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Positive decaying solutions to BVPs with mean curvature operator

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Dedicated to the 75th birthday of Professor Jean Mawhin

Abstract. *A boundary value problem on the whole half-closed interval* $[1,\infty)$*, associated to differential equations with the Euclidean mean curvature operator or with the Minkowski mean curvature operator is here considered. By using a new approach, based on a linearization device and some properties of principal solutions of certain disconjugate second-order linear equations, the existence of global positive decaying solutions is examined.*

Keywords: Second order nonlinear differential equation; Euclidean mean curvature operator, Minkowski mean curvature operator, Radial solution, Principal solution, Disconjugacy.
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1. Introduction

In this paper we deal with the following boundary value problems (BVPs) on the half-line for equations with the Euclidean mean curvature operator

$$\begin{cases} \left(a(t)\dfrac{x'}{\sqrt{1+x'^2}}\right)' + b(t)F(x) = 0, & t\in[1,\infty) \\ x(1)=1,\ x(t)>0,\ x'(t)\le 0 \text{ for } t\ge 1,\ \lim\limits_{t\to\infty} x(t)=0, & \end{cases} \tag{1}$$

and with the Minkowski mean curvature operator

$$\begin{cases} \left(a(t)\dfrac{x'}{\sqrt{1-x'^2}}\right)' + b(t)F(x) = 0, & t\in[1,\infty) \\ x(1)=1,\ x(t)>0,\ x'(t)\le 0 \text{ for } t\ge 1,\ \lim\limits_{t\to\infty} x(t)=0. & \end{cases} \tag{2}$$

Troughout the paper the following conditions are assumed:

(H_1) The function a is continuous on $[1,\infty)$, $a(t)>0$ in $[1,\infty)$, and

$$\int_1^\infty \frac{1}{a(t)}\,dt<\infty. \tag{3}$$

(H_2) The function b is continuous on $[1,\infty)$, $b(t) \geq 0$ and

$$\int_1^\infty b(t) \int_t^\infty \frac{1}{a(s)} ds dt < \infty. \tag{4}$$

(H_3) The function F is continuous on $\mathbb{R}$, $F(u)u > 0$ for $u \neq 0$, and

$$\limsup_{u \to 0^+} \frac{F(u)}{u} < \infty.$$

Define

$$\Phi_E(v) = \frac{v}{\sqrt{1+v^2}}, \quad \Phi_M(v) = \frac{v}{\sqrt{1-v^2}}.$$

The operator Φ_E arises in the search for radial solutions to partial differential equations which model fluid mechanics problems, in particular capillarity-type phenomena for compressible and incompressible fluids. The operator Φ_M originates from studying certain extrinsic properties of the mean curvature of hypersurfaces in the relativity theory. Therefore, it is called also the relativity operator.

For instance, the study of radial solutions for the problem

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 \pm |\nabla u|^2}}\right) + f(|x|, u) = 0, \quad x \in \Omega \subset \mathbb{R}^N$$
$$u(x) > 0 \text{ in } \Omega, \quad \lim_{|x|\to\infty} u(|x|) = 0,$$

where Ω is the exterior of a ball of radius $R > 0$, leads to the boundary value problem on the half-line

$$\big(r^{N-1}\frac{v'}{\sqrt{1 \pm v'^2}}\big)' + r^{N-1} f(r,v) = 0, \quad r \in [R,\infty)$$
$$v(r) > 0, \lim_{r\to\infty} v(r) = 0,$$

where $r = |x|$ and $v(r) = u(|x|)$. If $N > 2$, $f(r,v) = \hat{b}(r)F(v)$, with $\hat{b}(r) \geq 0$ in $[R,\infty)$ and $\int_R^\infty r\hat{b}(r)\, dr < \infty$, then assumptions ($H_1$) and ($H_2$) are satisfied. In particular, if $\hat{b}(r) \approx r^\delta$, then ($H_2$) reads as $\delta < -2$.

Boundary value problems associated to equations with the curvature operator in compact intervals are widely considered in the literature. We refer, in particular, to [3, 4, 5, 6, 7, 8, 11, 25, 27], and references therein. In unbounded domains, these equations have been considered in [14, 15], in which some asymptotic BVPs are studied, and in [1, 2, 13, 19], in which the search of ground state solutions, that is solutions which are globally positive on the whole half-line and tend to zero as $t \to \infty$, is examined.

Finally, equations with sign-changing coefficients are recently considered when the differential operator is the p-Laplacian, see, e.g. [9, 10, 22, 23] and references therein.

Here, our main aim is to study the solvability of the BVPs (1) and (2). As claimed, these BVPs originate from the search of ground state solutions for PDE with Euclidean or Minkowski mean curvature operator. Our approach is based on a fixed point theorem for operators defined in a Fréchet space by a Schauder's linearization device, see [16, Theorem 1.1]. This tool does not require the explicit form of the fixed point operator $\mathcal{T}$. Moreover, it simplifies the check of the topological properties of $\mathcal{T}$ in the noncompact interval $[1,\infty)$, since these properties become an immediate consequence of *a-priori* bounds for an associated linear equation. These bounds are obtained in an implicit form by means of the concepts of disconjugacy and principal solutions for second order linear equations. The main properties on this topic, needed in our arguments, are presented in Section 2. In Section 3 the solvability of (1) and (2) is given, by assuming some implicit conditions on functions a and b. Explicit conditions for the solvability of these BVPs, are derived in Section 4. Observe that also the BVP for equations with the Sturm-Liouville operator

$$\begin{cases}(a(t)x')' + b(t)F(x) = 0, & t \in [1,\infty)\\ x(1) = 1,\ x(t) > 0, \ \text{for } t \geq 1, \ \lim\limits_{t\to\infty} x(t) = 0, \end{cases} \tag{5}$$

can been treated by a similar method. Some examples and a discussion on these topics complete the paper.

2. Auxiliary results

To obtain a-priori bounds for solutions of BVPs (1) and (2), we employ a linearization method. Therefore, in this section we consider linear equations, we point out some properties of principal solutions, and we state new comparison results.

Consider the linear equation

$$(r(t)y')' + q(t)y = 0, \quad t \in [1,\infty), \tag{6}$$

where r, q are continuous functions, $r(t) > 0, q(t) \geq 0$ for $t \geq 1$.

The equation (6) is called nonoscillatory if all its solutions are nonoscillatory. In view of the Sturm theorem, see, e.g., [24, Chap. XI, Section 3], the existence of a nonoscillatory solution implies the nonoscillation of (6). When (6) is nonoscillatory, a powerful tool for studying the qualitative behavior of its solutions is based on the analysis of the corresponding Riccati equation

$$\xi' + q(t) + \frac{\xi^2}{r(t)} = 0, \tag{7}$$

see, e.g., [18, 24]. More precisely, if y is a non-vanishing solution of (6), then

$$\xi(t) = \frac{r(t)y'(t)}{y(t)}$$

is a solution of (7). Conversely, if ξ is a solution of (7), then any nontrivial solution y of the first order linear equation

$$y' = \frac{\xi(t)}{r(t)}y$$

is also a non-vanishing solution of (6). If (6) is nonoscillatory, then the corresponding Riccati equation (7) has a solution ξ_0, defined for large t, such that for any other solution ξ of (7), defined in a neighborhood I_ξ of infinity, we have $\xi_0(t) < \xi(t)$ for $t \in I_\xi$. The solution ξ_0 is called the *minimal solution* of (7) and any solution y_0 of

$$y' = \frac{\xi_0(t)}{r(t)}y \tag{8}$$

is called *principal solution* of (6). Clearly, y_0 is uniquely determined up to a constant factor and so by *the principal solution* of (6) we mean any solution of (8) which is eventually positive. The principal solution is, roughly speaking, the smallest solution of (6) near infinity. Indeed it holds

$$\lim_{t\to\infty} \frac{y_0(t)}{y(t)} = 0,$$

where y denotes any linearly independent solution of (6).

We recall that (6) is said to be *disconjugate* on an interval $I \subset [1,\infty)$, if any nontrivial solution of (6) has at most one zero on I. Equation (6) is disconjugate on $[1,\infty)$, if and only if it is disconjugate on $(1,\infty)$, see, e.g., [18, Theorem 2, Chap.1]. The relation between the notions of disconjugacy and principal solution is given by the following, see, e.g., [18, Chap. 1] or [24, Chap. XI, Section 6].

LEMMA 2.1. *The following statements are equivalent.*

(i_1) Equation (6) is disconjugate on $[1,\infty)$.

(i_2) The principal solution y_0 of (6) does not have zeros on $(1,\infty)$.

(i_3) The Riccati equation (7) has a solution defined throughout $(1,\infty)$.

The following characterization of principal solution of (6) holds, see [24, Chap. XI, Theorem 6.4].

LEMMA 2.2. *Let (6) be nonoscillatory. Then a nontrivial solution y_0 of (6) is the principal solution if and only if we have for large T*

$$\int_T^\infty \frac{1}{r(s)y_0^2(s)}ds = \infty.$$

Some asymptotic properties for solutions of (6) are summarized in the next lemma.

LEMMA 2.3. *Assume*

$$\int_1^\infty \frac{1}{r(t)}dt < \infty, \qquad \int_1^\infty q(t)\, R(t)dt < \infty,$$

where

$$R(t) = \int_t^\infty \frac{1}{r(s)}ds.$$

Then (6) is nonoscillatory, and the set of eventually nonincreasing positive solutions, with zero limit at infinity, is nonempty. Further, any such solution y satisfies

$$\lim_{t\to\infty} \frac{y(t)}{R(t)} = c_y, \tag{9}$$

where $0 < c_y < \infty$ is a suitable constant.

Proof. From [17, Theorem 1], see also [17, Lemma 2], we have the existence of eventually nonincreasing positive solutions, with zero limit at infinity. The asymptotic estimate (9) follows from [17, Theorem 2] and the l'Hopital rule. □

Under the assumptions of Lemma 2.3, the principal solution y_0 of (6) is nonincreasing for large t. However, y_0' can change sign on $[1,\infty)$, even if (6) is disconjugate on $[1,\infty)$, see, e.g., [20, Example 1]. Now, the question under what assumptions the principal solution is monotone on the whole interval $[1,\infty)$ arises. In the following we give conditions ensuring that $y_0(t)y_0'(t) \le 0$ on the whole interval $[1,\infty)$. To this end the following comparison criterion between two Riccati equations plays a crucial role, see [24, Chap. XI, Corollary 6.5].

Consider the linear equations

$$\big(r_2(t)y'\big)' + q_2(t)y = 0, \qquad t \ge 1, \tag{10}$$

and

$$\big(r_1(t)w'\big)' + q_1(t)w = 0, \qquad t \ge 1, \tag{11}$$

where r_i, q_i are continuous functions on $[1,\infty)$, $r_i(t) > 0, q_i(t) \ge 0$ for $t \ge 1, i = 1, 2$.

LEMMA 2.4. *Let (10) be a Sturm majorant of (11), that is, for $t \ge 1$*

$$r_1(t) \ge r_2(t), \qquad q_1(t) \le q_2(t). \tag{12}$$

Let (10) be disconjugate on $[T,\infty), T \ge 1$, and assume that a solution y of (10) exists, without zeros on $[T,\infty)$. Then (11) is disconjugate on $[T,\infty)$ and its principal solution w_0 satisfies for $t \ge T$

$$\frac{r_1(t)w_0'(t)}{w_0(t)} \le \frac{r_2(t)y'(t)}{y(t)}.$$

Using Lemma 2.4, we get the following comparison result, which will play a crucial role in the sequel.

LEMMA 2.5. *Let (10) be a majorant of (11), that is (12) holds for $t \geq 1$ and at least one of the inequalities in (12) is strict on a subinterval of $[1, \infty)$ of positive measure. If the principal solution of (10) is positive nonincreasing on $[1, \infty)$, then (11) has the principal solution which is positive nonincreasing on $[1, \infty)$.*

Proof. The assertion is an easy consequence of a well-known result on conjugate points for linear equations, see, e.g., [21, Theorem 4.2.3]. Since (10) is disconjugate on $[1, \infty)$, by Lemma 2.4 also (11) is disconjugate on the same interval. By Lemma 2.1 the principal solution w_0 of (11) is positive for $t > 1$. If $w_0(1) = 0$, using [21, Theorem 4.2.3], every solution of (10) should have a zero point on $(1, \infty)$, which contradicts the fact that the principal solution of (10) is positive on $(1, \infty)$. Thus $w_0(t) > 0$ on $[1, \infty)$. Using Lemma 2.4 we get $w_0'(t) \leq 0$ for $t \geq 1$, and the assertion follows. □

3. The existence results

Define
$$\bar{F} = \sup_{u \in (0,1]} \frac{F(u)}{u}. \tag{13}$$

We start by considering the BVP associated to the equation with the Euclidean mean curvature operator. The following holds.

THEOREM 3.1. *Let (H_i), i=1,2,3, be verified. Assume*
$$\alpha = \inf_{t \geq 1} \; a(t)A(t) > 1, \tag{14}$$
where
$$A(t) = \int_t^\infty \frac{1}{a(s)}\, ds. \tag{15}$$
If the principal solution z_0 of the linear equation
$$(a(t)z')' + \frac{\alpha}{\sqrt{\alpha^2 - 1}} \bar{F}\, b(t) z = 0, \quad t \geq 1, \tag{16}$$
is positive and nonincreasing on $[1, \infty)$, then the BVP (1) has at least one solution.

To prove this result, we use a general fixed point theorem for operators defined in the Fréchet space $C([1, \infty), \mathbb{R}^2)$, based on [16, Theorem 1.1]. We state the result in the form that will be used.

THEOREM 3.2. *Let S be a nonempty subset of the Fréchet space $C([1,\infty),\mathbb{R}^2)$. Assume that there exists a nonempty, closed, convex and bounded subset $\Omega \subset C([1,\infty),\mathbb{R}^2)$ such that, for any $(u,v) \in \Omega$, the linear equation*

$$\left(\sqrt{a^2(t)-v^2(t)}\; x'\right)' + b(t)\frac{F(u(t))}{u(t)}\, x = 0 \tag{17}$$

admits a unique solution x_{uv}, such that $(x_{uv}, x_{uv}^{[1]}) \in S$, where

$$x_{uv}^{[1]} = \sqrt{a^2(t)-v^2(t)}\, x'_{uv}$$

is the quasiderivative of x_{uv}.
Let $\mathcal{T}$ be the operator $\Omega \to S$, given by

$$\mathcal{T}(u,v) = (x_{uv}, x_{uv}^{[1]}).$$

Assume:

(i_1) $\mathcal{T}(\Omega) \subset \Omega$;

(i_2) if $\{(u_n,v_n)\} \subset \Omega$ is a sequence converging in Ω and $\mathcal{T}((u_n,v_n)) \to (x_1,x_2)$, then $(x_1,x_2) \in S$.

Then the operator $\mathcal{T}$ has a fixed point $(\overline{x},\overline{y}) \in \Omega \cap S$ and $\overline{x}$ is a solution of

$$\left(a(t)\frac{x'}{\sqrt{1+x'^2}}\right)' + b(t)F(x) = 0. \tag{18}$$

If the equation (17) is replaced by

$$\left(\sqrt{a^2(t)+v^2(t)}\, x'\right)' + b(t)\frac{F(u(t))}{u(t)}\, x = 0, \tag{19}$$

and (i_1), (i_2) are verified, then $\mathcal{T}$ has a fixed point $(\widetilde{x},\widetilde{y}) \in \Omega \cap S$ and $\widetilde{x}$ is a solution of

$$\left(a(t)\frac{x'}{\sqrt{1-x'^2}}\right)' + b(t)F(x) = 0.$$

Proof. Equation (17) can be written as the linear system

$$x_1' = \frac{1}{\sqrt{a^2(t)-v^2(t)}}\, x_2, \quad x_2' = -b(t)\frac{F(u(t))}{u(t)}\, x_1, \tag{20}$$

where $x_1 = x$ and $x_2 = x^{[1]}$. Hence, from [16, Theorem 1.1], the set $\mathcal{T}(\Omega)$ is relatively compact and $\mathcal{T}$ is continuous on Ω. The Schauder-Tychonoff fixed point theorem can now be applied to the operator $\mathcal{T} : \Omega \to \mathcal{T}(\Omega)$, since Ω

is bounded, closed, convex, $\mathcal{T}(\Omega)$ is relatively compact and $\mathcal{T}$ is continuous on Ω. Thus, $\mathcal{T}$ has a fixed point in Ω, say $(\overline{x},\overline{y})$, and $(\overline{x},\overline{y})=\mathcal{T}(\overline{x},\overline{y})$. Since $T(\Omega)\subset S$ and $T(\Omega)\subset\Omega$, we get $(\overline{x},\overline{y})\in\Omega\cap S$. From (20) we have

$$\overline{x}'(t)=\frac{\overline{y}(t)}{\sqrt{a^2(t)-\overline{y}^2(t)}},\qquad \overline{y}'(t)=-b(t)F(\overline{x}(t)),$$

Since

$$\overline{x}'(t)=\frac{\overline{y}(t)}{\sqrt{a^2(t)-\overline{y}^2(t)}}=\Phi_M\left(\frac{\overline{y}(t)}{a(t)}\right)$$

or

$$\Phi_E\left(\overline{x}'(t)\right)=\Phi_E\left(\Phi_M\left(\frac{\overline{y}(t)}{a(t)}\right)\right),$$

using the fact that $\Phi_E(\Phi_M(d))=d$, we obtain

$$a(t)\frac{\overline{x}'(t)}{\sqrt{1+(\overline{x}'(t))^2}}=\overline{y}(t),\qquad \overline{y}'(t)=-b(t)F(\overline{x}(t)).$$

Then $\overline{x}$ is a solution of (18). A similar argument holds when the operator $\mathcal{T}$ is defined via the linear equation (19). □

Proof of Theorem 3.1. In view of assumptions (H_1) and (H_2), Lemma 2.3 is applicable and (16) is nonoscillatory. Since the principal solution z_0 of (16) is positive nonincreasing on $[1,\infty)$, we can suppose also $z_0(1)=1$. Using Lemma 2.3 we have $\lim_{t\to\infty}z_0(t)=0$. From Lemma 2.1, equation (16) is disconjugate on $[1,\infty)$. Moreover, (16) is equivalent to

$$\left(\frac{\sqrt{\alpha^2-1}}{\alpha}\,a(t)z'\right)'+\bar{F}\,b(t)z=0,\quad t\geq 1, \tag{21}$$

which is a Sturm majorant of

$$(a(t)w')'=0,\quad t\geq 1, \tag{22}$$

whose principal solution is

$$w_0(t)=\frac{1}{A(1)}A(t), \tag{23}$$

where A is given in (15). Clearly, w_0 satisfies the boundary conditions:

$$w_0(1)=1,\ w_0(t)>0,\ w_0'(t)<0 \text{ on } [1,\infty),\ \lim_{t\to\infty}w_0(t)=0.$$

Put

$$\beta=\alpha\Phi_M(1/\alpha)=\frac{\alpha}{\sqrt{\alpha^2-1}}. \tag{24}$$

By applying Lemma 2.4, we get for $t \in [1, \infty)$

$$\frac{w_0'(t)}{w_0(t)} \le \frac{1}{\beta}\frac{z_0'(t)}{z_0(t)} \le 0,$$

or, taking into account that $0 < w_0(t) \le 1$,

$$w_0(t)^\beta \le w_0(t) \le z_0(t)^{1/\beta}.$$

In the Fréchet space $C([1, \infty), \mathbb{R}^2)$, consider the subsets given by

$$\Omega = \left\{(u, v) \in C([1, \infty), \mathbb{R}^2) : \, (w_0(t))^\beta \le u(t) \le (z_0(t))^{1/\beta}, \, |v(t)| \le \frac{1}{\alpha}\, a(t)\right\},$$

and

$$S = \left\{(x, y) \in C([1, \infty), \mathbb{R}^2) : x(1) = 1, \, x(t) > 0, \int_1^\infty \frac{1}{a(t)x^2(t)} dt = \infty\right\}. \quad (25)$$

Since $w_0(1) = z_0(1) = 1$ and $z_0(t) \le 1$, for any $(u, v) \in \Omega$ we get $u(1) = 1, u(t) \le 1$.

For any $(u, v) \in \Omega$, consider the linear equation

$$\left(\sqrt{a^2(t) - v^2(t)}\; x'\right)' + b(t)\frac{F(u(t))}{u(t)}\, x = 0. \quad (26)$$

Since

$$a(t) \ge \sqrt{a^2(t) - v^2(t)} \; \ge \frac{\sqrt{\alpha^2 - 1}}{\alpha}\, a(t), \quad (27)$$

equation (21) is a majorant of (26), and, by Lemma 2.4, (26) is disconjugate on $[1, \infty)$. Let x_{uv} be the principal solution of (26), such that $x_{uv}(1) = 1$. In virtue of Lemma 2.5, x_{uv} is positive nonincreasing on $[1, \infty)$. Put

$$x_{uv}^{[1]} = \sqrt{a^2(t) - v^2(t)} x_{uv}', \quad (28)$$

and let $\mathcal{T}$ be the operator which associates to any $(u, v) \in \Omega$ the vector $(x_{uv}, x_{uv}^{[1]})$, that is

$$\mathcal{T}(u, v)(t) = (x_{uv}(t), x_{uv}^{[1]}(t)) \; .$$

In view of Lemma 2.2 and (27), we have $\mathcal{T}(u, v) \in S$.

Equations (21) and (22) are a majorant and a minorant of (26), respectively. Applying Lemma 2.4 to (21) and (26), from (27), we obtain

$$a(t)\frac{x_{uv}'(t)}{x_{uv}(t)} \le \sqrt{a^2(t) - v^2(t)}\; \frac{x_{uv}'(t)}{x_{uv}(t)} \le \frac{\sqrt{\alpha^2 - 1}}{\alpha} a(t)\frac{z_0'(t)}{z_0(t)} \le 0.$$

Thus

$$x_{uv}(t) \le (z_0(t))^{1/\beta}.$$

Similarly, applying Lemma 2.4 to equations (22) and (26), we obtain

$$a(t)\frac{w_0'(t)}{w_0(t)} \le \sqrt{a^2(t)-v^2(t)}\frac{x_{uv}'(t)}{x_{uv}(t)} \le \frac{\sqrt{\alpha^2-1}}{\alpha}a(t)\frac{x_{uv}'(t)}{x_{uv}(t)}. \tag{29}$$

Hence

$$(w_0(t))^\beta \le x_{uv}(t),$$

where β is given in (24).

To prove that $\mathcal{T}$ maps Ω into itself, we have to show that

$$|x_{uv}^{[1]}(t)| \le \frac{1}{\alpha}a(t). \tag{30}$$

From (28) and (29) we obtain

$$\frac{|x_{uv}^{[1]}(t)|}{x_{uv}(t)} = \sqrt{a^2(t)-v^2(t)}\frac{|x_{uv}'(t)|}{x_{uv}(t)} \le a(t)\frac{|w_0'(t)|}{w_0(t)}. \tag{31}$$

In view of (23) we get

$$\frac{|w_0'(t)|}{w_0(t)} = \frac{1}{a(t)A(t)}.$$

Thus, from (31), since $0 < x_{uv}(t) \le 1$, we have

$$|x_{uv}^{[1]}(t)| \le \frac{1}{A(t)}x_{uv}(t) \le \frac{1}{A(t)},$$

and, in virtue of (14), the inequality (30) follows.

In order to apply Theorem 3.2, let us show that, if $\{(u_n, v_n)\}$ converges in Ω and $\{\mathcal{T}(u_n, v_n)\}$ converges to $(\overline{x}, \overline{y}) \in \Omega$, then $(\overline{x}, \overline{y}) \in S$. Clearly, $\overline{x}$ is positive for $t \ge 1$ and $\overline{x}(1) = 1$. Thus, it remains to prove that

$$\int_1^\infty \frac{1}{a(t)\overline{x}^2(t)}dt = \infty. \tag{32}$$

Since $\overline{\mathcal{T}(\Omega)} \subset \overline{\Omega} = \Omega$, we have $0 < \overline{x}(t) \le (z_0(t))^{1/\beta}$, and $\lim_{t\to\infty}\overline{x}(t) = 0$. Further, since $\{\mathcal{T}(u_n, v_n)\}$ converges to $(\overline{x}, \overline{y})$ uniformly in every compact of $[1, \infty)$, the function $\overline{x}$ is a solution of (26) for some $u = \overline{u}, v = \overline{v}$ such that $(\overline{u}, \overline{v}) \in \Omega$. Applying (27) and Lemma 2.3, there exist $T \ge 1$ and a constant $k > 0$ such that $\overline{x}(t) \le kA(t)$ on $[T, \infty)$, where A is given in (15). Thus

$$\int_T^t \frac{1}{a(s)\overline{x}^2(s)}ds \ge \frac{1}{k^2}\left(\frac{1}{A(T)} - \frac{1}{A(t)}\right)$$

and (32) is satisfied. Applying Theorem 3.2, the operator $\mathcal{T}$ has a fixed point $(\bar{x}, \bar{y}) \in \Omega \cap S$ and $\bar{x}$ is a solution of (1). $\square$

Now, we consider the case of the Minkowski curvature operator. The following holds.

THEOREM 3.3. *Assume that (H_i), i=1,2,3, are verified and let (14) be satisfied. If the linear equation*

$$(a(t)z')' + \bar{F}\, b(t)z = 0, \quad t \geq 1. \tag{33}$$

has the principal solution z_0 positive nonincreasing on $[1, \infty)$, then the BVP (2) has at least one solution.

Proof. The proof is similar to the one given in Theorem 3.1. Jointly with (33), consider the equation (22). Reasoning as in the proof of Theorem 3.1, we obtain $w_0(t) \leq z_0(t)$, where w_0 and z_0 are the principal solutions of (22) and (33), respectively, such that $w_0(1) = z_0(1) = 1$. Since z_0 is positive nonincreasing on $[1, \infty)$, we obtain

$$(w_0(t))^{\beta} \leq (z_0(t))^{1/\beta},$$

where $\beta = \alpha/\sqrt{\alpha^2 - 1} > 1$. Let $\Omega_1 \subset C([1, \infty), \mathbb{R}^2)$ be the set

$$\Omega_1 = \left\{(u, v) \in C([1, \infty), \mathbb{R}^2) : (w_0(t))^{\beta} \leq u(t) \leq (z_0(t))^{1/\beta},\ |v(t)| \leq \frac{\beta}{\alpha}\, a(t)\right\},$$

and for any $(u, v) \in \Omega_1$, consider the linear equation

$$\left(\sqrt{a^2(t) + v^2(t)}\ x'\right)' + b(t)\frac{F(u(t))}{u(t)}\, x = 0. \tag{34}$$

Let x_{uv} be the principal solution of (34) such that $x_{uv}(1) = 1$. Then $(x_{uv}, x_{uv}^{[1]}) \in S$, where S is given in (25). Since equation (22) is equivalent to

$$(\beta a(t)w')' = 0,$$

which is a minorant of (34), the assertion follows by using a similar argument to the one in the proof of Theorem 3.1, with minor changes. The details are left to the reader. $\square$

4. Applications and examples

Theorem 3.1 requires that the principal solution of (16) is positive nonincreasing on the whole half-line $[1, \infty)$. Lemma 2.5 can be used to assure this property if a majorant of (16) exists, whose principal solution is known. A similar argument holds for the conditions which are required in Theorem 3.3 for (33). In the following, some applications in this direction are presented.

Prototypes of a Sturmian majorant equation, for which the principal solution is positive nonincreasing on the whole interval $[1,\infty)$, can be obtained from the Riemann-Weber equation

$$v'' + \frac{1}{4(t+1)^2}\left(1+\frac{1}{\log^2(t+1)}\right)v = 0, \tag{35}$$

or from the Euler equation

$$v'' + \frac{1}{4t^2}v = 0. \tag{36}$$

Indeed, equation (35) is disconjugate on $(0,\infty)$, see [18, page 20]. Thus, from Lemma 2.1, the principal solution v_0 of (35) is positive on $[1,\infty)$. Since v_0 is concave for any $t \ge 1$, then $v_0'(t) > 0$ on $[1,\infty)$. Set

$$y_0(t) = v_0'(t),$$

a standard calculation shows that y_0 is solution of the linear equation

$$\left(\frac{4(t+1)^2\log^2(t+1)}{1+\log^2(t+1)}y'\right)' + y = 0. \tag{37}$$

Moreover, in view of [12, Theorem 1], y_0 is the principal solution of (37) and $y_0'(t) = v_0''(t) < 0$.

A similar argument holds for (36). Equation (36) is nonoscillatory and the principal solution is

$$v_0(t) = \sqrt{t},$$

see, e.g., [26, Chap. 2.1]. Hence, the function

$$y_0(t) = \frac{1}{2}\frac{1}{\sqrt{t}}$$

is the principal solution of the linear equation

$$(4t^2y')' + y = 0, \tag{38}$$

and $y_0'(t) < 0$.

Fix $\lambda > 0$. Equations (37) and (38) are equivalent to

$$\left(\lambda\frac{4(t+1)^2\log^2(t+1)}{1+\log^2(t+1)}y'\right)' + \lambda y = 0,$$

and

$$(4\lambda t^2y')' + \lambda y = 0,$$

respectively. Now, from Lemma 2.5 and Theorem 3.1, we obtain the following.

COROLLARY 4.1. *Let (H_i), i=1,2,3, be verified. Assume that there exists $\lambda > 0$ such that for $t \geq 1$*

$$a(t) \geq \min\left\{\lambda \frac{4(t+1)^2 \log^2(t+1)}{1+\log^2(t+1)},\quad 4\lambda t^2\right\}, \qquad \frac{\alpha}{\sqrt{\alpha^2-1}}\,\bar{F}b(t) \leq \lambda, \tag{39}$$

where $\bar{F}$ and α are defined in (13) and (14), respectively. If at least one of the inequalities in (39) is strict on a subinterval of $[1,\infty)$ of positive measure and (14) is verified, then the BVP (1) has at least one solution.

A similar result can be formulated for the problem (2).

COROLLARY 4.2. *Let (H_i), i=1,2,3, be verified. Assume that there exists $\lambda > 0$ such that for $t \geq 1$*

$$a(t) \geq \min\left\{\lambda \frac{4(t+1)^2 \log^2(t+1)}{1+\log^2(t+1)},\quad 4\lambda t^2\right\}, \qquad \bar{F}b(t) \leq \lambda, \tag{40}$$

where $\bar{F}$ is defined in (13). If at least one of the inequalities in (40) is strict on a subinterval of $[1,\infty)$ of positive measure and (14) is verified, then the BVP (2) has at least one solution.

Corollary 4.1 and Corollary 4.2 require the boundedness of b. Nevertheless, our results can be applied also when $\limsup\limits_{t\to\infty} b(t) = \infty$, as the following shows.

COROLLARY 4.3. *Let (H_i), i=1,2,3, be verified.*

(i_1) Assume that (14) holds, and that there exists $\lambda > 0$ such that for every $t \geq 1$ and some $n \geq 1$

$$a(t) \geq \lambda t^{n+2}, \qquad \frac{\alpha}{\sqrt{\alpha^2-1}}\,\bar{F}b(t) \leq n\lambda t^n, \tag{41}$$

where $\bar{F}$ is defined in (13). If at least one of the inequalities in (41) is strict on a subinterval of $[1,\infty)$ of positive measure, then the BVP (1) has at least one solution.

(i_2) Assume that (14) holds, and that there exists $\lambda > 0$ such that for every $t \geq 1$ and some $n \geq 1$

$$a(t) \geq \lambda t^{n+2}, \qquad \bar{F}b(t) \leq n\lambda t^n, \tag{42}$$

where $\bar{F}$ is defined in (13). If at least one of the inequalities in (42) is strict on a subinterval of $[1,\infty)$ of positive measure, then the BVP (2) has at least one solution.

Proof. Claim (i_1). For any $\lambda > 0$ the function $v_0(t) = t^{-n}$ is a solution of the linear equation

$$(\lambda t^{n+2} v')' + n\lambda t^n v = 0,\ t \geq 1. \tag{43}$$

Moreover, in view of Lemma 2.2, v_0 is the principal solution. Since, in view of (41), equation (43) is a Sturmian majorant of (16), from Lemma 2.5 the principal solution of (16) is positive nonincreasing on $[1, \infty)$.Thus, the assertion follows by Theorem 3.1. The proof of Claim *(i_2)* follows in the same way from Theorem 3.3. □

The following examples illustrate our results.

Example 4.4. Consider the equation with the Minkowski mean curvature operator

$$\big(2\pi(t+2)^2 \log^2(t+4)\Phi_M(x')\big)' + \frac{|\sin t|}{t} x^3 = 0, \quad t \geq 1. \tag{44}$$

It is easy to show that assumptions (3) and (4) are satisfied. Moreover, we have

$$\int_t^\infty \frac{1}{(s+2)^2 \log^2(s+4)} ds \geq \int_t^\infty \frac{1}{(s+2)^3} ds = \frac{1}{2(t+2)^2}.$$

Then

$$a(t)A(t) \geq \frac{1}{2} \log^2(t+4) \geq \frac{\log^2 5}{2} \simeq 1.2951$$

and (14) holds. Since

$$2\pi(t+2)^2 \log^2(t+4) \geq \frac{4(t+1)^2 \log^2(t+1)}{1+\log^2(t+1)}, \quad b(t) \leq \frac{1}{t} \leq 1,$$

conditions (40) hold with $\lambda = 1$. Thus, by Corollary 4.2, equation (44) has at least one solution x which satisfies the boundary conditions

$$x(1) = 1,\ x(t) > 0,\ x'(t) \leq 0,\ \lim_{t\to\infty} x(t) = 0. \tag{45}$$

Example 4.5. Consider the equation with the Euclidean mean curvature operator

$$\big(6(t+1)^2 \Phi_E(x')\big)' + \frac{|\sin t|}{t}\, x^3 = 0, \quad t \geq 1. \tag{46}$$

Assumptions (3) and (4) are satisfied. Further, we have $\bar{F} = 1$ and

$$a(t)A(t) = t + 1 \geq 2.$$

Thus, $\alpha = 2$ and (14) holds. Moreover, since $\beta = \alpha/\sqrt{\alpha^2 - 1} = 2/\sqrt{3}$, conditions (39) hold with $\lambda = 3/2$. Using Corollary 4.1 we get that the equation (46) has at least one solution x which satisfies the boundary conditions (45).

Example 4.6. Consider the equation with the Minkowski mean curvature operator

$$\big(3(t+3)^4\Phi_M(x')\big)' + 2t|\sin t + \cos t|\; x^{2n+1} = 0, \quad t \ge 1. \tag{47}$$

Similarly to Example 2, also for (47) assumptions (3) and (4) are satisfied. Further, we have $\bar{F} = 1$. Moreover

$$a(t)A(t) = \frac{t+3}{3} \ge \frac{4}{3}$$

and so (14) holds. Since

$$3(t+3)^4 \ge 3t^3, \quad 2t|\sin t + \cos t| \le 2\sqrt{2}t < 3\ t$$

and these inequalities are strict on a subinterval of $[1,\infty)$ of positive measure, then by Corollary 4.3-(i_2) with $n = 1$ and $\lambda = 3$, the equation (47) has at least one solution x which satisfies the boundary conditions (45). Observe that in equation (47) the function b is unbounded.

We close the section with some remarks concerning our assumptions.

Remark 4.7. If

$$\liminf_{t\to\infty} a(t) = 0, \tag{48}$$

then the BVP (1) is not solvable. Indeed, let x be a nonoscillatory solution of (18), $x(t) > 0$ for $t \ge t_0 \ge 1$. Then the function $a(t)\Phi_E(x'(t))$ is nonincreasing on $[t_0,\infty)$ and the limit

$$\lim_{t\to\infty} a(t)\Phi_E(x'(t))$$

exists. In virtue of (48), since Φ_E is bounded, we get

$$\lim_{t\to\infty} a(t)\Phi_E(x'(t)) = 0,$$

which implies $x'(t) > 0$ in a neighborhood of infinity. Thus, the BVP (1) is not solvable.

Remark 4.8. The assumption (4) guarantees that the principal solution y_0 of the majorant equation (16) satisfies

$$\lim_{t\to\infty} \frac{y_0(t)}{A(t)} = c, \quad 0 < c < \infty,$$

see Lemma 2.3. This property is needed for obtaining the continuity of the fixed point operator, see [16, Theorem 1]. If (3) holds and

$$\int_1^\infty b(t) \int_t^\infty \frac{1}{a(s)}\, ds dt = \infty,$$

then all solutions of (16) tend to zero as $t \to \infty$. In this situation, it seems hard to obtain the continuity of $\mathcal{T}$, since the solutions $\mathcal{T}(x_n)$ are principal solutions, but the sequence $\{\mathcal{T}(x_n)\}$ could converge to a nonprincipal solution.

Remark 4.9. BVPs on the half-line for equations involving the operator Φ_E or Φ_M with sign-changing coefficient have attracted very minor attention, especially when the boundary conditions concern the behavior of solutions on the whole half-line $[1,\infty)$. According to our knowledge, the only paper in this direction is [19], in which the existence of a global positive solution, bounded away from zero, is obtained. It should be interesting to extend Theorem 3.1 and Theorem 3.3 for obtaining the solvability of (1) and (2) when the function b does not have fixed sign.

Remark 4.10. Analougous results to the ones obtained in Theorems 3.1 and 3.3 can be formulated also for the BVP (5). Nevertheless the existence of solutions of (5) requires weaker assumptions than those in Theorems 3.1 or 3.3. Indeed, in this situation the operator $\mathcal{T}$ is defined via the linear equation

$$(a(t)x')' + b(t)\frac{F(u(t))}{u(t)}\,x = 0. \tag{49}$$

This fact permit us to simplify the above argument, by considering the set Ω as a subset of $C([1,\infty),\mathbb{R})$ instead of $C([1,\infty),\mathbb{R}^2)$, because *a-priori* bounds for the quasiderivative are not necessary. In addition, no assumptions on α are needed. The details are left to the reader.

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