



UNIVERSITÀ  
DEGLI STUDI  
FIRENZE

# FLORE

## Repository istituzionale dell'Università degli Studi di Firenze

### Regularity of Kobayashi metric

Questa è la Versione finale referata (Post print/Accepted manuscript) della seguente pubblicazione:

*Original Citation:*

Regularity of Kobayashi metric / Giorgio Patrizio; Andrea Spiro. - STAMPA. - (2018), pp. 335-349.  
[10.1007/978-981-13-1672-2\_24]

*Availability:*

The webpage <https://hdl.handle.net/2158/1128387> of the repository was last updated on 2021-03-29T08:03:01Z

*Publisher:*

Springer

*Published version:*

DOI: 10.1007/978-981-13-1672-2\_24

*Terms of use:*

Open Access

La pubblicazione è resa disponibile sotto le norme e i termini della licenza di deposito, secondo quanto stabilito dalla Policy per l'accesso aperto dell'Università degli Studi di Firenze (<https://www.sba.unifi.it/upload/policy-oa-2016-1.pdf>)

*Publisher copyright claim:*

La data sopra indicata si riferisce all'ultimo aggiornamento della scheda del Repository FloRe - The above-mentioned date refers to the last update of the record in the Institutional Repository FloRe

(Article begins on next page)

# REGULARITY OF KOBAYASHI METRIC

GIORGIO PATRIZIO AND ANDREA SPIRO

*Dedicated to Kang-Tae Kim for his sixtieth birthday*

**ABSTRACT.** We review some recent results on existence and regularity of Monge-Ampère exhaustions on the smoothly bounded strongly pseudoconvex domains, which admit at least one such exhaustion of sufficiently high regularity. A main consequence of our results is the fact that the Kobayashi pseudo-metric  $\kappa$  on an appropriate open subset of each of the above domains is actually a smooth Finsler metric. The class of domains to which our result apply is very large. It includes for instance all smoothly bounded strongly pseudoconvex complete circular domains and all their sufficiently small deformations.

## 1. INTRODUCTION

In this note, providing the necessary background, we survey some recent results about the existence of regular Monge-Ampère exhaustions which, in turn, imply regularity properties for the Kobayashi metric. More precisely, for domains  $D$  admitting a smooth Monge-Ampère exhaustion centered at a point  $z_o \in D$  (that is, a strictly plurisubharmonic  $\mathcal{C}^0$  exhaustions  $\tau : \overline{D} \rightarrow [0, 1]$ , which is  $\mathcal{C}^\infty$  at all points, with only possible exception at the minimum set  $\{\tau = 0\} = \{z_o\}$ , and such that  $u := \log \tau$  has a logarithmic singularity at  $z_o$  and satisfies the complex Monge-Ampère equation  $(\partial\bar{\partial}u)^n = 0$  at all other points), our results show that there exists an open neighborhood  $D' \subset D$  of  $z_o$  such that for each  $z \in D'$  there exists an analogous smooth Monge-Ampère exhaustion, centered at such point ([22]). One of the main consequence of this result is that the Kobayashi pseudo-metric on each such domain is actually a smooth complex Finsler metric on an open subset  $D' \subset D$ . Since any smoothly bounded strongly pseudoconvex complete circular domain and any of its sufficiently small smooth deformations have at least one Monge-Ampère exhaustion and our proof shows how to determine when  $D' = D$ , our result reveals that there exists a new large class of domains, on which the Kobayashi metric has extremely high regularity properties. In fact, the results in [22] are proven for closed strongly pseudoconvex domains with Monge-Ampère

---

2000 *Mathematics Subject Classification.* 32Q45, 32U35, 32G05, 32W20.

*Key words and phrases.* Monge-Ampère Equations, Pluricomplex Green Functions, Manifolds of Circular Type, Kobayashi metric, Deformations of Complex Structures.

*Acknowledgments.* This research was partially supported by the Project MIUR “Real and Complex Manifolds: Geometry, Topology and Harmonic Analysis” and by GNSAGA of INdAM.

exhaustions of class  $\mathcal{C}^{r,\alpha}$ ,  $r \geq 4$ ,  $\alpha > 0$ , and imply regularity for the Kobayashi pseudo-metric also under such weaker regularity assumptions.

The structure of the paper is the following. In §1, we recall a few basic properties of Monge-Ampère exhaustions and Kobayashi metrics. In §2 we present our results with a short description of their proofs. Finally, in §3 we present some open questions that might be addressed using the results in §2.

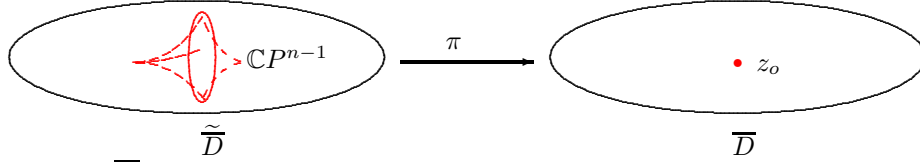
## 2. MONGE-AMPÈRE EXHAUSTIONS AND KOBAYASHI METRICS

**2.1. Monge-Ampère exhaustions and Monge-Ampère foliations.** Let  $D \subset \mathbb{C}^n$  be a bounded domain with boundary of class  $\mathcal{C}^{k,\alpha}$ ,  $k \geq 2$ ,  $\alpha > 0$ . A *Monge-Ampère exhaustion* for  $\overline{D}$  is a continuous exhaustion  $\tau : \overline{D} \rightarrow [0, 1]$  satisfying the following conditions:

- i) The boundary  $\partial D$  coincides with the level set  $\{\tau = 1\}$  while the level set  $\{\tau = 0\}$  consists of exactly one interior point  $z_o$ , called *center* of the exhaustion.
- ii) The map  $\tau$  is of class  $\mathcal{C}^{k,\alpha}$  on  $\overline{D} \setminus \{z_o\}$  and, in general, is only continuous at  $z_o$ . However, if  $\pi : \widetilde{D} \rightarrow \overline{D}$  denotes the blow up of  $\overline{D}$  at the center  $z_o$ , we assume that the  $\mathcal{C}^{k,\alpha}$  lifted map

$$\tilde{\tau} = \tau|_{\overline{D} \setminus \{0\}} \circ \pi : \widetilde{D} \setminus \{z_o\} \rightarrow (0, 1]$$

admits a  $\mathcal{C}^{k,\alpha}$ -extension to the whole  $\widetilde{D}$ .



- iii) On  $\overline{D} \setminus \{z_o\}$  the following differential conditions hold:

- (a)  $2i\partial\bar{\partial}\tau > 0$ ;
- (b)  $2i\partial\bar{\partial}\log\tau \geq 0$ ;
- (c)  $(\partial\bar{\partial}\log\tau)^n = 0$  (*homogeneous complex Monge-Ampère equation*).

Note that (a) - (c) imply that, at each point  $z$ , the kernel of the 2-form  $\partial\bar{\partial}\log\tau_z$  is 1-dimensional and that each level set  $\{\tau = c\}$  is a strongly pseudoconvex real hypersurface.

- iv) In proximity of the center,  $\log\tau$  goes as  $\log\tau(z) \simeq \log(\|z - z_o\|) + O(1)$ .

The closure of a domains  $D \subset \mathbb{C}^n$ , for which there is at least one Monge-Ampère exhaustion, is called (*closed*) *domain of circular type*.

**Remark 2.1.** The above definition, given here for domains in  $\mathbb{C}^n$ , can be easily extended and stated in full generality for complex manifolds with boundary. We refer to [22] for details.

Up to a few minor changes, the above notion of domain of circular type coincides with the one introduced by the first author in [16] to capture the most crucial properties of the following two important classes of domains.

**Class A.** Let  $\overline{D} \subset \mathbb{C}^n$  be the closure of a complete circular domain, i.e. of a domain  $D = \{z : \mu(z) < 1\}$  determined by a defining function  $\mu : \mathbb{C}^n \rightarrow [0, +\infty)$  satisfying the condition

$$\mu(\lambda z) = |\lambda| \mu(z) \quad \text{for all } \lambda \in \mathbb{C} .$$

Assume that the function  $\mu$ , called the *Minkowski function* of  $D$ , has the following two properties:

- (a) one (and, consequently, all) of the level sets  $\{\mu = c\}$  for  $0 < c \leq 1$  is a strongly pseudoconvex hypersurface,
- (b)  $\mu$  is of class  $\mathcal{C}^{k,\alpha}$ , with  $k \geq 2$ ,  $\alpha > 0$ , on  $\mathbb{C}^n \setminus \{0\}$ .

One can then directly see that the square  $\tau := (\mu|_{\overline{D}})^2$  is a Monge-Ampère exhaustion for  $\overline{D}$ , centered at  $z_o = 0$ , so that  $\overline{D}$  is of circular type.

**Class B.** Let  $\overline{D} \subset \mathbb{C}^n$  be the closure of a strictly linearly convex domain  $D$  with boundary of class  $\mathcal{C}^{k,\alpha}$  for some  $k \geq 4$ ,  $\alpha > 0$ . Let also  $\delta : D \times D \rightarrow \mathbb{R}$  be the Kobayashi pseudodistance of  $D$  and, for each given  $z_o \in D$ , set

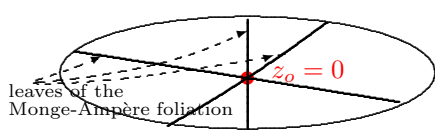
$$\tau^{(z_o)} : \overline{D} \rightarrow [0, 1] , \quad \tau^{(z_o)}(w) := \tanh^2(\delta(z_o, w)) . \quad (2.1)$$

Lempert's theory of Kobayashi distance on convex domains ([12]) shows that for each  $z_o \in D$  the corresponding real function (2.1) is a Monge-Ampère exhaustion for  $\overline{D}$ , centered at  $z_o$ . Thus, any such domain is of circular type. Note that, in contrast with the previous construction, for such domains *any*  $z_o$  occurs as the center of a Monge-Ampère exhaustion.

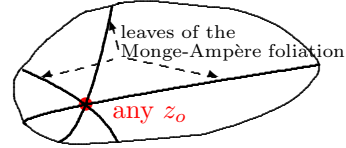
An arbitrary domain  $\overline{D}$  of circular type has always the following crucial features. For any  $z \in \overline{D} \setminus \{z_o\}$ , let  $\mathcal{Z}_z$  be the 1-dimensional kernel

$$\mathcal{Z}_z := \ker(2i\partial\bar{\partial} \log \tau)|_z \subset T_z D .$$

Being formed by the kernels of a closed 2-form, the distribution  $\mathcal{Z}$  is integrable and the (closures of) its integral leaves form a singular foliation of  $D$ , called *Monge-Ampère foliation*. The leaves are complex curves, each of them biholomorphic to the unit disk  $\Delta$  in  $\mathbb{C}$ , passing through the center  $z_o$  of the Monge-Ampère exhaustion ([16, 17, 19]).



strongly pseudoconvex circular domain



strictly linearly convex domain

The existence of such foliation has important consequences on the Kobayashi pseudo-metrics of domains of circular type, which we now review.

**2.2. Kobayashi pseudo-metrics.** Let  $D$  be a bounded domain of  $\mathbb{C}^n$  and  $z_o$  a point of  $D$ . We recall that the *Kobayashi (infinitesimal) pseudo-metric* of  $D$  at  $z_o$  is the real valued function on  $T_{z_o} D \simeq \mathbb{C}^n$ , defined by

$$\kappa_{z_o} : T_{z_o} D \setminus \{0\} \rightarrow [0, +\infty) , \quad \kappa_{z_o}(\underline{v}) := \inf_{f \in \mathcal{A}_{(z_o, \underline{v})}} \|f_*^{-1}(\underline{v})\| ,$$

where  $\mathcal{A}_{(z_o, \underline{v})}$  denotes the set of all holomorphic maps from  $\Delta$  to  $D$ , passing through  $z_o$  and tangent to  $\underline{v}$ , i.e.

$$\mathcal{A}_{(z_o, \underline{v})} := \left\{ f : \Delta \rightarrow D \text{ holom. with } f(0) = z_o, f_* \left( \lambda \frac{\partial}{\partial x} \Big|_0 \right) = \underline{v} \text{ for } \lambda \in \mathbb{R} \right\}.$$

The literature on the Kobayashi pseudometric is vast (see e.g. [10, 11, 9] and references therein). Here we just recall two simple – but crucial – facts.

- 1) It satisfies the so-called *distance decreasing property*, i.e. for any holomorphic map  $F : D \rightarrow D'$  between two domains  $D, D'$  and for each  $z_o \in D$ , the Kobayashi pseudo-metrics  $\kappa_{z_o}$  and  $\kappa_{F(z_o)}$  of  $D$  and  $D'$ , respectively, satisfy the inequality

$$\kappa_{F(z_o)}(F_*(\underline{v})) \leq \kappa_{z_o}(\underline{v}) \quad \text{for each } \underline{v} \in T_{z_o}D.$$

It follows immediately that if  $F$  is a biholomorphism, then the equality holds, meaning that the Kobayashi pseudo-metric is a (very important) biholomorphic invariant.

- 2) For each  $z_o \in D$  and  $\underline{v} \in T_{z_o}D$ ,

$$\kappa_{z_o}(\lambda \underline{v}) = |\lambda| \kappa_{z_o}(\underline{v}) \quad \text{for each } \lambda \in \mathbb{C}.$$

This yields that the *indicatrix* of the Kobayashi pseudo-metric at  $z_o$ , that is the set

$$I_{z_o} := \{\underline{v} \in T_{z_o}D \setminus \{0\} : \kappa_{z_o}(\underline{v}) < 1\} \cup \{0\}, \quad (2.2)$$

is always a balanced domain of  $T_{z_o}D \simeq \mathbb{C}^n$ .

The situations where  $I_{z_o}$  is a *strongly pseudoconvex domain* for each point  $z_o$  are of particular interest, because in those cases the Kobayashi pseudo-metric  $\kappa : TD \rightarrow [0, +\infty)$  is a (possibly non-smooth) complex Finsler metric.

**2.3. Finsler metrics.** Let us shortly recall the definitions of real and complex Finsler metrics. Given an  $n$ -dimensional real manifold  $M$ , with tangent bundle  $TM$ , a (smooth) *real Finsler metric* is a continuous map

$$F : TM \rightarrow [0, +\infty),$$

which is  $\mathcal{C}^\infty$  on  $TM^o := TM \setminus \{\text{zero section}\}$  and such that

- a) for every non-negative *real* number  $\lambda$  and every vector  $(x, \underline{v}) \in T_x M$  at some point  $x \in M$  one has that

$$F(x; \lambda \underline{v}) = |\lambda| F(x; \underline{v})$$

- b) for each  $x_o \in M$ , the indicatrix  $I_{x_o} := \{(x_o, \underline{v}) : F(x_o; \underline{v}) < 1\}$  is a strictly linearly convex domain of  $T_{x_o}M \simeq \mathbb{R}^n$ .

In other words, a smooth real Finsler metric is a norm function on all tangent spaces of  $M$ , which is smoothly depending on the base points and on the (non-zero) tangent vectors (when such dependence is not  $\mathcal{C}^\infty$ , it is usually said that  $F$  is a “non-smooth” Finsler metric). In fact, a very simple example of Finsler metric on a manifold  $M$  is given by the norm function

$$F(x; \underline{v}) := \sqrt{g_x(\underline{v}, \underline{v})},$$

determined by some fixed Riemannian metric  $g$  on  $M$ . But many other examples, for which there is no associated Riemannian metric, can be easily constructed. Indeed, in order to define a Finsler metric, it suffices to fix a linearly convex indicatrix  $I_x$  in each tangent space, smoothly depending on the base point  $x$ , and use it to define a norm function. If the assigned indicatrices are not linearly equivalent to quadrics, the corresponding Finsler metric cannot be associated with any Riemannian metric.

The notion of complex Finsler metric is very similar. If  $M$  is a complex manifold of complex dimension  $n$ , a (smooth) *complex Finsler metric* on  $M$  is a continuous real valued map

$$F : TM \longrightarrow [0, +\infty) ,$$

which is  $\mathcal{C}^\infty$  on  $TM^o := TM \setminus \{\text{zero section}\}$  and satisfies the following analogues of (a) and (b):

a') for every *complex* number  $\lambda$  and every vector  $(x, \underline{v}) \in T_x M$ ,  $x \in M$ ,

$$F(x; \lambda \underline{v}) = |\lambda| F(x; \underline{v});$$

b') for each  $x_o \in M$ , the indicatrix  $I_{x_o} := \{(x_o, \underline{v}) : F(x_o; \underline{v}) < 1\}$  is a strongly pseudoconvex domain of  $T_{x_o} M \simeq \mathbb{C}^n$ .

As before, a very simple example of complex Finsler metric on a complex manifold  $M$  is given by the norm function

$$F(x; \underline{v}) := \sqrt{h_x(\underline{v}, \underline{v})} ,$$

determined by an Hermitian metric  $h$  on  $M$ . But many other examples can be easily determined, for which there is no associated Hermitian metric. In perfect analogy with the real case, complex Finsler metrics are uniquely determined by the associated family of indicatrices.

As for the Kobayashi metric, also the literature on Finsler metrics is enormous. For an introduction to this important and interesting area, the reader might take a look at standard texts as [5, 3, 1] and references therein.

**2.4. Kobayashi metrics and Monge-Ampère foliations.** Coming back to the Kobayashi pseudo-metric  $\kappa$  of a domain  $D \subset \mathbb{C}^n$ , if the indicatrices (2.2) are smooth and strongly pseudoconvex and if they smoothly depend on their base points, then  $\kappa$  is a complex Finsler metric. But here come two of the most unfriendly features of Kobayashi pseudo-metrics.

- For a generic domain, the indicatrix (2.2) is usually *not strongly pseudoconvex* and the function  $\kappa : TD^o = TD \setminus \{\text{zero section}\} \rightarrow [0, +\infty)$  is often no better than  $\mathcal{C}^0$ .
- Domains, for which  $\kappa$  can be explicitly computed – hence analyzed in greater detail – are not easy to find.

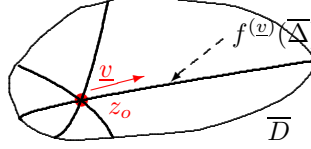
Nonetheless, for domains of circular type things are somehow nicer.

Let  $\overline{D} \subset \mathbb{C}^n$  be a strongly pseudoconvex domain of circular type, equipped with a Monge-Ampère exhaustion  $\tau$  with center  $z_o$ . As it is shown in [17], for each non-zero tangent vector  $\underline{v} \in T_{z_o} D$  at the center  $z_o$ , there exists a *unique*

proper holomorphic disk  $f^{(\underline{v})}:\overline{\Delta}\rightarrow\overline{D}$ , whose image  $f^{(\underline{v})}(\overline{\Delta})$  coincides with the closure of a leaf of the Monge-Ampère foliation of  $\tau$  and tangent to  $\underline{v}$  at the origin, i.e. satisfying the conditions

$$f(0) = z_o, \quad f_*\left(\lambda\frac{\partial}{\partial x}\Big|_0\right) = \underline{v} \quad \text{for some } \lambda \in \mathbb{R}. \quad (2.3)$$

Here  $\lambda$  is uniquely determined by  $\underline{v}$  and is non-zero. Let us denote it by  $\lambda^{(\underline{v})}$ .



A crucial relation between the Monge-Ampère foliation of  $\overline{D}$  and its Kobayashi pseudo-metric is represented by the following two facts ([18]):

- a) for each  $\underline{v} \in T_{z_o}D \setminus \{0\}$  one has that  $\kappa_{z_o}(\underline{v}) = \lambda^{(\underline{v})}$ ;
- b) the indicatrix  $I_{z_o}$  at the center of  $\kappa$  is strongly pseudoconvex.

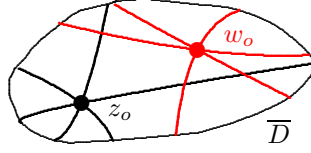
This yields to the following couple of nice properties.

- If  $D \subset \mathbb{C}^n$  is a smoothly bounded, strongly pseudoconvex circular domain, then at  $z_o = 0$  the indicatrix  $I_{z_o=0}$  is smooth and strongly pseudoconvex. In fact, it is linearly equivalent to the domain  $D$ .
- If  $D \subset \mathbb{C}^n$  is a smoothly bounded strictly linearly convex domain, then at each point  $z_o \in D$ , the indicatrix  $I_{z_o}$  is smooth and strongly pseudoconvex and it smoothly depends on the base point. In other words,  $\kappa$  is a smooth complex Finsler metric.

### 3. THE PHENOMENON OF PROPAGATION OF REGULARITY

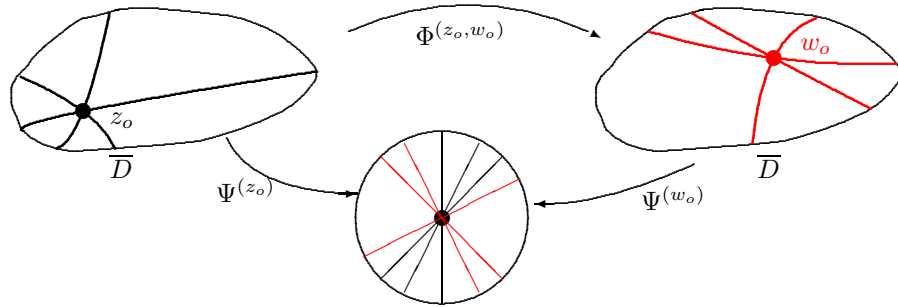
**3.1. Domains with a lot of Monge-Ampère exhaustions.** In this section, we study the properties of the domains of circular type admitting a one-parameter family of Monge-Ampère exhaustions, centered at the points of a given curve. We will shortly see that the existence of such Monge-Ampère exhaustions is equivalent to the existence of a special one-parameter family of homeomorphisms of the domain into itself, with nice regularity properties outside the curve and satisfying a special set of differential constraints. On the basis of these facts, all main results presented in this note will be built.

**3.1.1. Domains with two Monge-Ampère exhaustions.** Let  $\overline{D} \subset \mathbb{C}^n$  be a domain of circular type and assume that it admits at least *two* distinct Monge-Ampère exhaustions, the first centered at  $z_o$  and the second at  $w_o$ , and consequently with *two* Monge-Ampère foliations, one for  $z_o$ , the other for  $w_o$ .



Applying various results by the first author on the so-called *circular representation* ([17]), it is possible to show that the existence of a Monge-Ampère exhaustion  $\tau^{(z_o)}$  of class  $\mathcal{C}^{k,\alpha}$ ,  $k \geq 4$ ,  $\alpha > 0$ , off the center  $z_o$ , implies also the existence of a “straightening” homeomorphism  $\Psi^{(z_o)} : \overline{D} \rightarrow \overline{\mathbb{B}^n}$  onto the closed unit ball of  $\mathbb{C}^n$  centered at 0, with the following nice properties ([19, 22]):

- a)  $\Psi^{(z_o)}(z_o) = 0$  and each level set  $\{\tau^{(z_o)}(z) = c\}$ ,  $0 < c \leq 1$ , is mapped diffeomorphically onto the sphere  $\{|z| = c\}$  of radius  $c$ . The restriction of  $\Psi^{(z)}$  on each such level set is *in general not a CR map*, but nonetheless maps the real contact distributions underlying the two CR structures one into the other.
- b) The disks of the Monge-Ampère foliation through  $z_o$  are mapped bijectively onto the straight disks of  $\mathbb{B}^n$  through 0. Each restriction of  $\Psi^{(z_o)}$  along one such a disk is a biholomorphism.
- c)  $\Psi^{(z_o)}$  is of class  $\mathcal{C}^{k-2,\alpha}$  on  $\overline{D} \setminus \{z_o\}$ . Further, if we denote by  $\widetilde{\overline{D}}^{(z_o)}$  and  $\widetilde{\overline{\mathbb{B}^n}}$  the blow-ups at  $z_o$  and 0 of the domains, then the restriction  $\Psi^{(z_o)}|_{\overline{D} \setminus \{z_o\}}$  admits a unique  $\mathcal{C}^{k-2,\alpha}$ -extension to a map between  $\widetilde{\overline{D}}^{(z_o)}$  and  $\widetilde{\overline{\mathbb{B}^n}}$ .



Due to this, if we have *two* Monge-Ampère exhaustions, we may compose the *two* associated “straightening” homeomorphisms and get a homeomorphism from  $\overline{D}$  into itself  $\Phi^{(z_o, w_o)} := (\Psi^{(w_o)})^{-1} \circ \Psi^{(z_o)} : \overline{D} \rightarrow \overline{D}$  such that

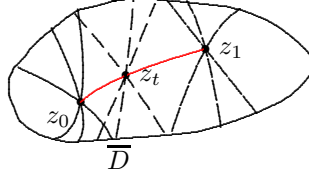
- $\alpha)$   $\Phi^{(z_o, w_o)}$  transforms the center  $z_o$  into the center  $w_o$  and maps each level set of  $\tau^{(z_o)}$  into the corresponding one of  $\tau^{(w_o)}$ . The restriction of  $\Phi^{(z_o, w_o)}$  on each level set maps one to the other the contact structures underlying the CR structures of those hypersurfaces.
- $\beta)$  The disks of the Monge-Ampère foliation through  $z_o$  are mapped bijectively onto the disks of the Monge-Ampère foliation through  $w_o$  and on each of them the restriction of  $\Phi^{(z_o, w_o)}$  is a biholomorphism.



$\gamma$ ) the map  $\Phi^{(z_o, w_o)}$  is of class  $\mathcal{C}^{k-2, \alpha}$  on  $\overline{D} \setminus \{z_o\}$  and, considering the blow-ups  $\widetilde{\overline{D}}^{(z_o)}$  and  $\widetilde{\overline{D}}^{(w_o)}$  at  $z_o$  and  $w_o$ , the restriction  $\Phi^{(z_o, w_o)}|_{\overline{D} \setminus \{z_o\}}$  admits a unique  $\mathcal{C}^{k-2, \alpha}$ -extension that goes from  $\widetilde{\overline{D}}^{(z_o)}$  to  $\widetilde{\overline{D}}^{(w_o)}$ .

### 3.1.2. Domains with a one-parameter family of Monge-Ampère exhaustions.

Let us now address the case of a closed domain of circular type  $\overline{D}$  admitting a *one-parameter family of Monge-Ampère exhaustions*, centered at the points  $z_t$ ,  $t \in [0, 1]$ , of a smooth curve of  $D$ .



The remarks of previous section imply that in this case  $\overline{D}$  is equipped with a 1-parameter family of homeomorphisms  $\Phi^{(t)} : \overline{D} \longrightarrow \overline{D}$ ,  $t \in [0, 1]$ , such that:

- 1)  $\Phi^{(t)}$  is  $\mathcal{C}^{k-2, \alpha}$  on  $\overline{D} \setminus \{z_0\}$ , with  $\Phi^{(t)}|_{\overline{D} \setminus \{z_0\}}$  with a  $\mathcal{C}^{k-2, \alpha}$ -extension to a map from the blow up  $\widetilde{\overline{D}}^{(z_0)}$  at  $z_0$  onto the blow up  $\widetilde{\overline{D}}^{(z_t)}$  at  $z_t$ ;
  - 2)  $\Phi^{(t)}$  maps  $z_0$  into  $z_t$ , and sends the level sets of  $\tau^{(z_0)}$  onto the corresponding level sets of  $\tau^{(z_t)}$  by contact transformations. In particular, it induces a  $\mathcal{C}^{k-2, \alpha}$  contact map from  $\partial D$  into itself and  $\tau^{(t)} = \tau^{(0)} \circ (\Phi^{(t)})^{-1}$ .
  - 3) The disks of the Monge-Ampère foliation through  $z_0$  are biholomorphically mapped by  $\Phi^{(t)}$  onto the disks of the Monge-Ampère foliation through  $z_t$ .
- Since each of the associated lifts between blow-ups is  $\mathcal{C}^{k-2, \alpha}$  with  $k-2 \geq 2$ , we may consider the pull-backed tensor fields on  $\widetilde{\overline{D}}^{(z_0)}$  defined by

$$J_t := (\Phi^{(t)-1})_*(J_o) , \quad t \in [0, 1] , \quad (3.4)$$

where  $J_o$  stands for the standard complex structure of the blow up  $\widetilde{\overline{D}}^{(z_t)}$  at  $z_t$ . The  $J_t$  are tensor fields of type  $(1, 1)$ , they verify the condition  $J_t^2|_z = -I$  and their Nijenhuis tensors are identically vanishing, being each  $J_t$  a pull-back of the *integrable* complex structure  $J_o$ . Further, each of them is of class  $\mathcal{C}^{k-1, \alpha}$  with  $k-1 \geq 1$  and  $\alpha > 0$ . Hence, by Newlander-Nirenberg Theorem ([14, 13, 15, 24, 8]), each  $J_t$  is a *non-standard* integrable complex structure. We call them *non-standard* simply because in general the  $\Phi^{(t)}$  are *not biholomorphisms* and, consequently, the pull-backs of the standard complex  $J_o$  by such maps are different from  $J_o$ .

If the curve of centers  $z_t$  is at least  $\mathcal{C}^1$  and if we select the diffeomorphisms  $\Phi^{(t)}$  in such a way that the family  $\Phi^{(t)}$  is differentiable with respect to the parameter  $t$ , we may also consider the one-parameter family of vector field on

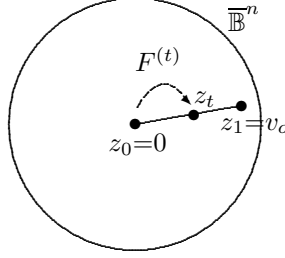
$\overline{D} \setminus \{z_0\}$ , defined by

$$X_t|_z := (\Phi^{(t)-1})_* \left( \frac{d\Phi^{(t)}}{dt} \Big|_{(t,z)} \right) \quad \text{for each } z \in \overline{D} \setminus \{z_0\}. \quad (3.5)$$

For getting a physical intuition of such vector fields, consider the one-parameter family of maps  $\Phi^{(t)}$  as a fluid motion and the coordinates of the points of  $\overline{D}$  as Lagrange coordinates for the fluid. In this way  $X_t$  can be interpreted as the *velocity field of the flow at time  $t$  in Lagrangian coordinates*.

We will shortly see that all crucial information about the  $\Phi^{(t)}$  is encoded in the one-parameter family of pairs  $(X_t, J_t)$ ,  $t \in [0, 1]$ , which we call the *fundamental pair* for the one-parameter family  $\Phi^{(t)}$ .

**3.1.3. A simple model example.** Assume that  $\overline{D} = \overline{\mathbb{B}^n}$ , let  $v_o \neq 0$  in  $\mathbb{B}^n$  and denote by  $z_t = t \cdot v_o$ ,  $t \in [0, 1]$ , the points of the segment between 0 and  $v_o$ . Since  $\mathbb{B}^n$  is homogeneous, we may consider a smooth family of automorphisms  $F^{(t)} \in \text{Aut}(\overline{\mathbb{B}^n}, J_o)$ , mapping the origin into the points  $z_t$ .



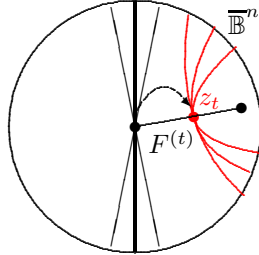
$\overline{\mathbb{B}^n}$  is clearly a circular domain with Minkowski function  $\|\cdot\|$  and

$$\tau^{(0)} : \overline{\mathbb{B}^n} \longrightarrow [0, 1], \quad \tau^{(0)}(w) := \|w\|^2$$

is a Monge-Ampère exhaustion for  $\mathbb{B}^n$  centered at 0. The corresponding Monge-Ampère foliation is made by the straight radial disks  $f^{(\underline{v})}(\overline{\Delta})$ , images of the maps  $f^{(\underline{v})}(\zeta) := \zeta \cdot \underline{v}$  with  $\|\underline{v}\| = 1$ . Using the fact that each  $F^{(t)}$  is a bi-holomorphism from  $\mathbb{B}^n$  into itself, one can directly check that the exhaustions

$$\tau^{(t)} = \tau^{(0)} \circ F^{(t)-1}$$

are all Monge-Ampère exhaustions, each of them centered at a different  $z_t$ . The corresponding Monge-Ampère foliations are made of the images under the maps  $F^{(t)}$  of the straight radial disks through the origin.



We are in the situation considered in the previous section: There is a one-parameter family of Monge-Ampère exhaustions  $\tau^{(t)}$ , centered at the points

of a curve  $z_t$ . In this special case, we may take as maps  $\Phi^{(t)} : \overline{\mathbb{B}}^n \rightarrow \overline{\mathbb{B}}^n$  the biholomorphisms  $\Phi^{(t)} = F^{(t)}$  and obtain as fundamental pair

$$J_t = (F^{(t)-1})_*(J_o) = J_o, \quad X_t = (F^{(t)-1})_* \left( \frac{dF^{(t)}}{dt} \right).$$

This situation is however somehow peculiar, because, in contrast with the generic case, here each map  $\Phi^{(t)} = F^{(t)}$  is a biholomorphism.

**3.2. The Propagation of Regularity Theorem.** We are now ready to state the main result and provide a short description of the tools which are used to obtain it. Basically, all is built upon the next two lemmas.

**3.2.1. The main lemmas.** If  $\Phi^{(t)} : \overline{D} \rightarrow \overline{D}$  is one of the families of homeomorphisms considered in §3.1.2, the associated fundamental pair  $(X_t, J_t)$ ,  $t \in [0, 1]$ , satisfies an important set of differential constraints, which we are now going to list. In order to prevent diversions of reader's attention from the most crucial aspects, we give here just intuitive descriptions of such constraints. The interested reader is referred to [22] for the detailed expressions.

The constraints that any fundamental pair satisfies are:

- (A) The restrictions of each  $J_t$  to the tangent spaces of the leaves of the Monge-Ampère foliations of  $\tau^{(0)}$  coincide with the restrictions to the same spaces of the standard complex structure  $J_o$ . This is due to the fact that the restriction of  $\Phi^{(t)}$  on each such a leaf is a biholomorphism.
- (B) The one-parameter families  $X_t$  and  $J_t$  satisfy  $\frac{dJ_t}{dt} = \mathcal{L}_{X_t} J_t$  with  $J_{t=0} = J_o$ .
- (C) Each  $J_t$  has identically vanishing Nijenhuis tensor, as pointed in §3.1.2.
- (D) Each vector field  $X_t$  satisfies special boundary conditions at  $\partial D$  and on its limit behavior near  $z_0$ . They are determined by the fact that each  $\Phi^{(t)}$  maps diffeomorphically  $\partial D$  into itself and, at the same time, it admits a  $\mathcal{C}^{k-2, \alpha}$  extension between the blow ups at  $z_0$  and  $z_t$ .

The explicit expression for the limit behavior of  $X_t$  at  $z_0$  mentioned in (D) depends in a non trivial way *on the tangent vector of the curve  $z_t$  at the time  $t$* . Such dependence is quite technical and we refer to [22] for explicit details.

All this motivates the next

**Definition 3.1.** Let  $z_t$ ,  $t \in [0, 1]$ , be a  $\mathcal{C}^1$  curve  $z_t$  in  $D$  starting from the center  $z_0$ . We call *abstract fundamental pair guided by  $z_t$*  any one-parameter family of pairs  $(X_t, J_t)$ , formed by vector fields on  $\overline{D} \setminus \{z_0\}$  and almost complex structures  $J_t$  on  $\widetilde{\overline{D}}^{(z_0)}$  satisfying the constraints (A) – (D).

If there exists a one-parameter family  $\tau^{(t)}$  of Monge-Ampère exhaustions, centered at the points  $z_t$  and with associated maps  $\Phi^{(t)}$  having  $(X_t, J_t)$  as fundamental pair, we call the pair a *concrete fundamental pair*.

We may now state our two main lemmas ([22]).

**Lemma 3.2.** *Let  $\tau = \tau^{(0)}$  be a Monge-Ampère exhaustion on  $\overline{D}$ , centered at  $z_0$  and of class  $\mathcal{C}^{k,\alpha}$ ,  $k \geq 4$ ,  $\alpha > 0$ , on  $\overline{D} \setminus \{z_0\}$ . Any abstract fundamental pair  $(X_t, J_t)$  of class  $\mathcal{C}^{k-2,\alpha}$  is concrete and is associated with the maps  $\Phi^{(t)}$  of a one-parameter family of Monge-Ampère exhaustions  $\tau^{(t)}$  of class  $\mathcal{C}^{k-2,\alpha}$  off the centers.*

**Lemma 3.3.** *Let  $\tau = \tau^{(0)}$  be a Monge-Ampère exhaustion on  $\overline{D}$  as in the previous lemma and denote by  $z_t$ ,  $t \in [0, 1]$ , the points of a radius of a holomorphic disk  $f(\Delta)$  of the Monge-Ampère foliation through  $z_0$  (i.e.  $z_t$  has the form  $z_t = f(tv)$  for some fixed  $v$  with  $\|v\| = 1$ ). Then there exists a value  $\lambda_o \in (0, 1]$  such that, for all  $\lambda \in (0, \lambda_o)$  there is an abstract (hence concrete) fundamental pair  $(X_t, J_t)$  of class  $\mathcal{C}^{k-2,\alpha}$  guided by the curve  $z_{\lambda t}$ .*

The proof of Lemma 3.2 essentially consists of two parts. One first shows that, for a given abstract pair  $(X_t, J_t)$ , one can solve the differential problem in  $\Phi^{(t)}$ , given by the equation (3.5) and the initial conditions  $\Phi^{(t=0)} = \text{Id}$ . For proving this, the key idea is to observe that the *non-linear* equation (3.5) is equivalent to a *quasi-linear* equation on the inverse maps  $\Psi^{(t)} = (\Phi^{(t)})^{-1}$ , for which the existence of solutions can be proved with little effort. Secondly, one tries to show that the compositions  $\tau^{(t)} := \tau^{(0)} \circ (\Phi^{(t)})^{-1}$  satisfy all conditions for being Monge-Ampère exhaustions. The only non immediate points of this check are reduced to prove that the maps  $\Phi^{(t)}$  have uniformly bounded Jacobians at the points where they are differentiable. This is first proved for the points in  $\partial D$  and then shown for all other points, using an argument based on the Maximum Principle for harmonic functions.

The starting point for Lemma 3.3 is given by a preliminary result, which shows that the constraints (A) and (D) are satisfied if and only if the vector fields  $X_t$  of an abstract fundamental pair must have a very special form, with very few degrees of freedom. With this the proof boils down to showing the existence of abstract pairs  $(X_t, J_t)$ , in which the  $X_t$  have the above mentioned special form and  $J_t$  has to satisfy the remaining constraints (B) and (C). Since (C) is actually a consequence of (B), everything reduces to proving the existence of a solution to the differential problem (B) with  $X_t$  in special form. Note that the explicit expression for a vector field  $X_t$  in special form involves the complex structure  $J_t$  and the curve  $z_t$ . This makes the differential equation in (B) *non-linear* in the tensor field  $J_t$ . The proof is obtained by first proving the existence of solutions in the real-analytic category and then getting the result by an approximation argument.

An immediate consequence of the above two lemmas is the fact that, for each straight segment  $z_t$  in a disk of a Monge-Ampère foliation of sufficiently high regularity, there is a one-parameter family of Monge-Ampère exhaustions  $\tau^{(t)} : \overline{D} \rightarrow [0, 1]$ , centered at the points  $z_t$ . Thus, the next theorem follows.

**Theorem 3.4** (Propagation of regularity). *Let  $\overline{D} \subset \mathbb{C}^n$  be a closed strongly pseudoconvex domain of circular type, with a Monge-Ampère exhaustion  $\tau : \overline{D} \rightarrow [0, 1]$ , with center  $z_o$  and of class  $\mathcal{C}^{k,\alpha}$  on  $\overline{D} \setminus \{z_o\}$  for some  $k \geq 4$  and  $\alpha > 0$ . Then there is an open neighborhood  $D' \subset D$  of  $z_o$  such that any other*

point  $z$  of  $D'$  is center of a Monge-Ampère exhaustion  $\tau^{(z)}: \overline{D} \rightarrow [0, 1]$  of class  $\mathcal{C}^{k-2, \alpha}$  on  $\overline{D} \setminus \{z\}$ . The dependence of  $\tau^{(z)}$  on  $z$  is  $\mathcal{C}^{k-2, \alpha}$ .

Combining this with the previously described properties of Kobayashi metric of domains of circular type, we immediately get the following

**Corollary 3.5.** *If  $\overline{D}$  admits a Monge-Ampère exhaustion, which is  $\mathcal{C}^\infty$  off the center, its Kobayashi pseudo-metric  $\kappa: TD \rightarrow [0, +\infty)$  is a smooth Finsler metric on an appropriate open subset  $D' \subset D$ .*

Note that the proof provides also an explicit description of the points of  $\partial D'$ , thus a method to determine when  $D' = D$ . From this we see that the class of domains in  $\mathbb{C}^n$ , for which the Kobayashi pseudo-metric is actually a Finsler metric not only contains all smoothly bounded, strictly linearly convex domains, as Lempert proved, but it is much larger than that. It includes for instance all smoothly bounded, strongly pseudoconvex circular domains satisfying appropriate conditions and, further, all sufficiently small deformations of such domains ([2, 20]). This is indeed a consequence of the property that any smooth deformation  $\overline{D}''$  of a given circular domain  $\overline{D}$  is biholomorphic to an (abstract) manifold with boundary, given by equipping  $\overline{D}$  with an appropriate deformed complex structure  $J \neq J_o$ . If the deformation is sufficiently small, stability properties of the equations for stationary disks imply that  $(\overline{D}, J)$  (hence, also  $\overline{D}''$ ) admits a special foliation, made of  $J$ -stationary disks passing through  $x_o = 0$  (see e.g. [20], Prop. 3.4). Then, the regularity of the data and the stability property of the condition which give  $D' = D$  imply that these disks form the Monge-Ampère foliation of an exhaustion  $\tau: \overline{D}'' \rightarrow [0, 1]$ , which makes  $\overline{D}''$  a domain of circular type to which Corollary 3.5 applies.

We conclude stressing that all proofs in [22] are actually given in the category of complex manifolds with boundary and are valid also for abstract strongly pseudoconvex manifolds, regardless of their embeddability in  $\mathbb{C}^n$ .

#### 4. CONCLUDING REMARKS

The above described results can be taken as starting points for various lines of further investigation. Some of them can be shortly described as follows.

(1) By Theorem 3.4, the existence of a single Monge-Ampère exhaustion  $\tau_o$  of class  $\mathcal{C}^{k, \alpha}$ ,  $k \geq 4$ ,  $\alpha > 2$ , off the center implies the existence of an infinity of other Monge-Ampère exhaustions of lower regularity, smaller by two orders. Such loss of regularity is due to a technical tool used in the proof, namely the use of the so-called *normalization maps*. Other than this, we do not see any intuitive reason for such loss of regularity and we expect that the main results can be refined on this aspect. Any such improvement would be quite valuable.

Note also that if  $\tau_o$  is a Monge-Ampère exhaustion, then the logarithm  $u_o = \log \tau_o$  is a pluricomplex Green function, with pole at the center of  $\tau_o$ . Hence, improvements of our results in the described direction might give useful information on pluricomplex Green functions and enrich the theory of such

functions that has been so far developed (for the known regularity properties of pluricomplex Green functions, see e.g. [6, 7, 4]).

(2) Corollary 3.5 implies that if  $\overline{D}$  is a closed domain of circular type with a smooth Monge-Ampère exhaustion and satisfying appropriate conditions, then  $\overline{D}$  is completely determined by the Finsler invariants of its Kobayashi metric, namely by the Finsler curvature and all Finsler covariant derivatives up to an appropriate order ([23]). On the other hand, the same domain is equipped also with another important sequence of invariants, the *Bland and Duchamp invariants*, which are tensor fields on the blow up of  $\overline{D}$  at the center of a Monge-Ampère exhaustion, which describe how the CR structures of the level sets  $\{\tau = c\}$  evolve when  $c$  varies between 0 and 1 ([2, 19]). Also these invariants completely determine  $\overline{D}$  up to biholomorphisms.

We feel that it is possible to determine explicit relations between the Finsler and the Bland and Duchamp invariants. Finding such relations would very likely lead to a deep insight on the intrinsic properties of strictly linearly convex domains and, more generally, of all domains of circular type.

(3) From Lempert theory, we know that on any closed, smoothly bounded, strictly linearly convex domain  $\overline{D}$ , the disks of the Monge-Ampère foliations are *complex geodesics* for the Kobayashi metric of the domain. For other types of closed domains of circular type with smooth Monge-Ampère exhaustions, the disks of the Monge-Ampère foliations are surely *extremal disks* for the Kobayashi metric at the centers but there is no manifest reason for them to be complex geodesics. Since the property of being a complex geodesic can be nicely described in terms of Finsler covariant derivatives (see e.g. [1]), writing down the explicit relations between the Bland and Duchamp invariants and the Finsler invariants might be helpful to characterize the convexifiable domains (i.e. those that are biholomorphic to some strictly linearly convex domain) in terms of their Bland and Duchamp invariants. Combining this with the so far known techniques for constructing domains with prescribed Bland and Duchamp invariants ([2, 21]), all this would pave the way towards a useful characterization of convexifiable domains of  $\mathbb{C}^n$ .

## REFERENCES

- [1] M. Abate & G. Patrizio, *Finsler Metrics - A Global Approach*, *Lecture Notes in Mathematics* **1591**, Springer-Verlag, 1994.
- [2] J. Bland and T. Duchamp, *Moduli for pointed convex domains*, *Invent. Math.* **104** (1991), 61–112.
- [3] D. Bao, S. S. Chern & Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer-Verlag, 2000.
- [4] Z. Błocki, *The  $C^{1,1}$  regularity of the pluricomplex Green function*, *Michigan Math. J.*, **47**, (2000), 211–215.
- [5] S.-S. Chern, *Finsler geometry is just Riemannian geometry without the quadratic restriction*, *Notices Amer. Math. Soc.* **43**, (1996), 959–963.
- [6] J.-P. Demailly, *Measures de Monge-Ampère et mesures pluriharmoniques*, *Math. Z.* **194** (1987), 519–564.
- [7] B. Guan, *The Dirichlet problem for complex Monge-Ampère equations and regularity of the pluri-complex Green function*, *Comm. Anal. Geom.* **6** (1998), 687–703.

- [8] C. D. Hill and M. Taylor, *Integrability of Rough Almost Complex Structures*, J. Geom. Anal. **13** (1) (2003), 163–172.
- [9] M. Jarnicki and P. Pflug, Invariant distances and metrics in complex analysis, *Walter de Gruyter GmbH & Co. KG, Berlin*, 2013.
- [10] S. Kobayashi, Hyperbolic manifolds and holomorphic mappings, *Dekker, New York*, 1970.
- [11] S. Kobayashi, Hyperbolic complex spaces, *Springer, New York*, 1998.
- [12] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France, **109** (1981), 427–474.
- [13] B. Malgrange, *Sur l'intégrabilité des structure presque-complex*, in Symposia Math. vol. II, 289–296, *Academic Press*, 1969.
- [14] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifold*, Ann. of Math. **65** (1957), 391–404.
- [15] A. Nijenhuis and W. Woolf, *Some integration problem in almost-complex manifolds*, Ann. of Math. **77** (1963), 424–489.
- [16] G. Patrizio, *Parabolic Exhaustions for Strictly Convex Domains*, Manuscripta Math. **47** (1984), 271–309.
- [17] G. Patrizio, *A characterization of complex manifolds biholomorphic to a circular domain*, Math. Z. **189** (1985), 343–363.
- [18] G. Patrizio, *Disques extrémaux de Kobayashi et équation de Monge-Ampère complex*, C. R. Acad. Sci. Paris, Série I, **305** (1987), 721–724.
- [19] G. Patrizio and A. Spiro, *Monge-Ampère equations and moduli spaces of manifolds of circular type*, Adv. Math. **223** (2010), 174–197.
- [20] G. Patrizio and A. Spiro, *Foliations by stationary disks of almost complex domains*, Bull. Sci. Math., **134** (2010), 215–234.
- [21] G. Patrizio and A. Spiro, *Modular data and regularity of Monge-Ampère exhaustions and of Kobayashi distance*, Math. Ann. **362** (2015), 425–449.
- [22] G. Patrizio and A. Spiro, *Propagation of regularity for Monge-Ampère exhaustions and Kobayashi metrics*, arXiv:1707.09041 (2017).
- [23] A. Spiro, *The structure equations of a complex Finsler manifold*, Asian J. Math. **5** (2001), 291–326.
- [24] S. Webster, *A new proof of the Newlander-Nirenberg theorem*, Math. Zeit. **201** (1989), 303–316.

GIORGIO PATRIZIO  
 DIP. MATEMATICA E INFORMATICA “U. DINI”  
 UNIVERSITÀ DI FIRENZE  
 & ISTITUTO NAZIONALE DI  
 ALTA MATEMATICA  
 “FRANCESCO SEVERI”  
 E-mail: patrizio@math.unifi.it

ANDREA SPIRO  
 SCUOLA DI SCIENZE E TECNOLOGIE  
 UNIVERSITÀ DI CAMERINO  
 VIA MADONNA DELLE CARCERI  
 I-62032 CAMERINO (MACERATA)  
 ITALY  
 E-mail: andrea.spiro@unicam.it