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# TORIC EXTREMAL KÄHLER-RICCI SOLITONS ARE KÄHLER-EINSTEIN

SIMONE CALAMAI AND DAVID PETRECCA

**ABSTRACT.** In this short note, we prove that a Calabi extremal Kähler-Ricci soliton on a compact toric Kähler manifold is Einstein. This solves for the class of toric manifolds a general problem stated by the authors that they solved only under some curvature assumptions.

## INTRODUCTION

Let  $M^{2n}$  be a compact Kähler manifold and let  $\Omega \in H^{1,1}(M)$  be a Kähler class. In the attempt to identify “special” representatives of  $\Omega$ , several notions of “canonical” Kähler metrics have been introduced. A natural choice are of course Kähler-Einstein metrics, generalized by *extremal* metrics and *Kähler-Ricci solitons* (KRS). Extremal metrics are defined to be critical points of the *Calabi functional*

$$\omega \mapsto \int_M s_\omega^2 \omega^n$$

that maps the Kähler metric  $\omega$  to the  $L^2$ -norm of its scalar curvature. The Euler-Lagrange equation of the Calabi functional is

$$(1) \quad \text{grad}_\omega(s_\omega) \text{ is holomorphic.}$$

Kähler-Ricci solitons are Kähler metrics that satisfy the relation

$$(2) \quad \rho + c\omega = \mathcal{L}_X \omega$$

with their Ricci form  $\rho$ , for some vector field  $X$  that is holomorphic and, in the compact case, is the gradient of a smooth function  $f: M \rightarrow \mathbb{R}$ . The KRS equation forces  $\omega$  to lie in the class  $2\pi c_1(M)$ .

In [2] we addressed the problem whether the same  $\omega \in 2\pi c_1(M)$  can be extremal and a KRS without being Einstein and we proved the following.

**Theorem 1** ([2]). *A compact extremal KRS with positive holomorphic sectional curvature is Kähler-Einstein.*

Toric manifolds are compact Kähler  $2n$ -manifolds admitting an effective Hamiltonian action of an  $n$ -torus  $\mathbb{T}$  by Kähler automorphism. Although in an algebraic geometric context, Fulton calls them a “remarkably fertile testing ground for general theories” and, also from the Kähler geometric point of view, their richness of symmetries makes them a large park of examples.

As compact symplectic manifolds, they are characterized by the image of their moment map, that is a *Delzant polytope*, i.e. a convex polytope  $\Delta \subset \mathbb{R}^n$  with

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certain combinatoric properties. Given a compact symplectic toric manifold with moment image  $\Delta$ , all possible compatible complex structure are described by a single function, as we explain below.

The  $\mathbb{T}$ -invariant Kähler geometry on a dense subset is well described in the coordinates given by the moment map itself. In these coordinates, the extremal condition (1) has a particularly simple description, see e.g. [1].

Separately, it is known that every toric Fano manifold admits a KRS, see e.g. [4] and references therein where, in addition, Donaldson explains also the relation between the soliton field  $X$  and the Delzant polytope. The existence of extremal metrics in the toric setting is discussed in [1].

The purpose of this note is to prove the following result.

**Theorem 2.** *A compact toric Calabi-extremal Kähler-Ricci soliton is Kähler-Einstein.*

This solves the problem stated in [2] for the class of toric Kähler metrics, that can have holomorphic sectional curvature of any sign and so are not included in Theorem 1.

The proof of Theorem 2 is based on the combinatoric properties of Delzant polytopes and the boundary behavior of the Abreu potential. The problem in its full generality remains open.

**Problem.** *Prove that every extremal Kähler-Ricci soliton is Einstein or find a counterexample.*

Another class of manifold related to toric Kähler manifolds is given by toric bundles, where the existence of KRS has been studied in [5]. It would be interesting to apply the techniques of toric geometry from [1, 4] to study the existence of extremal or constant scalar curvature Kähler metrics in this class of manifolds and establish an analogue of Theorem 2.

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## 1. PROOF OF THEOREM 2

Let  $(M, g, \omega)$  be a toric Kähler manifold, with moment map  $\mu: M \rightarrow \Delta = \mu(M) \subset \mathbb{R}^n$ . The moment image can be written as

$$(3) \quad \Delta = \{x \in \mathbb{R}^n : \ell_k(x) \geq b_k, 1 \leq k \leq d\}$$

as intersection of the  $d$  half-spaces  $\{x \in \mathbb{R}^n : \ell_k(x) - b_k \geq 0\}$ .

The linear functions  $\ell_k$  are defined by  $\ell_k(x) = \langle u_k, x \rangle$ , where  $u_k$  is the normal to the facet  $\{\ell_k(x) = 0\} \cap \Delta$ . The combinatoric property of being Delzant implies the following.

**Lemma 1.1.** *Let  $\Delta$  be a Delzant polytope in  $\mathbb{R}^n$ . Then the vertices of  $\Delta$  cannot lie on the any affine hyperplane.*

*Proof.* Let  $P$  be a vertex of  $\Delta$ . By definition of Delzant polytope, the exactly  $n$  edges meeting at  $P$  are of the form  $tv_i$  for  $t \in [0, a_i]$  and the  $v_i$  can be taken to be a basis of  $\mathbb{Z}^n$ . Further  $n$  vertices are of the form  $P_i = a_i v_i$  and they cannot lie on the same affine hyperplane of  $\mathbb{R}^n$  as the  $v_i$  are linearly independent over  $\mathbb{R}$ .  $\square$

Given a compact toric symplectic manifold  $(M, \omega)$  with Delzant polytope  $\Delta$ , consider the dense subset

$$M^0 = \{p \in M : \text{the } \mathbb{T}\text{-action is free at } p\} \simeq \Delta^0 \times \mathbb{T},$$

where  $\Delta^0$  is the interior of  $\Delta$  and  $(x, y) \in \Delta^0 \times \mathbb{T}$  are the *symplectic coordinates*. In these coordinates, the  $\mathbb{T}$ -action is just the group multiplication on the second component. In particular,  $\mathbb{T}$ -invariant tensor fields on  $M^0$  depend only on  $x \in \Delta^0$ .

All  $\mathbb{T}$ -invariant complex structures compatible with  $\omega$  are determined by the *Abreu potential*, a function  $g: \Delta^0 \rightarrow \mathbb{R}$  given by

$$(4) \quad 2g(x) = \sum \ell_k(x) \log \ell_k(x) + h(x),$$

on the interior of  $\Delta$ , where the  $\ell_k$  are from (3) and  $h$  is a smooth function on  $\Delta$ .

In the  $(x, y)$ -coordinates, the symplectic form is the canonical  $\omega = dx_i \wedge dy_i$  and the Kähler metric corresponding  $g$  as in (4) it is  $g_{ij}(x) dx_i \cdot dx_j$ , where  $G = (g_{ij})$  is the (Euclidean) Hessian of  $g$ . The matrix  $G$  has to be singular on the boundary of  $\Delta$  in order for the metric to extend smoothly on the whole  $M$ . However, it is possible to describe the behavior of  $G$  on the vertices of  $\Delta$ .

**Lemma 1.2.** *The inverse of the Hessian matrix  $G$  vanishes at the vertices of  $\Delta$ .*

*Proof.* Without loss of generality, up to translations and to a transformation of  $\text{SL}(n, \mathbb{Z})$ , we can assume that 0 is a vertex and that the edges meeting there are the coordinate axes  $x_1, \dots, x_n$ .

The transformed polytope is then given by

$$\Delta = \bigcap_{i=1}^n \{x \in \mathbb{R}^n : x_i \geq 0\} \cap \bigcap_{i=n+1}^d \{x \in \mathbb{R}^n : \ell_k(x) \geq 0\}$$

and the linear functions  $\ell_k$  do not vanish at zero.

The Abreu potential  $g$  is given by

$$2g(x) = \sum_{i=1}^n x_i \log x_i + \underbrace{\sum_{i=n+1}^d \ell_i(x) \log \ell_i(x)}_{=: \tilde{h}(x)} + h(x)$$

and its Hessian matrix is

$$(5) \quad G_{ij} = \frac{\delta_{ij}}{x_j} + \tilde{h}_{,ij}(x)$$

where the function  $\tilde{h}_{,ij}$  is given by

$$(6) \quad \tilde{h}_{,ij} = \sum_{k=n+1}^d \frac{\ell_{k,i}(x) \ell_{k,j}(x)}{\ell_k(x)} + h_{,ij}.$$

From [1, Thm. 2.8], the determinant of  $G$  is given by

$$\frac{1}{\det G} = \delta(x) x_1 \cdots x_n \cdot \ell_{n+1}(x) \cdots \ell_d(x)$$

for some function  $\delta$  *strictly positive and smooth on the whole  $\Delta$* .

The entry  $g^{ij}$  of  $G^{-1}$  is given by

$$g^{ij}(x) = \frac{1}{\det G} \operatorname{cof}(G)_{ij}$$

where  $\operatorname{cof}(G)$  is the cofactor matrix of  $G$ .

The conclusion follows from the claim that

$$\operatorname{cof}(G)_{ij} = o\left(\frac{1}{x_1 \cdots x_n}\right).$$

From (5) one can see that, after eliminating the  $i$ -th row and the  $j$ -th column, the variables  $x_i$  and  $x_j$  can appear at the denominator only in the derivatives of  $\tilde{h}$ , but from (6) we see that their limit for  $x \rightarrow 0$  is finite, so the claim is true.  $\square$

Abreu's characterization [1] of toric extremal metrics relies on the fact that a  $\mathbb{T}$ -invariant function has a holomorphic gradient if, and only if, it is an affine function in the symplectic coordinates. We use this on the scalar curvature and on the potential  $f$  of  $X$ .

*Proof of Theorem 2.* From the preserved quantity (in our notation)

$$s + |\nabla f|^2 + 2f = \text{const}$$

that holds for every Ricci soliton, see e.g. [3], plus the extremal assumption, it follows that both  $f$  and  $|\nabla f|^2$  are affine functions in the interior of  $\Delta$ .

If  $f = a \cdot x$ , then one has that

$$|\nabla f|^2 = a^T G^{-1}(x) a$$

is an affine function as well. If we consider its extension to the whole  $\mathbb{R}^n$ , it is zero in all the vertices of  $\Delta$  by the Lemma 1.2. On the other hand, the zeros of a nonzero affine function is a proper affine hyperplane, so by Lemma 1.1 we can conclude that the length of  $X$  must be the zero function. So  $X = 0$  and the metric is Einstein.  $\square$

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