two terms in (29) involving $D\sigma \cdot Du$ are dangerous, since $D\sigma$ is of order k: we need another estimate to control them.

Let us therefore continue by multiplying the PDE (5) by $-\operatorname{div}\left(\sigma^{q-1}|Du|^q Du\right)$ and thereby deriving the identity

$$\int_U \operatorname{div}(\sigma Du)\ \operatorname{div}(\sigma^{q-1}|Du|^q Du)\,dx = -\int_U f \operatorname{div}(\sigma^{q-1}|Du|^q Du)\,dx. \tag{30}$$

The term on the right equals

$$\begin{aligned}\int_U f\,\sigma^{q-2}|Du|^q\,(-\operatorname{div}(\sigma Du))\,dx - \int_U f\sigma Du \cdot D\left(\sigma^{q-2}|Du|^q\right)\,dx &= \int_U |f|^2\sigma^{q-2}|Du|^q\,dx \\ -(q-2)\int_U f\sigma^{q-2}|Du|^q Du\cdot D\sigma\,dx - q\int_U f\sigma^{q-1}|Du|^{q-1}Du\cdot(D|Du|)\,dx.\end{aligned}$$

We again recall (23) and deduce

$$\begin{aligned}-\int_U f \operatorname{div}(\sigma^{q-1}|Du|^q Du)\,dx &\le \int_U |f|^2\sigma^{q-2}|Du|^q\,dx \\ &+ (q-2)\int_U |f|\sigma^{q-2}|Du|^q|Du\cdot D\sigma|\,dx + \frac{q}{k}\int_U |f|\sigma^{q-2}|Du|^{q-2}|Du\cdot D\sigma|\,dx.\end{aligned} \tag{31}$$

The left hand term of (30) is

$$\begin{aligned}A :&= -\int_U \sigma u_i(\sigma^{q-1}|Du|^q u_j)_{ij}\,dx + \int_{\partial U}\sigma u_i\nu^i(\sigma^{q-1}|Du|^q u_j)_j\,d\mathcal{H}^{n-1} \\ &= \int_U (\sigma u_i)_j(\sigma^{q-1}|Du|^q u_j)_i\,dx \\ &\quad + \int_{\partial U}\sigma u_i\nu^i(\sigma^{q-1}|Du|^q u_j)_j - \sigma u_i\nu^j(\sigma^{q-1}|Du|^q u_j)_i\,d\mathcal{H}^{n-1}.\end{aligned} \tag{32}$$

Call the boundary term B. Then, almost exactly as in step 3 of the previous proof, we can show that

$$B = \frac{n-1}{R}\int_{\partial U}\sigma^q|Du|^{q+2}\,d\mathcal{H}^{n-1} \ge 0.$$

Consequently,

$$\begin{aligned}A &= \int_U (\sigma u_i)_i\left(\sigma^{q-1}|Du|^q u_j\right)_j\,dx \ge \int_U (\sigma u_i)_j\left(\sigma^{q-1}|Du|^q u_j\right)_i\,dx \\ &= \int_U (\sigma u_{ij}+\sigma_j u_i)\left(\sigma^{q-1}|Du|^q u_{ij} + (q-1)\sigma^{q-2}|Du|^q\sigma_i u_j + \frac{q}{k}\sigma^{q-2}|Du|^{q-2}\sigma_i u_j\right)\,dx \\ &= \int_U \sigma^q|Du|^q|D^2u|^2 + \frac{q}{k}\sigma^{q-2}|Du|^q|D\sigma|^2 + (q-1)\sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2 + \\ &\quad + \frac{q}{k^2}\sigma^{q-2}|Du|^{q-2}|D\sigma|^2 + \frac{q}{k}\sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\,dx.\end{aligned}$$

The first, the second and the fourth terms in the last expression are positive, and so we deduce

$$\begin{aligned}(q-1)\int_U \sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2\,dx + \frac{q}{k}\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\,dx \\ \le \int_U \operatorname{div}(\sigma Du)\operatorname{div}(\sigma^{q-1}|Du|^q Du)\,dx.\end{aligned} \tag{33}$$

Collecting (33), (30) and (31), we find

$$
\begin{aligned}
\int_U \sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2\,dx &+ \frac{1}{k}\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\,dx \\
&\le C\int_U |f|^2\sigma^{q-2}|Du|^q\,dx + C\int_U |f|\sigma^{q-2}|Du|^q|Du\cdot D\sigma|\,dx \\
&\quad + \frac{C}{k}\int_U |f|\sigma^{q-2}|Du|^{q-2}|Du\cdot D\sigma|\,dx.
\end{aligned} \tag{34}
$$

Take $\epsilon > 0$ to be a small constant, which will be fixed later on. Then

$$
\begin{aligned}
\int_U f^2\sigma^{q-2}|Du|^q\,dx &\le \epsilon\int_U \sigma^q|Du|^{q+2}\,dx + C\int_{\{|f|>\sigma|Du|\sqrt{\epsilon}\}} |f|^q|Du|^2\,dx \\
&\le \epsilon\int_U \sigma^q|Du|^{q+2}\,dx + C\int_U |f|^q\,dx,
\end{aligned} \tag{35}
$$

since $|Du|\le 2$ wherever $|f| > \sigma|Du|\sqrt{\epsilon}$, provided $k\ge K$ and K is large.

Recalling (35), we can likewise estimate for each $\delta > 0$ that

$$
\begin{aligned}
\int_U |f|\sigma^{q-2}|Du|^q|D\sigma\cdot Du|\,dx &\le \delta\int_U \sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2\,dx + C\int_U f^2\sigma^{q-2}|Du|^q\,dx \\
&\le \delta\int_U \sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2\,dx + C\left(\epsilon\int_U \sigma^q|Du|^{q+2}\,dx + \int_U |f|^q\,dx\right).
\end{aligned} \tag{36}
$$

Similarly,

$$
\begin{aligned}
\int_U |f|\sigma^{q-2}|Du|^{q-2}|Du\cdot D\sigma|dx &\le \delta\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2dx + C\int_U f^2\sigma^{q-2}|Du|^{q-2}dx \\
&\le \delta\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\,dx + C\left(\epsilon\int_U \sigma^q|Du|^q\,dx + C\int_U |f|^q\,dx\right).
\end{aligned} \tag{37}
$$

Since $\sigma\ge 1$ only if $|Du|\ge 1$, we have

$$
\int_U \sigma^q|Du|^q\,dx \le \int_U \sigma^q|Du|^{q+2}\,dx + C,
$$

and therefore

$$
\begin{aligned}
\int_U |f|\sigma^{q-2}|Du|^{q-2}|Du\cdot D\sigma|\,dx &\le \delta\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\,dx \\
&\quad + C\left(\epsilon\int_U \sigma^q|Du|^{q+2}\,dx + \int_U |f|^q\,dx + 1\right).
\end{aligned} \tag{38}
$$

Taking $\delta > 0$ small, we then derive from (34), (35), (36) and (38) our second main inequality

$$
\begin{aligned}
\int_U \sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2\,dx &+ \frac{1}{k}\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\,dx \\
&\le \epsilon\int_U \sigma^q|Du|^{q+2}\,dx + C\int_U |f|^q + C.
\end{aligned} \tag{39}
$$

5. Putting together inequalities (29) and (39) and fixing $\epsilon > 0$ small, we finally discover

$$
\int_U \sigma^q|Du|^{q+2}\,dx \le C + C\int_U |f|^q\,dx.
$$

As $\sigma\ge 1$ only if $|Du|\ge 1$, estimate (22) follows. $\square$

Theorem 4.1 concerns only smooth functions f. However, since the bound for the L^q norm of the transport density depends only upon the L^q norm of f, we can approximate:

Theorem 4.2. *For each $2 \leq q < \infty$ and $f \in L^q(U)$ the associated transport density a belongs to $L^q(U)$. Furthermore, there is a constant C, depending only upon n and U, such that*

$$\|a\|_{L^q} \leq C\left(\|f\|_{L^q} + 1\right). \tag{40}$$

Proof. **1.** Let us first define

$$f_j := f * \eta_{1/j},$$

the convolution of f with a standard mollifier. For each integer j, we then solve

$$\begin{cases} -\operatorname{div}\left(\sigma_{k,j} Du_{k,j}\right) = f_j & \text{in } U \\ u_{k,j} = 0 & \text{on } \partial U, \end{cases} \tag{41}$$

for

$$\sigma_{k,j} := e^{\frac{k}{2}\left(|Du_{k,j}|^2 - 1\right)}. \tag{42}$$

2. According to (22), we have the estimate

$$\int_U \sigma_{k,j}^q |Du_{k,j}|^q\, dx \leq C\left(\int_U |f_j|^q\, dx + 1\right) \leq C\left(\int_U |f|^q\, dx + 1\right) \tag{43}$$

for all k greater than or equal to some constant $K = k(j)$, depending only on the L^∞ norm of f_j. Now define

$$\sigma_j := \sigma_{k(j),j}, \quad u_j := u_{k(j),j}, \quad \mathbf{G}_j := \sigma_j Du_j.$$

Clearly $f_j \to f$ in L^q. Furthermore, (43) implies that $\mathbf{G}_j$ is bounded in L^q. We may therefore assume upon reindexing that

$$\mathbf{G}_j \rightharpoonup \mathbf{G} \quad \text{weakly in } L^q(U;\mathbb{R}^n).$$

Finally we may pass as necessary to a further subsequence to ensure u_j converges uniformly to a limit u. Apply Theorem 2.3. □

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