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Effective Field Theory Approach to Topological Insulators in two and three Dimensions

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A Pergine Valdarno,
il mio paese, il paese più bello del mondo.

Overview

A recent research activity in theoretical condensed matter physics concerns the *topological phases of matter*, that are many body collective states with strong quantum effects occurring e.g. at very low temperatures and/or under high magnetic fields. In these states, the interacting electrons give rise to fascinating quantum macroscopic phenomena, whose understanding involves fundamental aspects of quantum field theory, mathematical physics and geometry. Therefore, the topological phases of matter are interesting and relevant for theoretical physics in a very broad sense.

The Ginzburg–Landau theory provides the framework for understanding the phases of matter such as ferromagnets, superfluids and superconductors. In the early eighties other states have been found that cannot be explained by this theory. The topological states realize a new kind of order, the so called *Wen's topological order*, which measures the influence of the topology of space on the collective behavior of correlated electrons. In mathematics, topology concerns properties that are preserved under continuous deformations, such as stretching and bending, but not tearing or gluing. In condensed matter, topology specifies the robustness of the quantum phenomena under deformations of the Hamiltonian as well as the Aharonov-Bohm phases that are associated to excitations.

The best known example of a topological phase is given by the quantum Hall effect (QHE), a two-dimensional electron system subjected to strong magnetic field and placed at very low temperatures. In these conditions, the system shows constant plateaus in the transverse conductivity σ_H, called Hall conductivity, which takes very precise quantized values in units of e^2/h. These can be integer or fractional, corresponding to the integer and fractional QHE, respectively. The Hall states possess an energy gap for bulk excitations, but also massless excitations at the edge of the system that are chiral due to the magnetic field. Furthermore, the fractional states display anyon excitations with fractional charge and fractional exchange statistics.

Recently many other topological phases of matter have been discovered; these are also gapped in the bulk and possess massless boundary excitations. A classification of these states in ten universality classes has been achieved for free electron (band) systems. This follows from the analysis of general quadratic Hamiltonians, constrained by time-reversal symmetry, charge-conjugation and chiral symmetry. The ten possible phases are characterized by topological numbers. The present challenge is to extend the classification to interacting electron systems. For example, the integer $\mathbb{Z}$ classification of topological superconductors of class DIII in three spatial dimensions is known to reduce to $\mathbb{Z}_{16}$ in presence of interactions.

The aim of this thesis is to describe the topological phases of matter by means of effective field theories. In many cases, this approach provides an hydrodynamics picture of the topological excitations that is valid at energies below the bulk gap, realizes the underlying symmetries and is independent of microscopic details. The field theory approach reproduces the classification of non-interacting phases and, moreover, allows to discuss the effect of interactions.

We begin this thesis by reviewing the effective field theory description of the QHE. This involves the Chern-Simons theory, a topological gauge theory that accounts for the responses of the system, such as the Hall conductivity and the fractional charge and statistics of anyonic excitations. In presence of a boundary, the Chern-Simons theory is not gauge invariant and needs additional massless degrees of freedom located at the boundary. These are described by a conformal field theory (CFT) in $(1+1)$ dimensions, whose relation with the bulk Chern-Simons theory is well established. The boundary modes of the QHE states are stable: owing to their chirality, they cannot self-interact and acquire a mass.

We first review the edge theory of QHE states, the CFT of a compactified free boson, whose canonical quantization provides an exact description of interacting Hall states with Abelian fractional statistics. Moreover, we recall that the Hall current is described in the CFT by the non-conservation of the boundary charge, namely by a chiral anomaly. On the other hand, the charge is conserved in the whole theory of bulk and boundary: thus, the edge anomaly is cancelled by the bulk current of the Chern-Simons theory, through a mechanism called *anomaly inflow*. We remark that in this thesis we will often make use of anomalies, i.e. of classical symmetries broken by quantum effects. Being related to topological quantities, anomalies allow to explain and characterize the robustness of topological phases of matter.

The main topic of this thesis is to understand, by means of effective field theory, the stability of boundary excitations of two and three dimensional *time-reversal invariant topological insulators*. In these systems, the boundary massless states can self-interact and become massive, leading to a decay of topological phases onto trivial insulators. In literature, systems like these are called *symmetry protected topological phases*, due to the central role played by symmetry. In particular, time-reversal symmetry forbids interactions that give mass to excitations when certain conditions are met.

Two-dimensional topological insulators have been first analyzed in free fermion systems using band theory: they are characterized by a $\mathbb{Z}_2$ index $(-1)^N$, where N is the number of energy-level crossings between different bands. In the pioneering works of Fu, Kane and Mele, the odd (even) N cases were shown to be stable (unstable) by means of symmetry argument based on an adiabatic flux insertion. The stability analysis was then extended to interacting Abelian systems by Levin and Stern, which related the $\mathbb{Z}_2$ index to certain properties of edge excitations.

One result of this thesis is the generalization of the previous analysis. Using CFT techniques, we derive the partition functions of edge states in the space-time geometry of the torus. The antiperiodic and periodic boundary conditions in space and time give the four spin sectors of relativistic fermionic theories, such as the Neveu-Schwarz and Ramond sectors. Next, we reformulate the flux insertion argument due to Kane and coworkers in terms of transformation properties of partition functions. We then prove the general validity of the Levin-Stern $\mathbb{Z}_2$ index for interacting Abelian and non-Abelian topological insulators. In particular, our analysis clarifies that the stability is associated to the anomaly of the $\mathbb{Z}_2$ spin-parity, that is the fermion index of edge excitations. Furthermore, we point out that partition functions of stable topological insulators have interesting geometrical properties. Under modular transformations, the discrete coordinate changes respecting the periodicities of the torus, the four partition functions transform among themselves. We show that the stability is associated to the impossibility of combining these functions into a modular invariant expression. Thus, a discrete gravitational anomaly is accompanying the spin-parity anomaly.

Summarizing, an anomalous system possesses massless protected boundary excitations. This leaves open the question of whether a non-anomalous system does become fully gapped. In this thesis we find the interactions that completely gap the edge modes of unstable non-Abelian topological insulators: for example, we analyze the time-reversal invariant Pfaffian topological insulator, that is made of two copies of the Pfaffian QHE state with opposite chiralities.

Time-reversal reversal invariant topological insulators in three space dimensions have also been analyzed using band theory. Extending their flux argument, Fu, Kane and Mele showed that they are classified again by a $\mathbb{Z}_2$ topological index. These phases have a bulk energy gap and surface states protected by time-reversal symmetry consisting of an odd number of massless Dirac fermions in $(2+1)$ dimensions. At first glance this results might seem in contradiction with the known parity and time-reversal anomaly in $(2+1)$ dimensions. However, we verify that a cancellation between bulk and boundary terms of the effective action restores the symmetries. This cancellation is different from the anomaly inflow mechanism occurring in two dimensions.

Another result of this thesis is the reformulation of the $\mathbb{Z}_2$ stability criterion for three-dimensional topological insulators by studying the partition functions of boundary fermions. In the geometry of the three dimensional space-time torus we find eight partition functions, corresponding to periodic and antiperiodic boundary conditions. These are the spin sectors of the fermionic theory, among which we recognize the corresponding Neveu-Schwarz and Ramond sectors. Studying their transformations under the modular group and flux insertions, we show that the stability is again associated to the anomaly of the $\mathbb{Z}_2$ spin-parity. Furthermore, a modular invariant partition function cannot be constructed.

Recently, interacting three-dimensional topological insulators were also introduced and theoretically analyzed, showing that they possess fractional charge and vortex excitations. The final part of this thesis is dedicated to the effective field theory description of interacting topological insulators provided by the BF gauge theory. This is a time-reversal invariant topological gauge theory depending on two hydrodynamic fields, describing particles and vortex excitations. The theory depends on a coupling constant K, that is an odd integer. We first verify that the stable $K=1$ bosonic theory matches the fermionic description for the topological properties.

In presence of a boundary, an additional surface action should be introduced to compensate for the gauge non-invariance of the BF bulk theory, in full analogy to what happens in lower dimension. We thus study the corresponding bosonic surface theory and the dynamics it can support, respecting time-reversal invariance. We identify the fermionic excitations within the bosonic theory by studying the partition functions, and thus we are able to extend the stability argument to this theory. Of course, an exact map between fermions and bosons cannot be achieved in $(2+1)$ dimensions; nonetheless, we obtain some exact results that do not depend on the details of interactions.

After the canonical quantization of the compactified bosonic surface field, we calculate the partition functions on the three torus. We find a set of eight functions that transform under the modular group and flux insertions exactly as the fermionic functions. Although the bosonic and fermionic expressions are different in three dimensions, they become equal under dimensional reduction, owing to the exact map between bosons and fermions in $(1+1)$ dimensions. Upon comparison, we can assign fermionic numbers to the bosonic states and, thus, define the corresponding bosonic Neveu-Schwarz and Ramond sectors. Using this identification, we reformulate the $\mathbb{Z}_2$ stability criterion explained before and extend it for $K > 1$, namely for topological phases possessing fractionally charged particles and vortex excitations.

This thesis is organized as follows: In Chapter 1 we recall some general aspects of QHE and discuss the effective theories of bulk and boundary. In Chapter 2 we introduce the Fu-Kane-Mele stability argument of two dimensional topological insulators; then we present our reformulation in terms of transformations properties of partition functions for edge excitations (see work [1] in the following list of publications). Next, we discuss the extension of the stability criterion to Abelian and non-Abelian topological states [1]. In Chapter 3 we find the edge interactions that completely gap the edge spectrum of non-Abelian topological states, such as the Pfaffian and Read-Rezayi states (see work [2]). In Chapter 4 we discuss three-dimensional fermionic topological insulators and the anomaly cancellation between bulk and boundary. Next, we show our reformulation of the three-dimensional version of the Fu-Kane-Mele stability argument: we calculate the partition functions of the surface excitations and relate the $\mathbb{Z}_2$ stability criterion to their modular transformations (see work [3]). In Chapter 5 we discuss the topological BF theory in three space dimensions. We calculate the surface effective action, the bosonic partition functions and their modular transformations; thus, we present our results on exact properties of bosonization in (2+1) dimensions concerning fermion parity and spin sectors (see work [3]). Using these results, we extend the Fu-Kane-Mele stability analysis to interacting topological states in three space dimensions [3]. In Chapter 6 we give our conclusions.

List of publications

This thesis is based on the following publications

[1] A. Cappelli, E. Randellini,
"Partition Functions and Stability Criteria of Topological Insulators",
JHEP **1312** (2013) 101.

[2] A. Cappelli, E. Randellini,
"Stability of Topological Insulators with Non-Abelian Edge Excitations",
J. Phys. A: Math. Theor. **48** (2015) 105404.

[3] A. Cappelli, E. Randellini, J. Sisti,
"Three-dimensional Topological Insulators and Bosonization",
JHEP **05** (2017) 135.

Other publications not included in this thesis

[4] A. Cappelli, E. Randellini,
"Multipole expansion in the quantum Hall effect"
JHEP **1603** (2016) 105.

Contents

Chapter 1

The quantum Hall effect

This chapter is devoted to a brief introduction of the quantum Hall effect (QHE). After showing the main properties of this quantum phenomenon, we will focus on its low energy effective field theory description. We will see that the topological Chern-Simons gauge theory describes excitations with fractional charge and fractional statistics.

The QHE is a quantum system of electrons in a planar geometry which is characterized by a gapped insulating bulk and conducting massless degrees of freedom living at the boundary. We will analyze these boundary modes using conformal field theory methods and construct their partition function. We will show that these excitations have two descriptions, one bosonic and one fermionic, owing to the bosonization map in $(1+1)$ dimensions.

In the final part of this chapter we will show that a quantum anomaly is associated to the edge effective field theory. Actually, this provides another way to characterized the QHE and its relation to topological invariants. It turns out that similar arguments based on anomalies are also useful for other topological phases of matter and will be used very often throughout this thesis.

1.1 Integer and fractional QHE

The classical Hall effect is observed in a conducting strip subjected to an orthogonal magnetic field B_0. When an electric current I flows in the strip, a difference of voltage V_H, the Hall voltage, is detected in the direction orthogonal to the current, see Fig. 1.1. If ρ indicates the electron density in the strip, we can define the orthogonal resistance, or Hall resistance, as

$$R_H = \frac{V_H}{I} = \frac{B_0}{\rho e c}. \tag{1.1}$$

This linear dependence on the magnetic field is modified by quantum effects when the system is placed at extreme low temperatures (~ 100 mK) and high magnetic fields (~ 10 Tesla) [5]. In these conditions, the Hall resistance shows some constant plateaus at the values

$$R_H = \frac{h}{e^2}\frac{1}{\nu}, \qquad \nu = 1, 2, \cdots, \frac{1}{3}, \frac{2}{5}, \cdots \tag{1.2}$$

where h is the Planck constant and ν is the *filling fraction*, an integer or rational number, that is equal to the ratio between the number of electrons and the degeneracy of the Landau

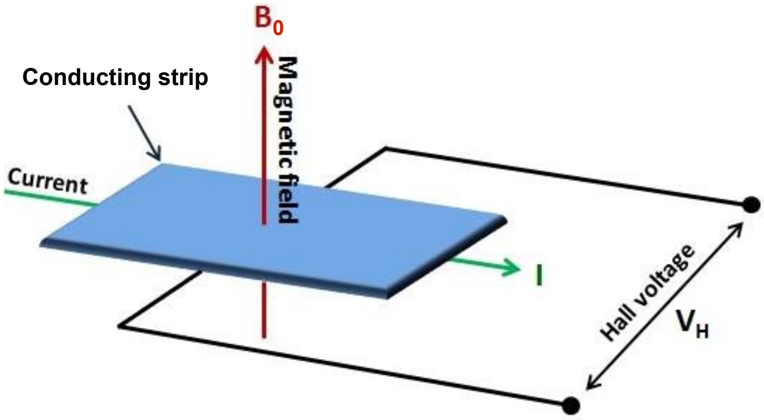


Figure 1.1: Set-up for the Hall effect

levels, see Fig. 1.2. At the same time, the longitudinal conductance vanishes, because the system is insulating in the bulk. At the beginning only integer values of ν were observed, but later fractional values of ν were found, among which the Laughlin sequence $\nu = 1/p$, p odd [5]. These phenomena were called, respectively, integer and fractional QHE and, in both cases, the experimental results showed an high precision of the quantized values, independently of the sample details (universality) [6, 7, 8].

To describe this kind of behavior, the main idea due to Laughlin [9] is that the electrons form an incompressible quantum fluid, namely that the density is constant in the bulk, $\rho(x) = \rho_0$, and there is a gap for density waves. In the following, we shall see this picture clearly applies to the case of n completely filled Landau levels, i.e. for integer QHE with $\nu = n$. For fractional fillings, the idea is that the Coulomb repulsion puts the electrons in the most symmetric configuration compatible with their density, such that the ground state has a gap and is again an incompressible fluid.

For integer filling, we can neglect the Coulomb interaction and also omit the constant Zeeman term for spin polarized electrons. Then, the hamiltonian H for electrons with charge e and mass m is

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2, \tag{1.3}$$

in the symmetric gauge $\mathbf{A} = \frac{1}{2}B(-y, x, 0)$. The classical motion of cyclotron frequency $\omega = eB/mc$ gives rise to quantization of the kinetic energy $E = \hbar\omega n$ and to quantized radii r_k. The corresponding orbits enclose a flux multiple of the quantum unit of flux $\Phi_0 = hc/e$, i.e. $\pi r_k^2 B = k\Phi_0$. The flux quantization implies that the magnetic field has associated a unit of length, the *magnetic length* $\ell_0 = \sqrt{2\hbar c/eB}$. In the following, we shall mainly use the units $\hbar = c = e = \ell_0 = 1$, which imply $\omega = 2$ and $\Phi_0 = 2\pi$.

At the quantum level, the hamiltonian and canonical angular momentum can be written

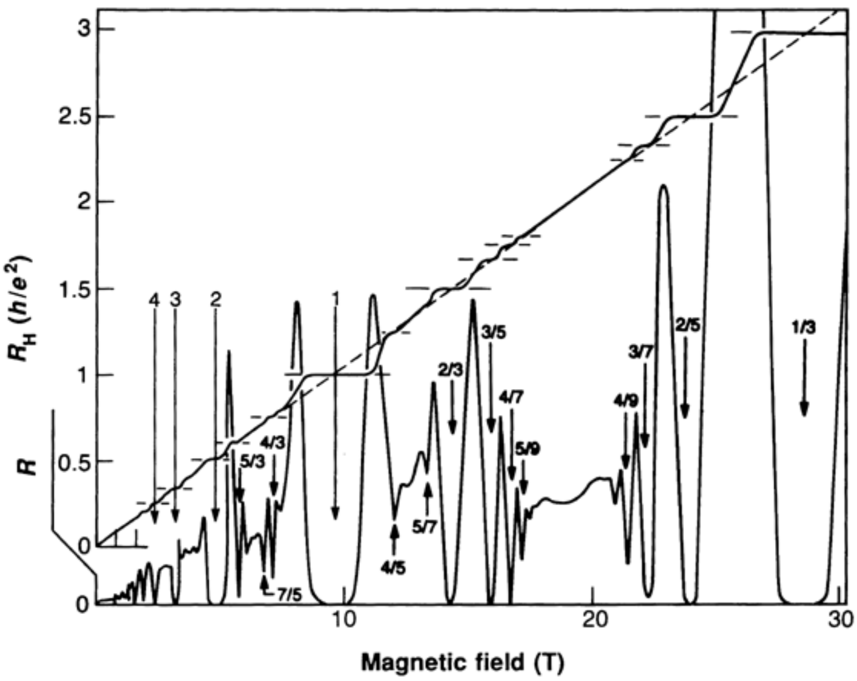


Figure 1.2: Longitudinal (R) and Hall (R_H) resistances as function of the applied magnetic field [8].

in terms of a pair of independent harmonic oscillators [10] [11]:

$$H = \omega \left(a^\dagger a + \frac{1}{2} \right), \qquad J = b^\dagger b - a^\dagger a, \tag{1.4}$$

satisfying the commutation relations $[a, a^\dagger] = 1, [b, b^\dagger] = 1$, with all other commutators vanishing. Starting from the vacuum, satisfying $a\,|0\rangle = b\,|0\rangle = 0$, one finds energy ($a^\dagger$) excitations, the Landau levels, which are degenerate with respect to the angular momentum ($b^\dagger$) excitations. The degeneracy N_g of levels occupying an area $\mathcal{A}$ is equal to the flux passing through it in quantum units, $N_g = \Phi/\Phi_0 = B\mathcal{A}/2\pi$.

When n Landau levels are completely filled, the number of electrons is an integer multiple of the degenerate states for each Landau level, $N = nN_g$. The filling fraction ν is then

$$\nu = \frac{N}{N_g} = N\frac{\Phi_0}{\Phi} = \frac{\rho}{B} = n. \tag{1.5}$$

In this case, when the Fermi energy is placed between the last occupied and the first empty Landau levels, the electrons can not jump in the higher level due to the energy gap $\omega \sim B$ and the system has vanishing longitudinal conductivity. From Eq.(1.5), we express ρ in term of ν, and substituting this in (1.1) we find the quantized values of the Hall resistance $R_H = 2\pi/\nu$.

For $\nu = 1$, the ground state wavefunction is given by the Slater determinant of N one-particle states of the lowest Landau level, that takes the form of the Vandermonde determinant [10, 11]

$$\Psi\big(\{z_i\},\{\bar{z}_i\}\big) = \prod_{0\leqslant i<j\leqslant N} (z_i - z_j)\, \exp\big(-\sum_i |z_i|^2/2\big), \tag{1.6}$$

where z_i are the positions of the n-th electrons. In this state, every orbital angular momentum l is filled. The total angular momentum L is related to the number of electrons N by $N = L + 1$, and the density is constant with the shape of a droplet of radius $R \simeq \sqrt{L}$. The angular momentum can be decreased by moving electrons to the second Landau level: these transitions are forbidden at low temperatures due to the large gap $\omega \propto B$. Therefore the fluid is incompressible.

The states corresponding to fractional values of ν are incompressible due to the Coulomb interaction between electrons, and the gap is a non perturbative effect. One should use effective approaches and trial wavefunctions. For $\nu = 1/p$, with p an odd integer, Laughlin proposed the wave function [12]

$$\Psi_p(\{z_i\},\{\bar{z}_i\}) = \prod_{0\leqslant i<j\leqslant N} (z_i - z_j)^p \exp\big(-\sum_i |z_i|/2\big), \tag{1.7}$$

that describes very well the physics of the fractional QHE states. It approximates accurately the numerical ground state for a large class of repulsive interactions, and the excitations have a finite gap. Laughlin developed the incompressibility picture, as well as the properties of the excitations, by interpreting $|\Psi|^2$ as the classical probability distribution for a two dimensional Coulomb gas of charges, the so called *plasma analogy*. For $\nu = 1/p$, it is know that this plasma is a liquid and that the charge is screened, thus providing a gap; furthermore, the excitations have fractional charge e/p [12]. We shall describe these properties by using effective field theories in the following subsections.

1.2 Chern-Simons bulk effective action

We have seen that the electrons gas forms a state of incompressible quantum fluid that is characterized by universal and robust quantities; in such a system, we expect that the physical properties can be deduced from general considerations of symmetries and conservation laws, that are independent of the detailed microscopic theory. It is then natural to formulate the problem in terms of the effective field theory for low-energy excitations [10, 11].

If we consider a system of N fully polarized electrons, with coordinates $\{\mathbf{x}_i\}$ and velocities $\{\mathbf{v}_i\}$, with $i = 1, \cdots, N$, we can define a local density $j_0(\mathbf{x})$ and current $\mathbf{j}_i(\mathbf{x})$ by the expressions

$$j_0(\mathbf{x}) = \sum_{i=1}^{N} \delta^{(2)}(\mathbf{x} - \mathbf{x}_i), \qquad \mathbf{j}_i(\mathbf{x}) = \sum_{i=1}^{N} \mathbf{v}_i \delta^{(2)}(\mathbf{x} - \mathbf{x}_i). \tag{1.8}$$

The condition of local charge conservation means that the 3-vector $j_\mu(x) = (j_0, \mathbf{j})$, with $x = (t, \mathbf{x})$, obeys the continuity equation

$$\partial_\mu j^\mu = 0 \iff \partial_t j_0 + \nabla \cdot \mathbf{j} = 0. \tag{1.9}$$

Since the current j^μ is conserved, it can be expressed as the dual of a gauge field $a_\mu(x)$, called the hydrodynamic field,

$$j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho, \tag{1.10}$$

where $\epsilon^{\mu\nu\rho}$ is the completely antisymmetric symbol. This current is invariant under a gauge transformation of the a_μ field, i.e. $a_\mu \;\to\; a_\mu + \partial_\mu\Lambda$.

In the effective field theory approach, the main step is to guess the form of the low energy action with the desired symmetries, in this case an Abelian gauge theory for the field a_μ (1.10). Since the external magnetic field breaks parity and time reversal symmetries, the leading term with lowest number of derivatives in the $(2+1)$ dimensional action is of the Chern-Simons type. This captures the main physical properties of the QHE with $\nu = 1/p$. We write:

$$S[a,A] = -\frac{p}{4\pi}\int_\Omega d^3x\ \epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \frac{1}{2\pi}\int_\Omega d^3x\ \epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho, \tag{1.11}$$

where the first term is the Chern-Simons action for a_μ and the second term is the coupling between the matter current j^μ and the external electromagnetic field A_μ. The Chern-Simons action for the a_μ field is a topological action, because it does not depend on the metric and does not describe propagating degrees of freedom in the bulk. Therefore, this description is valid for energies below the bulk gap in the Hall system.

Upon integrating the field a_μ, we find the induced effective action

$$S_{\rm ind}[A] = \frac{1}{4\pi p}\int_\Omega d^3x\ \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho, \tag{1.12}$$

that describes the response of the system to varying the external field A_μ. The induced electromagnetic current is given by

$$J^i = \frac{\delta S_{\rm ind}[A]}{\delta A_i} = \frac{1}{2\pi p}\epsilon^{ij}\mathcal{E}_j = \sigma_H\epsilon^{ij}\mathcal{E}_j, \qquad \rho = \frac{\delta S_{\rm ind}[A]}{\delta A_0} = \frac{1}{2\pi p}\mathcal{B} = \sigma_H\left(B_0 + \delta B\right), \tag{1.13}$$

where $\mathcal{E}_j = \partial_i A_0 - \partial_0 A_i$ and $\mathcal{B} = \partial_i A_j - \partial_j A_i$, are, respectively, the electric and magnetic field. One recovers the Hall conductivity $\sigma_H = \nu/2\pi$, $\nu = 1/p$, and the density of Laughlin states.

The quasiparticles/quasiholes are the low-energy excitations of the incompressible fluid, whose world-lines can be represented by a set of currents $\mathcal{J}_\mu$, that couple to the hydrodynamic field a_μ. Including these excitations, the effective action becomes

$$S[a,A] = -\frac{p}{4\pi}\int d^3x\ \epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \frac{1}{2\pi}\int d^3x\ \epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho + \int d^3x\ \mathcal{J}^\mu a_\mu. \tag{1.14}$$

Integrating out the a_μ field, we obtain the induced action for the excitations and the external field,

$$S_{\rm ind}[A] = \int_\Omega d^3x\left(\frac{1}{4\pi p}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{1}{p}A_\mu j^\mu - \frac{\pi}{p}\epsilon^{\mu\nu\rho}\mathcal{J}^\mu\frac{\partial_\nu}{\partial^2}\mathcal{J}^\rho\right). \tag{1.15}$$

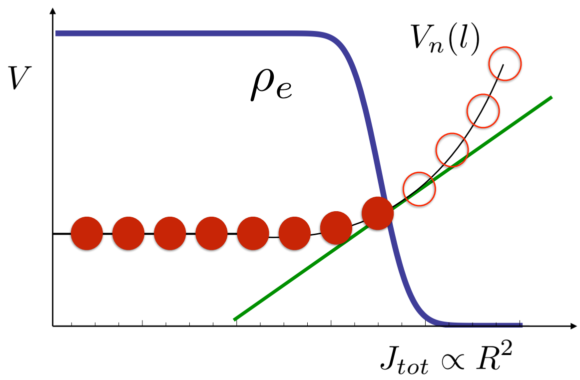


Figure 1.3: The confining potential modifies the Landau levels at the edge. The edge excitations (green line) are massless and chiral.

Computing the electromagnetic current for the case of one static quasiparticle at the origin, i.e. $\mathcal{J}^{\mu} = k(\delta(\mathbf{x}), 0)$, we obtain

$$\rho = \frac{\nu}{2\pi}\mathcal{B} + \frac{k}{p}\delta(\mathbf{x}), \tag{1.16}$$

where k is the charge of the quasiparticle/hole with respect to the hydrodynamic field. Besides the ground state density, the additional term corresponds to the increase of the electron density due to the excitation: its electric charge is $Q_{qp} = ek/p$ and, moreover, it carries k/p units of flux with respect to the a_μ field. The third term in the induced action (1.15) describes the Aharonov-Bohm phase θ_k that arises upon carrying one excitation around another. This is given by

$$\theta_k = 2\pi\frac{k^2}{p}. \tag{1.17}$$

For identical excitations, half of the monodromy defines the statistics phase of the quasiparticles. For $k = 1$, this is fractional, $\theta/\pi = 1/p$, thus showing that the effective Chern-Simons theory describes anyons [13, 14]. Next, the excitation with $k = p$ has a charge and statistics respectively equal to $Q = e$ and $\theta/\pi = p$. Thus for p odd this is nothing but that the electron. Experiments on fractional Hall states have accumulated a definitive evidence of fractionally charged excitations, see for example [15]; the observation of fractional statistics has been announced but it has not been definitely confirmed yet[16].

1.3 Bulk-edge correspondence in the QHE

In a finite geometry, like a disk or an annulus, the confining potential $V(|x|)$ modifies the structure of the Landau levels such that the energy eigenvalues are not longer degenerate, see Fig. 1.3 [17]. Due to incompressibility, a Fermi surface is created at the boundary. Upon expanding the energy around the Fermi surface, one obtains a linear dependence on l, the rescaled boundary angular momentum $J = L + l$, with $L = R^2/2\nu \gg l$ and R the disk radius, Fig. 1.3. The energy spectrum takes the form $E_l \sim vl/R = vk$ of massless excitations propagating on the edge in one direction only, i.e. they are chiral [17].

The presence of a non-trivial boundary physics can be explained using the effective field theory and the connection between bulk and boundary [10, 18]. We consider the QHE system in the geometry of a disk, with boundary circle of length $2\pi R$, that form the space-time cylinder $\mathcal{C} = S^1 \times \mathbb{R}$. In such a geometry, the Chern-Simons action $S[a, 0]$ (1.14) is not gauge invariant: under $a_\mu \to a_\mu + \partial_\mu \Lambda$, it gives the term

$$\delta S = -\frac{p}{4\pi} \int_{\mathcal{C}} d^2x \; \Lambda \left(\partial_0 a_1 - \partial_1 a_0\right), \tag{1.18}$$

where $x^1 = R\theta$ is the coordinate along the boundary of the disk. In order to cancel this term and make sure that the matter current is globally conserved, one introduces boundary degrees of freedom and a boundary action whose gauge transformation cancels δS in (1.18). At the edge of the system the gauge field a_μ can be expressed in terms of a scalar degree of freedom ϕ by $a_\mu = \partial_\mu \phi$, that acquires a dynamics $S_{\text{edge}}[\phi]$ [10]. The correct theory is that of the $(1+1)$ dimensional chiral and massless scalar field with Floreanini-Jackiw action [10, 19]

$$S_{\text{edge}}[\phi] = -\frac{p}{4\pi} \int_{\mathcal{C}} d^2x \;\; \left(\partial_0 + \partial_1\right) \phi \partial_1 \phi. \tag{1.19}$$

The equations of motion,

$$\left(\partial_0 + \partial_1\right) \partial_1 \phi = 0, \tag{1.20}$$

shows that the field is chiral as required (we fix the velocity $v_F = 1$). The gauge invariance of the complete system, $\delta S + \delta S_{\text{edge}} = 0$ is checked by transforming $\phi \to \phi + \Lambda$ and fixing the boundary gauge condition $a_0 + a_1 = 0$.

1.3.1 Conformal field theory of the compactified chiral boson

In this section we study the chiral boson action (1.19). This quadratic action, once quantized, gives rise to a sets of conformal fields with fractional dimensions appropriate to describe the universal long-range properties of the fractional QHE [20].

Rescaling the space-time coordinates as $x \to R\theta$ and $t \to Rt$, the action (1.19), the Hamiltonian and the higher moments read:

$$S_{\text{edge}}[\phi] = -\frac{p}{4\pi} \int_{-\infty}^{+\infty} dt \int_0^{2\pi} d\theta \;\; \left(\partial_0 + \partial_\theta\right) \phi \partial_\theta \phi, \tag{1.21}$$

$$H = \frac{v}{R} L_0, \qquad \text{with} \quad L_0 = \frac{p}{4\pi} \int_0^{2\pi} d\theta \left(\partial_\theta \phi\right)^2, \tag{1.22}$$

$$L_n \equiv \frac{p}{4\pi} \int_0^{2\pi} d\theta \;\left(\partial_\theta \phi\right)^2 \, \exp\left(-in\left(\theta - vt\right)\right). \tag{1.23}$$

We impose the following compactification condition

$$\phi(\theta, t) \equiv \phi(\theta, t) + 2\pi n r, \qquad n \in \mathbb{Z}. \tag{1.24}$$

Namely the field $\phi(\theta)$ maps the edge circle into another circle with radius r.

The following field expansion solves the equations of motion (1.20)

$$\phi(\theta, t) = \phi_0 - \alpha_0 (\theta - vt) + i \sum_{k \neq 0} \frac{\alpha_k}{k} \exp\left(ik(\theta - vt)\right), \tag{1.25}$$

with $\alpha_k^* = \alpha_{-k}$ and $\phi_0 \equiv \phi_0 + 2\pi r$ to satisfy the constraint (1.24). Note that the field expansion contains zero modes ϕ_0, α_0 and oscillating terms that are periodic for $\theta \to \theta + 2\pi$. Imposing canonical commutation relations of the field and its momentum $\Pi(\theta, t) = \delta\mathcal{L}/\delta\dot{\phi} = -p/4\pi\partial_\theta\phi$, i.e. [19, 20]

$$\left[\phi(\theta, t), \Pi(\theta', t)\right] = \frac{i}{2}\delta(\theta - \theta'), \tag{1.26}$$

we infer the corresponding commutation relations of the modes, that are

$$[\phi_0, \alpha_0] = \frac{i}{p}, \qquad [\alpha_n, \alpha_m] = \frac{n}{p}\delta_{n+m,0}. \tag{1.27}$$

Upon quantization, the coefficients ϕ_0, α_0 and α_n become operators acting on a bosonic Fock space, whose ground state $|\Omega\rangle$ is defined as

$$\alpha_n |\Omega\rangle = 0, \qquad n > 0. \tag{1.28}$$

Once defined the ground state, the Hamiltonian L_0 (1.22) takes the following normal ordered form in terms of the oscillating modes

$$L_0 = \frac{p}{2}\alpha_0^2 + p \sum_{k=1}^{\infty} \alpha_{-k}\alpha_k - \frac{1}{24}. \tag{1.29}$$

The higher moments of the Hamiltonian density (1.23) similarly read:

$$L_n = \frac{p}{2} \sum_{k=-\infty}^{\infty} \alpha_{n-k}\alpha_k. \tag{1.30}$$

By using (1.27), we find that the free compactified boson gives rise to a representation of a chiral algebra defined by the following commutation relations

$$[\alpha_n, \alpha_m] = \frac{n}{p}\delta_{n+m,0}, \tag{1.31}$$

$$[L_n, \alpha_m] = -m\alpha_{n+m}, \tag{1.32}$$

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}, \qquad c = 1. \tag{1.33}$$

The first relation is the $U(1)$ Kac-Moody algebra for the generators α_n; the third expression is the Virasoro algebra for the generators L_n of local conformal transformations [21, 22]. As is well known in the CFT literature, the $c-$number term in the right-hand side comes from the conformal anomaly and defines the central charge c, that takes the value $c = 1$ for this model [21, 22].

We now discuss the quantization conditions on the zero modes of the field ϕ, that at $t = 0$ takes the following form [20]

$$\phi(\theta, 0) = \phi_0 - \alpha_0\theta + i\sum_{k\neq 0} \frac{\alpha_k}{k} \exp(ik\theta) . \tag{1.34}$$

Owing to the compactification (1.24), ϕ_0 is periodic of $2\pi r$. Another condition comes from the fact that ϕ_0 and $p\alpha_0$ are canonically conjugate (1.27): actually, their wave function $\Psi(\phi_0) = \exp(ip\alpha_0\phi_0)$ should be periodic, implying the quantization

$$p\alpha_0 = \frac{m}{r}, \qquad m \in \mathbb{Z}. \tag{1.35}$$

Altogether, we obtain two periodicities

$$\phi(2\pi) = \phi_0 + 2\pi \left(rn + \frac{m}{pr} \right), \qquad n, m \in \mathbb{Z}, \tag{1.36}$$

whose commensurability requires pr^2 to take rational values [20].

A further physical condition is that the edge action $S_{\text{edge}}[\phi]$ in (1.21) should reproduce the bulk physics, in particular the excitations with fractional statistics (1.17), for $\nu = 1/p$. Therefore

$$\oint a_\mu dx^\mu = \phi(2\pi) - \phi(0) = \frac{2\pi n}{p}, \qquad n \in \mathbb{Z}. \tag{1.37}$$

This gives the spectrum of α_0 (1.35) and fixes the compactification radius to $r = 1$ due to (1.36). Thus, the fractional charges of the bulk excitations are [20]

$$Q\,|\alpha_0 = \frac{m}{p}\rangle = \frac{m}{p}\,|\alpha_0 = \frac{m}{p}\rangle, \qquad m \in \mathbb{Z}. \tag{1.38}$$

The corresponding conformal dimensions are given by the eigenvalues of L_0 (1.29) [20], i.e.

$$L_0\,|\alpha_0 = \frac{m}{p}\rangle = h_m\,|\alpha_0 = \frac{m}{p}\rangle, \qquad h_m = \frac{m^2}{2p}. \tag{1.39}$$

In particular, the spectrum contains electrons excitations with integer charge and odd integer statistics for p odd. It can be shown that more general quantizations of zero modes of this theory are possible for $r \neq 1$ but would not have electron excitations and thus should be discarded [20].

We now compute the euclidean grand canonical partition function at the outer edge of the annulus: this circle and the euclidean time period β realize the geometry of a torus (see Fig. 1.4). Owing to the knowledge of the spectrum of edge excitations for the Laughlin states with $\nu = 1/p$, (1.38) and (1.39), the trace on the Hilbert space can be decomposed into orthogonal sectors $\mathcal{H}^{(\lambda)}$, corresponding to the basic anyons plus any number of electrons [23]. There are p sectors, for $\lambda = 1, \ldots, p$, which contains representations with charges

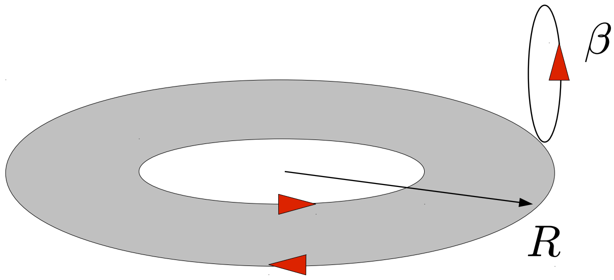


Figure 1.4: Space annulus with indicated the torus geometry of periods $(2\pi R, \beta)$.

$Q_n = \lambda/p + n$ of the $U(1)$ current algebra of the $c = 1$ CFT with conformal weights $h_n = (\lambda + pn)^2/2p$. The partition function for each sector takes the following form [23]:

$$\begin{aligned} K_\lambda(\tau, \zeta; p) &= \mathrm{Tr}_{\mathcal{H}^{(\lambda)}}[\exp(i2\pi\tau L_0 + i2\pi\zeta Q)] \\ &= \frac{F(\mathrm{Im}\tau, \mathrm{Im}\zeta)}{\eta(\tau)} \sum_{n\in\mathbb{Z}} \exp\left(i2\pi\left(\tau\frac{(np+\lambda)^2}{2p} + \zeta\frac{np+\lambda}{p}\right)\right), \end{aligned} \tag{1.40}$$

where $\eta(\tau)$ is the Dedekind function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n), \qquad \text{with} \quad q = \exp(2\pi i\tau), \tag{1.41}$$

and $F(\mathrm{Im}\tau, \mathrm{Im}\zeta)$ is a pre-factor explained in [23]. The function K_λ is parameterized by the two complex numbers,

$$\tau = \frac{i\beta}{2\pi R} + t, \qquad \zeta = \frac{\beta}{2\pi}(iV_o + \mu), \tag{1.42}$$

that are the modular parameter τ and the "coordinate" ζ. $\mathrm{Im}\tau > 0$ is related to the euclidean time period β, while $\mathrm{Re}\,\tau$ is the parameter conjugate to momentum P; ζ contains V_o and μ, respectively the electric and chemical potentials. These functions K_λ have the periodicity $K_{\lambda+p} = K_\lambda$, corresponding to the p anyon sectors [23].

The primary fields of the CFT with $c = 1$ are the vertex operators [21, 22]. These have the form [20]

$$V_m(z) = :\mathrm{e}^{im\phi(z)/p}:, \qquad m \in \mathbb{Z}, \qquad z = \exp(v\tau + i\theta), \tag{1.43}$$

and satisfy the following commutation relations

$$[L_0, V_m(z)] = \left(z\frac{\partial}{\partial z} + \frac{m^2}{2p}\right)V_m(z), \tag{1.44}$$

$$[\alpha_0, V_m(z)] = \frac{m}{p}\ V_m(z). \tag{1.45}$$

The vertex operators $V_m(z),\ m > 0$, describe the insertion at point z on the boundary of a quasi-hole excitations with fractional charge $Q_m = m/p$ and conformal dimension $h_m = m^2/2p$. Moreover, the operator product expansion (OPE) of two vertex operators is [21, 22]

$$\lim_{z_1 \to z_2} V_{m_1}(z_1) V_{m_2}(z_2) \simeq (z_1 - z_2)^{m_1 m_2/p} V_{m_1+m_2}(z_2), \tag{1.46}$$

whose phase gives the value of the fractional statistics [20]. The operators $V_m,\ m < 0$, describe the insertion of quasiparticle excitations at the boundary having opposite charge.

These results are in agreement with those obtained from the bulk effective action, i.e. Eq.(1.17). Because p is odd, the spectrum always allows the excitation V_p with the quantum numbers $Q = 1$ and $\theta_n = 2\pi p$, that is the electron. These results indicate that the $c = 1$ CFT of the chiral edge excitations is in agreement with the Laughlin's theory with $\nu = 1/p$ of Section 1.1, as it describes the long-range universal properties at the boundary of the fractional QHE.

We remark that the same bosonic CFT theory can also be applied to describe bulk wavefunctions [11]. Indeed, the Laughlin function (1.7) is basically the same function as the $N-$point correlator of vertex operators $V_p(z)$ for electrons (1.43), now located at the points $z = x + iy$ of the plane [24]. The description of quasi-hole and quasi-particle wavefunctions requires some modifications of the conformal fields that are described in the works [25, 26]. In this thesis, we will not discuss this subject because we will not make use of wavefunctions.

1.3.2 The free fermion theory: bosonization in $(1 + 1)$ dimension

The $\nu = 1$ integer QHE is a system of non-interacting electrons and the corresponding edge theory involves one Weyl fermion [27, 28]. Choosing the parameter $p = 1$, the CFT of the compactified boson is the same as that of the Weyl fermion, due to bosonization in $(1 + 1)$ dimensions [29]. The two theories have the same conformal charge $c = 1$, satisfy the same chiral algebra (1.31), and the bosonic vertex operators represent the fermion fields ψ and $\psi^\dagger$ as follows [20]:

$$\psi(\theta, t) = V_{-1} = :\mathrm{e}^{-i\phi(\theta,t)}:, \qquad \psi^\dagger(\theta, t) = V_{+1} = :\mathrm{e}^{i\phi(\theta,t)}:. \tag{1.47}$$

Indeed, these fields satisfy the usual anti-commutation relations:

$$\left\{\psi(\theta, t), \psi(\theta', t)\right\} = \left\{\psi^\dagger(\theta, t), \psi^\dagger(\theta', t)\right\} = 0, \tag{1.48}$$

$$\left\{\psi(\theta, t), \psi^\dagger(\theta', t)\right\} = 2\pi\delta(\theta - \theta'). \tag{1.49}$$

The charge density $\rho =: \psi^\dagger\psi :$ can be accordingly expressed in terms of the bosonic field once subtracted the short-distance divergent part using (1.46) [20]; the result is

$$\rho = -\frac{1}{2\pi}\partial_1\phi. \tag{1.50}$$

This implies the mapping $\rho_n = \alpha_n$ among the moments of the fields; in particular, the total charge ρ_0 is represented in the bosonic theory by α_0, in agreement with the previous results [20].

The equivalence between the bosonic and fermionic theories also amounts to the equivalence of their partition functions on the torus geometry and of the spectra of excitations. We first clarify a property of the normal ordering of the fermion field $\psi(\theta,t)$ (1.47) [29]. Using the relation $\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B}\mathrm{e}^{\frac{1}{2}[A,B]}$, if $[A,B]=c$ is a number, and the definition of the vacuum state (1.28), the normal ordered fermion field is expressed in terms of the bosonic modes as follows

$$\psi(\theta,t) = \exp\left(-\sum_{n<0}\frac{\alpha_n}{n}\exp\left(in\left(\theta-t\right)\right)\right)\mathrm{e}^{i\phi_0}\mathrm{e}^{-i\left(\alpha_0+\frac{1}{2}\right)(\theta-t)}\exp\left(-\sum_{n>0}\frac{\alpha_n}{n}\exp\left(in\left(\theta-t\right)\right)\right). \tag{1.51}$$

We see that the integer or half-integer quantizations of α_0 determines the antiperiodic (A) and periodic (P) spatial boundary conditions of the fermion field [29]:

$$\psi(\theta+2\pi,t) = \begin{cases} -\psi(\theta,t), & (A) \quad \text{if} \quad \alpha_0 \in \mathbb{Z}, \\ \psi(\theta,t), & (P) \quad \text{if} \quad \alpha_0 \in \mathbb{Z}+\frac{1}{2}. \end{cases} \tag{1.52}$$

In the double periodic geometry of a torus, a fermion may have two types of boundary conditions along the space and time directions, i.e. antiperiodic (A) and periodic (P). These correspond to the spin sectors known as the Neveu-Schwarz (NS) and Ramond (R) sectors and their tildes, as follows [21, 22]:

$$NS,\ \widetilde{NS},\ R,\ \widetilde{R}, \quad \text{respectively}: \quad (A,A),\ (A,P),\ (P,A),\ (P,P). \tag{1.53}$$

The partition functions of each sectors can be constructed via the operator formalism, namely expanding the Weyl field in terms of rising and lowering operators as follows

$$\psi(\theta,t) = \sum_k d_k \mathrm{e}^{ik(\theta-t)}, \tag{1.54}$$

with the modes operators satisfying the anti-commutation rules $\{b_k, b_l\} = \delta_{kl}$ and acting on the vacuum as

$$d_k \left|\Omega\right\rangle = 0, \qquad k>0, \tag{1.55}$$
$$d_k^\dagger \left|\Omega\right\rangle = 0, \qquad k \leqslant 0. \tag{1.56}$$

As shown in Appendix A, the Neveu-Schwarz sector corresponds to choosing $k \in \mathbb{Z}+1/2$; for $k\in\mathbb{Z}$ we obtain the Ramond one. The Neveu-Schwarz sector has a unique ground state $\left|\Omega\right\rangle_{NS}$. Instead, owing to the presence of the zero modes operators $d_0^\dagger$ and d_0, the Ramond sector has two degenerate ground states; they have eigenvalues ±1 with respect to the fermion number operator $(-1)^F$, that is $(-1)^F\left|\Omega\right\rangle_{\pm R} = \pm\left|\Omega\right\rangle_{\pm R}$.

The charge operator assumes the following normal ordered expression [20] (see Appendix A)

$$Q = \sum_k : d_k^\dagger d_k := \begin{cases} \sum_{k>0} \left(d_k^\dagger d_k - d_{-k} d_{-k}^\dagger \right), & \text{if} \quad k \in \mathbb{N} + \frac{1}{2}, \\ \\ \sum_{k>0} \left(d_k^\dagger d_k - d_{-k} d_{-k}^\dagger \right) + d_0^\dagger d_0 - \frac{1}{2}, & \text{if} \quad k \in \mathbb{N}, \end{cases} \tag{1.57}$$

from which follows that the Neveu-Schwarz ground state is neutral, that is $Q\,|\Omega\rangle_{NS} = 0$. On the other hand, the Ramond ground states are charged and satisfy $Q\,|\Omega\rangle_{\pm R} = \pm 1/2\,|\Omega\rangle_{\pm R}$.

On the space-time cylinder the hamiltonian operator L_0 is written in terms of the fermionic modes by the following normal ordered expression [20]

$$L_0 = \sum_k k : d_k^\dagger d_k := \sum_{k>0} k \left(d_k^\dagger d_k + d_{-k} d_{-k}^\dagger \right) + \begin{cases} -\frac{1}{24}, & \text{if} \quad k \in \mathbb{N} + \frac{1}{2}, \\ +\frac{1}{12}, & \text{if} \quad k \in \mathbb{N}, \end{cases} \tag{1.58}$$

where the vacuum energies are determined by the ζ-function regularization procedure.

Once we know the spectrum and the properties of the ground states of the fermionic theory, we can construct the corresponding partition functions on the torus geometry. Introducing the variable $w = \exp(2\pi i \zeta)$, with ζ the coordinate of the torus (2.11), the partition functions of the Weyl fermion are given by the following expressions in each spin sector [21, 22]

$$Z_W^{NS} = \mathrm{tr}_A q^{L_0} w^Q = \frac{1}{\eta(\tau)} \prod_{n=1} (1-q^n)(1+wq^{n-1/2})(1+w^{-1}q^{n-1/2}) = \frac{\theta_3(\tau,\zeta)}{\eta(\tau)}, \tag{1.59}$$

$$Z_W^{\widetilde{NS}} = \mathrm{tr}_A (-1)^F q^{L_0} w^Q = \frac{1}{\eta(\tau)} \prod_{n=1} (1-q^n)(1-wq^{n-1/2})(1-w^{-1}q^{n-1/2}) = \frac{\theta_4(\tau,\zeta)}{\eta(\tau)}, \tag{1.60}$$

$$Z_W^{R} = \mathrm{tr}_P q^{L_0} w^Q = \frac{1}{\eta(\tau)} w^{1/2} q^{1/8} \prod_{n=1} (1-q^n)(1+wq^{n})(1+w^{-1}q^{n-1}) = \frac{\theta_2(\tau,\zeta)}{\eta(\tau)}, \tag{1.61}$$

$$Z_W^{\widetilde{R}} = \mathrm{tr}_P (-1)^F q^{L_0} w^Q = \frac{1}{\eta(\tau)} w^{1/2} q^{1/8} \prod_{n=1} (1-q^n)(1-wq^{n})(1-w^{-1}q^{n-1}) = \frac{i\theta_1(\tau,\zeta)}{\eta(\tau)}. \tag{1.62}$$

Here, $\mathrm{tr}_{A(P)}$ means the trace over the states of the Hilbert space with antiperiodic (A) and periodic (P) spatial boundary conditions. The temporal periodic boundary conditions are obtained inserting the fermion operator $(-1)^F$ [21, 22].

The equivalence of the fermionic partition functions with the corresponding bosonic expressions follows from the Jacobi's triple product identity [21, 22]:

$$\prod_{n=1} (1-q^n)(1+wq^{n-1/2})(1+w^{-1}q^{n-1/2}) = \sum_{n\in\mathbb{Z}} q^{n^2/2} w^n. \tag{1.63}$$

This gives a representation of each theta functions $\theta(\tau,\zeta)$ in (1.59)-(1.62) in term of sums instead of products, leading to the bosonic representation in each spin sector as follows

$$Z_W^{NS} = \frac{\theta_3(\tau,\zeta)}{\eta(\tau)} = \frac{1}{\eta(\tau)} \sum_{\alpha_0\in\mathbb{Z}} q^{\alpha_0^2/2} w^{\alpha_0}, \tag{1.64}$$

$$Z_W^{\widetilde{NS}} = \frac{\theta_4(\tau,\zeta)}{\eta(\tau)} = \frac{1}{\eta(\tau)} \sum_{\alpha_0\in\mathbb{Z}} (-1)^{\alpha_0} q^{\alpha_0^2/2} w^{\alpha_0}, \tag{1.65}$$

$$Z_W^{R} = \frac{\theta_2(\tau,\zeta)}{\eta(\tau)} = \frac{1}{\eta(\tau)} \sum_{\alpha_0\in\mathbb{Z}+1/2} q^{\alpha_0^2/2} w^{\alpha_0}, \tag{1.66}$$

$$Z_W^{\widetilde{R}} = \frac{i\theta_1(\tau,\zeta)}{\eta(\tau)} = \frac{1}{\eta(\tau)} \sum_{\alpha_0\in\mathbb{Z}+1/2} (-1)^{\alpha_0-1/2} q^{\alpha_0^2/2} w^{\alpha_0}. \tag{1.67}$$

We make the following remarks:

i) The equivalence of the partition functions between the fermionic and bosonic picture shows that the parameter α_0 is quantized to integer (half-integer) values in the Neveu-Schwarz (Ramond) sectors, respectively. This result is in agreement with the expression of the fermion field in terms of bosonic modes, Eq.(1.51), where integer (half-integer) values of α_0 imply antiperiodic (periodic) spatial boundary condition.

ii) The bosonized version of the Neveu-Schwarz spin sector Z_W^{NS} in (1.64) coincides with the partition function K_λ in (1.40) with parameters $p=1$ and $\lambda=0$ corresponding to the $\nu=1$ QHE. In the following chapter we will see that the other spin sectors play an important role in the discussion of topological insulators.

1.3.3 Flux insertion and anomaly inflow

The full theory of bulk and boundary, i.e. the Chern-Simon and the chiral boson actions in (1.12) and (1.19), takes the following form in presence of an external electromagnetic gauge potential A_μ:

$$\begin{aligned} S_{\Omega+\partial\Omega}[A,\phi] = & \frac{1}{4\pi p}\int_\Omega d^3x\ \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \ - \frac{p}{4\pi}\int_{\partial\Omega} d^2x\ \left(\partial_0+\partial_1\right)\phi\partial_1\phi \ + \\ & + \frac{1}{2\pi}\int_{\partial\Omega} d^2x \left(A_0+A_1\right)\partial_1\phi \ - \ \frac{1}{4\pi p}\int_{\partial\Omega} d^2x \left(A_0+A_1\right)A_1. \end{aligned} \tag{1.68}$$

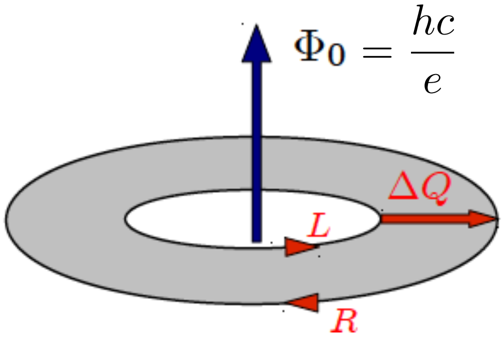


Figure 1.5: Laughlin's flux argument: the insertion of a flux Φ_0 inside the annulus moves a charge $\Delta Q = \nu e$ from the inner to the outer edge.

This action is left invariant by the gauge transformations $\phi \to \phi + \Lambda/p$ when $A_\mu \to A_\mu + \partial_\mu \Lambda$. The gauge non invariance of the bulk Chern-Simons term is cancelled by the non-conservation of the electric charge for the edge states, which is nothing but that the chiral anomaly [20, 28]. Altogether, the Hall current is conserved in the whole system. This mechanism by which an anomaly is cancelled by a classical effect in a higher-dimensional theory is called anomaly inflow [30]. The anomaly of the edge theory can be calculated by using the Hamiltonian equations of motion, which actually give the non-conservation of the charge density at the edge [20, 28]

$$(\partial_0 + \partial_1)\left(-\frac{1}{2\pi}\partial_1 \phi\right) = -\frac{\nu}{2\pi}\mathcal{E}_1. \tag{1.69}$$

Eq.(1.69) describes the edge density overflowing due to a tangential electric field $\mathcal{E}_1$, i.e. the $\nu = 1/p$ Hall effect from the point of view of the edge theory. We can also integrate the anomaly equation to obtain the adiabatic charge accumulation at one edge

$$Q(t=\infty) - Q(t=-\infty) = \int_{-\infty}^{+\infty} dt \int_0^{2\pi R} dx \partial_\mu J^\mu_B = -\frac{\nu}{2\pi}\int d^2x \mathcal{E}_1 = -\frac{\nu}{2\pi}\int \mathcal{F} = \nu n, \tag{1.70}$$

where $\mathcal{F}$ is the $(1+1)$ electromagnetic field, and $n \in \mathbb{Z}$. Eq.(1.70) relates the charge accumulated to the index of the Dirac operator in (1+1)-dimensions [31]. The index theorem establishes that the first Chern class, i.e. the quantity $1/2\pi \int \mathcal{F}$, is an integer topological number; namely, it is independent of continuous deformations of the geometry of the sample and of the background field. Therefore, this explains the robustness of the quantization of the Hall conductivity experimentally observed.

The anomaly inflow mechanism is actually equivalent to the Laughlin's flux insertion argument used to explain the charge transport from the inner to the outer edge in the annulus geometry, as shown in Fig. 1.5 [32]. The adiabatic insertion of a quantum unit of magnetic flux $\Phi_0 = hc/e$ through the hole induces a Faraday electric field $d\Phi/dt$ going around the annulus, which in turn generates a radial Hall current $I = \sigma_H d\Phi/dt$. At the end, a net charge $\sigma_H h/e$ has been transported from one edge to the other. When $\Phi = \Phi_0$, the vector potential can be eliminated by a gauge transformation, so the Hamiltonian has returned to its original form at $\Phi = 0$. Although the spectrum does not change, the states in the spectrum drift one into another, leading to the so-called *spectral flow* and the quantization of the transferred charge.

The fractional quantum Hall effect states are topological fluids made by strongly interacting electrons, that are different from standard crystals and liquids: they represent new states of matter [5, 10, 11]. As it is clear from the low-energy field theory description, there is no order parameter associated with broken symmetries as in the Landau-Ginzburg description. Every QHE state has the same symmetries, but they are characterized by different filling fractions ν and different Hall conductivities σ_H, that are topological and robust quantities. For these reasons, these new states of matter were called *topologically ordered* by Wen [10].

1.4 Transport properties and the Hall viscosity

The incompressible Hall fluids have been recently analyzed by coupling them to non-trivial metric backgrounds, in order to study the heat transport [33, 34] and the response of the fluid to strain. In particular, the Hall viscosity has been identified as a new universal quantity describing the non-dissipative transport [35, 36, 37].

In the study of the quantum Hall system, the low-energy effective action has been a very useful tool to describe and parameterize physical effects, and to discuss the universal features. Besides the earlier Chern-Simons term leading to the Hall current, the coupling to gravity was introduced by Fröhlich and collaborators [38] and by Wen and Zee [39]. The resulting Wen-Zee induced action takes the following form

$$S_{ind}[A, \omega] = \frac{\nu}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} \left(A_\mu \partial_\nu A_\rho + 2\bar{s} A_\mu \partial_\nu \omega_\rho + \bar{s}^2 \omega_\mu \partial_\nu \omega_\rho \right), \tag{1.71}$$

where ω_μ is the Abelian spin connection relative to the invariance of the system under local $O(2)$ rotations in space. This action describes the Hall viscosity and other transport effects in term of the parameter $\bar{s}$, corresponding to an intrinsic angular momentum of the low-energy excitations. Since $\bar{s}$ does not depend of the relativistic spin, it suggests a spatially extended structure of excitations. The predictions of the Wen-Zee action have been checked against the microscopic theory of electrons in Landau levels (in the case of integer Hall effect [40]) and corrections and improvements have been obtained [41, 42]. Further features have been derived under the assumption of local Galilean invariance of the effective theory, see for example [43, 44].

In our doctoral work, we studied the dependence on the metric background and we wrote the paper [4], that will be not reviewed in this thesis for lack of space. Summarizing, in this work we rederived the Wen-Zee action by using a different approach that employs the symmetry of Laughlin incompressible fluids under quantum area-preserving diffeomorphism

(W_∞ symmetry) [45, 46]. We studied the bulk excitations generated by W_∞ transformations in the lowest Landau level. We disentangled their inherent non-locality by using a power expansion in $(\hbar/B_0)^n$, where B_0 is the external magnetic field. Each term of this expansion defines an independent hydrodynamic field of spin $\sigma = 1, 2, \cdots$, that can be related to a multipole amplitude of the extended structure of excitations. The first term is just the Wen hydrodynamic gauge field, leading to the the Chern-Simons action (1.12) [10]. The next-to-leading term involves a traceless symmetric two-tensor field, that is a kind of dipole moment. Its independent coupling to the metric background gives rise to the Wen-Zee action and other effects found in the literature. The third-order term is also briefly analyzed. The structure of this expansion matches the non-relativistic limit of the theory of higher-spin fields in $(2+1)$ dimensions and the associated Chern-Simons actions developed in the Refs.[47].

Our approach allows to discuss the universality of quantities related to transport and geometric responses. We argued that the general expression of the effective action contains a series of universal coefficients, the first of which is the Hall conductivity and the second is the Hall viscosity. In principle, all the universal quantities can be observed once we probe the system with appropriate background fields, but so far our analysis is complete to second order in $\hbar/B_0$ only.

We believe that the multipole expansion developed in [4] offers the possibility of interpreting the physical models of dipoles [48] and vortices [49] developed by Haldane and Wiegmann, respectively.

Chapter 2

Two-dimensional topological insulators

In this chapter we will analyze the topological insulators in two space dimensions. These are *symmetry protected topological phases* occurring in systems that are invariant under time reversal symmetry. Actually, the presence of this symmetry allows the existence of massless counter-propagating spin polarized edge states.

We will construct the partition functions of edge modes, that turns out to be useful tools to analyze the stability of the topological phases. Moreover the stability will be associated to a discrete $\mathbb{Z}_2$ anomaly.

Among the models discussed in this chapter, there are the fractional Laughlin topological insulators and, moreover, the Pfaffian and Read-Rezayi topological insulators, i.e. phases of matter with non-Abelian statistics.

2.1 The quantum spin Hall effect

The quantum Hall effect is characterized by the breaking of time-reversal (TR) symmetry due to the presence of an external magnetic field. In the last ten years, it became clear that topological phases supporting edge states are also possible in absence of external field, that are TR invariant [50, 51]. The most relevant examples are the topological insulators. These phases of matter exist in two and three spatial dimensions and occur in certain materials with strong spin-orbit interactions.

The possible existence of $2d$ topological insulators was first noticed in a model of graphene by Kane and Mele [52]. They showed the existence of "spin filtered" edge states, where electrons with opposite spin orientation propagate in opposite directions. The edge states of this system are the same as those of two copies of a $\nu = 1$ Hall effect having opposite spin and chirality as pictorially shown in Fig. 2.1. Each chirality gives a quantized Hall conductivity $\pm e^2/h$ and an applied electric field leads to Hall currents for the opposite spins that chancel each other, but generate a net spin current $\mathbf{J}_S = 1/2(\mathbf{J}_\uparrow - \mathbf{J}_\downarrow)$ corresponding to a quantized spin Hall conductivity $\sigma_{SH} = 1$ (we use $\hbar = e = 1$) and the symmetry $U(1)_S$.

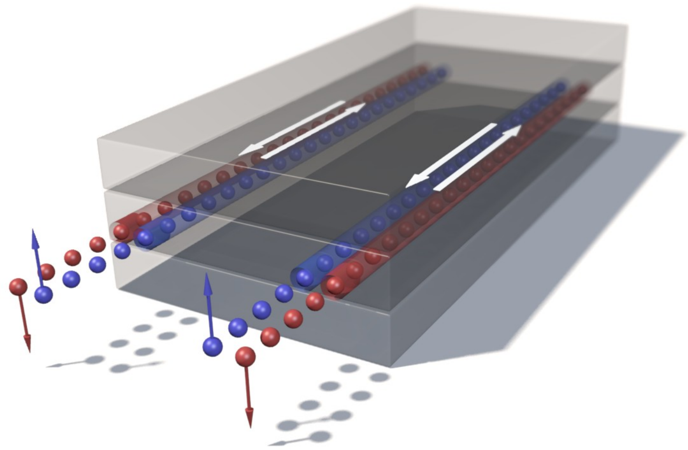


Figure 2.1: Edge states with opposite spin and chirality of a two dimensional topological insulator

This explain the name *quantum spin Hall effect*.

The original model of Kane and Mele was not realistic because graphene is made out of carbon, a light element whose spin-orbit interaction is too weak to generate a considerable energy gap. Subsequently, Bernevig, Hughes and Zhang considered materials made of heavy elements with strong spin-orbit interactions, such as compounds of Tellurium with Cadmium and Mercury [53].

These authors analyzed a quantum well structure where HgTe is sandwiched between layers of CdTe. They studied the band structure of the compound as a function of the thickness of the quantum well d_{QW}. The theoretical model predicted a phase transition from a conventional insulator, for $d_{QW} < 6.3$ nm, to a topological insulator with massless counter-propagating opposite spin edge states for $d_{QW} > 6.3$ nm. The authors showed that the phase transition is due to the inversion of the conducting and valence bands of the compound when the thickness of the quantum well exceeds the critical value $d_{QW} = 6.3$ nm, thus causing energy level crossing at the edge and massless excitations. Within a year of the theoretical proposal, the Molenkamp's group made the devices and performed transport experiments showing the first signature of the quantum spin Hall effect [54].

We remark that the quantum spin Hall effect is a rather academic model of topological insulator: in general the $U(1)_S$ symmetry is explicitly broken by relativistic effects and only the total angular momentum is conserved [51]. In these systems, however, the TR symmetry is still present and continues to map the two counter-propagating edge channels one into the other. We will see that this symmetry is crucial to the existence of the topological phase. When TR symmetry is present, it forbids some edge interactions that would lead to massive modes. If the symmetry is absent, as e.g. in presence of magnetic impurities, the edge modes interact and become massive, leading to a trivial insulating phase at low energies. Due to this property, these systems are called *symmetry protected topological phases* [55]. They must be contrasted with the quantum Hall effect which is absolutely stable, because its chiral edge states can never interact.

The main issue of TR symmetric topological phases, such as the topological insulators, is to establish the conditions of their stability, namely under which conditions TR symmetry acts to keep low-energy modes massless.

Topological insulators protected by TR symmetry have been first analyzed in free fermion systems using band theory [51, 52, 56, 57]. These systems were found to be characterized by a topological bulk quantity equal to the $\mathbb{Z}_2$ index $(-1)^N$, where N is the number of fermion edge modes of each chirality. It was proven that the odd (even) N case corresponds to a stable (unstable) topological phase.

For a quadratic Hamiltonian, this $\mathbb{Z}_2$ classification can be understood by studying the action of the anti-unitary TR operator $\mathcal{T}$ on edge electrons with up and down spin [58]:

$$\mathcal{T}: \qquad \psi_\uparrow \;\to\; \psi_\downarrow, \qquad\qquad \psi_\downarrow \;\to\; -\psi_\uparrow. \tag{2.1}$$

In the system in Fig. 2.1, made of two copies of $\nu = 1$ Hall effect with opposite spin and chiralities, the mass term coupling the two chiralities is odd under TR [50], namely

$$\mathcal{T}: \qquad H_{\text{int.}} = m \int \psi_\uparrow^\dagger \psi_\downarrow + h.c. \;\to\; -H_{\text{int.}}\ . \tag{2.2}$$

Therefore, a topological insulator with $N = 1$ is stable because the edge modes cannot become massive without breaking TR symmetry. In the case of two fermionic modes per spin, namely $N = 2$, a TR invariant mass term can be written that lets them interact and decouple from the low-energy spectrum. In general, in a system with N modes of each chirality, a single mode remains massless if N is odd [50]. Of course, if TR symmetry is broken all edge excitations become gapful and the insulator trivial.

2.2 The Fu, Kane and Mele flux argument

The analysis of more general, non-quadratic interactions compatible with TR can be done in some cases, but we describe here a criterion for stability that follows from a $\mathbb{Z}_2$ discrete symmetry that is valid for any TR invariant interactions [1]. This is the Fu-Kane-Mele flux insertion argument called the "spin pump" (a cyclic adiabatic process) [52, 56, 57]. We mostly follow the presentation of Ref.[59, 60].

The Kramers theorem

Here we briefly review the Kramers theorem in quantum mechanics, that is at the heart of the stability argument. The theorem establishes that states with half-integer spin are two fold degenerate in interacting systems that are TR symmetric [58]. This is called the Kramers degeneracy.

The theorem follows by studying the action of the TR transformations $\mathcal{T}$, that are anti-unitary and whose square turns out to be

$$\mathcal{T}^2 \left|\Psi\right\rangle = (-1)^F \left|\Psi\right\rangle . \tag{2.3}$$

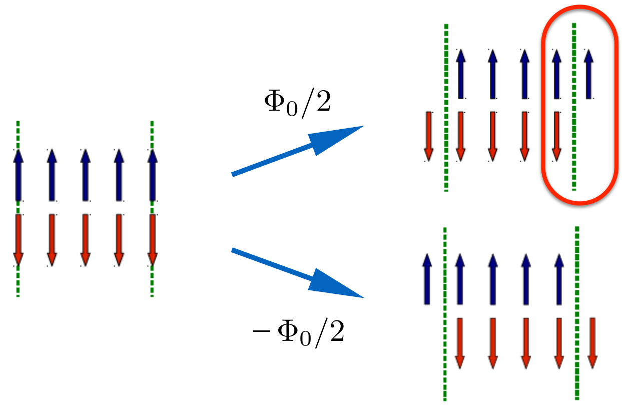


Figure 2.2: Flux insertion in the QSHE: up and down spins are displaced w.r.t. the Fermi surfaces at L and R edges of the annulus (dashed-dotted lines).

Here F is the fermion number, also equal to twice the spin of the state. The relevant case is when $\mathcal{T}^2 = -1$. In this case, if we consider the two states $|\Psi\rangle$ and its time reversal partner $\mathcal{T}|\Psi\rangle$, it turns out that they are independent states, orthogonal and degenerate. The pair of degenerate states $(|\Psi\rangle, \mathcal{T}|\Psi\rangle)$ is called Kramers doublet.

The flux insertion argument

Consider a system made of two copies of the $\nu = 1$ Hall effect having opposite spin and chiralities, as in Fig. 2.1, in the annulus geometry. The insertion of magnetic flux breaks TR symmetry, owing to:

$$\mathcal{T}H[\Phi]\mathcal{T}^{-1} = H[-\Phi] \ . \tag{2.4}$$

This relation together with the periodicity $H[\Phi] = H[\Phi + \Phi_0]$ (see Section. 1.3.3), implies that the bulk Hamiltonian is TR invariant for a discrete set of flux values:

$$\Phi = 0, \frac{\Phi_0}{2}, \Phi_0, \frac{3\Phi_0}{2}, \dots . \tag{2.5}$$

The Fu-Kane-Mele analysis of band insulators allows to define an index called “TR invariant polarization”, $(-1)^{P_\theta} = \pm 1$, that enjoys the following properties [56, 57]:

i) It is a bulk topological quantity, conserved by TR symmetry.

ii) Its value is equal to the spin parity (fermion number) at the edge,

$$(-1)^{P_\theta} = (-1)^{N_\uparrow + N_\downarrow} = (-1)^{2S}. \tag{2.6}$$

iii) In a stable topological insulator, it changes value between TR invariant points (2.5) separated by half flux $\Delta\Phi = \Phi_0/2$.

The stability argument goes as follows: adding a $\Phi_0/2$ flux in the center of the annulus moves spin-up (see Fig. 1.5) and spin-down electrons in opposite directions with respect to the Fermi surfaces at each edge (see Fig. 2.2). From the point of view of the theory at one

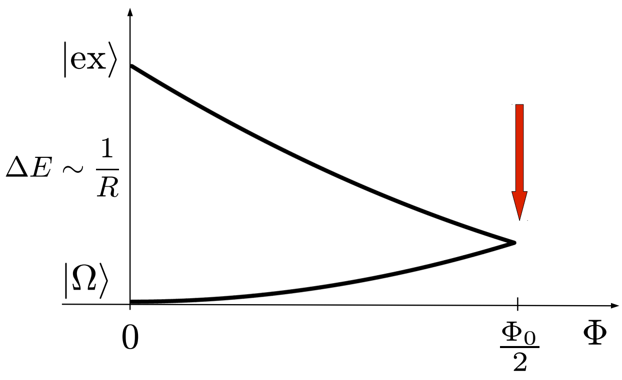


Figure 2.3: Kramers degeneracy at half flux quantum.

edge, say the outer one, the effect is to create a neutral excitations with spin one-half, i.e.

$$\Delta Q = \Delta Q^{\uparrow} + \Delta Q^{\downarrow} = 0, \qquad \Delta S = \frac{1}{2}\left(\Delta Q^{\uparrow} - \Delta Q^{\downarrow}\right) = \frac{1}{2}, \tag{2.7}$$

where $\Delta Q^{\uparrow} = -\Delta Q^{\downarrow} = \nu^{\uparrow}/2 = 1/2$ is the chiral charge moved by the flux insertion [61]. This excitation has a TR partner locally at the boundary, the configuration with flipped spin. Actually, the latter would be obtained with the insertion of the opposite flux $-\Phi_0/2$, (see Fig. 2.2). Upon using the Kramers theorem at the TR invariant point $\Phi = \Phi_0/2$, it follows that the two spin one-half edge excitations are degenerate in energy and orthogonal, this degeneracy being robust to addition of TR symmetric interactions.

The energy change of the edge ground state as the flux is varied from zero to $\Phi_0/2$ is shown in Fig. 2.3. At $\Phi_0/2$, the evolved ground state $|\Omega\rangle$ necessarily meets with one excited state $|\mathrm{ex}\rangle$ owing to Kramers theorem. Going back to $\Phi = 0$, the excited state must have an energy $O(1/R)$ equal to the work done by adding a flux quantum in a system of size R; thus, it cannot have a gap in the thermodynamic limit. It then follows that the existence of a Kramers (spin one-half) pair at the edge for $\Phi = \Phi_0/2$ implies the presence of a gapless excitation at $\Phi = 0$. In the case of two fermion modes, the corresponding spin one excitation created at the boundary would not be protected by Kramers theorem. The argument then extend to odd and even numbers of fermion modes, leading to the $\mathbb{Z}_2$ classification of topological insulators with $\nu^{\uparrow} = n$, with n integer. Note that our setting is that of non-interacting pairs of Hall states but we are discussing properties that remain valid in presence of interactions.

Let us add some remarks:

i) The existence of a Kramers pair is signaled by the change of spin parity of the ground state upon adding half flux,

$$\Phi = 0 : \ (-1)^{2S} = 1 \quad \longrightarrow \quad \Phi = \frac{\Phi_0}{2} : \ (-1)^{2S} = -1. \tag{2.8}$$

ii) The spin parity is conserved by TR symmetry, being just another way to state the Kramers theorem. This $\mathbb{Z}_2$ invariance is the remnant of the continuous $U(1)_S$ symmetry of the quantum spin Hall effect broken by relativistic effects.

iii) At the two TR symmetric points, $\Phi = 0, \Phi_0/2$, the spin parity takes different values while the Hamiltonian remains TR symmetric. Therefore, the quantity $(-1)^{2S}$ is no longer conserved at the quantum level and the associated $\mathbb{Z}_2$ symmetry is anomalous [1, 62].

2.3 Partition functions of topological insulators

The flux insertion argument was generalized by Levin and Stern to systems built from pairs of fractional Hall states with $\nu^{\uparrow} = -\nu^{\downarrow}$ that are generalizations of Laughlin states with Abelian fractional statistics of excitations [59] [60]. The $\mathbb{Z}_2$ index was show to extend as follows:

$$(-1)^{2\Delta S}, \qquad 2\Delta S = \frac{\sigma_{SH}}{e^*} = \frac{\nu^{\uparrow}}{e^*}, \tag{2.9}$$

where σ_{SH} is the spin Hall conductance, also equal to the filling fraction of the chiral component $\nu^{\uparrow}$, and e^* is the minimal fractional charge, in units of e. Therefore, an odd (even) ratio correspond to excitations with half-integer (integer) spin, generalizing the number of fermions of the previous case. Repeating the argument based on the Kramers degeneracy, one find corresponding stable (unstable) states [59] [60].

Here we will expose the main results of our work [1], that is the generalization of the stability analysis to any interacting topological insulators through the study of partition functions of the associated conformal field theory of edge excitations. We will recover the Levin and Stern index (2.9) for Abelian states. Moreover, we will extend the index to systems made of pairs of Hall states possessing excitations with non-Abelian fractional statistics.

2.3.1 Partition functions for Laughlin topological insulators

The chiral edge system

We recall from Section 1.3.1 the multiplet $K_\lambda(\tau, \zeta)$, $\lambda = 0, \cdots, p-1$ of chiral partition functions of $\nu = 1/p$ Laughlin states. These functions are defined on the double periodic geometry of the torus, see Fig. 1.4, and assume the following form [23]:

$$K_\lambda(\tau, \zeta; p) = \frac{F(\tau, \zeta)}{\eta(\tau)} \sum_{n \in \mathbb{Z}} \exp\left(i2\pi \left(\tau \frac{(np+\lambda)^2}{2p} + \zeta \frac{np+\lambda}{p} \right) \right). \tag{2.10}$$

They are parameterized by the two complex numbers,

$$\tau = \frac{i\beta}{2\pi R} + t, \qquad \zeta = \frac{\beta}{2\pi}(iV_o + \mu), \tag{2.11}$$

where τ is the modular parameter and ζ the "coordinate" of the torus. These functions K_λ have the the periodicity $K_{\lambda+p} = K_\lambda$, corresponding to the p anyon sectors.

From the CFT literature it is known that the torus geometry is left invariant by the modular transformations, that are discrete coordinate changes that respect the double periodicity [21]. These act on τ and ζ as follows [23]:

$$\tau \rightarrow \frac{a\tau+b}{c\tau+d}, \qquad \zeta \rightarrow \frac{\zeta}{c\tau+d}, \qquad a,b,c,d \in \mathbb{Z}, \qquad ad-bc=1, \tag{2.12}$$

and span the modular group $\Gamma = SL(2,\mathbb{Z})/\mathbb{Z}_2$. This group is generated by two transformations, $T : \tau \rightarrow \tau+1,\ \zeta \rightarrow \zeta$ and $S : \tau \rightarrow -1/\tau,\ \zeta \rightarrow -\zeta/\tau$, obeying the relations $S^2 = (ST)^3 = C$, where C is the charge conjugation matrix, $C^2 = 1$ [21]. In addition, there are the two periodicities of the coordinate ζ at τ fixed, respectively the $U : \zeta \rightarrow \zeta + 1$ and $V : \zeta \rightarrow \zeta + \tau$ transformations [23].

The modular transformations belong to the group of two-dimensional diffeomorphysms of the torus, being the global transformations not connected to the identity: they are the "large" gauge transformations of the theory placed in a gravitational background. In a similar way, the flux insertions discussed earlier are large gauge transformations of the electromagnetic background. In a generic system both symmetries are faithfully realized and the partition function is invariant. As shown in the following, stable topological states correspond to cases where the partition function transforms non-trivially, signaling the presence of gravitational and gauge anomalies. Anomaly in quantum field theory do not led to inconsistencies if the backgrounds are not quantized, as in our case, but actually can be used to characterize the topological universality classes [63] [64].

The multiplet of K_λ transforms linearly under the modular group; each generator has physical significance as we now review [23]. The S transformation reads:

$$S: \quad K_\lambda\left(\frac{-1}{\tau}, \frac{-\zeta}{\tau}\right) = e^{i\varphi} \sum_{\mu=1}^{p} S_{\lambda\mu} K_\mu(\tau,\zeta), \qquad S_{\lambda\mu} = \frac{1}{\sqrt{p}} \exp\left(i2\pi\frac{\lambda\mu}{p}\right), \tag{2.13}$$

where $S_{\lambda\mu}$ is the modular S-matrix and φ is an overall phase. This transformation of space and time implies a consistency condition on the spectrum [21].

The T^2 transformation do not change the sector K_λ but is represented by a phase factor as follows

$$T^2: \quad K_\lambda(\tau+2,\zeta) = \exp\left(i4\pi h_\lambda\right) K_\lambda(\tau,\zeta), \qquad h_\lambda = \frac{\lambda^2}{2p}. \tag{2.14}$$

From the expression of the conformal dimensions of the edge excitations (1.39), we see that this property follows from the fact that each anyon sector contains electron excitations having odd integer statistics, i.e. half-integer conformal dimension.

Finally, the V transformation,

$$V: \quad K_\lambda(\tau,\zeta+\tau) = e^{i\phi} K_{\lambda+1}(\tau,\zeta), \qquad \Delta\Phi = \Phi_0, \tag{2.15}$$

(with ϕ another global phase) realizes the change of the electric potential due to the addition of one flux quantum Φ_0. The change $\lambda \rightarrow \lambda + 1$ expresses the edge chiral anomaly, corresponding to the spectral flow $Q \rightarrow Q + \nu$ discussed in Section 1.3.3.

Altogether the single edge is described by a multiplet of partition functions K_λ, that is not modular invariant but covariant. This means that the chiral anomaly, i.e. the spectral flow, implies a discrete gravitational anomaly [23].

The non-chiral edge system

In our work [1], we have obtained the partition functions for edge excitations of topological insulators. We start by considering systems made of pairs of Laughlin states with $\nu = 1/p$, p odd, having chiral and antichiral sectors for up and down spins, respectively, thus realizing the quantum spin Hall effect. We found the expressions:

$$Z^{NS}(\tau, \zeta) = \sum_{\lambda=1}^{p} K_\lambda^{\uparrow} \, \overline{K}_{-\lambda}^{\downarrow} \, . \tag{2.16}$$

This partition function is invariant under S, T^2, V. It turns out that this quantity is formally equal to the QHE partition function for the two edges of the annulus [23]. The physical interpretation in the case of topological insulators is rather different, since it only describes a single edge.

The expression Z^{NS} is not fully modular invariant. As already said in Chapter 1, partition functions of fermionic systems always involve four terms corresponding to the four spins structures on the torus [29]. These amount to choosing antiperiodic (A) and periodic (P) boundary conditions for fermion fields in the space and time directions (in general there are 2^{2g} terms on a genus g surface). The expression (2.16) is identified with the Neveu-Schwarz sector since the natural fermionic boundary conditions are antiperiodic:

$$Z^{NS} = \mathrm{Tr}_A \left[\exp\left(i2\pi\tau L_0 + i2\pi\zeta Q + h.c. \right) \right] . \tag{2.17}$$

The other expressions are defined as:

$$\begin{aligned} Z^{R} &= \mathrm{Tr}_P \left[\exp\left(i2\pi\tau L_0 + i2\pi\zeta Q + h.c. \right) \right] , \\ Z^{\widetilde{NS}} &= \mathrm{Tr}_A \left[(-1)^{N_\uparrow + N_\downarrow} \exp\left(i2\pi\tau L_0 + i2\pi\zeta Q + h.c. \right) \right] , \\ Z^{\widetilde{R}} &= \mathrm{Tr}_P \left[(-1)^{N_\uparrow + N_\downarrow} \exp\left(i2\pi\tau L_0 + i2\pi\zeta Q + h.c. \right) \right] , \end{aligned} \tag{2.18}$$

where periodic conditions in time introduce the sign $(-1)^F = (-1)^{N_\uparrow + N_\downarrow}$. We note that for $p = 1$ these Z-functions become the modulus square of the bosonic functions given in Chapter 1, i.e. Eq.(1.64). Indeed a massless Dirac fermion is made of a pair of Weyl fermions with opposite chiralities [21, 22, 29].

The modular transformations among the four terms are depicted in Fig. 2.4(a) and are explicitly checked in the Appendix B.1. They form a triplet, $Z^{NS}, Z^{\widetilde{NS}}, Z^{R}$, and a singlet, $Z^{\widetilde{R}}$. Each one of the four spin sectors is made of p "anyonic" sectors:

$$Z^{s} = \sum_{\lambda=1}^{p} K_\lambda^{\uparrow s} \overline{K}_{-\lambda}^{\downarrow s}, \qquad s = NS, \widetilde{NS}, R, \widetilde{R}. \tag{2.19}$$

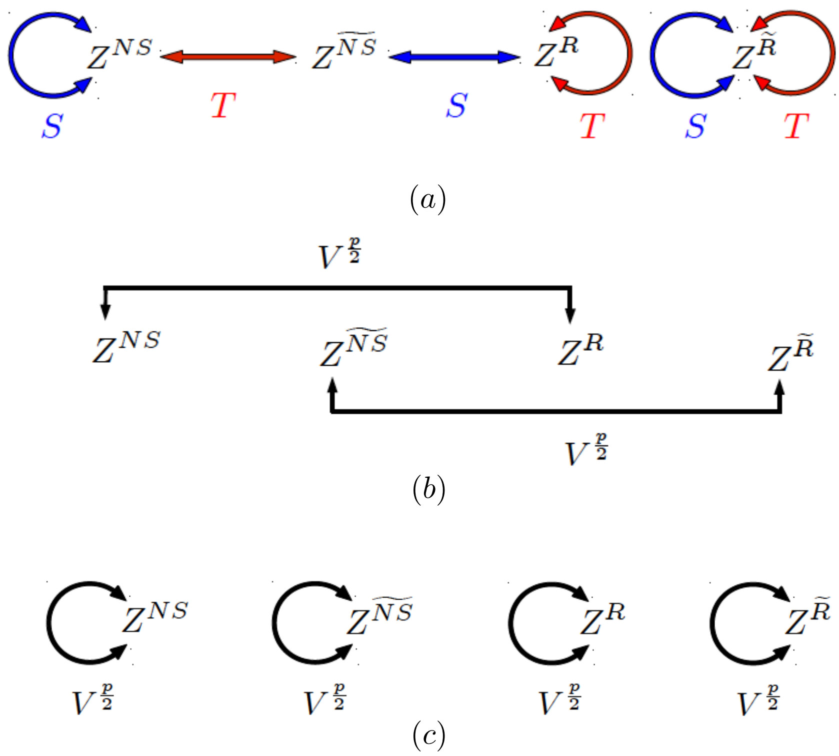


Figure 2.4: Actions on the four spin sectors $Z^{NS}, Z^{\widetilde{NS}}, Z^R, Z^{\widetilde{R}}$ of (a) modular transformations, (b) $p/2$ flux insertions for p odd, (c) for p even.

The $\widetilde{NS}$ sector is defined as follows,

$$\begin{aligned} Z^{\widetilde{NS}}(\tau,\zeta) &= Z^{NS}(\tau+1,\zeta)\,, \\ K_\lambda^{\widetilde{NS}}(\tau,\zeta) &= e^{i\theta_\lambda}\;K_\lambda(\tau+1,\zeta)\,. \end{aligned} \tag{2.20}$$

In this expression, the phase $\theta_\lambda = 2\pi\left(\frac{\lambda}{2}-\frac{\lambda^2}{2p}+\frac{1}{24}\right)$ is included for convenience in the following. We also write $K_\lambda^{NS} \equiv K_\lambda$ and omit spin arrows for simplicity.

The Ramond sector is similarly obtained by acting with ST on Z^{NS} and $Z^{\widetilde{R}}$ is defined by inserting the $(-1)^F$ sign into the Ramond expression.

2.3.2 Stability analysis

We observe that the addition of p fluxes creates an electron excitation within the same anyon sector, since it corresponds to a symmetry of each K_λ,

$$V^p : K_\lambda \;\to\; K_{\lambda+p} = K_\lambda, \qquad \Delta Q^\uparrow = \frac{p}{p} = 1\;. \tag{2.21}$$

On the other hand, the addition of $p/2$ fluxes is not a gauge transformation. It corresponds to a strong perturbation that modifies its spectrum and, in particular, creates a spin one-half excitation at the edge, $\Delta S = \Delta Q^\uparrow = 1/2$ [59, 60]. Under the action of $V^{\frac{p}{2}}$ the Neveu-Schwarz sector is mapped into the Ramond sector as shown in Fig. 2.4(b):

$$V^{\frac{p}{2}}: K_\lambda(\tau,\zeta) \to K_\lambda\left(\tau,\zeta+\frac{p\tau}{2}\right) \sim K_{\lambda+\frac{p}{2}}(\tau,\zeta) \equiv K^R_\lambda(\tau,\zeta)\,, \tag{2.22}$$

$$Z^{NS}(\tau,\zeta) \;\to\; Z^{NS}\left(\tau,\zeta+\frac{p\tau}{2}\right) = Z^R(\zeta,\tau)\,. \tag{2.23}$$

We can understand this kind of behavior looking to the non interacting case with $p = 1$. The insertion of one half-flux quantum can be reabsorbed in the Hamiltonian by a gauge transformation but, at the same time, gives a non trivial phase to the electron wave function changing the spatial boundary condition from anti-periodic to periodic and vice versa. As shown in Section 1.3.1, this corresponds to modify the values of α_0 from integer to half-integer, modifying the spectrum of the excitations and changing the spin sector from the Neveu-Schwarz to the Ramond one and vice versa.

We can now extend the stability analysis discussed in the $\nu^\uparrow = 1$ case. Upon applying $p/2$ fluxes, the Neveu-Schwarz ground state $|\Omega\rangle_{NS}$, the lowest state in $K_0\overline{K}_0$, evolves in the Ramond ground state $|\Omega\rangle_R$ present in $K^R_0\overline{K}^R_0 = K_{p/2}\overline{K}_{p/2}$. Inspecting $K^R_0\overline{K}^R_0$ to lowest order in $q\overline{q}$, with $q = \exp(i2\pi\tau)$, and checking the terms of $O(w^0\overline{w}^0)$, $w = \exp(i2\pi\zeta)$, i.e. not involving additional particles, it is easy to see the existence of the Kramers pair and the behavior of the spectrum shown in Fig. 2.3. Indeed, inspecting the Neveu-Schwarz partition function we find

$$Z^{NS} \sim \frac{1}{|\eta(q)|^2}\Big(1+\cdots\Big), \tag{2.24}$$

where the first term of the expansion corresponds to the ground state $|\Omega\rangle_{NS}$. Instead, in the Ramond case,

$$\begin{aligned} Z^R \sim \frac{1}{|\eta(q)|^2}\Big(& q^{p/8}w^{1/2}\bar{q}^{p/8}\bar{w}^{1/2} + q^{p/8}w^{1/2}\bar{q}^{p/8}\bar{w}^{-1/2} \\ &+ q^{p/8}w^{-1/2}\bar{q}^{p/8}\bar{w}^{1/2} + q^{p/8}w^{-1/2}\bar{q}^{p/8}\bar{w}^{-1/2} + \cdots\Big). \end{aligned} \tag{2.25}$$

In this expression, the first and fourth term correspond to the neutral and degenerate ground states $|\Omega\rangle_R$ and $|\Omega\rangle'_R$; they are TR partners, i.e. $|\Omega\rangle'_R = \mathcal{T}\,|\Omega\rangle_R$, and thus give rise to the Kramers doublet. We conclude that these models describe stable topological insulators for odd integer p values.

We remark that the spin parities of the Neveu-Schwarz and Ramond ground states change as follows:

$$(-1)^{2S}\,|\Omega\rangle_{NS} = |\Omega\rangle_{NS} \;\to\; (-1)^{2S}\,|\Omega\rangle_R = -\,|\Omega\rangle_R\,, \qquad \Delta S = \Delta Q^\uparrow = \frac{1}{2}\,. \tag{2.26}$$

The Levin-Stern stability index discussed at the beginning of Section 2.3 is thus reacquired:

$$2\Delta S = 2\Delta Q^\uparrow = \frac{\sigma_{sH}}{e^*} = 1\,, \qquad (-1)^{2\Delta S} = -1, \tag{2.27}$$

with $\sigma_{SH} = \nu^{\uparrow} = 1/p$ and $1/e^* = p$ is the number of charge sectors, i.e. the periodicity of K_λ. The (would-be) spin transport ΔS involved in this index, equal to the Hall current of one chiral component, is relative to half of the number of fluxes needed for creating an electron excitation within any given anyon sector.

In conclusion, we have found that the spin parity of the Ramond ground state is different from that of the Neveu-Schwarz ground state. This is the manifestation of the discrete $\mathbb{Z}_2$ anomaly: different sectors of the path integral (Eq.(2.17) and (2.18)) have associated different quantum numbers [1].

2.3.3 Stability and modular invariance

In a fermionic non-chiral system composed of the four spin sectors (2.17),(2.18) it is always possible to find a modular invariant partition function by summing over all sectors,

$$Z_{\text{Ising}} = Z^{NS} + Z^{\widetilde{NS}} + Z^{R} + Z^{\widetilde{R}}. \tag{2.28}$$

This is the so-called Ising partition function because it describes conformal field theories applied to statistical models like the Ising model, its supersymmetric generalizations etc. [21, 22]. The quantity Z_{Ising} is indeed $S, T, V^{\frac{p}{2}}$ invariant.

However, the theory defined by Z_{Ising} is not consistent with TR symmetry, that implies spin parity conservation. In presence of the $\mathbb{Z}_2$ anomaly (2.26), the partition function (2.28) sums spin sectors with different values of the ground state spin parity and violates TR symmetry. If we want to preserve it, we should not sum over spin sectors and let the partition function form a four-dimensional vector,

$$Z_{\text{TR}} = \left(Z^{NS}, Z^{\widetilde{NS}}, Z^{R}, Z^{\widetilde{R}}\right). \tag{2.29}$$

Since the modular group acts non trivially on these components, the path integral will depend on the coordinates chosen to describe the torus, namely the system possesses a gravitational anomaly. In this theory, the partition function Z^{NS} represents the unperturbed edge system, while the other functions, $Z^{\widetilde{NS}}, Z^{R}, Z^{\widetilde{R}}$, are excited states of the system in presence of electromagnetic or gravitational backgrounds. We thus obtain the following result [1]:

$$\begin{aligned} &TR\ symmetry + anomaly \ \leftrightarrow\ no\ modular\ invariance \ \leftrightarrow\ topological\ insulator, \\ &TR\ symmetry + modular\ invariance \ \leftrightarrow\ no\ anomaly \ \leftrightarrow\ trivial\ insulator. \end{aligned} \tag{2.30}$$

An analogous study of the modular invariance of the partition function was done by Ryu and Zhang in their analysis of the stability of two dimensional topological superconductors [64].

We remark that in a physics setting as the annulus geometry, there should not be any anomaly for the whole system, as in the case of the quantum Hall effect. Thus, the $\mathbb{Z}_2$ anomaly should cancel between the two edges by combining the relative partition functions in a global modular invariant expression.

2.4 Non-Abelian topological insulators

2.4.1 CFT and non-Abelian anyons

In Chapter 1, we saw that the fractional QHE supports excitations with fractional charge and fractional statistics; for example the $\nu = 1/p$ Laughlin states have $Q_n = n/p$ and $\theta_n/\pi = n^2/p$. These phases arise in the wave functions Ψ when excitations located at η_1, η_2 are exchanged in their positions. Thus, under $(\eta_1 - \eta_2) \to \mathrm{e}^{i\theta}(\eta_1 - \eta_2)$ the wave function transforms as

$$\Psi(\eta_1, \eta_2; z_1, \cdots, z_n) \ \to \ \mathrm{e}^{i\theta}\Psi(\eta_2, \eta_1; z_1, \cdots, z_n), \tag{2.31}$$

where $\{z_i\}$ are the electrons coordinates. For n excitations, those exchanges give rise to a one-dimensional representation of the group of n braids B_n.

Non-Abelian statistics is associated to higher dimensional representations of the braid group that can occur when there is a degenerate set of g states for the excitations at fixed positions $(\eta_1, \cdots, \eta_n)$. Let us define an orthonormal basis $\Psi_\alpha, \ \alpha = 1, 2, \cdots, g$ of these degenerate states: the element of the braid group σ_i that exchanges the particles i and $i+1$ is represented by a $g \times g$ unitary matrix $\rho(\sigma_i)$ acting on these states as

$$\Psi_\alpha \ \to \ [\rho(\sigma_i)]_{\alpha\beta}\Psi_\beta. \tag{2.32}$$

If the matrices $\rho(\sigma_i)$ and $\rho(\sigma_j)$ do not commute for $i \neq j$, the particles obey to non-Abelian braiding statistics (for a review see Ref.[65]).

The CFT literature offers a large number of models that realize this phenomenon [21] and, starting with the pioneering work of Moore and Read [24], they were applied to the QHE physics.

Briefly, to describe a CFT we need its conformal data, including the set of primary fields, each with a conformal dimension h, a table of fusion rules of these fields and a central charge c. The fusion rules are the selection rules for the Operator Product Expansion of the fields. Given a primary field ϕ_γ and indicating with $[\phi_\gamma]$ its representation containing all its descendant fields, the fusion rules encode the possible channels that can be created when two fields ϕ_α and ϕ_β are brought together to form a composite state, i.e. [21]

$$[\phi_\alpha] \times [\phi_\beta] = \sum_{\gamma \in P^k} N_{\alpha\beta}^{\gamma}[\phi_\gamma], \tag{2.33}$$

where P^k is the set of labels of primary fields and the integer $N_{\alpha\beta}^{\gamma}$ counts the number of time that the representation $[\phi_\gamma]$ appears in the OPE between ϕ_α and ϕ_β.

The OPE between vertex operators [21],

$$\lim_{z_1 \to z_2} V_{n_1}(z_1)V_{n_2}(z_2) = (z_1 - z_2)^{n_1 n_2/k}V_{n_1+n_2}(z_2), \tag{2.34}$$

is a fusion with only one term in the r.h.s, i.e. the case of Abelian statistics. In the CFT language, the edge theory of the compact chiral boson studied in Chapter 1 gives rise to a rational CFT, i.e. the spectrum of the charges is given by a multiple of a basic fraction, say

$1/p$, and the theory has a finite number of (generalized) primary fields V_n, with $n = 1, \cdots, p$ [21]. The OPE between the two vertex operators in (2.34) represents the addition of charges $Q_{n_1} + Q_{n_2}$ mod p since the theory, being invariant under the $U(1)$ (extended) symmetry, has to conserve the electric charge. Thus, in this case the fusion rules realized the Abelian group $\mathbb{Z}_p$ [65].

The conformal theories of general quantum Hall edge states possess not only charged excitations but also neutral modes that can be Abelian or non-Abelian. These theories have the affine symmetry $U(1) \times G/H$, where $U(1)$ is the charge symmetry and G is another (non-Abelian) symmetry characterizing the neutral part (possibly a coset G/H). The simplest example giving rise to non-Abelian statistics is for $G/H = \widehat{SU(2)}_2/U(1)$, that is the Ising model with central charge $c = 1/2$ [21, 24].

The primary fields of this model are (I, ψ, σ), respectively the identity with dimension $h_I = 0$, the Majorana fermion with $h_\psi = 1/2$ and the spin field with $h_\sigma = 1/16$. These fields satisfy the following fusion rules [21]

$$\psi \times \psi = I, \qquad \sigma \times \psi = \sigma, \qquad \sigma \times \sigma = I + \psi. \tag{2.35}$$

The non-Abelian statistics is due to the two possible channels in the third fusion rule and can be exemplified as follows [65]. Suppose to have four quasiparticle excitations of σ type. The correlator $\langle \sigma(\omega_1)\sigma(\omega_2)\sigma(\omega_3)\sigma(\omega_4)\rangle$ can be obtained by fusing $\sigma(\omega_1)$ with $\sigma(\omega_2)$ and $\sigma(\omega_3)$ with $\sigma(\omega_4)$, thus obtaining pairs of intermediate channels that should match. In the present example there are two intermediate channels corresponding to I and ψ. The correlator is given by the linear combination of the two possible resulting amplitudes, called conformal blocks. Putting the excitations at $w_1 = 0$, $w_2 = z$, $\omega_3 = 1$, and $w_4 \to \infty$ for convenience, this can be written as [21]

$$\lim_{w \to \infty} \langle \sigma(0)\sigma(z)\sigma(1)\sigma(\infty)\rangle = a_+ F_+(z) + a_- F_-(z), \tag{2.36}$$

where $F\pm$ are the two conformal blocks that are given by two independent Hypergeometric functions. Now, transporting the coordinate z around 0 or 1 by analytic continuation makes a rotation within the two dimensional vector space. It acts on the basis of the conformal blocks through the following non commuting matrices, i.e.

$$\begin{pmatrix} F_+ \\ F_- \end{pmatrix} (z\mathrm{e}^{2\pi i}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} F_+ \\ F_- \end{pmatrix} (z), \qquad \begin{pmatrix} F_+ \\ F_- \end{pmatrix} ((z-1)\mathrm{e}^{2\pi i}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_+ \\ F_- \end{pmatrix} (z), \tag{2.37}$$

hence the term non-Abelian.

2.4.2 General partition functions

In the CFT characterizing the quantum Hall edge states with symmetries $U(1) \times G/H$, the electron field is represented by the product of a chiral vertex operator for the charge part and a chiral neutral field ψ_e of the G/H theory:

$$\Psi_e = e^{i\alpha\varphi}\, \psi_e \ . \tag{2.38}$$

In any non-Abelian theory, the field ψ_e should have Abelian fusion rules with all fields in the theory; this property is needed for the electrons to have integer statistics with all excitations and non degenerate wavefunctions [66].

The field ψ_e, called a simple current in the CFT literature [21], can be used to build a modular invariant that couples neutral and charged parts non-trivially and fulfills the physical conditions on charge and statistics of the edge spectrum. The general expression of the partition function for the Hall edge states obtained in this way is determined uniquely by two inputs: the choice of neutral G/H theory and of the Abelian field ψ_e that represents the electron neutral part. These simple-current modular invariant partition functions were shown to reproduce earlier results obtained by physical insight in many models and to build new ones [66].

The construction starts from the partition sum of one anyon sector, generalizing the K_λ of the $c = 1$ theory introduced in Section 2.3.1 (Eq.(1.40)): this involves again a basic anyon plus any number of electrons added to it, with charge $Q = \lambda/p+n$, $n \in \mathbb{Z}$. It is characterized by λ, and the neutral quantum numbers (m, α). Such partition function takes the form [66]:

$$\Theta^{\alpha}_{\lambda}(\tau, \zeta) = \sum_{a=1}^{k} K_{\lambda+ap}\,(\tau, k\zeta; kp)\;\; \chi^{\alpha}_{\lambda+ap \bmod k}(\tau, 0). \tag{2.39}$$

The $K_\lambda(\tau, k\zeta; kp)$ are the earlier characters for the charge part (2.10), while the $\chi^{\alpha}_{m}(\tau, 0)$ are the G/H characters for the neutral part, that are labelled by the Abelian number m associated to the simple current and other, possibly non-Abelian, quantum numbers collectively denoted by α. The explicit form of the neutral characters χ^{α}_{m} is not needed, only their symmetries and modular transformations are relevant in the following.

Equation (2.39) can be explained as follows. The basic anyon has quantum numbers (λ, m, α), with m modulo k and λ modulo kp owing to the periodicity:

$$K_\lambda(\tau, k\zeta; kp) = K_{\lambda+kp}(\tau, k\zeta; kp)\ . \tag{2.40}$$

After adding one electron, the quantum numbers changes into $(\lambda + p, m + p, \alpha)$; then, after adding k electrons these numbers return to those of the basic anyon. This explains the k terms in the sum (2.39). The difference with respect to the $c = 1$ case (1.40) is that n-electron states couple to different neutral parts for n modulo k; actually, each $K_\lambda(\tau, k\zeta; kp)$ in (2.39) only sums electrons with $Q = \lambda/p + kn$, owing to the different ζ dependence.

We can use chiral-antichiral pairs of these edge theories to model interacting topological insulators. The functions $\Theta^{\alpha}_{\lambda}(\tau, \zeta)$ enjoy similar properties under modular transformations as the K_λ of Section (2.3.1) and the partition function Z^{NS} can be written accordingly that couples the up/down spin modes at one edge:

$$Z^{NS} = \sum_{\lambda,\alpha} \Theta^{\alpha}_{\lambda}\overline{\Theta}^{\alpha}_{-\lambda}. \tag{2.41}$$

In this sum, the allowed range of (λ, α) values gives the value of the Wen topological order.

Stability argument

The charge part K_λ of the sectors Θ_λ^α in (2.39) is parameterized by two independent numbers (k,p), whose meaning can be understood from the expression (1.40):

i) The values of the fractional charge are $Q = k\lambda/kp = \lambda/p$, $\lambda = 1,\dots,p$, and the minimal charge is equal to $e^* = 1/p$.

ii) the Hall current (spin current) is obtained by applying the V transformation on (2.39), that acts on the charge part K_λ, causing the shift of quantum numbers:

$$V : \zeta \to \zeta + \tau, \qquad \lambda \to \lambda + k, \qquad \Delta Q^\uparrow = \nu^\uparrow = \frac{k}{p}, \tag{2.42}$$

while the neutral characters in (2.39) are not affected.

As in Section 2.3.2, owing to the periodicity of K_λ in (2.40), the Fu-Kane-Mele flux argument is obtained by the insertion of $p/2$ fluxes, that gives rise to a variation of the spin of excitations as follows

$$V^{\frac{p}{2}} : \quad \Delta S = \Delta Q^\uparrow = \frac{p}{2}\,\nu^\uparrow = \frac{k}{2}. \tag{2.43}$$

Therefore, the Levin-Stern index (2.9) for the spin parity in this case is:

$$2\Delta S = \frac{\nu^\uparrow}{e^*} = k. \qquad (-1)^{2\Delta S} = (-1)^k. \tag{2.44}$$

The stability analysis then continues by observing that for odd values of k, the action of $V^{\frac{p}{2}}$ creates a Kramers pair at the edge that is protected by TR symmetry; then, the spectrum cannot be gapped and the topological insulator is stable.

The action on the anyon sectors (2.39) is,

$$V^{\frac{p}{2}} : \quad \Theta_\lambda^\alpha(\tau,\zeta) \ \to \ \sum_{a=1}^{k} K_{\lambda+ap+kp/2}(\tau, k\zeta; kp)\ \chi^\alpha_{\lambda+ap \bmod k}(\tau,0) \sim \Theta^{\alpha'}_{\lambda'}(\tau,\zeta), \tag{2.45}$$

where the values of (λ',α') depends on the specific theory considered through the symmetries of its characters. Looking at the expressions (2.39) and (2.45), it is clear that the neutral characters χ_m^α do not enter in the stability argument, i.e. in the determination of the index (2.44). Thus, the result (2.44) holds for topological insulators with TR symmetry having both Abelian and non-Abelian edge excitations [1].

Stability and modular non-invariance

The stability of general topological insulators, corresponding to the $\mathbb{Z}_2$ spin parity anomaly, is again accompanied by modular non-invariance of the partition function. However, electromagnetic and gravitational responses are not always equivalent as in the $c = 1$ case (neutral modes are clearly sensible to coordinate changes but not to flux additions).

We should distinguish the following cases, according to the parities of (k,p):

i) For p odd, the action of $V^{\frac{p}{2}}$ is not a symmetry of each spin sector and maps them one into another. The transformations between Neveu-Schwarz and Ramond sectors and among their tildes are the same as those of the $c = 1$ theory (see Fig. 2.4(b) and Eq.(2.22)). The anyon sector Θ_0^0 containing the NS ground state is naturally mapped into Θ_0^{R0} including the Ramond ground state. The modular invariant and non-invariant partition functions are constructed as in the $c = 1$ case and read:

$$\begin{aligned} Z_{\text{Ising}} &= Z^{NS} + Z^{\widetilde{NS}} + Z^{R} + Z^{\widetilde{R}}, & & k \text{ even, unstable,} \\ Z_{\text{TR}} &= \left(Z^{NS}, Z^{\widetilde{NS}}, Z^{R}, Z^{\widetilde{R}}\right), & & k \text{ odd, stable.} \end{aligned} \qquad (2.46)$$

ii) For p even, the action of $V^{\frac{p}{2}}$ maps each spin sector into itself and thus differs from the previous case (see Fig. 2.4(c)). For k odd, the Z_2 anomaly manifests itself within each spin sector, as a difference in spin parity between the ground state and another "anyon" ground state (actually degenerate). The TR symmetry of the theory then requires to splitting each spin sector in two sub-sectors, $Z^\sigma \to (Z_1^\sigma, Z_2^\sigma)$, $\sigma = NS, \widetilde{NS}, R, \widetilde{R}$, that are related by $V^{\frac{p}{2}}$: $Z_2^\sigma = V^{p/2}(Z_1^\sigma)$ and collect anyon sectors of same spin parity. These sub-sectors carry a eight-dimensional representation of the modular group instead of four-dimensional (2.46). Finally, for k and p both even, there is no anomaly and the Z_{Ising} partition function is consistent with TR symmetry. Summarizing, in all cases modular non-invariance is associated to stability and $\mathbb{Z}_2$ anomaly.

2.4.3 Pfaffian topological insulators

The Pfaffian state is the simplest example of non-Abelian quantum Hall states [24]. In this case the Hall conductivity and the minimal charge are

$$\nu^{\uparrow} = \frac{1}{2}, \qquad e^* = \frac{1}{4}, \qquad 2\Delta S = 2, \qquad (-1)^{2\Delta S} = 1. \qquad (2.47)$$

Thus the topological insulators made by pairs of these Hall states are unstable. The parameters entering the stability analysis are $(k, p) = (2, 4)$. In this model, the neutral and charge quantum number $(\lambda, m, \alpha) \equiv (\lambda, a, \ell)$ mentioned before are defined as: λ mod 8, a mod 4 and $\ell = 0, 1, 2$. Moreover, the Abelian charge λ and the neutral charge a obey the selection rule $\lambda = a$ mod 2, also called Parity Rule.

It turns out that the characters of the charge part are given by Eq.(2.10) and will be denoted as $K_\lambda = K_\lambda(\tau, 2\zeta; 8)$ with $\lambda = 1, \cdots, 8$. The neutral system, instead, is described by the characters of the Ising model, i.e. the Z_2 parafermions χ_a^ℓ, non-vanishing for $a = \ell$ modulo 2, and obeying $\chi_{a+2}^\ell = \chi_2^{2-\ell}$ [66]. The three independent character, denoted as their corresponding conformal field are $\chi_0^0 = \chi_2^2 = I$, $\chi_1^1 = \chi_3^1 = \sigma$ and $\chi_2^0 = \chi_0^2 = \psi$.

The construction of modular invariants obtained before [66] leads to the following anyon sectors (NS sector):

$$\Theta_a^\ell = K_a\, \chi_a^\ell + K_{a+4}\, \chi_{a+2}^\ell, \qquad a = 0, 1, 2, 3, \quad \ell = 0, 1, 2, \quad a = \ell \text{ mod } 2. \qquad (2.48)$$

The NS partition function reads [66]:

$$\begin{aligned} Z_{\text{Pf}}^{NS} &= \sum_{a=0,2} \left|K_a\, \chi_a^0 + K_{a+4}\, \chi_{a+2}^0\right|^2 + \left|K_a\, \chi_{a+2}^0 + K_{a+4}\, \chi_a^0\right|^2 + \\ &\qquad \left|(K_1 + K_{-3})\, \chi_1^1\right|^2 + \left|(K_3 + K_{-1})\, \chi_1^1\right|^2 . \end{aligned} \qquad (2.49)$$

Note that each charge sector appears twice, coupling to two neutral states, and the Wen topological order is equal to 8. The expressions of the other spin sectors for the Pfaffian state are given in the Appendix B.2.

As explained earlier, this unstable theory is characterized by (k, p) both even, and flux insertions and modular transformations act on the four spin sectors as shown in Fig. 2.4 (a) and (c). There is no $\mathbb{Z}_2$ anomaly, and the standard modular invariant Z_{Ising} partition function (2.46) is consistent with TR symmetry.

2.4.4 Read-Rezayi parafermionic topological insulators

In our work [1], we also considered the stability of topological states built by the Read-Rezayi states [67]. These are generalization of the Pfaffian state involving neutral modes of the $\mathbb{Z}_k$ parafermions, that can be described by the coset $\widehat{SU(2)}_k/U(1)$ [66]. The quantities entering in the stability index (2.9) are

$$\nu^{\uparrow} = \frac{k}{kM+2}, \qquad e^* = \frac{1}{kM+2}, \qquad 2\Delta S = k, \qquad (-1)^{2\Delta S} = (-1)^k, \tag{2.50}$$

where $k = 3, 4\ldots$ and $M = 1, 3, \ldots$. In this case, $(k, p) = (k, kM + 2)$, thus the topological insulators made by pairs of these states are stable (unstable) for k odd (even) [1]. Note that k and p have the same parity:

i) For k and p odd, the flux insertions and modular transformations follow the same pattern of the stable, odd p Laughlin states and of the $c = 1$ theory (Fig. 2.4 (a) and (b)). The modular non-invariant partition function takes the form Z_{TR} in (2.46).

ii) For k and p even, they have a common factor of 2 and the flux insertions and modular transformations are the same as those of the Pfaffian case (Fig. 2.4 (a) and (c)). The modular invariant partition function is Z_{Ising}.

All partition functions and modular transformations are described in Appendix B.3. Let us briefly discuss their expression in the Neveu-Schwarz sector, taken from Ref.[66]. These read:

$$\Theta^{\ell}_a = \sum_{b=1}^{k} K_{a+bp}\left(\tau, k\zeta; kp\right) \chi^{\ell}_{a+2b}(\tau, 0, 2k), \qquad a = \ell \mod 2, \quad p = 2 + kM. \tag{2.51}$$

The charge characters K_λ with periodicity kp, are coupled to the $\mathbb{Z}_k$ parafermion characters χ^{ℓ}_m, that are specified by the $SU(2)_k$ quantum number $\ell = 0, 1, \ldots, k$, and the Abelian number m modulo $2k$. There is a $\mathbb{Z}_k$ parity rule between the two Abelian numbers, that is $\lambda = m$ modulo k (note that $p = 2$ modulo k). The parafermion characters obey the periodicities $\chi^{\ell}_m = \chi^{\ell}_{m+2k} = \chi^{k-\ell}_{m+k}$ and vanish for $m + \ell = 1$ modulo 2. Taking into account these properties, one finds the periodicity $\Theta^{\ell}_{a+p} = \Theta^{k-\ell}_a$, implying $p(k + 1)/2$ independent anyon sectors, the value of the topological order.

The Read-Rezayi parafermion theory with $k = 3$ is, then, the first non trivial stable topological insulator supporting non-Abelian excitations, owing to the fusion rules of the parafermionic field ϵ with itself, i.e. $\epsilon\times\epsilon = I+\epsilon$. The non-Abelian anyons are called Fibonacci anyons, since the dimension of the braiding matrices grows as the Fibonacci number F_n for the correlator of n ϵ excitations [67].

The system of Fibonacci anyons is very important: it realizes the simplest unitary transformations that can model universal quantum gates, the building blocks of quantum computation algorithms [68] [69]. An advantage of using topological insulators for the realization of a quantum computer is given by the fact that topological excitations do not decay due to the local interactions with the environment and thus they are coherent for long time. This field of research is called "topological quantum computation" [68]. The challenge of finding experimental realizations of non-Abelian topological insulators is very important; these systems could actually be simpler to realize than the corresponding quantum Hall states, owing to the larger gaps and the absence of strong magnetic fields [65].

Chapter 3

Edge interactions of non-Abelian topological insulators

In Chapter 2 we extended the validity of the Levin-Stern index (2.9) to non-Abelian topological phases through the study of their partition functions [1]. We found that the stability of the topological phases is associated to the presence of a $\mathbb{Z}_2$ anomaly and depends on the $\mathbb{Z}_2$ index (2.39)

$$(-1)^{2\Delta S}, \qquad 2\Delta S = \frac{\sigma_{sH}}{e^*} = \frac{\nu^\uparrow}{e^*} = k. \tag{3.1}$$

For k odd the index $(-1)^{2\Delta S} = -1$ indicates the presence of degenerate Kramers pairs at half-integer fluxes, implying a gapless spectrum. For k even, instead, an open question of the previous analysis is whether a non-anomalous system with index $(-1)^{2\Delta S} = 1$ does become fully gapped.

In a series of papers, two different groups answered this question. Levin and Stern in Ref.[59, 60], and Neupert at al. in Ref.[70, 71], analyzed the possible TR invariant electron interactions of the general multicomponent Abelian topological insulators discussed in Section 2.3.1. They found that the stability is based again on the index (3.1): when this is positive, that is k even, there are enough interactions for gapping all edge modes; otherwise one mode remains gapless.

In this chapter we present the corresponding analysis of interactions for non-Abelian topological insulators that we derived in Ref.[2]. We have not found a general result valid for all non-Abelian topological phases, but analyzing some well-known non-Abelian models, we were able to find a sufficient set of interactions satisfying all physical tests, that gap all edge excitations of topological insulators characterized by the index $(-1)^k = 1$.

Before explaining our result, we will review the study of interactions that gap the multicomponent Abelian topological insulators [59, 60, 70, 71]. After that, we shall introduce a "projection" that maps Abelian to non-Abelian states and use it to obtain the corresponding gapping interactions [2].

3.1 Time-reversal invariant interactions in Abelian theories

In Chapter 2 we modeled the fractional topological insulators with $\nu^{\uparrow} = 1/p$ by pairs of Laughlin Hall states carrying opposite spin and chirality. These states can be generalized to $2N$ layers with chiral central charge $c^{\uparrow} = N$ and their interactions can be described by using the so called K-matrix formalism [59, 60]. To this end we introduce the $2N \times 2N$ symmetric invertible matrix $\mathcal{K}$ with integer entries, that has the following form

$$\mathcal{K} = \begin{pmatrix} K & W \\ W^T & -K \end{pmatrix}, \tag{3.2}$$

where K is a $N \times N$ matrix and $W^T = -W$. This matrix parametrizes the couplings of the N components chiral-antichiral bosonic theory that is TR symmetric [70]. The spectrum of the excitations is characterized by a $2N$-dimensional Lorentian lattice of conformal weights and charges whose Gram matrix is given by (3.2). The electron excitations are specified by vectors Λ with $2N$ integer components, such that their exchange statistics and charges, given by $\theta/\pi = \Lambda^T \mathcal{K} \Lambda$ and $Q = \Lambda^T \rho$ respectively, are integer valued. Note that ρ is the so-called charge vector, made of two N-dimensional vectors, $\rho = (\rho^{\uparrow}, \rho^{\downarrow})$, $\rho^{\uparrow} = \rho^{\downarrow} = (1, \ldots, 1)$ within our conventions of lattice coordinates. The elementary electron excitations correspond to the basis vectors $\Lambda = e_i$, that are equal to one in the i-th position and zero elsewhere, $i = 1, \ldots, 2N$. The electron fields are given by normal ordered vertex operators of the $2N$-component bosonic field $\Phi(t, x) = (\phi_1(x, t), \cdots, \phi_{2N}(x, t))$, as follows:

$$\Psi_i^{\dagger}(t, x) =: \exp\left(-i e_i^T \mathcal{K} \Phi(t, x)\right):, \qquad i = 1, \cdots, 2N. \tag{3.3}$$

If $W = 0$ in (3.2), the first N operators, $i = 1, \cdots, N$, represent chiral spin-up electrons and the second N ones antichiral spin-down electrons; if $W \neq 0$, the first (resp. second) N operators describe electrons with spin up (down) with mixed chiralities.

The time-reversal $\mathcal{T}$ transformations act on the bosonic field as follows [60]:

$$\mathcal{T}\ \Phi(t, x)\ \mathcal{T}^{-1} = \Sigma_1 \Phi(-t, x) + \pi \mathcal{K}^{-1} \Sigma_{\downarrow} \rho, \tag{3.4}$$

where

$$\Sigma_1 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \qquad \Sigma_{\downarrow} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \tag{3.5}$$

are $2N \times 2N$ block matrices. Time-reversal symmetry implies $\mathcal{K} = -\Sigma_1\ \mathcal{K}\ \Sigma_1$, and $\rho = \Sigma_1 \rho$. From (3.4) we derive the following TR transformation of the basic fermionic fields (3.3) [60]

$$\mathcal{T}: \quad \Psi_i^{\dagger} \ \to\ \Psi_{i+N}^{\dagger}, \qquad\qquad \Psi_{i+N}^{\dagger} \ \to\ -\Psi_i^{\dagger}, \qquad i = 1, \ldots, N, \tag{3.6}$$

generalizing the one-component case of Eq.(2.1).

The Hamiltonian H_{int} of electronic edge interactions is expressed in terms of vertex operators U_{Λ_i} as follows:

$$\begin{aligned} H_{\mathrm{int}} &= \int dt\ \sum_i g_i\, U_{\Lambda_i} + \mathrm{h.c.}, \\ U_{\Lambda_i}(t, x) &=: \exp\big(-i \Lambda_i^T \mathcal{K} \Phi(t, x)\big):, \end{aligned} \tag{3.7}$$

where Λ_i are integer vectors subjected to the conditions specified below. The coupling constant g_i can be complex and space dependent to account for interactions at impurities, possibly leading to $g_i \to \infty$, such that both relevant and irrelevant interactions, in the renormalization group sense, should be considered.

The condition on Λ_i for realizing admissible TR symmetric interactions are [60, 70]:

i) charge neutrality,

$$Q = \Lambda_i^T \rho = 0; \tag{3.8}$$

ii) mutual locality of interactions (Haldane null vector criterion [72]),

$$\frac{\theta}{\pi} = \Lambda_i^T \mathcal{K} \Lambda_j = 0, \quad \forall i, j; \tag{3.9}$$

iii) time-reversal invariance of H_{int} [60],

$$\Sigma_1 \Lambda_i = \pm \Lambda_i, \qquad \Lambda_i^T \ \Sigma_\downarrow \ \rho = \text{even}, \quad \forall i; \tag{3.10}$$

iv) linear independence of the Λ_i and, more strongly, the following minimality, or "primitivity", condition, requiring that products of interactions are not polynomials of simpler interactions [60]:

$$n_1 \Lambda_1 + \cdots + n_k \Lambda_k \neq m \Lambda, \quad \text{with} \quad m > 1. \tag{3.11}$$

Actually, solutions to this equation with integer Λ vector and $m > 1$ could imply spontaneous symmetry breaking of TR symmetry. For example, in the one component case, the square of the mass term $U = (\overline{\Psi}^\dagger \Psi)^2$ is TR invariant, but it would also induce an expectation value for the more primitive interactions $\langle \overline{\Psi}^\dagger \Psi \rangle \neq 0$ that is TR breaking. Since we do not allowed explicit and spontaneous breaking of TR symmetry, these kinds of interactions are discarded.

The stability analysis of Refs.[60, 70] showed that for the general non-anomalous Abelian theory, there always exist N interactions that obey the previous conditions and gap all N edge modes. Let us recall the main steps of this proof.

First, observe that $(N-1)$ gapping interactions can always be found for any matrix $\mathcal{K}$, such that only one massless mode is possible, at most. These $(N-1)$ solutions to conditions (3.8)-(3.11) are the eigenvectors of the Σ_1 matrix (3.5) with eigenvalue one, that can be taken of the form:

$$\Lambda_1 = \left(\Lambda_1^\uparrow, \Lambda_1^\downarrow\right) = \big(\underbrace{1, -1, 0, \cdots, 0}_{N}, \underbrace{1, -1, 0, \cdots, 0}_{N}\big), \tag{3.12}$$

$$\vdots$$

$$\Lambda_{N-1} = \left(\Lambda_{N-1}^\uparrow, \Lambda_{N-1}^\downarrow\right) = \big(\underbrace{1, 0, \cdots, 0, -1}_{N}, \underbrace{1, 0, \cdots, 0, -1}_{N}\big). \tag{3.13}$$

These vectors are globally neutral, $Q_i = \Lambda_i^T \rho = 0$, but they also have neutral chiral components, $\Lambda_i^{\uparrow T} \rho^\uparrow = \Lambda_i^{\downarrow T} \rho^\downarrow = 0$.

The N-th solution for the gapping interactions depends on the form of $\mathcal{K}$. We consider, for simplicity, the diagonal case, i.e. with $W = 0$, and define the vector [70]

$$\overline{\Lambda} = r \begin{pmatrix} K^{-1} \rho^\uparrow \\ -K^{-1} \rho^\downarrow \end{pmatrix}, \tag{3.14}$$

where r is the smallest integer such that all components of $\overline{\Lambda}$ are integers. It turns out that (3.14) is a eigenvector of Σ_1 with eigenvalue -1 obeying the conditions (3.8), (3.9) and (3.11). There remains the condition (3.10), that is actually related to the Levin-Stern index (3.1) [70]

$$2\Delta S = -\overline{\Lambda}^T\ \Sigma_\downarrow\ \rho = r\,\rho^{\downarrow T} K^{-1} \rho^\downarrow = \text{even}. \tag{3.15}$$

Namely, the condition for the remaining $N-$th interaction to be TR invariant is the same as that coming from the flux insertion argument. It then follows that the $\mathbb{Z}_2$ anomalous topological insulators ($(-1)^{2\Delta S} = -1$) posses a massless edge spectrum due to the flux insertion argument, while the non-anomalous systems ($(-1)^{2\Delta S} = 1$) have enough interactions to become gapful.

3.2 Invariant interactions in the Pfaffian topological insulator

In this section we review the analysis of TR interactions in non-Abelian topological insulators [2]. Our approach uses the known result that some non-Abelian Hall states can be obtained from certain Abelian systems, called "parent states", by a projection of degrees of freedom [73]. Since this projection does not spoil the TR symmetry, one can use it to export the general analysis of interactions in the Abelian states to the non-Abelian models. Let us start from recalling the relation between the Pfaffian state and its Abelian parent state, the so called Halperin (331) state. We shall describe the map of fields and interactions between the two theories.

3.2.1 From the (331) to the Pfaffian state

The (331) and the Pfaffian states have the same filling fraction, minimal electric charge and Levin-Stern index, i.e.

$$\nu^\uparrow = \frac{1}{2}, \qquad e^* = \frac{1}{4}, \qquad 2\Delta S = 2. \qquad (-1)^{2\Delta S} = 1, \tag{3.16}$$

from which follows that both are expected to be unstable topological states. Let us show how the Pfaffian state can be described as the projection from its "parent" Abelian (331) state [73].

The (331) ground state wavefunction is

$$\Psi_{(331)}(z_i; w_j) = \prod_{i<j}^N z_{ij}^3 \prod_{i<j}^N w_{ij}^3 \prod_{i,j}^N (z_i - w_j), \tag{3.17}$$

where $z_{ij} = z_i - z_j$, $w_{ij} = w_i - w_j$ and it corresponds to the following K-matrix

$$K = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}. \tag{3.18}$$

The two sets of coordinates z_i and w_i, $i = 1, \ldots, N$, pertain to electrons that are distinct by an additional quantum number, say isospin up and down, and thus the wavefunction is only antisymmetric for exchanges of coordinates of the same kind [73, 74].

The Abelian theory possesses two kinds of electrons (3.3). We shall use the basis where neutral and charged components are separated [73, 74]; both parts are expressed by vertex operators of chiral bosonic fields, φ and ϕ, that are linear combinations of earlier field Φ in (3.3):

$$V = \exp\left(i\alpha\varphi\right), \qquad F = \exp\left(i\phi\right), \tag{3.19}$$

with $\alpha = \sqrt{2}$. The dimensions of the fields are $h_V = \alpha^2/2 = 1$ and $h_F = 1/2$. The field F is actually a Weyl fermion whose charge does not contribute to the electric charge but accounts for the isospin [73]. The two edge electrons with spin up, can be written in the decomposition as follows

$$\Psi_1 = V\,F, \qquad \Psi_2 = V\,F^\dagger. \tag{3.20}$$

The corresponding topological insulators are obtained as usual by doubling the edge modes; thus the two antichiral electrons with spin-down are represented by the following vertex operators

$$\Psi_3 = \overline{V}\,\overline{F}, \qquad \Psi_4 = \overline{V}\,\overline{F}^\dagger, \tag{3.21}$$

where $\overline{V} = \exp(-i\alpha\overline{\varphi})$ and $\overline{F} = \exp(-i\alpha\overline{\phi})$. Note that in our notation the bar denotes antichirality, e.g. $\varphi = \varphi(z)$, $\overline{\varphi} = \overline{\varphi}(\bar{z})$, while the dagger refers to Fock space operators. The two TR invariant interactions of the (331) Abelian theory are obtained by the methods described in Section 3.1; the first one is associated to the lattice vector Λ_1 (3.12) and the second one is obtained by specializing the expression of the vector $\overline{\Lambda}$ (3.14) for the K matrix (3.18). They read:

$$\Lambda_1 = (1, -1, 1, -1), \qquad \overline{\Lambda} = (1, 1, -1, -1). \tag{3.22}$$

These vectors determine the edge interactions (3.7) that can written in terms of normal-ordered product of fermionic fields:

$$U_N =: \Psi_1^\dagger\,\Psi_2\,\Psi_3^\dagger\,\Psi_4 : + \text{h.c.}, \tag{3.23}$$
$$U_C =: \Psi_1^\dagger\,\Psi_2^\dagger\,\Psi_3\,\Psi_4 : + \text{h.c.}. \tag{3.24}$$

The labels N and C in (3.24) refer to the fact that U_N exchanges a particle and an antiparticle $(Q, \overline{Q}) = (0, 0)$ in each chirality (chiral neutral), while U_C exchanges two chiral particles and two antichiral antiparticles $(Q, \overline{Q}) = (2, -2)$ (chiral charged). According to the earlier discussion, the two TR invariant interactions U_N and U_C are sufficient to completely gap the (331) state.

We now introduce the projection that leads to the Pfaffian state [24]. The wavefunction of the Pfaffian theory can be obtained from the (331) wavefunction (3.17) by antisymmetrizing with respect to all $2N$ electron coordinates, such that the isospin quantum number is washed out. Indeed, the following relation holds [73]:

$$\Psi_{\mathrm{Pfaff}}(z_i, z_{i+n}) = \mathcal{A}\left[\Psi_{(331)}(z_i; w_j)\right] = \prod_{i<j}^{2N} z_{ij}^2 \ \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right), \tag{3.25}$$

where $\mathcal{A}[..]$ denotes antisymmetrization over all the $2N$ coordinates $(z_i, w_j = z_{N+j},\ i, j = 1, \cdots, N)$.

At the operator level, the projection from the Abelian to the Pfaffian states is obtained by identifying the two species of Abelian fermions $\Psi_1 \sim \Psi_2$ [73]. This amounts to projecting the Weyl fermion to a neutral Majorana fermion, $F \to \psi$ and $F^\dagger \to \psi$:

$$\Psi_1 \ \to \ V\,\psi, \qquad \Psi_2 \ \to \ V\,\psi. \tag{3.26}$$

The corresponding map between conformal theories relates the Abelian $U(1) \times U(1)$ to the non-Abelian $U(1) \times$ Ising theory, with central charges $c = 2$ and $c = 3/2$, respectively [24]. The analogous map between the Abelian and Pfaffian topological insulators is done remembering that, besides the chiral spin-up Hall states discussed so far, there are corresponding antichiral spin-down states, whose electrons fields Ψ_{i+2}, $i = 1, 2$ are similarly projected into antichiral Pfaffian fields: $\Psi_{3,4} \ \to \ \overline{V}\,\overline{\psi}$. Summarizing, we have the following map between electrons in the two theories [2],

$$\begin{aligned}
\Psi_i^\dagger \quad &\to \quad :\exp(-i\alpha\varphi):\psi \ = \ V^\dagger\,\psi, \qquad i = 1, 2, \\
\Psi_i \quad &\to \quad :\exp(i\alpha\varphi):\psi \ = \ V\,\psi, \\
\Psi_{i+2}^\dagger \quad &\to \quad :\exp(i\alpha\overline{\varphi}):\overline{\psi} \ = \ \overline{V}^\dagger\,\overline{\psi}, \\
\Psi_{i+2} \quad &\to \quad :\exp(-i\alpha\overline{\varphi}):\overline{\psi} \ = \ \overline{V}\,\overline{\psi}.
\end{aligned} \tag{3.27}$$

The TR transformations of the electron fields (3.6) are left invariant by the projection (3.26), and act on the fields of the Pfaffian theory as follows:

$$\begin{aligned}
\mathcal{T}:\quad \Psi_i^\dagger \ &= \ V^\dagger\,\chi \quad \to \quad \Psi_{i+2}^\dagger = \overline{V}^\dagger\,\overline{\chi}, \qquad i = 1, 2, \\
\Psi_{i+2}^\dagger \ &= \ \overline{V}^\dagger\,\overline{\chi} \quad \to \quad -\Psi_i^\dagger = -V^\dagger\,\chi.
\end{aligned} \tag{3.28}$$

These correspond to the following transformations of the charged V and neutral ψ fields [2]

$$\begin{aligned}
\mathcal{T}:\quad V^\dagger \ &\to \ \overline{V}^\dagger, \qquad \overline{V}^\dagger \ \to \ -V^\dagger, \\
\psi \ &\to \ \overline{\psi}, \qquad \overline{\psi} \ \to \ \psi.
\end{aligned} \tag{3.29}$$

3.2.2 Projected interactions

The projection (3.27) does not affect the TR symmetry of states and operators, then, following our work [2], it will be employed to find the gapping interactions for the Pfaffian state from the known expressions for the Abelian (331) state, i.e. (3.23) and (3.24).

Since the maps (3.27) apply to individual fermion fields, we should first undo the normal ordering in (3.24) by point splitting, then apply the projection and finally re-normal order the result in the Pfaffian theory. Let us consider the two interactions U_N and U_C in turn. Upon using the normal-ordering (2.34), the point splitting of U_N reads:

$$U_N = \lim_{z_1 \to z_2} z_{12}^M \, \overline{z}_{12}^M \; \Psi_1^\dagger(z_1)\, \Psi_2(z_2)\, \Psi_3^\dagger(\overline{z}_1)\, \Psi_4(\overline{z}_2) + \text{h.c.}. \tag{3.30}$$

Applying the projection (3.27) to each field in this expression and using once again (2.34) to normal-order the vertex operators $VV^\dagger$, we obtain:

$$U_N \;\to\; U_N^{\text{Pfaff}} = \lim_{z_1 \to z_2} \left[\frac{1}{z_{12}} \; : V^\dagger(z_1)\, V(z_2) : \; \psi(z_1)\, \psi(z_2) \right]_{\text{reg.}} \times \left[\; z \to \bar{z} \; \right]. \tag{3.31}$$

We now consider the expansions of the two vertex operators and the Majorana fields for $z_1 \to z_2$. They are (omitting constants),

$$: V^\dagger(\varepsilon)\, V(0) := 1 + \varepsilon \partial\varphi + \varepsilon^2 \left((\partial\varphi)^2 + \partial^2\varphi \right) + O(\varepsilon^3), \tag{3.32}$$

and

$$\chi(\varepsilon)\, \chi(0) = \frac{1}{\varepsilon} + \varepsilon : \chi\partial\chi : + O(\varepsilon^3). \tag{3.33}$$

Owing to the fusion rules $V^\dagger \cdot V \sim I$ and $\psi \cdot \psi = I$, taking the first finite terms in U_N^{Pfaff} for $z_1 \to z_2$ correspond to take descendant fields in the conformal representation (sector) of the identity field I of both the charged $c = 1$ and neutral Majorana $c = 1/2$ theories [21]: schematically, $U_{\Lambda_1}^{\text{Pfaff}} = [I]_{c=1}\, [I]_{c=1/2}$.

The final expression of $U_{\Lambda_1}^{\text{Pfaff}}$ is obtained by selecting the finite terms for $\varepsilon \to 0$ in the product of (3.32) and (3.33), that is the normal-ordering procedure for general CFT. Neglecting total derivatives, we finally obtain [2]:

$$U_N^{\text{Pfaff}} = \left(2T_n + \alpha^2\, T_c \right)\left(2\overline{T}_n + \alpha^2\, \overline{T}_c \right), \tag{3.34}$$

where $T_n = -\psi\partial\psi/2$ and $T_c = -(\partial\varphi)^2/2$ are the stress tensors of the Majorana fermion and bosonic theory, respectively [21].

Following similar steps, for the interaction U_C in (3.24) we find the expression:

$$U_C \;\to\; U_C^{\text{Pfaff}} = \lim_{z_1 \to z_2} \left[z_{12} \; : V(z_1)^\dagger\, V(z_2)^\dagger : \; \psi(z_1)\, \psi(z_2) \right]_{\text{reg.}} \times \left[\; z \to \bar{z} \; \right] + \text{h.c.}. \tag{3.35}$$

The conformal sectors now involved are $U_C^{\text{Pfaff}} = \left[V^{\dagger 2} \right]_{c=1} [I]_{c=1/2}$, since the original Abelian interaction had charged chiral/antichiral parts, i.e. $Q = (2, -2)$. After re-normal ordering, we finally obtain [2]:

$$U_C^{\text{Pfaff}} =: \left(V^\dagger \right)^2 \left(\overline{V} \right)^2 : + \text{h.c.}. \tag{3.36}$$

3.2.3 Properties of non-Abelian interactions

Using Eq.(3.29) we can verify that both U_N^{Pfaff} and U_C^{Pfaff} are TR invariant expressions. Actually, they cannot break TR symmetry spontaneously either. As discussed in Section 3.1, the quartic interaction $\left(\Psi^\dagger \Psi \right)^2$ must be discarded, because it would imply the symmetry breaking expectation value of the "square-root" $\langle \Psi^\dagger \Psi \rangle \neq 0$ [59, 60, 70]. Note that $U_N^{\rm Pfaff}$ is quartic in the Majorana field, but it is the minimal or "primitive" interaction, since the simpler quadratic interaction in the (331) theory [59], $U = \Psi_1^\dagger \Psi_4 - \Psi_2^\dagger \Psi_3 + {\rm h.c.}$, vanishes when projected to the Pfaffian. $U_C^{\rm Pfaff}$ is quartic as well, but the corresponding TR breaking quantity $V^\dagger \overline{V}$ cannot acquire an expectation value owing to the non locality with respect to some excitations. For example, we consider the chiral quasiparticle $V_\sigma \sigma$ of minimal charge $Q = 1/4$. The operator product expansion between the charge part $V_\sigma = \exp(i1/\sqrt{8}\varphi)$ and $V^\dagger \overline{V}$ is given by $V_\sigma V^\dagger \sim z^{-1/2}$. Therefore, the following correlator involving two σ has a square-root branch-cut and nontrivial monodromy:

$$\begin{aligned} \left\langle V^\dagger(0)\overline{V}(0)\, V_\sigma\left(z e^{i2\pi}\right) V_\sigma(w) \right\rangle &= -\left\langle V^\dagger(0)\overline{V}(0)\, V_\sigma(z) V_\sigma(w) \right\rangle \\ &\neq \left\langle V^\dagger(0)\overline{V}(0) \right\rangle \langle V_\sigma(z) V_\sigma(w) \rangle . \end{aligned} \tag{3.37}$$

Equation (3.37) implies $\langle V^\dagger \overline{V} \rangle = 0$.

The interaction $U_C^{\rm Pfaff}$, with chiral and antichiral charges $(Q, \overline{Q}) = (2, 2)$, is non-vanishing for the correlator that describes scattering with charge transfer between the two chiralities, as e.g.

$$\langle \chi^\dagger(z_1)\, \chi^\dagger(z_2) \overline{\chi}(\overline{z}_3)\, \overline{\chi}(\overline{z}_4)\, U_N^{\rm Pfaff} \rangle \neq 0. \tag{3.38}$$

The excitations χ realizing this process must be charged electrons or quasiparticles. It follows that all them acquire mass.

The Pfaffian theory further possesses neutral quasiparticle excitations that do not couple to $U_C^{\rm Pfaff}$. Indeed, their scattering processes do not involve charge transfer and correspond to vanishing chiral correlators, as e.g. $\langle \chi^\dagger(z_1)\, \chi(z_2)\, U_\Lambda^{\rm Pfaff} \rangle = 0$. Neutral excitations couple to the other interaction $U_N^{\rm Pfaff}$ with $(Q, \bar{Q}) = (0, 0)$, since $\langle \chi^\dagger(z_1)\, \chi(z_2)\, U_\Lambda^{\rm Pfaff} \rangle \neq 0$. However, this coupling cannot give mass to neutral excitations; this is an established fact in the literature of perturbations of the Ising model [75]. Actually the interaction $gU_N^{\rm Pfaff} = gT_n\overline{T}_n$, with $T_n \sim \psi\partial\psi$, describes the renormalization group flow from the tricritical to the critical Ising model, as viewed from the low-energy end point. All along this flow, the Majorana field ψ stays massless, thus this interaction breaks conformal invariance but leave a massless neutral state at low-energy. Therefore $U_N^{\rm Pfaff}$ cannot cause the instability of the Pfaffian topological insulators.

In Section 2.4.3, we described the partition function of the Pfaffian topological insulator; after summation of the Neveu-Schwarz and Ramond sectors (B.13), it takes the form

$$\begin{aligned} Z_{\rm Pf\ TI} &= Z_{NS} + Z_{\widetilde{NS}} + Z_R + Z_{\widetilde{R}} \\ &= 2 \sum_{a=-3}^{4} \left(\left|K_a I\right|^2 + \left|K_a \psi\right|^2 + \left|K_a \sigma\right|^2 \right). \end{aligned} \tag{3.39}$$

In this expression, the K_a are characters of $U(1)$ representations corresponding to the Abelian parts of excitations, carrying charge $Q = a/4 + 2\mathbb{Z}$, while the characters I, ψ and σ describe the neutral non-Abelian parts, being the identity, fermion and spin of the Ising model, respectively [21].

In presence of the charged interaction U_C^{Pfaff} with large coupling, all charged excitations become highly massive, such that $K_a \to \delta_{a,0}$ in (3.39) (up to an irrelevant factor). Therefore, there remain the neutral excitations of the Ising model,

$$Z_{\mathrm{Pf\ TI}} \to Z_{\mathrm{Ising}} = \left|I\right|^2 + \left|\psi\right|^2 + \left|\sigma\right|^2, \tag{3.40}$$

that are time-reversal invariant and non-chiral.

We now remark that the Ising model (3.40) possesses another relevant interaction,

$$U_{qp} = m\overline{\psi}\psi, \tag{3.41}$$

that generically gives mass to the theory. This corresponds to a quasiparticle interaction in the original Pfaffian topological insulator that was not considered before. Actually, earlier discussions were limited to electron interactions, because they are local with respect to all chiral excitations of topological insulators and can easily implemented in microscopic models. Quasiparticle interactions, such as (3.41), were discarded because they can be non-local with some chiral quasiparticles. However, in the reduced theory (3.40), electrons and charged chiral quasiparticles have disappeared, thus U_{qp} is local with respect to the remaining neutral (non-chiral) excitations and is acceptable.

In conclusion, in the Pfaffian topological insulator we introduced a quasiparticle interaction for gapping the neutral non-Abelian modes that is allowed when the charged excitations are infinitely massive. This argument requires a separation of scales between heavy charged excitations and light neutral excitations, that is not required in the corresponding analysis of Abelian systems. Moreover, such quasiparticle interaction is generically unavoidable, but its microscopic origin is yet unclear.

3.3 Invariant interactions in the Read-Rezayi topological insulators

3.3.1 Projected interactions

The Read-Rezayi Hall states describe the binding of identical electrons in clusters of k elements, extending the $k = 2$ case of the Pfaffian [67]. In the conformal field theory description, the electrons are represented by the $\mathbb{Z}_k$ parafermion field χ_1, whose k-th fusion with itself yields the identity, $(\chi_1)^k \sim I$, leading to a non-vanishing correlator at coincident points. The parafermion conformal field theory can be realized by the coset $\widehat{SU(2)}_k/U(1)$ [66]. As usual, excitations also have a charge part expressed by vertex operators, leading to a theory with central charge $c = 1 + c_k$, $c_k = 2(k-1)/(k+2)$. The fusion of n parafermions χ_1 define the parafermion field χ_n; these fields obey Abelian fusion rules among themselves:

$$\chi_i \cdot \chi_j \sim \chi_\ell, \qquad \ell = i + j \bmod k, \tag{3.42}$$

that conserve a $\mathbb{Z}_k$ quantum number. Moreover, they obey $\chi_n^\dagger = \chi_{k-n}$.

The parent Abelian theory of the Read-Rezayi state is a k-fluid generalization of the (331) state with the following K matrix [67, 73]:

$$K_{ij} = \begin{cases} M+2 & i=j=1,\cdots,k, \\ M & i\neq j\,. \end{cases} \tag{3.43}$$

The Abelian and non-Abelian systems share the same spectrum of charges: the filling fraction and minimal charge are,

$$\nu^\uparrow = \frac{k}{kM+2}, \qquad e^* = \frac{1}{kM+2}, \qquad 2\Delta S = k, \qquad (-1)^{2\Delta S} = (-1)^k. \tag{3.44}$$

Thus, the Levin-Stern index tells us that the topological insulators made by pairs of Read-Rezayi states, as well as their parent states, are stable (unstable) for k odd (even).

The general study of interactions of the Abelian theory described earlier yields k TR invariant gapping interactions specified by $(k-1)$ vectors Λ_i and by $\overline{\Lambda}$, given in (3.12)-(3.13) and (3.14), respectively. In particular, the interaction corresponding to $\overline{\Lambda}$ is TR invariant only for k even, in agreement with the Levin-Stern index.

We now proceed to describe the projection to the non-Abelian state by studying its action on the conformal fields. In the present case the projection maps k different chiral species into a single one and the corresponding electron fields Ψ_i, $i=1,\ldots,k$ in (3.3) go into the Read-Rezayi electron $\psi = V\chi_1$, where V is the charged vertex operator and χ_1 the first parafermion. More precisely, the correspondence is as follows:

$$\begin{aligned}
&\Psi_i^\dagger && \to \quad :\exp\big(-i\alpha\varphi\big):\chi_1 && = V^\dagger\,\chi_1, \qquad i=1,\ldots,k \\
&\Psi_i && \to \quad :\exp\big(i\alpha\varphi\big):\chi_1^\dagger && = V\,\chi_{k-1}, \\
&\Psi_{i+k}^\dagger && \to \quad :\exp\big(i\alpha\overline{\varphi}\big):\overline{\chi}_1 && = \overline{V}^\dagger\,\overline{\chi}_1, \\
&\Psi_{i+k} && \to \quad :\exp\big(-i\alpha\overline{\varphi}\big):\overline{\chi}_1^\dagger && = \overline{V}\,\overline{\chi}_{k-1},
\end{aligned} \tag{3.45}$$

where $\alpha^2 = (2+kM)/k$ and $\varphi = \varphi(z)$, $\overline{\varphi} = \overline{\varphi}(\bar{z})$.

From the map of the fields we obtain the expressions of the non-Abelian interactions as follows. The U_{Λ_i} are quartic in the fermion fields as in the $k=2$ case, Eq. (3.24), and their projection follows similar steps. After point splitting and projection, one obtains:

$$U_{\Lambda_i} \;\to\; U_\Lambda^{\rm RR} = \lim_{z_1\to z_2}\left[z_{12}^{-2/k}\;:V^\dagger(z_1)\,V(z_2):\;\chi_1(z_1)\,\chi_{k-1}(z_2)\right]_{\rm reg.} \times \left[\;z\to\bar{z}\;\right]. \tag{3.46}$$

These interactions, for $i=1,\ldots,k-1$, are all projected into the same expression $U_\Lambda^{\rm RR}$, that involves the identity sectors for both the charged and neutral $\mathbb{Z}_k$ parafermion theories, owing to the fusion rules $V^\dagger\cdot V\sim I$ and $\chi_1\cdot\chi_{k-1}\sim I$, respectively. The normal ordering of vertex operators is the same as in the Pfaffian case. For the parafermions we use the general operator expansion of descendant fields in the identity sector [21],

$$\chi_1(\varepsilon)\,\chi_{k-1}(0) = \varepsilon^{-2+2/k} + \frac{2h_1}{c_k}\varepsilon^{2/k}\;T_n(0) + :\chi_1(0)\chi_{k-1}(0): + \cdots\,, \tag{3.47}$$

where T_n is the stress tensor of the parafermion theory, c_k its central charge and $h_1 = (k-1)/k$ the conformal dimension of χ_1. Combining all together, we obtain:

$$U_{\Lambda}^{RR} = \left(\frac{2h_1}{c_k}T_n + \alpha^2\, T_c\right)\left(\frac{2h_1}{c_k}\overline{T}_n + \alpha^2\, \overline{T}_c\right). \tag{3.48}$$

This interaction takes the same $T\overline{T}$ form of descendent of the identity already found in the Pfaffian case, and fulfills the same properties. In particular, it cannot provide a mass for neutral excitations of these topological insulators.

The projection of the Abelian interaction corresponding to $\overline{\Lambda}$ is slightly more difficult, because it involves $2k$ fermionic fields:

$$\begin{aligned} U_{\overline{\Lambda}} &= :\prod_{i=1}^{k} \Psi_i^{\dagger} :\, :\prod_{i=1}^{k} \Psi_{k+i} : +\text{h.c.} \\ &= :\prod_{i=1}^{k} \exp\Big(-ie_i \mathcal{K}\Phi(z)\Big) :\, :\prod_{i=1}^{k} \exp\Big(i\bar{e}_i \mathcal{K}\Phi(z)\Big) : +\text{h.c.}. \end{aligned} \tag{3.49}$$

The operators in each chiral part should be split in k different points $\{z_1, \ldots, z_k\}$, with $|z_i - z_j| = \varepsilon \ \ \forall i, j$, and later brought back to a common point, $\varepsilon \to 0$. We use the formula for the normal ordering of k vertex operators [21],

$$\prod_{i=1}^{k} : \exp\left(-ie_i\mathcal{K}\Phi(z_i)\right) := \prod_{i<j}^{k}(z_i - z_j)^M \ \ : \exp\left[-i\left(\sum_{i=1}^{k} e_i\right)\mathcal{K}\Phi(z_i)\right] :, \tag{3.50}$$

where the exponent M is given by the K-matrix element. Upon performing the projection (3.45) on individual fermion fields, we re-normal order the k vertex operators $V^{\dagger}$, and obtain:

$$U_{\overline{\Lambda}} \to U_{\overline{\Lambda}}^{RR} = \lim_{\varepsilon\to 0}\left[\prod_{i<j}^{k} z_{ij}^{2/k} : \Big(V^{\dagger}(z)\Big)^k : \prod_{i=1}^{k}\chi_i(z_i)\right] \times \Big[z \to \bar{z}\Big] + \text{h.c.}. \tag{3.51}$$

The normal ordering of parafermion fields uses the operator product expansions [21],

$$\begin{aligned} \chi_{\ell}(z)\chi_{\ell'}(0) &\sim z^{-2\ell\ell'/k}\,\chi_{\ell+\ell'}(0) + \cdots, \qquad (\ell+\ell' < k), && (3.52)\\ \chi_{\ell}(z)\chi_{k-\ell}(0) &\sim z^{-2\ell(k-\ell)/k}\left(1 + z^2\frac{2h_{\ell}}{c_k}T_n(0) + \cdots\right), && (3.53) \end{aligned}$$

where $h_{\ell} = \ell(k-\ell)/k$ is the dimension of χ_{ℓ}. The coincidence limit of the first $(k-1)$ coordinates $z_i - z = \varepsilon \to 0$, creates the parafermion field χ_{k-1} with singular behavior given by the sum of exponents in (3.52); for the k-th limit we use (3.53) involving the stress tensor, and obtain,

$$\lim_{\varepsilon\to 0}\prod_{i=1}^{k}\chi_i(z_i) = \varepsilon^{1-k}\left(1 + \varepsilon^2\frac{2h_1}{c_k}T_n(z)\right). \tag{3.54}$$

This singularity exactly cancels that coming from the product of vertex operators in (3.51), leading to final result:

$$U_{\overline{\Lambda}}^{RR} =: V^{\dagger k}(z)\overline{V}^k(\bar{z}) + \text{h.c.}. \tag{3.55}$$

3.3.2 Properties of interactions

The TR transformations of Read-Rezayi fields is again inherited from the Abelian fields (3.6) through the projection (3.45) and read:

$$\begin{aligned} \mathcal{T}: \quad & V^\dagger \to \overline{V}^\dagger, \qquad && \overline{V}^\dagger \to -V^\dagger, \\ & \chi_i \to \overline{\chi}_i, && \overline{\chi}_i \to \chi_i, \end{aligned} \tag{3.56}$$

in complete analogy with (3.29). It follows that the neutral interaction $U_\Lambda^{RR} \sim \overline{T}T$ is time-reversal invariant for any k, while the charged interaction $U_{\overline{\Lambda}}^{RR}$ is only invariant for k even, as in the parent Abelian theory. We stress that this result is in agreement with the Levin-Stern index (3.44) [1]: for k odd, $U_{\overline{\Lambda}}^{RR}$ is forbidden by TR symmetry and some charged edge excitations remain massless.

For k even, the Read-Rezayi topological insulators are unstable according to the flux argument [1]. The charged interaction $U_{\overline{\Lambda}}^{RR}$ is allowed and gaps all charged excitations. We should again consider the possibility that $U_{\overline{\Lambda}}^{RR}$ breaks spontaneously the symmetry, but this cannot happen because $V^\dagger\overline{V}$ is non-local, for example, with respect the field operator of the smallest charge e^* in (3.44). Regarding the neutral modes, $U_\Lambda^{RR} \sim \overline{T}T$ it is not sufficient to give them mass, owing to the arguments discussed in Section 3.2.3 for $k = 2$. We then consider a quasiparticle interaction that is allowed in the reduced neutral theory where all charged states have acquired very large masses. In this limit, the single-edge partition function of Read-Rezayi topological insulators becomes the following expression [1]:

$$Z^{RR} \to \sum_{\ell=0,\ \text{even}}^{k} \left|\chi_0^\ell\right|^2 + \sum_{\ell=0,\ \ell-\frac{k}{2}\ \text{even}}^{k} \left|\chi_{\frac{k}{2}}^\ell\right|^2, \tag{3.57}$$

that contains a subset of the excitations of the Z_k parafermion statistical model [66]. The parafermionic characters χ_m^ℓ describe neutral excitations with quantum numbers $(\ell, m) \equiv (\ell, m+2k) \equiv (k-\ell, m+k)$ and $m = \ell \bmod 2$.

The quasiparticle interaction,

$$U_{qp}^{RR} = \overline{\chi}_0^2 \chi_0^2 + \text{h.c.}, \tag{3.58}$$

with dimension $2h_0^2 = 4/(k+2) < 2$, is relevant and couples to all sectors, since the fusion rules $\chi_0^2\chi_0^\ell$ and $\chi_0^2\chi_{\frac{k}{2}}^\ell$ are different from zero for any allowed value of ℓ [21]. It turns out that this interaction drives the systems with k even into a completely massive phase.

In our paper, the analysis of gapping interactions was also extended to the non-Abelian spin singlet states (NASS) [76]. Applying the same procedure of projection from their parents Abelian states, we found that there exists enough interactions to completely gap the edge of these systems [2]. It follows that these topological insulators are all unstable.

Chapter 4

Three-dimensional fermionic topological insulators

In this chapter we will discuss three dimensional topological band insulators, namely gapped systems of non-interacting fermions that are characterized by massless Dirac fermions at their boundaries. We can divide this chapter in three parts.

In the first part we will analyze the effective field theory description of these TR invariant topological phases. We will show the dimensional reduction argument that gives rise to massless boundary degrees of freedom and explain the anomaly cancellation occurring between bulk and boundary.

In the second part we shall discuss the stability of the fermionic topological phases using the partition functions of the surface fermions and the generalization of the flux insertion argument discussed in the previous chapter.

In the third part we shall study the $3D$ modular transformations of the partition functions. Once again, we will associate the stability of the topological insulators to the impossibility of writing a modular invariant partition function that is consistent with TR symmetry.

4.1 Ten-fold way classification

In the above chapters we discussed two different topological phases of matter, the quantum Hall effect and TR invariant topological insulators. In literature, the analysis of these phases was first developed in the case of non-interacting electrons through the use of band theory [11]. The analysis showed the existence of topological quantum number characterizing each phase. Indeed, the integer quantum Hall effect is classified by the first Chern number, namely the Hall conductivity, that is an integer number [77]; TR invariant topological insulators, instead, are characterized by the $\mathbb{Z}_2$ TR invariant polarization [56].

In Chapter 1 and Chapter 2 we discussed the generalization of these two-dimensional phases to the interacting case. In presence of interactions, band theory is not longer valid; thus, using effective field theory methods [10, 11, 59, 78], we found that both systems continue to be classified by the same $\mathbb{Z}$ and $\mathbb{Z}_2$ topological numbers.

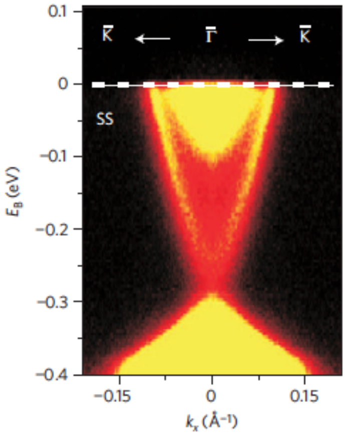
class\δ	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Figure 4.1: Periodic Table of topological insulators and superconductors. The ten symmetry classes are labeled in the first column using the notation by Atland and Zirnbauer [83]; the following three column specify the $\mathcal{T}$symmetry, $\mathcal{C}$ symmetry and $\mathcal{S}$ chiral symmetry: (± 1) and (0) denote the presence and absence of the symmetry, respectively, with ($\pm$) specifying the values of $\mathcal{T}^2$ and $\mathcal{C}^2$ equal to ± 1.

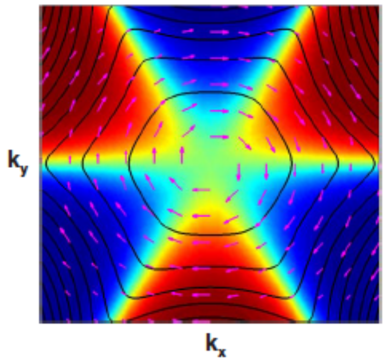
In nature there are many other topological phases of matter beyond the quantum Hall effect and topological insulators in two dimensions; all together fit into an elegant mathematical structure called *ten-fold way classification of topological insulators and superconductors*[79, 80, 81, 82]. The symmetry class depends on the presence ("± 1") or absence ("0") of time-reversal symmetry $\mathcal{T}$ with $\mathcal{T}^2 = \pm 1$, and/or charge conjugation $\mathcal{C}$ with $\mathcal{C}^2 = \pm 1$ and chirality $\mathcal{S} = \mathcal{T}\mathcal{C}$, that can be present ("1") or absent ("0"). There are ten distinct classes of gapped non-interacting fermionic systems; their classification, shown into the Periodic Table in Fig. 4.1, is closely related to the classification of random matrices by Atland and Zirnbauer [83] and is periodic in the space dimension d for $d \to d+8$. The symbols in the table "$\mathbb{Z}$", "$\mathbb{Z}_2$", "$2\mathbb{Z}$" and "0" represent whether or not the phase exists for a given symmetry class in a given dimension, and if exists, what kind of topological invariant characterizes the topological phases. For example, "$2\mathbb{Z}$" means the topological phase is characterized by an even integral topological invariant, and "0" means the topological phase is absent.

The quantum Hall effect belongs to the class A in two space dimension and is characterized by $\mathbb{Z}$, namely by the first Chern number [77]. TR invariants topological insulators have $\mathcal{T}^2 = -1$, belong to the class AII and in $d = 2$ are classified by a topological $\mathbb{Z}_2$ number [56]. As shown in Chapter 2 and Chapter 3 the $\mathbb{Z}_2$ classification continues to be valid also in presence of interactions.

In this chapter we will discuss TR invariants topological insulators of class AII in $d = 3$ that, as shown in Fig. 4.1, are classified by a $\mathbb{Z}_2$ number. In the last chapter of this thesis we will show that this classification remains valid in presence of interactions.



(*a*)



(*b*)

Figure 4.2: (a): Dirac cone between the valence and conduction bands in Bi_2Se_3 found by using ARPES technology [89]. (b): Spin texture of the conduction band of the surface states in momentum space. The arrow represent the $x-y$ planar spin polarizations while the color indicates the z component of the spin polarization [91].

4.2 Massless surface fermions

In 2006 three groups independently found that the characterization of topological insulators state has a natural generalization in three dimensions [84, 85, 86]. They showed that the stability is described in $3D$ by four $\mathbb{Z}_2$ invariants $(\nu_0; \nu_1, \nu_2, \nu_3) = \pm 1$, which are determined by the topology of the band structure.

The three indices $\nu_1, \nu_2, \nu_3 = \pm 1$ characterize the *weak* topological insulators; these phases are not particularly relevant in the following because they can be represented by stacks of $2D$ topological insulators, whose surface states interact in pairs and are usually gapped in presence of disorder.

The index $\nu_0 = \pm 1$ characterizes the *strong* topological insulators that correspond to the novel $3D$ topological states. These systems have a gap in the bulk and gapless surface states, that are protected by TR symmetry and consisting of an odd number of $(2+1)$ dimensional massless Dirac fermions, or Dirac cones. The case where there is only one of such cones is the simplest non trivial surface state, see Fig. 4.2(a). The existence of an odd number of stable massless Dirac cones on the surface is ensured by the $\mathbb{Z}_2$ topological invariant of the bulk, introduced by Fu, Kane and Mele extending their flux insertion argument to the $3D$ case [84]. In the same work, the authors show that the surface state is an *helical* fermion, namely the spin is located on the surface and is orthogonal to the momentum direction, see Fig. 4.2(b).

Kane and coworkers also indicated a candidate system of $3D$ topological insulators as the crystal Bi_xSb_{1-x}; its Dirac cone and spin texture were observed experimentally by the use of the ARPES technology (*angle-resolved photoemission spectroscopy*) [87, 88]. The Bi_xSb_{1-x} system is however too complicated to study, both theoretically and experimentally. For these reasons other crystals have been then considered, such as Bi_2Se_3, Sb_2Te_3 [89, 90, 91]. In particular, the Bi_2Se_3 has a large band gap of ~ 0.3 eV (3600 K), and can exhibit the topological behavior at room temperature, greatly increasing the potential for applications [51].

4.3 Effective field theory of fermionic topological insulators

The $\mathbb{Z}_2$ classification of $3D$ topological insulators can be understood by means of the field theory of massless fermions at the surface. In this theory a mass term breaks explicitly the TR symmetry [92]. We can have a TR invariant massive theory for pairs of fermions for which an invariant mass term can be written. In the odd case, at least one excitations remains massless.

A well understood result is that the massless surface fermion interacting with the electromagnetic field generates a Chern-Simons induced action in $(2+1)$ dimensions that is odd under TR transformations [93, 94, 95]. This TR (and parity) $\mathbb{Z}_2$ anomaly is cancelled by the bulk theory by a mechanism that is different from the anomaly inflow of Chapter 1, and will be explained in the following [3].

4.3.1 Jackiw-Rebbi dimensional reduction

The bulk states of $3D$ topological insulators can be described by a continuous field theory of a massive $(3+1)$ Dirac fermion at low energies with respect to the bulk energy gap, whose mass m vanishes near the boundary. If the surface is given by the plane $z = 0$, separating the bulk of the material ($z < 0$) from empty space ($z > 0$), we can take a mass profile of the form $M(z) = -M_0 \tanh(z/\ell)$, where M_0 is of the order of the bulk gap and ℓ of the lattice spacing (see the blue line in Fig. 4.3).

The Dirac theory with this mass profile possesses massless fermionic surface excitations that are obtained by the so-called Jackiw-Rebbi dimensional reduction [96]. To show how the method works, we consider the following representation of the Dirac γ matrices in in $(3+1)$ dimensions

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad \gamma^1 = i\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \qquad \gamma^2 = i\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad \gamma^3 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.1}$$

where the σ's are the Pauli matrices. The Dirac hamiltonian takes the form

$$H = -i\gamma^0\gamma^1\partial_x - i\gamma^0\gamma^2\partial_y - i\gamma^0\gamma^3\partial_z + \gamma^0 M(z) \equiv H_0 + H_z. \tag{4.2}$$

The surface fermion corresponds to a low-energy solutions localized near $z = 0$. These are the zero-energy eigenstates of H_z, obeying the equation

$$\left(i\partial_z + \gamma^3 M(z)\right)\psi = 0. \tag{4.3}$$

We look for a solution of the type

$$\psi(x, y, z) = f_\pm(z)u_\pm(x, y). \tag{4.4}$$

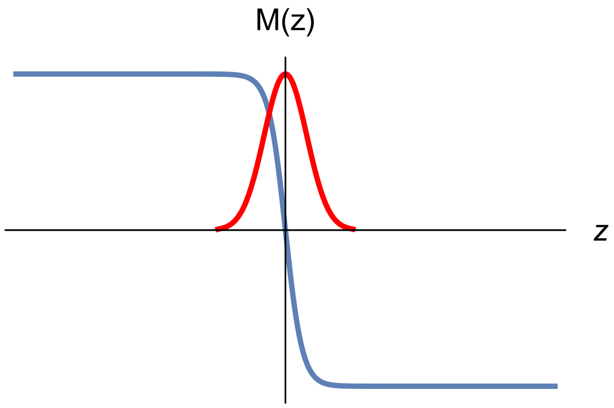


Figure 4.3: (blue line): Mass profile $M = M(z)$; (red line): amplitude probability of the zero modes of the Hamiltonian H_z.

If $u_\pm$ is such that $\gamma^3 u_\pm = \pm i u_\pm$, then Eq. (4.3) becomes

$$(\partial_z \pm M(z))\, f_\pm(z) = 0. \tag{4.5}$$

The normalizable solution to Eq.(4.3) and (4.5) are those corresponding to the $(-)$ negative eigenvalue of γ^3, i.e.

$$\psi(x, y, z) = \exp\left(-\int_0^z dz'\ M(z')\right) u_-(x, y). \tag{4.6}$$

This zero mode is localized to the surface, where the mass $m(z)$ changes sign, i.e. the red line in Fig. 4.3, thus realizing the dimensional reduction.

The surface dynamics is governed by the hamiltonian H_0 acting on spinors of the form $(u_-)^T = (0, \chi_-)$, with the lowest component χ_- a bi-component spinor. Projecting H_0 in the subspace of χ_- through the projector operator

$$P_- = \frac{1 + i\gamma^3}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.7}$$

we find the massless Dirac Hamiltonian in $(2 + 1)$ dimensions.

$$(k_y \sigma_1 - k_x \sigma_2)\chi_- = E\chi_-. \tag{4.8}$$

This is the expected Hamiltonian of a massless Dirac excitation in $(2 + 1)$ dimensions. Of course, this results holds for low energies $E \ll M_0$.

In a physical setup, the system will have two boundaries along the z axis, with the second surface described by the inverted mass profile $M(z) \to -M(z + z_0)$. Performing the same steps as before, the normalizable zero mode is now given by the positive eigenvector in (4.6), i.e. $(u_+)^T = (\chi_+, 0)$. It turns out that the bi-spinor χ_+ obeys the same Dirac equation (4.8).

The Jackiw and Rebbi method does not allow to determine the helical texture of the spin of the boundary fermions. It results, indeed, that the $(3 + 1)$ spin operators given by the Clifford algebra, i.e. $S_i = i/4\epsilon_{ijk}[\gamma_j, \gamma_k]$ with $i, j, k = 1, 2, 3$, have non-vanishing projection only for S_3. Actually, in the study of the microscopic model of band topological insulators, the authors of Ref.[91] considered the expansion of the Hamiltonian for momenta near the band crossing point and obtained the result (4.8) at low-energy. In that approach, the spin of the surface states was also discussed, involving the electron spin as well as a $L = 1$ contribution coming from the p-wave orbitals near the surface. It turns out that the low-energy surface excitations have spin one-half given by $S_i \sim \sigma_i$, where the σ_i are the same Pauli matrices occurring in (4.1). Therefore, the surface excitations solutions of (4.8) have associated the values:

$$\langle S_3 \rangle = 0, \qquad \langle S_x k_x + S_y k_y \rangle = 0. \tag{4.9}$$

Namely, the spin lies on the surface and is orthogonal to momentum (helical spin excitation).

4.3.2 One loop corrections and parity anomaly

In this section we consider the fermion at the boundary of $3D$ topological insulators interacting with an external electromagnetic gauge potential A_μ and analyze the induced action [92, 93, 95].

In Minkowski space, the Dirac Lagrangian of a two-components spinor field in $(2+1)$ dimensions is

$$\mathcal{L} = \overline{\psi} \left(i\partial\!\!\!/ + eA\!\!\!/ - m \right) \psi, \tag{4.10}$$

where the mass m can be positive or negative and the 2×2 Dirac matrices are [92]

$$\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^2, \quad \text{with} \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{4.11}$$

The classical Lagrangian (4.10) is parity and TR symmetric when the fermion mass vanishes [92]. We will restore the massless limit at the end of the calculations.

The analytical continuation to the Euclidean time is performed by the substitutions

$$x_0 \rightarrow ix_0, \quad A_0 \rightarrow -iA_0, \quad \gamma_k \rightarrow -i\gamma_k, \quad k = 1, 2, \tag{4.12}$$

in such a way that the Euclidean γ matrices coincide with the σ Pauli matrices. Integrating out the fermion degrees of freedom we find the induced effective action, that it is given by the fermionic determinant [97]

$$\mathrm{e}^{-S_{\text{ind}}[A]} = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \exp\left(-\int d^3x \overline{\psi} \left(\partial\!\!\!/ - ieA\!\!\!/ + m \right) \psi \right) = \frac{\det\left(\partial\!\!\!/ - ieA\!\!\!/ + m \right)}{\det\left(\partial\!\!\!/ + m \right)}. \tag{4.13}$$

The induced action at the second order in A_μ is simply obtained by the one-loop Feynman diagram of the vacuum polarization of Fig. 4.4, which gives [92, 93]

$$S^{(2)}_{ind}[A] = -\frac{e^2}{2} \int \frac{d^3k}{(2\pi)^3} A_\mu(k) \Pi_{\mu\nu}(k, m) A_\nu(-k), \tag{4.14}$$

where

$$\Pi^{(2)}_{\mu\nu}(k, m) = \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr}\left[\gamma_\mu \frac{i\left(k\!\!\!/ + q\!\!\!/ \right) + m}{(k+q)^2 + m^2} \gamma^\nu \frac{iq\!\!\!/ + m}{q^2 + m^2} \right]. \tag{4.15}$$

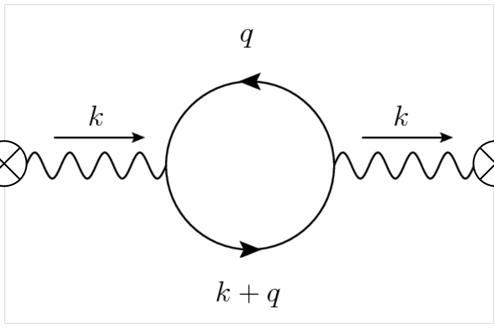


Figure 4.4: One loop vacuum polarization diagram

The calculation use dimensional regularization formulas to eliminate the linear divergence in (4.15) [97] and should be further regularized by a Pauli-Villars subtraction. Thus, the regularized vacuum polarization is defined as

$$\Pi^{(2)REG}_{\mu\nu}(k,m) = \Pi^{(2)}_{\mu\nu}(k^2,m) - \Pi^{(2)}_{\mu\nu}(k^2, M\to\infty), \tag{4.16}$$

where M is the mass of the Pauli-Villars regulator with sign(M)=sign(m). Finally, the result takes the following form

$$\begin{aligned}\Pi^{(2)REG}_{\mu\nu}(k,m) = &\frac{1}{4\pi} k_\alpha \epsilon^{\alpha\mu\nu} \frac{m}{|m|}\left(\frac{\arctan(x)}{x} - 1\right)\\ &- \left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right)\frac{1}{8\pi|k|}\left(\frac{1}{x} - \frac{1-x^2}{x^2}\arctan(x)\right), \qquad x = \frac{|k|}{2|m|}.\end{aligned} \tag{4.17}$$

Inserting $\Pi^{(2)REG}_{\mu\nu}(k,m)$ in Eq (4.14), the second order induced action takes the following forms for different regimes of the mass m:

- $|m| \to 0$, i.e. $x \to \infty$:

$$S^{(2)}_{ind}[A] = \frac{ie^2}{8\pi}\frac{m}{|m|}\int d^3x \epsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \frac{e^2}{64}\int d^3x F_{\mu\nu}\frac{1}{(-\Box)^{1/2}}F_{\mu\nu}. \tag{4.18}$$

  The first term is imaginary and amounts to the topological Chern-Simons term breaking both parity and TR symmetry; it gives rise to the parity anomaly in (2+1) dimensions [93, 94]. The second term, instead, is dynamical and non-local in the external potential [92].

- $|k| \ll |m|$, i.e. $x \to 0$:

$$S^{(2)}_{ind}[A] = \frac{e^2}{48|m|}\int d^3x F_{\mu\nu}F_{\mu\nu}. \tag{4.19}$$

  In the limit of large mass $m \to \infty$, the anomalous term cancels. The induced action is left with a subleading local term of order $\mathcal{O}(1/|m|)$.

We will see later how these regimes are relevant for the physics of $3D$ topological insulators.

4.3.3 Bulk θ-term and parity anomaly cancellation

Our first problem is to solve the apparent contradiction between TR invariance and the presence of a massless fermion at the boundary of a $3D$ topological insulator; as shown above, the induced action acquires a Chern-Simons term breaking TR symmetry. In what follow we explain how the anomaly is cancelled between bulk and boundary [3].

Our argument uses the Jackiw and Rebbi model discussed in Sec. 4.3.1. We focus on a topological insulator with two boundaries placed at the planes $z = \pm z_0$, and introduce the following TR breaking term in the bulk Hamiltonian (4.2)

$$H(\tilde{m}) = i\gamma^0\gamma^5\tilde{m}. \tag{4.20}$$

Once projected at the two boundaries $z = \pm z_0$, this term behaves as a mass for the surface fermions, leading to two Dirac Lagrangians in $(2+1)$ dimensions

$$\mathcal{L}_- = \overline{\chi}_-(i\partial\!\!\!/ - \tilde{m})\chi_-, \qquad \mathcal{L}_+ = \overline{\chi}_+(i\partial\!\!\!/ + \tilde{m})\chi_+. \tag{4.21}$$

Owing to the coupling with the vector potential A_μ, according to (4.18), the limit $\tilde{m} \to 0$ leads to the corresponding Chern-Simons actions with opposite couplings $\pm 1/2$ on the two boundaries, namely

$$S^{(2)+}_{ind}[A] = +\frac{e^2}{8\pi}\int d^3x\ \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho, \qquad S^{(2)-}_{ind}[A] = -\frac{e^2}{8\pi}\int d^3x\ \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho. \tag{4.22}$$

Here we disregarded the non-topological terms and worked in Minkowski space-time.

The second step of our argument takes into account the results of Ref.[98], where it is carried out an interesting analysis on the effective topological field theories of TR invariant topological insulators in various dimensions. The authors show that the induced bulk effective action is given by the so called *Abelian θ-term* or *magneto-electric term*, that is:

$$S_\theta[A] = -\frac{\theta e^2}{32\pi^2}\int d^4x\ \epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} = \frac{\theta e^2}{4\pi^2}\int d^4x\ \mathbf{E}\cdot\mathbf{B} = \theta\ \mathcal{C}_2. \tag{4.23}$$

$\mathcal{C}_2$ is the integral of the second Chern class of the electromagnetic gauge field, which is quantized to integer values for closed space-time manifolds. Since $\mathcal{C}_2$ is integer-valued, the path-integral is invariant under $\theta \to \theta + 2\pi$. Moreover, $\mathcal{C}_2$ is odd under TR. Thus, in a TR invariant theory, θ must equal 0 or π (equivalent to $-\pi$) [98] [99].

In our previous discussion of the topological insulator the manifold has two boundary surfaces. In this case $\mathcal{C}_2$ is not quantized and $S_\theta[A]$, being a total derivative, it reduces on the boundary $\partial\mathcal{M}$ to the Chern-Simons term

$$S_\theta[A] = \pm\frac{e^2}{8\pi}\int_{\partial\mathcal{M}} d^3x\ \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho, \qquad \theta = \pi. \tag{4.24}$$

Therefore, $S_\theta[A]$ with $\theta = \pi$ assumes at the boundaries $z = \pm z_0$ the same form, but with opposite coupling with respect to the fermionic induced actions $S^{(2)\pm}_{ind}[A]$ in (4.22). The complete effective action of the topological phase is obtained by the sum of the bulk and boundary actions, that is

$$S_{TOT}[A] = S_\theta[A] + S^{(2)+}_{ind}[A] + S^{(2)-}_{ind}[A] = 0, \qquad \theta = \pi. \tag{4.25}$$

Since the full action is vanishing, we obtain the desired parity and TR anomaly cancellation. We remark that similar arguments were discussed in [99, 100, 101].

This argument is generalized to the case of N_f surface fermions; in this case the cancellation is realized by choosing $\theta = N_f\pi$ in Eq.(4.23). However, for $N_f > 1$ pairs of fermions can interact by a TR invariant mass term and disappear from the low-energy theory. The significative cases are the trivial insulator $\theta = 0$ and the topological insulator $\theta = \pi$, leading to the $\mathbb{Z}_2$ classification.

We remark that the anomaly cancellation between the $3D$ bulk and the $2D$ boundary is different from the anomaly inflow mechanism of the QHE of Section 1.3.3. Here the anomaly cancellation is not accompanied by current flow from the bulk to the surface and no observable bulk effects are generated.

4.3.4 Bulk θ-term and surface QHE

The effective action (4.23) gives another interesting phenomenon when the boundary of a topological insulator is connected to a magnetic material with local magnetization [98]. In this case TR symmetry is explicitly broken at the boundary but is still present in the bulk.

The interaction between the boundary fermion and the magnetic field is given by the usual Zeeman term, that corresponds to a mass term for the fermion: more intense is the magnetic field greater is the mass of the fermion. In this regime, the induced action at one of the two boundaries is once again the sum of the bulk effective action $S_\theta[A]$ with $\theta = \pi$, written as a surface integral, and the boundary induced action (4.19) in the limit of large mass, i.e.

$$S_{TOT}[A] = \frac{e^2}{8\pi}\int d^3x\ \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{e^2}{48|m|}\int d^3x F_{\mu\nu}F_{\mu\nu}. \tag{4.26}$$

In this case, there is no longer a cancellation of the Chern-Simons terms and the second term can also be neglected for large masses. The remaining Chern-Simons term with coupling 1/2 gives rise to a surface Hall effect with quantized Hall conductivity $\sigma_H = 1/2$ (in units on e^2/h). In the physical setup where the system have two boundaries, the global Hall conductivity is $\sigma_H = 1$.

The discovery of the surface Hall effect at the boundaries of a $3D$ topological insulator it is a recent experimental result [102]. The topological material was made in contact to a topological superconductor and a global quantized Hall conductivity $\sigma_H = 1$ was measured. The experiment also showed that each surface carries a quantized Hall conductivity $\sigma_H = 1/2$.

4.4 Free fermion partition functions

In this section, we derive the partition functions of the massless fermion at the surface of $3D$ topological insulators [3]. Extending our earlier analysis, we use the partition functions to reformulate the Fu-Kane-Mele stability argument for the existence of massless fermionic surface states in presence of TR invariant interactions (the 'strong topological insulators' of Ref.[57, 84]). The advantage of this formulation of the stability argument is that it can be extended to any interacting system for which the partition function is known, as we shall see in next chapter where we discuss interacting $3D$ topological insulators [3]. We remark that the same argument has been applied also to exactly soluble lattice models of non interacting and interacting $3D$ phases [101, 103].

We consider the spatial geometry of a 'Corbino donut' (see Fig. 4.6), whose internal and external surfaces are two-torii. The space-time three-torus $\mathbb{T}^3$ is obtained by considering one surface and the Euclidean time with period the temperature T. The partition functions for periodic (P) and anti-periodic (A) boundary conditions in space and time form a eight-dimensional multiplet, corresponding to the eight spin sectors. Our derivation of the partition functions uses results of Ref.[104].

The torus geometry

The T^3 torus is the quotient of the flat space $\mathbb{R}^3$ by the lattice Λ^3 generated by the vector moduli $\boldsymbol{\omega}_\mu, \mu = 0,1,2$, see (Fig. 4.5), where the vectors $\boldsymbol{\omega}_i, i = 1,2$ belong to the (x_1, x_2) plane. We define the matrix $\boldsymbol{\omega}$ whose rows are the components of the vector moduli in $\mathbb{R}^3$,

$$\boldsymbol{\omega} = \begin{pmatrix} \boldsymbol{\omega}_0 \\ \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \end{pmatrix} = \begin{pmatrix} \omega_{00} & \omega_{01} & \omega_{02} \\ 0 & \omega_{11} & \omega_{12} \\ 0 & \omega_{21} & \omega_{22} \end{pmatrix}, \tag{4.27}$$

and also introduce the dual matrix $\boldsymbol{k}$ defined as

$$\boldsymbol{k}_\mu \boldsymbol{\omega}_\nu = \delta_{\mu\nu} \Longrightarrow \boldsymbol{k} = (\boldsymbol{\omega}^{-1})^T. \tag{4.28}$$

The spatial components of the $\boldsymbol{\omega}$ and $\boldsymbol{k}$ vectors are also written as

$$\vec{\omega}_i = (\omega_{i1}, \omega_{i2}), \qquad \vec{k}_i = (k_{i1}, k_{i2}), \qquad i = 1,2. \tag{4.29}$$

The volumes of the $3D$ space-time cell and $2D$ space cell are given by, respectively,

$$V^{(3)} = \det(\boldsymbol{\omega}), \qquad V^{(2)} = \det \vec{\omega} = \omega_{11}\omega_{22} - \omega_{12}\omega_{21} = |\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|. \tag{4.30}$$

The relations between the components of $\vec{k}$ and $\vec{\omega}$ are

$$k_{11} = \frac{\omega_{22}}{V^{(2)}}, \qquad k_{12} = -\frac{\omega_{21}}{V^{(2)}}, \qquad k_{21} = -\frac{\omega_{12}}{V^{(2)}}, \qquad k_{22} = \frac{\omega_{11}}{V^{(2)}}, \tag{4.31}$$

moreover

$$\omega_{00} = \frac{V^{(3)}}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|}, \qquad \frac{\omega_{00}}{V^{(2)}} = \frac{V^{(3)}}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|^2}. \tag{4.32}$$

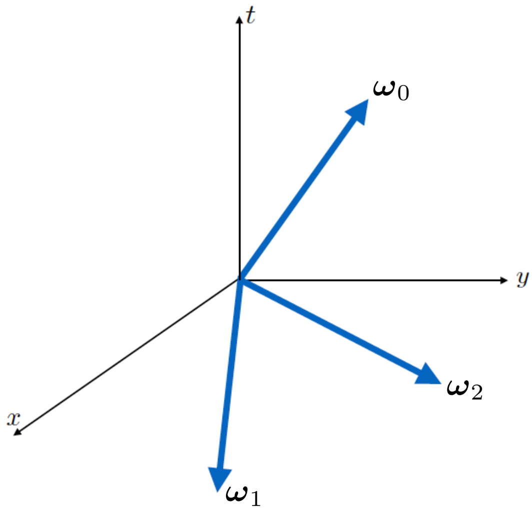


Figure 4.5: Vector moduli of $\mathbb{T}^3$ torus in the flat space $\mathbb{R}^3$

The fermionic spectrum

The partition function is given in terms of the on-shell data of the free fermion, namely its spectrum of energy, momentum, charge and fermion number. The usual creation and annihilation operators of particles $(a^\dagger_{\mathbf{n}}, a_{\mathbf{n}})$ and antiparticles $(b^\dagger_{\mathbf{n}}, b_{\mathbf{n}})$, where $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$, obey the following anti-commutation relations

$$\left\{a_{\mathbf{n}}, a^\dagger_{\mathbf{n}'}\right\} = \delta_{\mathbf{n}\mathbf{n}'}, \qquad \left\{b_{\mathbf{n}}, b^\dagger_{\mathbf{n}'}\right\} = \delta_{\mathbf{n}\mathbf{n}'}, \tag{4.33}$$

and satisfy the vacuum conditions [97]

$$a_{\mathbf{n}} \left|\Omega\right\rangle = b_{\mathbf{n}} \left|\Omega\right\rangle = 0, \qquad n_1, n_2 \in \mathbb{Z}. \tag{4.34}$$

In the torus geometry, the energy and momentum of the excitations are given by the following normal ordered expressions:

$$E = \sum_{\mathbf{n}} E_{\mathbf{n}} \left(a^\dagger_{\mathbf{n}} a_{\mathbf{n}} + b^\dagger_{\mathbf{n}} b_{\mathbf{n}}\right) - \sum_{\mathbf{n}} E_{\mathbf{n}}, \tag{4.35}$$

$$P_i = \sum_{\mathbf{n}} p_{\mathbf{n}} \left(a^\dagger_{\mathbf{n}} a_{\mathbf{n}} + b^\dagger_{\mathbf{n}} b_{\mathbf{n}}\right), \qquad i = 1, 2, \tag{4.36}$$

where

$$E_{\mathbf{n}} = 2\pi \big|(n_1 + \alpha_1)\vec{k}_1 + (n_2 + \alpha_2)\vec{k}_2\big|, \tag{4.37}$$

$$p_{\mathbf{n}} = 2\pi \left((n_1 + \alpha_1) k_{1i} + (n_2 + \alpha_2) k_{2i}\right), \qquad i = 1, 2. \tag{4.38}$$

In the expression of the energy (4.35), the infinite sum $(-\sum_{\mathbf{n}} E_{\mathbf{n}})$ represents the contribution

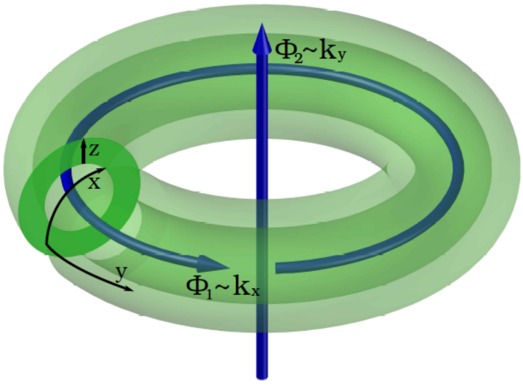


Figure 4.6: $3D$ "Corbino donut". The fluxes Φ_1 and Φ_2 play the role of the k_x and k_y momenta, respectively.

of the Casimir vacuum energy and the parameters α_i specify the boundary conditions along the $i = 1, 2$ spatial directions: $\alpha_i = 0, 1/2$ means, respectively, periodic (P) and anti-periodic (A) boundary conditions.

The charge operator is given by the following normal ordered expression [97]

$$Q = \sum_{\mathbf{n}} \left(a^\dagger_{\mathbf{n}} a_{\mathbf{n}} - b^\dagger_{\mathbf{n}} b_{\mathbf{n}} \right) , \tag{4.39}$$

where the renormalization condition has been chosen such that the ground state $|\Omega\rangle$ is electrically neutral, namely $Q\,|\Omega\rangle = 0$. Finally, the fermion number is defined as

$$(-1)^F = (-1)^{\sum_{\mathbf{n}} a^\dagger_{\mathbf{n}} a_{\mathbf{n}} + b^\dagger_{\mathbf{n}} b_{\mathbf{n}}}, \tag{4.40}$$

also equal to the parity of the number of particles and antiparticles.

The partition functions

It is useful to find the partition functions with general boundary conditions because the insertion of fluxes Φ_1 and Φ_2 across the spatial circles of the torus modifies the quantization of the momenta k_x and k_y, as pictorially reported in Fig. 4.6. As happens in $2D$, when $\Phi_i = \Phi_0/2$ the spatial boundary condition along the orthogonal direction is changed from periodic to anti-periodic and vice versa.

The Euclidean partition function of a massless Dirac fermion with (A) boundary condition along the time direction and interacting with a constant scalar potential A_0 is given by the following expression

$$Z^F_{A,\alpha_1,\alpha_2} = \mathrm{Tr}\left[\exp\left(-T(E + QA_0) + i\omega_{01} P_1 + i\omega_{02} P_2 \right) \right]. \tag{4.41}$$

As usual, one implements the (P) temporal boundary condition by inserting the fermion number operator $(-1)^F$. This gives

$$Z^F_{P,\alpha_1,\alpha_2} = \mathrm{Tr}\Big[(-1)^F \exp\Big(- T(E + QA_0) + i\omega_{01}P_1 + i\omega_{02}P_2\Big)\Big]. \tag{4.42}$$

The trace over the fermionic Fock space is evaluated using the anti-commutation relations (4.33). The final result, once we substituted the $\mathbf{k}_i$ vectors in terms of the $\boldsymbol{\omega}_i$, introduced the phase $\alpha_0 = 0, 1/2$ for the (P) or (A) temporal boundary conditions and relabeled for the antiparticles case $n_i \to -n_i$ and $\alpha_i \to -\alpha_i$, gives

$$\begin{aligned} Z^F_{\alpha_0,\alpha_1\alpha_2} = e^{-V^{(3)}F_0} \prod_{n_1,n_2\in\mathbb{Z}} & \left\{1 - \exp\left(-2\pi\mathcal{E}^{\alpha_1,\alpha_2}_{n_1,n_2} + 2\pi i\mathcal{P}^{\alpha_1,\alpha_2}_{n_1,n_2} - 2\pi i\mathcal{A}\right)\right\} \\ & \times \left\{1 - \exp\left(-2\pi\mathcal{E}^{\alpha_1,\alpha_2}_{n_1,n_2} - 2\pi i\mathcal{P}^{\alpha_1,\alpha_2}_{n_1,n_2} + 2\pi i\mathcal{A}\right)\right\}. \end{aligned} \tag{4.43}$$

In this formula

$$\mathcal{A} = \alpha_0 - i\frac{V^{(3)}A_0}{2\pi|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|}, \tag{4.44}$$

$$\mathcal{E}^{\alpha_1,\alpha_2}_{n_1,n_2} = \frac{V^{(3)}}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|^2}|(n_1+\alpha_1)\boldsymbol{\omega}_2 - (n_2+\alpha_2)\boldsymbol{\omega}_1|, \tag{4.45}$$

$$\mathcal{P}^{\alpha_1,\alpha_2}_{n_1,n_2} = \frac{(\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2)}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|^2}\left[(n_1+\alpha_1)(\boldsymbol{\omega}_0 \times \boldsymbol{\omega}_2) - (n_2+\alpha_2)(\boldsymbol{\omega}_0 \times \boldsymbol{\omega}_1)\right], \tag{4.46}$$

$$F_0 = \frac{1}{2\pi}\sum_{\mathbf{n}\in\mathbb{Z}^2}{}' \frac{e^{-2\pi i(\alpha_2 n_1 - \alpha_1 n_2)}}{|n_1\boldsymbol{\omega}_2 - n_2\boldsymbol{\omega}_1|^3}, \tag{4.47}$$

where in the sum $\sum'$ are excluded the values $(n_1, n_2) = (0, 0)$. In this expression F_0 is the regularized vacuum energy obtained with the Epstein's analytical continuation formula [105, 106], see Appendix (C.1).

4.5 Stability criteria

In this section we reformulate the $3D$ version of the flux insertion argument discussed by Kane and coworkers [57, 84] using the fermionic partition functions derived above [3]. We shall show how the ground state of the system is adiabatically deformed by adding half magnetic fluxes and a neutral spin one-half excitation is created at the boundary. Thus, the Kramers theorem is invoked, saying that this excitation is part of a doublet that remains degenerate in presence of any TR invariant interaction. Finally, the partner state of the doublet is evolved back to zero flux, where it is found to be an excited state with energy $O(1/R)$, where R is the size of the system, thus proving that there is no gap in the thermodynamic limit.

The Neveu-Schwarz sector

In the unperturbed configuration without fluxes, the natural boundary conditions for the fermion field are antiperiodic both in space and time, i.e we take $(\alpha_0, \alpha_1, \alpha_2) = (1/2, 1/2, 1/2)$

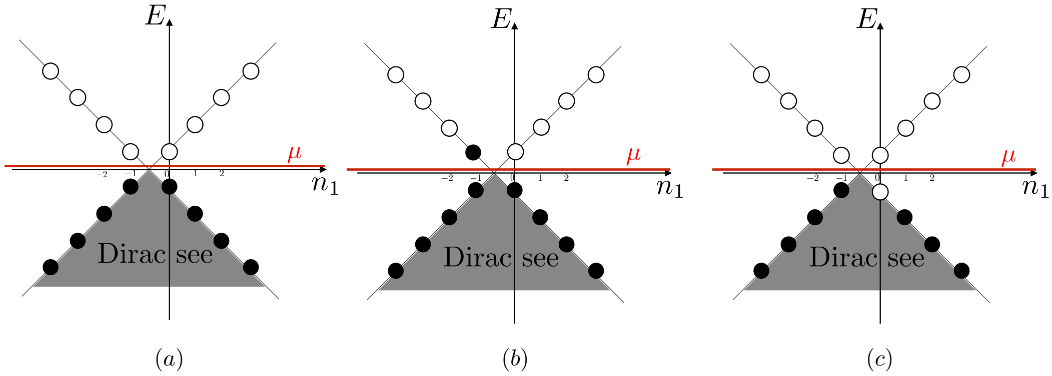


Figure 4.7: Low-lying surface states of the Neveu-Schwarz sector as a function of n_1 (cf.(4.49)). In (a) the ground state $|\Omega\rangle_{\frac{1}{2}\frac{1}{2}}$ without excitations over or below the Fermi energy μ. In (b) and (c) examples of particle and hole excitations.

in the general partition function (4.43). In analogy with (1+1) dimensions, this can be called Neveu-Schwarz sector.

The stability of the surface excitations can be discussed in the simpler geometry of a rectangular torus where $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are displaced respectively along the x_1 and x_2 Cartesian axes (see Fig. 4.5), that is taking $\omega_{12} = \omega_{21} = 0$. In this case the energies of the excitations in (4.45), (neglecting the global Casimir energy (4.47)), assume the form

$$\mathcal{E}^{\frac{1}{2},\frac{1}{2}}_{n_1,n_2} = \omega_{00}\sqrt{\left(n_1+\frac{1}{2}\right)^2\frac{1}{\omega_{11}^2}+\left(n_2+\frac{1}{2}\right)^2\frac{1}{\omega_{22}^2}}. \tag{4.48}$$

There are four low-lying degenerate energy levels, for $(n_1, n_2) = (0,0), (0,-1), (-1,0), (-1,-1)$. The expansion of the partition function gives

$$Z^F_{\frac{1}{2},\frac{1}{2}\frac{1}{2}} \sim 1 + \sum_{(n_1,n_2)=(0,0),(0,-1),(-1,0),(-1,-1)} e^{-2\pi\mathcal{E}_{\mathbf{n}}+2\pi i\mathcal{P}_{\mathbf{n}}-\omega_{00}A_0} + e^{-2\pi\mathcal{E}_{\mathbf{n}}-2\pi i\mathcal{P}_{\mathbf{n}}+\omega_{00}A_0}. \tag{4.49}$$

Therefore, the low-lying states are the ground state $|\Omega\rangle_{\frac{1}{2}\frac{1}{2}}$ plus four particle and four antiparticle excitations. The ground state corresponds to a completely filled Dirac see, without excitations over or below the Fermi energy (see Fig. 4.7); its charge and fermion number assignment can be read from the definition (4.40); they are

$$Q\,|\Omega\rangle_{\frac{1}{2}\frac{1}{2}} = 0, \qquad (-1)^F\,|\Omega\rangle_{\frac{1}{2}\frac{1}{2}} = |\Omega\rangle_{\frac{1}{2}\frac{1}{2}}\,. \tag{4.50}$$

In order to discuss the stability argument, we need to distinguish between surface excitations with integer and half-integer spin, thus we shall introduce the 'spin parity' index $(-1)^{2S}$ that,

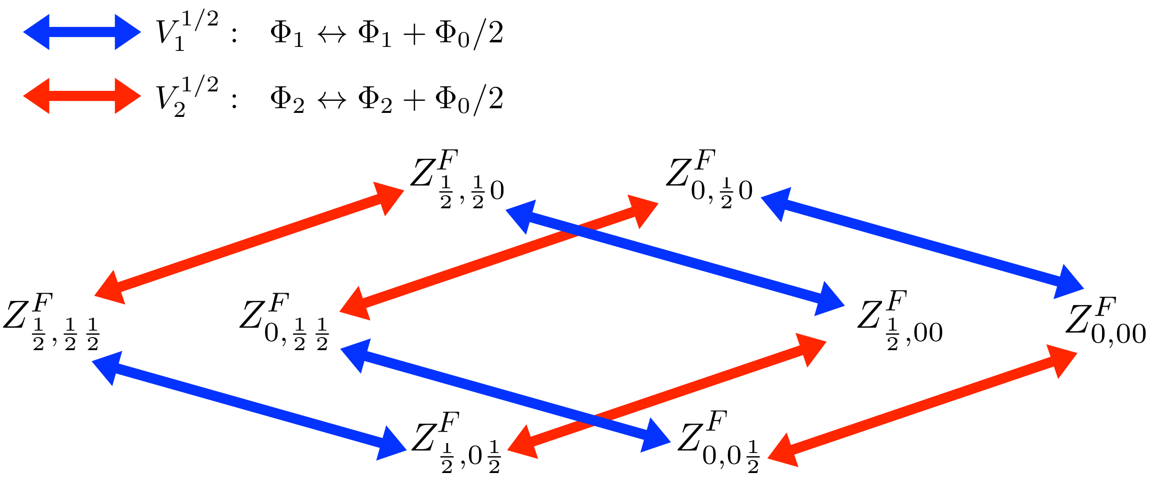


Figure 4.8: Action of $V_1^{1/2}$ and $V_2^{1/2}$ on the eight partition functions $Z^F_{\alpha_0,\alpha_1\alpha_2}$, where $\alpha_\mu = 0$ $(1/2)$ corresponds to periodic (antiperiodic) boundary conditions.

as discussed in Appendix A, is equal to the fermion number of the $(2+1)$-dimensional theory [3], namely

$$(-1)^{2S} = (-1)^F, \qquad (-1)^{2S} \left|\Omega\right\rangle_{\frac{1}{2}\frac{1}{2}} = \left|\Omega\right\rangle_{\frac{1}{2}\frac{1}{2}} . \tag{4.51}$$

Our conclusions are in agreement with recent discussions on condensed matter fermionic systems in $(2+1)$ dimensions [107, 108]. In these works, indeed, the fermionic path integrals are defined on particular manifolds, called spin$_c$ manifolds, in order to capture the spin-charge relation satisfied by the unperturbed ground state of a gapped fermionic system, namely the fact that states with odd electric charge have half integral spin and states of even electric charge have integral spin.

The Ramond sector

The first step of the stability argument consists on the adiabatic insertion of a $\Phi_0/2$ flux through the holes of the torus. We call $V_i^{1/2}$, with $i = 1, 2$, the corresponding transformations. The result of the insertion is to change the spatial boundary conditions $\alpha_i \to \alpha_i + 1/2$. Fig. 4.8 shows the transformations of the partition functions under $V_1^{1/2}$ and $V_2^{1/2}$. Starting from the Neveu-Schwarz sector $Z^F_{\frac{1}{2},\frac{1}{2}\frac{1}{2}}$ and applying $V_1^{1/2}$ and $V_2^{1/2}$ as follows

$$V_1^{1/2} : \Phi_1 \to \Phi_1 + \Phi_0/2, \qquad \text{then} \qquad Z^F_{\frac{1}{2},\frac{1}{2}\frac{1}{2}} \longrightarrow Z^F_{\frac{1}{2},0\frac{1}{2}}, \tag{4.52}$$

$$V_2^{1/2} : \Phi_2 \to \Phi_2 + \Phi_0/2, \qquad \text{then} \qquad Z^F_{\frac{1}{2},0\frac{1}{2}} \longrightarrow Z^F_{\frac{1}{2},00}, \tag{4.53}$$

we reach the periodic-periodic sector that can be called the $(2+1)$ dimensional Ramond sector, with partition function $Z^F_{\frac{1}{2},00}$. In a rectangular torus the spectrum of the Ramond

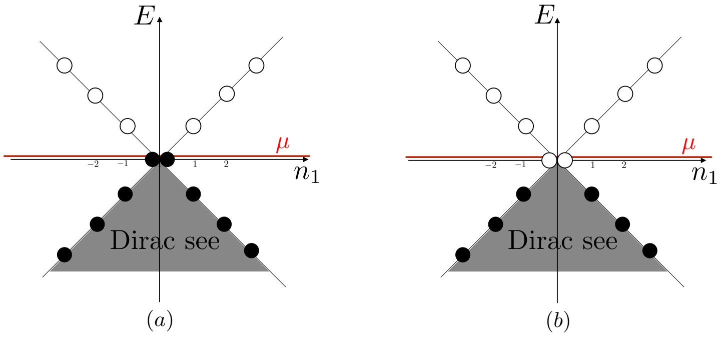


Figure 4.9: Degenerate ground states of the Ramond sector. In (a) the particle excitation with charge $Q = 1$ corresponding to $|\Omega\rangle_{00}^{(2)}$. In (b) the hole excitation with charge $Q = -1$ corresponding to $|\Omega\rangle_{00}^{(3)}$.

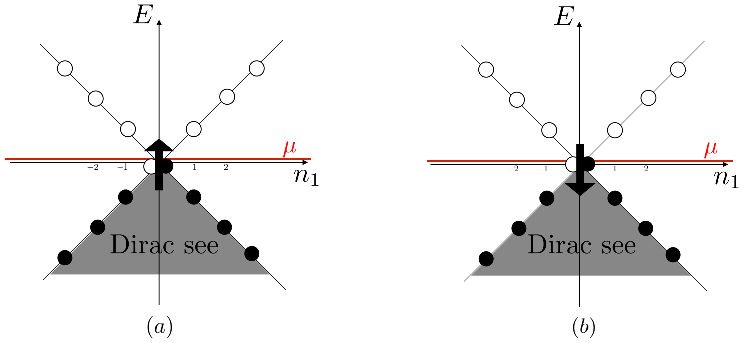


Figure 4.10: Degenerate ground states of the Ramond sector. (a) and (b) represent the neutral excitations corresponding to $|\Omega\rangle_{00}^{(0)}$ and $|\Omega\rangle_{00}^{(4)}$, respectively. They have negative spin-parity index, i.e. $(-1)^{2\Delta S} = -1$, and give rise to the Kramers doublet.

sector has the following energy

$$\mathcal{E}_{n_1,n_2}^{0,0} = \omega_{00}\sqrt{\frac{n_1^2}{\omega_{11}^2} + \frac{n_2^2}{\omega_{22}^2}}. \tag{4.54}$$

In this sector, the energy spectrum (4.54), as well as the momentum spectrum, possesses one vanishing value for $(n_1, n_2) = (0, 0)$, i.e $\mathcal{E}_{0,0}^{0,0} = \mathcal{P}_{0,0}^{0,0} = 0$, up to an overall ground state energy. Upon expanding the Ramond partition function, we find four degenerated states exactly located at the Fermi level (see Fig. 4.9 and Fig. 4.10),

$$Z_{\frac{1}{2},00}^{F} \propto 1 + e^{-\omega_{00}A_0} + e^{\omega_{00}A_0} + e^{-\omega_{00}A_0}e^{\omega_{00}A_0} + \dots, \tag{4.55}$$

that we call $|\Omega\rangle_{00}^{(i)}$, $i = 1, \cdots, 4$. Let us analyze their quantum numbers. Two of them have charge $Q = \pm 1$ (see Fig. 4.9),

$$e^{-\omega_{00}A_0} \leftrightarrow |\Omega\rangle_{00}^{(2)}, \qquad e^{\omega_{00}A_0} \leftrightarrow |\Omega\rangle_{00}^{(3)}, \tag{4.56}$$

the other states, instead, are neutral (see Fig. 4.10),

$$1 \leftrightarrow |\Omega\rangle_{00}^{(1)}\,, \qquad e^{-\omega_{00}A_0}e^{\omega_{00}A_0} \leftrightarrow |\Omega\rangle_{00}^{(4)}\,, \qquad Q\,|\Omega\rangle_{00}^{(1)} = Q\,|\Omega\rangle_{00}^{(4)} = 0. \tag{4.57}$$

Upon following the evolution of the spectrum while adding the fluxes, i.e. $\alpha_i : 1/2 \to 0$, we can see that the Neveu-Schwarz ground state is mapped into the following Ramond state: $|\Omega\rangle_{\frac{1}{2}\frac{1}{2}} \to |\Omega\rangle_{00}^{(1)}$. Being neutral, we are led to identify the state $|\Omega\rangle_{00}^{(4)}$ as the expected partner of the Kramers pair, i.e. $|\Omega\rangle_{00}^{(4)} = \mathcal{T}\,|\Omega\rangle_{00}^{(1)}$. As shown in Fig. 4.9, in the Ramond sector there is an ambiguity on the identification of the particles and antiparticles characterizing the ground state because two charged excitations are exactly located at the Fermi level. Assuming particle-hole symmetry, as discussed in Appendix A, we obtain the following assignments for the spin parity and fermion number indices on the Ramond sector [3]:

$$(-1)^{2S} = (-1)^F = -1 \qquad \text{on} \qquad |\Omega\rangle_{00}^{(1)}\,,\ |\Omega\rangle_{00}^{(4)}\,, \tag{4.58}$$

$$(-1)^{2S} = (-1)^F = 1 \qquad \text{on} \qquad |\Omega\rangle_{00}^{(2)}\,,\ |\Omega\rangle_{00}^{(4)}\,. \tag{4.59}$$

Actually, the neutral states $|\Omega\rangle_{00}^{(1)}$ and $|\Omega\rangle_{00}^{(4)}$ are identified with the Kramers doublet that we are looking for. Since the Ramond sector corresponds to a TR invariant point for the Hamiltonian, this degeneracy is protected from any TR invariant perturbations. To conclude the stability argument we return to the zero flux configuration: while the Ramond ground state $|\Omega\rangle_{00}^{(1)}$ goes back to the Neveu-Schwarz one $|\Omega\rangle_{\frac{1}{2}\frac{1}{2}}$, its Kramers partner flows into the following neutral excited state

$$|ex\rangle_{\frac{1}{2}\frac{1}{2}} \ \leftrightarrow\ e^{-2\pi\mathcal{E}_{-1,-1}+\omega_{00}A_0}e^{-2\pi\mathcal{E}_{-1,-1}-\omega_{00}A_0}. \tag{4.60}$$

The energy of this excitation is $\mathcal{O}(1/R)$, where R is the typical dimension of the system; being vanishing in the thermodynamic limit, this proves that the spectrum is gapless for any TR invariant interaction [3].

We remark that the neutral $S = 1/2$ excitation created by adding half fluxes is a non-perturbative excitation in the fermionic theory with respect to the Neveu-Schwarz ground state,

$$|\Omega\rangle_R = \sigma(0)\,|\Omega\rangle_{\frac{1}{2}\frac{1}{2}}\,. \tag{4.61}$$

In the $(1+1)$-dimensional theory, this is know as the 'spin field' and its properties are well understood, e.g. within the fermionic description of the Ising model [22, 21].

$\mathbb{Z}_2$ spin parity anomaly

The stability of the surface excitations can be related to a $\mathbb{Z}_2$ anomaly, as discussed in $2D$ (see Chapter 2). Indeed, the Neveu-Schwarz and Ramond ground states are eigenstates of a TR invariant Hamiltonian, but possess different spin-parity index, i.e.

$$(-1)^{2S}\,|\Omega\rangle_{\frac{1}{2}\frac{1}{2}} = |\Omega\rangle_{\frac{1}{2}\frac{1}{2}}\,, \qquad (-1)^{2S}\,|\Omega\rangle_{00}^{(1)} = -\,|\Omega\rangle_{00}^{(1)}\,. \tag{4.62}$$

The spin-parity is conserved by TR symmetry, but changes between two invariant ground states, without any breaking of the symmetry either explicit or spontaneous. Therefore, we interpret this change as a discrete $\mathbb{Z}_2$ anomaly, which is equivalent to the $\mathbb{Z}_2$ index of stability [3].

4.6 Stability and modular invariance

In this section we study the behavior of the eight partition functions under the discrete changes of coordinates that map the three-torus into itself. The pattern of transformations will further characterize the different sectors [3]. We note that a similar study was done in Ref.[104]. Moreover, we shall associate the stability of the topological insulators to the impossibility of writing a modular invariant partition function that is consistent with all physical requirements: that is the same result for $2D$ systems discussed in Chapter 2.

In what follows we will take for simplicity $A_0 = 0$, then the partition function (4.43) takes the form

$$Z^F_{\alpha_0,\alpha_1\alpha_2} = e^{-VF_0} \prod_{n_1,n_2\in\mathbb{Z}} \left| 1 - \exp\left(-2\pi\mathcal{E}_{n_1,n_2} + 2\pi i\mathcal{P}_{n_1,n_2} - 2\pi i\alpha_0\right) \right|^2 . \tag{4.63}$$

The generators

The modular group $SL(3,\mathbb{Z})$ is generated by T_1 and $U_1 = S_1P_{12}$. With (T_1, S_1) we indicate the generators of the subgroup $SL(2,\mathbb{Z})$ acting on the subspace (x_0, x_1); P_{12} is the permutation of the spatial vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. On the basis $(\boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$, their action is

$$T_1 : \begin{pmatrix}\boldsymbol{\omega}_0\\ \boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix} \longrightarrow \begin{pmatrix}\boldsymbol{\omega}_0+\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix}, \qquad S_1 : \begin{pmatrix}\boldsymbol{\omega}_0\\ \boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix} \longrightarrow \begin{pmatrix}-\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_0\\ \boldsymbol{\omega}_2\end{pmatrix}, \qquad P_{12} : \begin{pmatrix}\boldsymbol{\omega}_0\\ \boldsymbol{\omega}_1\\ \boldsymbol{\omega}_2\end{pmatrix} \longrightarrow \begin{pmatrix}\boldsymbol{\omega}_0\\ -\boldsymbol{\omega}_2\\ \boldsymbol{\omega}_1\end{pmatrix}. \tag{4.64}$$

The generators (T_2, S_2) of the subgroup $SL(2,\mathbb{Z})$ acting on the subspace (x_0, x_2) are clearly found from T_1 and S_1 as follows:

$$T_2 = P_{12}T_1P_{12}, \qquad S_2 = P_{12}S_1P_{12}. \tag{4.65}$$

P_{12} transformation

The action of P_{12} on the partition functions (4.63) is manifest. $Z^F_{\alpha_0,\frac{1}{2}\frac{1}{2}}$ and $Z^F_{\alpha_0,00}$ are left invariant, while the others are exchanged: $Z^F_{\alpha_0,\frac{1}{2}0} \leftrightarrow Z^F_{\alpha_0,0\frac{1}{2}}$. This result implies that it is sufficient to study the modular transformations given by T_1 and S_1 and then apply (4.65) to find the action of the entire group.

T_1 transformation

Also the action of T_1 is simple to find. If $\alpha_1 = 0$ the partition functions are left invariant, i.e. $Z^F_{\alpha_0,0\alpha_2} \leftrightarrow Z^F_{\alpha_0,0\alpha_2}$. If $\alpha_1 = 1/2$, T_1 changes the temporal boundary conditions from (A) to (P) and vice versa, i.e $Z^F_{\frac{1}{2},\frac{1}{2}\alpha_2} \leftrightarrow Z^F_{0,\frac{1}{2}\alpha_2}$.

S_1 transformation

The action of the transformation S_1 requires some calculations. Following [104], it is useful to choose the particular geometry in which the $\boldsymbol{\omega}$ matrix assumes the following form

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_{00} & \omega_{01} & \omega_{02} \\ 0 & \omega_{11} & \omega_{12} \\ 0 & 0 & \omega_{22} \end{pmatrix} = \begin{pmatrix} 2\pi R_0 & -2\pi\alpha R_1 & -2\pi\gamma R_2 \\ 0 & 2\pi R_1 & -2\pi\beta R_2 \\ 0 & 0 & 2\pi R_2 \end{pmatrix}. \tag{4.66}$$

With this choice, the fermionic partition functions (4.41) and (4.42), before making the regularization of the vacuum energy, takes the following form

$$\begin{aligned} Z^F_{\alpha_0;\alpha_1\alpha_2} = \prod_{n_2} \Bigg\{ \prod_{n_1} \Bigg| 1 - \exp\Big(& -2\pi r_{01}\sqrt{[(n_1+\alpha_1)+\beta(n_2+\alpha_2)]^2 + [r_{12}(n_2+\alpha_2)]^2} \\ & + 2\pi i\left[\alpha(n_1+\alpha_1) + (n_2+\alpha_2)(\alpha\beta+\gamma)\right] - 2\pi i\alpha_0 \Big) \Bigg|^2 \times \\ \times \exp & \left[2\pi r_{01} \sum_{n_1} \sqrt{\left[(n_1+\alpha_1)+\beta(n_2+\alpha_2)\right]^2 + \left[r_{12}(n_2+\alpha_2)\right]^2} \right] \Bigg\}, \end{aligned} \tag{4.67}$$

where we splitted the products on n_1 and n_2 and introduced the two quantities $r_{01} = R_0/R_1$ e $r_{12} = R_1/R_2$. In Appendix C.2 is shown how to regularize the last exponent through the Mellin transform. For convenience, we introduce the “massive Θ functions” of Ref.[104, 109]

$$\begin{aligned} \Theta_{[a,b]}(\tau; m) = \prod_{n\in\mathbb{Z}} & \left| 1 - \exp\left[-2\pi\mathrm{Im}(\tau)\sqrt{(n+a)^2 + m^2} + 2\pi i\mathrm{Re}(\tau)(n+a) + 2\pi i b \right] \right|^2 \times \\ & \times \exp\left[4\pi\mathrm{Im}(\tau)\Delta(m; a)\right], \end{aligned} \tag{4.68}$$

where $a, b, m \in \mathbb{R}$, $\tau \in \mathbb{C}$, and Δ is the following function

$$\Delta(m; a) = -\frac{1}{2\pi^2} \sum_{l>0} \int_0^{+\infty} dt\; e^{-\frac{\pi^2 m^2}{t} - tl^2} \cos(2\pi l a). \tag{4.69}$$

Identifying $a = \alpha_1 + \beta(n_2 + \alpha_2)$, $b = \gamma(n_2 + \alpha_2) + \alpha_0$, $m = r_{12}(n_2 + \alpha_2)$ and $\tau = -\omega_{01}/\omega_{11} + i\omega_{00}/\omega_{11} = \alpha + ir_{01}$, the fermionic partition function (4.67) becomes

$$Z^F_{\alpha_0,\alpha_1,\alpha_2} = \prod_{n_2\in\mathbb{Z}} \Theta_{[\alpha_1+\beta(n_2+\alpha_2),\gamma(n_2+\alpha_2)+\alpha_0]}\left(\alpha + ir_{01}; r_{12}(n_2+\alpha_2)\right). \tag{4.70}$$

We observe that the boundary conditions α_μ appear in the partition function through the parameters of the Θ and Δ functions.

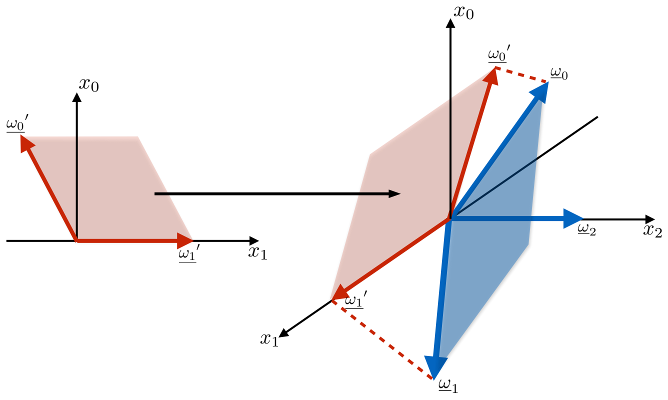


Figure 4.11: Projection of the parallelogram between $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_1$ on the plane (x_0, x_1).

On the basis (4.66), the action (4.64) of S_1 becomes

$$S_1: \quad \boldsymbol{\omega}_0 = 2\pi \begin{pmatrix} R_0 \\ -\alpha R_1 \\ -\gamma R_2 \end{pmatrix} \longrightarrow -\boldsymbol{\omega}_1 = 2\pi \begin{pmatrix} 0 \\ -R_1 \\ \beta R_2 \end{pmatrix}, \tag{4.71}$$

$$S_1: \quad \boldsymbol{\omega}_1 = 2\pi \begin{pmatrix} 0 \\ R_1 \\ -\beta R_2 \end{pmatrix} \longrightarrow \boldsymbol{\omega}_0 = 2\pi \begin{pmatrix} R_0 \\ -\alpha R_1 \\ -\gamma R_2 \end{pmatrix}. \tag{4.72}$$

from which follows that

$$S_1: \quad \gamma \to -\beta, \qquad \beta \to \gamma, \qquad R_2 \to R_2. \tag{4.73}$$

Moreover, since the first two components of $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_1$ do not mix to the third, without loss of generality, we can analyze the reduced vectors $\boldsymbol{\omega}_0'$ and $\boldsymbol{\omega}_1'$ made of the first two components. This coincides to projecting the parallelogram between $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_1$ on the plane (x_0, x_1) as shown in Fig. 4.11.

We define the modular parameter

$$\tau' = -\alpha + i r_{01}, \qquad S_1 \quad : \tau' \to -\frac{1}{\tau'}. \tag{4.74}$$

Since the partition function (4.70) depends on $\tau = \alpha + i r_{01}$, then

$$S_1 \quad : \alpha \longrightarrow -\frac{\alpha}{\alpha^2 + r_{01}^2}, \qquad r_{01} \longrightarrow \frac{r_{01}}{\alpha^2 + r_{01}^2}. \tag{4.75}$$

Finally, imposing that S_1 exchanges the moduli $|\boldsymbol{\omega}_0'| \leftrightarrow |\boldsymbol{\omega}_1'|$, we obtain its action on the two radii R_0 and R_1:

$$S_1 \quad : R_1 \longrightarrow R_1|\tau|, \qquad R_0 \longrightarrow \frac{R_0}{|\tau|}. \tag{4.76}$$

In Appendix D it is shown that the massive Θ function satisfies the relation:

$$\Theta_{[a,b](\tau,m)} = \Theta_{[b,-a]}\left(-\frac{1}{\tau}, m|\tau|\right). \tag{4.77}$$

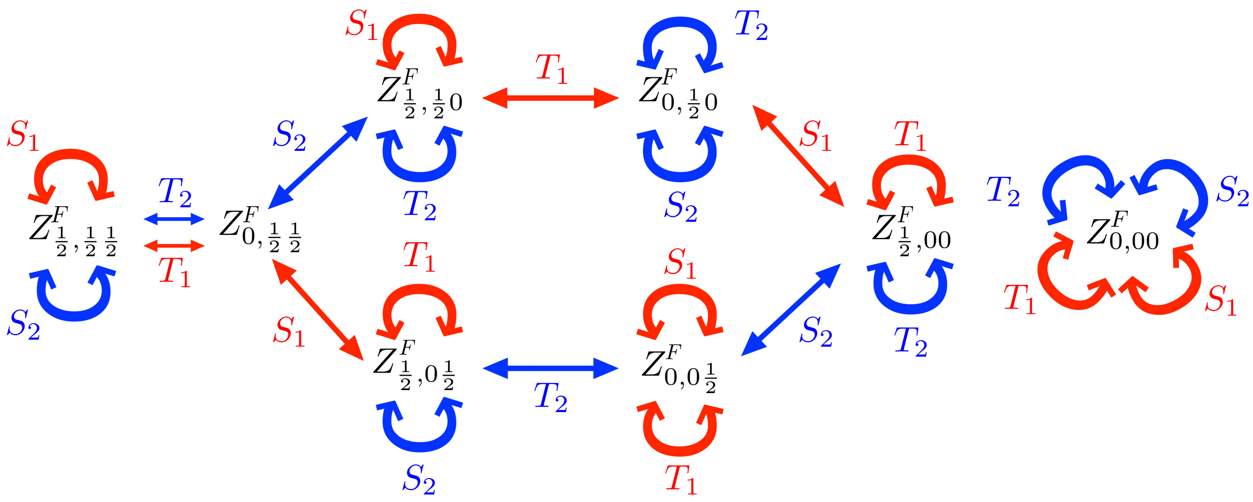


Figure 4.12: Action of the modular transformations T_1, T_2, S_1 and S_2 on the eight partition functions $Z^F_{\alpha_0,\alpha_1,\alpha_2}$. Recall that $\alpha_\mu = 0(1/2)$ for periodic (antiperiodic) boundary conditions.

Acting with S_1 on the partition function (4.70), and making use of the main relation (4.77), we obtain

$$\begin{aligned} S_1: \quad & \Theta_{[\alpha_1+\beta(n_2+\alpha_2),\gamma(n_2+\alpha_2)+\alpha_0]}(\tau; r_{12}(n_2+\alpha_2)) \longrightarrow \\ & \Theta_{[\alpha_1+\gamma(n_2+\alpha_2),-\beta(n_2+\alpha_2)+\alpha_0]}\left(-\frac{1}{\tau}; r_{12}(n_2+\alpha_2)|\tau|\right) = \\ & \Theta_{[-\alpha_0+\beta(n_2+\alpha_2),\gamma(n_2+\alpha_2)+\alpha_1]}(\tau; r_{12}(n_2+\alpha_2)), \end{aligned} \tag{4.78}$$

from which, finally it follows the transformation

$$S_1: \quad Z^F_{\alpha_0,\alpha_1\alpha_2}(\boldsymbol{\omega}_0,\boldsymbol{\omega}_1,\boldsymbol{\omega}_2) \quad \to \quad Z^F_{\alpha_0,\alpha_1\alpha_2}(-\boldsymbol{\omega}_1,\boldsymbol{\omega}_0,\boldsymbol{\omega}_2) = Z^F_{\alpha_1,\alpha_0\alpha_2}(\boldsymbol{\omega}_0,\boldsymbol{\omega}_1,\boldsymbol{\omega}_2). \tag{4.79}$$

that verifies the expectations. All together, the transformations T_1, T_2, S_1 and S_2 acts on the set of partition functions $Z^F_{\alpha_0,\alpha_1,\alpha_2}$ as shown in Fig. 4.12.

Stability and modular invariance

By the use of the maps in Fig. 4.8 and Fig. 4.12, the sum over the eight spin sectors

$$Z^F_{\text{Ising}} = \sum_{\alpha_0,\alpha_1,\alpha_2=0,\frac12} Z^F_{\alpha_0,\alpha_1,\alpha_2}, \tag{4.80}$$

is invariant under $V_1^{1/2}$, $V_2^{1/2}$ and the group $SL(3,\mathbb{Z})$. We call it Z^F_{Ising} being the $3D$ generalization of the $2D$ modular invariant partition function (2.28). The $3D$ Ising partition function, as its $2D$ version, is not consistent with TR symmetry because the single free fermion theory suffers of the $\mathbb{Z}_2$ spin parity anomaly. The function (4.80) sums spin sectors with different values of the ground state spin parity, e.g. $(\frac{1}{2}, 00)$ and $(\frac{1}{2}, \frac{1}{2}\frac{1}{2})$, thus explicitly violating TR symmetry.

As we did in two dimensions, we should not sum over the different spin sectors if we insist on preserving TR symmetry. In the theory described by this set of eight partition functions, $Z^F_{\frac{1}{2},\frac{1}{2}\frac{1}{2}}$ represents the unperturbed TR invariant surface system, while the other functions contain excited states due to changes of electromagnetic and gravitational backgrounds. Again, the stability of $3D$ free topological insulators is related to the $\mathbb{Z}_2$ spin parity anomaly which is accompanied by the modular non-invariance of the surface partition function, that is a discrete gravitational $\mathbb{Z}_2$ anomaly [3].

We mentions that, recently, also other authors, see for example [64, 110], related the modular non-invariance of the boundary partition function to the existence of a non trivial topological phase in the system.

4.7 Dimensionally reduced partition functions

In this section we further characterize the eight fermionic partition functions by performing a dimensional reduction from two to one spatial dimension, where we recover well-known formulas [3].

Let us consider the partition functions for a rectangular torus, i.e. $\omega_{12} = \omega_{21} = 0$, and vanishing scalar potential $A_0 = 0$, for simplicity. We perform a dimensional reduction of the Kaluza-Klein type, namely take the limit $R_2 \to 0$ of the Corbino donut, such that the modes of energy $O(n_2/R_2)$ are never excited, corresponding to $n_2 \to 0$. The remaining geometry is that of two-torus in the plane (x^0, x^1); about the energy spectrum (4.35), there are two possibilities: i) for periodic boundary condition along x_2, i.e. $\alpha_2 = 0$, the spectrum becomes exactly that of a massless fermion in a two-dimensional torus; ii) for antiperiodic conditions, i.e. $\alpha_2 = 1/2$, there remains the constant $1/(4\pi R_2)^2$ that plays the role of a mass in $(1+1)$-dimensions.

We start from the expressions (4.41) and (4.42) before regularization of the ground state energy, and rewrite them as follows:

$$
\begin{aligned}
Z^F_{\alpha_0,\alpha_1,\alpha_2} = &\prod_{n_2} \left\{ \prod_{n_1} \left| 1 - \exp\left(-\frac{2\pi\omega_{00}}{\omega_{11}} \sqrt{(n_1+\alpha_1)^2 + (n_2+\alpha_2)^2 \frac{\omega_{11}^2}{\omega_{22}^2}} \right.\right.\right. \\
&\left.\left. + \frac{2\pi i\omega_{01}}{\omega_{11}}(n_1+\alpha_1) + \frac{2\pi i\omega_{02}}{\omega_{22}}(n_2+\alpha_2) + 2\pi i\alpha_0 \right) \right|^2 \times \\
&\left. \times \exp\left[\frac{2\pi\omega_{00}}{\omega_{11}} \sum_{n_1} \sqrt{(n_1+\alpha_1)^2 + (n_2+\alpha_2)^2 \frac{\omega_{11}^2}{\omega_{22}^2}} \right] \right\} .
\end{aligned} \tag{4.81}
$$

The regularized form of the sum on the integer n_1 in second exponent is written again in terms of the Δ function (4.69)(see Appendix C.2) :

$$
\frac{2\pi\omega_{00}}{\omega_{11}} \sum_{n_1} \sqrt{(n_1+\alpha_1)^2 + (n_2+\alpha_2)^2 \frac{\omega_{11}^2}{\omega_{22}^2}} = \frac{4\pi\omega_{00}}{\omega_{11}} \Delta\left[\frac{\omega_{11}}{\omega_{22}}(n_2+\alpha_2); \alpha_1 \right] . \tag{4.82}
$$

The remaining two-torus is specified by the modular parameter

$$\tau = \tau_1 + i\tau_2 = -\frac{\omega_{01}}{\omega_{11}} + i\frac{\omega_{00}}{\omega_{11}}. \tag{4.83}$$

A further convenient simplification is setting $\omega_{02} = 0$, i.e. no bending of this torus in three dimensions. In the limit $\omega_{22} \to 0$, the leading behavior of (4.81) is given by the factor with $n_2 = 0$, that reads

$$Z^F_{\alpha_0\alpha_1\alpha_2} \longrightarrow Z^F_{\alpha_0\alpha_1\big|\alpha_2} = \Theta_{[\alpha_1,\alpha_0]}\left(\tau;\ \frac{\omega_{11}}{\omega_{22}}\alpha_2\right), \tag{4.84}$$

where the massive Θ function is defined in (4.68).

Massless case $\alpha_2 = 0$

In this case

$$\Delta(0;\alpha_1 = 0) = -\frac{1}{12}, \qquad \Delta(0;\alpha_1 = \frac{1}{2}) = \frac{1}{24}, \tag{4.85}$$

and thus the ground state energy pre-factor becomes

$$\exp\left(4\pi \mathrm{Im}(\tau)\Delta(0;\alpha_1)\right) = \begin{cases} \left(\frac{1}{q\bar{q}}\right)^{-1/12}, & \alpha_1 = 0, \\ \\ \left(\frac{1}{q\bar{q}}\right)^{1/24}, & \alpha_1 = \frac{1}{2}, \end{cases} \tag{4.86}$$

where $q = \exp 2\pi i\tau$. Remembering that $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind function, the reduced partition functions (4.84) assume the following expressions:

$$Z^F_{\frac{1}{2},\frac{1}{2}\big|0} = \left|\frac{1}{\eta(\tau)}\prod_{n=1}(1-q^n)(1+q^{n-1/2})^2\right|^2 = \left|\frac{1}{\eta(\tau)}\sum_{m\in\mathbb{Z}} q^{m^2/2}\right|^2 = Z^{NS}, \tag{4.87}$$

$$Z^F_{0,\frac{1}{2}\big|0} = \left|\frac{1}{\eta(\tau)}\prod_{n=1}(1-q^n)(1-q^{n-1/2})^2\right|^2 = \left|\frac{1}{\eta(\tau)}\sum_{m\in\mathbb{Z}}(-1)^m q^{m^2/2}\right|^2 = Z^{\widetilde{NS}}, \tag{4.88}$$

$$Z^F_{\frac{1}{2},0\big|0} = \left|\frac{1}{\eta(\tau)}2q^{1/8}\prod_{n=1}(1-q^n)(1+q^n)^2\right|^2 = \left|\frac{1}{\eta(\tau)}\sum_{m\in\mathbb{Z}} q^{(m+1/2)^2/2}\right|^2 = Z^{R}, \tag{4.89}$$

$$Z^F_{0,0\big|0} = \left|\frac{1}{\eta(\tau)}q^{1/8}\prod_{n=1}(1-q^n)(1-q^n)(1-q^{n-1})\right|^2 = Z^{\widetilde{R}} = 0. \tag{4.90}$$

These are the well-known partition functions of the $(1+1)$-dimensional Dirac fermion that describes the edge of the two-dimensional $\nu = 1$ topological insulator (see Section 2.3.1).

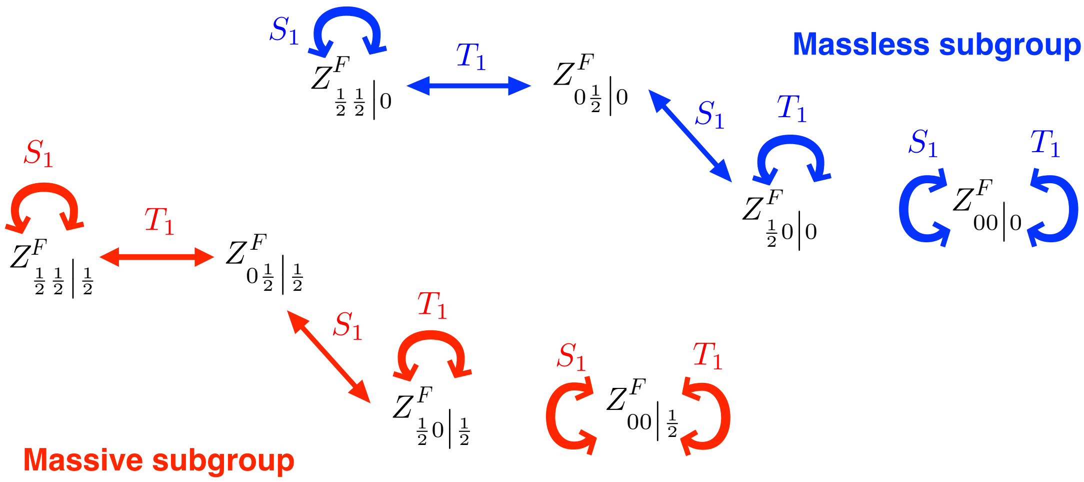


Figure 4.13: Action of the modular group $SL(2,\mathbb{Z})$ over of the eight fermionic partition functions (4.63) reduced on the (x^0, x^1) plane.

In these formulas, we identify the sectors (AA), (PA), (AP), (PP) as $NS, \widetilde{NS}, R, \widetilde{R}$, respectively. Applying the bosonization formulas obtained in Section 1.3.2, we also wrote the bosonic version of these expressions for later use. The $SL(2,\mathbb{Z})$ modular transformations of these partition functions is denoted as the 'massless subgroup' shown in Fig. 4.13.

Massive case $\alpha_2 = 1/2$

As anticipated, in this case a mass term $M = \omega_{11}/2\omega_{22} \to \infty$ remains in the energy spectrum in $(1+1)$ dimensions. Therefore, the reduction leads to the following four partition functions of a massive fermion, that read

$$Z^F_{\frac{1}{2},\frac{1}{2}\left|\frac{1}{2}\right.} = \Theta_{[\frac{1}{2},\frac{1}{2}]}(\tau; M), \qquad Z^F_{0,\frac{1}{2}\left|\frac{1}{2}\right.} = \Theta_{[\frac{1}{2},0]}(\tau; M),$$

$$Z^F_{\frac{1}{2},0\left|\frac{1}{2}\right.} = \Theta_{[0,\frac{1}{2}]}(\tau; M), \qquad Z^F_{0,0\left|\frac{1}{2}\right.} = \Theta_{[0,0]}(\tau; M). \tag{4.91}$$

These functions transform under the subgroup $SL(2,\mathbb{Z})$ in the same way as the massless ones, and are indicated as the 'massive subgroup' in Fig. 4.13.

Chapter 5

Three-dimensional bosonic topological insulators

In this Chapter we will discuss the effective field theory description of topological insulators in $(3+1)$ dimensions provided by the topological BF gauge theory and the associated bosonic surface theory. We shall recall some known facts, derive the action at the boundary, its quantization and the partition function on the three-torus. We shall then compare these results with those found in the previous section for the fermionic theory and discuss the insight they provide on the issue of bosonization in $(2+1)$ dimensions.

5.1 Hydrodynamic BF effective field theory

In Chapter 1 we introduced bosonic degrees of freedom to explain interacting topological states in $(2+1)$ dimensions. The effective Chern-Simons theory (1.11) allows to describe universal long range features of the fractional quantum Hall effect, providing a complementary view to wavefunction approaches [10, 11]. Moreover, the canonical quantization of the compactified free boson theory in (1+1) dimensions of Section 1.3.1 gives an exact description of interacting Hall edge states with Abelian fractional statistics [20].

In Chapter 2 we generalized this approach to topological insulators by taking pairs of quantum Hall effect systems with opposite spin and chirality [59]. Actually, two coupled Chern-Simons theories with opposite chiralities are equivalent to a TR invariant theory, the Abelian BF theory, whose action takes the form

$$S_{BF}^{(2D)}[a,b] = \frac{k}{2\pi}\int d^3x \left(\epsilon^{\mu\nu\rho}a_\mu\partial_\nu b_\rho - \mathcal{J}_a^\mu a_\mu - \mathcal{J}_b^\mu b_\mu\right), \tag{5.1}$$

where $\mathcal{J}_a^\mu$ and $\mathcal{J}_b^\mu$ are the sources of the two fields a and b. It turns out that (5.1) captures the main properties of $2D$ Abelian topological insulators [70] [71], namely those properties that we discussed in Chapter 2 and Chapter 3 by means of partition functions.

In Chapter 4 we discussed the main features of $3D$ topological band insulators. These are gapped phases of matter made of non-interacting fermions whose surface states consist of an odd number of massless Dirac fermions. Furthermore, we showed that when TR symmetry is explicitly broken at the boundary owing to the presence of magnetic fields, the surface states become massive and a surface quantum Hall effect is generated with conductivity $\sigma_H = 1/2$.

Recently, the study on $3D$ topological insulators went beyond the band insulators involving non-interacting electrons, and other TR invariant interacting $3D$ phases with fractionally charged excitations were introduced and theoretically analyzed [103, 111]. Band theory cannot capture the consequences coming from strong electronic interactions, and other methods have been introduced.

In their paper [112], Cho and Moore proposed the Abelian BF theory in (3+1) dimensions as the effective hydrodynamic theory of $3D$ topological insulators. Although this theory is less obvious with respect to its $2D$ counterpart, it allows to recover some features of non-interacting topological phases discussed before, such as the surface quantum Hall effect with conductivity $\sigma_H = 1/2$.

This theory, as shown by other research groups [113, 114, 115, 116], gives an effective description of interacting electron systems. In a similar way as the Chern-Simons theory for the fractional quantum Hall effect, the BF theory in $(3+1)$ dimensions allows to describe interacting $3D$ topological insulators having a richer spectrum of excitations with fractionally charged particles and vortices [112, 117].

5.1.1 Basics of BF theory in $(3+1)$ dimensions

In a $(3+1)$-dimensional manifold $\mathcal{M}$, we consider a theory with matter fluctuations that are described by conserved currents for quasiparticles and vortices, respectively:

$$J^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu b_{\rho\sigma}, \qquad J^{\mu\nu} = \frac{1}{2\pi}\varepsilon^{\mu\nu\rho\sigma}\partial_\rho a_\sigma, \tag{5.2}$$

defining two hydrodynamic gauge fields, the two-form $b = 1/2\, b_{\mu\nu}\, dx^\mu \wedge dx^\nu$ and the one-form $a = a_\mu dx^\mu$. The topological effects between static sources in TR invariant topological states at energies below the bulk gap can be described by the following BF action [112]

$$S_{BF}[a,b,A] = \int_{\mathcal{M}} \frac{K}{2\pi} b\, da + \frac{1}{2\pi} b\, dA + \frac{\theta}{8\pi^2}\, da\, dA - a_\mu \mathcal{J}^\mu - \frac{1}{2} b_{\mu\nu}\mathcal{J}^{\mu\nu}, \tag{5.3}$$

where $A = A_\mu dx^\mu$ is the electromagnetic background, $\mathcal{J}^\mu$ and $\mathcal{J}^{\mu\nu}$ are, respectively, the particles and vortex sources of the fields a_μ and $b_{\mu\nu}$, and the coupling K is an odd integer for fermionic systems [111]. The canonical dimensions of the fields are $[a] = 1$ and $[b] = 2$ in units of mass, and the invariance of the action under TR symmetry, apart from the term proportional to θ, follows from the field transformations

$$\mathcal{T}: \quad (a_0, \vec{a}) \;\to\; (a_0, -\vec{a}), \tag{5.4}$$

$$\mathcal{T}: \quad (A_0, \vec{A}) \;\to\; (A_0, -\vec{A}), \tag{5.5}$$

$$\mathcal{T}: \quad (b_{0i}, b_{ij}) \;\to\; (-b_{0i}, b_{ij}). \tag{5.6}$$

Finally, for a compact manifold $\mathcal{M}$ the BF action (5.3) is invariant under the following gauge transformations

$$a \to a + d\lambda, \qquad b \to b + d\xi, \tag{5.7}$$

where λ and ξ are, respectively, a scalar function and a one-form.

The current $\mathcal{J}^\mu$ represents a quasiparticle source, namely

$$\mathcal{J}^\mu(x) = N_0 \int_L d\tau \frac{dX^\mu}{d\tau} \delta^{(4)}[x - X(\tau)], \tag{5.8}$$

where $X^\mu(\tau)$ is the the world-line L of a particle with charge N_0. Similarly, the current $\mathcal{J}^{\mu\nu}$ is a vortex source, i.e.

$$\mathcal{J}^{\mu\nu}(x) = N_1 \int_\Sigma d^2\sigma \epsilon^{\alpha\beta} \frac{dX^\mu}{d\sigma^\alpha} \frac{dX^\nu}{d\sigma^\beta} \delta^{(4)}[x - X(\sigma_\alpha)], \tag{5.9}$$

where $X^\mu(\sigma_1, \sigma_2)$ is the world-sheet Σ of the string with "electromagnetic" flux N_1.

For $A_\mu = 0$, the action (5.3) gives the following equations of motion

$$\mathcal{J}^{\mu\nu} = \frac{k}{2\pi} \epsilon^{\mu\nu\lambda\rho} \partial_\lambda a_\rho, \tag{5.10}$$

$$\mathcal{J}^{\mu} = \frac{k}{4\pi} \epsilon^{\mu\lambda\rho\nu} \partial_\lambda b_{\rho\nu}. \tag{5.11}$$

A non-trivial effect of the BF theory is the particle-string holonomy, the analog of fractional statistics in $(3+1)$ dimensions [117]. In three space dimensions point-like particles can braid with vortex lines. When a particle of charge N_0 encircles a vortex of flux N_1 once, the wave function of the system acquires the phase factor [117]

$$\Psi_{BF} \;\to\; \exp\left(-\frac{2\pi i}{K} N_0 N_1\right) \Psi_{BF}. \tag{5.12}$$

It follows that the wavefunctions carry a one-dimensional unitary representation of the fractional particle-vortex linking.

5.1.2 Abelian θ-term and surface QHE

Integrating out the matter fields a and b of the BF action (5.3) with $\mathcal{J}^\mu = \mathcal{J}^{\mu\nu} = 0$, we obtain the induced action for the electromagnetic background

$$S_{\text{ind}}[A] = -\frac{\theta}{8\pi^2 K} \int_{\mathcal{M}} dA\, dA = -\frac{\theta}{32\pi^2 K} \int d^4x\ \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}. \tag{5.13}$$

This is the *Abelian θ-term* or *magnetoelectric term* [98] [112], already discussed in the fermion case for $K = 1$ in Section 4.3.3. The case $\theta = 0$ corresponds to TR invariant systems, where bulk and boundary contributions cancel each other. For $\theta = \pi$, TR symmetry is broken at the surface, leading to the induced Chern-Simons term,

$$S_{\text{ind}}[A] = -\frac{1}{8\pi K} \int_{\partial\mathcal{M}} A\, dA \tag{5.14}$$

implying a surface quantum Hall effect with filling fraction $\nu = 1/2K$. In particular, for $K = 1$ the fermion result (4.26) is recovered. This is the first indication that the bosonic theory for $K = 1$ matches the fermionic description of $3D$ topological insulators, at least for the topological properties [112]. Therefore, other values of K should describe, in the low energy approximation, interacting fermionic theories with quasiparticle and vortex excitations possessing non-trivial braiding statistics (5.12) [111, 112].

5.1.3 Surface BF theory and fermion-boson duality

In this section we will discuss the surface massless excitations provided by the bulk BF theory and introduce a dynamics for them that respects TR symmetry [3].

Several times in this thesis we encountered the problem of bosonization, namely the mapping of quantum field theories of interacting fermions onto an equivalent theory of interacting bosons. These mappings, as shown in Chapter 1, are well established in the context of $(1+1)$ dimensional theories, and are powerful tools to study the non-perturbative behavior of both quantum field theories and condensed matter systems. However, in dimensions higher then $(1+1)$, much less is known.

The first attempts to understanding a possible bosonization approach to fermionic models were done in the past by Luther and Aratyn Ref.[118, 119]. The authors constructed Fermi fields out of Bose operators via the so called tomographic transformation [119]. Later, Marino [120], based in an order-disorder duality, was able to express a $(2+1)$ massless fermion field in terms of a bosonic vector field [120]. In the last case, the resulting induced action, expanded at the quadratic order in the electromagnetic background field, is equivalent to the fermionic result in (4.18): it contains the parity violating Chern-Simons term and the non-local Maxwell term.

In the recent literature, the problem of bosonization has been reconsidered in three spacetime dimensions and new relations have been proposed [114, 115, 121, 122, 123, 124]. Among these new results, the particle-vortex duality plays an important role in describing the surface states of interacting topological insulators as shown in Ref.[125, 126]. Further dualities between bosonic and fermionic field theories are suggested by flux attachment [127, 128, 129], an idea that was first developed for non-relativistic particles in order to transmute their statistics from fermionic to bosonic and vice versa [13, 130].

In our work [3] we recall that the surface BF theory allows to reproduce the duality relations proposed by Aratyn in Ref.[119]. Moreover, introducing a non-local dynamics, we show that BF theory with $K = 1$ is able to reproduce the induced effective action for the external electromagnetic field of the massless fermion to quadratic order. These results highlight the possibility that the $3D$ BF theory is the hydrodynamic effective field theory of TR invariant topological insulators in three space dimensions as suggested by Cho and Moore [112].

Gauge invariance and boundary degrees of freedom

In a manifold $\mathcal{M}$ with boundary the BF action (5.3) is not longer invariant under the gauge transformations (5.7). The symmetry can be recovered by introducing degrees of freedom living at the boundary $\partial\mathcal{M}$ that compensate the gauge non-invariance of the bulk. As shown in Ref.[112], these degrees of freedom can be viewed as pure gauge configurations reproducing the bulk loop observables, namely $b = d\zeta$ with ζ a vector field. Therefore, we introduce surface terms as follows

$$S_{BF}^{\text{Bulk+Boundary}}[a, b, \zeta, A, \theta] = \int_{\mathcal{M}} \frac{K}{2\pi} bda + \frac{1}{2\pi} bdA + \frac{\theta}{8\pi^2} dadA \tag{5.15}$$

$$- \int_{\partial\mathcal{M}} \frac{K}{2\pi} \zeta da + \frac{1}{2\pi} \zeta dA,$$

whose combined gauge transformations are

$$a \to a + d\lambda, \qquad b \to b + d\xi, \tag{5.16}$$

$$A \to A + d\Lambda, \qquad \zeta \to \zeta + \xi. \tag{5.17}$$

Surface theory in absence of electromagnetic coupling

Let us focus our attention to the boundary terms of the gauge invariant BF action (5.15) in absence of the electromagnetic background, that is putting $A_\mu = 0$. The topological theory has vanishing Hamiltonian: in the gauge $\zeta_0 = a_0 = 0$, the boundary action becomes

$$S_{\text{surf}}[\zeta, \phi] = \frac{K}{2\pi} \int d^3x \, \epsilon^{ij} \partial_i \zeta_j \, \dot{\phi}, \tag{5.18}$$

where we also take $a = d\phi$, a pure gauge configuration at the boundary. The transformations under TR symmetry of the surface fields are inferred from those of the bulk fields (5.4); they are

$$\mathcal{T}: \qquad \phi \;\to\; -\phi, \tag{5.19}$$

$$\mathcal{T}: \qquad \zeta_i \;\to\; \zeta_i, \tag{5.20}$$

from which follows the invariance of the boundary action (5.18).

The action (5.18) shows the symplectic structure made by two scalar degrees of freedom that are canonically conjugate, ϕ and χ, being the longitudinal part $a_i = \partial_i\phi$, and the transverse part $\zeta_i = \epsilon_{ik}\partial_k\chi$, respectively. Therefore Eq.(5.18) becomes

$$S_{\text{surf}}[\zeta, \phi] = \int d^3x \, \pi \, \dot{\phi}, \qquad \pi = \frac{K}{2\pi} \epsilon^{ij} \partial_i \zeta_j = -\frac{K}{2\pi} \Delta\chi. \tag{5.21}$$

As suggested for example in [112, 113, 131], we can introduce a relativistic dynamics by adding the Hamiltonian of the free scalar field in $(2+1)$ dimensions, as follows

$$S_{\rm surf} \to \int d^3x \left(\pi\dot{\phi} - \mathcal{H}(\pi,\phi)\right) = \int d^3x \left(\pi\dot{\phi} - \frac{1}{2m}\pi^2 - \frac{m}{2}(\partial_i\phi)^2\right). \qquad (5.22)$$

In this equation, we introduced a mass parameter for adjusting the mismatch of dimensions between bulk and boundary: indeed, the bulk gauge fields imply the mass dimensions $[\phi]=0$ and $[\pi]=2$, which are different from those of the three-dimensional theory, i.e. 1/2 and 3/2, respectively. A dimensionless coupling could also be introduced for the third term in the action (5.22), that would determine the Fermi velocity of excitations; we fix it conventionally to one. The equations of motion following the action (5.22) are

$$\pi = m\dot{\phi}, \qquad \dot{\pi} = m\Delta\phi \qquad (5.23)$$

and the Lagrangian form of the action is clearly

$$S_{\rm surf}[\phi] = \frac{m}{2}\int d^3x\ (\partial_\mu\phi)^2 . \qquad (5.24)$$

The Hamiltonian equations of motion (5.23) can be recast into a duality relation between the boundary scalar and vector fields, that can be written in covariant form (with $\zeta_0 = 0$):

$$\frac{K}{2\pi}\,\epsilon^{\mu\nu\rho}\partial_\nu\zeta_\rho = m\,\partial_\mu\phi. \qquad (5.25)$$

This relation is just the electric-magnetic duality in (2+1) dimensions. It plays an important role in the bosonization by Aratyn using the tomographic representation [112, 113, 119], and in the functional bosonization approach [114, 115, 123]. We remark that in our context this is just the first-order Hamiltonian description of the relativistic wave equation, that is inherited from the first-order bulk theory. We also stress that the main motivation for the quadratic Hamiltonian (5.22) is its simplicity: we know that the surface fermion discussed in Chapter 4 cannot be exactly matched to a free boson (for $K=1$).

Surface theory with electromagnetic coupling

Restoring the electromagnetic coupling at the boundary, we can infer other interesting conclusions to test the correspondence between the bosonic and fermionic theories. The coupling inherited from the bulk theory is shown in (5.15) and it amounts to the shift $a_\mu \to a_\mu + A_\mu/K$. This can be implemented in the symplectic form (5.21) in the gauge $\partial_i A_i = 0$, and in the Hamiltonian (5.22) by enforcing gauge invariance, namely by means of the Higgs-like substitution $\partial_\mu\phi \to \partial_\mu\phi - A_\mu/K$. We thus obtain the following action

$$S_{\rm surf}[\zeta,\phi,A] = \int d^3x \left[\pi\left(\dot{\phi} - \frac{A_0}{K}\right) - \frac{1}{2m}\pi^2 - \frac{m}{2}\left(\partial_i\phi - \frac{A_i}{K}\right)^2\right], \qquad (5.26)$$

Upon integrating the scalar fields, we obtain the induced action

$$S^B_{\rm ind}[A] = -\frac{m}{4K^2}\int d^3x\ F_{\mu\nu}\frac{1}{\Box}F_{\mu\nu}. \qquad (5.27)$$

This result should be compared for $K = 1$ with the fermionic induced action computed in Section 4.3.2, Eq.(4.18); this contains the parity violating Chern-Simons term that is cancelled by the bulk contribution and should not be reproduced by the bosonic theory. The other non-local Maxwell term in the fermionic action is different from (5.27) owing to the mass scale m of the bosonic theory. Note that this parameter could be eliminated by the field redefinition

$$\widetilde{\phi} = \sqrt{m}\phi, \qquad \widetilde{\zeta}_i = \frac{\zeta_i}{\sqrt{m}}, \qquad \widetilde{\pi} = \frac{1}{\sqrt{m}}\pi, \tag{5.28}$$

but this would not change the induced action (5.27).

Nonetheless, the fermionic induced action can be reproduced by introducing another dynamics for the bosonic theory [3]. Let us reconsider the duality relation between vector and scalar fields (5.25) and modify it as follows

$$\frac{K}{2\pi}\,\epsilon^{\mu\nu\rho}\partial_\nu\zeta_\rho = \Box^{1/2}\,\partial_\mu\phi, \tag{5.29}$$

namely by replacing the mass with a Lorentz invariant non-local operator. This modified duality corresponds to the following Hamiltonian equations of motion

$$\pi = \frac{K}{2\pi}\varepsilon^{ij}\partial_i\zeta_j = \Box^{1/2}\dot{\phi}, \qquad \dot{\pi} = \Box^{1/2}\Delta\phi, \tag{5.30}$$

that follow from the action (in absence of electromagnetic coupling)

$$S'_{\text{surf}}[\zeta,\phi] = \int d^3x \left(\pi\dot{\phi} - \frac{1}{2}\pi\frac{1}{\Box^{1/2}}\pi - \frac{1}{2}\partial_i\phi\,\Box^{1/2}\partial_i\phi\right). \tag{5.31}$$

Integrating (5.31) on ζ, we find that the Lagrangian formulation reads

$$S'_{\text{surf}}[\phi] = -\frac{1}{2}\int \phi\,\Box^{3/2}\phi = \frac{1}{2}\int \left(\partial_\mu\widetilde{\phi}\right)^2, \qquad \text{with} \quad \phi = \Box^{1/4}\widetilde{\phi}, \tag{5.32}$$

that is again the free bosonic theory in the rescaled variable $\widetilde{\phi}$. The coupling to the electromagnetic field implied by the bulk theory is still given by the Higgs-like substitution, $\partial_\mu\phi \to \partial_\mu\phi - A_\mu/K$, leading to the action

$$S'_{\text{surf}}[\phi, A] = \frac{1}{2}\int d^3x \left(\partial_\mu\phi - \frac{A_\mu}{K}\right)\Box^{1/2}\left(\partial_\mu\phi - \frac{A_\mu}{K}\right). \tag{5.33}$$

Upon integrating in the ϕ field, for $K = 1$ we obtain the following induced action

$$S^B_{\text{ind}}[A] = -\frac{1}{4}\int d^3x F_{\mu\nu}\frac{1}{\Box^{1/2}}F_{\mu\nu}. \tag{5.34}$$

Up to coefficient, this action coincides with the non-local TR invariant Maxwell term in (4.18) responsible for non-topological but universal subleading effects [3]. Although possible for a massless theory, a non-local effective action usually means that further massless excitations have been integrated out. Thus, the present analysis may not be a complete description of bosonization in $(2 + 1)$ dimensions.

In conclusion, we have shown that the surface degrees of freedom of $(3+1)$ dimensional topological insulators amount to a Hamiltonian conjugate pair of scalar fields. The simpler quadratic Hamiltonian for them is not able to reproduce the fermionic theory to leading order, while a modified non-local dynamics does work. The two bosonic theories are identical on-shell, since both imply the free wave equation for a suitably rescaled field variable, but may differ in the solitonic excitations, to be discussed later.

We also remark that the simple action (5.22) with coupling to electromagnetic field given by (5.26) is equivalent to the Abelian Higgs model in $(2+1)$ dimensions in the deep infrared limit of the spontaneously broken phase [3]. Namely, the scalar field ϕ can be interpreted as the Goldstone mode of a complex scalar

$$\Phi = \rho\, \mathrm{e}^{i\phi}, \qquad \langle \rho^2 \rangle = m, \tag{5.35}$$

where the mass parameter m fixes the vacuum expectation value, and the Higgs field is frozen. We conclude that in cases where the electromagnetic field could be considered dynamical, we would have a surface superconducting phase. On the other hand, the nonlocal dynamics (5.33) would keep the photon massless as in the fermionic theory. These issues are left for further investigations. Before proceeding, we remark that the following analysis will deal with properties that are independent of the specific dynamics and that the topological data in $(3+1)$ dimensions given by the BF action (5.3) do not determine a unique dynamics for the surface states, contrary to the $(2+1)$-dimensional case.

5.2 Canonical quantization of the surface theory

In this section we consider the canonical quantization of the surface BF theory (5.22) with compactified boson field. Following Ref.[116, 131] and our work [3], in the quantization of the action (5.22) we shall pay particular attention to the properties of solitonic modes of the ϕ and ζ_i fields, in such a way that they consistently reproduce the topological properties of the bulk BF theory.

Bulk topological sectors and boundary observables

The quantization of the bulk BF theory (5.3) on the spatial three-torus $\mathcal{M} = \mathbb{T}^3 \times \mathbb{R}$, leads to the topological order of K^3 'anyon' sectors, for odd integer values of the coupling K. The proof of this results is very simple [11, 117]: one considers the integrals of the gauge fields on the surfaces and cycles of the torus

$$\pi_{ij} = \int_{\Sigma_{ij}} b, \quad i \neq j, \qquad q_i = \int_{\gamma_i} a, \qquad i,j,k = 1,2,3 \tag{5.36}$$

Once inserted into the BF action, these global quantities become three pairs of canonically conjugate variables $(\pi_{ij}(t), q_k(t))$, leading to the following commutation relations

$$[\pi_{ij}(t), q_k(t)] = i\frac{2\pi}{K}\varepsilon_{ijk} \tag{5.37}$$

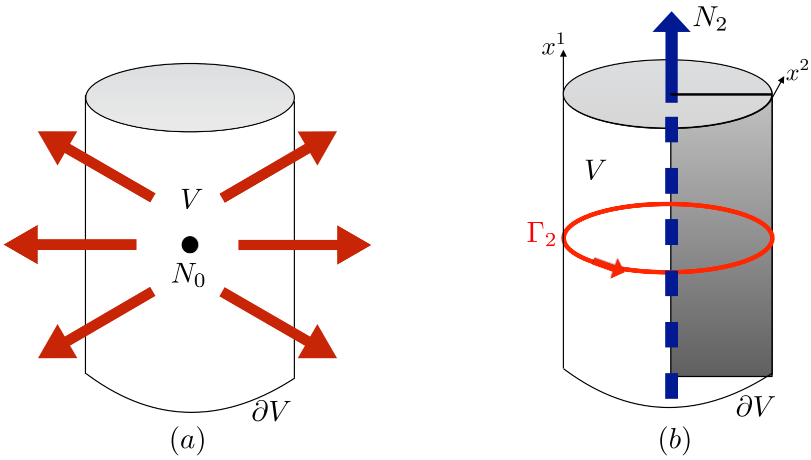


Figure 5.1: The thick two-torus $V = D^2 \times S^1$ is represented as a filled cylinder with identified faces. (a): The bulk quasiparticle with charge N_0 creates a flux of the b field across the boundary surface ∂V. (b): In blue the bulk vortex of charge N_2 along the non-trivial cycle x^1, in red the closed line Γ_2 encircling on the surface the vortex excitation, in grey the branch cut surface from the vortex excitation to the boundary surface ∂V.

A basis of holonomies on the torus is given by the operators $U_i = \exp(iq_i)$ and $V_{ij} = \exp(i\pi_{ij})$, that form three pairs of K-dimensional clock and shift matrices [11], thus leading to the topological order K^3. In the following, we find a corresponding symplectic structure among the solitonic modes of the boundary fields ϕ and ζ_μ, that are defined on the space-time three-torus.

The relation between bulk and boundary observables can be studied on the spatial geometry of the thick two-torus $V = D^2 \times S^1$ shown in Fig. 5.1, whose boundary is the two-torus $\partial V = S^1 \times S^1$. We first consider a bulk quasiparticle with charge N_0 at rest in $\vec{x} = \vec{x}_0$, whose current is $\mathcal{J}^0(\vec{x}) = N_0 \delta^{(3)}(\vec{x} - \vec{x}_0)$. The solution of the equations of motion (5.11) leads to a flux of the b field across the surface enclosing the charge (see Fig. 5.1(a)), that becomes the following expression on the boundary surface

$$\frac{2\pi N_0}{K} = \int_{\partial V} d^2x\, \epsilon^{ij} \partial_i \zeta_j. \tag{5.38}$$

Next, a static vortex in the bulk stretched along the non-trivial cycle as shown in Fig. 5.1(b) with magnetic charge N_2 corresponds to the current $\mathcal{J}^{01}(\vec{x}) = N_2 \delta^{(2)}(\vec{x} - \vec{x}_0)$. The presence of the vortex propagates from the bulk V to the boundary surface ∂V as a branch cut surface. The equations of motion (5.10) imply a non-vanishing integral of the a field along a closed path encircling the vortex; for the path Γ_2 on the boundary surface it reads (see Fig. 5.1(b))

$$\frac{2\pi}{K} N_2 = \oint_{\Gamma_2} dx^2 \partial_2 \phi, \tag{5.39}$$

An analogous relation holds for the vortex stretched along the other non-trivial cycle of the boundary, i.e.

$$\frac{2\pi}{K}N_1 = \oint_{\Gamma_1} dx^1 \partial_1 \phi. \tag{5.40}$$

The bulk-boundary relations (5.38) and (5.39) together with the TR transformations of the boundary fields (5.19) and (5.20) provide the following transformations for the bulk charges under TR symmetry

$$\mathcal{T}: \qquad N_0 \;\rightarrow\; N_0, \tag{5.41}$$

$$\mathcal{T}: \qquad N_i \;\rightarrow\; -N_i, \qquad i=1,2, \tag{5.42}$$

that are consistent with the usual transformation rules of an electric charge for N_0 and a magnetic flux for N_i.

Canonical quantization

The simpler surface bosonic action (5.22)

$$S_{\text{surf}}[\zeta,\phi] = \int_{\partial\mathcal{M}} d^3x \, \left(\pi\dot{\phi} - \frac{1}{2m}\pi^2 - \frac{m}{2}(\partial_i\phi)^2 \right), \tag{5.43}$$

is considered on the spatial two-torus, i.e. $\partial\mathcal{M} = S^1 \times S^1 \times \mathbb{R}$. The conjugate momentum π is

$$\pi = \frac{K}{2\pi}\epsilon^{ij}\partial_i\zeta_j. \tag{5.44}$$

and the Hamiltonian equations of motion are

$$\frac{K}{2\pi}\epsilon^{ij}\partial_i\zeta_j = m\dot{\phi}, \qquad\qquad \frac{K}{2\pi}\epsilon^{ij}\partial_i\dot{\zeta}_j = m\Delta\phi. \tag{5.45}$$

The canonical quantization proceeds by expanding the fields in terms of solutions of the equations of motion, with boundary conditions of the spatial two-torus specified by the periods $\vec{\omega}_i$ (4.29) and dual vectors $\vec{k}_i$ (4.28), $i=1,2$ discussed in Section 4.4. Let us write the field expansions and then explain them [116]:

$$\begin{aligned} \phi(\vec{x},t) &= \phi_0 + 2\pi\Lambda_i\vec{k}_i\cdot\vec{x} + \frac{K\Lambda_0 t}{mV^{(2)}} \\ &+ \frac{1}{\sqrt{mV^{(2)}}}\sum_{\vec{n}\neq\vec{0}}\frac{1}{\sqrt{2E_{\{\vec{n}\}}}}\left[a_{\vec{n}}\,\mathrm{e}^{-iE_{\{\vec{n}\}}t+2\pi i\vec{k}_{\{\vec{n}\}}\cdot\vec{x}} + a^\dagger_{\vec{n}}\,\mathrm{e}^{iE_{\{\vec{n}\}}t-2\pi i\vec{k}_{\{\vec{n}\}}\cdot\vec{x}}\right], \end{aligned} \tag{5.46}$$

$$\begin{aligned} \zeta_j(\vec{x},t) &= \frac{\epsilon_{ji}}{V^{(2)}}\left(\omega_{2i}\gamma_1 - \omega_{1j}\gamma_2 - \pi\Lambda_0 x_i\right) \\ &+ \frac{8\pi^2}{K}\sqrt{\frac{m}{V^{(2)}}}\sum_{\vec{n}\neq\vec{0}}\frac{\epsilon_{jm}\left(n_1k_{1m}+n_2k_{2m}\right)}{\left(2E_{\{\vec{n}\}}\right)^{3/2}}\left[a_{\vec{n}}\,\mathrm{e}^{-iE_{\{\vec{n}\}}t+2\pi i\vec{k}_{\{\vec{n}\}}\cdot\vec{x}} + a^\dagger_{\vec{n}}\,\mathrm{e}^{iE_{\{\vec{n}\}}t-2\pi i\vec{k}_{\{\vec{n}\}}\cdot\vec{x}}\right]. \end{aligned} \tag{5.47}$$

These expressions involve oscillating functions specified by energies and momenta

$$E_{\{\vec{n}\}} = 2\pi \left| n_1 \vec{k}_1 + n_2 \vec{k}_2 \right| = \frac{2\pi}{V^{(2)}} \left| n_1 \vec{\omega}_2 - n_2 \vec{\omega}_1 \right|, \tag{5.48}$$

$$k_{1\{\vec{n}\}} = \frac{1}{V^{(2)}} \left(n_1 \omega_{22} - n_2 \omega_{12} \right), \tag{5.49}$$

$$k_{2\{\vec{n}\}} = \frac{1}{V^{(2)}} \left(-n_1 \omega_{21} + n_2 \omega_{11} \right), \qquad \vec{n} = (n_1, n_2) \in \mathbb{Z}^2. \tag{5.50}$$

The field expansions (5.46),(5.47) also contain constant and linear terms, almost unconstrained by the equations of motion, that are needed for specifying the solitonic modes. Actually, upon inserting these expressions in the boundary observables (5.38)-(5.39), we find the following values of the ζ_i flux and $\partial_i \phi$ circulations

$$\Lambda_\alpha = \frac{N_\alpha}{K}, \qquad \alpha = 0, 1, 2 \tag{5.51}$$

that explain the normalizations adopted for such terms.

The commutation relations between the fields ϕ and π,

$$\left[\phi(\vec{x}, t), \varepsilon^{ij} \partial_i \zeta_j(\vec{y}, t) \right] = i \frac{2\pi}{K} \delta^{(2)}(\vec{x} - \vec{y}), \tag{5.52}$$

imply the following non-vanishing commutators

$$\left[a_{\vec{n}}, a^\dagger_{\vec{k}} \right] = \delta_{\vec{n}, \vec{k}}, \qquad [\phi_0, \Lambda_0] = \frac{i}{K}. \tag{5.53}$$

Moreover, integrating by parts the symplectic term in the action (5.43), we can also consider ζ_i and $\varepsilon^{ij} \partial_i \phi$, for $i = 1, 2$, as two pairs of coordinates and momenta, leading to two further commutation relations

$$\left[\zeta_i(\vec{x}, t), \epsilon_{ij} \partial_j \phi(\vec{y}, t), \right] = -\frac{2\pi i}{k} \delta^{(2)}(\vec{x} - \vec{y}), \qquad i = 1, 2. \tag{5.54}$$

These should be consider as independent relations for the solitonic modes only; they imply the earlier quantizations plus the following ones

$$[\gamma_1, \Lambda_2] = -\frac{i}{K}, \qquad [\gamma_2, \Lambda_1] = \frac{i}{K}. \tag{5.55}$$

These two commutation relations together with that of Λ_0 in (5.53) represent the bulk degrees of freedom (5.36) within the boundary theory: one quantity in each pair is actually the same bulk observable evaluated at the boundary, while the conjugate variable is a field zero mode. After quantization, the eigenvalues of Λ_μ can be identified with the spectra (5.51), that are consistent with the periodicities of the field zero modes

$$\phi_0 \equiv \phi_0 + 2\pi r, \qquad \gamma_i \equiv \gamma_i + 2\pi r_i, \quad i = 1, 2, \tag{5.56}$$

with compactification radii $r = r_1 = r_2 = 1$. It is also immediate to see that these periodicities are commensurate with those of the fields $\phi(\vec{x})$ and $\zeta_i(\vec{x})$ winding around the cycles of the torus. In conclusion, the Λ_μ spectra given by (5.51) are both suggested by the bulk theory and consistently obtained by quantization of the boundary theory. The same conclusion follows from the better-known study of the bosonic edge theory in $(1+1)$ dimensions discussed in Section 1.3.2 [20]. In both cases, there could be more general consistent quantizations of the bosonic theories and solitonic spectra, but they would not relate to the bulk topological data.

The Hamiltonian and momenta obtained by the surface bosonic theory (5.43) are, respectively,

$$H = \frac{m}{2}\int d^2x \left(\dot{\phi} + (\partial_1\phi)^2 + (\partial_2\phi)^2\right). \tag{5.57}$$

and

$$P^i = -m\int d^2x \dot{\phi}\partial_i\phi, \qquad i = 1, 2, \tag{5.58}$$

We substitute the field expansions (5.46) and (5.47), and use the Fock ground state $|\Omega\rangle$ definition

$$a_{\vec{n}}\,|\Omega\rangle = 0, \qquad n_1, n_2 \in \mathbb{Z}, \tag{5.59}$$

to obtain the following expressions in terms of Fock and solitonic operators

$$\begin{aligned} H = {} & \frac{K^2\Lambda_0^2}{2mV^{(2)}} + \frac{(2\pi)^2 m}{2V^{(2)}}\left[(\Lambda_1\omega_{22} - \Lambda_2\omega_{12})^2 + (\Lambda_1\omega_{21} - \Lambda_2\omega_{11})^2\right] \\ & + \sum_{\vec{n}\neq\vec{0}} E_{\{\vec{n}\}}\left(a^\dagger_{\vec{n}} a_{\vec{n}} + \frac{1}{2}\right), \end{aligned} \tag{5.60}$$

$$P^1 = \frac{2\pi k\Lambda_0}{V^{(2)}}\left(-\Lambda_1\omega_{22} + \Lambda_2\omega_{12}\right) + 2\pi\sum_{\vec{n}\neq\vec{0}} k_{1\{\vec{n}\}}\ a^\dagger_{\vec{n}} a_{\vec{n}}, \tag{5.61}$$

$$P^2 = \frac{2\pi k\Lambda_0}{V^{(2)}}\left(\Lambda_1\omega_{21} - \Lambda_2\omega_{11}\right) + 2\pi\sum_{\vec{n}\neq\vec{0}} k_{2\{\vec{n}\}}\ a^\dagger_{\vec{n}} a_{\vec{n}}, \tag{5.62}$$

where the energies $E_{\{\vec{n}\}}$ are momenta $k_{i\{\vec{n}\}}$ are given, respectively, in (5.48) and (5.49)-(5.50).

5.3 Torus partition functions

In this section we will compute the partition functions of the surface BF theory by compactifying the time direction. In the first part, following the canonical quantization studied above, we will separate the contributions to the partition functions coming from the oscillating and the solitonic modes. In the second part, in order to perform the flux insertion argument and obtain the stability criteria of fermionic topological insulators, will require to enlarge the spectrum of the solitonic modes to half-integer values, leading to eight partition functions and spin sectors for the bosonic theory [3].

The BF partition function

In the $\mathbb{T}^3$ space-time torus described by the $\boldsymbol{\omega}$-matrix in (4.27) of periods $\boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2$, the Euclidean partition function is defined as

$$Z^B = \sum_{\Lambda_\mu \in \mathbb{Z}^3/K} \mathrm{Tr}\, \exp\left(-\omega_{00} H + i\omega_{01} P_1 + i\omega_{02} P_2\right), \tag{5.63}$$

where ω_{00} is the Euclidean time period, equal to the inverse temperature β.

In the trace over the states, the sums over the Fock space and solitonic modes can be done independently, because their contributions add up in the expressions of Hamiltonian (5.60) and momentum (5.61)-(5.62). Thus, the partition function can be factorized into oscillator and solitonic parts Z_{HO} and $Z^{(0)}$, respectively

$$Z^B = Z_{HO} Z^{(0)}. \tag{5.64}$$

Performing the trace over the states of the Fock space, the oscillator part reads

$$\begin{aligned} Z_{HO} = &\prod_{n_1 n_2 \neq (0,0)} \left(1 - \exp\left(-\omega_{00} E_{\{\vec{n}\}} + i\omega_{01} 2\pi k_{1\{\vec{n}\}} + i\omega_{02} 2\pi k_{2\{\vec{n}\}}\right)\right)^{-1} \\ &\times \exp\left(-\frac{\omega_{00}}{2} \sum_{n_1, n_2 \neq (0,0)} E_{\{\vec{n}\}}\right), \end{aligned} \tag{5.65}$$

where the second exponential factor involves the infinite vacuum energy. For the solitonic part we obtain

$$\begin{aligned} Z^{(0)} = \sum_{\Lambda_\mu \in \mathbb{Z}^3/K} \exp\Bigg[&- \frac{\omega_{00}}{V^{(2)}} \left(\frac{K^2 \Lambda_0^2}{2m} + 2\pi^2 m \left[(\Lambda_1 \omega_{22} - \Lambda_2 \omega_{12})^2 + (-\Lambda_1 \omega_{21} + \Lambda_2 \omega_{11})^2\right]\right) \\ &- i\omega_{01} \frac{2\pi k \Lambda_0}{V^{(2)}} \left(\Lambda_1 \omega_{22} - \Lambda_2 \omega_{12}\right) - i\omega_{02} \frac{2\pi k \Lambda_0}{V^{(2)}} \left(-\Lambda_1 \omega_{21} + \Lambda_2 \omega_{11}\right) \Bigg]. \end{aligned} \tag{5.66}$$

We can recast these expressions in covariant $(2+1)$-dimensional notation as functions of the moduli $\boldsymbol{\omega}_\mu$ by means of (4.32). We obtain

$$Z_{HO} = \exp(F) \prod_{(n_1, n_2) \neq (0,0)} \left(1 - \exp\left(-2\pi \mathcal{E}_{\{\vec{n}\}} + 2\pi i \mathcal{K}_{\{\vec{n}\}}\right)\right)^{-1}, \tag{5.67}$$

where

$$\mathcal{E}_{\{\vec{n}\}} = V^{(3)} \frac{|n_1 \boldsymbol{\omega}_2 - n_2 \boldsymbol{\omega}_1|}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|^2}, \tag{5.68}$$

$$\mathcal{K}_{\{\vec{n}\}} = \frac{(\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2)}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|^2} \left(n_1 \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_2 - n_2 \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_1\right), \tag{5.69}$$

$$F = \frac{V^{(3)}}{4\pi} \sum_{n_1, n_1 \neq (0,0)} \frac{1}{|n_1 \boldsymbol{\omega}_2 - n_2 \boldsymbol{\omega}_1|^3}, \tag{5.70}$$

with the vacuum energy F regularized by analytical continuation as in the fermionic case (see Appendix C.1) [105, 106]; and

$$Z^{(0)} = \sum_{\Lambda_\mu \in \mathbb{Z}^3/K} \exp\Bigg[- \frac{V^{(3)}}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|^2} \left(\frac{K^2 \Lambda_0^2}{2m} + 2\pi^2 m |\Lambda_1 \boldsymbol{\omega}_2 - \Lambda_2 \boldsymbol{\omega}_1|^2 \right) \\ - \frac{i 2\pi K \Lambda_0}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|^2} (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2) \cdot (\Lambda_1 \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_2 - \Lambda_2 \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_1) \Bigg] \qquad (5.71)$$

The spin sectors of the bosonic theory

The experience with fermionic topological insulators in $(2+1)$ dimensions and bosonization suggests some properties regarding the results (5.67) and (5.71) just found:

- The partition function should split into the sum over K^3 terms, each one pertaining to an anyon sector with given fractional values of the charges of the theory.
- Further partition functions should be found that are associated to the eight fermionic spin sectors of the three-torus.
- Similar to the fermionic cases discussed in Sections 2.3 and (4.4), they can be related one to another by adding half-flux quanta and by performing modular transformations.

Let us gradually derive these results in the $(3+1)$-dimensional theory [3]. The anyon sectors can be identified by splitting the summations over the charge lattice $\Lambda_\mu \in \mathbb{Z}^3/K$ in $Z^{(0)}$ into integer and fractional values, by substituting

$$\Lambda_\mu = M_\mu + \frac{m_\mu}{K}, \qquad M_\mu \in \mathbb{Z}, \qquad m_\mu = 0, 1, \cdots, k-1, \qquad \mu = 0, 1, 2. \qquad (5.72)$$

In this way we get the K^3 terms, each one involving summations over integer-spaced charges

$$Z^{(0)} = \sum_{\Lambda_\mu \in \mathbb{Z}^3/K} \cdots = \sum_{m_0, m_1, m_2 = 0}^{k-1} \sum_{M_0, M_1, M_2 \in \mathbb{Z}} \cdots = \sum_{m_0, m_1, m_2 = 0}^{k-1} Z^{(0)\, m_0, m_1, m_2}. \qquad (5.73)$$

In Chapter 4, we formulated the flux insertion argument for the stability of fermionic topological insulators in terms of fermionic partition functions. Starting from the Neveu-Schwarz sector, we added half fluxes through the donut and obtained the other spin sectors of the theory. In the bosonic theory, adding fluxes clearly modify the values of the loop observable (5.39) and (5.40), such that one flux Φ_0 adds one unit of magnetic charge to the corresponding vortex, causing $N_i \to N_i + 1$.

For $K = 1$ adding one flux is clearly a symmetry of the Hamiltonian and of the partition function (5.71), owing to the summation over $\Lambda_i \in \mathbb{Z}$; thus, we should consider adding half fluxes by the transformations [3]

$$V_i^{1/2}: \quad \Phi_i \to \Phi_i + \frac{\Phi_0}{2} \qquad \Lambda_i \to \Lambda_i + \frac{1}{2}, \quad i = 1, 2, \qquad (5.74)$$

changing the summation values. For $K > 1$ odd, guided by the experience in $(2+1)$ dimensions, we should add fluxes without changing the anyon sectors, and thus consider the transformations

$$V_i^{K/2}: \qquad \Lambda_i = M_i + \frac{m_i}{2} \ \to \ \Lambda_i + \frac{1}{2}, \quad i = 1, 2. \tag{5.75}$$

We shall introduce three labels $\alpha_\mu = 0, 1/2$, $\mu = 0, 1, 2$ for the partition function (5.71), as follows

$$Z^B_{\alpha_0,\alpha_1\alpha_2} = Z_{HO}\, Z^{(0)}_{\alpha_0,\alpha_1\alpha_2}, \qquad \alpha_0, \alpha_1, \alpha_2 = 0, \frac{1}{2}. \tag{5.76}$$

Two of them, α_1, α_2, specify the half-integer values taken by the variables M_1, M_2 after half-flux insertions, while $\alpha_0 = 1/2$ amounts to adding the sign $(-1)^{K\Lambda_0}$ to the summand for reasons that will be clear in the following. Note that the oscillator part Z_{HO} stays invariant under the changes (5.74)-(5.75). In conclusion, we have the following eight partition functions

$$\begin{aligned} Z^B_{\alpha_0,\alpha_1\alpha_2} = Z_{HO} \sum_{m_\mu \in \mathbb{Z}^3_K} \sum_{M_\mu \in \mathbb{Z}^3} & (-1)^{2\alpha_0 K \Lambda_0} \\ & \exp\Bigg[- \frac{V^{(3)}}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|^2} \left(\frac{K^2 \Lambda_0^2}{2m} - 2\pi^2 m |\Lambda_1 \boldsymbol{\omega}_2 - \Lambda_2 \boldsymbol{\omega}_1|^2 \right) \\ & - \frac{i 2\pi K \Lambda_0}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|^2} (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2) \cdot (\Lambda_1 \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_2 - \Lambda_2 \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_1) \Bigg] \end{aligned} \tag{5.77}$$

$$\begin{aligned} & \Lambda_0 = M_0 + \frac{m_0}{K}, \quad \Lambda_1 = M_1 + \frac{m_1}{K} + \alpha_1, \quad \Lambda_2 = M_2 + \frac{m_2}{K} + \alpha_2, \\ & \alpha_0, \alpha_1, \alpha_2 = 0, \frac{1}{2}. \end{aligned} \tag{5.78}$$

These partition functions are mapped one into another by the flux insertions $V_i^{K/2}$, $i = 1, 2$ as shown in Fig.5.2. A characterization of these functions as the bosonic analogues of the fermionic spin sectors will become clear in the following discussion.

5.4 Bosonization in $(2 + 1)$ dimensions

In this section we focus on the set of eight bosonic partition functions $Z^B_{\alpha_0,\alpha_1\alpha_2}$ for $K = 1$. We show that they have the same modular transformations and other properties of the fermionic functions $Z^F_{\alpha_0,\alpha_1\alpha_2}$. We then argue that these quantities are actually describing a fermionic theory, although different from the free theory of Chapter 4. Our results provide an exact instance of bosonization in $(2+1)$ dimensions, namely on the correspondence between (interacting) fermionic and bosonic theories. It concerns the transformation properties of the spectrum under changes of backgrounds, that are actually independent of the dynamics and thus can be studied in the free limits of the two theories.

The fermionic nature of the bosonic partition functions will be based on the following properties [3]:

- The bosonic $Z^B_{\alpha_0,\alpha_1\alpha_2}$ and fermionic $Z^F_{\alpha_0,\alpha_1\alpha_2}$ behave in the same way under modular transformations and flux insertions.

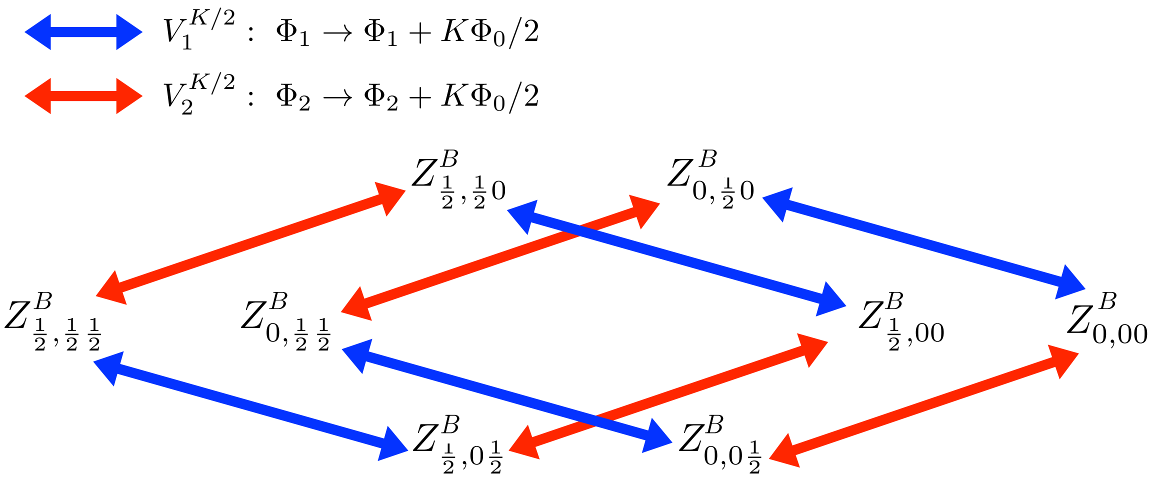


Figure 5.2: $V_1^{k/2}$ and $V_2^{k/2}$ transformations on the bosonic partition functions $Z^B_{\alpha_0,\alpha_1\alpha_2}$.

- For each sector, they become equal under dimensional reduction to $(1+1)$ dimensions, where the free bosonic and fermionic theories match exactly.
- The fermion number is associated to the states of the bosonic theory, and checked under dimensional reduction.
- The stability argument for fermionic topological states of Section 4.12 is formulated in the bosonic theory; it reproduce the results for $K = 1$, and also proves the stability of $K > 1$ topological states, for K odd integer.

5.4.1 Modular transformation

We recall from Section 4.6 that the action of the modular group $SL(3, \mathbb{Z})$ is described by the generators T_1, S_1 and P_{12}. Their action on the oscillator Z_{HO} and solitonic $Z^{(0)}_{\alpha_0,\alpha_1,\alpha_2}$ factors of the partition functions will be described in turn.

P_{12} transformation

Since

$$P_{12}: \qquad \boldsymbol{\omega}_0 \ \to \ \boldsymbol{\omega}_0, \qquad \boldsymbol{\omega}_1 \ \to \ -\boldsymbol{\omega}_2, \qquad \boldsymbol{\omega}_2 \ \to \ \boldsymbol{\omega}_1, \tag{5.79}$$

we find that Z_{HO} (5.67) is manifestly invariant, as can be checked by relabelling $(m_1 \to -m_2,\ m_2 \to m_1)$. The action on the solitonic part $Z^{(0)}$ (5.77) is equivalent to the relabeling of the variables $(M_1 \to -M_2,\ M_2 \to M_1)$, whose values are integer or half integer depending on the values of α_1, α_2. Thus, we find

$$P_{12}: \qquad Z^B_{\alpha_0,\alpha_1\alpha_2} \ \to \ Z^B_{\alpha_0,\alpha_2,\alpha_1}\ , \qquad \alpha_1, \alpha_2 = 0, \frac{1}{2} \tag{5.80}$$

T_1 transformation

Since

$$T_1: \quad \boldsymbol{\omega}_0 \;\to\; \boldsymbol{\omega}_0 + \boldsymbol{\omega}_1, \tag{5.81}$$

the oscillating function Z_{HO} is left invariant. The solitonic function do not change under T_1 if M_1 takes integer values; for half-integer values, the sums acquire the factor $(-1)^{M_0}$, thus changing the value of the index α_0 from zero to 1/2 in $Z^B_{\alpha_0,\alpha_1\alpha_2}$.

S_1 transformation

The action of the transformation S_1 is obtained by following the same strategy of the fermionic case in Section 4.6 as well as explained in Ref.[116]. First we choose the reference frame given by Eq.(4.66). Before applying S_1, we find the expressions of the solitonic and oscillating partition functions in this frame. We find that the solitonic partition function (5.77) with $K = 1$ and indices $\alpha_0 = \alpha_1 = \alpha_2 = 0$ acquires the following form

$$\begin{aligned} Z^{(0)}_{0,00}(\tau, R_0, R_1, R_2, \beta, \gamma) = \sum_{M_\mu \in \mathbb{Z}^3} \exp\bigg(&- \frac{\tau_2 M_0^2}{4\pi m R_2} - \frac{\tau_2 m (2\pi)^3}{2R_2} \Big[R_2^2 \left(M_1 + \beta M_2\right)^2 + R_1^2 M_2^2 \Big] \\ &+ 2\pi i \alpha M_0 \left(M_1 + \beta M_2\right) + 2\pi i \gamma M_0 M_2 \bigg), \end{aligned} \tag{5.82}$$

where the parameters are defined as

$$\tau = \tau_1 + i\tau_2 = -\frac{\omega_{01}}{\omega_{11}} + i\frac{\omega_{00}}{\omega_{11}} = \alpha + i r_{01}, \qquad \frac{\omega_{12}}{\omega_{22}} = -\beta, \qquad \frac{\omega_{02}}{\omega_{22}} = -\gamma, \tag{5.83}$$

with $r_{01} = R_0/R_1$. All the other solitonic functions are recovered taking $M_1 \to M_1 + 1/2$, $M_2 \to M_2 + 1/2$ and inserting the factor $(-1)^{M_0}$.

To find the expression of the oscillating partition functions in the same frame we need some calculations. We start from (5.65) without performing regularizations of the vacuum energy; introducing the "mass" parameter $r_{12} = R_1/R_2$, we find

$$\begin{aligned} Z_{HO} = \prod_{m_1 m_2 \neq (0,0)} \bigg[&\left(1 - \exp\left(-2\pi\tau_2\sqrt{(m_1+\beta m_2)^2 + (m_2 r_{12})^2} - 2\pi i(\alpha m_1 + m_2(\gamma + \tau_1\beta))\right)\right)^{-1} \\ &\times \exp\left(-\pi\tau_2\sqrt{(m_1+\beta m_2)^2 + (m_2 r_{12})^2}\right)\bigg]. \end{aligned} \tag{5.84}$$

The double product over (m_1, m_2) can be separated in two products; the first with ranges $(m_1 \in \mathbb{Z} \neq 0, m_2 = 0)$, the second with $(m_1 \in \mathbb{Z}, m_2 \in \mathbb{Z} \neq 0)$. The first product, once introduced the parameter $q = \exp(2\pi i\tau)$ and regularized the infinite sum at the exponent using the ζ-function regularization, gives the modulus square of the Dedekind function, i.e.

$$\left|\frac{1}{q^{1/24}\prod_{n>0}(1-q^n)}\right|^2=\left|\frac{1}{\eta(\tau)}\right|^2. \tag{5.85}$$

The second product, instead, can be rewritten in the following way

$$\prod_{m_2\in\mathbb{Z}\neq 0}\Bigg(\prod_{m_1\in\mathbb{Z}}\left[1-\exp\left(-2\pi\tau_2\sqrt{(m_1+\beta m_2)^2+(m_2r_{12})^2}-2\pi i\left(\tau_1 m_1+m_2(\gamma+\tau_1\beta)\right)\right)\right]^{-1}$$
$$\exp\Bigg(-\pi\tau_2\sum_{m_1\in\mathbb{Z}}\sqrt{(m_1+\beta m_2)^2+(m_2r_{12})^2}\Bigg)\Bigg). \tag{5.86}$$

Once regularized the infinite sum at the second exponent (see Appendix. C.2), and introduced the function $\Delta(m;a)$

$$\Delta(m;a)=-\frac{1}{2\pi^2}\sum_{\ell}\int_0^\infty dt\,\exp\left(\frac{-\pi^2m^2}{t}-t\ell^2\right)\cos(2\pi\ell a), \tag{5.87}$$

and the so called “massive” theta function [109]

$$\Theta_{[a,b]}(\tau,m)=e^{4\pi\tau_2\Delta(m,a)}\prod_{n\in\mathbb{Z}}\left|1-\exp\left(-2\pi\tau_2\sqrt{m^2+(n+a)^2}+2\pi i\tau_1(n+a)+2\pi ib\right)\right|^2, \tag{5.88}$$

we finally find the partition function of the oscillating modes in the frame (4.66), i.e.

$$Z_{HO}(\tau,R_0,R_1,R_2,\beta,\gamma)=\left|\frac{1}{\eta(q)}\right|^2\prod_{m_2\in\mathbb{Z}^+}\Theta^{-1}_{[\beta m_2,m_2\gamma]}(\tau,r_{12}m_2). \tag{5.89}$$

We are now ready to discuss the S_1 transformation. As found in the fermionic case of Section 4.6, S_1 acts on the parameters in the following way

$$\begin{aligned}&\tau\to-\frac{1}{\tau};\qquad \alpha\to-\frac{\alpha}{\alpha^2+r_{02}^2};\qquad r_{01}\to\frac{r_{01}}{\alpha^2+r_{01}^2};\\ &R_0\to\frac{R_0}{|\tau|};\qquad R_1\to R_1|\tau|;\qquad R_2\to R_2;\qquad \gamma\to-\beta;\qquad \beta\to\gamma.\end{aligned} \tag{5.90}$$

Therefore, the solitonic function $Z^{(0)}_{0,00}$ transforms as follows

$$S_1:\qquad Z^{(0)}_{0,00}\left(\tau,R_0,R_1,R_2,\beta,\gamma\right)\to Z^{(0)}_{0,00}\left(-\frac{1}{\tau},\frac{R_0}{|\tau|},R_1|\tau|,R_2,\gamma,-\beta\right). \tag{5.91}$$

The first step is to apply the Poisson formula on M_0, i.e.

$$\sum_{M_0\in\mathbb{Z}}\exp\left(-\pi AM_0^2+2\pi iM_0B\right)=\frac{1}{\sqrt{A}}\sum_{M_0'\in\mathbb{Z}}\exp\left(-\frac{\pi}{A}\left(M_0'-B\right)^2\right). \tag{5.92}$$

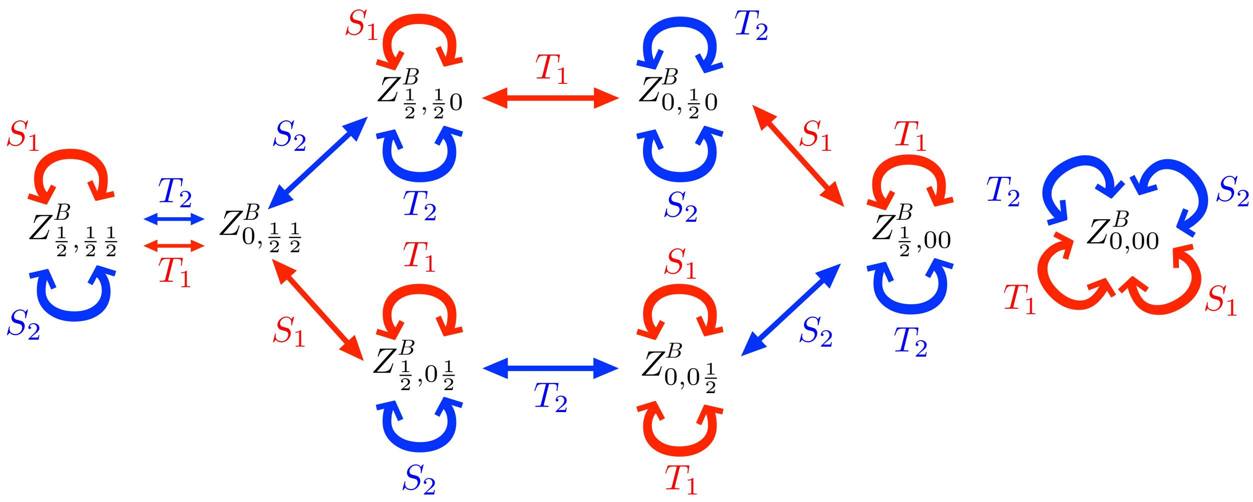


Figure 5.3: Action of $SL(3,\mathbb{Z})$ over the bosonic partition functions $Z^B_{\alpha_0,\alpha_1\alpha_2}$.

Another resummation (5.92) is done on the M_1 index and, once made the relabelling $M_0' \to M_1$ and $M_1' \to M_0$, we finally reconstruct the starting function $Z^{(0)}_{0,00}$ up to a factor $|\tau|$, i.e.

$$S_1: \qquad Z^{(0)}_{0,00}\left(\tau, R_0, R_1, R_2, \beta, \gamma\right) \quad \to \quad |\tau| Z^{(0)}_{0,00}\left(\tau, R_0, R_1, R_2, \beta, \gamma\right). \tag{5.93}$$

Regarding the oscillating partition function (5.89), we use the identity

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \tag{5.94}$$

and the "massive" theta function transformation (see Appendix D) [109]

$$\Theta_{[a,b](\tau,m)} = \Theta_{[b,-a]}\left(-\frac{1}{\tau}, m|\tau|\right). \tag{5.95}$$

Putting all results together, we find

$$\begin{aligned} S_1: \quad Z_{HO}(\tau, R_0, R_1, R_2, \beta, \gamma) \quad \to \quad & Z_{HO}\left(-\frac{1}{\tau}, \frac{R_0}{|\tau|}, R_1|\tau|, R_2, \gamma, -\beta\right) \\ &= \left|\frac{1}{\eta\left(-1/\tau\right)}\right|^2 \prod_{m_2\in\mathbb{Z}^+} \Theta_{[\gamma m_2, -\beta n_2]}\left(-\frac{1}{\tau}; r_{12}m_2|\tau|\right) \\ &= \frac{1}{|\tau|}\left|\frac{1}{\eta(q)}\right|^2 \prod_{m_2\in\mathbb{Z}^+} \Theta_{[\beta m_2, \gamma m_2]}(\tau; r_{12}m_2) \\ &= \frac{1}{|\tau|} Z_{HO}(\tau, R_0, R_1, R_2, \beta, \gamma). \end{aligned} \tag{5.96}$$

Combining the solitonic (5.93) and the oscillating functions (5.96), we obtain the final result

$$S_1: \qquad Z^B_{0,\ 00} \quad \to \quad Z^B_{0,\ 00}. \tag{5.97}$$

Following similar steps, we find the transformations of the other partition functions that lead to the expected results

$$S_1 : \quad Z^B_{\alpha_0,\alpha_1\alpha_2}(\boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2) \quad \to \quad Z^B_{\alpha_0,\alpha_1\alpha_2}(-\boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_2) = Z^B_{\alpha_1,\alpha_0\alpha_2}(\boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2). \tag{5.98}$$

Finally, all the modular transformations are shown in Fig. 5.3 [3]. This pattern as well as that given by flux insertions in Fig. 5.2 is identical to those of the fermionic theory given in Fig. 4.12 and Fig. 4.8, once the $Z^B_{\alpha_0,\alpha_1\alpha_2}$ are put in suitable positions. Before establishing a correspondence term to term we still need two steps: the dimensional reduction and a change of basis [3].

5.4.2 Dimensional reduction

In a similar way as done for the fermionic case in Section 4.7, we further characterize the eight bosonic partition functions by performing a dimensional reduction from two to one spatial dimension. The mapping to well know relations of two-dimensional bosonization will give useful informations on the nature of the $(2+1)$-dimensional bosonic sectors and their ability to describe (interacting) fermionic systems.

Kaluza-Klein dimensional reduction

Let us consider the bosonic partition functions (5.77) with $K = 1$ for a rectangular torus in the spatial directions, i.e. $\omega_{12} = \omega_{21} = 0$, for simplicity. We perform again the Kaluza-Klein dimensional reduction, namely take the limit $R_2 \to 0$ of the Corbino donut, such that the oscillating and solitonic modes of energy, respectively, $O(m_2/R_2)$ and $O(M_2/R_2)$, are never excited, corresponding to $m_2, M_2 \to 0$. Upon setting $\omega_{02} = 0$, the remaining geometry is that of two-torus in the plane (x^0, x^1) with modular parameter

$$\tau = \tau_1 + i\tau_2 = -\frac{\omega_{01}}{\omega_{11}} + i\frac{\omega_{00}}{\omega_{11}}. \tag{5.99}$$

Applying this reduction to the oscillating partition functions Z_{HO} (5.65) before having regularized the infinite vacuum energy, we obtain

$$\begin{aligned} Z_{HO}(\omega_{00},\, \omega_{01},\, \omega_{11},\, \omega_{22} \to 0)\Big|_{m_2=0} &= \prod_{m_1 \in \mathbb{Z} \neq 0} \left(1 - \exp\left(-2\pi \frac{\omega_{00}}{\omega_{11}} |m_1| + 2\pi\, m_1 \frac{\omega_{01}}{\omega_{11}}\right)\right)^{-1} \\ &\quad \times \exp\left(-\pi \frac{\omega_{00}}{\omega_{11}} \sum_{m_1 \in \mathbb{Z} \neq 0} |m_1|\right). \end{aligned} \tag{5.100}$$

Once introduced the modular parameter (5.99) and regularized the infinite sum at the exponent of (5.100) through the Riemann ζ-function, we obtain

$$Z_{HO}(\tau)\Big|_{m_2=0} = \left|\frac{1}{q^{1/24} \prod_{m>0}(1-q^m)}\right|^2 = \left|\frac{1}{\eta(\tau)}\right|^2, \tag{5.101}$$

where $q = \exp(2\pi i \tau)$.

The solitonic factors $Z^{(0)}_{\alpha_0;\alpha_1\alpha_2}$ are similarly expanded for $\omega_{22} \to 0$, fixing the summand $M_2 = 0$. Since the classical action (5.43) would vanish in this limit, we should also let the mass $m \to \infty$ so that their product stay finite

$$m\omega_{22} = r^2/\pi, \qquad \text{finite.} \tag{5.102}$$

We find the $(1+1)$-dimensional limit

$$Z^{(0)}_{00|0} = \sum_{M_0 M_1 \in \mathbb{Z}} q^{\frac{1}{2}\left(\frac{M_0}{2r}+rM_1\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{M_0}{2r}-rM_1\right)^2}. \tag{5.103}$$

Thus r becomes the compactification radius of the scalar field in two dimensions, that we fix to $r = 1$ for mapping to the free fermion as well known in CFT literature [21, 22]. Similar expressions are obtained for the reductions of the other functions $Z^{(0)}_{\alpha_0\alpha_1|0}$; once fixed $r = 1$ they read

$$Z^B_{\frac{1}{2}\frac{1}{2}\big|0} = \left|\frac{1}{\eta(\tau)}\right|^2 \sum_{M_0 M_1 \in \mathbb{Z}} (-1)^{M_0} q^{\frac{1}{2}\left(\frac{M_0}{2}+\left(M_1+\frac{1}{2}\right)\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{M_0}{2}-\left(M_1+\frac{1}{2}\right)\right)^2} \tag{5.104}$$

$$Z^B_{0\frac{1}{2}\big|0} = \left|\frac{1}{\eta(\tau)}\right|^2 \sum_{M_0 M_1 \in \mathbb{Z}} q^{\frac{1}{2}\left(\frac{M_0}{2}+\left(M_1+\frac{1}{2}\right)\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{M_0}{2}-\left(M_1+\frac{1}{2}\right)\right)^2} \tag{5.105}$$

$$Z^B_{\frac{1}{2}0\big|0} = \left|\frac{1}{\eta(\tau)}\right|^2 \sum_{M_0 M_1 \in \mathbb{Z}} (-1)^{M_0} q^{\frac{1}{2}\left(\frac{M_0}{2}+M_1\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{M_0}{2}-M_1\right)^2} \tag{5.106}$$

$$Z^B_{00\big|0} = \left|\frac{1}{\eta(\tau)}\right|^2 \sum_{M_0 M_1 \in \mathbb{Z}} q^{\frac{1}{2}\left(\frac{M_0}{2}+M_1\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{M_0}{2}-M_1\right)^2}. \tag{5.107}$$

The reduction of the bosonic partition function with $\alpha_2 = 1/2$ leads to the results

$$Z^B_{\frac{1}{2}\frac{1}{2}\big|\frac{1}{2}} = (q\bar{q})^{\omega_{11}^2/8\omega_{22}^2} \left|\frac{1}{\eta(\tau)}\right|^2 \sum_{M_0 M_1 \in \mathbb{Z}} (-1)^{N_0} q^{\frac{1}{2}\left(\frac{M_0}{2}+\left(M_1+\frac{1}{2}\right)\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{M_0}{2}-\left(M_1+\frac{1}{2}\right)\right)^2} \tag{5.108}$$

$$Z^B_{0\frac{1}{2}\big|\frac{1}{2}} = (q\bar{q})^{\omega_{11}^2/8\omega_{22}^2} \left|\frac{1}{\eta(\tau)}\right|^2 \sum_{M_0 M_1 \in \mathbb{Z}} q^{\frac{1}{2}\left(\frac{M_0}{2}+\left(M_1+\frac{1}{2}\right)\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{M_0}{2}-\left(M_1+\frac{1}{2}\right)\right)^2} \tag{5.109}$$

$$Z^B_{\frac{1}{2}0\big|\frac{1}{2}} = (q\bar{q})^{\omega_{11}^2/8\omega_{22}^2} \left|\frac{1}{\eta(\tau)}\right|^2 \sum_{M_0 M_1 \in \mathbb{Z}} (-1)^{M_0} q^{\frac{1}{2}\left(\frac{M_0}{2}+M_1\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{M_0}{2}-M_1\right)^2} \tag{5.110}$$

$$Z^B_{00\big|\frac{1}{2}} = (q\bar{q})^{\omega_{11}^2/8\omega_{22}^2} \left|\frac{1}{\eta(\tau)}\right|^2 \sum_{M_0 M_1 \in \mathbb{Z}} q^{\frac{1}{2}\left(\frac{M_0}{2}+M_1\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{M_0}{2}-M_1\right)^2}. \tag{5.111}$$

These expressions differ from (5.104)-(5.107) for an overall ground state energy $E_0 = O(1/R^2) \to \infty$, that should be subtracted for a finite limit. Note that this shift in the spectrum due to the minimal energy of waves with twisted boundary conditions actually corresponded to a mass in the dispersion relation of reduced fermions (4.91). In both cases, the modular transformations split into two $SL(2,\mathbb{Z})$ subgroups, as shown in Fig. 5.4.

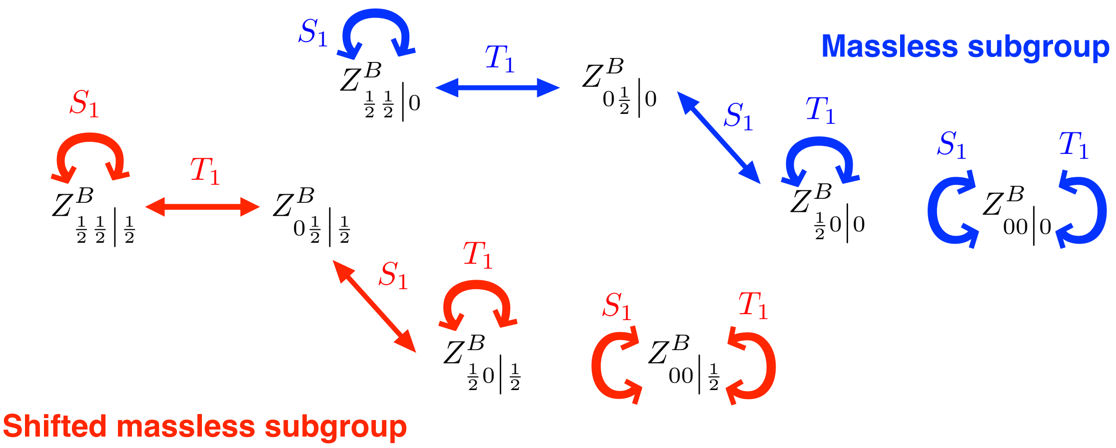


Figure 5.4: Action of the two-dimensional modular group $SL(2,\mathbb{Z})$ over of the eight bosonic partition functions $Z^B_{\alpha_0\alpha_1|\alpha_2}$ reduced on the (x^0, x^1) plane.

Bosonization in $(1+1)$ dimensions and fermionic spin sectors

For later use, it is convenient to rewrite the expressions (5.104)-(5.107) and (5.108)-(5.111) in terms of the fermionic spin sectors Z^{NS}, $Z^{\widetilde{NS}}$, Z^R and $Z^{\widetilde{R}}$ in (4.87)-(4.90) by using the bosonization formulae discussed in Section 1.3.2. The rewriting of $Z^B_{00|0}$ requires the following standard manipulations. First we split it in two equal part as follows

$$Z^B_{00|0} = \frac{1}{2}\left|\frac{1}{\eta}\right|^2 \sum_{M_0 M_1 \in \mathbb{Z}} q^{\frac{1}{2}\left(\frac{M_0}{2}+M_1\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{M_0}{2}-M_1\right)^2} + \frac{1}{2}\left|\frac{1}{\eta}\right|^2 \sum_{M_0 M_1 \in \mathbb{Z}} q^{\frac{1}{2}\left(\frac{M_0}{2}+M_1\right)^2} \bar{q}^{\frac{1}{2}\left(\frac{M_0}{2}-M_1\right)^2}. \tag{5.112}$$

In the first term we replace

$$2\ell = M_0, \qquad n = M_1, \tag{5.113}$$

while in the second part

$$2\ell - 1 = M_0, \qquad n = M_1, \tag{5.114}$$

thus obtaining

$$Z^B_{00|0} = \left|\frac{1}{\eta}\right|^2 \sum_{\ell,n\in\mathbb{Z}} \left[q^{\frac{1}{2}(\ell+n)^2} \bar{q}^{\frac{1}{2}(\ell-n)^2} + q^{\frac{1}{2}\left(\ell+n-\frac{1}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\ell-n-\frac{1}{2}\right)^2} \right]. \tag{5.115}$$

Making the further replacement

$$\ell + n = \alpha_0, \qquad \ell - n = -\overline{\alpha}_0, \tag{5.116}$$

the constraint $\alpha_0 - \overline{\alpha}_0 = 2\ell$ can be enforced by inserting the projector $\left(1 + (-1)^{\alpha_0 + \bar{\alpha}_0}\right)/2$ into the sum, finally obtaining

$$\begin{aligned} Z^B_{00\left|0\right.} = \frac{1}{2}\left|\frac{1}{\eta}\right|^2 \sum_{\alpha_0, \overline{\alpha}_0 \, \in \mathbb{Z}} \Bigg(& q^{\frac{1}{2}\alpha_0^2} \bar{q}^{\frac{1}{2}\overline{\alpha}_0^2} + (-1)^{\alpha_0 + \overline{\alpha}_0} q^{\frac{1}{2}\alpha_0^2} \bar{q}^{\frac{1}{2}\overline{\alpha}_0^2} \\ & + q^{\frac{1}{2}\left(\alpha_0 + \frac{1}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\overline{\alpha}_0 + \frac{1}{2}\right)^2} + (-1)^{\alpha_0 + \overline{\alpha}_0} q^{\frac{1}{2}\left(\alpha_0 + \frac{1}{2}\right)^2} \bar{q}^{\frac{1}{2}\left(\overline{\alpha}_0 + \frac{1}{2}\right)^2} \Bigg), \end{aligned} \tag{5.117}$$

namely

$$Z^B_{00\left|0\right.} = \frac{1}{2}\left(Z^{NS} + Z^{\widetilde{NS}} + Z^R + Z^{\widetilde{R}}\right) = Z_{\text{Dirac}}. \tag{5.118}$$

The other one-dimensional limits of the partition functions become, following similar steps

$$Z^B_{\frac{1}{2}0\left|0\right.} = \frac{1}{2}\left(Z^{NS} + Z^{\widetilde{NS}} - Z^R - Z^{\widetilde{R}}\right) \quad \sim Z^B_{\frac{1}{2},0\left|\frac{1}{2}\right.}, \tag{5.119}$$

$$Z^B_{0\frac{1}{2}\left|0\right.} = \frac{1}{2}\left(Z^{NS} - Z^{\widetilde{NS}} + Z^R - Z^{\widetilde{R}}\right) \quad \sim Z^B_{0,\frac{1}{2}\left|\frac{1}{2}\right.}, \tag{5.120}$$

$$Z^B_{\frac{1}{2}\frac{1}{2}\left|0\right.} = \frac{1}{2}\left(-Z^{NS} + Z^{\widetilde{NS}} + Z^R - Z^{\widetilde{R}}\right) \sim Z^B_{\frac{1}{2},\frac{1}{2}\left|\frac{1}{2}\right.}. \tag{5.121}$$

In these formulae, we removed the zero-point energies from the partition functions with $\alpha_2 = 1/2$.

Therefore, exact bosonization in (1+1) dimensions establishes the existence of two different bases, the bosonic $Z^B_{\alpha_0\alpha_1\left|0\right.} = \left(Z^B_{\frac{1}{2}\frac{1}{2}\left|0\right.}, Z^B_{0\frac{1}{2}\left|0\right.}, Z^B_{\frac{1}{2}0\left|0\right.}, Z^B_{00\left|0\right.}\right) \sim Z^B_{\alpha_0\alpha_1\left|\frac{1}{2}\right.}$ and the fermionic one $Z^F = \left(Z^{NS}, Z^{\widetilde{NS}}, Z^R, Z^{\widetilde{R}}\right)$. They are related by the following matrix

$$Z^B_{\alpha_0\alpha_1\left|0\right.} = O Z^F, \qquad \text{with} \qquad O = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \tag{5.122}$$

which leaves invariant the patterns of modular transformations $SL(2,\mathbb{Z})$, namely the change of basis represented by the matrix O (5.122) is an isometry with respect to the action of the modular group, see Fig. 2.4 and Fig. 5.4.

Fermion number in the bosonic theory

Equation (5.122) shows that under dimensional reduction the eight bosonic partition functions $Z^B_{\alpha_0,\alpha_1\alpha_2}$ become sums of fermionic functions of the sectors $NS, \widetilde{NS}, R, \widetilde{R}$. The linear combinations (5.118)-(5.121) are characterized by having definite fermion number $(-1)^{\mathcal{F}}$ in $(1+1)$ dimensions: the relative sum (difference) between NS and $\widetilde{NS}$ as well as R and $\widetilde{R}$ involves a projector over even (odd) number of fermionic excitations, thus over states with positive (negative) definite fermion number. The results are listed in Table 5.1, where we also specify partition functions that possess coefficients with positive and indefinite sign, corresponding to values $\alpha_0 = 0$ (resp. $\alpha_0 = 1/2$), owing to the signs $(-1)^{M_0}$ in the definition (5.117).

	NS	R	positive Z	indefinite Z
$(-1)^{\mathcal{F}}$	1	1	$Z^B_{00\|0}\,,\ Z^B_{00\|\frac{1}{2}}$	$Z^B_{\frac{1}{2},0\|0}\,,\ Z^B_{\frac{1}{2},0\|\frac{1}{2}}$
$(-1)^{\mathcal{F}}$	-1	-1	$Z^B_{0,\frac{1}{2}\|0}\,,\ Z^B_{0,\frac{1}{2}\|\frac{1}{2}}$	$Z^B_{\frac{1}{2},\frac{1}{2}\|0}\,,\ Z^B_{\frac{1}{2},\frac{1}{2}\|\frac{1}{2}}$

Table 5.1: Fermion number of reduced bosonic partition functions in $(1+1)$ dimensions

From these results, we can assign a $(2+1)$-dimensional fermion number $(-1)^F$ to the bosonic partition functions $Z^B_{\alpha_0,\alpha_1\alpha_2}$, that is consistent with dimensional reduction. As shown

	$(-1)^F$
$Z^B_{000}\,,\quad Z^B_{\frac{1}{2},00}$	1
$Z^B_{0,\frac{1}{2}\frac{1}{2}}\,,\quad Z^B_{\frac{1}{2},\frac{1}{2}\frac{1}{2}}$	-1
$Z^B_{0,\frac{1}{2}0}\,,\quad Z^B_{0,0\frac{1}{2}}$	/
$Z^B_{\frac{1}{2},\frac{1}{2}0}\,,\quad Z^B_{\frac{1}{2},0\frac{1}{2}}$	/

Table 5.2: Fermion number of bosonic partition functions in $(2+1)$ dimensions

in Table 5.2, four partition functions have states with definite fermion number, while four others have no assignment, that is denoted by (/). For example, $Z^B_{0\frac{1}{2}0}$ would have fermionic states according to the dimensional reduction $x_2 \to 0$, i.e. $Z^B_{0\frac{1}{2}|0}$ in Table 5.1, and bosonic states in the reduction $x_1 \to 0$, i.e. $Z^B_{00|\frac{1}{2}}$ in the same Table 5.1. The nature of the solitonic bosonic states in the latter four sectors is not clear at the moment: they might correspond to non-local degrees of freedom in the fermionic theory [3]. On the other hand, as we shall see in the following, only the states with definite fermionic number participate in the discussion on the stability and characterization of interacting topological insulators.

5.4.3 Bosonic Neveu-Schwarz and Ramond sectors in $(2+1)$ dimensions

In the analysis of the fermionic theory of Section 4.5, we were able to identify $(2+1)$-dimensional analogs of the partition functions for Neveu-Schwarz and Ramond sectors, that are sums of positive terms and possess the low-energy expansions

$$NS: \quad Z^F_{\frac{1}{2},\frac{1}{2}\frac{1}{2}} \sim 1 + \cdots\,; \qquad\qquad R: \quad Z^F_{\frac{1}{2},00} \sim 2 + \cdots\,. \tag{5.123}$$

The first state in the Neveu-Schwarz sector is the ground state and is bosonic, namely with positive fermion and spin parity indices, while the doublet of the Ramond sector is fermionic. These sectors are mapped one into another by half flux insertions according to Fig.4.8, and they occupy a definite position in the pattern of modular transformations, as shown in Fig.4.12.

In the following, we want to identify bosonic functions that possess these same characteristics and, moreover, became equal to the corresponding fermionic functions under reduction to $(1+1)$ dimensions.

According to the pattern of bosonic modular transformations shown in Fig. 5.3, one would be led to the identification $Z^B_{\alpha_0,\alpha_1\alpha_2} \sim Z^F_{\alpha_0,\alpha_1\alpha_2}$; however, the bosonic functions go into sums of fermionic sectors under dimensional reduction, and moreover, the would-be Neveu-Schwarz sector $Z^B_{\frac{1}{2},\frac{1}{2}\frac{1}{2}}$ would not be a sum of positive terms because to the factor $(-1)^{M_0}$. Note also that dimensional reduction and fermion number assignment would favor $Z^B_{0,00}$ as the candidate Neveu-Schwarz sector; but, unfortunately, being a singlet under $SL(3,\mathbb{Z})$, its modular properties do not mach those of $Z^F_{\frac{1}{2},\frac{1}{2}\frac{1}{2}}$.

The solution to this puzzle is found by considering a change of basis among the bosonic functions that is an isometry with respect to the action of the modular group in Fig. 5.3 and the $V_i^{1/2}$ transformations for $i = 1,2$ in Fig. 5.2 [3]. Let us first write this transformation and then discuss its features. The map between the original eight-dimensional basis

$$Z^B = (Z^B_{\frac{1}{2},\frac{1}{2}\frac{1}{2}},\ Z^B_{0,\frac{1}{2}\frac{1}{2}},\ Z^B_{\frac{1}{2},\frac{1}{2}0},\ Z^B_{0,\frac{1}{2}0},\ Z^B_{\frac{1}{2},0\frac{1}{2}},\ Z^B_{0,0\frac{1}{2}},\ Z^B_{\frac{1}{2},00},\ Z^B_{0,00}). \tag{5.124}$$

and the new basis $Z'^B = MZ^B$ is given by the following matrix

$$M = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}. \tag{5.125}$$

This transformation leaves invariant the patterns of modular transformations and flux insertions given in Fig. 5.3 and Fig. 5.2, respectively; namely, M commutes with $T_i, S_i, V_i^{1/2}$, for $i = 1,2$. Furthermore, it is unique up to exchange of space coordinates $x_1 \leftrightarrow x_2$, that is the up-down reflection of the patterns of transformations.

The idea behind the derivation of (5.125) is very simple [3]: the action of the modular group in Fig. 5.3 shows that there are two invariants given by the sum of the first seven elements of the multiplet in (5.124) and by the last element $Z^B_{0,00}$. In the new basis, the eighth component, namely the new singlet, should be either the sum or the difference of the two previous invariants; the second choice is correct and then the other components of the matrix M follow by the action of flux and modular transformations on $Z'^B_{0,00}$.

The bosonic Neveu-Schwarz sector

In the new basis, we are ready to identify the bosonic analogue of the fermionic Neveu-Schwarz sector of Section 4.5, that is

$$Z'^{B}_{\frac{1}{2},\frac{1}{2}\frac{1}{2}} \;\leftrightarrow\; Z^{F}_{\frac{1}{2},\frac{1}{2}\frac{1}{2}}, \tag{5.126}$$

as suggested by the position occupied in the patterns of transformations. According to (5.125), the superposition of bosonic functions is given by

$$Z'^{B}_{\frac{1}{2},\frac{1}{2}\frac{1}{2}} = \frac{1}{2}\left(-Z^{B}_{\frac{1}{2},\frac{1}{2}\frac{1}{2}} + Z^{B}_{0,\frac{1}{2}\frac{1}{2}} + Z^{B}_{\frac{1}{2},\frac{1}{2}0} + Z^{B}_{0,\frac{1}{2}0} + Z^{B}_{\frac{1}{2},0\frac{1}{2}} + Z^{B}_{0,0\frac{1}{2}} + Z^{B}_{\frac{1}{2},00} + Z^{B}_{0,00}\right). \tag{5.127}$$

This expression involves sums of terms with positive integer coefficients, owing to the presence of the projectors $(-(-1)^{M_0}+1)/2$ in the first pair of functions and $((-1)^{M_0}+1)/2$ in the other pairs.

The low-energy expansion of the partition functions is done by inspecting the energy spectrum of solitonic modes (5.60)

$$\mathcal{E}^{\alpha_1,\alpha_2}_{M_0,M_1,M_2} = \frac{M_0^2}{2mV^{(2)}} + \frac{(2\pi)^2 m}{2V^{(2)}}\left|(M_1+\alpha_1)\,\boldsymbol{\omega}_2 - (M_2+\alpha_2)\,\boldsymbol{\omega}_1\right|^2. \tag{5.128}$$

This vanishes for $\alpha_1 = \alpha_2 = 0$ and $M_0 = M_1 = M_2 = 0$ and the corresponding state is found in the term $(Z^{B}_{\frac{1}{2},00} + Z^{B}_{0,00})/2$ in (5.127). It gives

$$Z'^{B}_{\frac{1}{2},\;\frac{1}{2}\frac{1}{2}} \sim 1 + \cdots\,. \tag{5.129}$$

The first term can be identified with the Neveu-Schwarz, i.e. unperturbed, ground state of the fermionic system. This state is neutral, since $M_0 = 0$, and bosonic owing to the fermion number assignments in Table (5.2) to the functions $Z^{B}_{\frac{1}{2},00}$ and $Z^{B}_{0,00}$. The identification of the ground state is further confirmed by dimensional reduction. Applying the limits (5.118)-(5.121) to the linear combination in (5.127), we see that the reductions for $x_2 \to 0$ or $x_1 \to 0$ gives the same result by construction. We find

$$Z'^{B}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}} \;\to\; \frac{1}{2}\left(Z^{NS} + Z^{\widetilde{NS}} + Z^{R} - Z^{\widetilde{R}}\right) + (\exp(-E_0))\,Z_{NS}, \tag{5.130}$$

where E_0 is the energy shift for $\alpha_i = 1/2$ in the i-th dimension going to zero. In summary, we found the following properties of the bosonic Neveu-Schwarz ground state in $(2+1)$ dimensions,

$$1 \quad\leftrightarrow\quad |\Omega\rangle_{NS}\,, \tag{5.131}$$

$$H\,|\Omega\rangle_{NS} = Q\,|\Omega\rangle_{NS} = 0, \tag{5.132}$$

$$(-1)^F\,|\Omega\rangle_{NS} = (-1)^{2S}\,|\Omega\rangle_{NS} = |\Omega\rangle_{NS}\,. \tag{5.133}$$

The bosonic Ramond sector

According to the analysis of the fermionic theory of Section 4.5, the (2 + 1)-dimensional Ramond sector is found for half-flux insertions $V_i^{1/2}$, $i = 1, 2$, that map

$$V_1^{1/2} \cdot V_2^{1/2} : \qquad Z'^B_{\frac{1}{2},\frac{1}{2}\frac{1}{2}} \quad \to \quad Z'^B_{\frac{1}{2},00}. \tag{5.134}$$

In agreement with (5.125), the corresponding linear combination of bosonic partition functions is given by

$$Z'^B_{\frac{1}{2},00} = \frac{1}{2}\left(Z^B_{\frac{1}{2},\frac{1}{2}\frac{1}{2}} + Z^B_{0,\frac{1}{2}\frac{1}{2}} + Z^B_{\frac{1}{2},\frac{1}{2}0} + Z^B_{0,\frac{1}{2}0} + Z^B_{\frac{1}{2},0\frac{1}{2}} + Z^B_{0,0\frac{1}{2}} - Z^B_{\frac{1}{2},00} + Z^B_{0,00}\right). \tag{5.135}$$

We make the following observations:

- This expression is again a sum of terms with positive integer coefficients.
- Owing to the projector $(-Z^B_{\frac{1}{2},00} + Z^B_{0,00})/2$, the previous Neveu-Schwarz ground state is cancelled.
- Under flux insertion, the Neveu-Schwarz ground state has flowed in the term $(Z^B_{\frac{1}{2},\frac{1}{2}\frac{1}{2}} + Z^B_{0,\frac{1}{2}\frac{1}{2}})/2$. The corresponding energy is obtained choosing $\alpha_1 = \alpha_2 = 1/2$ and $M_0 = M_1 = M_2 = 0$ in (5.128). This is a degenerate state, indeed $\mathcal{E}^{\frac{1}{2}\frac{1}{2}}_{000} = \mathcal{E}^{\frac{1}{2}\frac{1}{2}}_{0-1-1}$.
- From the above assignments of fermion number in Table (5.2), the functions $Z^B_{\frac{1}{2},\frac{1}{2}\frac{1}{2}}$ and $Z^B_{0,\frac{1}{2}\frac{1}{2}}$ possess fermionic states with $(-1)^F = (-1)^{2S} = -1$. Thus, the two degenerate states in the partition function (5.135) form a Kramers pair under time reversal transformations,

$$Z'^B_{\frac{1}{2},00} \sim \cdots + \exp\left(-\mathcal{E}^{\frac{1}{2}\frac{1}{2}}_{000}\right) + \exp\left(-\mathcal{E}^{\frac{1}{2}\frac{1}{2}}_{0-1-1}\right) + \cdots . \tag{5.136}$$

- The lowest energy states in $Z'^B_{\frac{1}{2},00}$ are found in the terms $(Z^B_{\frac{1}{2},\frac{1}{2}0} + Z^B_{0,\frac{1}{2}0})/2$ or $(Z^B_{\frac{1}{2},0\frac{1}{2}} + Z^B_{0,0\frac{1}{2}})/2$, depending on the torus geometry. This can be checked taking a rectangular torus with $\omega_{12} = \omega_{21} = 0$ for simplicity.
- The lowest energy states have not definite fermion number as it follows from the Table 5.2. Thus, although they appear degenerate in energy, they do not correspond to a Kramers pair under TR transformations.
- The two dimensional reductions of $Z'^B_{\frac{1}{2},00}$ are again equivalent by construction and gives the result

$$Z'^B_{\frac{1}{2},00} \;\to\; Z^R + (\exp(-E_0))\,\frac{1}{2}\left(Z^{NS} + Z^{\widetilde{N}S} + Z^R - Z^{\widetilde{R}}\right). \tag{5.137}$$

Summarizing, the properties of the two degenerate ground states in (5.136), obtained by the evolution of the Neveu-Schwarz ground state under the insertion of two fluxes, are

$$\exp(-\mathcal{E}^{\frac{1}{2}\frac{1}{2}}_{000}) \leftrightarrow |\Omega\rangle_R\,, \qquad \exp(-\mathcal{E}^{\frac{1}{2}\frac{1}{2}}_{0-1-1}) \leftrightarrow |\Omega\rangle'_R\,, \qquad \mathcal{E}^{\frac{1}{2}\frac{1}{2}}_{000} = \mathcal{E}^{\frac{1}{2}\frac{1}{2}}_{0-1-1}, \tag{5.138}$$

$$Q\,|\Omega\rangle_R = Q\,|\Omega\rangle'_R = 0, \qquad (-1)^{2S}\,|\Omega\rangle_R = (-1)^{2S}\,|\Omega\rangle'_R = -1, \quad \mathcal{T}\,|\Omega\rangle_R = |\Omega\rangle'_R\,.$$

Therefore, since the fermionic Ramond sector $Z^F_{\frac{1}{2},00}$ satisfies exactly these properties (see Section 4.5), we can identify $Z'^B_{\frac{1}{2},00}$ as the bosonic analogue, namely

$$Z'^B_{\frac{1}{2},00} \;\sim\; Z^F_{\frac{1}{2},00}. \tag{5.139}$$

This correspondence is valid although the Kramers pair $(|\Omega\rangle_R\,, |\Omega\rangle'_R)$ does not coincide with the lowest energy states of the corresponding Ramond sector and, thus, there is not a correspondence between Kramers pairs in two and three dimensions. Nevertheless, as we shall see in the following, the stability argument continues to be hold.

5.4.4 Stability of bosonic topological insulators

The previous analysis has shown that the surface theory with coupling constant $K = 1$ possesses fermionic degrees of freedom: these are not free particles, owing to the differences in the bosonic and free fermionic spectra, but nonetheless their partition functions show the characteristic eight spin sectors, that are mapped one into the other by the addition of half fluxes through the two loops of the Corbino geometry and by modular transformations [3].

The entire analysis regarding modular transformations, dimensional reductions, change of basis and fermion number assignments can be extended to the bosonic theory with odd integer values of $K > 1$. A few clarifications are needed:

- The maps between sectors are found by adding $K/2$ fluxes instead of half fluxes, as shown in Fig. 5.2.
- Each partition function splits into K^3 anyon sectors for $m_\mu = 0, 1, \ldots, K-1$ and $\mu = 0, 1, 2$, as shown in Eq.(5.77)

  $$Z^B_{\alpha_0,\alpha_1\alpha_2} = \sum_{m_\mu \in \mathbb{Z}^3_K} Z^{B\,m_0,m_1,m_2}_{\alpha_0,\alpha_1\alpha_2}, \tag{5.140}$$

  where the indices m_μ are the fractional parts of solitonic numbers, $M_\mu \to M_\mu + m_\mu/K$.
- The analysis of states and energies for $K = 1$ is also valid for $K > 1$, since it applies to the electron spectrum that is contained in the sub-partition function (5.140) with $m_\mu = 0$ for each spin sector.

- The pattern of modular transformations is again given by Fig. 5.3; there appears phase factors among the anyonic sectors that do not affect the results and will specified later.

The strategy to prove the stability of bosonic (fractional) topological insulators will be the following: repeat the Fu-Kane-Mele stability argument for fermionic insulators of Section 4.5, by addressing the fermionic states identified within the bosonic theory by the previous analysis.

Upon following the evolution of the low-lying states of $Z'^{B}_{\frac{1}{2},\frac{1}{2}\frac{1}{2}}$ in (5.127) under continuos change of the fluxes Φ_i from zero to $K\Phi_0/2$, one can check that the Ramond state $|\Omega\rangle_R$ is the evolution of the Neveu-Schwarz one $|\Omega\rangle_{NS}$. The Ramond state possesses a Kramers partner $|\Omega\rangle'_R$, that remains degenerate upon adding any time-reversal invariant interaction to the Hamiltonian. Then, following the evolution back to zero flux of $|\Omega\rangle'_R$, one finds that this matches the following excited state of the Neveu-Schwarz sector

$$|ex\rangle_{NS} \leftrightarrow \exp(-\mathcal{E}^{00}_{0-1-1}), \tag{5.141}$$

whose energy is of order $O(1/R_1, 1/R_2)$. According to the stability argument discussed in Section 4.5, it follows that the bosonic spectrum remains gapless in the thermodynamic limit in presence of TR invariant interactions. This completes the proof of stability of bosonic topological insulators for any odd integer value of the coupling K.

It is worth stressing the usefulness of the effective field theory approach for interacting topological states. The stability argument originally using band theory was first translated into the language of fermionic surface states and then reformulated in terms of properties of partition functions. Then, the map between fermionic and bosonic partition functions was used to extend the argument to interacting topological states (hydrodynamic approach) which cannot be described by band theory.

The stability of the surface excitations can be again related to a $\mathbb{Z}_2$ anomaly. Indeed, the bosonic Neveu-Schwarz and Ramond states related by the insertion of half- fluxes are eigenstates of a TR invariant Hamiltonian, but possess different spin-parity index, i.e.

$$(-1)^{2S}\,|\Omega\rangle_{NS} = |\Omega\rangle_{NS}\,, \qquad (-1)^{2S}\,|\Omega\rangle_{R} = -\,|\Omega\rangle_{R}\,, \tag{5.142}$$

although this quantity is conserved by TR symmetry. Therefore, similarly to the fermionic case, we interpret this change as being a $\mathbb{Z}_2$ anomaly, which is equivalent to the $\mathbb{Z}_2$ index of stability.

5.4.5 Stability and modular invariance

We now determine the transformations under $SL(3,\mathbb{Z})$ of the bosonic partition functions (5.77) with $K > 1$ [3]. The oscillator part of partition functions Z_{HO} does not depend on K and its transformations were already described in Section 5.4.1. The K^3 anyon sectors $Z^{B\, m_0 m_1 m_2}_{\alpha_0,\alpha_1\alpha_2}$ within each spin sector carry a unitary linear representation of the modular group. This is just the generalization of the K^2 sectors of topological insulators in $(2+1)$ dimensions, called $K_\lambda(\tau)\overline{K(\tau)}_{\lambda'}$, $\lambda, \lambda' \in \mathbb{Z}_K$ in Chapter 2 [1]; the only difference is that there is no chiral factorization. The action of T_1 reads

$$
\begin{aligned}
T_1: \quad & Z^{B\,m_0m_1m_2}_{\alpha_0,0\alpha_2} \to \exp\left(-2\pi i\frac{m_0m_1}{K}\right) Z^{B\,m_0m_1m_2}_{\alpha_0,0\alpha_2} \\
& Z^{B\,m_0m_1m_2}_{\frac{1}{2},\frac{1}{2}\alpha_2} \to \exp\left(-2\pi i\frac{m_0m_1}{K}\right) Z^{B\,m_0m_1m_2}_{0,\frac{1}{2}\alpha_2} \\
& Z^{B\,m_0m_1m_2}_{0,\frac{1}{2}\alpha_2} \to \exp\left(-2\pi i\frac{m_0m_1}{K}\right) Z^{B\,m_0m_1m_2}_{\frac{1}{2},\frac{1}{2}\alpha_2}
\end{aligned}
\tag{5.143}
$$

while S_1 is represented by

$$
S_1: \qquad Z^{B\,m_0m_1m_2}_{\alpha_0,\alpha_1\alpha_2} \to \sum_{\widetilde{m}_0,\widetilde{m}_1\in\mathbb{Z}_K} \frac{1}{K}\exp\left(2\pi i\frac{\widetilde{m}_1m_0+\widetilde{m}_0m_1}{K}\right) Z^{B\,\widetilde{m}_0\widetilde{m}_1m_2}_{\alpha_1,\alpha_0\alpha_2} \tag{5.144}
$$

The map between spin sectors for $K>1$ is equal to that of $K=1$ shown in Fig.5.3. As in earlier discussions, the action of T_2 and S_2 can be found with the help of the parity P_{12}, leading to the matrices

$$
\begin{aligned}
(T_2)_{m_\mu,\widetilde{m}_\mu} &= \delta^{(3)}_{m_\mu,\widetilde{m}_\mu}\exp\left(2\pi i\frac{m_0m_2}{K}\right), \\
(S_2)_{m_\mu,\widetilde{m}_\mu} &= \frac{1}{K}\delta_{m_1,\widetilde{m}_1}\exp\left(2\pi i\frac{\widetilde{m}_2m_0+\widetilde{m}_0m_2}{K}\right).
\end{aligned}
\tag{5.145}
$$

We remark that these results have been first found in Ref. [116] for the case of $Z^B_{0,00}$.

In Section 5.2 we recalled the quantization of the global degrees of freedom of the BF theory on the spatial three-torus $\mathcal{M}=\mathbb{T}^3\times\mathbb{R}$. We then discussed the relation between bulk and boundary observables, and how the bulk spectra is reproduced in the quantization of the surface bosonic theory, through the quantum numbers of solitonic states. We note that the matrices T_1, T_2 reproduce the statistical phases coming from braiding anyons around vortex lines [117, 132, 133, 134]. This is another instance of the relation between bulk and boundary observables, that has been stressed in Ref.[116], and further investigated for more general hydrodynamic theories representing the three-loop braiding statistics [132, 133, 134, 135, 136, 137].

More precisely, in the geometry of the thick spatial two-torus of Fig.5.1, the conservation of charge and flux between bulk and boundary implies that the partition function of anyon indices (m_0, m_1m_2) describes the edge theory in presence of bulk charge $-m_0$ and bulk fluxes $(-m_1,-m_2)$. Modular invariant partition functions are obtained as usual by taking linear combinations of anyon sectors. We should consider the case of vanishing bulk charge $m_0=0$, otherwise there is no symmetry of exchanging space and time. The following expression summing over all fractional values of the fluxes (m_1,m_2),

$$
\mathcal{Z}^B_{\alpha_0,\alpha_1\alpha_2} = \sum_{m_1,m_2\in\mathbb{Z}_K} Z^{B'\,0\,m_1m_2}_{\alpha_0,\alpha_1\alpha_2} \tag{5.146}
$$

is left invariant by the modular transformations, apart from the usual maps between spin sectors of Fig 5.3. Of course this expression matches earlier results under the dimensional reduction of Section 5.4.2

We remark that the stability of the bosonic topological insulators is again related to the the impossibility of writing a modular invariant partition function that is consistent with the physical requirements. The expression that is invariant under $V_1^{K/2}$, $V_2^{K/2}$ and the modular group is the sum over the eight bosonic spin sectors of (5.146)

$$\mathcal{Z}^B_{\text{INV}} = \sum_{\alpha_0,\alpha_1,\alpha_2=0,\frac{1}{2}} \mathcal{Z}^B_{\alpha_0\,\alpha_1\alpha_2}. \tag{5.147}$$

In analogy with the fermionic case in Section 4.6, this partition function is not consistent with TR symmetry due to the presence of the $\mathbb{Z}_2$ anomaly, the change of the spin parity index between the the bosonic Neveu-Schwarz and Ramond states related by the insertion of half- fluxes. Therefore, TR symmetry requires not to sum over the sectors, leaving a set of eight functions $\mathcal{Z}^B_{\alpha_0,\alpha_1\alpha_2}$ that are modular covariant.

Chapter 6

Conclusions and perspectives

In this thesis we have analyzed time-reversal invariant topological insulators in two and three spatial dimensions. We have discussed their effective actions, computed the partition functions of the edge and surface excitations, recovered the $\mathbb{Z}_2$ classification and extended it in presence of interactions.

Our analysis clarified that the stability of these topological phases is associated to an anomaly of the boundary theories. Indeed, the states of the Neveu-Schwarz and Ramond sectors possess different values of the $\mathbb{Z}_2$ spin-parity index, that is a time-reversal invariant quantity. It turns out that anomalous system possesses gapless excitations protected by the symmetry.

Furthermore, the partitions functions of topological insulators have interesting geometrical properties connected to stability. Indeed, studying the behavior under two and three dimensional modular transformations, we found that the stability is associated to the impossibility of having a modular invariant partition function that is consistent with time-reversal symmetry. We have interpreted this result as a discrete gravitational anomaly accompanying the $\mathbb{Z}_2$ spin-parity anomaly.

In this thesis, we have also answered the question of whether non-anomalous two dimensional systems does become fully gapped. By using CFT methods, we found the interactions that completely gap the edge modes of unstable Abelian and non-Abelian topological insulators in two dimensions.

In order to find these results, we have analyzed the low energy effective field theories of both two and three dimensional topological insulators. In two dimensions, this analysis took full advantage of the exact bosonization, while in $(2+1)$ dimensions an analogous exact mapping between fermions and bosons cannot be found. Our analysis clarified some aspects of the recent discussions on effective field theories for three dimensional topological insulators as well as added some insight on the general problem of bosonization in $(2+1)$ dimensions.

We considered the $(3+1)$ dimensional topological BF gauge theory as the hydrodynamic effective field theory for topological insulators. First, studying the surface effective field theory, we introduced another non-local dynamics for the bosonic field, that reproduces the fermionic induced action in presence of the background gauge field, to quadratic order.

Furthermore, by quantizing the compactified scalar field we determined the surface partition functions on the torus geometry. We found eight functions that are different from those of the fermionic spin sectors, but transform in the same way for 'large gauge transformations', i.e. for magnetic flux insertions and modular transformations. Moreover, bosonic and fermionic functions become equal under dimensional reduction to $(1+1)$ dimensions. We have assigned fermion numbers to the bosonic states and, thus, we have defined the corresponding three dimensional bosonic Neveu-Schwarz and Ramond sectors. By means of this correspondence, we reformulated the flux insertion argument and the $\mathbb{Z}_2$ stability criterion for every interacting topological insulators with $K \geqslant 1$, with K odd integer.

Our study of the bosonic theory is related with recent conjectures of bosonization in $(2+1)$ dimensions. The basic picture underlying these correspondences is that of 'attaching flux tubes to particles', that changes the statistics from fermionic to bosonic and viceversa [13] [130]. One flux per particle can be attached by coupling matter to a 'statistical' gauge field $\mathcal{A}_\mu$ with Chern-Simons action of coupling constant $K = 1$: this interaction can be removed by a (singular) gauge transformation that changes the statistics of wave functions from fermionic to bosonic and viceversa. Recently, several authors have suggested that flux attachment also holds for relativistic excitations and have proposed a web of dualities between fermionic and bosonic theories [128, 129, 124]. In our setting, we can argue that the flux attachment is represented by the choice of boundary conditions for the soliton excitations in the Ramond sector, corresponding to half fluxes added along the two spatial cycles of the torus.

The compactified bosonic theory is an interesting exactly solvable model of interacting fermions that could be further analyzed by computing correlation functions and other observables. In our comparison of bosonic and fermionic surface theories, we discussed properties that are rather independent of interactions; thus, we did not address the problem of a precise map between fermionic and bosonic dynamics. Nonetheless, the quantization of the other non-local bosonic action introduced in Section 5.1.3 and the study of partition functions could be future steps in understanding the relation between interacting bosons and fermions in $(2+1)$ dimensions.

In this thesis we discussed the simplest $(3+1)$ dimensional BF theory involving particle and vortex excitations. Recently, some authors have pointed out that vortex excitations in three spatial dimensions may possess a new effect, the so called three-loop braiding [132, 133, 134, 135]. In order to capture this statistical phase, one needs more BF theories coupled together [116, 136, 137]. Some progress in studying the quantization of these theories has been done [116, 137]. It would be interesting to study the relations between these theories and interacting fermionic states. Upon repeating the quantization of the solitonic modes of this thesis and introducing the corresponding Ramond sectors, we could extend the $\mathbb{Z}_2$ stability criterion to these theories.

We conclude this thesis by stressing that the effective field theory approach has many interesting direction to explore in the study of topological states. Recently, new phases have been introduced called *symmetry enriched topological phases* [138]. In these cases, the surface excitations are gapped but supports intrinsic two dimensional topological order, thus the systems cannot be adiabatically continued into the trivial phase [139] [140]. A well-known example is the *T-Pfaffian* state, a time-reversal state that possesses non-Abelian excitations similar to those of the Pfaffian state in the QHE [141, 142, 143]. To understand these phases new duality relations between fermionic and bosonic theories have been recently proposed [125, 126].

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Appendix A

Fermion quantum numbers

In this appendix we shall give some details about the fermion quantum numbers discussed in the effective theories of the quantum Hall effect and topological insulators in two and three dimensions. First we will discuss the Weyl and the Dirac fermion in two dimensions, thus we will generalize these results to the three dimensional case.

Weyl fermion in two dimensions

To discuss the Dirac theory in two dimensions we take advantage of the chiral/antichiral decomposition that occurs in the massless case, namely the fact that the massless Dirac fermion can be decomposed in two massless Weyl fermions left and right moving, respectively. We focus our attention on one Weyl fermion defined on the circle parametrized by the angle θ and decompose it in terms of rising and lowering operators as follows

$$\psi(\theta, t) = \sum_k d_k \mathrm{e}^{ik(\theta - t)}. \tag{A.1}$$

The fermionic modes satisfy the anti-commutation rules $\{d_k, d_l\} = \delta_{kl}$ and act on the vacuum as

$$d_k \left|\Omega\right\rangle = 0, \qquad k > 0, \tag{A.2}$$

$$d_k^\dagger \left|\Omega\right\rangle = 0, \qquad k \leqslant 0. \tag{A.3}$$

It follows that d_k with $k > 0$ and $d_k^\dagger$ with $k \leqslant 0$ annihilate, respectively, particle and antiparticle excitations.

Imposing antiperiodic (periodic) spatial boundary conditions we find that the field expansion (A.1) takes half-integer (integer) values defining the Neveu-Schwarz (Ramond) sector as follows

$$\psi(\theta + 2\pi, t) = \begin{cases} -\psi(\theta, t), & \text{if} \quad k \in \mathbb{Z} + \frac{1}{2} \quad \text{Neveu-Schwarz}, \\ \psi(\theta, t), & \text{if} \quad k \in \mathbb{Z} \quad \text{Ramond}. \end{cases} \tag{A.4}$$

Thus, the Ramond sector has a zero energy mode satisfying the anti-commutation rule $\{d_0^\dagger, d_0\} = 1$ and acting on the vacuum as $d_0^\dagger \left|\Omega\right\rangle = 0$.

The normal ordered expression of the charge operator takes the form [20]

$$Q = \sum_k : d_k^\dagger d_k := \sum_{k>0} \left(d_k^\dagger d_k - d_{-k} d_{-k}^\dagger \right) \qquad \text{if} \quad k \in \mathbb{N} + \frac{1}{2}, \tag{A.5}$$

$$Q = \sum_k : d_k^\dagger d_k := \sum_{k>0} \left(d_k^\dagger d_k - d_{-k} d_{-k}^\dagger \right) + : d_0^\dagger d_0 : \quad \text{if} \quad k \in \mathbb{N}. \tag{A.6}$$

In these expressions the renormalized charge was defined in such a way that the Neveu-Schwarz ground state is neutral, $Q \left|\Omega\right\rangle_{NS} = 0$.

To find the charge of the Ramond ground state we have to determine the normal ordered prescription of the zero modes operators. First, let us show that the Ramond sector possesses two degenerate ground states. Following standard CFT arguments [22] [21], we introduce the operator $(-1)^F$, that anti-commutes with the fermion field (A.1), $(-1)^F \psi(\theta, t) = -\psi(\theta, t)(-1)^F$. In terms of modes, this means that

$$\{(-1)^F, d_k\} = 0 \quad \text{for all } k, \tag{A.7}$$

that is $(-1)^F$ has eigenvalues ± 1 acting on states with even and odd numbers of fermion creation operators, respectively. Since the zero modes anti-commutes with $(-1)^F$ also, it follows that the Ramond sector possesses two degenerate ground states $\left|\Omega\right\rangle_{\pm R}$, such that

$$(-1)^F \left|\Omega\right\rangle_{\pm R} = \pm \left|\Omega\right\rangle_{\pm R}. \tag{A.8}$$

This condition can be satisfied taking $\left|\Omega\right\rangle_{+R}$ such that

$$d_0^\dagger \left|\Omega\right\rangle_{+R} = 0, \qquad \left|\Omega\right\rangle_{-R} = d_0 \left|\Omega\right\rangle_{+R}. \tag{A.9}$$

The normal ordering of the zero modes is written in the general form

$$: d_0^\dagger d_0 := d_0^\dagger d_0 - x, \qquad x = \left\langle d_0^\dagger d_0 \right\rangle, \tag{A.10}$$

where $0 \leqslant x < 1$ is a parameter expressing the partial filling of the ground state located at the Fermi level. It can be shown that the higher moments of charge and Hamiltonian operators satisfy the Virasoro and Kac-Moody (see Chapter 1), consistently with the fermionic anti-commutation relations only for $x = 1/2$ [20]. With this choice, the charge operator of the Ramond sector takes the following form

$$Q = \sum_{k>0} \left(d_k^\dagger d_k - d_{-k} d_{-k}^\dagger \right) + d_0^\dagger d_0 - \frac{1}{2}, \tag{A.11}$$

from which follows that

$$Q \left|\Omega\right\rangle_{\pm R} = \pm \frac{1}{2} \left|\Omega\right\rangle_{\pm R}. \tag{A.12}$$

In Table (A.1) we summarize the indices and charges assigned to ground states of the Weyl theory. These values are those obtained by the insertion of half-flux $\Phi_0/2$ described in Chapter 2.

ground state	Q	$(-1)^F$
$\|\Omega\rangle_{NS}$	0	1
$\|\Omega\rangle_{+R}$	1/2	1
$\|\Omega\rangle_{-R}$	−1/2	−1

Table A.1: Indices and charges of the Neveu-Schwarz and Ramond ground states of the Weyl theory.

Dirac fermion in two dimensions

In two space-time dimensions, the massless Dirac fermion is obtained combining together one chiral and antichiral Weyl fermion. This realizes the edge theory of $2D$ topological insulator with $\nu^{\uparrow} = -\nu^{\downarrow} = 1$, also called quantum spin Hall effect (see Section 2.1). The global charge operator is given by the sum of the charge operators, respectively, of the chiral (spin up) and anti-chiral (spin down) components, namely

$$Q = Q_{\uparrow} + Q_{\downarrow}. \tag{A.13}$$

The charge $Q_{\downarrow}$ is defined with opposite sign w.r.t. $Q_{\uparrow}$, as shown by the coupling to the electromagnetic background leading to vanishing total Hall current. Therefore, the antichiral charge $Q_{\downarrow}$ is defined by the same expressions in (A.5) and (A.11) changed in sign, for the antichiral Neveu-Schwarz and Ramond sectors, respectively. It turns out that the total charge has no shift in the Ramond sector, leading to

$$Q\,|\Omega\rangle_{+\uparrow}\,|\Omega\rangle_{+\downarrow} = 0, \tag{A.14}$$

as shown in Table A.2.

In Chapter 2 we introduce the spin-parity index that is related to the difference of charges $Q_{\uparrow} - Q_{\downarrow}$,

$$(-1)^{2S} = (-1)^F = (-1)^{Q_{\uparrow} - Q_{\downarrow}}. \tag{A.15}$$

In this expression, the normal ordering shift in the Ramond sector adds up for the two chiralities, giving the result

$$(-1)^{2S}\,|\Omega\rangle_{+\uparrow}\,|\Omega\rangle_{+\downarrow} = -\,|\Omega\rangle_{+\uparrow}\,|\Omega\rangle_{+\downarrow}\,. \tag{A.16}$$

The values of charge and spin-parity for all ground states of the Neveu-Schwarz and Ramond sectors are summarized in Table A.2 and match those obtained in Chapter 2.

Dirac fermion in three dimensions

In $(2 + 1)$ dimensions the Clifford algebra (4.11) does not have a chiral-antichiral decomposition; thus, it is not obvious how to generalize the previous results. Therefore, we should reconsider the problem of defining the spin quantum number, and then the spin-parity index, of the ground states of Neveu-Schwarz and Ramond sectors.

ground state	$Q_\uparrow$	$Q_\downarrow$	Q	$(-1)^{2S}=(-1)^F$
$\|\Omega\rangle_{NS} = \|\Omega\rangle_\uparrow \|\Omega\rangle_\downarrow$	0	0	0	1
$\|\Omega\rangle_R^{(1)} = \|\Omega\rangle_{+\uparrow} \|\Omega\rangle_{+\downarrow}$	1/2	−1/2	0	−1
$\|\Omega\rangle_R^{(2)} = \|\Omega\rangle_{-\uparrow} \|\Omega\rangle_{+\downarrow}$	−1/2	−1/2	−1	1
$\|\Omega\rangle_R^{(3)} = \|\Omega\rangle_{+\uparrow} \|\Omega\rangle_{-\downarrow}$	1/2	1/2	1	1
$\|\Omega\rangle_R^{(4)} = \|\Omega\rangle_{-\uparrow} \|\Omega\rangle_{-\downarrow}$	−1/2	1/2	0	−1

Table A.2: Charges and indices of the Neveu-Schwarz $|\Omega\rangle_{NS}$ and Ramond $|\Omega\rangle_R^{(i)}$, $i=1,\dots,4$ ground states of the Dirac theory in (1+1) dimensions.

Let us start again from the $(2+1)$ dimensional Dirac field involving creation and annihilation operators of particles $(a^\dagger_{\mathbf{n}}, a_{\mathbf{n}})$ and antiparticles $(b^\dagger_{\mathbf{n}}, b_{\mathbf{n}})$, where $\mathbf{n}=(n_1,n_2)\in\mathbb{Z}^2$ [97]. These operators obey anti-commutation relations

$$\left\{a^\dagger_{\mathbf{n}}, a_{\mathbf{n}'}\right\} = \delta_{\mathbf{n},\mathbf{n}'}, \qquad \left\{b^\dagger_{\mathbf{n}}, b_{\mathbf{n}'}\right\} = \delta_{\mathbf{n},\mathbf{n}'}, \tag{A.17}$$

and satisfy the vacuum conditions

$$a_{\mathbf{n}}|\Omega\rangle = b_{\mathbf{n}}|\Omega\rangle = 0 \qquad\qquad n_1,n_2\in\mathbb{Z}. \tag{A.18}$$

In $(2+1)$ dimensions, the charge and fermion operators are defined as, respectively,

$$Q=\sum_{\mathbf{n}}\left(a^\dagger_{\mathbf{n}}a_{\mathbf{n}} - b^\dagger_{\mathbf{n}}b_{\mathbf{n}}\right), \tag{A.19}$$

$$(-1)^F = (-1)^{\sum_{\mathbf{n}} a^\dagger_{\mathbf{n}}a_{\mathbf{n}} + b^\dagger_{\mathbf{n}}b_{\mathbf{n}}}. \tag{A.20}$$

Requiring antiperiodic and periodic spatial boundary conditions, we can define the (2+1) dimensional equivalents of the Neveu-Schwarz and Ramond sectors, respectively. As shown in Chapter 4, the Neveu-Schwarz sector takes $n_1,n_2\in\mathbb{Z}+1/2$ and has a unique ground state $|\Omega\rangle_{\frac{1}{2}\frac{1}{2}}$; this is neutral and its spin parity index, equal to the fermion number, is given by

$$Q|\Omega\rangle_{\frac{1}{2}\frac{1}{2}} = 0, \tag{A.21}$$

$$(-1)^{2S}|\Omega\rangle_{\frac{1}{2}\frac{1}{2}} = (-1)^F|\Omega\rangle_{\frac{1}{2}\frac{1}{2}} = |\Omega\rangle_{\frac{1}{2}\frac{1}{2}}. \tag{A.22}$$

As shown in Chapter 4, the Ramond sector possesses indices $n_1,n_2\in\mathbb{Z}$. Thus, using the zero modes operators $(a_{00},a^\dagger_{00})$ and $(b_{00},b^\dagger_{00})$, one obtains four degenerate ground states $|\Omega\rangle_{00}^{(i)}$, with $i=1,2,3,4$.

In Chapter 4, upon following the evolution of the spectrum under the insertion of half fluxes, we saw that the Neveu-Schwarz ground state is mapped in the following Ramond ground state: $|\Omega\rangle_{\frac{1}{2}\frac{1}{2}} \to |\Omega\rangle_{00}^{(1)}$, while the $|\Omega\rangle_{00}^{(4)}$ is identified as the partner of the Kramers pair, namely $|\Omega\rangle_{00}^{(4)} = \mathcal{T}|\Omega\rangle_{00}^{(1)}$. Both states are neutral, i.e.

$$Q|\Omega\rangle_{00}^{(1)} = Q|\Omega\rangle_{00}^{(4)} = 0. \tag{A.23}$$

Then, the Fock space identification of the four states in the Ramond sector of the fermionic partition function is:

$$|\Omega\rangle_{00}^{(1)} \equiv |\Omega\rangle_R\,, \qquad |\Omega\rangle_{00}^{(2)} = a_{00}^\dagger\,|\Omega\rangle_R\,, \tag{A.24}$$

$$|\Omega\rangle_{00}^{(3)} = b_{00}^\dagger\,|\Omega\rangle_R\,, \qquad |\Omega\rangle_{00}^{(4)} = b_{00}^\dagger a_{00}^\dagger\,|\Omega\rangle_R\,, \tag{A.25}$$

where $|\Omega\rangle_R$ is the Ramond ground state.

Let us reconsider the normal ordering of the charge and fermion number given in (A.19) and (A.20), starting from the expansion around the Fermi level of a non-relativistic spectrum at finite volume. In the case of the Neveu-Schwarz sector, as showed in Fig. 4.7, the Fermi level is located in between the empty and filled states, because the energy spectrum (4.35) is strictly positive. This gives a clear identification of particles and antiparticles and determines the standard normal-ordering of the relativistic expressions written in (A.21) and (A.22).

In the Ramond sector, instead, there is an ambiguity because two charged excitations are exactly located at the Fermi level (see Fig. 4.9). Thus, we shall assume that they are partially filled as in the previous case of two dimensions:

$$\langle a_{00}^\dagger a_{00}\rangle = x, \qquad \langle b_{00}^\dagger b_{00}\rangle = 1-x, \qquad 0\le x<1 \tag{A.26}$$

It turns out that the normal-ordered expressions of charge and fermion number in (A.19) and (A.20) should be modified in the term $(n_1,n_2)=(0,0)$ of the sums, as follows

$$Q = \sum_{\mathbf{n}} a_{\mathbf{n}}^\dagger a_{\mathbf{n}} - b_{\mathbf{n}}^\dagger b_{\mathbf{n}} + 1 - 2x, \tag{A.27}$$

$$(-1)^F = (-1)^{\sum_{\mathbf{n}} a_{\mathbf{n}}^\dagger a_{\mathbf{n}} + b_{\mathbf{n}}^\dagger b_{\mathbf{n}} + 1} \tag{A.28}$$

Upon further assuming the particle-hole symmetric filling $x=1/2$, we obtain the following quantum number assignments

$$(-1)^{2S} = (-1)^F = -1 \qquad \text{on} \qquad |\Omega\rangle_{00}^{(1)}\,,\ |\Omega\rangle_{00}^{(4)}\,, \tag{A.29}$$

$$(-1)^{2S} = (-1)^F = 1 \qquad \text{on} \qquad |\Omega\rangle_{00}^{(2)}\,,\ |\Omega\rangle_{00}^{(4)}\,. \tag{A.30}$$

These are the generalizations of the results in $(1+1)$ dimensions discussed before. The values of charge, fermion number and spin-parity indices of the Neveu-Schwarz and Ramond ground states are summarized in Table (A.3).

The expressions for Q and $(-1)^F$ in (A.27) and (A.28) are equivalent to the previous results for the $(1+1)$ dimensional Dirac theory (A.11) taking into account the pairing of chiralities explained earlier and the particle vs antiparticle identification.

ground state	Q	$(-1)^{2S} = (-1)^F$
$\vert\Omega\rangle_{\frac{1}{2}\frac{1}{2}}$	0	1
$\vert\Omega\rangle_{00}^{(1)}$	0	-1
$\vert\Omega\rangle_{00}^{(2)}$	1	1
$\vert\Omega\rangle_{00}^{(3)}$	-1	1
$\vert\Omega\rangle_{00}^{(4)}$	0	-1

Table A.3: Charges and indices of the Neveu-Schwarz and Ramond ground states of the massless Dirac theory in $(2+1)$ dimensions.

Appendix B

Modular transformation and spin structures of $2D$ topological insulators

In the following we give the expressions of partition functions for the four spin sectors NS, $\widetilde{NS}$, R, $\widetilde{R}$, that describe the edge excitations of topological insulator models examined in the main text. We describe their behavior under flux insertion and modular transformations [1].

B.1 Laughlin states

The Laughlin states discussed in Section 2.3.1 are described by the $c = 1$ CFT of the chiral boson [21]. The anyon sectors for the chiral modes of the NS and $\widetilde{NS}$ spin sectors are, for $\lambda = 1, \cdots, p$, p odd, [23]:

$$K_\lambda^{NS}(\tau,\zeta;k) = \frac{F(\tau,\zeta)}{\eta(\tau)} \sum_{n\in\mathbb{Z}} \exp\left(i2\pi\left(\tau\frac{(np+\lambda)^2}{2p} + \zeta\frac{np+\lambda}{p}\right)\right), \tag{B.1}$$
$$K_\lambda^{\widetilde{NS}}(\tau,\zeta;k) = \frac{F(\tau,\zeta)}{\eta(\tau)} \sum_{n\in\mathbb{Z}} (-1)^{pn} \exp\left(i2\pi\left(\tau\frac{(np+\lambda)^2}{2p} + \zeta\frac{np+\lambda}{p} + \frac{\lambda}{2}\right)\right),$$

with $F = \exp\left[-\pi(\mathrm{Im}\zeta)^2/p\,\mathrm{Im}\tau\right]$ is a non-holomorphic prefactor and $\eta(\tau)$ the Dedekind function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), \qquad q = \exp(i2\pi\tau). \tag{B.2}$$

The anyon sectors of the R and $\widetilde{R}$ spin sectors are defined by:

$$K_\lambda^{R} = K_{\lambda+\frac{p}{2}}^{NS},$$
$$K_\lambda^{\widetilde{R}} = K_{\lambda+\frac{p}{2}}^{\widetilde{NS}}. \tag{B.3}$$

The edge partition functions for each spin sector (Eq.(2.17) and (2.18)) are obtained by matching the fractional charge of the chiral and antichiral anyon sectors locally at the edge

$$Z^{(\sigma)} = \sum_{\lambda=1}^{p} K_{\lambda}^{(\sigma)} \overline{K_{-\lambda}^{(\sigma)}}, \qquad \sigma = NS, \widetilde{NS}, R, \widetilde{R}. \tag{B.4}$$

The transformation of the anyon sectors ((B.1) and (B.3)) under the modular group, generated by S and T, and for the insertion of one and $p/2$ fluxes through the annulus, the V and $V^{\frac{p}{2}}$ transformations, respectively, are obtained by extending the calculations of Ref. [23] [66]. Altogether they are:

- S

$$\begin{aligned}
K_{\lambda}^{NS}(-\frac{1}{\tau}, \frac{-\zeta}{\tau}) &= e^{i\varphi} \sum_{\lambda'=1}^{p} S_{\lambda\lambda'}\ K_{\lambda'}^{NS}(\tau, \zeta), \\
K_{\lambda}^{\widetilde{NS}}(-\frac{1}{\tau}, \frac{-\zeta}{\tau}) &= e^{i\varphi} \sum_{\lambda'=1}^{p} S_{\lambda\lambda'} K_{\lambda'}^{R}(\tau, \zeta), \\
K_{\lambda}^{R}(-\frac{1}{\tau}, \frac{-\zeta}{\tau}) &= e^{i\varphi} \sum_{\lambda'=1}^{p} S_{\lambda\lambda'}\ K_{\lambda'}^{\widetilde{NS}}(\tau, \zeta), \\
K_{\lambda}^{\widetilde{R}}(-\frac{1}{\tau}, \frac{-\zeta}{\tau}) &= \exp\bigg(2\pi i \frac{p}{4}\bigg)\ e^{i\varphi} \sum_{\lambda'=1}^{p} S_{\lambda\lambda'} K_{\lambda'}^{\widetilde{R}}(\tau, \zeta),
\end{aligned} \tag{B.5}$$

with

$$S_{\lambda\lambda'} = \frac{1}{\sqrt{p}}\bigg(2\pi i \frac{\lambda\lambda'}{p}\bigg), \qquad e^{i\varphi} = \exp\bigg(\frac{i\pi}{p}\mathrm{Re}\bigg(\frac{\zeta^2}{\tau}\bigg)\bigg). \tag{B.6}$$

- T

$$\begin{aligned}
K_{\lambda}^{NS}(\tau+1, \zeta) &= \exp\bigg(-2\pi i \frac{\lambda}{2}\bigg) T_a K_{\lambda}^{\widetilde{NS}}(\tau, \zeta), \\
K_{\lambda}^{\widetilde{NS}}(\tau+1, \zeta) &= \exp\bigg(2\pi i \frac{\lambda}{2}\bigg) T_a K_{\lambda}^{NS}(\tau, \zeta), \\
K_{\lambda}^{R}(\tau+1, \zeta) &= T_a T_b K_{\lambda}^{R}(\tau, \zeta), \\
K_{\lambda}^{\widetilde{R}}(\tau+1, \zeta) &= T_a T_b K_{\lambda}^{\widetilde{R}}(\tau, \zeta),
\end{aligned} \tag{B.7}$$

with

$$T_a = \exp\bigg(2\pi i \bigg(\frac{\lambda^2}{2p} - \frac{1}{24}\bigg)\bigg), \qquad T_b = \exp\bigg(2\pi i \bigg(\frac{p}{8} + \frac{\lambda}{2}\bigg)\bigg). \tag{B.8}$$

- V

$$\begin{aligned}
K_{\lambda}^{NS}(\tau, \zeta+\tau) &= V_{\Phi_0} K_{\lambda+1}^{NS}(\tau, \zeta), \\
K_{\lambda}^{\widetilde{NS}}(\tau, \zeta+\tau) &= -V_{\Phi_0} K_{\lambda+1}^{\widetilde{NS}}(\tau, \zeta), \\
K_{\lambda}^{R}(\tau, \zeta+\tau) &= V_{\Phi_0} K_{\lambda+1}^{R}(\tau, \zeta), \\
K_{\lambda}^{\widetilde{R}}(\tau, \zeta+\tau) &= -V_{\Phi_0} K_{\lambda+1}^{\widetilde{R}}(\tau, \zeta),
\end{aligned} \tag{B.9}$$

with

$$V_{\Phi_0}(\tau, \zeta) = \exp\bigg(-2\pi i \frac{1}{p}\bigg(Re\frac{\tau}{2} + Re\zeta\bigg)\bigg) \tag{B.10}$$

- $V^{\frac{p}{2}}$

$$\begin{aligned}
&K_\lambda^{NS}(\tau,\zeta+\frac{p\tau}{2}) = V_{\frac{p}{2}\Phi_0}\ K_\lambda^{R}, \\
&K_\lambda^{\widetilde{NS}}(\tau,\zeta+\frac{p\tau}{2}) = V_{\frac{p}{2}\Phi_0}\exp\Big(-2\pi i\frac{p}{4}\Big)K_\lambda^{\widetilde{R}}, \\
&K_\lambda^{R}(\tau,\zeta+\frac{p\tau}{2}) = V_{\frac{k}{2}\Phi_0}K_\lambda^{NS}, \\
&K_\lambda^{\widetilde{R}}(\tau,\zeta+\frac{p\tau}{2}) = V_{\frac{p}{2}\Phi_0}\exp\Big(-2\pi i\frac{p}{4}\Big)K_\lambda^{\widetilde{NS}},
\end{aligned} \tag{B.11}$$

with

$$V_{\frac{p}{2}\Phi_0} = \exp\bigg(-2\pi i\Big(\frac{p}{8}Re\tau+\frac{1}{2}Re\zeta\Big)\bigg). \tag{B.12}$$

Upon using these formulas, we obtain that the transformations of partition functions (B.4) illustrated in Fig. 2.4.

B.2 Pfaffian states

The edge partition function for the Pfaffian states in the Neveu-Schwarz spin sector is given in (2.49) [66]. We obtain the partition functions of the other spin sectors acting with T and ST on Z_{Pf}^{NS} (2.49) as described in Section 2.4.3. They read:

$$\begin{aligned}
Z_{\mathrm{Pf}}^{NS/\widetilde{NS}} =&\big|K_0 I \pm K_4\psi\big|^2 + \big|K_0\psi \pm K_4 I\big|^2 + \big|(K_1 \pm K_{-3})\sigma\big|^2 \\
&\big|K_2 I \pm K_{-2}\psi\big|^2 + \big|K_2\psi \pm K_{-2} I\big|^2 + \big|(K_3 + K_{-1})\sigma\big|^2,
\end{aligned} \tag{B.13}$$

$$\begin{aligned}
Z_{\mathrm{Pf}}^{R/\widetilde{R}} =&\big|K_3 I \pm K_{-1}\psi\big|^2 + \big|K_3\psi \pm K_{-1} I\big|^2 + \big|(K_0 \pm K_4)\sigma\big|^2 \\
&\big|K_{-3} I \pm K_1\psi\big|^2 + \big|K_{-3}\psi \pm K_1 I\big|^2 + \big|(K_2 + K_{-2})\sigma\big|^2.
\end{aligned}$$

We have written the neutral characters with the same symbol of the Ising fields

$$\chi_0^0 = \chi_2^2 = I, \qquad \chi_1^1 = \chi_3^1 = \sigma, \qquad \chi_2^0 = \chi_0^2 = \psi, \tag{B.14}$$

that model the neutral excitations of this system [21] [24].

B.3 Read-Rezayi states

The Read-Rezayi models [67] are based on the neutral $\mathbb{Z}_k$ parafermion conformal theories with central charge $c = 2(k-1)/(k+2)$, and are described by the coset construction $\widehat{SU(2)}_k/\widehat{U(1)}_{2k}$ [66]. For these theories the values of the stability parameters are $p = kM+2$ with $M = 1, 3, 5, \cdots$ and $k = 2, 3, \cdots$. The expressions of partition functions depend on the parity of k.

Partition functions for RR states with even k

The charge characters are given by the functions (1.40) $K_\lambda(\tau, k\zeta, kp)$ with periodicities $K_{\lambda+kp} = K_\lambda$. The $\mathbb{Z}_k$ parafermionic characters that describe the neutral part are denoted by $\chi^\ell_m(\tau; 2k)$, and have the following periodicities and modular transformations [66],

$$\begin{aligned}
&\chi^\ell_m = \chi^\ell_{m+2k} = \chi^{k-\ell}_{m+k}, \qquad m = \ell \bmod 2, \qquad\qquad (B.15)\\
&\chi^\ell_m = 0 \qquad\qquad\qquad\qquad\quad m = \ell + 1 \bmod 2,\\
&S: \quad \chi^\ell_m(-\frac{1}{\tau}, 0; 2k) = \frac{1}{\sqrt{2k}} \sum_{\ell'=0}^{k} \sum_{m'=1}^{2k} \exp\Big(-2\pi i \frac{mm'}{2k}\Big) s_{\ell,\ell'}\ \chi^{\ell'}_{m'}(\tau; 2k), \qquad (B.16)\\
&\qquad s_{\ell,\ell'} = \sqrt{\frac{2}{k+2}} \sin\Big(\frac{\pi(\ell+1)(\ell'+1)}{k+2}\Big),\\
&T: \quad \chi^\ell_m(\tau+1, 0; 2k) = \exp\Big(2\pi i\Big(\frac{\ell(\ell+2)}{4(k+2)} - \frac{m^2}{4k} + \frac{1}{24}\Big)\Big)\chi^\ell_m(\tau; 2k).
\end{aligned}$$

Starting from the Neveu-Schwarz sector given in Ref.[66] and acting with T and ST, we find those of the other spin sectors. Altogether they read:

$$\begin{aligned}
\Theta^{\ell NS}_a(\tau, \zeta; k) &= \sum_{b=1}^{k} K_{a+bp}(\tau, k\zeta; kp)\ \chi^\ell_{a+2b}(\tau; 2k), \qquad (B.17)\\
\Theta^{\ell \widetilde{NS}}_a(\tau, \zeta; k) &= \sum_{b=1}^{k} (-1)^b K_{a+bp}(\tau, k\zeta; kp)\ \chi^\ell_{a+2b}(\tau; 2k),\\
\Theta^{\ell R}_a(\tau, \zeta; k) &= \sum_{b=1}^{k} K_{a+bp}(\tau, k\zeta; kp)\ \chi^\ell_{a+2b+\frac{k}{2}}(\tau; 2k),\\
\Theta^{\ell \widetilde{R}}_a(\tau, \zeta; k) &= \sum_{b=1}^{k} (-1)^b K_{a+bp}(\tau, k\zeta; kp)\ \chi^\ell_{a+2b+\frac{k}{2}}(\tau; 2k),
\end{aligned}$$

where $a = 0, 1, \cdots, p-1$, and $\ell = 0, 1, \cdots, k$. The partition functions in the corresponding spin sectors are:

$$Z^{(\sigma)}_{RR} = \sum_{\ell=0}^{k} \sum_{\substack{a=1\\ a=\ell \bmod 2}}^{p} \Theta^{\ell(\sigma)}_a\ \overline{\Theta^{\ell(\sigma)}_{-a}}, \qquad \sigma = NS, \widetilde{NS}, R, \widetilde{R}. \qquad (B.18)$$

We note that the partition functions of the Pfaffian state (Eq.B.13) are obtained by choosing $M = 1$ and $k = 2$ in the previous formulas. The transformations of the anyon sectors (B.17) and the partition functions (B.18) under the insertion of fluxes and the modular group are the same of the Pfaffian state, represented in Fig. 2.4 (a) and (c).

Partition functions for RR states with odd k

In this case, the charge characters are different for each spin sector. For the NS and $\widetilde{NS}$ sectors they are:

$$K_\lambda^{NS}(\tau, k\zeta; kp) = K_\lambda(\tau, k\zeta; kp), \tag{B.19}$$
$$K_\lambda^{\widetilde{NS}}(\tau, k\zeta; kp) = \frac{F(\tau,\zeta)}{\eta(\tau)} \sum_{n\in\mathbb{Z}} (-1)^{nkp} \exp\left[2\pi i \left(\frac{\tau}{2kp}\left(nkp+\lambda\right)^2 + \frac{\zeta}{p}\left(nkp+\lambda\right) + \frac{\lambda}{2}\right)\right].$$

Those of the R and $\widetilde{R}$ spin sectors are defined by:

$$K_\lambda^{R} = K^{NS}_{\lambda+\frac{kp}{2}}, \tag{B.20}$$
$$K_\lambda^{\widetilde{R}} = K^{\widetilde{NS}}_{\lambda+\frac{kp}{2}}.$$

These charge characters have the same periodicities (2.40), i.e. $K_\lambda = K_{\lambda+kp}$. Making use of $\mathbb{Z}_k$ parafermionic characters $\chi^\ell_m(\tau; 2k)$ introduced in (B.15), the anyon sectors for four spin sector take the usual form of simple current invariants:

$$\Theta_a^{\ell(\sigma)}(\tau,\zeta;k) = \sum_{b=1}^{k} K^{(\sigma)}_{a+bp}(\tau, k\zeta; kp)\ \chi^\ell_{a+2b}(\tau; 2k), \qquad \sigma = NS, \widetilde{NS}, R, \widetilde{R}. \tag{B.21}$$

The edge partition functions of the four spin sector have the same expression of those in the even k case (B.18). Because the values of (k,p) are odd, as discussed in Section 2.4.4, it is easy to show that under the $V^{\frac{p}{2}}$ transformation and the modular group the partition functions transform as represented in Fig. 2.4 (a) and (b).

Appendix C

Regularized vacuum energy

C.1 Regularization through the Epstein's formula

We define the generalized Epstein's Zeta function [105][106]:

$$\zeta_\phi[\boldsymbol{g},\mathbf{h}](s) = \sum_{m_1,\ldots,m_p\in\mathbb{Z}} \frac{e^{2\pi i \mathbf{m}\cdot\mathbf{h}}}{[\phi(\mathbf{m}+\mathbf{g})]^{s/2}}, \tag{C.1}$$

where $\mathbf{g} = (g_1,\ldots,g_p)$, $\mathbf{h} = (h_1,\ldots,h_p)$, $\mathbf{m} = (m_1,\ldots,m_p)$ e $\phi(\mathbf{g}) = \mathbf{g}^T C\mathbf{g}$ and C a $p\times p$ invertible and definite positive matrix. This function can be rewritten in the following form

$$\begin{aligned}\zeta_\phi[\mathbf{g},\mathbf{h}](s) &= \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}\right)}\int_0^{+\infty}\frac{dz}{z}z^{s/2}\sum_{m_1,\ldots,m_p\in\mathbb{Z}} e^{-\pi z\phi(\mathbf{g}+\mathbf{m})+2\pi i(\mathbf{m}\cdot\mathbf{h})}\\ &\equiv \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}\right)}\int_0^{+\infty}\frac{dz}{z}z^{s/2}\Theta_\phi[\mathbf{g},\mathbf{h}](z).\end{aligned} \tag{C.2}$$

For $s > 2$ the integral in (C.2) converges, while diverges if $s \leq 2$. To find an analytic continuation of the function ζ_ϕ we split the integral as: $\int_0^{+\infty} = \int_0^1 + \int_1^{+\infty}$. The first integral can be rewritten using the generalized Gauss sum

$$\sum_{m_1,\ldots,m_p\in\mathbb{Z}} e^{-\pi z\phi(\mathbf{g}+\mathbf{m})+2\pi i\mathbf{h}\cdot\mathbf{m}} = \frac{e^{-2\pi i(\mathbf{g}\cdot\mathbf{h})}}{z^{p/2}\sqrt{\det C}}\sum_{m_1,\ldots,m_p\in\mathbb{Z}} e^{-\frac{\pi}{z}\Phi(\mathbf{h}+\mathbf{m})-2\pi i(\mathbf{g}\cdot\mathbf{m})}, \tag{C.3}$$

where $\Phi(\mathbf{g}) = \mathbf{g}^T C^{-1}\mathbf{g}$. Substituting $z\to 1/z$ in the first integral and then adding the two integrals we obtain

$$\begin{aligned}&(\det C)^{1/4}e^{i\pi\mathbf{g}\cdot\mathbf{h}}\ \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta_\phi[\mathbf{g},\mathbf{h}](s)\\ &= (\det C)^{1/4}e^{i\pi\mathbf{g}\cdot\mathbf{h}}\int_1^{+\infty}\frac{dz}{z}\ z^{\frac{s}{2}}\Theta_\phi[\mathbf{g},\mathbf{h}](z)\ +\ \frac{e^{-i\pi\mathbf{g}\cdot\mathbf{h}}}{(\det C)^{1/4}}\int_1^{+\infty}\frac{dz}{z}\ z^{\frac{p-s}{2}}\Theta_\Phi[\mathbf{h},-\mathbf{g}](z).\end{aligned} \tag{C.4}$$

This last equation, since that $\Theta_\phi[-\mathbf{g},-\mathbf{h}](z) = \Theta_\phi[\mathbf{g},\mathbf{h}](z)$, is left invariant if we make the following substitutions

$$s\longrightarrow p-s, \qquad [\mathbf{g},\mathbf{h}]\longrightarrow[\mathbf{h},-\mathbf{g}], \qquad C\longrightarrow C^{-1}. \tag{C.5}$$

The analytic continuation of the ζ_ϕ function for $s < p-2$ is then obtained:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta_\phi[\mathbf{g},\mathbf{h}](s) = \frac{e^{-2\pi i\mathbf{g}\cdot\mathbf{h}}}{\sqrt{\det C}}\pi^{-\frac{p-s}{2}}\Gamma\left(\frac{p-s}{2}\right)\zeta_{\Phi=\phi^{-1}}[\mathbf{h},-\mathbf{g}](p-s). \tag{C.6}$$

We are interested to $\mathbf{h}=0$ and $\mathbf{g}=(\alpha_1,\ldots,\alpha_d)$ with $\alpha_\mu = 0, 1/2$ for $\mu = 1,\cdots,d$. In this case, in order to avoid divergences on the generalized Zeta function (C.1), we have to exclude the zero modes $\mathbf{m}=(0,\ldots,0)$ on the r.h.s of (C.6); is possible to show that the analytic continuations (C.6) is still valid.

Application for $d=2$ and $s=-1$

In the two formulas for the partition function (4.41) and (4.42) we have to regularized the divergent vacuum energy

$$\begin{aligned}\sum_{\mathbf{k}} k_0 &= 2\pi\sum_{n_1,n_2}|(n_1+\alpha_1)\mathbf{k_1}+(n_2+\alpha_2)\mathbf{k_2}| \qquad \text{(C.7)}\\ &= 2\pi\sum_{n_1,n_2}\left[((n_1+\alpha_1)k_{11}+(n_2+\alpha_2)k_{21})^2+((n_1+\alpha_1)k_{12}+(n_2+\alpha_2)k_{22})^2\right]^{1/2}\\ &= 2\pi\sum_{n_1,n_2}\left[(\boldsymbol{n}+\boldsymbol{\alpha})^T kk^T(\boldsymbol{n}+\boldsymbol{\alpha})\right]^{1/2} = 2\pi\sum_{n_1,n_2}\left[(\boldsymbol{n}+\boldsymbol{\alpha})^T C(\boldsymbol{n}+\boldsymbol{\alpha})\right]^{1/2}\\ &= 2\pi\sum_{n_1,n_2}\left(\phi[\boldsymbol{n}+\boldsymbol{\alpha}]\right)^{1/2},\end{aligned}$$

where $C=kk^T$ and k is the spatial sub-matrix of the general $\mathbf{k}$ matrix. Using the analytic continuation formula (C.6) with $\mathbf{h}=0$, $\mathbf{g}=(\alpha_1,\alpha_2)$, $p=2$, $s=-1$ and $\det C = (\det k)^2$, we obtain the following relation

$$2\pi\sum_{\mathbf{k}} k_0 = -\frac{1}{2\pi\det k}\sum_{n_1,n_2}{}' \frac{\mathrm{e}^{(-2\pi i\boldsymbol{\alpha}\cdot\mathbf{n})}}{|n_1\mathbf{k_1}'+n_2\mathbf{k_2}'|^3}, \tag{C.8}$$

where in the sum $\sum'$ are excluded the values $(n_1,n_2)=(0,0)$. It is easy to show that the components of vectors $\mathbf{k_1}'$ and $\mathbf{k_2}'$ are related to those of $\mathbf{k_1}$ and $\mathbf{k_2}$ through the following matrix relation

$$\begin{pmatrix}\mathbf{k_1}'\\ \mathbf{k_2}'\end{pmatrix} = \begin{pmatrix}k'_{11} & k'_{12}\\ k'_{21} & k'_{22}\end{pmatrix} = \frac{1}{\det k}\begin{pmatrix}k_{22} & -k_{21}\\ -k_{12} & k_{11}\end{pmatrix}. \tag{C.9}$$

Substituting these components in (C.8) and relabelling $(n_1\to n_2,\ n_2\to -n_1)$ we obtain

$$2\pi\sum_{\mathbf{k}} k_0 = -\frac{(\det k)^2}{2\pi}\sum_{n_1,n_2}{}' \frac{\mathrm{e}^{-2\pi i(\alpha_1 n_2-\alpha_2 n_1)}}{|n_1\mathbf{k_1}+n_2\mathbf{k_2}|^3}. \tag{C.10}$$

If now we transpose the k's in terms of the ω's using the relations (4.31), we obtain the desired result in (4.47)

$$2\pi\sum_{\mathbf{k}} k_0 = -V^{(3)}F_0 = -\frac{V^{(3)}}{2\pi}\sum_{n_1,n_2}{}' \frac{e^{-2\pi i(\alpha_2 n_1-\alpha_1 n_2)}}{|n_1\boldsymbol{\omega}_2-n_2\boldsymbol{\omega}_1|^3}. \tag{C.11}$$

C.2 Regularization through the Mellin transform

The partition function (4.67) defined in the special geometry (4.66) possesses the following divergent exponent

$$2\pi r_{01}\sum_{n}\sqrt{[(n+\alpha_1)+\beta(n_2+\alpha_2)]^2+[r_{12}(n_2+\alpha_2)]^2}=2\pi r_{01}\sum_{n}|n+\zeta|\,, \tag{C.12}$$

where we introduced the complex variable $\zeta = \alpha_1 + \beta(n_2 + \alpha_2) + ir_{12}(n_2 + \alpha_2)$. Let us consider the following Mellin transform

$$\frac{1}{2}\sum_{n}\frac{1}{|n+\zeta|^{2s}}=\frac{\sqrt{\pi}}{2\Gamma(s)}\sum_{l\in\mathbb{Z}}\int_0^{+\infty}dt\;t^{s-\frac{3}{2}}e^{-t(\mathrm{Im}(\zeta))^2+\frac{\pi^2l^2}{t}+2\pi il\mathrm{Re}(\zeta)}. \tag{C.13}$$

Note that for $s = -1/2$ we recover the divergent sum in (C.12). Separating the mode $l = 0$ from those with $l \neq 0$ and making the substitution $t \to \pi^2/t$ in the latter case, we obtain

$$\begin{aligned}\frac{1}{2}\sum_{n}\frac{1}{|n+\zeta|^{2s}}&=\frac{\sqrt{\pi}}{2\Gamma(s)}(\mathrm{Im}(\zeta))^{1-2s}\Gamma(s-1/2)+\\&\quad+\frac{\pi^{-1/2+2s}}{\Gamma(s)}\sum_{l>0}\int_0^{+\infty}dt\;t^{-(s+1/2)}e^{-\frac{\pi^2}{t}(\mathrm{Im}(\zeta))^2+tl^2}\cos(2\pi l\mathrm{Re}(\zeta)).\end{aligned} \tag{C.14}$$

Since $\Gamma(x)$ has a pole in $x = -1$, the first term on the right hand side is divergent for $s = -1/2$. Our regularization procedure is to remove the divergent term, obtaining

$$\frac{1}{2}\sum_{n}|n+\zeta|=-\frac{1}{2\pi^2}\sum_{l>0}\int_0^{+\infty}dt\;e^{-\frac{\pi^2}{t}(\mathrm{Im}(\zeta))^2+tl^2}\cos(2\pi l\mathrm{Re}(\zeta)). \tag{C.15}$$

Introducing the Δ function

$$\Delta(m;a)=-\frac{1}{2\pi^2}\sum_{l>0}\int_0^{+\infty}dt\;e^{-\frac{\pi^2m^2}{t}-tl^2}\cos(2\pi la), \tag{C.16}$$

we arrived to the desired result for the regularized expression (C.12)

$$\begin{aligned}&2\pi r_{01}\sum_{n_1}\sqrt{[(n_1+\alpha_1)+\beta(n_2+\alpha_2)]^2+[r_{12}(n_2+\alpha_2)]^2}\\&=4\pi r_{01}\Delta[r_{12}(n_2+\alpha_2);\alpha_1+\beta(n_2+\alpha_2)].\end{aligned} \tag{C.17}$$

Appendix D

Massive Θ functions

The massive Θ functions are defined by the following expression [104] [109]

$$\Theta_{[a,b]}(\tau;m) = \prod_{n\in\mathbb{Z}} \left|1 - \exp\left[-2\pi\mathrm{Im}(\tau)\sqrt{(n+a)^2+m^2} + 2\pi i\mathrm{Re}(\tau)(n+a) + 2\pi i b\right]\right|^2 \times \\ \times \exp\left[4\pi\mathrm{Im}(\tau)\Delta(m;a)\right], \tag{D.1}$$

where $a, b, m \in \mathbb{R}$ and $\tau \in \mathbb{C}$; taking the logarithm we have

$$\log\Theta_{[a,b]}(\tau,m) = \sum_n \log\left[1 - \exp\left\{-2\pi\tau_2\sqrt{(n+a)^2+m^2} + 2\pi i\tau_1(n+a) + 2\pi i b\right\}\right] + c.c. \\ + 4\pi\tau_2\Delta(m,a). \tag{D.2}$$

Now we expand the logarithm as a Taylor series. We have

$$\log\Theta_{[a,b]}(\tau,m) = -\sum_n\sum_{p=1}^{+\infty}\left\{\frac{1}{p}\left[\exp\left\{-2\pi\tau_2\sqrt{(n+a)^2+m^2} + 2\pi i\tau_1(n+a) + 2\pi i b\right\}\right] + c.c.\right\} \\ + 4\pi\tau_2\Delta(m,a). \tag{D.3}$$

The following identity is used for the square-root at the exponent

$$e^{-z} = \frac{1}{\sqrt{\pi}}\int_0^{+\infty} ds\ s^{-1/2}e^{-s-\frac{z^2}{4s}}. \tag{D.4}$$

After the substitution $s \to p^2 s$, we obtain

$$\begin{aligned} &\log\Theta_{[a,b]}(\tau,m) \\ &= -\frac{1}{\sqrt{\pi}}\sum_n\sum_{p=1}^{+\infty}\int_0^{+\infty} ds\ \left\{\exp\left[-\pi\left(\frac{\pi\tau_2^2}{s}\right)n^2 + 2\pi i\left(\frac{i\pi\tau_2^2 a}{s} + \tau_1 p\right)n + \right.\right. \\ &\quad \left.\left. -\pi^2\frac{\tau_2^2p^2a^2}{s} - \frac{\pi^2\tau_1^2}{s}m^2 + 2\pi i p a\tau_1 + 2\pi i b p - p^2 s\right] + c.c.\right\} + 4\pi\tau_2\Delta(m,a). \end{aligned} \tag{D.5}$$

Applying on the sum over n the following Poisson formula

$$\sum_{n\in\mathbb{Z}} e^{-\pi A n^2+2\pi i B n} = \frac{1}{\sqrt{A}}\sum_{n\in\mathbb{Z}} e^{-\frac{\pi}{A}(n-B)^2}, \tag{D.6}$$

we have

$$\begin{aligned}
&\log\Theta_{[a,b]}(\tau,m)\\
&=-\frac{1}{\pi\tau_2}\sum_{n\in\mathbb{Z}}\sum_{p=1}^{+\infty}\int_0^{+\infty} ds\ \left\{\exp\left[-sp^2\frac{|\tau|^2}{\tau_2^2}-\frac{\pi^2\tau_2^2 m^2}{s}+2\pi ibp-\frac{sn^2}{\tau_2^2}+2nsp+\right.\right.\\
&\left.\left.+2\pi ina\frac{\tau_1}{\tau_2^2}\right]+c.c.\right\}+4\pi\tau_2\Delta(m,a).
\end{aligned} \tag{D.7}$$

Now we can split the sum over n in (D.7) in three intervals: $(n\in(-\infty-1], n=0, n\in[1,+\infty))$, and observe that the two intervals $(n\in(-\infty-1], n\in[1,+\infty))$ can be collect in a unique interval with $n\in[1,+\infty)$ but with the sum over p extended to $p\in(-\infty,+\infty)$, to which we must remove the case $p=0$. We have

$$\begin{aligned}
&\log\Theta_{[a,b]}(\tau,m)\\
&=-\frac{1}{\pi\tau_2}\sum_{n=1}^{+\infty}\sum_{p\in\mathbb{Z}}\int_0^{+\infty} ds\left\{\exp\left[-sp^2\frac{|\tau|^2}{\tau_2^2}-\frac{\pi^2\tau_2^2m^2}{s}+2\pi ibp-\frac{sn^2}{\tau_2^2}+2ns\frac{\tau_1}{\tau_2^2}p+2\pi ina\right]+c.c.\right\}\\
&+\frac{1}{\pi\tau_2}\sum_{n=1}^{+\infty}\int_0^{+\infty} ds\left\{\exp\left[-\frac{\pi^2\tau_2^2m^2}{s}-\frac{sn^2}{\tau_2^2}-2\pi ina\right]+c.c.\right\}\\
&-\frac{1}{\pi\tau_2}\sum_{p=1}^{+\infty}\int_0^{+\infty} ds\left\{\exp\left[-sp^2\frac{|\tau|^2}{\tau_2^2}-\frac{\pi^2\tau_2^2m^2}{s}+2\pi ibp\right]+c.c.\right\}+4\pi\tau_2\Delta(m,a).
\end{aligned} \tag{D.8}$$

Using the definition (C.16) of the Δ function, we can check that the second term cancels with the last one, while the third term gives $4\pi\tau_2/|\tau|^2\Delta(m|\tau|,b)$. Moreover, applying the Poisson formula (D.6) to the sum over p in the first integral and making the substitution $s\to sk^2/|\tau|^2$ we obtain

$$\begin{aligned}
&\log\Theta_{[a,b]}(\tau,m)\\
&=-\frac{1}{\sqrt{\pi}}\sum_{p\in\mathbb{Z}}\sum_{k=1}^{+\infty}\frac{1}{k}\int_0^{+\infty} ds\ s^{-1/2}\left\{\exp\left[-s-\frac{\pi^2k^2\tau_2^2}{s|\tau|^4}\left((p-b)^2+m^2|\tau|^2\right)+\right.\right.\\
&\left.\left.+2\pi ik\left(-a+(p-b)\frac{\tau_1}{|\tau|^2}\right)\right]+c.c.\right\}+4\pi\frac{\tau_2}{|\tau|^2}\Delta(m|\tau|,b).
\end{aligned} \tag{D.9}$$

Apply once again the identity (D.4) and reconstructing the Taylor expansion for the logarithm, we finally obtain the desired result (4.77) [104] [109]

$$\Theta_{[a,b]}(\tau,m)=\Theta_{[b,-a]}\left(-\frac{1}{\tau},m|\tau|\right). \tag{D.10}$$

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