A NOTE ON HIERONYMI'S THEOREM: EVERY DEFINABLY COMPLETE STRUCTURE IS DEFINABLY BAIRE

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Abstract. We give an exposition and strengthening of P. Hieronymi's Theorem: if C is a nonempty closed set definable in a definably complete expansion of an ordered field, then C satisfies an analogue of Baire's Category Theorem.

1. Introduction

The real line is Dedekind Complete: every subset of $\mathbb{R}$ has a least upper bound in $\mathbb{R} \cup \{ \pm\infty \}$. The notion of being Dedekind Complete is clearly not first-order. People have been studying a weaker, but *first-order*, version of Dedekind Completeness: a structure $\mathbb{K}$ expanding an ordered field is **Definably Complete** (**DC**) if every *definable* subset of $\mathbb{K}$ has a least upper bound in $\mathbb{K} \cup \{ \pm\infty \}$.

Examples of DC structures are: all expansions of the real field, o-minimal structures, and ultra-products of DC structures.

DC structure were introduced in [Mil01], where it was further observed that definable completeness is equivalent to the intermediate value property for definable functions; it is also shown in [Mil01, Ser08, Fra08, DMS10, FS10, For11] that most results of elementary real analysis can be generalized to DC structures (see §2 for some examples). Several people have also proved definable versions of more difficult results: for instance, in [AF11] they transfered a theorem on Lipschitz functions by Kirszbraun and Helly, in [FS10, FS11] we considered Wilkie's and Speissegger's theorems on o-minimality of Pfaffian functions (see also [FS12] for a more expository version), while in [FH15] we considered Hieronymi's dichotomy theorem and Lebesgue's differentiation theorem for monotone functions (and some other results from measure theory).

On the other hand, not every first-order property of structures expanding the real field can be generalized to DC structures: for instance, [HP07] show that there exists a first-order sentence which true in any expansion of the real field but false in some o-minimal structures (see also [Ren14] for a related result).

In this note we will focus on a first-order version of Baire Category Theorem. Tamara Servi and I conjectured in [FS10] that every DC structure is definably Baire (see Definition 1.1). In [Hie13], Philipp Hieronymi proved our conjecture. The aim of this note is to give an alternative proof of Hieronymi's Theorem, together with a generalization of Hieronymi's and Kuratowski-Ulam's theorems to definable closed subsets of $\mathbb{K}^n$.

We recall the relevant definitions.

Definition 1.1 ([FS10])**.** Let $A \subseteq B \subseteq \mathbb{K}^n$ be definable sets.
A is nowhere dense in B if the closure of A has interior (inside B); otherwise, A is somewhere dense in B.
A is **definably meager** in B if there exists a definable increasing family $\big(Y_t : t \in \mathbb{K}\big)$

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of nowhere dense subsets of B, such that $A \subseteq \bigcup_t Y_t$).
A is **definably residual** in B if $B \setminus A$ is definably meager in B.
B is **definably Baire** if every nonempty open definable subset of B is not definably meager in B (or, equivalently, in itself).
A is an $\mathcal{F}_\sigma$ subset of B if there exists a definable increasing family $\big(Y_t : t \in \mathbb{K}\big)$ of closed subsets of B, such that $A = \bigcup_t Y_t$; $A \subseteq B$ is a $\mathcal{G}_\delta$ subset of B if $B \setminus A$ is an $\mathcal{F}_\sigma$ subset of B; if we don't specify the ambient space B, we mean that $B = \mathbb{K}^n$.

Fact 1.2 ([FS10, §3]). *A finite Boolean combination of closed definable subsets of $\mathbb{K}^n$ is $\mathcal{F}_\sigma$ in $\mathbb{K}^n$. Moreover, for every $n \in \mathbb{N}$, the family of $\mathcal{F}_\sigma$ subsets of $\mathbb{K}^n$ is closed under finite union and intersections. Besides, if $X \subseteq \mathbb{K}^n$ is $\mathcal{F}_\sigma$ and $g : \mathbb{K}^n \to \mathbb{K}^m$ is a definable continuous function, then $g(X)$ is also $\mathcal{F}_\sigma$.*

The main result of this note is the following.

Theorem 1.3 (Baire Category). *Let $C \subseteq \mathbb{K}^n$ be a nonempty $\mathcal{G}_\delta$ subset of $\mathbb{K}^n$. Then, C is definably Baire.*

Notice that the case when $C = \mathbb{K}^n$ in Theorem 1.3 is exactly Hieronymi's Theorem (see [Hie13]). We will prove Theorem 1.3 in §5.

We denote by $\Pi^{m+n}_m : \mathbb{K}^{n+m} \to \mathbb{K}^m$ the projection onto the first m coordinates and, given $C \subseteq \mathbb{K}^{n+m}$ and $\bar{x} \in \mathbb{K}^m$, by $C_{\bar{x}} := \{\, \bar{y} \in \mathbb{K}^n : \langle \bar{x}, y \rangle \in C \,\}$ the fiber of C at $\bar{x}$.

Theorem 1.4 (Kuratowski-Ulam). *Let $C \subseteq \mathbb{K}^{m+n}$ be definable and $E := \Pi^{n+m}_m(C)$. Let $F \subseteq C$ be a definable set. Let*

$$T := T^m_C(F) := \{\, \bar{x} \in E : F_{\bar{x}} \text{ is definably meager in } C_{\bar{x}} \,\}.$$

Assume that F is definable meager in C. If either F or C is $\mathcal{F}_\sigma$ in $\mathbb{K}^{m+n}$, then T contains some $T' \subseteq C$ such that T' is definable and definably residual in E.

Notice that the case when $C = \mathbb{K}^{m+n}$ in Theorem 1.4 is [FS10, Theorem 4.1]. We will prove Theorem 1.4 in §4.

As an application, we prove the following results.

Corollary 1.5. *Let C be a nonempty, closed, bounded, and definable subset of $\mathbb{K}^{m+n}$, and $A := \Pi^{n+m}_m(C)$. Define $f : A \to \mathbb{K}^n$, $f(\bar{x}) := \operatorname{lex\,min}(C_{\bar{x}})$. Let E be the set of $\bar{x} \in A$ such that either $\bar{x}$ is an isolated point of A, or f is continuous at $\bar{x}$. Then, E is definably residual in A, and therefore it is dense in A.*

Proof. By [DMS10, 1.9] (see Fact 2.9), E is definably residual in A. By Theorem 1.3, A is definably Baire, and every definably residual subset E of a definably Baire set A is dense in A. □

Corollary 1.6. *Let $F \subseteq C \subseteq \mathbb{K}^{m+n}$ be nonempty definable, closed subsets of $\mathbb{K}^{m+n}$. Let $E := \Pi^{n+m}_m(C)$. Assume that E is closed inside $\mathbb{K}^m$, and that the set*

$$T' := {T'}^m_C(F) := \{\, \bar{x} \in E : F_{\bar{x}} \text{ has no interior inside } C_{\bar{x}} \,\}$$

is not dense in E. Then, F has interior inside C.

Proof. By Theorem 1.3 (applied to each fiber $C_{\bar{x}}$), $T' = T^m_C(F)$. By Theorem 1.3 again (applied to the set E), T' is not definably residual inside E. Thus, by Theorem 1.4, F is not meager inside C; therefore F is somewhere dense inside C, and thus it has interior inside C. □

Question 1.7. What is the most general form of Theorem 1.4? For instance, can we drop the assumption that either F or C are $\mathcal{F}_\sigma$ subsets of $\mathbb{K}^{m+n}$? Can we prove that the set $T^m_C(F)$ is definable?

Definition 1.8 ([For11, §4])**.** A pseudo-$\mathbb{N}$ set is a set $\mathcal{N} \subset \mathbb{K}_{\geq 0}$, such that $\mathcal{N}$ is definable, closed, discrete, and unbounded.

A quasi-order $\langle D, \trianglelefteq \rangle$ is a forest if, for every $a \in D$, the set $\{ c \in D : c \trianglelefteq a \}$ is totally ordered by $\triangleleft$.

The following lemma is at the core of the proof: we hope it may be of independent interest.

Lemma 1.9 (Leftmost Branch)**.** *Let $\mathcal{N}$ be a pseudo-$\mathbb{N}$ set. Let $\trianglelefteq$ be a definable quasi-order of $\mathcal{N}$ (i.e., $\trianglelefteq$ is a reflexive and transitive binary relation on $\mathcal{N}$, whose graph is definable). Assume that $\trianglelefteq$ is a forest. Then, there exists a definable set $E_0 \subseteq \mathcal{N}$, such that:*

(a) the minimum of $\mathcal{N}$ is in E_0;
(b) for every $d \in E_0$, the successor of d in E_0 (if it exists) is

$$n(d) := \min\{ e \in \mathcal{N} : d < e \ \&\ d \triangleleft e \};$$

conversely, if $n(d)$ exists, then it is the successor of d in E_0.

Furthermore, E_0 is unique, satisfying the above conditions. Besides, $\leq$ and $\trianglelefteq$ coincide on E_0 (and, in particular, E_0 is linearly ordered by $\trianglelefteq$).

If moreover we have

(*) For every $d \in \mathcal{N}$ there exists $e \in \mathcal{N}$ such that $d \triangleleft e$,

then E_0 is unbounded (and hence cofinal in $\mathcal{N}$).

We call the set E_0 defined in the above lemma the leftmost branch of $\trianglelefteq$ (inside $\mathcal{N}$); notice $\mathcal{N}$ has a minimum, and every element of $\mathcal{N}$ has a successor in $\mathcal{N}$ (see [For11, §4]).

The proof of a particular case of the above lemma is given in [Hie13, Def. 12, Lemma 14, Def. 15, Lemma 16, Lemma 17], where $d \triangleleft e$ if "$f(e)$ extends $f(d)$" (in [Hie13] terminology). We will give a sketch of the proof in §3. P. Hieronymi pointed out a mistake in a previous version of these notes, when we did not require the condition that $\trianglelefteq$ is a forest in Lemma 1.9 (see §3.1).

Let $\mathcal{N}$ be a pseudo-$\mathbb{N}$ set. While it is quite clear how to prove statements about elements of $\mathcal{N}$ by (a kind of) induction (see [For11, Remark 4.15]), *a priori* it is not clear how to construct (definable) sets by recursion: Lemma 1.9 gives a way to produce a definable set E_0 whose definition is recursive; this will allow us to prove that $\mathbb{K}$ is definably Baire (see §5.1). However, to prove that a $\mathcal{G}_\delta$ set $C \subseteq \mathbb{K}^n$ is definably Baire we need to use a different method (since, for technical reason, the proof in §5.1 requires the assumption that C contains a dense pseudo-enumerable set, and we do not know if the assumption holds for C), that relies on Theorem 1.4, used inductively (see §5.2).

In [FH15] we gave a completely different proof of Hieronymi's Theorem, based on our Dichotomy Theorem: either $\mathbb{K}$ is "unrestrained" (i.e., $\mathbb{K}$ is, in a canonical way, a model of the first-order formulation of second-order arithmetic, and therefore any of the classical proof of Baire's Category Theorem generalize to $\mathbb{K}$), or $\mathbb{K}$ is "restrained" (and many "tameness" results from o-minimality hold in $\mathbb{K}$, allowing a relatively straightforward proof of Hieronymi's Theorem). When we are in the unrestrained situation, the same reasoning gives a proof of Theorem 1.3. However, when $\mathbb{K}$ is restrained, it was not clear how to prove Theorem 1.3 for $C \neq \mathbb{K}^n$.

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2. Preliminaries

Fact 2.1 ([FS10, Proposition 2.11])**.** *Let $U \subseteq \mathbb{K}^n$ be open and definable. U is definably meager in $\mathbb{K}^n$ iff U is definably meager in itself.*

Definition 2.2. A d-compact set is a definable, closed, and bounded subset of $\mathbb{K}^n$ (for some n).

The following fact will be used many times without mentioning it explicitly.

Fact 2.3 ([Mil01])**.** (1) *Let $X \subseteq \mathbb{K}^n$ be a d-compact set, and $f : X \to \mathbb{K}^m$ be a definable and continuous function. Then, $f(X)$ is d-compact.*
(2) *Let $\big(X(t) : t \in \mathbb{K}\big)$ be a definable decreasing family of nonempty d-compact subsets of $\mathbb{K}^n$. Then, $\bigcap_t X(t)$ is nonempty.*

Definition 2.4. Let $a \in \mathbb{K}^n$, $X \subseteq \mathbb{K}^n$, and $r > 0$. Define

$$\begin{aligned} B(a;r) &\coloneqq \{\, x \in \mathbb{K}^n : |x - a| < r \,\}; \\ \overline{B}(a;r) &\coloneqq \{\, x \in \mathbb{K}^n : |x - a| \leq r \,\}; \\ B_X(a;r) &\coloneqq X \cap B(a;r); \\ \overline{B}_X(a;r) &\coloneqq X \cap \overline{B}(a;r). \end{aligned}$$

Given $Y \subseteq X$, denote by $\mathrm{cl}_X(Y)$ (resp., $\mathrm{int}_X(Y)$) the topological closure (resp., the interior) of Y inside X, and denote $\mathrm{cl}(Y) \coloneqq \mathrm{cl}_{\mathbb{K}^n}(Y)$.

Remark 2.5. Let X be a topological space and $A \subseteq X$. A is somewhere dense in X iff there exists $V \neq \emptyset$ an open subset of X, such that, for every $W \neq \emptyset$ open subset of W, $W \cap A \neq \emptyset$.

Lemma 2.6. (1) *Let X be a topological space, U be a dense subset of X, and $A \subseteq X$ be any subset. T.f.a.e.:*
i) *A is nowhere dense in X;*
ii) *$A \cap U$ is nowhere dense in X;*
iii) *$A \cap U$ is nowhere dense in U.*
(2) *Let $X \subseteq \mathbb{K}^n$ be a definable, U be a dense open definable subset of X, and $A \subseteq X$ be any definable subset. T.f.a.e.:*
i) *A is definably meager in X;*
ii) *$A \cap U$ is definably meager in X;*
iii) *$A \cap U$ is definably meager in U.*
(3) *Let $X \subseteq \mathbb{K}^n$ be a definable set and $U \subseteq X$ be a definable dense open subset of X. Then, X is definably Baire iff U is definably Baire.*

Proof. (1) follows from Remark 2.5.
(2) follows from (1).
(3) follows from (2). □

Corollary 2.7. *Let $A \subseteq X \subseteq \mathbb{K}^n$ be definable nonempty sets. Assume that A is a dense subset of X.*
(1) *If A is definably Baire, then X is also definably Baire.*
(2) *If X is definably Baire and A is a $\mathcal{G}_\delta$ subset of X, then A is also definably Baire.*

Proof. (1) is clear from Lemma 2.6(1).

(2) Assume, for a contradiction, that A is not definably Baire. We can easily reduce to the case when A is definably meager in itself. Let $F \coloneqq X \setminus A$. By our assumption on A, F is an $\mathcal{F}_\sigma$ subset of X with empty interior; thus, F is definably meager in X. Since X is definably Baire and $X = A \cup F$, A is not definably meager in X. Since A is definably meager in itself, $A = \bigcup_t C_t$, for some definable increasing

family $\big(C_t : t \in \mathbb{K}\big)$ of nowhere dense subsets of A. By Lemma 2.6(1), each C_t is also nowhere dense in X, contradicting the fact that A is not definably meager in X. $\square$

Definition 2.8. Let $E \subseteq \mathbb{K}^m$ and $f : E \to \mathbb{K}^n$ be definable. Given $\varepsilon > 0$, define

$$\mathcal{D}(f;\varepsilon) := \{\, \bar{a} \in E : \forall \delta > 0\ f(B_E(\bar{a};\delta)) \not\subseteq B(f(\bar{a});\varepsilon) \,\}.$$

Fact 2.9 ([DMS10, 1.9])**.** *Let $C \subset \mathbb{K}^{m+n}$ be a nonempty d-compact set and $E := \Pi^{m+n}_m(C) \subset \mathbb{K}^m$. Define $f : E \to \mathbb{K}$, $f(x) := \operatorname{lex}\min(C_{\bar{x}})$. Then, for every $\varepsilon > 0$, $\mathcal{D}(f;\varepsilon)$ is nowhere dense in E.*

Conjecture 2.10. *Let X_1, X_2 be definable subsets of $\mathbb{K}^n$, and $X := X_1 \cup X_2$. If both X_1 and X_2 are definably Baire, then X is also definably Baire.*

3. Proof of Lemma 1.9

We will proceed by various reductions. Define

$$E := \{\, d \in \mathcal{N} : (\forall e \in \mathcal{N})\ e < d \to d \ntrianglelefteq e \,\}.$$

Let $e_0 := \min(\mathcal{N})$. Define

$$E_1 := \{\, d \in E : e_0 \trianglelefteq d \,\}.$$

Define E_2 as the set of elements $d \in E_1$, such that d is the minimum of the set $\{\, d' \in E_2 : d' \trianglelefteq d \ \&\ d \trianglelefteq d' \,\}$. Notice that E_2 satisfies the following conditions, for all $d, d' \in E_2$:

(i) $e_0 \in E_2$;
(ii) $e_0 \leq d$;
(iii) $n(d) \in E_2$;
(iv) $d \trianglelefteq d' \to d \leq d'$;
(v) $\trianglelefteq$ is a partial order on E_2;
(vi) $\langle E_2, \trianglelefteq \rangle$ is a forest.

For every $a, b \in E_2$, define $a \perp b$ if $a \ntrianglelefteq b$ and $b \ntrianglelefteq a$. Given $a, b \in E_2$ such that $a \perp b$, define $c(a,b) := \min\{\, a' \in E_2 : a' \trianglelefteq a \ \&\ a' \perp b \,\}$ (where the minimum is taken w.r.t. $\leq$). Notice that if $a \perp b$, $a' \trianglelefteq a$, and $a' \perp b$, then $c(a,b) \leq a'$ and, by (iv), either $c(a,b) \trianglelefteq a'$, or $c(a,b) \perp a'$; moreover, $c(a,b) \leq a$, $c(a,b) \trianglelefteq a$, and $c(a,b) \perp b$.

Finally, define E_0 as the set of $a \in E_2$, such that, for every $b \in E_2$, if $b \perp a$, then $c(a,b) < b$.

We have to show that E_2 is the leftmost branch of $\trianglelefteq$ inside $\mathcal{N}$. W.l.o.g., we can assume that $\mathcal{N} = E_2$.

Claim 1. For every $a \in E_0$, $n(a) \in E_0$.

Assume not. Let $a \in E_0$ be such that $b := n(a) \notin E_0$. Thus, by definition, there exists $d \in \mathcal{N}$ such that $c := c(b,d) \geq d$. If $d \trianglelefteq a$, then $d \trianglelefteq b$, absurd. If $a \trianglelefteq d$, then $d \geq b$ because $b = n(a)$, also absurd. If $d \perp a$, then, since $a \in \mathcal{N}$, we have that $c' := c(a,d) < d$. Moreover, $c' \trianglelefteq a \triangleleft b$ and $c' \perp d$; thus, by definition, $c \leq c'$, and therefore $c' > d$, absurd.

The next claim is the only place where we use the fact that $\mathcal{N}$ is a forest.

Claim 2. Let $b \in E_0$ and $a \in \mathcal{N}$ with $a \trianglelefteq b$. Then, $a \in E_0$.

Assume not. Let $d \in \mathcal{N}$ such that $d \perp a$ and $c := c(a,d) \geq d$. Since $c \leq a$ and $a \neq d$, we have $d < a$, and therefore $d < b$. If $d \trianglelefteq b$, then, since $\langle \mathcal{N}, \trianglelefteq \rangle$ is a forest, we have $d \trianglelefteq a$, absurd. Thus, we have $d \perp b$. Since $b \in E_0$, we have $c' := c(b,d) < d$. Moreover, since $c \perp d$ and $c \trianglelefteq b$, the definition of c' implies $c' \leq c$, and therefore, since $\langle \mathcal{N}, \trianglelefteq \rangle$ is a forest, $c' \trianglelefteq c$. Conversely, since $c' \trianglelefteq a$ and $c' \perp d$, the definition of c implies $c \leq c'$, and therefore $c = c' < d$, absurd.

Claim 3. $\leq$ and $\trianglelefteq$ coincide on E_0.

Assume not. By (iv), there exist $a, b \in E_0$, such that $a \perp b$; let $a \in E_0$ be minimal such there exists $b \in E_0$ with $a \perp b$. Let $c \coloneqq c(b, a)$. By Claim 2, since $c \trianglelefteq b$, we have $c \in E_0$. Since $b \in E_0$, we have $c < a$, contradicting the minimality of a. □

3.1. **A counterexample: the forest hypothesis is necessary.** The following counterexample is due to P. Hieronymi.([1]) We show that the conclusion of Lemma 1.9 may fail if we remove the assumption that $\langle \mathcal{N}, \trianglelefteq \rangle$ is a forest, even under the assumption that $\mathbb{K}$ expands the reals and satisfies strong "tameness" condition (i.e., d-minimality).

Let $P \coloneqq \{ 2^{2^n} : n \in \mathbb{N} \}$ and let $\mathbb{K} \coloneqq \langle \mathbb{R}, P \rangle$ be the expansion of the real field by a predicate for P; by [MT06], $\mathbb{K}$ is d-minimal. It is quite clear that P is a pseudo-$\mathbb{N}$ set. We now define a partial ordering on P. For every $a \in P$, denote by $s(a)$ the successor of a in P, i.e. $s(a) = a^2$. Let $a, b \in P$; define $a \trianglelefteq b$ iff either $a = b$ or $b > s(a)$ (that is, if $b \geq a^4$). It is clear that $\langle P, \trianglelefteq \rangle$ is a partially ordered set, satisfying (*), and that the leftmost branch Q of $\langle P, \trianglelefteq \rangle$ is the set of elements with even index, i.e. $Q \coloneqq \{ 2^{4^n} : n \in \mathbb{N} \}$. However, the set Q is not definable in $\mathbb{K}$: see [For12, Lemma 2.2] and [MT06] for the details.

4. Proof of Theorem 1.4

Lemma 4.1. *Let $m, n \in \mathbb{N}_{\geq 1}$. Let $\pi \coloneqq \Pi^{m+n}_m$. Let $C \subseteq \mathbb{K}^{m+n}$ be definable and $E \coloneqq \pi(C)$. Let $F \subseteq C$ be a d-compact definable set. Define*

$$T' \coloneqq {T'}^m_C(F) \coloneqq \{ \bar{x} \in E : \operatorname{int}_{C_{\bar{x}}}(F_{\bar{x}}) = \emptyset \}.$$

If $\operatorname{int}_C(F) = \emptyset$, then T' is definably residual in E.

Proof. The proof of the Lemma is similar to [FS10, §4, Case 1]. Fix $\varepsilon > 0$; define

$$F(\varepsilon) \coloneqq \{ \langle \bar{x}, \bar{y} \rangle \in F : B_C(\bar{y}; \varepsilon) \subseteq F_{\bar{x}} \};$$
$$X(\varepsilon) \coloneqq \operatorname{cl}_C(F(\varepsilon)) = \operatorname{cl}_F(F(\varepsilon));$$
$$Y(\varepsilon) \coloneqq \pi(X(\varepsilon)) \subseteq E.$$

Since $E \setminus T' \subseteq \bigcup_{\varepsilon > 0} Y(\varepsilon)$, we only have to prove the following:

Claim 4. $Y(\varepsilon)$ is nowhere dense in E.

Since F is d-compact and $X(\varepsilon)$ is closed in F, $X(\varepsilon)$ is d-compact. Since $Y(\varepsilon) = \pi(X(\varepsilon))$, $Y(\varepsilon)$ is also d-compact, and therefore it is closed in E. Assume, for a contradiction, that $Y(\varepsilon)$ is somewhere dense in A: thus, $U \coloneqq \operatorname{int}_E(Y(\varepsilon)) \neq \emptyset$. Define $f : U \to \mathbb{K}^n$, $\bar{x} \mapsto \operatorname{lex}\min(X(\varepsilon)_{\bar{x}})$; notice that $\Gamma(f) \subseteq X(\varepsilon)$. By Fact 2.9, $\mathcal{D}(f; \varepsilon/4)$ is nowhere dense in U. Thus, there exist $\bar{a} \in U$ and $\delta > 0$ such that $B_E(\bar{a}, \delta) \subseteq U \setminus \operatorname{cl}_E(\mathcal{D}(f; \varepsilon/4))$, and $\delta < \varepsilon/4$. Let $\bar{b} \coloneqq f(\bar{a})$; thus, $\langle \bar{a}, \bar{b} \rangle \in \Gamma(f) \subseteq X(\varepsilon) \subseteq F$. The following Claim 5 contradicts the fact that F is nowhere dense in C, and therefore Claim 4 will follow.

Claim 5. $B_C(\langle \bar{a}, \bar{b} \rangle; \delta_1) \subseteq F$, for some $\delta_1 > 0$.

Choose $\delta_1 > 0$ such that $\delta_1 < \delta$ and $f(B_E(\bar{a}; \delta_1)) \subseteq B(\bar{b}; \delta)$ (δ_1 exists because $\bar{a} \notin \mathcal{D}(f; \varepsilon/4)$). Let $\langle \bar{x}, \bar{y} \rangle \in B_C(\langle \bar{a}, \bar{b} \rangle; \delta_1)$. Thus, $\bar{x} \in E$, $|\bar{x} - \bar{a}| < \delta_1$, $y \in C_{\bar{x}}$, and $|\bar{y} - \bar{b}| < \delta_1$. Therefore, $\bar{x} \in B_E(\bar{a}; \delta) \subseteq U \setminus \operatorname{cl}_A(\mathcal{D}(f; \varepsilon/4))$. Thus,

$$|\bar{y} - f(\bar{x})| \leq |\bar{y} - \bar{b}| + |\bar{b} - f(\bar{x})| \leq \delta_1 + \delta < 2\delta < \varepsilon,$$

and therefore $\bar{y} \in B_{C_{\bar{x}}}(f(\bar{x}); \varepsilon)$. Since $\langle \bar{x}, f(\bar{x}) \rangle \in X(\varepsilon)$, we have $B_{C_{\bar{x}}}(f(\bar{x}); \varepsilon) \subseteq F_{\bar{x}}$; thus, Claim 5 is proven.

([1]) Private communication.

Claim 6. $E \setminus T' \subseteq \bigcup_{\varepsilon 0} Y(\varepsilon)$.

Let $\bar{x} \in E \setminus T'$. Since $F_{\bar{x}}$ is d-compact, we have, $\operatorname{int}_{C_{\bar{x}}}(F_{\bar{x}}) \neq \emptyset$. Let $\bar{y} \in \operatorname{int}_{C_{\bar{x}}}(F_{\bar{x}})$ and $\varepsilon > 0$ such that $B_{C_{\bar{x}}}(\bar{y}; \varepsilon) \subseteq F_{\bar{x}}$. Thus, $\langle \bar{x}, \bar{y} \rangle \in F(\varepsilon) \subseteq X(\varepsilon)$, and $\bar{x} \in Y(\varepsilon)$.

Thus, by claims 4 and 6, T' is definably residual in E. □

Proof of Theorem 1.4. The proof of the Lemma is as in [FS10, §4, Case 2].

Case 1: F is $\mathcal{F}_\sigma$ in $\mathbb{K}^{m+n}$. Then, $F = \bigcup_{s>0} F(s)$, for some definable increasing family $\big(F(s) : s \in \mathbb{K}\big)$ of d-compact sets. By the proof of Lemma 4.1, for each $s \in \mathbb{K}$, $E \setminus T^m(F(s)) \subseteq \bigcup_{\varepsilon>0} Y(s, \varepsilon)$, where $Y(s, \varepsilon)$ is a definable family of nowhere dense subsets of E, which is increasing in t and decreasing in ε. Thus, $T^m(F)$ is definably residual in E.

Case 2: C is $\mathcal{F}_\sigma$ in $\mathbb{K}^{m+n}$. By definition, there exists $F' \subseteq C$, such that F' is a definably meager $\mathcal{F}_\sigma$ subset of C, and $F \subseteq F'$; thus, by replacing F with F', w.l.o.g. we can assume that F is $\mathcal{F}_\sigma$ in C. Then, since C is an $\mathcal{F}_\sigma$ set, F is $\mathcal{F}_\sigma$ also in $\mathbb{K}^{m+n}$, and we can apply Case 1. □

5. Proof of Theorem 1.3

Lemma 5.1. *Let $m \in \mathbb{N}$ and $C \subseteq \mathbb{K}^n$ be a definable nonempty set. Assume that, for every $a \in C$, there exists $U \subseteq C$, such that U is a definable neighborhood (in C) of a which is definably Baire. Then, C is definably Baire.*

Proof. Let $V \subseteq C$ be a definable open nonempty subset of C. Assume, for a contradiction, that V is definably meager in itself. Let $a \in V$ and let U be a definable neighborhood (in C) of a which is definably Baire. Let $W := \operatorname{int}_C(U \cap V)$. Notice that W is a nonempty open subset of C. Since V is definably meager in itself and W is an open subset of V, W is also definably meager in itself. Since W is open in U, W is meager in U. Since W is a nonempty open subset of U and U is definably Baire, W is not definably meager in U, absurd. □

Lemma 5.2. *Let $C \subseteq \mathbb{K}^n$ be definable, closed (in $\mathbb{K}^n$), and nonempty. If C is not definably Baire, then there exists $E \subseteq C$, such that E is definable, nonempty, d-compact, and definably meager in itself.*

Proof. By Lemma 5.1, there exists a d-compact B such that $C' := B \cap C$ is not definably Baire; thus, by replacing C with C', w.l.o.g. we can assume that C is d-compact. Let $U \subseteq C$ be definable, nonempty, and open in C, such that U is definably meager in itself. Let $E := \operatorname{cl}(U) = \operatorname{cl}_C(U)$. By assumption, U is an open and dense subset of E; thus, by Lemma 2.6(2) (applied to $A = X = E$), E is definably meager in itself. □

5.1. **The case** $n = 1$**.** The first step in the proof of Theorem 1.3 is the case when $m = 1$ and C is closed. Thus, we have to prove the following lemma.

Lemma 5.3. *Let $C \subseteq \mathbb{K}$ be definable, nonempty, and closed. Then, C is definably Baire.*

The remainder of this subsection is the proof of the above lemma.

Definition 5.4 ([For11, §4])**.** Let $C \subseteq \mathbb{K}^n$ be a definable set. C is at most pseudo-enumerable if there exists a pseudo-$\mathbb{N}$ set $\mathcal{N}$ and a definable surjective function $f : \mathcal{N} \to C$. C is pseudo-finite if it is closed, discrete, and bounded. C is pseudo-enumerable if it is at most pseudo-enumerable but not pseudo-finite. A family of sets $\big(C(t) : t \in T\big)$ is pseudo-enumerable (resp. pseudo-finite, resp. at most pseudo-enumerable) if it is a definable family and its index set T is pseudo-enumerable (resp. pseudo-finite, resp. at most pseudo-enumerable).

We need a few results on pseudo-enumerable sets and families.

Fact 5.5 ([For11, For13])**.** (1) *The union of two pseudo-enumerable (resp. pseudo-finite, resp. at most pseudo-enumerable) sets is pseudo-enumerable (resp. pseudo-finite, resp. at most pseudo-enumerable).*
(2) *Every definable discrete subset of $\mathbb{K}^n$ is at most pseudo-enumerable.*
(3) *The image of a pseudo-finite set under a definable function is also pseudo-finite.*
(4) *If $\big(C(t) : t \in T\big)$ is an pseudo-finite family of nowhere dense sets, then $\bigcup_{t\in T} C(t)$ is also nowhere dense.*

Lemma 5.6. *Let $C \subseteq \mathbb{K}$ be definable, nonempty, and closed. If C is definably meager in itself, then there exists a pseudo-enumerable set $P \subseteq C$, such that P is dense in C.*

Proof. Assume that C is definably meager in itself. Let $U \coloneqq \operatorname{int}_{\mathbb{K}}(C)$ and $E \coloneqq C \setminus U$. Notice that E is definable, closed, and nowhere dense in $\mathbb{K}$. If U is nonempty, then, since U is open in C and definably meager in itself, then, by [For11, Proposition 6.4], there is a pseudo-enumerable set $P_0 \subseteq U$, such that P_0 is dense in U. Let P_1 be the set of endpoints of $\mathbb{K} \setminus E$; notice that P_1 is at most pseudo-enumerable. Since E is nowhere dense in $\mathbb{K}$, P_1 is dense in E. Define $P \coloneqq P_0 \cup P_1$. Since P is the union of two at most pseudo-enumerable sets, P is also pseudo-enumerable pseudo-enumerable, and it is dense in C. □

Lemma 5.7. *Let $C \subseteq \mathbb{K}^n$ be definable, nonempty, and definably meager in itself. Then, C has no isolated points.*

Let $U \subseteq C$ be a nonempty definable open subset of C. Then, U is not pseudo-finite: that is, there is no discrete and d-compact subset D of $\mathbb{K}$, such that there is a definable surjective function $f : D \to U$.

Proof. Clear. □

Definition 5.8. Let $C \subseteq \mathbb{K}^n$ be a nonempty definable set. Let $\mathcal{A} \coloneqq \big(A_i : i \in I\big)$ be a definable family of subsets of C. We say that $\mathcal{A}$ is a **weak basis** for C if:
(1) for every $i \in I$, $\operatorname{int}_C(A_i) \neq \emptyset$;
(2) if $U \subseteq C$ is a nonempty open subset of C, then there exists $i \in I$ such that $A_i \subseteq U$.

Lemma 5.9. *Let $C \subseteq \mathbb{K}$ be definable, nonempty, and closed. Assume that C is definably meager in itself. Then, C has a pseudo-enumerable weak basis of d-compact sets.*

Proof. By Lemma 5.6, there exists $P \subseteq C$ which is pseudo-enumerable and dense in C; thus, we can write $P \coloneqq \{\, p_i : i \in \mathcal{N} \,\}$, for some pseudo-$\mathbb{N}$ set $\mathcal{N} \subset \mathbb{K}_{\geq 1}$ and some definable function $i \mapsto p_i$. For every $i \in \mathcal{N}$, define $A_i \coloneqq \overline{B}_C(p_i; 1/i)$. Let $\mathcal{A} \coloneqq \big(A_i : i \in \mathcal{N}\big)$. The lemma follows from the following claim.

Claim 7. Let $U \subseteq C$ be a definable open nonempty subset of C. Then, there exists $i \in \mathcal{N}$ such that $C_i \subseteq U$.

Choose $i_0 \in \mathcal{N}$ such that $q \coloneqq p_{i_0} \in U$. Choose $r > 0$ such that $B_C(q; 3r) \subseteq U$.

Claim 8. There exists $i \in \mathcal{N}$ such that $p_i \in B_C(q; r)$ and $i > 1/r$

We know that C has no isolated points. Let $F \coloneqq \{q\} \cup \{\, j \in \mathcal{N} : j \leq r \,\}$; notice that F is d-compact and discrete; thus, $V \coloneqq B_C(q; r) \setminus F$ is open in C, and, since C is not pseudo-finite, V is nonempty. Thus, by density, there exists $i \in \mathcal{N}$ such that $p_i \in V$, proving the claim.

Then,
$$\overline{B}_C(p_i; 1/i) \subseteq B_C(p_i; 2r) \subseteq B_C(q; 3r) \subseteq U.$$
□

We also need a choice function for open sets.

Lemma 5.10. (1) *Let $\bar{b}$ be a set of parameters and let $X \subseteq \mathbb{K}^n$ be a $\bar{b}$-definable set. Assume that X is nonempty and open. Then, there exists $a \in X$ which is $\bar{b}$-definable.*
(2) *Let $\big(X(t) : t \in T\big)$ be a definable family of subsets of $\mathbb{K}^n$. Assume that each $X(t)$ is nonempty and open. Then, there is a definable function $f : T \to \mathbb{K}$ such that, for every $t \in T$, $f(t) \in C(t)$.*

The above lemma remains true if we weaken the hypothesis from "X open" (or each $X(t)$ open) to "X constructible" (or each $X(t)$ constructible, i.e., a finite Boolean combination of open sets), see [For15]; however, the proof is more involved and we won't use the more general version.

Proof. (2) follows from (1) and standard compactness arguments. Thus, we only have to show (1). It is trivial to see that it suffices to do the case when $n = 1$, and therefore we will assume that $n = 1$.

W.l.o.g., we may assume that X is bounded. For every $r > 0$, let $U(r) := \{ x \in X : \overline{B}(x; r) \subseteq X \}$; since X is open, $X = \bigcup_{r>0} U(r)$. Define $r_0 := \inf\{ r > 0 : U(r) \neq \emptyset \}$. Notice that $U(\frac{1}{2} r_0)$ is $\bar{b}$-definable, d-compact, nonempty, and contained in X; thus, it has a minimum element a, which is therefore $\bar{b}$-definable and in X. □

We now turn to the proof of Lemma 5.3 proper. Assume, for a contradiction, that $C \subseteq \mathbb{K}$ is nonempty, definable, closed, but it is not definably Baire. Let $E \subseteq C$ be as in Lemma 5.2. By replacing C with E, w.l.o.g. we can also assume that C is also d-compact and definably meager in itself.

By Lemma 5.9, there exists a pseudo-$\mathbb{N}$ set $\mathcal{N} \subset \mathbb{K}_{\geq 1}$ and a definable family $\mathcal{A} := \big(A_i : i \in \mathcal{N}\big)$, such that $\mathcal{A}$ is a weak basis for C of d-compact sets. Moreover, since C is definably meager in itself, there exists a definable decreasing family $\big(U_j : j \in \mathcal{N}\big)$, such that each U_j is a dense open subset of C, and $\bigcap_j U_j = \emptyset$.

Since each U_d is open and dense, and A_d has nonempty interiors, $A_d \cap U_d$ has nonempty interior. Since $\mathcal{A}$ is a weak basis, there exists $e \in \mathcal{N}$ such that $A_e \subseteq U_d \cap A_d$; since moreover C has no isolated points, we can find e as above such that $e \geq d$. For every $d \in \mathcal{N}$, let $g(d)$ be the minimum element of $\mathcal{N}$, such that $g(d) \geq d$ and $A_{g(d)} \subseteq U_d \cap A_d$. Notice that $\mathcal{A}' := \big(A_{g(d)} : d \in \mathcal{N}\big)$ is also a weak basis; thus, by replacing $\mathcal{A}$ with $\mathcal{A}'$ (and each A_d with $A_{g(d)}$), we can assume that $A_d \subseteq U_d$ for every $d \in \mathcal{N}$.

For every $d \in \mathcal{N}$, notice that $E_d := \bigcup_{e \leq d, e \in \mathcal{N}} \operatorname{bd}(A_d)$ is a pseudo-finite union of closed nowhere dense subsets of C; thus, $A_d \setminus E_d$ is non-empty and open, and therefore, by Lemma 5.10, there is a definable function $f : \mathcal{N} \to \mathbb{K}$ such that $f(d) \in A_d \setminus E_d$ for every $d \in \mathcal{N}$.

For every $a \in \mathcal{N}$, define

$$T(a) := \{ d \in \mathcal{N} : d \leq a \ \&\ f(a) \in A_d \}.$$

Notice that each $T(a)$ is a pseudo-finite set, and that $a = \max(T(a))$.

We now define the following partial order on $\mathcal{N}$: $a \trianglelefteq b$ if $T(a)$ is an initial segment of $T(b)$, that is:

$$\forall c \leq a \quad c \in T(a) \leftrightarrow c \in T(b).$$

Lemma 5.11. $\big(\mathcal{N}, \trianglelefteq\big)$ *is a partially ordered set, which is a forest and satisfies condition (*) in Lemma 1.9.*

Proof. $a \trianglelefteq a$ by definition.

Notice that $a \trianglelefteq b$ implies $a \leq b$, by definition. Moreover, if $a \trianglelefteq b$, then $a \in T(b)$, since $a \in T(a)$.

Claim 9. If $a \trianglelefteq b$ and $b \trianglelefteq c$, then $a \trianglelefteq c$.

In fact, let $d \leq a$. Then, $d \in T(a)$ iff $d \in T(b)$ iff $d \in T(c)$.
Thus, $\trianglelefteq$ is a partial order.
The fact that $\langle \mathcal{N}, \trianglelefteq \rangle$ is a forest is clear.
We now prove that $\trianglelefteq$ satisfies (*). Let $a \in \mathcal{N}$ and $b \leq a$. For every $a \in \mathcal{N}$, let

$$J(a) \coloneqq \bigcap_{b \in T(a)} \operatorname{int} A_b \setminus \bigcup_{b \leq a \ \& \ b \notin T(a)} A_b.$$

Since each A_b is closed, and the set $\{ b \in \mathcal{N} : b \leq a \ \& \ b \notin T(a) \}$ is pseudo-finite, we have that $J(a)$ is an open set. We claim that $J(a)$ is nonempty. It suffices to prove the following claim.

Claim 10. $f(a) \in J(a)$.

In fact, by our choice of f, we have that, for every $b \leq a$, $b \in T(a)$ iff $f(a) \in A_b$ iff $f(a) \in \operatorname{int} A_b$; the claim is then obvious from the definition of $J(a)$.

Since C has no isolated points, the set $\{ f(b) : b \leq a \}$ is pseudo-finite, and $J(a)$ is open and nonempty, the set $J'(a) \coloneqq J(a) \setminus \{ f(b) : b \leq a \}$ is also open and nonempty; thus, there exists $b \in \mathcal{N}$ such that $A_b \subseteq J'(a)$. The lemma then follows from the following claim.

Claim 11. $a \triangleleft b$.

The fact that $b > a$ is clear from the fact that $f(c) \notin A_b$ for every $c \leq a$.

Let $c \leq a$. We have to show that $c \in T(b)$ iff $c \in T(a)$. If $c \in T(a)$, then $J(a) \subseteq \operatorname{int} A_c$, therefore $f(b) \in \operatorname{int}(A_c)$, and thus $c \in T(b)$. Conversely, if $c \in T(b)$, then $f(b) \in \operatorname{int} A_c \cap J(a)$; thus, $J(a) \cap \operatorname{int} A_c \neq \emptyset$, and therefore, by definition of $J(a)$, we have $c \in T(a)$. □

We now continue the proof of Lemma 5.3. By Lemma 5.11, we can apply Lemma 1.9 to the partial order $\trianglelefteq$: denote by E_0 the leftmost branch of $\trianglelefteq$ inside $\mathcal{N}$.

For every $a \in E_0$, let $F_a \coloneqq \bigcap_{d \in E_0 \ \& \ d \leq a} A_d$. Then, $f(a) \in F_s$, the family $\big(F_a : a \in E_0\big)$ is a definable decreasing family of d-compact nonempty sets. Therefore, by Fact 2.3

$$\emptyset \neq \bigcap_{a \in E_0} F_a = \bigcap_{d \in E_0} A_d \subseteq \bigcap_{d \in E_0} U_d = \bigcap_{d \in \mathcal{N}} U_d = \emptyset,$$

absurd. □

5.2. **The inductive step.**

Lemma 5.12. *Let $m \in \mathbb{N}$. Let $C \subseteq \mathbb{K}^m$ be nonempty and d-compact. Then, C is definably Baire.*

Proof. We will prove the lemma by induction on m.

Let $1 \leq m \in \mathbb{N}$. We denote by $(5.12)_m$ the instantiation of Lemma 5.12 at m. Notice that $(5.12)_1$ follows from Lemma 5.3. Thus, we assume that we have already proven $(5.12)_m$ and $(5.12)_1$; we need to prove $(5.12)_{m+1}$.

Let $C \subseteq \mathbb{K}^{m+1}$ be d-compact and nonempty. We have to show that C is definably Baire. Assume not. Let F be a definable nonempty open subset of C, such that F is definably meager in C. Define $\pi \coloneqq \Pi^{m+1}_m$, and $E \coloneqq \pi(C)$. By Theorem 1.4, the set $S \coloneqq \{ \bar{x} \in E : F_{\bar{x}} \text{ is not definably meager in } C_{\bar{x}} \}$ is definably meager in E. Since F is open in C, $F_{\bar{x}}$ is open in $C_{\bar{x}}$ for every $\bar{x} \in E$; thus, by $(5.12)_1$, $S = \pi(F)$. Notice that E is also d-compact and nonempty; thus, by $(5.13)_m$, E is definably Baire. Since moreover S is open in E, the fact that S is definably meager in E imply that S is empty, contradicting the fact that F is nonempty. □

Lemma 5.13. *Let $m \in \mathbb{N}$. Let $C \subseteq \mathbb{K}^m$ be closed, nonempty, and definable. Then, C is definably Baire.*

Proof. We want to apply Lemma 5.1; thus, given $a \in C$; it suffices to find $A \subseteq C$, such that A is a definable neighborhood of a inside C, and A is definably Baire. Fix $r > 0$ (e.g., $r = 1$); Let $A \coloneqq \overline{B}_C(a; r)$. It is clear that A is a definable neighborhood of a inside C. Moreover, A is d-compact; thus, by Lemma 5.12, A is definably Baire. □

Proof of Theorem 1.3. Let $Y \coloneqq \mathrm{cl}(C)$. By Lemma 5.13, Y is definably Baire. Since C is dense in Y, the conclusion follows from Corollary 2.7. □

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