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to the memory of Jim Wiegold

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Abstract

A group \(G\) is a \(cn\)-group if for each subgroup \(H\) of \(G\) there exists a normal subgroup \(N\) of \(G\) such that the index \(|HN : (H \cap N)|\) is finite. The class of \(cn\)-groups contains properly both the well-known classes of core-finite groups and of finite-by-abelian groups. In the present paper it is shown that a \(cn\)-group whose periodic images are locally finite is finite-by-abelian-by-finite. Then such groups are described into some details by considering automorphisms of abelian groups. Finally, it is shown that if \(G\) is a locally graded group with the property that the above index is bounded independently of \(H\), then \(G\) is finite-by-abelian-by-finite. \(^1\)

1 Introduction

In a celebrated paper, B.H.Neumann [9] showed that for a group \(G\) the property that each subgroup \(H\) has finite index in a normal subgroup of \(G\) (i.e. \(|H^G : H|\) is finite) is equivalent to the fact that \(G\) has finite derived subgroup (\(G\) is finite-by-abelian).

The class of groups with a dual property was considered in [1]. A group \(G\) is said a \(cf\)-group (core-finite) if each subgroup \(H\) contains a normal subgroup of \(G\) with finite index in \(H\) (i.e. \(|H : H_G|\) is finite). As Tarski groups are \(CF\), a complete classification of \(cf\)-groups seems to be much difficult. However, in [1] and [11] it has been proved that a \(cf\)-group \(G\) whose periodic quotients are locally finite is abelian-by-finite and there exists an integer \(n\) such that \(|H : H_G| \leq n\) for all \(H \leq G\) (say that \(G\) is \(bcf\), boundedly \(cf\)).

\(^1\)Key words and phrases: locally finite, core-finite, subnormal, inert, \(cf\)-group.
2010 Mathematics Subject Classification: Primary 20F24, Secondary 20F18, 20F50, 20E15
Moreover a locally graded BCF-group is abelian-by-finite. Furthermore, an easy example of a metabelian (and even hypercentral) group which is CF but not BCF is given. It seems to be a still open question whether every locally graded CF-group is abelian-by-finite. Recall that a group is said abelian-by-finite if has an abelian subgroup with finite index and that a group is locally finite (locally graded, resp.) if each finitely generated subgroup is finite (has a proper subgroup with finite index, resp.).

With the aim of considering the above two classes in a common framework, recall that two subgroups $H$ and $K$ of a group $G$ are said commensurable if and only if $H \cap K$ has finite index in both $H$ and $K$. This is an equivalence relation and will be denoted by $\sim$. Clearly, if $H \sim K$, then $(H \cap L) \sim (K \cap L)$ and $HM \sim KM$ for each $L \leq G$ and $M \in G$.

In the present paper we consider the class of $cn$-groups, that is groups in which each subgroup is commensurable to a normal subgroup. Into details, for a subgroup $H$ of a group $G$ define $\delta_G(H)$ to be the minimum index $|HN : (H \cap N)|$ with $N \lhd G$. Then $G$ is a $cn$-group if and only if $\delta_G(H)$ is finite for all $H \leq G$. Clearly both finite-by-abelian and CF groups are CN. Moreover, the class of CN-groups is both subgroup and quotient closed.

Note that if a subgroup $H$ of a group $G$ is commensurable with a normal subgroup $N$, then $S := (H \cap N)_N$ has finite index in $H$. Thus the class of $cn$-groups is contained in the class of sbyf-groups, that is, groups in which each subgroup $H$ is subnormal-by-finite; that is to say that $H$ contains a subnormal subgroup $S$ of $G$ such that the index $|H : S|$ is finite. It is known that locally finite sbyf-groups are (locally nilpotent)-by-finite (see [3]) and nilpotent-by-Chernikov (see [6]).

Recall also that from results in [4] it follows that for an abelian-by-finite group properties $cn$ and $cf$ are equivalent. However, for each prime $p$ there is a nilpotent $p$-group with property $cn$ which is neither finite-by-abelian nor abelian-by-finite, see Proposition 2.2 below.

Our main result is the following.

**Theorem A** Let $G$ be a $cn$-group such that every periodic image of $G$ is locally finite. Then $G$ is finite-by-abelian-by-finite.
for which there is \( n \in \mathbb{N} \) such that \( \delta_G(H) \leq n \) for all \( H \leq G \). We will show the following theorem.

**Theorem B** Let \( G \) be a finite-by-abelian-by-finite group.

i) if \( G \) is \( \text{cn} \), then the \( \text{fc} \)-center of \( G \) has finite index and is finite-by-abelian;

ii) \( G \) is \( \text{cn} \) if and only if it is finite-by-\( \text{cf} \).

iii) \( G \) is \( \text{bcn} \) if and only if it is finite-by-\( \text{bcf} \).

It follows that that if the group \( G \) is periodic and finite-by-abelian-by-finite, then \( G \) is \( \text{bcn} \) if and only if it is \( \text{cn} \). Then we consider non-periodic finite-by-abelian-by-finite \( \text{bcf} \)- and \( \text{bcn} \)-groups by Proposition 4.4.

The more restrictive property \( \text{bcn} \) reveals fruitful when we consider the wider class of locally graded groups.

**Corollary** A locally graded \( \text{bcn} \)-group is finite-by-abelian-by-finite.

Our notation is mostly standard and we refer to [10].

## 2 Preliminaries

We point out a sufficient condition for a group to be \( \text{cn} \) (or even \( \text{bcn} \)).

**Proposition 2.1** Let \( G \) be a group with a normal series \( G_0 \leq G_1 \leq G \), where \( G_0 \) and \( G/G_1 \) have finite order \( m \) and \( n \) resp. If \( H \leq G \), then \( H \) is commensurable with \( H_1 := (H \cap G_1)G_0 \leq G_1 \) and \( \delta_G(H) \leq mn \cdot \delta_{G/G_0}(H_1/G_0) \).

In particular, if each subgroup of \( G_1/G_0 \) is commensurable with a normal subgroup of \( G/G_0 \), then \( G \) is a \( \text{cn} \)-group.

Now we give examples of non trivial \( \text{cn} \)-groups.

**Proposition 2.2** For each prime \( p \) there is a nilpotent \( p \)-group with property \( \text{bcn} \), which is not abelian-by-finite nor finite-by-abelian.

**Proof.** Consider a sequence \( P_n \) of isomorphic groups with order \( p^4 \) defined by \( P_n := \langle x_n, y_n \mid x_n^{p^3} = y_n^p = 1, x_n^{p^n} = x_n^{1+p^2} \rangle = \langle x_n \rangle \times \langle y_n \rangle \) where clearly \( P'_n = \langle x_n^{p^2} \rangle \) has order \( p \). Let \( P := \text{Dr}_{n \in \mathbb{N}} P_n \) and consider the automorphism \( \gamma \) of \( P \) such that \( x_n^{\gamma} = x_n^{1+p} \) and \( y_n^{\gamma} = y_n \), for each \( n \in \mathbb{N} \). Clearly, \( \gamma \) has order \( p^2 \), acts as the automorphism \( x \mapsto x^{1+p} \) on \( P/P' \) (which has exponent \( p^2 \)) and acts trivially on \( P' \) (which is elementary abelian). Finally let \( N := \langle x_0^{p^2} x_n^{p^2} \mid n \in \mathbb{N} \rangle \). Then \( N \) is a subgroup of \( P' \) with index \( p \). Thus the \( p \)-group
\( G := (P \rtimes \langle \gamma \rangle)/N \) is a bcn-group by Proposition 2.1 applied to the series \( P'/N \leq P/N \leq G \).

We have that \( G' \) is infinite, since for each \( n \) we have \( x^n = [x, \gamma] \in [P_n, \gamma] > P'_n \). Moreover, we have that \( gN \in Z(P/N) \) if and only if \( \forall i \ (g, P_i) \leq N \), and \( N \cap P_i = 1 \). Thus \( Z(P/N) = Z(P)/N \) where \( Z(P) = \text{Dr}_n \langle x^n \rangle \) has infinite index in \( P \).

If, by contradiction, \( G \) is abelian-by-finite, then there is an abelian normal subgroup \( A/N \) of \( P/N \) with finite index. Then for some \( m \in \mathbb{N} \) we have \( P = AF \), where \( F = \text{Dr}_{n<m} P_n \) is a finite normal subgroup of \( P \). Therefore \( P/N \) is center-by-finite, a contradiction. \( \square \)

3 Automorphisms of abelian groups

As in [4], for the action of a group \( \Gamma \) on a group \( A \), we consider the following properties:

- \( \forall H \leq A \ H = H^\Gamma \); (P)
- \( \forall H \leq A \ |H/H^\Gamma| < \infty \); (AP)
- \( \forall H \leq A \ |H^\Gamma/H| < \infty \); (BP)
- \( \forall H \leq A \ \exists K = K^\Gamma \leq A \) such that \( H \sim K \), (H, K are commensurable). (CP)

When \( P \) holds, one says that \( \Gamma \) acts on \( A \) by means of power automorphisms or that \( A \) is \( \Gamma \)-hamiltonian ([10],[1]). Recall that if \( \gamma \) is a power automorphism of an abelian \( p \)-group \( A \), then there exists a \( p \)-adic integer \( \alpha \) such that \( a^\gamma = a^\alpha \) for all \( a \in A \) (see [10] for details). Here \( a^\alpha \) stands for \( a^n \), where \( n \in \mathbb{N} \) is congruent to \( \alpha \) modulo the order of \( a \). On the other hand, a power automorphism of a non-periodic abelian group is either the identity or the inversion map.

Obviously both AP and BP imply CP. Moreover, these three properties are equivalent, provided \( A \) is abelian and \( \Gamma \) is finitely generated, while they are in fact different in the general case even when \( A \) and \( \Gamma \) are elementary abelian \( p \)-groups (see [4]). On the other hand, the properties AP and BP have previously characterized in [5] and [2] resp., as we are going to recall. To shorten statements we define a further property:

- \( \tilde{P} \): \( \Gamma \) has \( P \) on the factors of a \( \Gamma \)-series \( 1 \leq V \leq D \leq A \) where
  - \( i) \ V \) is free abelian with finite rank,
  - \( ii) \ D/V \) is divisible periodic with finite total rank,
  - \( iii) \ A/D \) is periodic and has finite \( p \)-exponent for each prime \( p \in \pi(D/V) \).

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Theorem 3.1 [5],[2] Let $\Gamma$ be group acting on an abelian group $A$. Then:

a) $\Gamma$ has $\text{AP}$ on $A$ if and only if there is a $\Gamma$-subgroup $A_1$ such that $A/A_1$ is finite and $\Gamma$ has either $P$ or $\tilde{P}$ on $A_1$.

b) $\Gamma$ has $\text{BP}$ on $A$ if and only if there is a $\Gamma$-subgroup $A_0$ such that $A_0$ is finite and $\Gamma$ has either $P$ or $\tilde{P}$ on $A/A_0$.

By next statement we give a caracteration of the property $\text{CP}$ along the same lines.

Theorem 3.2 Let $\Gamma$ be group acting on an abelian group $A$. Then:

c) $\Gamma$ has $\text{CP}$ on $A$ if and only if there are $\Gamma$-subgroups $A_0 \leq A_1 \leq A$ such that $A_0$ and $A/A_1$ are finite and $\Gamma$ has either $P$ or $\tilde{P}$ on $A_1/A_0$.

The proof of Theorem 3.2 is at the end of this section. Here we deduce a corollary.

Corollary 3.3 For a group $\Gamma$ acting on an abelian group $A$, the following are equivalent:

a) $\Gamma$ has $\text{AP}$ on $A/A_0$ for a finite $\Gamma$-subgroup $A_0$ of $A$,

b) $\Gamma$ has $\text{BP}$ on a finite index $\Gamma$-subgroup $A_1$ of $A$,

c) $\Gamma$ has $\text{CP}$ on $A$.

Let us recall some basic facts from [4] where inertial automorphisms of abelian groups have been introduced. These are automorphisms $\gamma$ of a group $G$ such that $H^\gamma \sim H$ for all $H \leq G$. Clearly, if $\Gamma$ has $\text{CP}$ on $G$ and $\gamma \in \Gamma$, then $\gamma$ is inertial.

Proposition 3.4 Let $\Gamma$ be group acting on a locally nilpotent periodic group $A$. Then $\Gamma$ has $\text{AP}$, $\text{BP}$, $\text{CP}$ resp. on $A$ if and only if $\Gamma$ has $\text{AP}$, $\text{BP}$, $\text{CP}$ resp. on finitely many primary components of $A$ and $P$ on all the other ones.

Lemma 3.5 Let $\Gamma$ be a group acting on an abelian group $A$. If $\Gamma$ has $\text{CP}$, then:

i) $\Gamma$ has $P$ on the maximum periodic divisible subgroup of $A$.

ii) if $A$ is torsion-free, then each $\gamma \in \Gamma$ acts by conjugation on $A$ by either the identity or the inversion map.

Now we prove some lemmas. In the first one we do not require that the group $A$ is abelian.
Lemma 3.6 Let $\Gamma$ be a group acting on a group $A$. If $\Gamma$ has $CP$, then $\Gamma$ has $BP$ on the subgroup $X := \{ a \in A \mid \langle a \rangle^\Gamma \text{ is finite} \}$ of $A$.

Proof. For any $H \leq X$ there is $K$ such that $H \sim K = K^\Gamma \leq A$. Then there is finite subgroup $F \leq X$ such that $H \leq KF$. Thus $H^\Gamma \leq KF^\Gamma$ and $|H^\Gamma : H| \leq |F^\Gamma| \cdot |HK : H|$ is finite. $\square$

Lemma 3.7 Let $\Gamma$ be a group acting on a $p$-group $A$ which is the direct product of cyclic groups. If $\Gamma$ has $CP$, then the following subgroup has finite index in $A$:

$$X := \{ a \in A \mid \langle a \rangle^\Gamma \text{ is finite} \}$$

Proof. Assume by contradiction that $A/X$ is infinite.

Let us see that, by elementary facts, there is a sequence $(a_n)$ of elements of $A$ such that
1) $\langle a_n \mid n \in \mathbb{N} \rangle = Dr_{n \in \mathbb{N}} \langle a_n \rangle$,
2) $A_I/A_I \cap X$ is infinite, for each infinite subset $I$ of $\mathbb{N}$, where $A_I := \langle a_i \mid i \in I \rangle$.

In fact, if $A/X$ has finite rank, it has a Prüfer subgroup $Q/X$. Let $Y$ be a countable subgroup such that $Q = YX$. By Kulikov Theorem (see [10]) $Y$ is the direct product of cyclic groups, so that we may choose elements $a_n \in Y$ such that $\langle a_n \mid n \in \mathbb{N} \rangle = Dr_{n \in \mathbb{N}} \langle a_n \rangle \leq Y$ and $|a_nX| < |a_{n+1}X|$. The claim holds. Similarly, if $A/X$ has infinite rank, we may consider its socle $S/X$ and consider a countable subgroup $Y$ such that $S = YX$. Then we may choose elements $a_n \in Y$ which are independent mod $X$ and generate their direct product as in (1).

We claim now that there are sequences of infinite subsets $I_n$, $J_n$ of $\mathbb{N}$ and $\Gamma$-subgroups $K_n \leq A$ such that for each $n \in \mathbb{N}$:
3) $I_n \cap J_n = \emptyset$ and $I_{n+1} \subseteq J_n$
4) $K_n \sim A_{I_n}$
5) $(K_1 \ldots K_i) \cap (A_{I_1} \ldots A_{I_i}) \leq (A_{I_1} \ldots A_{I_i})$, $\forall i \leq n$.

Proceed by induction on $n$. Choose an infinite subset $I_1$ of $\mathbb{N}$ such that $J_1 := \mathbb{N} \setminus I_1$ is infinite. By $CP$-property there exists $K_1 = K_1^\Gamma$ commensurable with $A_{I_1}$.

Suppose we have defined $I_j$, $J_j$ $K_j$ for $1 \leq j \leq n$ such that 3-5 holds. Since $(K_1 \ldots K_n) \sim (A_{I_1} \ldots A_{I_n})$, there is $m \in \mathbb{N}$ such that
6) $(K_1K_2 \ldots K_n) \cap A_N \leq (A_{I_1}A_{I_2} \ldots A_{I_n})\langle a_1, \ldots, a_m \rangle$.

Let $I_{n+1}$ and $J_{n+1}$ be disjoint infinite subsets of $J_n \setminus \{1, \ldots, m\}$. By $CP$-property there exists $K_{n+1} = K_{n+1}^\Gamma$ commensurable with $A_{I_{n+1}}$. By the choice
of $I_{n+1}$ it follows that

7) \((K_1 \ldots K_i) \cap (A_{I_1} \ldots A_{I_{n+1}}) \leq (K_1 \ldots K_i) \cap (A_{I_1} \ldots A_{I_n}) \ \forall i \leq n\)

and so (5) holds for $n+1$, as whished. The claim is proved.

Note that by (2) and (5) it follows that $A_{I_n}/A_{I_n} \cap X$ is infinite for each $n \in \mathbb{N}$ and that also the following property holds

8) \((K_1 K_2 \ldots K_n) \cap \tilde{A} \leq (A_{I_1} A_{I_2} \ldots A_{I_n}) \ \forall n\), where $\tilde{A} := \text{Dr}_{n \in \mathbb{N}} A_{I_n}$.

Now for each $n \in \mathbb{N}$, choose an element $b_n \in (A_{I_n} \cap K_n) \setminus X$. Then we have $B := \langle b_n \mid n \in \mathbb{N} \rangle = \text{Dr}_n \langle b_n \rangle$, where $\langle b_n \rangle^\Gamma$ is infinite and $\langle b_n \rangle^\Gamma \leq K_n \sim A_{I_n}$, so that

9) \(\langle b_n \rangle^\Gamma \cap A_{I_n}\) is infinite for each $n$.

Since there exists $B_0 = B_0^\Gamma \sim B$, we may take
\(- B_* := (B_0 \cap B)^\Gamma = (B_0 \cap B)^\Gamma \leq B^\Gamma $ where $B_* \sim B$.

Now $B_*/(B_* \cap B)$ and $B/(B_0 \cap B)$ are both finite and there is $n \in \mathbb{N}$ such that if $B_n := \langle b_1, \ldots, b_n \rangle$ we have
\(- (B_* \cap B)^\Gamma = B_* \leq (B_* \cap B)B_n^\Gamma \and
\(- B = (B_* \cap B)B_n^\Gamma$.

Since $b_n \in K_n$ for each $n$, we have $B_n \leq \bar{K}_n := K_1 K_2 \ldots K_n$ and
\(- B^\Gamma = (B_* \cap B)^\Gamma B_n^\Gamma \leq (B_* \cap B)B_n^\Gamma \leq (B_* \cap B)\bar{K}_n \leq B\bar{K}_n$, so that
\(- B^\Gamma \cap \tilde{A} \leq B\bar{K}_n \cap A = B(\bar{K}_n \cap \tilde{A}) \leq BA_{I_1} A_{I_2} \ldots A_{I_n}$ by (8) above.

Thus
\(- \langle b_{n+1} \rangle^\Gamma \cap A_{I_{n+1}} \leq B^\Gamma \cap A_{I_{n+1}} \leq (BA_{I_1} A_{I_2} \ldots A_{I_n}) \cap A_{I_{n+1}} = \langle b_{n+1} \rangle$ is finite,

a contradiction with (9). \qed

**Lemma 3.8** Let $\Gamma$ be a group acting on an abelian periodic reduced group $A$. If $\Gamma$ has CP, then there are $\Gamma$-subgroups $A_0 \leq A_1 \leq A$ such that $A_0$ and $A/A_1$ are finite and $\Gamma$ has $\text{P}$ on $A_1/A_0$.

**Proof.** By Proposition 3.4 it is enough to consider the case when $A$ is a $p$-group. If $A$ is the direct product of cyclic groups, by Lemma 3.7 we have that $A_1 := \{ a \in A \mid \langle a \rangle^\Gamma \text{ is finite} \}$ has finite index in $A$. Further, by Lemma 3.6, $\Gamma$ has BP on $A_1$. Then the statement follows from Theorem 3.1.

Let $A$ be any reduced $p$-group and $B_*$ be a basic subgroup of $A$. Then there is $B = B^\Gamma \sim B_*$. Since $A/B_*$ is divisible, then the divisible radical of $A/B$ has finite index. Thus we may assume that $A/B$ is divisible. By Kulikov Theorem (see [10]), also $B$ is a direct product of cyclic groups, therefore by the above there are $\Gamma$-subgroups $B_0 \leq B_1 \leq B$ such that $B_0$ and $B/B_1$ are
finite and Γ has p on $B_1/B_0$. We may assume $B_0 = 1$. Also, since $A/B_1$ is finite-by divisible, it is divisible-by finite and we may assume it is divisible.

Let $γ \in Γ$ and $α$ be a p-adic integer such that $x^γ = x^α$ for all $x \in B_1$. Consider the endomorphism $γ - α$ of $A$ and note that $B_1 \leq \ker(γ - α)$. Thus $A/\ker(γ - α) \cong \text{im}(γ - α)$ is both divisible and reduced, hence trivial. It follows $γ = α$ on the whole $A$. □

**Proof of Theorem 3.2** For the sufficiency of the condition note that for any subgroup $H \leq A$ we have $H \sim H \cap A_1$ and the latter is in turn commensurable with a Γ-subgroup since Γ has BP on $A_1$ by Theorem 3.1.

Concerning necessity, we first prove the statement when $A$ is periodic. Let $A = D \times R_1$, where $D$ is divisible and $R_1$ is reduced. Then there is a $R = R^Γ \sim R_1$. Thus $DR$ and $D \cap R$ are Γ-subgroups of $A$ with finite index and order resp. Then we can assume $A = D \times R$. Let $X := \{ a \in A \mid \langle a \rangle^Γ$ is finite$\}$. Clearly $D \leq X$, as Γ has p on $D$ by Lemma 3.5. On the other hand, $X \cap R$ has finite index in $R$ by Lemma 3.8. It follows $A/X$ is finite and by Lemma 3.6 and Theorem 3.1 the statement holds.

In the non-periodic case, note that if $V_0$ is a maximal free subgroup of $A$ (hence $A/V_0$ is periodic), then there is $V_1 = V_1^Γ \sim V_0$. Let $n := |V_1/(V_0 \cap V_1)|$. Thus by applying Lemma 3.5 we have - there is a free abelian Γ-subgroup $V := V_1^n$ such that $A/V$ is periodic and each $γ \in Γ$ acts on $V$ by either the identity or the inversion map.

Suppose that $V$ has finite rank. Consider now the action of Γ on the periodic group $A/V$ and apply the above. Then there is a series $V \leq A_0 \leq A_1 \leq A$ such that $A_0/V$ and $A/A_1$ are finite and Γ has either p or ̄p on $A_1/A_0$. Since $A_0$ has finite torsion subgroup $T$ we can factor out $T$ and assume $A_0 = V$. Then Γ has either p or ̄p on $A_1$ as straightforward verification shows.

Suppose finally that $V$ has infinite rank. Let $V_2 \leq V$ be such that $V/V_2$ is divisible periodic and its p-component has infinite rank for each prime p. We may assume $V := V_2$. By the above case when $A$ is periodic, there is a Γ-series $V \leq A_0 \leq A_1 \leq A$ such that $A_0/V$ and $A/A_1$ are finite and Γ has p on $A_1/A_0$. We may factor out the torsion subgroup of $A_0$, as it is finite, and assume $A_0 = V$.

Again let $V_2 \leq V$ be such that $V/V_2$ is divisible periodic and its p-component has infinite rank for each prime p. Let $γ \in Γ$ and $α_p$ be a p-adic integer such that $x^γ = x^{α_p}$ for all $x$ in the p-component of $A_1/V$. Let $ε = \pm 1$ be such that $x^γ = x^ε$ for all $x \in V$. By Lemma 3.5, $γ$ has p on
the maximum divisible subgroup $D_p/V_2$ of the $p$-component of $A_1/V$. Thus $D_p = V_2$. Therefore $x^n = x^r$ for all $x \in V$ and for all $x \in A_1/V$. We claim that $a^n = a^r$ for each $a \in A_1$. To see this, for any $a \in A_1$ consider $n \in \mathbb{N}$ such that $a^n \in V$. Then there is $v \in V$ such that $a^n = a^r v$. Hence $a^{n^2} = (a^n)^r = (a^n)^n = (a^r v)^n = a^{n^2} v^n$. Thus $v^n = 1$. Therefore, as $V$ is torsion-free, we have $v = 1$, as whished.

4 Abelian-by-finite CN-groups and Theorem B

Locally finite CF-groups are known to be abelian-by-finite and BCF (see [1]).

**Proposition 4.1** Let $G$ be an abelian-by-finite group.

i) if $G$ is CN, then $G$ is CF;

ii) if $G$ is BCN, then $G$ is BCF.

**Proof.** Let $A$ be a normal abelian subgroup with finite index $r$. Then each $H \leq A$ has at most $r$ conjugates in $G$. If $\delta_G(H) \leq n < \infty$ then for each $g \in G$ we have $|H : (H \cap H^g)| \leq 2\delta_G(H) \leq 2n$ hence $|H/H_G| \leq (2n)^r$. More generally, if $H$ is any subgroup of $G$, then $|H/H_G| \leq r(2n)^r$. □

We state now a key fact about non-periodic CN-grups.

**Lemma 4.2** Let $G$ be a CN-group and $A = A(G)$ its subgroup generated by all infinite cyclic normal subgroups. Then $G/A$ is periodic, $A$ is abelian and each $g \in G$ acts on $A$ by either the identity or the inversion map, hence $|G/C_G(A)| \leq 2$.

**Proof.** For any $x \in G$ there is $N \triangleleft G$ which is commensurable with $\langle x \rangle$. Then $n := |N : (N \cap \langle x \rangle)|$ is finite. Thus $N^{n!} \leq \langle x \rangle$ where $N^{n!} \triangleleft G$. Hence $G/A$ is periodic.

It is clear that $A$ is abelian. Let $g \in G$. If $\langle a \rangle \triangleleft G$ and $a$ has infinite order, then there is $\epsilon_a = \pm 1$ such that $a^\epsilon = a^{\epsilon_a}$. On the other hand, by Lemma 3.5, there is $\epsilon = \pm 1$ such that for each $a \in A$ there is a periodic element $t_a \in A$ such that $a^\epsilon = a^\epsilon t_a$. It follows $a^{\epsilon_a - \epsilon} = t$. Therefore $\epsilon_a = \epsilon$ is independent of $a$, as wished. □

**Lemma 4.3** Let $G$ be an FC-group. If $G$ is a CN-group then $G$ is finite-by-abelian.
Proof. Let $H$ be any subgroup of $G$. We shall prove that $|H^G : H|$ is finite. Consider $A = A(G)$ as in Lemma 4.2. Then $H \cap A \triangleleft G$ and $H/A \cap H$ is periodic. Hence we may assume $H$ is periodic, that is, $H$ contained in the torsion subgroup of the fc-group $G$. Our claim follows then from Lemma 3.6. □

Proof of Theorem B Let $G$ be a cn-group and $G_0 \leq G_1 \leq G$ be a normal series such that $G_1/G_0$ is abelian and both $G_0$ and $G/G_1$ are finite. Then $G$ has cp on $G_1/G_0$. By Corollary 3.3, the group $G$ has bp on a subgroup $A_1/G_0 \leq G_1/G_0$ with finite index in $G_1/G_0$. Thus $A_1/G_0$ is contained in the fc-centre of $G/G_0$. Hence $A_1$ is contained in the fc-centre of $F$ of $G$. So that $G/F$ is finite. On the other hand, from Lemma 4.3 it follows that $F'$ is finite.

Finally, (ii) and (iii) follow from Proposition 2.1 and Proposition 4.1. □

Let us characterize bcf-groups among abelian-by-finite cf-groups.

Proposition 4.4 Let $G$ be a non-periodic group with an abelian normal subgroup $A$ with finite index. Then the following are equivalent:

(i) $G$ is a bcf-group;

(ii) $G$ is a cf-group and there is $B \leq A$ such that $B$ has finite exponent, $B \triangleleft G$ and each $g \in G$ acts by conjugation on $A/B$ by either the identity or the inversion map.

Proof. Let $T$ be the torsion subgroup of $A$. By Lemma 3.5, each $g \in G$ acts on $A/T$ as the automorphism $x \mapsto x^{\epsilon_g}$ where $\epsilon_g = \pm 1$. Then the equivalence of (i) and (ii) holds with $B := \langle A^g \epsilon_g \mid g \in G \rangle$, by Theorem 3 of [4]. □

5 Proof of Theorem A

Our first statement in this section is a reduction to nilpotent groups.

Lemma 5.1 A soluble $p$-group $G$ with the property cn is nilpotent-by-finite.

Proof. By Theorem 3.2, one may refine the the derived series of $G$ to a finite $G$-series $S$ such that $G$ has $P$ on each infinite factor of $S$. Recall that a $p$-group of power automorphisms of an abelian $p$-group is finite (see [10]). Then the stability group $S \leq G$ of the series $S$, that is, the intersection of
the centralizers in $G$ of the factors of the series, has finite index in $G$. On
the other hand, by a theorem of Ph.Hall, $S$ is nilpotent. □

We recall now an elementary property of nilpotent groups.

**Lemma 5.2** Let $G$ be a nilpotent group with class $c$. If $G'$ has finite exponent $e$, then $G/Z(G)$ has finite exponent dividing $e^c$.

**Proof.** Argue by induction on $c$, the statement being clear for $c = 1$. Assume $c > 1$ and that $G/Z$ has exponent dividing $e^{c-1}$, where $Z/G/\gamma_c(G) := Z(G/\gamma_c(G))$. Then for all $g, x \in G$ we have $[g^{e^{c-1}}, x] \in \gamma_c(G) \leq G' \cap Z(G)$. Therefore $1 = [g^{e^{c-1}}, x]^e = [g^e, x]$, and $g^e \in Z(G)$, as claimed. □

Next lemma follows easily from Lemma 6 in [8].

**Lemma 5.3** Let $G$ be a nilpotent $p$-group and $N$ a normal subgroup such that $G/N$ is an infinite elementary abelian group. If $H$ and $U$ are finite subgroup of $G$ such that $H \cap U = 1$, there exists a subgroup $V$ of $G$ such that $U \leq V$, $H \cap V = 1$ and $V/N \cap V$ is infinite. □

We deduce a technical lemma which is a tool for our pourpose.

**Lemma 5.4** Let $G$ be a nilpotent $p$-group and $N$ be a normal subgroup such that $G/N$ is an infinite elementary abelian group. If $N$ contains the FC-center of $G$ and $G'$ is abelian with finite exponent, then there are subgroups $H, U$ of $G$ such that $H \cap U = 1$, with injective maps $n \mapsto h_n \in H$ and $(i, n) \mapsto u_{i,n} \in [G, h_i^{-1}h_n] \cap U$, where $i, n \in \mathbb{N}, i < n$.

**Proof.** Let us show that for each $n \in N$ there is an $(n + 1)$-uple $v_n := (h_n, u_{0,n}, u_{1,n}, \ldots, u_{n-1,n})$ of elements of $G$ such that:

1. $\{h_1, \ldots, h_n\}$ is linearly independent modulo $N$;
2. $u_{i,n} \in [G, h_i^{-1}h_n]$ $\forall i \in \{0, \ldots, n-1\}$;
3. $\{u_{j,h} \mid 0 \leq j < k \leq n\}$ is $\mathbb{Z}$-independent in $G'$;
4. $H_n \cap U_n = 1$, where $H_n := \langle h_1, \ldots, h_n \rangle$ and $U_n := \langle u_{j,h} \mid 0 \leq j < k \leq n \rangle$.  

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Then the statement is true for $H := \bigcup_{n \in \mathbb{N}} H_n$ and $U := \bigcup_{n \in \mathbb{N}} U_n$.

Let $h_0 := 1$ and choose $h_1 \in G \setminus N$. Since $N \geq F$, the rF-center of $G$, we have that $\eta_1$ has an infinite numbers of conjugates in $G$, hence $[G, h_1]$ is infinite and residually finite. Thus we may choose $u_{0,1} \in [G, h_1]$ such that $\langle u_{0,1} \rangle \cap \langle h_1 \rangle = 1$.

Assume then that we have defined $v_i$ for $i \leq n$, that is, we have elements $h_0, \ldots, h_n$, $u_{j,k}$, with $0 \leq j < k \leq n$ such that conditions 1-4 hold. To define an adequate $v_{n+1}$, note that by Lemma 5.3 we have that there exists $V_n \leq G$ such that $H_n \leq V_n$, $U_n \cap V_n = 1$ and $V_nN/N$ is infinite. Then choose

\begin{itemize}
  \item[i)] $h_{n+1} \in V_n \setminus NU_nH_n$.
  \item[ii)] $u_{n,n} \in [G, h_i^{-1}h_n]$
  \item[iii)] $\langle u_{i,n+1} \rangle \cap U_n \langle u_{h,n+1} \mid 0 \leq h < i \rangle H_{n+1} = 1$
\end{itemize}

Then properties 1-3 hold for $v_{n+1}$. Finally suppose there are $h \in H_n$, $u \in U_n$, $s, t_0, \ldots, t_n \in \mathbb{Z}$ such that

\begin{itemize}
  \item[iii)] $a = hh_{n+1}^s = uu_{t_0,n+1} \cdots u_{t_n,n+1}^1 H_{n+1} \cap U_{n+1}$.
\end{itemize}

Then from (iii) it follows $u_{t_0,n+1}^1 = \cdots = u_{t_n,n+1}^1 = 1$. Hence $a = hh_{n+1}^s = u \in V_n \cap U_n = 1$ and 4 holds.

\begin{lemma}
Let $G$ be a nilpotent p-group. If $G$ is CN, then $G'$ has finite exponent.
\end{lemma}

\begin{proof}
If, by contradiction, $G'$ has infinite exponent, then the same happens to the abelian group $G'/\gamma_3(G)$ and there is $N$ such that $G' \geq N \geq \gamma_3(G)$ and $G'/N$ is a Prüfer group. We may assume $N = 1$, that is, $G'$ itself is a Prüfer group and $G' \leq Z(G)$. Let us show that for any $H \leq G$ we have $|H^G : H| < \infty$, hence $G'$ is finite, a contradiction. In fact we have that, by CN-property there is $K < G$ such that $K \sim H$. Thus $H$ has finite index in $HK$ and we can also assume $H = HK$, that is, $H/HG$ is finite. Thus, we can assume $H_G = 1$ and $H \cap G' = 1$, that is, $H$ is finite with order $p^n$ and $HG'$ is an abelian Chernikov group. It follows that $H$ is contained in the $n$-th socle $S$ of $HG' < G$, where $S$ is finite and normal in $G$, as whished.

\begin{lemma}
Let $G$ be a nilpotent p-group. If $G$ is CN, then $G$ is finite-by-abelian-by-finite.
\end{lemma}
Proof. Assume, by contradiction, $G$ is a counterexample. Then both $G'$ and $G/Z(G)$ are infinite. However, they have finite exponent by Lemmas 5.5 and 5.2. Moreover, even the fc-center $F$ of $G$ has infinite index by Lemma 4.3. On the other hand, $G/F$ has finite exponent, since $F \trianglerighteq Z(G)$.

Then $N := FG^pG'$ has infinite index in $G$, otherwise the abelian group $G/FG'$ has finite rank and finite exponent, hence it is finite. This implies that the nilpotent group $G/F$ is finite, a contradiction.

If $G'$ is abelian we are in a condition to apply Lemma 5.4 and get infinite elements and subgroups $h_n \in H$, $u_{i,n} \in U$ as in that statement. By cn-property there is $K$ such that $H \sim K \triangleleft G$. So that the set $\{h_n(H \cap K) / n \in \mathbb{N}\}$ is finite. Hence there is $i \in \mathbb{N}$ and an infinite set $I \subseteq \mathbb{N}$ such that for each $n \in I$ we have $h_i^{-1}h_n \in H \cap K$ and $u_{i,n} \in U \cap [G,H \cap K] \leq U \cap K$. Therefore $U \cap K$ is infinite, in contradiction with $U \cap K \sim U \cap H = 1$.

For the general case, proceed by induction on the nilpotency class $c > 1$ of $G$ and assume that the statement is true for $G/Z(G)$ and even that this is finite-by-abelian. Then there is a subgroup $L \leq G$ such that $G/L$ is abelian and $L/Z(G)$ is finite. Thus $L'$ is finite and, by the above, $G/L'$ is finite-by-abelian-by-finite, a contradiction. $\square$

Proof of Theorem A. Recall from the Introduction that all subgroups of $G$ are subnormal-by-finite. Thus, by above quoted results in [6] and [3] resp., we may assume that $G$ is locally nilpotent and soluble. Assume first $G$ is periodic. Then, by Lemma 3.4, only finitely many primary components are non-abelian. Thus we may assume $G$ is a $p$-group and apply Lemma 5.1 and Lemma 5.6. It follows that $G$ is finite-by-abelian-by-finite.

To treat the general case, consider $A = A(G)$ as in Lemma 4.2. We may assume $A$ is central in $G$. Let $V$ be a torsion-free subgroup of $A$ such that $A/V$ is periodic. Then $G/V$ is locally finite and we may apply the above. Thus there is a series $V \leq F \leq G_1 \leq G$ such that $G$ acts trivially on $V$, $G_1/G_0$ is abelian, while $G_0/V$ and $G/G_1$ are finite. Then we can assume $G = G_1$ and note that the stabilizer $S$ of the series has now finite index. Since $S$ is nilpotent (by Ph.Hall Theorem) we can assume that $G = S$ is nilpotent. If $T$ is the torsion subgroup of $G$, then $VT/T$ is contained in the center of $G/T$. Since all factor of the upper central series of $G/T$ are torsion-free we have $G/T$ is abelian. Thus $G' \leq T \cap G_0$ is finite. $\square$

Proof of Corollary. If the statement is false, by Theorem A we may assume there is a counterexample $G$ periodic and not locally finite. Also we may
assume $G$ is finitely generated and infinite. Let $R$ be the locally finite radical of $G$. By Theorem A again, $R$ is finite-by-abelian-by-finite. By Theorem B(i), there is a finite subgroup $G_0 \lhd G$ such that $R/G_0$ is abelian-by-finite. We may assume $G_0 = 1$, so that $R$ is abelian-by-finite.

We claim that $\bar{G} := G/R$ has finite exponent at most $(n + 1)!$ where $n$ is such that $n \geq \delta_G(H)$ for each $H \leq G$. In fact, for each $x \in \bar{G}$, there is $\bar{N} \triangleleft \bar{G}$ such that $|\bar{N} : (\bar{N} \cap \langle x \rangle)| \leq n$. Thus $\bar{N}^{n!} \leq \langle x \rangle$ and $\bar{N}^{n!} \lhd \bar{G}$. Hence $\bar{N}^{n!} = 1$ and $x^{n!} = 1$.

By the positive answer (for all exponents) to the Restricted Burnside Problem, there is a positive integer $k$ such that every finite image of $\bar{G}$ has order at most $k$. Since $\bar{G}$ is finitely generated, this means that the finite residual $\bar{K}$ of $\bar{G}$ has finite index and is finitely generated as well. Since also $\bar{G}$ is locally graded (see [7]), we have $\bar{K} = 1$ and $\bar{G}$ is finite. Therefore $G$ is abelian-by-finite, a contradiction. \[\square\]

References


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