



UNIVERSITÀ
DEGLI STUDI
FIRENZE

FLORE

Repository istituzionale dell'Università degli Studi di Firenze

On rank range of interval matrices

Questa è la Versione finale referata (Post print/Accepted manuscript) della seguente pubblicazione:

Original Citation:

On rank range of interval matrices / Elena Rubei. - In: LINEAR ALGEBRA AND ITS APPLICATIONS. - ISSN 0024-3795. - STAMPA. - 561:(2019), pp. 81-97. [10.1016/j.laa.2018.09.018]

Availability:

The webpage <https://hdl.handle.net/2158/1135368> of the repository was last updated on 2021-03-23T19:17:41Z

Published version:

DOI: 10.1016/j.laa.2018.09.018

Terms of use:

Open Access

La pubblicazione è resa disponibile sotto le norme e i termini della licenza di deposito, secondo quanto stabilito dalla Policy per l'accesso aperto dell'Università degli Studi di Firenze (<https://www.sba.unifi.it/upload/policy-oa-2016-1.pdf>)

Publisher copyright claim:

La data sopra indicata si riferisce all'ultimo aggiornamento della scheda del Repository FloRe - The above-mentioned date refers to the last update of the record in the Institutional Repository FloRe

(Article begins on next page)

On rank range of interval matrices

Elena Rubei

Dipartimento di Matematica e Informatica “U. Dini”, viale Morgagni 67/A, 50134 Firenze, Italia

E-mail address: elena.rubei@unifi.it

Abstract

An interval matrix is a matrix whose entries are intervals in \mathbb{R} . Let $p, q \in \mathbb{N} \setminus \{0\}$ and let $\alpha = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times q$ interval matrix; given a $p \times q$ matrix A with entries in \mathbb{R} , we say that $A \in \alpha$ if $a_{i,j} \in [\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}]$ for any i, j . We establish a criterion to say if an interval matrix contains a matrix of rank 1. Moreover we determine the maximum rank of the matrices contained in a given interval matrix. Finally, for any interval matrix α with no more than 3 columns, we describe a way to find the range of the ranks of the matrices contained in α .

1 Introduction

Let $p, q \in \mathbb{N} \setminus \{0\}$; a $p \times q$ interval matrix is a $p \times q$ matrix whose entries are intervals in \mathbb{R} ; let $\alpha = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times q$ interval matrix, where $\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}$ are real numbers with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ for any i and j ; given a $p \times q$ matrix A with entries in \mathbb{R} , we say that $A \in \alpha$ if $a_{i,j} \in [\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}]$ for any i, j . In this paper we investigate about the range of the ranks of the matrices contained in α .

There are several papers studying when an interval $p \times q$ matrix α has full rank, that is when all the matrices contained in α have rank equal to $\min\{p, q\}$. For any $p \times q$ interval matrix $\alpha = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$, let $\text{mid}(\alpha)$, $\text{rad}(\alpha)$ and $|\alpha|$ be respectively the midpoint, the radius and the modulus of α , that is the $p \times q$ matrices such that

$$\text{mid}(\alpha)_{i,j} = \frac{\underline{\alpha}_{i,j} + \overline{\alpha}_{i,j}}{2}, \quad \text{rad}(\alpha)_{i,j} = \frac{\overline{\alpha}_{i,j} - \underline{\alpha}_{i,j}}{2}, \quad |\alpha|_{i,j} = \max\{|\underline{\alpha}_{i,j}|, |\overline{\alpha}_{i,j}|\}$$

for any i, j . The following theorem characterizes full-rank square interval matrices:

Theorem 1. (Rohn, [7]) *Let $\alpha = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times p$ interval matrix, where $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ for any i, j . Let $Y_p = \{-1, 1\}^p$ and, for any $x \in Y_p$, denote by T_x the diagonal matrix whose diagonal is x .*

Then α is a full-rank interval matrix if and only if, for each $x, y \in Y_p$,

$$\det(\text{mid}(\alpha)) \det(\text{mid}(\alpha) - T_x \text{rad}(\alpha) T_y) > 0.$$

2010 Mathematical Subject Classification: 15A99, 15A03, 65G40

Key words: interval matrices, rank

See [7] and [8] for other characterizations. The following theorem characterizes full-rank $p \times q$ interval matrices, see [9], [10], [11]:

Theorem 2. (Rohn) *A $p \times q$ interval matrix α with $p \geq q$ has full rank if and only if the system of inequalities*

$$|\text{mid}(\alpha) x| \leq \text{rad}(\alpha) |x|, \quad x \in \mathbb{R}^q$$

has only the trivial solution $x = 0$.

A problem which can be connected with the quoted ones is the one of the partial matrices: let K be a field; a partial matrix over K is a matrix where only some of the entries are given and they are elements of K ; a completion of a partial matrix is a specification of the unspecified entries. We say that a submatrix of a partial matrix is specified if all its entries are given. The problem of determining whether, given a partial matrix, a completion with some prescribed property exists and related problems have been widely studied: we quote, for instance, the papers [1], [2], [13]. In particular there is a wide literature about the problem of determining the maximal and the minimal rank of the completions of a partial matrix.

In [2], Cohen, Johnson, Rodman and Woerdeman determined the maximal rank of the completions of a partial matrix in terms of the ranks and the sizes of its maximal specified submatrices; see also [1] for the proof. The problem of determining the minimal rank of the completions of a partial matrix seems more difficult and it has been solved only in some particular cases, see for instance the paper [12] for the case of triangular matrices. We quote also the paper [3], where the authors establish a criterion to say if a partial matrix has a completion of rank 1.

Also for interval matrices, the problem of determining the minimal rank of the matrices contained in a given interval matrix seems much more difficult than the problem of determining the maximal rank. In this paper we establish a criterion to say if an interval matrix contains a matrix of rank 1; the criterion is analogous to the one to establish if a partial matrix has a completion of rank 1 found in [3]. Moreover, we determine the maximum rank of the matrices contained in a given interval matrix. Finally, for any interval matrix α with no more than 3 columns, we describe a way to find the range of the ranks of the matrices contained in α .

2 Notation and first remarks

- Let $\mathbb{R}_{>0}$ be the set $\{x \in \mathbb{R} \mid x > 0\}$ and let $\mathbb{R}_{\geq 0}$ be the set $\{x \in \mathbb{R} \mid x \geq 0\}$; we define analogously $\mathbb{R}_{<0}$ and $\mathbb{R}_{\leq 0}$.
- Throughout the paper let $p, q \in \mathbb{N} \setminus \{0\}$.
- Let Σ_p be the set of the permutations on $\{1, \dots, p\}$. For any $\sigma \in \Sigma_p$ we denote the sign of the permutation σ by $\epsilon(\sigma)$.
- For any ordered multiset $J = (j_1, \dots, j_r)$, a **multiset permutation** $\sigma(J)$ of J is an ordered arrangement of the multiset $\{j_1, \dots, j_r\}$ where each element appears as often as it does in J .
- Let $M(p \times q, \mathbb{R})$ denote the set of the $p \times q$ matrices with entries in \mathbb{R} . For any $A \in M(p \times q, \mathbb{R})$, let $\text{rk}(A)$ denote the rank of A and let $A^{(j)}$ be the j -th column of A .

- For any vector space V over a field K and any $v_1, \dots, v_k \in V$, let $\langle v_1, \dots, v_k \rangle$ be the span of v_1, \dots, v_k .
- Let $\alpha = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times q$ interval matrix, where $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ for any i and j . As we have already said, given a matrix $A \in M(p \times q, \mathbb{R})$, we say that $A \in \alpha$ if and only if $a_{i,j} \in [\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}]$ for any i, j . We define

$$\text{mrk}(\alpha) = \min\{\text{rk}(A) \mid A \in \alpha\},$$

$$\text{Mrk}(\alpha) = \max\{\text{rk}(A) \mid A \in \alpha\}.$$

We call them respectively **minimal rank** and **maximal rank** of α . Moreover, we define

$$\text{rkRange}(\alpha) = \{\text{rk}(A) \mid A \in \alpha\};$$

we call it the **rank range** of α .

We say that the entry i, j of α is a **degenerate** if $\underline{\alpha}_{i,j} = \overline{\alpha}_{i,j}$.

Remark 3. Let α be an interval matrix. Observe that

$$\text{rkRange}(\alpha) = [\text{mrk}(\alpha), \text{Mrk}(\alpha)] \cap \mathbb{N}.$$

Proof. Obviously $\text{rkRange}(\alpha) \subseteq [\text{mrk}(\alpha), \text{Mrk}(\alpha)] \cap \mathbb{N}$. We have to show the other inclusion.

There exist $A, B \in \alpha$ with $\text{rk}(A) = \text{mrk}(\alpha)$ and $\text{rk}(B) = \text{Mrk}(\alpha)$; let $(i_1, j_1), \dots, (i_k, j_k)$ be the entries of A different from the corresponding entries of B and, for every $r \in \{1, \dots, k\}$, let C_r be the matrix obtained from A by changing the entries $(i_1, j_1), \dots, (i_r, j_r)$ of A into the corresponding entries of B . Obviously $C_k = B$, $C_1, \dots, C_k \in \alpha$ and, for every i , the absolute value of the difference between the rank of C_i and the rank of C_{i+1} is less than or equal to 1.

Therefore the set $\{\text{rk}(A), \text{rk}(C_1), \dots, \text{rk}(C_{k-1}), \text{rk}(B)\}$ is an interval of \mathbb{N} and thus it must contain $[\text{rk}(A), \text{rk}(B)] \cap \mathbb{N}$, that is $[\text{mrk}(\alpha), \text{Mrk}(\alpha)] \cap \mathbb{N}$; moreover, it is contained in $\text{rkRange}(\alpha)$ (since $A, C_1, \dots, C_{k-1}, B \in \alpha$), so we can conclude. \square

We defer to some classical books on interval analysis, such as [4], [6] and [5] for the definition of sum and multiplication of two intervals.

Definition 4. Let α be an interval matrix. We say that another interval matrix α' is obtained from α by an **elementary row operation** if it is obtained from α by one of the following operations:

- I) interchanging two rows,
- II) multiplying a row by a nonzero real number,
- III) adding to a row the multiple of another row by a real number.

In an analogous way we may define **elementary column operations**.

Remark 5. Let α and α' be two interval matrices such that α' is obtained from α by elementary row (or column) operations. Then, obviously,

$$\text{rkRange}(\alpha) \subseteq \text{rkRange}(\alpha'). \tag{1}$$

Moreover, if α' is obtained from α only by elementary row (or column) operations of kind I or II, we have the equality in (1).

3 Interval matrices containing rank-zero or rank-one matrices

Remark 6. Let $\alpha = ([\underline{\alpha}_{i,j}, \bar{\alpha}_{i,j}])_{i,j}$, with $\underline{\alpha}_{i,j} \leq \bar{\alpha}_{i,j}$ for any $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$, be an interval matrix. Obviously $\text{mrk}(\alpha) = 0$ if and only if, for any $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$, the interval $[\underline{\alpha}_{i,j}, \bar{\alpha}_{i,j}]$ contains 0.

Moreover, if α' is the interval matrix obtained from α by deleting the columns and the rows such that all their entries contain 0, we have that $\text{mrk}(\alpha) = \text{mrk}(\alpha')$.

Definition 7. We say that an interval matrix α is **reduced** if every column and every row has at least one entry not containing 0. Reducing an interval matrix means eliminating every row and every column such that each of its entries contains 0.

Remark 8. Let α be a $p \times q$ interval matrix. Let $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$ be such that $\underline{\alpha}_{i,j} \leq 0 \leq \bar{\alpha}_{i,j}$. Define α' and α'' to be the interval matrices such that

$$\alpha'_{i,j} = [\underline{\alpha}_{i,j}, 0], \quad \alpha''_{i,j} = [0, \bar{\alpha}_{i,j}]$$

and $\alpha'_{t,s} = \alpha''_{t,s} = \alpha_{t,s}$ for any $(t, s) \neq (i, j)$. Let $r \in \mathbb{N}$. Then obviously there exists $A \in \alpha$ with $\text{rk}(A) = r$ if and only if either there exists $A \in \alpha'$ with $\text{rk}(A) = r$ or there exists $A \in \alpha''$ with $\text{rk}(A) = r$. In particular, to study whether an interval matrix α contains a rank- r matrix, it is sufficient to consider the case where, for any i, j , either $\alpha_{i,j} \subseteq \mathbb{R}_{\geq 0}$ or $\alpha_{i,j} \subseteq \mathbb{R}_{\leq 0}$.

Moreover, by Remark 5, we can suppose $\alpha_{i,j} \subseteq \mathbb{R}_{\geq 0}$ for every (i, j) such that either i or j is equal to 1.

Remark 9. Let α be a reduced interval matrix such that $\alpha_{i,j} \subseteq \mathbb{R}_{\geq 0}$ for every (i, j) with either i or j is equal to 1. Obviously, if there exists (i, j) such that $\alpha_{i,j} \subseteq \mathbb{R}_{< 0}$, then α does not contain a rank-one matrix. Otherwise, that is $\bar{\alpha}_{i,j} \geq 0$ for any i, j , define $\hat{\alpha}$ to be the interval matrix such that

$$\hat{\alpha}_{i,j} = [\max\{0, \underline{\alpha}_{i,j}\}, \bar{\alpha}_{i,j}]$$

for any i, j . Obviously α contains a rank-one matrix if and only if $\hat{\alpha}$ contains a rank-one matrix.

So to study when a reduced interval matrix contains a rank-one matrix it is sufficient to study the problem for a reduced interval matrix α , with $\alpha_{i,j} \subseteq \mathbb{R}_{\geq 0}$ for every i, j . The following theorem gives an answer for such a problem in this case.

Theorem 10. Let $\alpha = ([\underline{\alpha}_{i,j}, \bar{\alpha}_{i,j}])_{i,j}$ be a $p \times q$ reduced interval matrix with $p, q \geq 2$ and $0 \leq \underline{\alpha}_{i,j} \leq \bar{\alpha}_{i,j}$ for any $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$. There exists $A \in \alpha$ with $\text{rk}(A) = 1$ if and only if, for any $h \in \mathbb{N}$ with $2 \leq h \leq 2^{\min\{p,q\}-1}$, for any $i_1, \dots, i_h \in \{1, \dots, p\}$, for any $j_1, \dots, j_h \in \{1, \dots, q\}$ and for any $\sigma \in \Sigma_h$, we have:

$$\underline{\alpha}_{i_1, j_1} \cdots \underline{\alpha}_{i_h, j_h} \leq \bar{\alpha}_{i_1, j_{\sigma(1)}} \cdots \bar{\alpha}_{i_h, j_{\sigma(h)}}. \quad (2)$$

Proof. We can suppose $q \leq p$.

\implies Let $A \in \alpha$ with $\text{rk}(A) = 1$. For any $i_1, \dots, i_h \in \{1, \dots, p\}$, for any $j_1, \dots, j_h \in \{1, \dots, q\}$ and for any $\sigma \in \Sigma_h$, we have:

$$\underline{\alpha}_{i_1, j_1} \dots \underline{\alpha}_{i_h, j_h} \leq a_{i_1, j_1} \dots a_{i_h, j_h} = a_{i_1, j_{\sigma(1)}} \dots a_{i_h, j_{\sigma(h)}} \leq \overline{\alpha}_{i_1, j_{\sigma(1)}} \dots \overline{\alpha}_{i_h, j_{\sigma(h)}}.$$

\Leftarrow Observe that, since α is reduced, there exists $A \in \alpha$ with $\text{rk}(A) = 1$ if and only if there exist $x_1, \dots, x_p, c_2, \dots, c_q$ in \mathbb{R} such that

$$x_i \in [\underline{\alpha}_{i,1}, \overline{\alpha}_{i,1}], \quad c_j x_i \in [\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}]$$

for any i, j . Since α is reduced and $\underline{\alpha}_{i,j} \geq 0$, this is equivalent to ask that there exist $x_1, \dots, x_p, c_2, \dots, c_q$ in $\mathbb{R}_{>0}$ such that, for any i, j ,

$$x_i \in [\underline{\alpha}_{i,1}, \overline{\alpha}_{i,1}], \quad c_j x_i \in [\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}].$$

If we define $c_1 = 1$, this is equivalent to the following condition: there exist $x_1, \dots, x_p, c_2, \dots, c_q$ in $\mathbb{R}_{>0}$ such that, for any i, j ,

$$x_i \in \bigcap_{j \in \{1, \dots, q\}} \left[\frac{\underline{\alpha}_{i,j}}{c_j}, \frac{\overline{\alpha}_{i,j}}{c_j} \right].$$

Obviously this is equivalent to say that there exist c_2, \dots, c_q in $\mathbb{R}_{>0}$ such that

$$\frac{\underline{\alpha}_{i,t}}{c_t} \leq \frac{\overline{\alpha}_{i,k}}{c_k} \quad (3)$$

for any $i \in \{1, \dots, p\}$, $t, k \in \{1, \dots, q\}$. We will use the following notation:

$$\underline{\alpha} \left(\begin{matrix} i_1, \dots, i_s \\ j_1, \dots, j_s \end{matrix} \right) := \underline{\alpha}_{i_1, j_1} \dots \underline{\alpha}_{i_s, j_s}, \quad \overline{\alpha} \left(\begin{matrix} i_1, \dots, i_s \\ j_1, \dots, j_s \end{matrix} \right) := \overline{\alpha}_{i_1, j_1} \dots \overline{\alpha}_{i_s, j_s}.$$

Let us prove, by induction on r , that, for any $r \in \{2, \dots, q\}$, there exist $c_2, \dots, c_r \in \mathbb{R}_{>0}$ such that, for any $k, t \in \{1, \dots, r\}$, $b \in [1, 2^{q-r}] \cap \mathbb{N}$, $i_1, \dots, i_b \in \{1, \dots, p\}$, $j_1, \dots, j_{b-1} \in \{1, \dots, q\}$, σ multiset permutation of (j_1, \dots, j_{b-1}, k) , we have:

$$\underline{\alpha} \left(\begin{matrix} i_1, \dots, i_{b-1}, i_b \\ j_1, \dots, j_{b-1}, t \end{matrix} \right) c_k \leq \overline{\alpha} \left(\begin{matrix} i_1, \dots, i_{b-1}, i_b \\ \sigma(j_1, \dots, j_{b-1}, k) \end{matrix} \right) c_t, \quad (4)$$

where c_1 is defined to be 1. Observe that the condition above implies (3) (take $b = 1$ and $r = q$) and also that the case $k = t$ follows from (2).

As to the induction base case $r = 2$, we have to prove that there exists $c_2 \in \mathbb{R}_{>0}$ such that for any $b, d \in [1, 2^{q-2}] \cap \mathbb{N}$, $i_1, \dots, i_b, u_1, \dots, u_d \in \{1, \dots, p\}$, $j_1, \dots, j_{b-1}, v_1, \dots, v_{d-1} \in \{1, \dots, q\}$, σ multiset permutation of $(j_1, \dots, j_{b-1}, 1)$, γ multiset permutation of $(v_1, \dots, v_{d-1}, 2)$,

$$\underline{\alpha} \left(\begin{matrix} i_1, \dots, i_{b-1}, i_b \\ j_1, \dots, j_{b-1}, 2 \end{matrix} \right) \leq \overline{\alpha} \left(\begin{matrix} i_1, \dots, i_{b-1}, i_b \\ \sigma(j_1, \dots, j_{b-1}, 1) \end{matrix} \right) c_2 \quad (5)$$

and

$$\underline{\alpha} \left(\begin{array}{c} u_1, \dots, u_{d-1}, u_d \\ v_1, \dots, v_{d-1}, 1 \end{array} \right)^{c_2} \leq \overline{\alpha} \left(\begin{array}{c} u_1, \dots, u_{d-1}, u_d \\ \gamma(v_1, \dots, v_{d-1}, 2) \end{array} \right). \quad (6)$$

Observe that, if

$$\underline{\alpha} \left(\begin{array}{c} i_1, \dots, i_{b-1}, i_b \\ j_1, \dots, j_{b-1}, 2 \end{array} \right) = 0,$$

then the inequality (5) is implied by the condition $c_2 > 0$ and, if

$$\underline{\alpha} \left(\begin{array}{c} u_1, \dots, u_{d-1}, u_d \\ v_1, \dots, v_{d-1}, 1 \end{array} \right) = 0,$$

then the inequality (6) is always true. So we can consider only $b, d \in [1, 2^{q-2}] \cap \mathbb{N}$, $i_1, \dots, i_b, u_1, \dots, u_d \in \{1, \dots, p\}$, $j_1, \dots, j_{b-1}, v_1, \dots, v_{d-1} \in \{1, \dots, q\}$ such that the terms $\underline{\alpha}$ in (5), (6) are positive. So a c_2 as we search for exists if and only if for any $b, d \in [1, 2^{q-2}] \cap \mathbb{N}$, $i_1, \dots, i_b, u_1, \dots, u_d \in \{1, \dots, p\}$, $j_1, \dots, j_{b-1}, v_1, \dots, v_{d-1} \in \{1, \dots, q\}$ such that

$$\underline{\alpha} \left(\begin{array}{c} i_1, \dots, i_{b-1}, i_b \\ j_1, \dots, j_{b-1}, 2 \end{array} \right) \quad \text{and} \quad \underline{\alpha} \left(\begin{array}{c} u_1, \dots, u_{d-1}, u_d \\ v_1, \dots, v_{d-1}, 1 \end{array} \right)$$

are positive, σ multiset permutation of $(j_1, \dots, j_{b-1}, 1)$, γ multiset permutation of $(v_1, \dots, v_{d-1}, 2)$, we have that

$$\underline{\alpha} \left(\begin{array}{c} u_1, \dots, u_{d-1}, u_d, i_1, \dots, i_{b-1}, i_b \\ v_1, \dots, v_{d-1}, 1, j_1, \dots, j_{b-1}, 2 \end{array} \right) \leq \overline{\alpha} \left(\begin{array}{c} u_1, \dots, u_{d-1}, u_d, i_1, \dots, i_{b-1}, i_b \\ \gamma(v_1, \dots, v_{d-1}, 2) \sigma(j_1, \dots, j_{b-1}, 1) \end{array} \right)$$

and this follows from our assumption (2).

Let us prove the induction step. By induction assumption we can suppose there exist $c_2, \dots, c_{r-1} \in \mathbb{R}_{>0}$ such that for any $k, t \in \{1, \dots, r-1\}$, $b \in [1, 2^{q-r+1}] \cap \mathbb{N}$, $i_1, \dots, i_b \in \{1, \dots, p\}$, $j_1, \dots, j_{b-1} \in \{1, \dots, q\}$, σ multiset permutation of (j_1, \dots, j_{b-1}, k) , the inequality (4) holds. We want to show that there exists $c_r \in \mathbb{R}_{>0}$ such that for any $t, l \in \{1, \dots, r-1\}$, $b, d \in [1, 2^{q-r}] \cap \mathbb{N}$, $i_1, \dots, i_b, u_1, \dots, u_d \in \{1, \dots, p\}$, $j_1, \dots, j_{b-1}, v_1, \dots, v_{d-1} \in \{1, \dots, q\}$, σ multiset permutation of $\{j_1, \dots, j_{b-1}, r\}$, γ multiset permutation of $\{v_1, \dots, v_{d-1}, l\}$,

$$\underline{\alpha} \left(\begin{array}{c} i_1, \dots, i_{b-1}, i_b \\ j_1, \dots, j_{b-1}, t \end{array} \right)^{c_r} \leq \overline{\alpha} \left(\begin{array}{c} i_1, \dots, i_{b-1}, i_b \\ \sigma(j_1, \dots, j_{b-1}, r) \end{array} \right)^{c_t} \quad (7)$$

and

$$\overline{\alpha} \left(\begin{array}{c} u_1, \dots, u_{d-1}, u_d \\ \gamma(v_1, \dots, v_{d-1}, l) \end{array} \right)^{c_r} \geq \underline{\alpha} \left(\begin{array}{c} u_1, \dots, u_{d-1}, u_d \\ v_1, \dots, v_{d-1}, r \end{array} \right)^{c_l}. \quad (8)$$

Observe that when

$$\underline{\alpha} \begin{pmatrix} i_1, \dots, i_{b-1}, i_b \\ j_1, \dots, j_{b-1}, t \end{pmatrix} = 0,$$

the inequality (7) is always verified and, if

$$\underline{\alpha} \begin{pmatrix} u_1, \dots, u_{d-1}, u_d \\ v_1, \dots, v_{d-1}, r \end{pmatrix} = 0,$$

then the inequality (8) is implied by the condition $c_r > 0$. So we can consider only $t \in \{1, \dots, r-1\}$, $b, d \in [1, 2^{q-r}] \cap \mathbb{N}$, $i_1, \dots, i_b, u_1, \dots, u_d \in \{1, \dots, p\}$, $j_1, \dots, j_{b-1}, v_1, \dots, v_{d-1} \in \{1, \dots, q\}$ such that the terms $\underline{\alpha}$ appearing in (7) and in (8) are positive. Thus a c_r as we search for exists if and only if for any $t, l \in \{1, \dots, r-1\}$, $b, d \in [1, 2^{q-r}] \cap \mathbb{N}$, $i_1, \dots, i_b, u_1, \dots, u_d \in \{1, \dots, p\}$, $j_1, \dots, j_{b-1}, v_1, \dots, v_{d-1} \in \{1, \dots, q\}$ such that

$$\underline{\alpha} \begin{pmatrix} i_1, \dots, i_{b-1}, i_b \\ j_1, \dots, j_{b-1}, t \end{pmatrix} \quad \text{and} \quad \underline{\alpha} \begin{pmatrix} u_1, \dots, u_{d-1}, u_d \\ v_1, \dots, v_{d-1}, r \end{pmatrix}$$

are positive, σ multiset permutation of (j_1, \dots, j_{b-1}, r) , γ multiset permutation of (v_1, \dots, v_{d-1}, l) , we have:

$$\underline{\alpha} \begin{pmatrix} i_1, \dots, i_{b-1}, i_b, u_1, \dots, u_{d-1}, u_d \\ j_1, \dots, j_{b-1}, t, v_1, \dots, v_{d-1}, r \end{pmatrix}^{c_l} \leq \bar{\alpha} \begin{pmatrix} i_1, \dots, i_{b-1}, i_b, u_1, \dots, u_{d-1}, u_d \\ \sigma(j_1, \dots, j_{b-1}, r) \gamma(v_1, \dots, v_{d-1}, l) \end{pmatrix}^{c_t}$$

and this is true by induction assumption. □

Example. Let

$$\boldsymbol{\alpha} = \begin{pmatrix} [2, 3] & [1, 6] & [-2, 2] & [-3, -1] \\ [1, 2] & [2, 3] & [-2, 3] & [-2, 3] \\ [1, 4] & [0, 2] & [3, 4] & [-1, 0] \end{pmatrix}.$$

By Remarks 5 and 8, the interval matrix $\boldsymbol{\alpha}$ contains a rank-one matrix if and only if at least one of the following interval matrices contains a rank-one matrix:

$$\boldsymbol{\alpha}' := \begin{pmatrix} [2, 3] & [1, 6] & [0, 2] & [1, 3] \\ [1, 2] & [2, 3] & [-2, 3] & [-3, 2] \\ [1, 4] & [0, 2] & [3, 4] & [0, 1] \end{pmatrix}, \quad \boldsymbol{\alpha}'' := \begin{pmatrix} [2, 3] & [1, 6] & [-2, 0] & [1, 3] \\ [1, 2] & [2, 3] & [-2, 3] & [-3, 2] \\ [1, 4] & [0, 2] & [3, 4] & [0, 1] \end{pmatrix};$$

by Remark 5, this is equivalent to ask that at least one of the interval matrices $\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}'''$ contains a rank-one matrix, where

$$\boldsymbol{\alpha}''' := \begin{pmatrix} [2, 3] & [1, 6] & [0, 2] & [1, 3] \\ [1, 2] & [2, 3] & [-3, 2] & [-3, 2] \\ [1, 4] & [0, 2] & [-4, -3] & [0, 1] \end{pmatrix}.$$

By Remark 9, the interval matrix α''' does not contain a rank-one matrix since $\alpha'''_{i,j} \subset \mathbb{R}_{\geq 0}$ for any i, j such that either i or j is equal to 1, but $\alpha'''_{3,3} \subset \mathbb{R}_{< 0}$. Hence α contains a rank-one matrix if and only if α' contains a rank-one matrix and, by Remark 9, this is equivalent to ask that $\hat{\alpha}'$ contains a rank-one matrix, where

$$\hat{\alpha}' := \begin{pmatrix} [2, 3] & [1, 6] & [0, 2] & [1, 3] \\ [1, 2] & [2, 3] & [0, 3] & [0, 2] \\ [1, 4] & [0, 2] & [3, 4] & [0, 1] \end{pmatrix}.$$

By Theorem 10, the interval matrix $\hat{\alpha}'$ does not contain a rank-one matrix because for instance the product of the minima of the intervals $\hat{\alpha}'_{1,1}$, $\hat{\alpha}'_{2,2}$ and $\hat{\alpha}'_{3,3}$ is greater than the product of the maxima of the intervals $\hat{\alpha}'_{1,3}$, $\hat{\alpha}'_{2,1}$ and $\hat{\alpha}'_{3,2}$. So we can conclude that α does not contain any rank-one matrix.

Remark 11. It is not true that, given a $p \times q$ interval matrix, $\alpha = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$, with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ for any $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$, then there exists $A \in \alpha$ with $\text{rk}(A) = 1$ if and only if, for any $h \in [2, 2^{\min\{p,q\}-1}] \cap \mathbb{N}$, for any $i_1, \dots, i_h \in \{1, \dots, p\}$, for any $j_1, \dots, j_h \in \{1, \dots, q\}$ and for any $\sigma \in \Sigma_h$, we have

$$\alpha_{i_1, j_1} \cdots \alpha_{i_h, j_h} \cap \alpha_{i_1, j_{\sigma(1)}} \cdots \alpha_{i_h, j_{\sigma(h)}} \neq \emptyset. \quad (9)$$

In fact, for instance the interval matrix

$$\alpha = \begin{pmatrix} [-3, \frac{1}{3}] & [2, 4] & [0, 1] \\ 1 & [-1, 3] & 1 \end{pmatrix}$$

satisfies condition (9), but it is easy to see that it does not contain any rank-one matrix: in fact, suppose by contradiction that there exists a matrix $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \in \alpha$ with $\text{rk} \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = 1$; then there exists $k \in \mathbb{R}$ such that $\begin{pmatrix} a & b & c \end{pmatrix} = k \begin{pmatrix} d & e & f \end{pmatrix}$ and obviously we must have $d = 1$, $f = 1$; so $a = c = k \in [-3, \frac{1}{3}] \cap [0, 1] = [0, \frac{1}{3}]$ and $b = k e$; since k must be contained in $[0, \frac{1}{3}]$ and e must be contained in $[-1, 3]$, then b must be contained in $[-\frac{1}{3}, 1]$; but b must be contained also in $[2, 4]$, so we get a contradiction.

4 Maximal rank of matrices contained in an interval matrix

Definition 12. Given a $p \times p$ interval matrix, α , a **partial generalized diagonal** (**pg-diagonal** for short) of length k of α is a k -uple of the kind

$$(\alpha_{i_1, j_1}, \dots, \alpha_{i_k, j_k})$$

for some $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_k\}$ subsets of $\{1, \dots, p\}$.

Its **complementary matrix** is defined to be the submatrix of α given by the rows and columns whose indices are respectively in $\{1, \dots, p\} \setminus \{i_1, \dots, i_k\}$ and in $\{1, \dots, p\} \setminus \{j_1, \dots, j_k\}$.

We say that a pg-diagonal is **totally nondegenerate** if and only if all its entries are not degenerate.

We define $\det^c(\alpha)$ to be

$$\sum_{\sigma \in \Sigma_p \text{ s.t. } \alpha_{1,\sigma(1)}, \dots, \alpha_{p,\sigma(p)} \text{ are degenerate}} \epsilon(\sigma) \alpha_{1,\sigma(1)} \cdot \dots \cdot \alpha_{p,\sigma(p)}$$

if there exists $\sigma \in \Sigma_p$ such that $\alpha_{1,\sigma(1)}, \dots, \alpha_{p,\sigma(p)}$ are degenerate; we define $\det^c(\alpha)$ to be equal to 0 otherwise.

For every pg-diagonal of length p , say $\alpha_{1,\sigma(1)} \cdot \dots \cdot \alpha_{p,\sigma(p)}$ for some $\sigma \in \Sigma_p$, we call $\epsilon(\sigma)$ also the sign of the pg-diagonal.

Theorem 13. Let α be a $p \times p$ interval matrix. Then $\text{Mrk}(\alpha) < p$ if and only if the following conditions hold:

- (1) in α there is no totally nondegenerate pg-diagonal of length p ,
- (2) the complementary matrix of every totally nondegenerate pg-diagonal of length between 0 and $p - 1$ has \det^c equal to 0 (in particular $\det^c(\alpha) = 0$).

Proof. \implies We prove the statement by induction on p . For $p = 1$ the statement is obvious. Suppose $p \geq 2$ and that the statement is true for $(p - 1) \times (p - 1)$ interval matrices. Let α be a $(p \times p)$ interval matrix such that $\text{Mrk}(\alpha) < p$, that is $\det(A) = 0$ for every $A \in \alpha$.

If α contained a totally nondegenerate pg-diagonal of length p , say $\alpha_{i_1,j_1}, \dots, \alpha_{i_p,j_p}$, then $\alpha_{\hat{i}_1,\hat{j}_1}$ would have obviously a totally nondegenerate pg-diagonal of length $p - 1$, so by induction assumption, $\text{Mrk}(\alpha_{\hat{i}_1,\hat{j}_1}) = p - 1$. Thus there would exist $B \in \alpha_{\hat{i}_1,\hat{j}_1}$ with $\det(B) \neq 0$. So, for any choice of elements $x_{i_1,j} \in \alpha_{i_1,j}$ for $j \neq j_1$ and $x_{i,j_1} \in \alpha_{i,j_1}$ for $i \neq i_1$, we could find $x \in \alpha_{i_1,j_1}$ such that the determinant of the matrix X defined by $X_{\hat{i}_1,\hat{j}_1} = B$, $X_{i_1,j_1} = x$, $X_{i,j_1} = x_{i,j_1}$ for any $i \neq i_1$ and $X_{i_1,j} = x_{i_1,j}$ for any $j \neq j_1$ is nonzero, which is absurd. So we have proved that (1) holds.

Moreover, by contradiction, suppose (2) does not hold. Thus in α there exists a totally nondegenerate pg-diagonal of length k with $0 \leq k \leq p - 1$ whose complementary matrix has \det^c nonzero. If there exists such a diagonal with $k \geq 1$, say $\alpha_{i_1,j_1}, \dots, \alpha_{i_k,j_k}$, then also $\alpha_{\hat{i}_1,\hat{j}_1}$ does not satisfy (2), so, by induction assumption, there exists $B \in \alpha_{\hat{i}_1,\hat{j}_1}$ with $\det(B) \neq 0$ and we conclude as before. On the other hand, suppose that $\det^c(\alpha) \neq 0$ but the complementary matrix of every totally nondegenerate pg-diagonal of length k with $1 \leq k \leq p - 1$ has \det^c equal to zero; we call this assumption (*).

Let $A \in \alpha$. By (1), we can write $\det(A)$ as the sum of:

- the sum (with sign) of the product of the entries of the pg-diagonals of A of length p such the corresponding entries of α are all degenerate,
- the sum (with sign) of the product of the entries of the pg-diagonals of A of length p such all the corresponding entries of α apart from one are degenerate,
-
- the sum (with sign) of the product of the entries of the pg-diagonals of A of length p such all the corresponding entries of α apart from $p - 1$ are degenerate.

The first sum coincides with $\det^c(\alpha)$, so it is nonzero by the assumption (*); we can write the second sum by collecting the terms containing the same entry corresponding to the nondegenerate entry of α ; so, by assumption (*), we get that this sum is zero; we argue analogously for the other sums. So we can conclude that $\det(A)$ is nonzero, which is absurd.

\Leftarrow Let α be a matrix satisfying (1) and (2) and let $A \in \alpha$. By assumption (1), we can write $\det(A)$ as the sum of:

- the sum (with sign) of the product of the entries of the pg-diagonals of A of length p such the corresponding entries of α are all degenerate,
- the sum (with sign) of the product of the entries of the pg-diagonals of A of length p such all the corresponding entries of α apart from one are degenerate,

....

- the sum (with sign) of the product of the entries of the pg-diagonals of A of length p such all the corresponding entries of α apart from $p - 1$ are degenerate.

The first sum is zero by assumption; we can write the second sum by collecting the terms containing the same entry corresponding to the nondegenerate entry of α ; so by assumption we get that also this sum is zero. We argue analogously for the other sums. \square

Corollary 14. *Let α be an interval matrix. Then $\text{Mrk}(\alpha)$ is the maximum of the natural numbers t such that there is a $t \times t$ submatrix of α either with a totally nondegenerate pg-diagonal of length t or with a totally nondegenerate pg-diagonal of length between 0 and $t - 1$ whose complementary matrix has $\det^c \neq 0$.*

Proof. Let \hat{t} be the maximum of the natural numbers t such that there is a $t \times t$ submatrix of α with Mrk equal to t and let $\tilde{t} = \text{Mrk}(\alpha)$.

Obviously $\tilde{t} \geq \hat{t}$. On the other hand, let $A \in \alpha$ such that $\text{rk}(A) = \tilde{t}$. Then there exists a $\tilde{t} \times \tilde{t}$ submatrix of A with nonzero determinant. Hence the corresponding submatrix of α is a $\tilde{t} \times \tilde{t}$ submatrix with Mrk equal to \tilde{t} , so $\hat{t} \geq \tilde{t}$. So $\hat{t} = \tilde{t}$. By Theorem 13 we can conclude. \square

5 The case of the matrices with three columns

Let $\alpha = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$, with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ for any $i \in \{1, \dots, p\}$ and $j \in \{1, 2, 3\}$, be an interval matrix. By Remarks 6, 8, 9 and Theorem 10 we can see easily whether $\text{mrk}(\alpha) \in \{0, 1\}$ and by Corollary 14 we can calculate $\text{Mrk}(\alpha)$. So if $\text{mrk}(\alpha) \in \{0, 1\}$ we can calculate $\text{rkRange}(\alpha)$. Moreover, obviously, if $\text{mrk}(\alpha) \notin \{0, 1\}$ and $\text{Mrk}(\alpha) = 2$, we must have $\text{mrk}(\alpha) = 2$ and we get that every matrix contained in α has rank 2. So at the moment, we do not know $\text{rkRange}(\alpha)$ only if $\text{mrk}(\alpha) \notin \{0, 1\}$ and $\text{Mrk}(\alpha) = 3$. In this case, $\text{mrk}(\alpha)$ can be only 2 or 3. In this section we give a criterion to establish if, in this case, $\text{mrk}(\alpha)$ is equal to 2 or to 3.

Lemma 15. (a) *Let $a_i, c_j \in \mathbb{R}_{\geq 0}$ and $b_i, z_i, d_j, u_j \in \mathbb{R}$ for $i = 1, \dots, k$ and $j = 1, \dots, h$ such that a_i and c_j are nonzero for some $\hat{i} \in \{1, \dots, k\}$ and $\hat{j} \in \{1, \dots, h\}$. Then there exist $\lambda, \gamma \in \mathbb{R}_{\geq 0}$ such that, for every i and j ,*

$$\begin{cases} \lambda a_i + \gamma b_i \leq z_i, \\ \lambda c_j + \gamma d_j \geq u_j \end{cases}$$

if and only if there exist $\lambda, \gamma \in \mathbb{R}_{\geq 0}$ such that, for every i and j ,

$$\begin{cases} \lambda a_i c_j + \gamma b_i c_j \leq z_i c_j, \\ \lambda c_j a_i + \gamma d_j a_i \geq u_j a_i. \end{cases}$$

(b) Let $a_i \in \mathbb{R}_{\geq 0}$ and $b_i, c_i, z_i, u_i \in \mathbb{R}$ for $i = 1, \dots, k$; suppose there exists \hat{i} such that $a_{\hat{i}} \neq 0$. Then there exist $\lambda, \gamma \in \mathbb{R}_{\geq 0}$ such that, for every i ,

$$\begin{cases} \lambda a_i + \gamma b_i \leq z_i, \\ \lambda a_i + \gamma c_i \geq u_i \end{cases} \quad (10)$$

if and only if there exists $\gamma \in \mathbb{R}_{\geq 0}$ such that, for every $i, j \in \{1, \dots, k\}$,

$$a_j(z_i - \gamma b_i) \geq a_i(u_j - \gamma c_j)$$

and

$$z_i - \gamma b_i \geq 0.$$

(c) Let $x_i, y_i \in \mathbb{R}$ for $i = 1, \dots, k$. Then there exists $\gamma \in \mathbb{R}_{\geq 0}$ such that, for every i ,

$$\gamma x_i \geq y_i \quad (11)$$

if and only if, for every i , $x_i \leq 0$ implies $y_i \leq 0$ and

$$y_j x_i \geq y_i x_j$$

for every i, j such that $x_j > 0$ and $x_i < 0$.

Proof. Part (a) is obvious. Let us prove (b). There exist $\lambda, \gamma \in \mathbb{R}_{\geq 0}$ such that, for every i , (10) holds if and only if there exist $\lambda, \gamma \in \mathbb{R}_{\geq 0}$ such that

$$\begin{cases} 0 \leq z_i - \gamma b_i & \forall i \text{ s.t. } a_i = 0, \\ 0 \geq u_i - \gamma c_i & \forall i \text{ s.t. } a_i = 0, \\ \lambda \leq (z_i - \gamma b_i)/a_i & \forall i \text{ s.t. } a_i \neq 0, \\ \lambda \geq (u_i - \gamma c_i)/a_i & \forall i \text{ s.t. } a_i \neq 0, \end{cases}$$

and this is equivalent to the existence of $\gamma \in \mathbb{R}_{\geq 0}$ such that

$$\begin{cases} 0 \leq z_i - \gamma b_i & \forall i \text{ s.t. } a_i = 0, \\ 0 \geq u_i - \gamma c_i & \forall i \text{ s.t. } a_i = 0, \\ (u_j - \gamma c_j)/a_j \leq (z_i - \gamma b_i)/a_i & \forall i, j \text{ s.t. } a_i a_j \neq 0, \\ z_i - \gamma b_i \geq 0 & \forall i \text{ s.t. } a_i \neq 0, \end{cases}$$

and we can easily prove that this is equivalent to the existence of $\gamma \in \mathbb{R}_{\geq 0}$ such that, for every i, j , we have that $a_j(z_i - \gamma b_i) \geq a_i(u_j - \gamma c_j)$ and $z_i - \gamma b_i \geq 0$.

Let us prove (c). There exists $\gamma \in \mathbb{R}_{\geq 0}$ such that, for every i , (11) holds if and only if there exists $\gamma \in \mathbb{R}_{\geq 0}$ such that

$$\begin{cases} \gamma \geq y_i/x_i & \forall i \text{ s.t. } x_i > 0, \\ \gamma \leq y_i/x_i & \forall i \text{ s.t. } x_i < 0, \\ y_i \leq 0 & \forall i \text{ s.t. } x_i = 0, \end{cases}$$

and this is equivalent to the following condition: for every i , $x_i \leq 0$ implies $y_i \leq 0$ and $y_j x_i \geq y_i x_j$ for every i, j such that $x_j > 0$ and $x_i < 0$. \square

Corollary 16. Let $a_i, c_j \in \mathbb{R}_{\geq 0}$ and $b_i, z_i, d_j, u_j \in \mathbb{R}$ for $i, j \in \{1, \dots, k\}$ such that a_i and c_j are nonzero for some $\hat{i}, \hat{j} \in \{1, \dots, k\}$. Then there exist $\lambda, \gamma \in \mathbb{R}_{\geq 0}$ such that, for every i and j ,

$$\begin{cases} \lambda a_i + \gamma b_i \leq z_i, \\ \lambda c_j + \gamma d_j \geq u_j \end{cases} \quad (12)$$

if and only if, for any i, j, r, s :

- $b_i \geq 0$ implies $z_i \geq 0$,
- $b_i > 0$ and $b_j < 0$ imply $b_i z_j \geq b_j z_i$,
- $a_i d_r - b_i c_r \leq 0$ implies $a_i u_r - c_r z_i \leq 0$,
- $a_i d_r - b_i c_r < 0$ and $a_j d_s - b_j c_s > 0$ imply

$$(a_i d_r - b_i c_r)(a_j u_s - c_s z_j) \geq (a_j d_s - b_j c_s)(a_i u_r - c_r z_i),$$

- $b_j < 0$ and $a_i d_r - b_i c_r < 0$ imply $z_j(a_i d_r - b_i c_r) \leq b_j(a_i u_r - c_r z_i)$,
- $b_j > 0$ and $a_i d_r - b_i c_r > 0$ imply $z_j(a_i d_r - b_i c_r) \geq b_j(a_i u_r - c_r z_i)$.

Proof. By Remark 15 (a), there exist $\lambda, \gamma \in \mathbb{R}_{\geq 0}$ such that, for every i and j , (12) holds if and only if there exist $\lambda, \gamma \in \mathbb{R}_{\geq 0}$ such that, for every i and j ,

$$\begin{cases} \lambda a_i c_j + \gamma b_i c_j \leq z_i c_j, \\ \lambda c_j a_i + \gamma d_j a_i \geq u_j a_i. \end{cases}$$

By Remark 15 (b), this is true if and only if there exists $\gamma \in \mathbb{R}_{\geq 0}$ such that, for every i, j, r, s ,

$$\begin{cases} \gamma b_i c_j \leq z_i c_j, \\ a_s c_r (z_i c_j - \gamma b_i c_j) \geq a_i c_j (u_r a_s - \gamma d_r a_s), \end{cases}$$

which is obviously true if and only if there exists $\gamma \in \mathbb{R}_{\geq 0}$ such that, for every i, r ,

$$\begin{cases} \gamma (-b_i) \geq -z_i, \\ \gamma (a_i d_r - b_i c_r) \geq a_i u_r - c_r z_i. \end{cases}$$

By Remark 15 (c), this is true if and only if the following conditions hold for any i, j, r, s :

- $b_i \geq 0$ implies $z_i \geq 0$,
- $b_i > 0$ and $b_j < 0$ imply $b_i z_j \geq b_j z_i$,
- $a_i d_r - b_i c_r \leq 0$ implies $a_i u_r - c_r z_i \leq 0$,
- $a_i d_r - b_i c_r < 0$ and $a_j d_s - b_j c_s > 0$ imply

$$(a_i d_r - b_i c_r)(a_j u_s - c_s z_j) \geq (a_j d_s - b_j c_s)(a_i u_r - c_r z_i),$$

- $b_j < 0$ and $a_i d_r - b_i c_r < 0$ imply $z_j(a_i d_r - b_i c_r) \leq b_j(a_i u_r - c_r z_i)$,
- $b_j > 0$ and $a_i d_r - b_i c_r > 0$ imply $z_j(a_i d_r - b_i c_r) \geq b_j(a_i u_r - c_r z_i)$. □

Theorem 17. Let $\alpha = ([\underline{\alpha}_{i,j}, \bar{\alpha}_{i,j}])_{i,j}$, with $\underline{\alpha}_{i,j} \leq \bar{\alpha}_{i,j}$ for any $i \in \{1, \dots, p\}$ and $j \in \{1, 2, 3\}$, be a reduced interval matrix. Suppose $\underline{\alpha}_{i,1} \geq 0$ for every $i \in \{1, \dots, p\}$. Then $\text{mrk}(\alpha)$ is less than or equal to 2 if and only if, for $(v, w) = (2, 3)$ or for $(v, w) = (3, 2)$, we have that at least one of (1), (2), (3), (4) holds:

(1) for any $i, j, r, s \in \{1, \dots, p\}$, we have:

- $\underline{\alpha}_{i,v} \geq 0$ implies $\bar{\alpha}_{i,w} \geq 0$,
- $\underline{\alpha}_{i,v} > 0$ and $\underline{\alpha}_{j,v} < 0$ imply $\underline{\alpha}_{i,v}\bar{\alpha}_{j,w} \geq \underline{\alpha}_{j,v}\bar{\alpha}_{i,w}$,
- $\underline{\alpha}_{i,1}\bar{\alpha}_{r,v} - \underline{\alpha}_{i,v}\bar{\alpha}_{r,1} \leq 0$ implies $\underline{\alpha}_{i,1}\underline{\alpha}_{r,w} - \bar{\alpha}_{r,1}\bar{\alpha}_{i,w} \leq 0$,
- $\underline{\alpha}_{i,1}\bar{\alpha}_{r,v} - \underline{\alpha}_{i,v}\bar{\alpha}_{r,1} < 0$ and $\underline{\alpha}_{j,1}\bar{\alpha}_{s,v} - \underline{\alpha}_{j,v}\bar{\alpha}_{s,1} > 0$ imply

$$(\underline{\alpha}_{i,1}\bar{\alpha}_{r,v} - \underline{\alpha}_{i,v}\bar{\alpha}_{r,1})(\underline{\alpha}_{j,1}\underline{\alpha}_{s,w} - \bar{\alpha}_{s,1}\bar{\alpha}_{j,w}) \geq (\underline{\alpha}_{j,1}\bar{\alpha}_{s,v} - \underline{\alpha}_{j,v}\bar{\alpha}_{s,1})(\underline{\alpha}_{i,1}\underline{\alpha}_{r,w} - \bar{\alpha}_{r,1}\bar{\alpha}_{i,w}),$$

- $\underline{\alpha}_{j,v} < 0$ and $\underline{\alpha}_{i,1}\bar{\alpha}_{r,v} - \underline{\alpha}_{i,v}\bar{\alpha}_{r,1} < 0$ imply

$$\bar{\alpha}_{j,w}(\underline{\alpha}_{i,1}\bar{\alpha}_{r,v} - \underline{\alpha}_{i,v}\bar{\alpha}_{r,1}) \leq \underline{\alpha}_{j,v}(\underline{\alpha}_{i,1}\underline{\alpha}_{r,w} - \bar{\alpha}_{r,1}\bar{\alpha}_{i,w}),$$

- $\underline{\alpha}_{j,v} > 0$ and $\underline{\alpha}_{i,1}\bar{\alpha}_{r,v} - \underline{\alpha}_{i,v}\bar{\alpha}_{r,1} > 0$ imply

$$\bar{\alpha}_{j,w}(\underline{\alpha}_{i,1}\bar{\alpha}_{r,v} - \underline{\alpha}_{i,v}\bar{\alpha}_{r,1}) \geq \underline{\alpha}_{j,v}(\underline{\alpha}_{i,1}\underline{\alpha}_{r,w} - \bar{\alpha}_{r,1}\bar{\alpha}_{i,w}).$$

(2) the same conditions as in (1) with $-\bar{\alpha}_{\cdot,v}$ instead of $\underline{\alpha}_{\cdot,v}$ and vice versa hold;

(3) the same conditions as in (1) with $-\bar{\alpha}_{\cdot,1}$ instead of $\underline{\alpha}_{\cdot,1}$ and vice versa hold;

(4) the same conditions as in (1) with $-\bar{\alpha}_{\cdot,1}$ instead of $\underline{\alpha}_{\cdot,1}$ and vice versa and with $-\bar{\alpha}_{\cdot,v}$ instead of $\underline{\alpha}_{\cdot,v}$ and vice versa hold.

Proof. Obviously $\text{mrk}(\alpha) \leq 2$ if and only if there exists $A \in \alpha$ and c_1, c_2, c_3 not all zero such that $c_1 A^{(1)} + c_2 A^{(2)} + c_3 A^{(3)} = 0$. Since α is reduced, it is not possible that $c_2 = c_3 = 0$. Hence $\text{mrk}(\alpha) \leq 2$ if and only if there exists $A \in \alpha$ such that either $A^{(3)} \in \langle A^{(1)}, A^{(2)} \rangle$ or $A^{(2)} \in \langle A^{(1)}, A^{(3)} \rangle$.

Clearly there exists $A \in \alpha$ such that $A^{(w)} \in \langle A^{(1)}, A^{(v)} \rangle$ (with $(v, w) \in \{(2, 3), (3, 2)\}$) if and only if at least one of the following holds:

(1) there exist $\lambda, \gamma \in \mathbb{R}_{\geq 0}$ such that, for any $i, j \in \{1, \dots, p\}$,

$$\begin{cases} \lambda \underline{\alpha}_{i,1} + \gamma \underline{\alpha}_{i,v} \leq \bar{\alpha}_{i,w}, \\ \lambda \bar{\alpha}_{j,1} + \gamma \bar{\alpha}_{j,v} \geq \underline{\alpha}_{j,w}, \end{cases}$$

(2) there exist $\lambda \in \mathbb{R}_{\geq 0}, \gamma \in \mathbb{R}_{\leq 0}$ such that, for any $i, j \in \{1, \dots, p\}$,

$$\begin{cases} \lambda \underline{\alpha}_{i,1} + \gamma \bar{\alpha}_{i,v} \leq \bar{\alpha}_{i,w}, \\ \lambda \bar{\alpha}_{j,1} + \gamma \underline{\alpha}_{j,v} \geq \underline{\alpha}_{j,w}, \end{cases}$$

(3) there exist $\lambda \in \mathbb{R}_{\leq 0}, \gamma \in \mathbb{R}_{\geq 0}$ such that, for any $i, j \in \{1, \dots, p\}$,

$$\begin{cases} \lambda \bar{\alpha}_{i,1} + \gamma \underline{\alpha}_{i,v} \leq \bar{\alpha}_{i,w}, \\ \lambda \underline{\alpha}_{j,1} + \gamma \bar{\alpha}_{j,v} \geq \underline{\alpha}_{j,w}, \end{cases}$$

(4) there exist $\lambda, \gamma \in \mathbb{R}_{\leq 0}$ such that, for any $i, j \in \{1, \dots, p\}$,

$$\begin{cases} \lambda \bar{\alpha}_{i,1} + \gamma \bar{\alpha}_{i,v} \leq \bar{\alpha}_{i,w}, \\ \lambda \underline{\alpha}_{j,1} + \gamma \underline{\alpha}_{j,v} \geq \underline{\alpha}_{j,w}. \end{cases}$$

Clearly condition (2) is equivalent to the existence of $\lambda, \gamma \in \mathbb{R}_{\geq 0}$ such that, for any $i, j \in \{1, \dots, p\}$,

$$\begin{cases} \lambda \underline{\alpha}_{i,1} + \gamma (-\bar{\alpha}_{i,v}) \leq \bar{\alpha}_{i,w}, \\ \lambda \bar{\alpha}_{j,1} + \gamma (-\underline{\alpha}_{j,v}) \geq \underline{\alpha}_{j,w}. \end{cases}$$

Condition (3) is equivalent to the existence of $\lambda, \gamma \in \mathbb{R}_{\geq 0}$ such that, for any $i, j \in \{1, \dots, p\}$,

$$\begin{cases} \lambda (-\bar{\alpha}_{i,1}) + \gamma \underline{\alpha}_{i,v} \leq \bar{\alpha}_{i,w}, \\ \lambda (-\underline{\alpha}_{j,1}) + \gamma \bar{\alpha}_{j,v} \geq \underline{\alpha}_{j,w}. \end{cases}$$

Finally, condition (4) is equivalent to the existence of $\lambda, \gamma \in \mathbb{R}_{\geq 0}$ such that, for any $i, j \in \{1, \dots, p\}$,

$$\begin{cases} \lambda (-\bar{\alpha}_{i,1}) + \gamma (-\bar{\alpha}_{i,v}) \leq \bar{\alpha}_{i,w}, \\ \lambda (-\underline{\alpha}_{j,1}) + \gamma (-\underline{\alpha}_{j,v}) \geq \underline{\alpha}_{j,w}. \end{cases}$$

So, by Corollary 16, for $i = 1, 2, 3, 4$, condition (i) is equivalent to condition (i) in the statement of the theorem. □

The website <http://web.math.unifi.it/users/rubei/> contains some programs (in octave) which use the theorems and the remarks in the present paper. In particular the main programs are the following: rkmax.m (which says if a $p \times p$ interval matrix has Mrk equal to p), rk01.m (which says if mrk of an interval matrix is 0, 1 or greater than 1) and mrkpx3.m (which calculates mrk of any $p \times 3$ interval matrix).

The complexity of the first two is very high; in fact to check the conditions of Theorem 13 we need at least $O(p^{2p})$ operations and to choose $i_1, \dots, i_h, j_1, \dots, j_h$ as in Theorem 10 we have $p^{2p} p^{2p}$ possibilities.

Consider now the algorithm to find the minimal rank of a $p \times 3$ interval matrix α . First observe that, if the entries of the first column of α are contained in the set of nonnegative real numbers, then, to see if $\text{mrk}(\alpha) \geq 2$, we have to check the conditions of Theorem 17, which require $O(p^4)$ operations. Moreover, to see if a $p \times 3$ interval matrix α with all its entries contained in the set of nonnegative real numbers contains a rank one matrix, we need $O(p^4)$ operations (in fact we have to check the conditions of Theorem 10 and in our case we have that $2 \leq h \leq 4$, so we have at most p^4 choices for i_1, \dots, i_h). Finally, by Remark 9 (see also the example after Theorem 10), observe that, given a generic $p \times 3$ interval matrix α , in order to calculate its minimal rank, we have to consider at most $O(2^p)$ interval matrices (obtained from α as in the example after Theorem 10) with all the entries of the first row and the first column contained in the set of the nonnegative real numbers.

Acknowledgments. This work was supported by the National Group for Algebraic and Geometric Structures, and their Applications (GNSAGA-INdAM).

We wish to thank the anonymous referee for his/her valuable suggestions, which helped to improve the paper.

References

- [1] Cohen, N.; Dancis, J. *Maximal Ranks Hermitian Completions of Partially specified Hermitian matrices*. Linear Algebra Appl. 244 (1996), 265-276.
- [2] Cohen, N.; Johnson, C.R.; Rodman, Leiba; Woerdeman, H. J. *Ranks of completions of partial matrices*. The Gohberg anniversary collection, Vol. I (Calgary, AB, 1988), 165-185, Oper. Theory Adv. Appl., 40, Birkhäuser, Basel, 1989.
- [3] Hadwin, D.; Harrison J.; Ward, J.A. *Rank-one completions of partial matrices and completing rank-nondecreasing linear functionals*. Proc. A.M.S. 134 (2006), no. 8, 2169-2178.
- [4] Moore, R. Methods and applications of interval analysis, SIAM Studies in Applied Mathematics, 1979
- [5] Moore, R.E., Kearfott, R.B., Cloud, M. Introduction to interval analysis, SIAM, Philadelphia, 2009
- [6] Neumaier, A. Interval methods for systems of equations, Cambridge University Press, 1990
- [7] Rohn, J. *Systems of Linear Interval Equations*. Linear Algebra Appl. 126 (1989) 39-78.
- [8] Rohn, J. *Forty necessary and sufficient conditions for regularity of interval matrices: A survey*. Electronic Journal of Linear Algebra 18 (2009).
- [9] Rohn, J. *A Handbook of Results on Interval Linear Problems*, Prague: Institute of Computer Science, Academy of Sciences of the Czech Republic, 2012.
- [10] Rohn, J. *Enclosing solutions of overdetermined systems of linear interval equations*, Reliable Computing, 2 (1996), 167-171.
- [11] Shary, S.P. *On Full-Rank Interval Matrices* Numerical Analysis and Applications 7 (2014), no. 3, 241-254.
- [12] Woerdeman, H. J. *The lower order of lower triangular operators and minimal rank extensions*. Integral Equations and Operator Theory 10 (1987), 859-879.
- [13] Woerdeman, H. J. *Minimal rank completions for block matrices*. Linear algebra and applications (Valencia, 1987). Linear Algebra Appl. 121 (1989), 105-122.