A distributed Kalman filter with event-triggered communication and guaranteed stability

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Abstract

The paper addresses Kalman filtering over a peer-to-peer sensor network with a careful eye towards data transmission scheduling for reduced communication bandwidth and, consequently, enhanced energy efficiency and prolonged network lifetime. A novel consensus Kalman filter algorithm with event-triggered communication is developed by enforcing each node to transmit its local information to the neighbors only when this is considered as particularly significant for estimation purposes, in the sense that it notably deviates from the information that can be predicted from the last transmitted one. Further, it is proved how the filter guarantees stability (mean-square boundedness of the estimation error in each node) under network connectivity and system collective observability. Finally, numerical simulations are provided to demonstrate practical effectiveness of the distributed filter for trading off estimation performance versus transmission rate.

Key words: Distributed Kalman filtering; sensor networks; event-triggered communication; sensor fusion.

1 Introduction

Nowadays, *wireless sensor networks* (WSNs) are getting an ever increasing usage in a wide range of on-line monitoring tasks (e.g. navigation, tracking, environmental and power system monitoring, etc.) that require recursive estimation of the state of a linear or nonlinear dynamical system. Since the individual nodes of the sensor network are usually low-cost, battery-supplied devices with scarce energy resources, it becomes of paramount importance for networked state estimation to limit as much as possible data transmission which represents by far the most energy consuming node task.

In this respect, simple ways to limit the communication bandwidth are *periodic* and *random* transmission at a prescribed rate, whose effects on distributed Kalman filter stability and estimation performance are analyzed in Battistelli et al. (2012b) for a *centralized network* wherein all nodes transmit their local data (either measurements or estimates) to a fusion center. It is natural to expect however that a data-driven transmission strategy could easily outperform periodic and random scheduling. These considerations motivate the growing interest towards the development of data-driven (or event-triggered) logics for scheduling data communication. Interested readers are referred to Suh et al. (2007); Marck and Sijs (2010); Shi et al. (2011); Battistelli et al. (2012a); Shi et al. (2014); Sijs et al. (2014); Trimpe and D'Andrea (2014); Han et al. (2015); Shi et al. (2016), and references therein, for an overview on the stability properties and performance achievable by these strategies on a centralized network. A great deal of work (Olfati-Saber, 2009; Stankovic et al., 2009; Farina et al., 2010; Kamal et al., 2013; Ugrinovskii, 2013; Battistelli and Chisci, 2014; Battistelli et al., 2015; Noack et al., 2016) has concerned distributed state estimation over a *peer-to-peer network* wherein there is no fusion center and each node (peer) operates in the same way and can only exchange data with a limited subset of neighbors. All these references, however, have considered the situation wherein each node broadcasts data to neighbors after each update of the local information. In this respect, recent work (Yan et al., 2013; Liu et al., 2015; Li et al., 2012; Meng and Chen, 2014; Wu et al., 2015; Li et al., 2016) has addressed distributed state estimation with event-triggered communication. In particular Yan et al. (2013) and Liu et al. (2015) proposed

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measurement-based transmission tests on the distance between the current and latest transmitted measurements and, respectively, on the innovation. Conversely (Li et al., 2012; Meng and Chen, 2014; Wu et al., 2015; Li et al., 2016) developed event-triggered distributed state estimators all relying on the consensus Kalman filter of Olfati-Saber (2009) but differing for the adopted triggering condition. This paper presents a novel event-triggered distributed state estimator based on a different consensus Kalman filtering approach (Battistelli and Chisci, 2014; Battistelli et al., 2015) as well as on a different transmission triggering condition which essentially requires that the local estimate and/or covariance of a given node be sufficiently far away from the ones that could be computed by neighbors, exploiting only the transmitted data. It is proved that the proposed distributed Kalman filter algorithm with event-triggered communication enjoys nice stability properties (i.e., mean-square boundedness of the state estimation error in all nodes) under minimal requirements of network connectivity and collective system observability extending, in a non trivial way, the results of Battistelli and Chisci (2014); Battistelli et al. (2015); Battistelli and Chisci (2016) already available for the full transmission case. This paper extends preliminary work carried out in (Battistelli et al., 2016) with the stability analysis.

2 Distributed State Estimation Setting

This paper addresses distributed state estimation (DSE) over a network in which each node can process local data as well as exchange data with neighbors. Further, some nodes (called sensor nodes) have also sensing capabilities, i.e., they can sense data from the environment. Notice that the presence of nodes without sensing capabilities serves only the purpose of improving network connectivity. In the sequel, the sensor network will be denoted as $(\mathcal{N}, \mathcal{A}, \mathcal{S})$ where: $\mathcal{N} = \{1, \ldots, N\}$ is the set of nodes; $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of arcs (edges); $\mathcal{S} \subseteq \mathcal{N}$ is the subset of sensor nodes. A directed edge $(i, j) \in \mathcal{A}$ from node i to node j means that i can send messages to j, and we say that j is an out-neighbor of i and i is an in-neighbor of j. For each node $i \in \mathcal{N}$, $\mathcal{N}_i \subseteq \mathcal{N}$ will denote the set of its in-neighbors, i.e. $\mathcal{N}_i \stackrel{\triangle}{=} \{j : (j, i) \in \mathcal{A}\}$. The network does not contain self-loops so that $i \notin \mathcal{N}_i$.

The DSE problem can be formulated as follows. Each node $i \in \mathcal{N}$ must estimate at each time $k \in \mathbb{Z}^+ = \{0, 1, \ldots\}$ the state x_k of the dynamical system

$$x_{k+1} = Ax_k + w_k \tag{1}$$

given local measurements

$$y_k^i = C^i x_k + v_k^i\,, \qquad i \in \mathcal{S}, \tag{2}$$

and data received from all in-neighbors $j \in \mathcal{N}_i$.

It is assumed that w_k and v_k^i, $i \in \mathcal{S}$, are zero-mean white noises with positive definite covariance matrices Q and R^i, $i \in \mathcal{S}$, respectively. Further, the process disturbance and the measurement noises are supposed to be uncorrelated, i.e., $\mathbb{E}\{v_k^i\, w_\tau^\top\} = 0$, for any $k, \tau \in \mathbb{Z}^+$, and $i \in \mathcal{S}$.

In this setting, it was recently shown that there exist families of consensus-based DSE algorithms (Kamal et al., 2013; Battistelli and Chisci, 2014; Battistelli et al., 2015) able to guarantee stability of the estimation error in each network node under the minimal requirements of collective detectability and network connectivity. Generally speaking, these algorithms require that each node i transmits its local estimate $\hat{x}_{k|k}^i$ and covariance matrix $P_{k|k}^i$ to all its out-neighbors such that $i \in \mathcal{N}_j$ at least once for each sampling interval. However, in many contexts, it is desirable to reduce data transmission as much as possible while preserving performance. The objective of this paper is precisely that of developing a strategy for controlling transmission in existing DSE algorithms, so that each node i selectively transmits only the most relevant data, without compromising stability properties.

To this end, let us introduce for each node i binary variables c_k^i such that $c_k^i = 1$ if node i transmits at time k or $c_k^i = 0$ otherwise. The focus is on data-driven (or event-triggered) transmission strategies in which the variable c_k^i is a function of the information currently available in node i and of the information most recently transmitted by node i.

3 Distributed Kalman-Filtering with Event-triggered Communication

In this paper, we focus on a DSE algorithm wherein each node $i \in \mathcal{N}$ runs a local Kalman filter and then, in order to improve its local estimate, fuses the local information with the one received from its in-neighbors $j \in \mathcal{N}_i$. Concerning the local Kalman filter, it is convenient for the presentation of the algorithm to consider the *information form* of the Kalman filter recursion which, instead of the estimate $\hat{x}_{k|k}^i$ and of the covariance matrix $P_{k|k}^i$, propagates the *information matrix* $\Omega_{k|k}^i = (P_{k|k}^i)^{-1}$ and the *information vector* $q_{k|k}^i = \Omega_{k|k}^i \hat{x}_{k|k}^i$. Hereafter, the steps of the proposed DSE algorithm are described in some detail.

Correction: Let $(q_{k|k-1}^i, \Omega_{k|k-1}^i)$ denote the predicted *information pair* available in node i at time k. Then, for any sensor node $i \in \mathcal{S}$, the local information pair is updated by means of the standard Kalman filter correction step

$$q_{k|k}^i = q_{k|k-1}^i + (C^i)^\top (R^i)^{-1} y_k^i\,, \tag{3}$$

$$\Omega_{k|k}^i = \Omega_{k|k-1}^i + (C^i)^\top (R^i)^{-1} C^i\,. \tag{4}$$

In all the remaining nodes $i \in \mathcal{N} \setminus \mathcal{S}$, since no local measurement is available, we simply set $(q^i_{k|k}, \Omega^i_{k|k}) = (q^i_{k|k-1}, \Omega^i_{k|k-1})$.

Information exchange: Notice preliminarily that, after the correction step, the currently available information is represented by the local posterior information pair $(q^i_{k|k}, \Omega^i_{k|k})$. Let now n^i_k be the number of discrete time instants elapsed from the most recent transmission of node i, so that the most recently transmitted data is $(q^i_{k-n^i_k|k-n^i_k}, \Omega^i_{k-n^i_k|k-n^i_k})$. Such data can be propagated in time by repeatedly applying the Kalman filter prediction step so as to obtain the information pair $(\bar{q}^i_k, \bar{\Omega}^i_k)$ (see equations (13)-(14) in the prediction step below). Accordingly, $\bar{x}^i_k = (\bar{\Omega}^i_k)^{-1}\, \bar{q}^i_k$ represents a prediction of the system state based on the data most recently transmitted by node i.

Notice that $\left(\bar{q}^i_k, \bar{\Omega}^i_k\right)$ can be computed also by the out-neighbors of node i. Then, the idea is to selectively transmit only in case the discrepancy between $(q^i_{k|k}, \Omega^i_{k|k})$ and $(\bar{q}^i_k, \bar{\Omega}^i_k)$ is large, which means that the data $(\bar{q}^i_k, \bar{\Omega}^i_k)$ currently computable by the out-neighbors of node i is no longer consistent with the data locally available in node i. More formally, the following event-triggered transmission strategy is adopted

$$c^i_k = \begin{cases} 0, \text{ if } \|\hat{x}^i_{k|k} - \bar{x}^i_k\|^2_{\Omega^i_{k|k}} \leq \alpha \\ \qquad \text{and } \frac{1}{1+\beta}\, \Omega^i_{k|k} \leq \bar{\Omega}^i_k \leq (1+\delta)\, \Omega^i_{k|k} \\ 1, \text{ otherwise} \end{cases} \tag{5}$$

where α, β, and δ are positive scalars and, given a positive definite matrix M, $\|\cdot\|_M$ denotes the corresponding weighted Euclidean norm.

The three scalars α, β, and δ can be seen as design parameters which can be tuned so as to achieve a desired behavior in terms of transmission rate and performance. In particular, the transmission test in (5) is designed so as to ensure that, in the case of no transmission, the data $(\bar{q}^i_k, \bar{\Omega}^i_k)$ currently computable by the out-neighbors of node i are close to the data locally available in node i both in terms of mean and covariance.

Remark 1 *From the information-theoretic point of view, the proposed event-triggered transmission strategy can be analyzed by resorting, for instance, to the Kullback-Leibler divergence (KLD). To see this, let us denote by $\mathcal{G}_{k|k}(x)$ and $\overline{\mathcal{G}}_k(x)$ the Gaussian densities associated with the information pairs $(q^i_{k|k}, \Omega^i_{k|k})$ and, respectively, $(\bar{q}^i_k, \bar{\Omega}^i_k)$. Then, it turns out that when no transmission occurs, i.e., $c^i_k = 0$, the KLD between such two densities can be bounded as*

$$D_{KL}(\overline{\mathcal{G}}_k || \mathcal{G}_{k|k}) \leq \frac{1}{2}[\alpha + \beta\, n + n \log(1+\delta)] \tag{6}$$

where n is the dimension of the system state x_k and $D_{KL}(p||q) = \int p(x) \log \frac{p(x)}{q(x)} dx$. Hence, the proposed approach is related to the idea, already proposed in the literature in the context of event-based estimation (Marck and Sijs, 2010; Li et al., 2012), of defining triggering conditions directly expressed in terms of KLD. Test (5) however allows to weight differently the performance objectives in terms of estimate/covariance. For a formal proof of (6) as well as for further discussion on this issue, the interested reader is referred to (Battistelli et al., 2016).

Information fusion: When node i receives data from all its in-neighbors $j \in \mathcal{N}_i$, a viable way to perform fusion amounts to computing the fused information pair as a convex combination of the local information pairs:

$$q^{i,F}_{k|k} = \pi_{i.i}\, q^i_{k|k} + \sum_{j \in \mathcal{N}_i} \pi_{i,j}\; q^j_{k|k} \tag{7}$$

$$\Omega^{i,F}_{k|k} = \pi_{i,i}\, \Omega^i_{k|k} + \sum_{j \in \mathcal{N}_i} \pi_{i,j}\; \Omega^j_{k|k}\, , \tag{8}$$

where the combination weights $\pi_{i,j}$ in (7)-(8) are taken all strictly positive and satisfy the condition

$$\pi_{i,i} + \sum_{j \in \mathcal{N}_i} \pi_{i,j} = 1, \forall i \in \mathcal{N}\, . \tag{9}$$

In fact, DSE algorithms based on (7)-(8), which is the well-known *covariance intersection* (Julier and Uhlmann, 2001) fusion rule, have been shown to enjoy interesting properties such as robustness with respect to the unknown correlation of the estimates to be fused as well as stability of the estimation error dynamics (see Battistelli and Chisci, 2014).

When, however, for some neighbor j it happens that $c^j_k = 0$, then $(q^j_{k|k}, \Omega^j_{k|k})$ is not available and the fusion rule has to be modified. In this case, thanks to the adopted event-triggered transmission strategy (5), node i is still able to compute $(\bar{q}^j_k, \bar{\Omega}^j_k)$ and to infer that such an information pair is close to the true one $(q^j_{k|k}, \Omega^j_{k|k})$. Then, such a knowledge can be exploited by replacing, in the information fusion step, $(q^j_{k|k}, \Omega^j_{k|k})$ with

$$\tilde{q}^j_k = \frac{1}{1+\delta}\, \bar{q}^j_k\, , \quad \tilde{\Omega}^j_k = \frac{1}{1+\delta}\, \bar{\Omega}^j_k\, . \tag{10}$$

The introduction of the factor $1/(1+\delta)$ in (10) is intended to reduce the weight of the contribution of node j in the convex combination, so as to account for the additional uncertainty due to the discrepancy between $(\bar{q}^j_k, \bar{\Omega}^j_k)$ and $(q^j_{k|k}, \Omega^j_{k|k})$. In particular, it can be readily seen that the adopted choice enjoys the following positive features: (i) the information matrix $\tilde{\Omega}^j_k$ always satisfies

Table 1
Algorithm 1 - Distributed Kalman Filtering with Event-Triggered Communication

At each time $k = 0, 1, \ldots$, for each node $i \in \mathcal{N}$:

(1) **Correction:**
if $i \in \mathcal{S}$, collect the local measurement y_k^i and update the local *information pair* $(q^i_{k|k-1}, \Omega^i_{k|k-1})$ via equations (3)-(4) to obtain the local posterior information pair $(q^i_{k|k}, \Omega^i_{k|k})$; otherwise, for any $i \in \mathcal{N} \setminus \mathcal{S}$, set $(q^i_{k|k}, \Omega^i_{k|k}) = (q^i_{k|k-1}, \Omega^i_{k|k-1})$;

(2) **Information exchange:**
if $k = 0$ set $c_k^i = 1$, otherwise determine c_k^i as in (5);
if $c_k^i = 1$ transmit $(q^i_{k|k}, \Omega^i_{k|k})$ to the out-neighbors;
receive $(q^j_{k|k}, \Omega^j_{k|k})$ from all the in-neighbors $j \in \mathcal{N}_i$ for which $c_k^j = 1$;

(3) **Information fusion:**
compute the fused information pair $(q^{i,F}_{k|k}, \Omega^{i,F}_{k|k})$ by

$$q^{i,F}_{k|k} = \pi_{i.i}\, q^i_{k|k} + \sum_{j \in \mathcal{N}_i} \pi_{i,j} \left[c_k^j q^j_{k|k} + (1 - c_k^j)\tilde{q}_k^j \right]$$
$$\Omega^{i,F}_{k|k} = \pi_{i,i}\, \Omega^i_{k|k} + \sum_{j \in \mathcal{N}_i} \pi_{i,j} \left[c_k^j \Omega^j_{k|k} + (1 - c_k^j)\tilde{\Omega}_k^j \right] ,$$

where $(\tilde{q}_k^j, \tilde{\Omega}_k^j)$ are computed as in (10).

(4) **Prediction:**
compute the local prior information pair $(q^i_{k+1|k}, \Omega^i_{k+1|k})$ from $(q^{i,F}_{k|k}, \Omega^{i,F}_{k|k})$ via (11)-(12);
compute $(\bar{q}^i_{k+1}, \bar{\Omega}^i_{k+1})$ and $(\bar{q}^j_{k+1}, \bar{\Omega}^j_{k+1})$, $j \in \mathcal{N}_i$, via (13)-(14).

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the inequality $\tilde{\Omega}_k^j \le \Omega^j_{k|k}$, thus ensuring that the information matrix after fusion is never larger than the one which would be obtained in case all the nodes transmit (this property is important in order to ensure the consistency of the distributed estimator); (ii) the estimate used in the fusion is not modified by the introduction of such a factor in that $(\tilde{\Omega}_k^j)^{-1}\tilde{q}_k^j = (\bar{\Omega}_k^j)^{-1}\bar{q}_k^j = \bar{x}_k^j$.

Prediction: In each network node $i \in \mathcal{N}$, the fused information pair $(q^{i,F}_{k|k}, \Omega^{i,F}_{k|k})$ is propagated in time by applying the Kalman filter prediction step

$$q^i_{k+1|k} = \Omega^i_{k+1|k}\, A \left(\Omega^{i,F}_{k|k}\right)^{-1} q^{i,F}_{k|k} , \tag{11}$$
$$\Omega^i_{k+1|k} = Q^{-1} - Q^{-1} A \left(\Omega^{i,F}_{k|k} + A^\top Q^{-1} A\right)^{-1} A^\top Q^{-1} . \tag{12}$$

Further, the information pairs $(\bar{q}^i_{k+1}, \bar{\Omega}^i_{k+1})$, to be used in the transmission tests at time $k+1$, are computed as

$$\bar{q}^i_{k+1} = \bar{\Omega}^i_{k+1}\, A \left(\breve{\Omega}_k^i\right)^{-1} \breve{q}_k^i , \tag{13}$$
$$\bar{\Omega}^i_{k+1} = Q^{-1} - Q^{-1} A \left(\breve{\Omega}_k^i + A^\top Q^{-1} A\right)^{-1} A^\top Q^{-1} \tag{14}$$

where $\breve{q}_k^i = c_k^i q^i_{k|k} + (1 - c_k^i)\, \bar{q}_k^i$ and $\breve{\Omega}_k^i = c_k^i \Omega^i_{k|k} + (1 - c_k^i)\, \bar{\Omega}_k^i$.

Summing up, the above-described approach to DSE with event-triggered communication gives rise to the algorithm of Table 1.

The algorithm is initialized at time $k = 0$ with some a priori information pairs $(q^i_{0|-1}, \Omega^i_{0|-1})$. By taking the initial information matrix $\Omega^i_{0|-1}$ positive definite in all the network nodes we ensure, by construction, that all the information matrices $\Omega^i_{k|k}$ are positive definite for any k and i so that the covariance matrices $P^i_{k|k} = (\Omega^i_{k|k})^{-1}$ are always well-defined and positive definite as well.

4 Stability analysis

In this section, the stability properties of the proposed algorithm are analyzed. For the reader's convenience, all the proofs are given in the Appendix. The following preliminary assumptions are needed.

A1. The system matrix A is invertible.
A2. The system is *collectively observable*, i.e., the pair (A, C) is observable where $C := \mathrm{col}\left(C^i;\, i \in \mathcal{S}\right)$.
A3. The network is strongly connected, i.e., there exists a directed path between any pair of nodes $i, j \in \mathcal{N}$.

Notice that assumption A1 is automatically satisfied in sampled-data systems wherein the matrix A is obtained by discretization of a continuous-time system matrix. As for assumption A2, the collective observability requirement can be relaxed to collective detectability by resorting to an observability decomposition in each network node as discussed for example in Battistelli and Chisci (2014). These are the very same assumptions under which stability of the Distributed Kalman filter with full transmission rate has been proved (Battistelli and Chisci, 2014; Battistelli et al., 2015). Hereafter, we show that similar stability properties are enjoyed also by the Distributed Kalman filter with Event-Triggered Communication of Table 1.

Let us denote by Π the consensus matrix, whose elements are the consensus weights $\pi_{i,j}$, $i, j \in \mathcal{N}$ (in case $j \neq i$ does not belong to $\mathcal{N}_i$ we simply set $\pi_{i,j} = 0$). Recall that a non-negative square matrix M is row stochastic if all its rows sum up to 1. Further, it is primitive if there exists an integer ℓ such that all the elements of M^ℓ are strictly positive. By construction, the consensus matrix Π is row stochastic since all the combination weights in (7)-(8) are strictly positive and satisfy (9). It is an easy matter to verify that, under assumption A3, Π is also primitive. In turn, by the Perron-Frobenius theorem, this implies the existence of a vector p having strictly positive components $p_i, i \in \mathcal{N}$, and satisfying the equation $p^\top \Pi = p^\top$, i.e., $\sum_{j\in\mathcal{N}} p_j \pi_{j,i} = p_i$.

In order to prove the stability of the estimation error dynamics, the quadratic function $\mathcal{V}_k(e_k) = \sum_{i\in\mathcal{N}} p_i \|e_k^i\|^2_{\Omega^i_{k|k}}$ is considered as candidate Lyapunov function, where $e_k = \mathrm{col}\left(e_k^i,\, i\in\mathcal{N}\right)$ is the collective estimation error vector and $e_k^i = x_k - \hat{x}^i_{k|k}$.

As a first step, the following lemma shows that the time-varying function $\mathcal{V}_k(e_k)$ is a well-defined candidate Lyapunov function since the matrices $\Omega^i_{k|k}$ are uniformly bounded from both above and below.

Lemma 1 *Let assumptions A1-A3 hold. Further, let the matrices $\Omega^i_{k|k}$, $i\in\mathcal{N}$, be generated according to Algorithm 1 starting from some initial values $\Omega^i_{0|-1}$, $i\in\mathcal{N}$, with $\Omega^i_{0|-1} > 0$. Then, there exist positive reals $\underline{\omega}$ and $\overline{\omega}$, such that $0 < \underline{\omega} I \le \Omega^i_{k|k} \le \overline{\omega} I$ for any $i\in\mathcal{N}$.*

An inspection of the proof of Lemma 1 shows that the lower bound $\underline{\omega}$ on the information matrix decreases monotonically with increasing parameters β and δ in the triggering condition (5). Hence, as β and δ increase (that is the transmission rate decreases) the bound $1/\underline{\omega}$ on the covariance matrix increases.

We now derive an upper bound on each element $\|e^i_{k+1}\|^2_{\Omega^i_{k+1|k+1}}$ of $\mathcal{V}_{k+1}(e_{k+1})$ which depends on the estimation errors at time k, on the process disturbance and measurement noises, and on the discrepancy between the information $(q^j_{k|k}, \Omega^j_{k|k})$ currently available in the neighboring nodes and the information predicted on the basis of the most recently received data.

Lemma 2 *Let assumptions A1-A3 hold. Further, let the information pairs $(q^i_{k|k}, \Omega^i_{k|k})$, $i\in\mathcal{N}$, be generated according to Algorithm 1 with all the initial information matrices $\Omega^i_{0|-1}$, $i\in\mathcal{N}$, positive definite. Then, for any $i\in\mathcal{N}$, the following bound holds*

$$\|e^i_{k+1}\|^2_{\Omega^i_{k+1|k+1}} \le \gamma^2 \Bigg(\pi_{i,i} \|e^i_k + \xi^i_k\|^2_{\Omega^i_{k|k}} + \sum_{j\in\mathcal{N}_i} \pi_{i,j} \|e^j_k + \xi^i_k + \eta^j_k\|^2_{\Omega^j_{k|k}} \Bigg)$$

where $\gamma \in (0,1)$ and

$$\xi^i_k = A^{-1}\left[w_k - \mathbf{1}_{\mathcal{S}}(i) (\Omega^i_{k+1|k})^{-1} (C^i)^\top (R^i)^{-1} v^i_{k+1}\right] \quad (15)$$

$$\eta^j_k = (1 - c^j_k)\cdot(\hat{x}^i_{k|k} - \bar{x}^i_k) \quad (16)$$

where $\mathbf{1}_{\mathcal{S}}(i)$ is the indicator function taking value 1 if $i\in\mathcal{S}$ and 0 otherwise.

Notice that, in view of Lemma 1, the quantities ξ^i_k, $i\in\mathcal{N}$, are bounded (in mean square) since the process disturbance and measurement noises are supposed to be bounded (in mean square). Hence, there exists a positive real ρ^2_ξ such that $\mathbb{E}\left\{\|\xi^i_k\|^2\right\} \le \rho^2_\xi$ for any $k\in\mathbb{Z}^+$ and $i\in\mathcal{N}$. Further, the quantities η^i_k, $i\in\mathcal{N}$, can be bounded in view of the fact that, when node i does not transmit at time k, the difference $\hat{x}^i_{k|k} - \bar{x}^i_k$ necessarily satisfies the non-transmission conditions in (5) so that $\|\eta^i_k\|^2_{\Omega^i_{k|k}} \le \alpha$ for any $k\in\mathbb{Z}^+$ and $i\in\mathcal{N}$. Hence, by exploiting Lemmas 1 and 2, the following stability result can be derived.

Theorem 1 *Let the same assumptions of Lemma 2 hold. Then, in each network node $i\in\mathcal{N}$, the estimation error e^i_t is uniformly bounded in mean square, in that*

$$\limsup_{k\to\infty} \mathbb{E}\{\|e^i_k\|^2\} \le \left(\frac{\gamma}{1-\gamma} \frac{\sum_{i\in\mathcal{N}} \sqrt{p_i}\,(\sqrt{\overline{\omega}}\rho_\xi + \sqrt{\alpha})}{\sqrt{\underline{\omega}}\min_{i\in\mathcal{N}} \sqrt{p_i}} \right)^2 .$$

5 Simulations

In this section, a single-target tracking problem is considered, with the target motion modeled by means of an integrated Ornstein-Uhlenbeck process (Stone et al., 2014):

$$dx(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -a \end{bmatrix} x(t)dt + \begin{bmatrix} 0 & 0 \\ \sigma & 0 \\ 0 & 0 \\ 0 & \sigma \end{bmatrix} dw(t)\,. \quad (17)$$

The unknown target state vector $x = [p_x,\, \dot{p}_x,\, p_y,\, \dot{p}_y]^\top$ includes the position and velocity components along the coordinate axes; further, $w(t)$ is a Wiener process with zero mean and unit rate of variance. Following Stone et al. (2014), $1/a$ is interpreted as the mean time between velocity changes, while $\sigma/\sqrt{a}$ as the root mean squared speed of the limiting velocity distribution. The model in (17) is discretized with sampling interval $\Delta = 1\,s$.

We consider a network composed of 20 linear sensor nodes and 80 communication nodes located in a square region of $5000\,m$ side length (see Fig. 1). Each node can

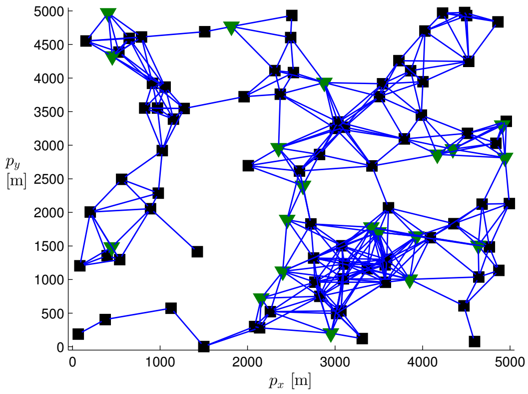

Fig. 1. Network composed of 20 linear sensor nodes (triangles) and 80 communication nodes (squares).

send its local information to all the nodes whose distance is less than a given communication radius equal to 839 m. Measurements of the target state are provided by each sensor node in Cartesian coordinates as

$$y_k^i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_k + v_k^i \,. \tag{18}$$

The standard deviation of each component of the measurement noise v_k^i is set to 10 m. The *Metropolis* weights are employed as defined in Xiao et al. (2005).

In all the considered scenarios, the simulation tests are carried out by performing Monte Carlo simulations with 200 independent runs obtained by varying the measurement noise realizations. The results are evaluated in terms of both the transmission rate $\overline{r}$, averaged over the whole network and over all the Monte Carlo trials, and the performance, expressed in terms of Root Mean Square Error (RMSE).

A simple scenario is first considered involving a target which moves with constant velocity following the trajectory depicted in Fig. 2 (top). For the sake of simplicity, δ is set equal to β. Fig. 2 (bottom) shows the evolution of the RMSE for different values of α, while keeping $\beta = \delta$ fixed to 30. The corresponding transmission rate $\overline{r}$ is also indicated. As expected, increasing α results in decreasing $\overline{r}$ and in increasing the RMSE. A similar behavior is obtained for increasing values of $\beta = \delta$ while keeping α fixed (this latter test is not reported due to space limitations).

We further consider a second scenario involving a target moving along the trajectory depicted in Fig. 3 (top), and compare the performance obtained by applying both Algorithm 1 achieving a certain rate $\overline{r}$, and a periodic-based transmission strategy at the same rate $\overline{r}$. Notice that, in the case of periodic transmission, since there are

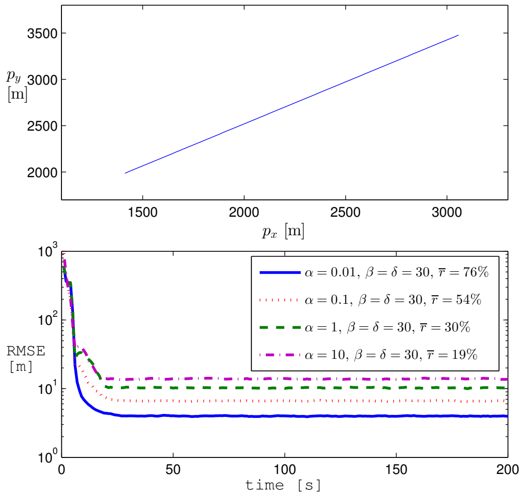

Fig. 2. First scenario. (top) Trajectory employed to evaluate the effect of the parameters α, β and δ (network of Fig. 1). (bottom) RMSE evaluated for $\beta = \delta = 30$, and for different values of α (logarithmic scale).

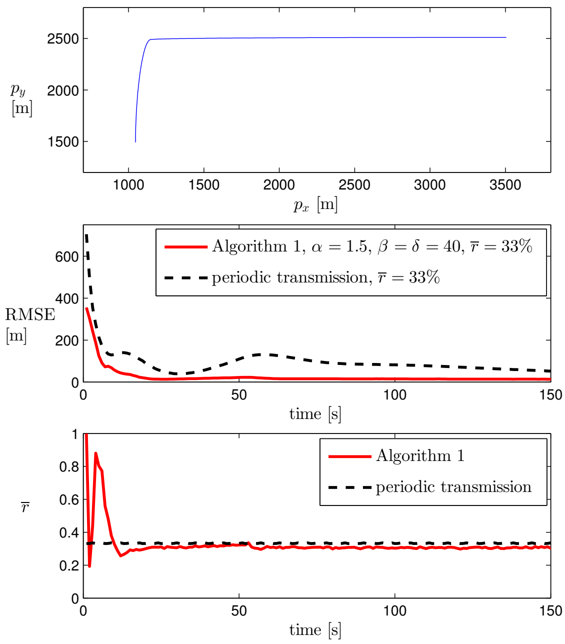

Fig. 3. Second scenario. (top) Trajectory employed to compare the performance of Algorithm 1 achieving a certain transmission rate $\overline{r}$, and a periodic-based strategy at the same rate $\overline{r}$ (network of Fig. 1). (middle) Comparison, in terms of RMSE, between Algorithm 1 with $\alpha = 1.5$, $\beta = \delta = 40$, achieving a transmission rate of $\overline{r} = 33\%$, and a periodic-based strategy at the same rate $\overline{r} = 33\%$. (bottom) Time evolution of the overall transmission rate $\overline{r}$: comparison between Algorithm 1 and periodic transmission.

no guarantees regarding the distance between the current data and the last transmitted one, in the fusion rule each node i simply discards the neighbors which have not transmitted their information at that time, and $|\mathcal{N}_i|$

(where $|\cdot|$ denotes set cardinality) is reduced accordingly, while the consensus weights must be reset so that they satisfy condition (9). Further, since this applies also to $|\mathcal{N}_j|$ for all $j \in \mathcal{N}_i$, the Metropolis weights cannot be used in this case. Accordingly, the consensus weights are replaced by uniform weights, i.e., $\pi_{i,j} = 1/(|\mathcal{N}_i|+1)$ for $j \in \mathcal{N}_i$ and $j = i$. Fig. 3 (middle) shows the comparison, in terms of RMSE, between Algorithm 1 with $\alpha = 1.5$ and $\beta = \delta = 40$, achieving an overall transmission rate $\overline{r} = 33\%$, and periodic transmission at the same rate $\overline{r} = 33\%$. In Fig. 3 (bottom) the time evolution of $\overline{r}$ is shown for both Algorithm 1 and periodic transmission; it is worth underlining that, while by construction both strategies perform the same number of transmissions in the considered time interval $[0, 150]s$, the transmission rate of the proposed algorithm is not distributed uniformly over time. In fact, it is higher at the beginning, when the estimation error is large, and increases again in correspondence of the maneuver. As it can be seen from Fig. 3 (middle), this results in a faster convergence rate as well as in a much prompter response to the target maneuver.

6 Conclusions

In this paper, a novel distributed Kalman filter with event-triggered communication over a peer-to-peer sensor network has been developed. A stability analysis of the filter has been carried out showing that it ensures mean-square boundedness of the state estimation error in all nodes provided that the network is strongly connected and the system collectively observable. Simulation tests on a tracking case-study show how the proposed strategy can achieve a desired trade off between estimation performance and energy efficiency.

Appendix

Proof of Lemma 1: Observe first that, in view of the event-triggered transmission strategy (5), in each network node the fused information matrix $\Omega^{i,F}_{k|k}$ can be bounded as follows

$$\Omega^{i,F}_{k|k} \geq \pi_{i,i}\Omega^i_{k|k} + \frac{1}{(1+\beta)(1+\delta)} \sum_{j \in \mathcal{N}_i} \pi_{i,j}\Omega^j_{k|k} \tag{19}$$

$$\Omega^{i,F}_{k|k} \leq \pi_{i,i}\Omega^i_{k|k} + \sum_{j \in \mathcal{N}_i} \pi_{i,j}\Omega^j_{k|k} \tag{20}$$

irrespectively of the data transmission pattern, i.e. of the time evolution of the variables c^i_k, $i \in \mathcal{N}$.

Observe also that a uniform upper bound on $\Omega^i_{k|k}$ can be readily obtained in view of (4) and of the fact that $\Omega^i_{k|k-1} \leq Q^{-1}$ for any k. Hence, only the existence of a positive definite lower bound on $\Omega^i_{k|k}$ remains to be proved. In this respect, we note that, when the initial information matrices $\Omega^i_{0|-1}$ are positive definite, then the matrices $\Omega^i_{k|k}$ are always positive definite for any k. This property can be proved by induction. To this end, observe that for a sensor node $i \in \mathcal{S}$, we can write

$$\begin{aligned}
&(\Omega^i_{k|k})^{-1} = (\Omega^i_{k|k-1} + (C^i)^\top R^i C^i)^{-1} \\
&= (\Omega^i_{k|k-1})^{-1} - (\Omega^i_{k|k-1})^{-1}(C^i)^\top \\
&\times \left[(R^i)^{-1} + C^i(\Omega^i_{k|k-1})^{-1}(C^i)^\top\right]^{-1} C^i(\Omega^i_{k|k-1})^{-1},
\end{aligned}$$

while, for any communication node $i \in \mathcal{N} \setminus \mathcal{S}$, we simply have $(\Omega^i_{k|k})^{-1} = (\Omega^i_{k|k-1})^{-1}$. This implies that $\Omega^i_{0|0}$ is invertible and, hence, positive definite. Further, when $\Omega^i_{k|k}$ is positive definite, $\Omega^i_{k+1|k}$ is positive definite as well. In fact, we have $(\Omega^i_{k+1|k})^{-1} = A(\Omega^{i,F}_{k|k})^{-1}A^\top + Q$ where the fused information matrix $\Omega^{i,F}_{k|k}$ is positive definite by virtue of (19). Hence, the positive definiteness of $\Omega^i_{k+1|k+1}$ follows at once.

Then in order to conclude the proof, it is sufficient to show that, even in the limit for k going to infinity, a positive definite lower bound on $\Omega^i_{k|k}$ can be found. In order to show this, notice first that, by applying fact ii) in Lemma 1 of Battistelli and Chisci (2014), we obtain the lower bound $\Omega^i_{k|k} \geq \upsilon A^{-\top}\,\Omega^{i,F}_{k-1|k-1}\,A^{-1} + \mathbf{1}_{\mathcal{S}}(i)(C^i)^\top R^i C^i$ for some positive real υ, where $\mathbf{1}_{\mathcal{S}}(i)$ is the indicator function taking value 1 if $i \in \mathcal{S}$ and 0 otherwise. Then, application of inequality (19) yields

$$\begin{aligned}
\Omega^i_{k|k} \geq \frac{\upsilon}{(1+\beta)(1+\delta)} A^{-\top} \left(\sum_{j \in \mathcal{N}} \pi_{i,j}\Omega^j_{k-1|k-1} \right) A^{-1} \\
+ \mathbf{1}_{\mathcal{S}}(i)(C^i)^\top R^i C^i .
\end{aligned}$$

For sufficiently large values of k, by recursively applying the latter inequality L times, it is possible to write

$$\begin{aligned}
\Omega^i_{k|k} \geq & \left(\frac{\upsilon}{(1+\beta)(1+\delta)}\right)^L (A^L)^{-\top} \\
& \times \left(\sum_{j \in \mathcal{N}} \Pi^L_{[i,j]}\,\Omega^j_{k-L|k-L} \right) (A^L)^{-1} \\
& + \sum_{\tau=k-L+1}^{k} \left(\frac{\upsilon}{(1+\beta)(1+\delta)}\right)^{k-\tau} (A^{k-\tau})^{-\top} \\
& \times \left(\sum_{j \in \mathcal{S}} \Pi^{k-\tau}_{[i,j]} (C^j)^\top R^j C^j \right) (A^{k-\tau})^{-1}
\end{aligned}$$

where $\Pi^L_{[i,j]}$ denotes the element (i,j) of the matrix Π^L.

Since the consensus matrix Π is primitive, the elements $\Pi^{k-\tau}_{[i,j]}$ are all positive provided that $k-\tau$ is greater than a certain integer, say M. Then, it is an easy matter to see that, for sufficiently large values of k, there exists a positive real $\bar{\upsilon}$ such that

$$\begin{aligned}\Omega^i_{k|k} &\geq \bar{\upsilon} \sum_{\tau=k-L+1}^{k-M} (A^{k-\tau})^{-\top} \left(\sum_{j\in\mathcal{S}} (C^j)^\top R^j C^j\right) (A^{k-\tau})^{-1} \\ &= \bar{\upsilon} \sum_{\tau=M}^{L-1} (A^{\tau})^{-\top} \left(\sum_{j\in\mathcal{S}} (C^j)^\top R^j C^j\right) (A^{\tau})^{-1}. \qquad (21)\end{aligned}$$

Observe now that, under collective observability (assumption A2) and provided that L is chosen so that $L-M$ is larger that the plant order, the right-hand side of (21) is positive definite and represents a uniform lower bound on $\Omega^i_{k|k}$ for sufficiently large values of k. ∎

Proof of Lemma 2: Notice first that the following identity holds

$$\begin{aligned}\hat{x}^i_{k+1|k+1} &= (\Omega^i_{k+1|k+1})^{-1}\big[\Omega^i_{k+1|k}\hat{x}^i_{k+1|k} \\ &\quad + \mathbf{1}_{\mathcal{S}}(i)(C^i)^\top (R^i)^{-1} y^i_{k+1}\big] \\ &= (\Omega^i_{k+1|k+1})^{-1}\big[\Omega^i_{k+1|k}\hat{x}^i_{k+1|k} \\ &\quad + \mathbf{1}_{\mathcal{S}}(i)(C^i)^\top (R^i)^{-1} C^i x_{k+1} \\ &\quad + \mathbf{1}_{\mathcal{S}}(i)(C^i)^\top (R^i)^{-1} v^i_{k+1}\big].\end{aligned}$$

Further, recalling (4), we can write

$$\begin{aligned}x_{k+1} &= (\Omega^i_{k+1|k+1})^{-1}\big[\Omega^i_{k+1|k} x_{k+1} \\ &\quad + \mathbf{1}_{\mathcal{S}}(i)(C^i)^\top (R^i)^{-1} C^i x_{k+1}\big]\end{aligned}$$

which implies

$$\begin{aligned}e^i_{k+1} &= x_{k+1} - \hat{x}^i_{k+1|k+1} \\ &= (\Omega^i_{k+1|k+1})^{-1}\Omega^i_{k+1|k}\big(x_{k+1} - \hat{x}^i_{k+1|k} + \bar{v}^i_{k+1}\big)\end{aligned}$$

where $\bar{v}^i_{k+1} = -\mathbf{1}_{\mathcal{S}}(i)(\Omega^i_{k+1|k})^{-1}(C^i)^\top (R^i)^{-1} v^i_{k+1}$. In turn, the latter identity implies

$$\begin{aligned}&\|e^i_{k+1}\|^2_{\Omega^i_{k+1|k+1}} \\ &= \|\Omega^i_{k+1|k}\big(x_{k+1} - \hat{x}^i_{k+1|k} + \bar{v}^i_{k+1}\big)\|^2_{(\Omega^i_{k+1|k+1})^{-1}} \\ &\leq \|x_{k+1} - \hat{x}^i_{k+1|k} + \bar{v}^i_{k+1}\|^2_{\Omega^i_{k+1|k}} \qquad (22)\end{aligned}$$

in view of the fact that $(\Omega^i_{k+1|k+1})^{-1} \leq (\Omega^i_{k+1|k})^{-1}$. Observe now that, in view of Lemma 1 and assumption A3, we can apply fact iii) of Lemma 1 of Battistelli and Chisci (2014) so as to derive the bound $\Omega^i_{k+1|k} \leq \gamma^2 A^{-\top}\Omega^{i,F}_{k|k}A^{-1}$ for some scalar $\gamma \in (0,1)$. Moreover, we have

$$x_{k+1} - \hat{x}^i_{k+1|k} = A(x_k - \hat{x}^{i,F}_{k|k}) + w_k$$

where $\hat{x}^{i,F}_{k|k} = (\Omega^{i,F}_{k|k})^{-1} q^{i,F}_{k|k}$. Hence, from (22) we can obtain

$$\begin{aligned}\|e^i_{k+1}\|^2_{\Omega^i_{k+1|k+1}} &\leq \gamma^2 \|A(x_k - \hat{x}^{i,F}_{k|k}) + w_k + \bar{v}^i_{k+1}\|^2_{A^{-\top}\Omega^{i,F}_{k|k}A^{-1}} \\ &= \gamma^2 \|x_k - \hat{x}^{i,F}_{k|k} + \xi^i_k\|^2_{\Omega^{i,F}_{k|k}} \qquad (23)\end{aligned}$$

where ξ^i_k is defined as in (15).

For the sake of compactness, let us now rewrite the information fusion step of Algorithm 1 as

$$\begin{aligned}q^{i,F}_{k|k} &= \pi_{i,i}\, q^i_{k|k} + \sum_{j\in\mathcal{N}_i} \pi_{i,j}\check{q}^j_{k|k} \\ \Omega^{i,F}_{k|k} &= \pi_{i,i}\, \Omega^i_{k|k} + \sum_{j\in\mathcal{N}_i} \pi_{i,j}\check{\Omega}^j_{k|k}\end{aligned}$$

where $\check{q}^j_{k|k} = c^j_k q^j_{k|k} + (1-c^j_k)\tilde{q}^j_k$ and $\check{\Omega}^j_{k|k} = c^j_k \Omega^j_{k|k} + (1-c^j_k)\tilde{\Omega}^j_k$. By exploiting these definitions, we can write

$$\begin{aligned}&\Omega^{i,F}_{k|k}\left(x_k - \hat{x}^{i,F}_{k|k} + \xi^i_k\right) \\ &\quad = \big(\pi_{i,i}\, \Omega^i_{k|k} + \sum_{j\in\mathcal{N}_i} \pi_{i,j}\check{\Omega}^j_{k|k}\big)\left(x_k + \xi^i_k\right) - \Omega^{i,F}_{k|k}\hat{x}^{i,F}_{k|k} \\ &\Omega^{i,F}_{k|k}\hat{x}^{i,F}_{k|k} = \pi_{i,i}\, q^i_{k|k} + \sum_{j\in\mathcal{N}_i} \pi_{i,j}\check{q}^j_{k|k} \\ &\quad = \pi_{i,i}\, \Omega^i_{k|k}\hat{x}^i_{k|k} + \sum_{j\in\mathcal{N}_i} \pi_{i,j}\check{\Omega}^j_{k|k}\check{x}^j_{k|k}\end{aligned}$$

where $\check{x}^j_{k|k} = (\check{\Omega}^j_{k|k})^{-1}\check{q}^j_{k|k}$. As a consequence, inequality (23) can be rewritten as

$$\begin{aligned}\|e^i_{k+1}\|^2_{\Omega^i_{k+1|k+1}} \leq \gamma^2 \Bigg\| &\pi_{i,i}\, \Omega^i_{k|k}(x_k - \hat{x}^i_{k|k} + \xi^i_k) \\ &+ \sum_{j\in\mathcal{N}_i} \pi_{i,j}\check{\Omega}^j_{k|k}(x_k - \check{x}^j_{k|k} + \xi^i_k)\Bigg\|^2_{(\Omega^{i,F}_{k|k})^{-1}} \qquad (24)\end{aligned}$$

By applying Lemma 2 of Battistelli and Chisci (2014), from (24) we can obtain

$$\begin{aligned}\|e^i_{k+1}\|^2_{\Omega^i_{k+1|k+1}} \leq \gamma^2 \Bigg(&\pi_{i,i}\|x_k - \hat{x}^i_{k|k} + \xi^i_k\|_{\Omega^i_{k|k}} \\ &+ \sum_{j\in\mathcal{N}_i} \pi_{i,j}\|x_k - \check{x}^j_{k|k} + \xi^i_k\|^2_{\check{\Omega}^j_{k|k}}\Bigg). \qquad (25)\end{aligned}$$

Notice now that, thanks to the transmission test, we have $\check{\Omega}^j_{k|k} \leq \Omega^j_{k|k}$. Thus, (24) yields

$$\begin{aligned}\|e^i_{k+1}\|^2_{\Omega^i_{k+1|k+1}} \leq \gamma^2 \Bigg(&\pi_{i,i}\|e^i_k + \xi^i_k\|_{\Omega^i_{k|k}} \\ &+ \sum_{j\in\mathcal{N}_i} \pi_{i,j}\|e^j_k + \xi^i_k + (\hat{x}^j_{k|k} - \check{x}^j_{k|k})\|^2_{\Omega^j_{k|k}}\Bigg).\end{aligned}$$

Then, the proof can be concluded by defining $\eta_k^j = \hat{x}_{k|k}^j - \check{x}_{k|k}^j$ which can be written as in (16), since $\check{x}_{k|k}^j$ coincides either with $\hat{x}_{k|k}^j$ when $c_k^k = 1$ or with $\bar{x}_{k|k}^j$ when $c_k^k = 0$. ■

Proof of Theorem 1: In view of Lemma 2, we can bound the function $\mathcal{V}_{k+1}(e_{k+1})$ as follows

$$\mathcal{V}_{k+1}(e_{k+1}) \le \gamma^2 \sum_{i\in\mathcal{N}} \Bigg(p_i\, \pi_{i,i} \| e_k^i + \xi_k^i \|^2_{\Omega^i_{k|k}} + \sum_{j\in\overline{\mathcal{N}}_i} p_j\, \pi_{j,i} \| e_k^i + \xi_k^j + \eta_k^i \|^2_{\Omega^i_{k|k}} \Bigg) \tag{26}$$

where $\overline{\mathcal{N}}_i$ denotes the set of out-neighbors of node i, i.e. $\overline{\mathcal{N}}_i = \{j : (i,j) \in \mathcal{A}\}$. For any $i \in \mathcal{N}$, let us now denote by $\overline{\omega}_k^i$ the vector in the set $\{\xi_k^i\} \cup \{\xi_k^j + \eta_k^i, j \in \overline{\mathcal{N}}_i\}$ for which the weighted norm $\|\cdot\|_{\Omega^i_{k|k}}$ is maximal. Further, let us define the vector ω_k^i as follows

$$\omega_k^i = \begin{cases} \overline{\omega}_k^i, & \text{if } e_k^i = 0 \\ \left(\|\overline{\omega}_k^i\|_{\Omega^i_{k|k}} / \|e_k^i\|_{\Omega^i_{k|k}} \right) e_k^i, & \text{otherwise}. \end{cases}$$

Then, it is possible to show that

$$\| e_k^i + \xi_k^i \|^2_{\Omega^i_{k|k}} \le \| e_k^i + \omega_k^i \|^2_{\Omega^i_{k|k}} \tag{27}$$

$$\| e_k^i + \xi_k^j + \eta_k^i \|^2_{\Omega^i_{k|k}} \le \| e_k^i + \omega_k^i \|^2_{\Omega^i_{k|k}}. \tag{28}$$

In fact, by construction, both (27) and (28) trivially hold when $e_k^i = 0$. Further, when $e_k^i \neq 0$, we have

$$\begin{aligned} \| e_k^i + \xi_k^i \|^2_{\Omega^i_{k|k}} &= \| e_k^i \|^2_{\Omega^i_{k|k}} + \| \xi_k^i \|^2_{\Omega^i_{k|k}} + 2 (e_k^i)^\top \Omega^i_{k|k} \xi_k^i \\ &\le \| e_k^i \|^2_{\Omega^i_{k|k}} + \| \xi_k^i \|^2_{\Omega^i_{k|k}} + 2 \| e_k^i \|_{\Omega^i_{k|k}} \| \xi_k^i \|^2_{\Omega^i_{k|k}} \\ &\le \| e_k^i \|^2_{\Omega^i_{k|k}} + \| \omega_k^i \|^2_{\Omega^i_{k|k}} + 2 \| e_k^i \|_{\Omega^i_{k|k}} \| \omega_k^i \|^2_{\Omega^i_{k|k}} \\ &= \| e_k^i \|^2_{\Omega^i_{k|k}} + \| \omega_k^i \|^2_{\Omega^i_{k|k}} + 2 (e_k^i)^\top \Omega^i_{k|k} \omega_k^i \\ &= \| e_k^i + \omega_k^i \|^2_{\Omega^i_{k|k}}, \end{aligned}$$

where we have exploited the facts that: ω_k^i and $\overline{\omega}_k^i$ have the same weighted norm; ω_k^i and e_k^i are collinear vectors. Inequality (28) can be proved in a similar way. By combining inequalities (27)-(28) with (26), we obtain

$$\begin{aligned} \mathcal{V}_{k+1}(e_{k+1}) &\le \gamma^2 \sum_{i\in\mathcal{N}} \Bigg(p_i\, \pi_{i,i} + \sum_{j\in\overline{\mathcal{N}}_i} p_j\, \pi_{j,i} \Bigg) \| e_k^i + \omega_k^i \|^2_{\Omega^i_{k|k}} \\ &= \gamma^2 \sum_{i\in\mathcal{N}} p_i\, \| e_k^i + \omega_k^i \|^2_{\Omega^i_{k|k}} \end{aligned}$$

where the latter equality follows from the fact that p is a left eigenvector of the consensus matrix Π with eigenvalue 1. Hence, by defining $\omega_k = \mathrm{col}(\omega_k^i, i \in \mathcal{N})$, we can write $\mathcal{V}_{k+1}(e_{k+1}) \le \gamma^2\, \mathcal{V}_k(e_k + \omega_k)$. Since the latter inequality holds for any realization of disturbance/measurement noises and thanks to the linearity of the expected value, this also implies

$$\mathbb{E}\{\mathcal{V}_{k+1}(e_{k+1})\} \le \gamma^2\, \mathbb{E}\{\mathcal{V}_k(e_k + \omega_k)\}. \tag{29}$$

Notice now that $(\mathbb{E}\{\mathcal{V}_k(\cdot)\})^{1/2}$ is a norm since all the matrices $\Omega^i_{k|k}$ are positive definite and all the components of the Perron-Frobenius left eigenvector p are strictly positive. As a consequence, we can take the square root of both sides of (29) and apply the triangular inequality so as to obtain

$$\sqrt{\mathbb{E}\{\mathcal{V}_{k+1}(e_{k+1})\}} \le \gamma \sqrt{\mathbb{E}\{\mathcal{V}_k(e_k)\}} + \gamma \sqrt{\mathbb{E}\{\mathcal{V}_k(\omega_k)\}}. \tag{30}$$

Observe now that the term $\sqrt{(\mathbb{E}\{\mathcal{V}_k(\omega_k)\})}$ can be uniformly bounded as

$$\begin{aligned} \sqrt{\mathbb{E}\{\mathcal{V}_k(\omega_k)\}} &\le \sum_{i\in\mathcal{N}} \sqrt{p_i} \sqrt{\mathbb{E}\{\|\omega_k^i\|^2_{\Omega^i_{k|k}}\}} \\ &\le \sum_{i\in\mathcal{N}} \sqrt{p_i} \max_{j\in\mathcal{N}} \left[\sqrt{\mathbb{E}\{\|\xi_k^j\|^2_{\Omega^i_{k|k}}\}} + \sqrt{\mathbb{E}\{\|\eta_k^i\|^2_{\Omega^i_{k|k}}\}} \right] \\ &\le \sum_{i\in\mathcal{N}} \sqrt{p_i} (\sqrt{\overline{\omega}}\, \rho_\xi + \sqrt{\alpha}) \end{aligned}$$

where the latter inequality follows from the available bounds on ξ_k^i and η_k^i (see the discussion before the statement of the theorem). Then, since the scalar γ belongs to the interval $(0,1)$, inequality (30) implies that

$$\limsup_{k\to\infty} \sqrt{\mathbb{E}\{\mathcal{V}_k(e_k)\}} \le \frac{\gamma}{1-\gamma} \sum_{i\in\mathcal{N}} \sqrt{p_i} (\sqrt{\overline{\omega}}\, \rho_\xi + \sqrt{\alpha}).$$

In order to conclude the proof, it is sufficient to note that

$$\mathbb{E}\left\{ \sum_{i\in\mathcal{N}} p_i \|e_k^i\|^2_{\Omega^i_{k|k}} \right\} \ge \left(\underline{\omega} \min_{i\in\mathcal{N}} p_i \right) \mathbb{E}\left\{ \sum_{i\in\mathcal{N}} \|e_k^i\|^2 \right\},$$

with $\underline{\omega}$ the same constant as in Lemma 1, which implies $\mathbb{E}\{\|e_k\|^2\} \le (\underline{\omega} \min_{i\in\mathcal{N}} p_i)^{-1}\, \mathbb{E}\{\mathcal{V}_k(e_k)\}$. ■

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