

On the uniqueness of limit cycle for certain Liénard systems without symmetry

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

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Abstract. The problem of the uniqueness of limit cycles for Liénard systems is investigated in connection with the properties of the function $F(x)$. When α and β ($\alpha < 0 < \beta$) are the unique nontrivial solutions of the equation $F(x) = 0$, necessary and sufficient conditions in order that all the possible limit cycles of the system intersect the lines $x = \alpha$ and $x = \beta$ are given. Therefore, in view of classical results, the limit cycle is unique. Some examples are presented to show the applicability of our results in situations with lack of symmetry.

Keywords: Liénard system, uniqueness, limit cycles, invariant curves.

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1 Introduction

In this paper we consider the well-known Liénard system

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x) \tag{L}$$

with the aim to propose a necessary and sufficient condition in for the uniqueness of the limit cycle. Throughout the paper, our basic assumptions will be the following:

(C1) $F(x)$ and $g(x)$ are locally Lipschitz continuous functions;

(C2) $F(0) = g(0) = 0$ and $g(x)x > 0$ for $x \neq 0$;

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(C3) There exist α and β with $\alpha < 0 < \beta$ such that $F(x)$ is strictly increasing for $x \leq \alpha$ and $x \geq \beta$ and, moreover, $xF(x) < 0$ for $x \in (\alpha, \beta) \setminus \{0\}$.

We see under the above assumptions that the uniqueness of solutions of system (L) for initial value problems is guaranteed and the only equilibrium point $(0,0)$ is unstable. This system has been widely investigated in the literature (for instance see [5] or [18]) and plays an important role in the qualitative analysis of planar ODEs and applications.

As a starting point, we recall the classical uniqueness results due to Liénard [8], Levinson–Smith [7] and Sansone [10] from which the following result can be deduced.

Proposition 1.1. *Under the conditions (C1), (C2) and (C3) there is at most a limit cycle intersecting both the lines $x = \alpha$ and $x = \beta$.*

In view of Proposition 1.1, it can be useful to introduce the following terminology.

Definition 1.2. System (L) has the property (A) if all its limit cycles intersect the lines $x = \alpha$ and $x = \beta$.

In the above quoted papers, the authors assume some symmetry conditions ensuring that property (A) is fulfilled. If no symmetry condition on $F(x)$ and $g(x)$ is assumed, it is necessary to produce sufficient conditions for the validity of (A) and hence for the uniqueness. In this light, sufficient conditions for property (A) were proposed by several researchers (cf. [2, 6, 11] and the recent paper [15]). For related results, in a more general environment, see also [1, 9, 17].

In [6], the relation between the magnitude of $F(x)$ and the unique existence of a limit cycle of system (L) has been investigated and the following condition was introduced.

$$(C4) \quad \begin{cases} G(\alpha) > G(\beta), \exists x_2 \in (0, \beta] \text{ such that } \frac{1}{2}F(x_2)^2 + G(\beta) < G(\alpha) \\ \text{or} \\ G(\alpha) \leq G(\beta), \exists x_1 \in [\alpha, 0) \text{ such that } \frac{1}{2}F(x_1)^2 + G(x_1) \geq G(\beta), \end{cases}$$

where $G(x) = \int_0^x g(\xi) d\xi$.

Proposition 1.3 ([6]). *Under the conditions (C1), (C2), (C3) and (C4), property (A) holds and therefore system (L) has at most one limit cycle which is stable.*

The case in which system (L) does not satisfy condition (C4) has been recently investigated in [15]. Further results in this direction will be presented in this paper. In particular, in Section 2 we give necessary and sufficient conditions in this direction. Finally, in Section 4 we present some concrete examples which involve new applications and show the effectiveness of this approach.

It is worth to note that in the same context, but with a different approach, in a very recent paper [16], the monotonicity assumption on $F(x)$, and therefore the sign assumption on $f(x)$, were strongly relaxed.

2 Main results

As observed in the previous section, the values of the function $G(x)$ at the points α and β play a crucial role. For this reason, we investigate in detail the situation. If $G(\alpha) = G(\beta)$, we

note that there exists $x_1 = \alpha$ satisfying the condition (C4) (see also Example 4.1) and hence Proposition 1.3 applies. So we focus our attention to the case

$$G(\alpha) \leq G(\beta), \quad (2.1)$$

the case $G(\beta) < G(\alpha)$ being treated in a symmetric way. Therefore, from now on, we take (2.1) as a basic assumption for the rest of the paper.

For $x \in [\alpha, 0]$ we consider the points in which the function $\Phi(x) := F(x)^2 + 2G(x)$ takes its maximum and let a be the maximum of such points. In other words,

$$a = \max \left\{ x \in [\alpha, 0] \mid \Phi(x) = \max_{\xi \in [\alpha, 0]} [F(\xi)^2 + 2G(\xi)] \right\}.$$

Clearly, $a < 0$ and, by definition,

$$F(a)^2 + 2G(a) = \max_{\xi \in [\alpha, 0]} \Phi(\xi) \geq \Phi(\alpha) = F(\alpha)^2 + 2G(\alpha) = 2G(\alpha). \quad (2.2)$$

In virtue of (2.1) and (2.2), only the following two cases can occur and so they will be taken into consideration:

Case 1. $F(a)^2 + 2G(a) \geq 2G(\beta)$,

Case 2. $2G(\alpha) \leq F(a)^2 + 2G(a) < 2G(\beta)$,

The following result holds.

Theorem 2.1 (Case 1). *Let $F(a)^2 + 2G(a) \geq 2G(\beta)$ or $G(\alpha) = G(\beta)$. If system (L) satisfies the conditions (C1), (C2), (C3), then the system satisfies the property (A) and therefore it has at most one limit cycle.*

Remark 2.2. The case 1 can be rewritten in the following form

$$\max_{x \in [\alpha, 0]} \left\{ F(x)^2 + 2G(x) \right\} \geq 2G(\beta).$$

We consider now Case 2 or and let $y(x)$ be the solution of system (L) with

$$y(0) = \sqrt{F(a)^2 + 2G(a)}. \quad (2.3)$$

Let also $\gamma \in (0, \beta)$ be such that $2G(\gamma) = 2G(\beta) - y(0)^2$, that is

$$F(a)^2 + 2G(a) = 2G(\beta) - 2G(\gamma). \quad (2.4)$$

In this case, the following result holds.

Theorem 2.3 (Case 2). *Assume that system (L) satisfies the conditions (C1), (C2), (C3) and $F(a)^2 + 2G(a) < 2G(\beta)$. Then the system satisfies the property (A) and therefore it has at most one limit cycle if and only if there exists $c \in (\gamma, \beta)$ such that*

$$(C5) \quad y(0) = \sqrt{2(G(\beta) - G(c))} + \int_0^c \frac{g(x)}{y(x) - F(x)} dx.$$

Condition (C5), even if necessary and sufficient, is of implicit type because it require the knowledge of the solution $y(x)$. Accordingly, it may be useful to state some corollaries which guarantee that such a condition is automatically satisfied. Taking into account that $y(x) > 0$, $g(x) > 0$ and $F(x) < 0$ for all $x \in (0, c)$, we have the following consequence of Theorem 2.3

Corollary 2.4. *Under the conditions in Theorem 2.4 if there exists $c \in (\gamma, \beta)$ satisfying the inequality*

$$(C6) \quad y(0) \geq \sqrt{2(G(\beta) - G(c))} + \int_0^c \frac{g(x)}{y(0) - F(x)} dx,$$

then the system satisfies the property (A) and therefore it has at most one limit cycle.

Remark 2.5. We just observe that if, instead of (2.1), we assume the dual hypothesis $G(\alpha) > G(\beta)$, the condition (C6) will be replaced by

$$(C6)^* \quad y(0) \leq \sqrt{2(G(\alpha) - G(c'))} + \int_0^{c'} \frac{g(x)}{y(0) - F(x)} dx,$$

where $c' \in (\alpha, \gamma')$, $\gamma' < 0$ and $2G(\gamma') = 2G(\alpha) - y(0)^2$. This inequality will be used in Example 4.3.

As previously observed, the presented results guarantee the existence of at most one limit cycle. In order to obtain the existence, it is necessary to add some other assumption, which will guarantee that the orbits of the systems will be ultimately bounded and the classical Poincaré–Bendixson theorem can be applied. Typical condition make use of assumptions on F and/or G at infinity. In this light, following [3, 5, 12, 14], we have the following theorem.

Theorem 2.6. *If system (L) satisfies the conditions in Theorem 2.1, or Corollary 2.4, and*

$$(C7) \quad \limsup_{x \rightarrow \pm\infty} \{G(x) \pm F(x)\} = +\infty,$$

then it has a unique limit cycle which is stable and hyperbolic.

For condition (C7), see [12].

3 Proofs of the results

At first we consider Theorem 2.1, which is essentially the result in [6] and also recalled in Proposition 1.3. We give quick details for the sake of completeness.

Let $F(a)^2 + 2G(a) \geq 2G(\beta)$. The case $G(\alpha) = G(\beta)$ is well known (see [3, 6, 10]). We introduce the energy function

$$V(x, y) := (1/2)y^2 + G(x)$$

and define the plane curve Γ constructed by six curves as follows.

$$\Gamma \begin{cases} \Gamma_1 : y = F(x) & \text{for } x \in [\alpha, a] \\ \Gamma_2 : V(x, y) = (1/2)F(a)^2 + G(a) & \text{for } x \in [a, 0], y \geq 0 \\ \Gamma_3 : x = 0, \sqrt{2G(\beta)} \leq y \leq \sqrt{F(a)^2 + 2G(a)} \\ \Gamma_4 : V(x, y) = G(\beta) & \text{for } x \in [0, \beta], y \geq 0 \\ \Gamma_5 : y = F(x) & \text{for } x \in [a^*, \beta] \\ \Gamma_6 : V(x, y) = G(\alpha) & \text{for } x \in [\alpha, a^*], y \leq 0, \end{cases}$$

where a^* is a positive number satisfying the equation $V(x, F(x)) = G(\alpha)$. Such construction may be found in [6]. See Figure 3.1 below. Keep following [6], one can check that the domain

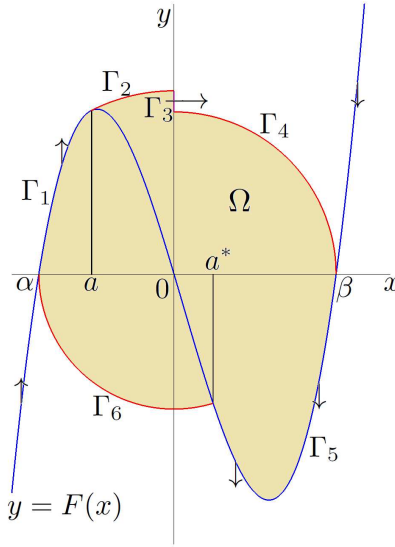


Figure 3.1: (Case 1). For the example considered in the figure, we have chosen $g(x) = x$ and $F(x) = k(x + \frac{5}{2})(x - 3)$, with $k = 9/20$. The shadowed region represents the set Ω . We have $\alpha = -5/2$, $\beta = 3$ and we can estimate $a \doteq -1.537662135$ and $a^* \doteq .7246695283$. We also find that $F(a)^2 + 2G(a) \doteq 11.49431131 > 2G(\beta) = 9$, so that we are in the situation of [6] and Theorem 2.1 applies.

Ω surrounded by the closed curve Γ and including the equilibrium point O has the vector field pointing outside at the boundary. Moreover, it is proved that no limit cycles of the system can exist in the interior of Ω . Thus, any possible limit cycle must lie outside Ω , and therefore it will intersect the lines $x = \alpha$ and $x = \beta$. Hence the property (A) is fulfilled and this completes the proof of Theorem 2.1. \square

For the proof of Theorem 2.3 it is useful to premise the following statement whose trivial proof is omitted.

Lemma 3.1. *Let $\sqrt{2(G(\alpha))} \leq y(0) \leq \sqrt{2(G(\beta))}$, and γ as in (2.3) and (2.4). The orbit starting from $(0, y(0))$ intersects the plane curve $y = \sqrt{2(G(\beta) - G(x))}$ if and only if there exists $c \in (\gamma, \beta]$ satisfying the equality $y(c) = \sqrt{2(G(\beta) - G(c))}$.*

Now we are in position to give a proof for Theorem 2.3.

Let $y(0) = \sqrt{F(a)^2 + 2G(a)}$ as in (2.3). If the solution orbit $y = y(x)$ starting from $(0, y(0))$ intersects the plane curve Γ_4 on the boundary of the domain Ω at $x \in (c, \beta)$, then we have from Lemma 3.1

$$y(0) = y(c) - \int_0^c \frac{dy(x)}{dx} = \sqrt{2(G(\beta) - G(c))} - \int_0^c \frac{-g(x)}{y(x) - F(x)} dx$$

and hence (C5). Moreover, since the solution orbit cannot return in Ω , it will intersect the line $x = \beta$. On the other hand, if we start from a point on the segment Γ_3 , namely $x = 0$ and $\sqrt{F(a)^2 + 2G(a)} \leq y \leq \sqrt{2G(\beta)}$, back in time, the orbit must have intersected the $x = \alpha$ and, forward in time, it will stay above the previously considered orbit $(x, y(x))$ $x = \beta$ and therefore it will intersect $x = \beta$, giving the desired property (A). See Figure 3.2

The converse is trivial.

We note that if $y(0) = \sqrt{2G(\alpha)}$, then the two curves Γ_1 and Γ_2 are replaced by the curve C^* which is the energy level $V(x, y) = G(\alpha)$, for $x \in [\alpha, 0]$, $y \geq 0$. \square

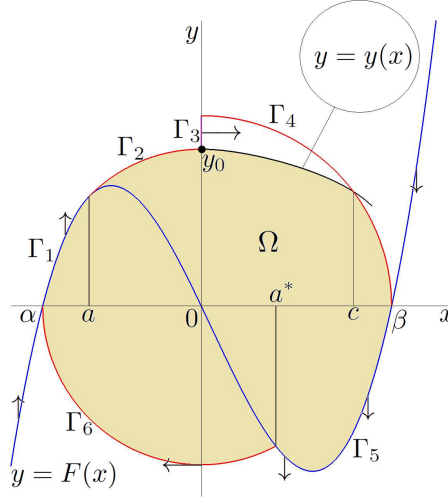


Figure 3.2: (Case 2). For the example considered in the figure, we have chosen $g(x) = x$ and $F(x) = k(x + \frac{5}{2})(x - 3)$, with $k = 7/25$. The shadowed region represents the part of the set Ω bounded by the orbit path passing through $(0, y_0)$ with $y_0 = y(0)$. We have $\alpha = -5/2$, $\beta = 3$ and we can estimate $a \doteq -1.769233452$ and $a^* \doteq 1.175057813$. We also find that $F(a)^2 + 2G(a) \doteq 6.111041847 < 2G(\beta) = 9$ and in this case Theorem 2.1 does not apply.

For a previous step in this direction, see [15] where only a sufficient condition was given, where $\int_0^c \frac{g(x)}{y(x)-F(x)} dx$ was replaced by $\int_0^c \frac{g(x)}{-F(x)} dx$.

Finally, we prove Theorem 2.6. We observe that, recently, M. Cioni and G. Villari ([3]) have proved the existence of the limit cycles for the system under the condition (C7). Other conditions may be found in the literature and appear, for instance, in the references of the above mentioned paper. Consequently, putting together the existence results coming from (C7) and the uniqueness of the limit cycle following from Theorem 2.1 and Corollary 2.4, we get the thesis. \square

4 Examples

We shall present concrete systems for system (L) as applications of our results. The first one is very elementary and classic, while the other two show the concrete applicability of our results.

Example 4.1. We consider the well-known Van der Pol system $F(x) = \lambda(\frac{1}{3}x^3 - x)$ and $g(x) = x$. It is classical fact that such system has exactly one limit cycle. It is worth to note that, among the others, there is a very elegant uniqueness proof due to Massera (see, for instance, [13] and the references therein). However, it is possible to get the same result, just observing that, due to the symmetry properties, $G(\alpha) = G(\beta)$.

Example 4.2. Consider system (L) with $F(x) = \sqrt{3}x(x+1)(x-3)$ and $g(x) = x$. By the choice of g and F we have that $G(x) = x^2/2$ and $\alpha = -1$, $\beta = 3$. The value of the constant a for which

we obtain the maximum of the function $F(x)^2 + 2G(x)$ for $x \in [\alpha, 0]$ is

$$a \doteq -0.5646076839 \quad (4.1)$$

for which we have

$$F(a)^2 + 2G(a) \doteq 2.622343392 \quad \text{and} \quad y(0) \doteq 1.619365120.$$

Now we observe that, if we wish to apply Corollary 2.4, we can start also from a constant a which is not the optimal one as in (2.2) (which, for our case, would be the one computed in (4.1)). Indeed, if we chose another constant $a \in (\alpha, 0)$, we will produce a lower value with respect to the optimal choice of $y(0)$. However, if we are able to get (C6) satisfied for a lower value of $y(0)$, then (C6) also will hold for the optimal one.

To show how our result can be applied even if we not use the optimal value of the constant a given above, let us choose $a = -1/2$ and observe that

$$F(a)^2 + 2G(a) = \frac{163}{64} = 2.546875 < 2G(\beta) = 9.$$

As a next step, we can compute $y(0)$ using (2.3), as

$$y(0) = \sqrt{F(a)^2 + 2G(a)} = \frac{\sqrt{413}}{8} \doteq 1.595893166$$

and also $\gamma \in (0, 3) = (0, \beta)$ using (2.4), from which we have

$$\gamma = \sqrt{2G(\beta) - 2G(a) - F(a)^2} = \frac{\sqrt{413}}{8} \doteq 2.540300179.$$

After these preliminaries, we can easily find (with a little help of some numerical integration), that there exists $c \in (\gamma, \beta]$ satisfying the inequality (C6) in Corollary 2.4. In fact, taking

$$c = \frac{14}{5} = 2.8 \in (\gamma, \beta),$$

we have

$$\begin{aligned} K &:= \sqrt{2(G(\beta) - G(c))} + \int_0^c \frac{g(x)}{y(0) - F(x)} dx \\ &= \frac{1}{5}\sqrt{29} + \int_0^{14/5} \frac{x}{\frac{\sqrt{413}}{8} - \sqrt{3}x(x+1)(x-3)} dx \doteq 1.519184294. \end{aligned}$$

Thus we have find that $y(0) > K$ and Corollary 2.4 applies, ensuring that the system

$$\dot{x} = y - \sqrt{3}x(x+1)(x-3), \quad \dot{y} = -x$$

has a unique limit cycle. A numerical simulation is shown in Figure 4.1

We point out that the condition (C4) cannot be applied to this example since

$$G(\alpha) < G(\beta) \quad \text{and} \quad 2G(\alpha) < \max_{\alpha \leq x \leq 0} F(x)^2 + 2G(x) < 2G(\beta)$$

This example shows that in this case Theorem 2.1 and the results in [6] cannot be applied, while Corollary 2.4 finds its applicability.

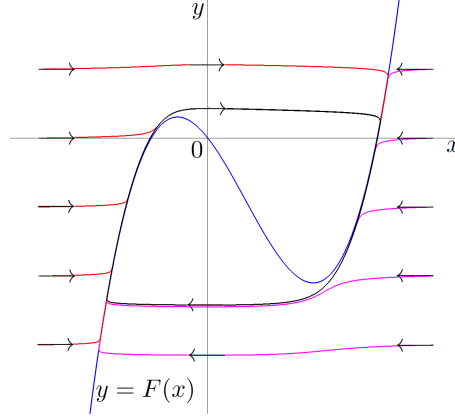


Figure 4.1: Phase-portrait for Example 4.2, with the unique limit cycle.

Example 4.3 (Duff and Levinson [4]). The following system was studied in [6] as an example for the unique existence of the limit cycle with the property (A).

$$\dot{x} = y - \lambda \left(\frac{64}{35\pi} x^7 - \frac{112}{5\pi} x^5 + \frac{196}{3\pi} x^3 - \frac{C}{2} x^2 - \frac{36}{\pi} x \right), \quad \dot{y} = -x \quad (\text{DL})$$

This example is crucial, because in the fundamental paper [4] it was proved that the system has three limit cycles when λ is sufficiently small and C is large. The importance of this result lies on the fact that before the work of Duff and Levinson it was believed that the condition that $F(x)$ has three zeros $\alpha < 0 < \beta$ was actually ensuring the uniqueness.

Let $C = 47$. In [6], it was proved that the system has the property (A) if $\lambda \geq 2.86896$. Slight different examples appear also in [3, 15].

In virtue of Corollary 2.4 and Remark 2.5 the estimate obtained [6] in is now replaced and improved by the following statement.

Proposition 4.4. *System (DL) has the property (A) and hence a unique limit cycle if $C = 47$, for every $\lambda \geq 0.862$.*

Proof. In fact, solving the equation $F(x) = 0$, we have $\alpha \doteq -3.18941$, $\beta \doteq 0.3715$. Moreover $g(x) = x$.

To prove (C6)*, namely

$$y(0) \leq -\sqrt{2(G(\alpha) - G(c'))} + \int_0^{c'} \frac{g(x)}{y(0) - \lambda F(x)} dx,$$

we set $\lambda = 0.862$ and choose $c' = -3.18$. Then, for $y(0) = -\sqrt{2G(\beta)} = -\beta$, we have

$$\begin{aligned} -0.3715 &\leq -\sqrt{\alpha^2 - c'^2} + \int_0^{c'} \frac{x}{-\beta - \lambda F(x)} dx \\ &\doteq -0.244818602 - 0.126575 = -0.371393602, \end{aligned}$$

thus proving (C6)*. □

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