Nonuniformly elliptic energy integrals with p,q-growth

Elsevier Editorial System(tm) for Nonlinear Analysis

Manuscript Draft

Manuscript Number:

Title: Nonuniformly elliptic energy integrals with p,q-growth

Article Type: SI:Nonlinear PDEs

Keywords: nonuniformly elliptic equations and systems, p,q-growth conditions, regularity, local boundedness.

Corresponding Author: Professor Paolo Marcellini,

Corresponding Author's Institution: Università di Firenze

First Author: Giovanni Cupini

Order of Authors: Giovanni Cupini; Paolo Marcellini; Elvira Mascolo

Abstract: We study the local boundedeness of minimizers of a nonuniformly energy integral of the form $\int_{}f(x,Dv)dx$ under p,q-growth conditions of the type

$(x)||^{p}\leq f(x,)\leq(x)(1+||^{q})$

for some exponents $q\geq p>1$ and with nonnegative functions , satisfying some summability conditions. We use here the original notation introduced in 1971 by Trudinger <cite>Trudinger</cite>, where (x) and (x) had the role of the minimum and the maximum eigenvalues of an $n\times n$ symmetric matrix $(a_{ij}(x))$ and

f(x,)=<K1.1/>$\sum_{i,j=1}^{n}a_{ij}(x)_{i}_{j}$

<K1.1 ilk="TEXTOBJECT" >
<screen-nom>displaystyle</screen-nom>
<LaTeX>\displaystyle</LaTeX></K1.1>

was the energy integrand associated to a linear nonuniformly elliptic equation in divergence form. In this paper we consider a class of energy integrals, associated to nonlinear nonuniformly elliptic equations and systems, with integrands f(x,) satisfying the general growth conditions above.

Suggested Reviewers: Giuseppe Mingione
giuseppe.mingione@unipr.it
He is expert in this subject

Michela Eleuteri
michela.eleuteri@unimore.it
She is expert in this subject

Vicentiu Radulescu
radulescu@inf.ucv.ro
He knows this subject

Cover Letter

Click here to download Cover Letter: invitation-letter_to_Paolo_Marcellini_for_C_Sbordone.pdf

Editors:

Siegfried Carl

Enzo Mitidieri

Guest Editors:

Nicola Fusco

Giuseppe Mingione

Important Deadlines:

Agreement to send a paper: March 31, 2017

Paper submission: March 30, 2018

Publication: End of 2018

Submission Instructions:

Submission starts March 30, 2017 Manuscripts and any supplementary material should be submitted through Elsevier Editorial System (EES). The authors must select "**SI: Nonlinear PDEs"** when they reach the "Article Type" step in the submission process. The EES website is located at http://ees.elsevier.com/na.

CALL FOR PAPERS

Special Issue dedicated to Prof. Carlo Sbordone 70th birthday

Nonlinear Analysis

http://www.elsevier.com/locate/na

Dear Professor,

On the occasion of Professor Carlo Sbordone's 70th birthday, we are planning a special issue of Nonlinear Analysis honoring Carlo and titled "Nonlinear PDEs and Geometric Function Theory, in honor of Carlo Sbordone on his 70th Birthday".

We would like to invite you to contribute to this special issue dedicated to Professor Carlo Sbordone, and would be very pleased and honored if you would accept our invitation. If you would like to contribute a paper, we would be grateful if you could let us know your intention before **March 31, 2017.** You will find further details and deadlines in this letter (see the left-hand side). The final deadline for submission is **March 31, 2018.** Both research and survey high quality papers are welcome.

We are looking forward to hearing from you soon.

Sincerely yours

Submission Format:

The submitted papers must be written in English and describe original research which is not published nor currently under review by other journals or conferences. Author guidelines for preparation of manuscript can be found at /www.elsevier.com/locate/na. For more information, please contact Professor Enzo Mitidieri (mitidieri@units.it).

NONUNIFORMLY ELLIPTIC ENERGY INTEGRALS WITH p, q-GROWTH

GIOVANNI CUPINI – PAOLO MARCELLINI – ELVIRA MASCOLO

Carlo Sbordone is a leader in Calculus of Variations and PDEs, as well as a master in boosting all aspects of mathematical projects and events. He was among the first one to appreciate Neil Trudinger's work [25] *on nonuniformly linear elliptic operators and to find application - for instance - to homogenization* [20], [21] *and not only to this subject* [5]. *It is an honor for us to dedicate this work to Carlo on the occasion of his seventieth birthday.*

Abstract. We study the local boundedeness of minimizers of a nonuniformly energy integral of the form $\int_\Omega f(x, Dv)\, dx$ under p, q-growth conditions of the type

$$\lambda(x)|\xi|^p \leq f(x, \xi) \leq \mu(x)\,(1 + |\xi|^q)$$

for some exponents $q \geq p > 1$ and with nonnegative functions λ, μ satisfying some summability conditions. We use here the original notation introduced in 1971 by Trudinger [25], where $\lambda(x)$ and $\mu(x)$ had the role of the minimum and the maximum eigenvalues of an $n \times n$ symmetric matrix $(a_{ij}(x))$ and

$$f(x, \xi) = \sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j$$

was the energy integrand associated to a linear nonuniformly elliptic equation in divergence form. In this paper we consider a class of energy integrals, associated to nonlinear nonuniformly elliptic equations and systems, with integrands $f(x, \xi)$ satisfying the general growth conditions above.

1. Introduction

In the recent mathematical literature a large interest received the energy integral

$$\int_\Omega \{|Du(x)|^p + |x|^\alpha |Du(x)|^q\}\, dx, \tag{1.1}$$

where $1 < p \leq q$, $\alpha > 0$, Ω is an open set in $\mathbb{R}^n$, $n \geq 2$, $0 \in \Omega$, $u : \Omega \to \mathbb{R}^m$, $m \geq 1$, and Du is the gradient of u. See for instance [1], [3], [4], [13], [14], [15], [23], [26]. To explain the point of view of this research, we denote as ξ the gradient variable and

$$f(x, \xi) := |\xi|^p + |x|^\alpha |\xi|^q\,;$$

then $f : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a convex function with respect to $\xi \in \mathbb{R}^{m\times n}$ which satisfies the growth conditions

$$|\xi|^p \leq f(x, \xi) \leq c\,(1 + |\xi|^q) \tag{1.2}$$

and also

$$|x|^\alpha ||\xi|^q \leq f(x, \xi) \leq c\,(1 + |\xi|^q) \tag{1.3}$$

for some positive constant c and for every $\xi \in \mathbb{R}^{m\times n}$. With respect to the condition (1.2) we can say that $f(x, \xi)$ enters in the p, q-growth regularity theory and, by the recent results in [6], [8], [10],

2010 *Mathematics Subject Classification.* Primary: 49N60; Secondary: 35J60.

Key words and phrases. nonuniformly elliptic equations, nonuniformly elliptic systems, p, q-growth conditions, regularity, local boundedness.

Acknowledgement: The authors are members of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) .

we know that any local minimizer of the energy integral in (1.1) is locally bounded in Ω if $p<n$ and

$$q<p^* =: \frac{np}{n-p}$$

(with $q \le p^*$ if $m=1$). However for the model integral (1.1) our point of view here is to take under consideration also condition (1.3). With the symbols described more in details below, we have that, if $\alpha<q$, then $\lambda^{-1} := |x|^{-\alpha} \in L^r_{\mathrm{loc}}(\Omega)$ for some r such that $\frac{n}{q}<r<\frac{n}{\alpha}$. This allows us to apply Theorem 1.1 below and to obtain that every local minimizer of the energy integral (1.1) is locally bounded.

Let us summarize the discussion above: any local minimizer of the energy integral (1.1) is locally bounded in Ω if either the integrand $f(x,\xi)$ satisfies the p,q-growth in (1.2) and $q<p^*$, or if $f(x,\xi)$ is nonuniformly elliptic as in (1.3) and $\alpha<q$. Of course here it would be interesting to unify the two cases and to give a sole condition. This is one of our aims in what we propose below.

In this paper we consider the general case of nonuniformly elliptic energy integrals of p,q-growth of the type

$$\mathcal{F}(v;\Omega)=\int_\Omega f(x,Dv)\,dx \tag{1.4}$$

with

$$\begin{cases} \lambda(x)|\xi|^p \le f(x,\xi) \le \mu(x)\left(|\xi|^q+1\right) \\ \lambda^{-1}\in L^r_{\mathrm{loc}}(\Omega), \quad \mu \in L^s_{\mathrm{loc}}(\Omega) \end{cases} \tag{1.5}$$

for $1<p\le q$, for some exponents $r\in[1,\infty]$, $s\in(1,\infty]$ and for every ξ in $\mathbb{R}^{m\times n}$. The integrand $f(x,\xi)$ also satisfies the condition

$$f(x,\xi)=g(x,|\xi|)\,, \tag{1.6}$$

where $g:\Omega\times[0,\infty)\to[0,\infty)$ is a convex Δ_2 function of class C^1 with respect to the last variable, with $g(x,0)=g_t(x,0)=0$.

Theorem 1.1. *Let us consider the energy integral* (1.4) *under the assumptions* (1.5), (1.6), *with* $r\in[1,\infty]$, $s\in(1,\infty]$ *(if* $p\le 2$ *then we also require* $r>\frac{1}{p-1}$*) and*

$$\frac{1}{pr}+\frac{1}{qs}+\frac{1}{p}-\frac{1}{q}<\frac{1}{n}\,; \tag{1.7}$$

then every local minimizer of the energy integral (1.4) *is locally bounded.*

We observe that in the relevant case $r=s=\infty$ then condition (1.7) can be equivalently written in the form $q<p^*$. We also point out the connection of our result with the pioneering and celebrated paper by Trudinger [25], where it is studied the linear case of nonuniformly elliptic second order equations (of course with $p=q=2$). Indeed, if $p=q$, then (1.7) becomes

$$\frac{1}{r}+\frac{1}{s}<\frac{p}{n}\,,$$

which is exactly the assumption on the summability exponents appearing in [25] in the particular case $p=q=2$.

We refer to the next Section 2 for more details. There we also fix the notation and we give the complete statement of our main result, see Theorem 2.1, including a precise estimate for the L^∞ norm. In Section 3 we give some preliminary results, while Section 4 is devoted to the proof of Theorem 2.1.

For completeness, the regularity theory under p,q-growth conditions have been considered nowadays by many authors. See for instance [1], [2], [3], [4], [11], [13], [14], [16], [17], [19]; see also the survey [22] on regularity under non standard growth.

Our research collaboration on regularity in the calculus of variations under p, q-growth conditions was originated in 2009 and focused in particular on the local boundedness of local minimizers ([6], [7], [8], [9], [10]). A comparison with the previous papers and in particular with the most recent [10] shows that here we consider the vector-valued case (with respect to the scalar one in [10]) and we assume here quite weaker summability conditions on the coefficients.

2. The main results

Let us consider the functional

$$\mathcal{F}(v;\Omega) = \int_\Omega f(x, Dv)\, dx, \tag{2.1}$$

with $\Omega \subset \mathbb{R}^n$, $n \geq 2$, open set, and $f : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$, $m \geq 1$, a Carathéodory function. We assume that there exist measurable functions $\lambda, \mu : \Omega \to [0, \infty)$ such that

$$\lambda(x)|\xi|^p \leq f(x,\xi) \leq \mu(x)\left(|\xi|^q + 1\right), \qquad 1 < p \leq q, \tag{2.2}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{m\times n}$, where

$$\lambda^{-1} \in L^r_{\mathrm{loc}}(\Omega), \quad \mu \in L^s_{\mathrm{loc}}(\Omega),$$

for some exponents $r \in [1, \infty]$ and $s \in (1, \infty]$.

With simple computations (see Proposition 3.1 below), for every open set Ω' compactly contained in Ω we deduce that

$$\left\|\lambda^{-1}\right\|^{-1}_{L^r(\Omega')} \|Dv\|^p_{L^{\frac{pr}{r+1}}(\Omega';\mathbb{R}^{m\times n})} \leq \int_{\Omega'} f(x, Dv)\, dx \leq \|\mu\|_{L^s(\Omega')} \|Dv\|^q_{L^{\frac{qs}{s-1}}(\Omega';\mathbb{R}^{m\times n})} + \|\mu\|_{L^1(\Omega')}\ , \tag{2.3}$$

and therefore we have

$$W^{1,\frac{pr}{r+1}}\left(\Omega';\mathbb{R}^m\right) \supset W^{1,\mathcal{F}}(\Omega';\mathbb{R}^m) \supset W^{1,\frac{qs}{s-1}}\left(\Omega';\mathbb{R}^m\right),$$

where $W^{1,\mathcal{F}}(\Omega';\mathbb{R}^m)$ denotes the set of maps u of finiteness of the integral; i.e.,

$$W^{1,\mathcal{F}}(\Omega';\mathbb{R}^m) := \{u \in W^{1,1}(\Omega';\mathbb{R}^m)\ :\ \mathcal{F}(u;\Omega') < \infty\}.$$

It is clear that if we fix appropriate conditions at the boundary of a fixed Ω', then from standard direct methods of the calculus of variations we derive existence of minimizers in $W^{1,\frac{pr}{r+1}}(\Omega';\mathbb{R}^m)$; at this stage we need the condition $\frac{pr}{r+1} > 1$.

Of course any minimizer belongs also to $W^{1,\mathcal{F}}(\Omega';\mathbb{R}^m)$.

In the vector-valued case, as suggested by well known counterexamples by De Giorgi [12], Giusti-Miranda [18], Šverák-Yan [24], generally some structure conditions are required for everywhere regularity. Thus, as usual in this theory, we assume that f is a radial function with respect to the gradient variable ξ. Precisely:

there exists $g : \Omega \times [0, \infty) \to [0, \infty)$ such that, for a.e. $x \in \Omega$,

$$f(x,\xi) = g(x, |\xi|) \ \text{ is a convex and } C^1 \text{ function with respect to } \xi \in \mathbb{R}^{m\times n}. \tag{2.4}$$

We also assume the so-called Δ_2-condition holds:

there exists $\gamma \geq 1$ such that

$$f(x, t\xi) \leq t^\gamma f(x,\xi) \tag{2.5}$$

for every $t > 1$, for a.e. x and every ξ. This condition implies that $W^{1,\mathcal{F}}(\Omega';\mathbb{R}^m)$ is a vector space.

We study the regularity of local minimizers of $\mathcal{F}$. We recall that a function $u \in W^{1,\frac{pr}{r+1}}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ is a local minimizer of $\mathcal{F}$ if

$$\mathcal{F}(u;\Omega') \leq \mathcal{F}(u + \varphi;\Omega')$$

for every open set Ω' compactly contained in Ω and for all $\varphi \in W^{1,\mathcal{F}}(\Omega;\mathbb{R}^m)$ with $\operatorname{supp}\varphi \Subset \Omega'$.

Before stating our regularity result, we recall that, given a real number $\ell \geq 1$, ℓ^* is its Sobolev exponent, i.e.

$$\ell^* := \begin{cases} \frac{n\ell}{n-\ell} & \text{if } \ell < n, \\ \text{any number} > \ell & \text{if } \ell \geq n \end{cases}$$

and ℓ' is the conjugate exponent of ℓ, i.e., $\frac{1}{\ell} + \frac{1}{\ell'} = 1$. Moreover, as usual, $\dfrac{1}{\infty}$ has to be read as 0.

Theorem 2.1. *Let us assume that* (2.2), (2.4) *and* (2.5) *hold under the summability conditions*

$$\lambda^{-1} \in L^r_{\mathrm{loc}}(\Omega), \qquad \mu \in L^s_{\mathrm{loc}}(\Omega),$$

for some $r \in [1,\infty]$, $s \in (1,\infty]$ such that

$$\frac{1}{pr} + \frac{1}{qs} + \frac{1}{p} - \frac{1}{q} < \frac{1}{n}\,. \tag{2.6}$$

If $p \leq 2$ we also require $r > \frac{1}{p-1}$. Then, every local minimizer $u \in W^{1,\frac{pr}{r+1}}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ of $\mathcal{F}$ is locally bounded and there exists a positive constant c, depending on p,q,n,m,γ, such that

$$\|u\|_{L^\infty(B_{R_0/2}(x_0);\mathbb{R}^m)} \leq c\left\{R_0^{-q}\|\lambda^{-1}\|_{L^r(B_{R_0})}\,\|\mu\|_{L^s(B_{R_0})}\right\}^{\vartheta_1}\,\||u|+1\|^{\vartheta_2}_{L^{qs'}(B_{R_0}(x_0))}\,, \tag{2.7}$$

for every $x_0 \in \Omega$ and $B_{R_0}(x_0) \Subset \Omega$, $0 < R_0 \leq 1$, where $\sigma := \frac{pr}{r+1}$ ($\sigma := p$ if $r = \infty$) and

$$\vartheta_1 := \frac{\sigma^*}{p(\sigma^* - qs')}, \qquad \vartheta_2 := \frac{q(\sigma^* - ps')}{p(\sigma^* - qs')}.$$

Remark 2.2. If $r = s = \infty$ inequality (2.6) reduces to $q < p^*$, see e.g. [7], [8], [10]. Moreover, we observe that the assumption on the summability exponent of μ can be written as

$$\left(\frac{\sigma^*}{q}\right)' < s \leq \infty, \tag{2.8}$$

and that, if $p = q$, then it becomes

$$\frac{1}{s} + \frac{1}{r} < \frac{p}{n}.$$

Remark 2.3. Trudinger [25] considered nonuniformly elliptic second order differential equations, with $p = q = 2$, under the following assumptions on λ and μ

$$\lambda^{-1} \in L^r_{\mathrm{loc}}(\Omega), \qquad \lambda^{-1}\mu^2 \in L^s_{\mathrm{loc}}(\Omega) \qquad \frac{1}{s} + \frac{1}{r} < \frac{2}{n}\,.$$

His condition on μ is slightly stronger than ours; in fact in our context $\lambda(x) \leq 2\mu(x)$ a.e. in Ω (by (2.2); also in [25] $\lambda(x) \leq \mu(x)$) and thus

$$\lambda^{-1}\mu^2 \in L^s_{\mathrm{loc}}(\Omega) \Rightarrow \mu \in L^s_{\mathrm{loc}}(\Omega).$$

3. Preliminary results

We start this section with a precise statement and proof of (2.3).

Proposition 3.1. *Assume that* (2.2) *holds, where $\lambda^{-1} \in L^r_{\mathrm{loc}}(\Omega)$, $\mu \in L^s_{\mathrm{loc}}(\Omega)$ for some exponents $r \in [1,\infty]$ and $s \in (1,\infty]$ (if $p \leq 2$ then we also require $r \geq \frac{1}{p-1}$). Then for every $v \in W^{1,\frac{pr}{r+1}}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ and every open $\Omega' \Subset \Omega$ the inequalities* (2.3) *hold; i.e.,*

$$\left\|\lambda^{-1}\right\|^{-1}_{L^r(\Omega')}\|Dv\|^p_{L^{\frac{pr}{r+1}}(\Omega';\mathbb{R}^{m\times n})} \leq \int_{\Omega'} f(x,Dv)\,dx \leq \|\mu\|_{L^s(\Omega')}\,\|Dv\|^q_{L^{\frac{qs}{s-1}}(\Omega';\mathbb{R}^{m\times n})} + \|\mu\|_{L^1(\Omega')}\,.$$

Proof. Consider $r \in [1, \infty)$, $v \in W^{1,\frac{pr}{r+1}}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ and an open set $\Omega' \Subset \Omega$. By Hölder inequality with exponents $\frac{r+1}{r}$ and $r+1$,

$$\int_{\Omega'} |Dv|^{\frac{pr}{r+1}}\, dx = \int_{\Omega'} \lambda^{\frac{r}{r+1}} |Dv|^{\frac{pr}{r+1}} \lambda^{-\frac{r}{r+1}}\, dx \le \left(\int_{\Omega'} \lambda |Dv|^p\, dx \right)^{\frac{r}{r+1}} \left(\int_{\Omega'} (\lambda^{-1})^r\, dx \right)^{\frac{1}{r+1}}.$$

Therefore, using the left inequality in (2.2)

$$\int_{\Omega'} f(x, Dv)\, dx \ge \int_{\Omega'} \lambda |Dv|^p\, dx \ge \left(\int_{\Omega'} |Dv|^{\frac{pr}{r+1}}\, dx \right)^{\frac{r+1}{r}} \left(\int_{\Omega'} (\lambda^{-1})^r\, dx \right)^{-\frac{1}{r}}$$

and the left inequality in (2.3) follows.

If $r = \infty$, that is $\frac{r}{r+1} = 1$, consider $v \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$. The left inequality in (2.3) follows by

$$\int_{\Omega'} |Dv|^p\, dx \le \|\lambda^{-1}\|_{L^\infty(\Omega')} \int_{\Omega'} \lambda |Dv|^p\, dx$$

and the left inequality in (2.2).

Let us now prove the right inequality in (2.3).

Let us first consider $s \in (1, \infty)$. If $v \notin W^{1,\frac{qs}{s-1}}(\Omega')$ the inequality trivially holds. Let us assume $v \in W^{1,\frac{qs}{s-1}}(\Omega')$. By Hölder inequality with exponents s and $\frac{s}{s-1}$, we obtain

$$\int_{\Omega'} \mu |Dv|^q\, dx \le \left(\int_{\Omega'} \mu^s\, dx \right)^{\frac{1}{s}} \left(\int_{\Omega'} |Dv|^{\frac{qs}{s-1}}\, dx \right)^{\frac{s-1}{s}}.$$

Using the right inequality in (2.2) we conclude.

Let us now consider $s = \infty$, that is $\frac{s}{s-1} = 1$. If $v \notin W^{1,q}(\Omega')$ the inequality trivially holds. Let us assume $v \in W^{1,q}(\Omega')$. We obtain

$$\int_{\Omega'} \mu |Dv|^q\, dx \le \|\mu\|_{L^\infty(\Omega')} \int_{\Omega'} |Dv|^q\, dx.$$

Using the right inequality in (2.2) we conclude. □

The next lemma is about a Poincaré-Sobolev type inequality.

Lemma 3.2. *Consider a bounded open set $\Omega \subset \mathbb{R}^n$ and let p and r be such that $p > 1$ and $r \in [1, \infty]$ (if $p \le 2$ then we also require $r \ge \frac{1}{p-1}$). Let $v \in W^{1,\frac{pr}{r+1}}_0(\Omega; \mathbb{R}^m)$ ($v \in W^{1,p}_0(\Omega; \mathbb{R}^m)$ if $r = \infty$) and let $\lambda : \Omega \to [0, \infty)$ be a measurable function such that $\lambda^{-1} \in L^r(\Omega)$. Then there exists a positive constant c such that*

$$\left\{ \int_\Omega |v|^{\sigma^*}\, dx \right\}^{\frac{p}{\sigma^*}} \le c \|\lambda^{-1}\|_{L^r(\Omega)} \int_\Omega \lambda\, |Dv|^p\, dx, \tag{3.1}$$

where $\sigma := \frac{pr}{r+1}$ ($\sigma := p$ if $r = \infty$).

Proof. First, assume that $r < \infty$. By applying the Poincaré inequality and Hölder inequality with exponents $\frac{r+1}{r}, r+1$ we obtain

$$\begin{aligned} \left\{ \int_\Omega |v|^{\sigma^*}\, dx \right\}^{\frac{p}{\sigma^*}} &\le c \left\{ \int_\Omega |Dv|^\sigma\, dx \right\}^{\frac{p}{\sigma}} \le c \left\{ \int_\Omega \left(\lambda^{\frac{r}{r+1}} |Dv|^\sigma \right) \lambda^{-\frac{r}{r+1}}\, dx \right\}^{\frac{p}{\sigma}} \\ &\le c \|\lambda^{-1}\|_{L^r(B_R)} \int_\Omega \lambda |Dv|^p\, dx. \end{aligned}$$

Let us now discuss the case $r=\infty$, that implies $\sigma = p$. We have

$$\left\{ \int_\Omega |v|^{\sigma^*}\, dx \right\}^{\frac{p}{\sigma^*}} \leq c \int_\Omega |Dv|^p\, dx \leq c \int_{B_R} \lambda^{-1} \left(\lambda |Dv|^p\right)\, dx$$
$$\leq c \|\lambda^{-1}\|_{L^\infty(\Omega)} \int_\Omega \lambda |Dv|^p\, dx.$$

This concludes the proof of (3.1). □

The following lemma deals with well known properties of the convex functions satisfying (2.2), (2.4) and (2.5).

Lemma 3.3. *Assume* (2.2), (2.4), (2.5). *Then*

(i) $f(x, t\xi) \leq \max\{1, t^\gamma\} f(x,\xi)$ *for every* $t>0$ *and every* $\xi \in \mathbb{R}^{m\times n}$,
(ii) $f(x, \xi+\eta) \leq 2^{\gamma-1} \left(f(x,\xi) + f(x,\eta)\right)$ *for every* $\xi, \eta \in \mathbb{R}^{m\times n}$,
(iii) $g_t(x,t)t \leq \gamma g(x,t)$ *for every* $t \geq 0$.

We also remark that a Euler's equation holds true.

Proposition 3.4. *Assume* (2.2), (2.4), (2.5) *and let* u *be a local minimizer of (2.1). Then*

$$\int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^m \frac{\partial f}{\partial \xi_i^\alpha}\,(x, Du)\; \varphi^\alpha_{x_i}\, dx = 0 \tag{3.2}$$

for all $\varphi \in W^{1,\mathcal{F}}(\Omega; \mathbb{R}^m)$, $\operatorname{supp}\varphi \Subset \Omega$.

Proof. Let $\varphi \in W^{1,\mathcal{F}}(\Omega; \mathbb{R}^m)$, $\operatorname{supp}\varphi \Subset \Omega$. By (2.4) also $-\varphi$ is in $W^{1,\mathcal{F}}(\Omega; \mathbb{R}^m)$. By Lemma 3.3 we get $u + t\varphi \in W^{1,\mathcal{F}}(\Omega; \mathbb{R}^m)$ for every $t \in \mathbb{R}$. By the local minimality of u,

$$\mathcal{F}(u) \leq \mathcal{F}(u+t\varphi) \qquad \forall t \in \mathbb{R}.$$

To prove (3.2) it suffices to prove that

$$\frac{d}{dt}\mathcal{F}(u+t\varphi)\Big|_{t=0} = \int_\Omega \frac{d}{dt} f(x, Du(x) + tD\varphi(x))\Big|_{t=0}\, dx.$$

To prove this, we need to prove that

$$\Big| \sum_{i=1}^n \sum_{\alpha=1}^m \frac{\partial f}{\partial \xi_i^\alpha}(x, Du + tD\varphi)\varphi^\alpha_{x_i} \Big| \leq H(x) \qquad \forall\, t \in (-1,1)$$

with $H \in L^1(\operatorname{supp}\varphi)$. By the convexity,

$$f(x,\xi_0) - f(x, 2\xi_0 - \xi) \leq \sum_{i=1}^n \sum_{\alpha=1}^m \frac{\partial f}{\partial \xi_i^\alpha}(x,\xi_0)(\xi_i^\alpha - (\xi_0)_i^\alpha) \leq f(x,\xi) - f(x,\xi_0).$$

If $\xi_0 = Du(x) + tD\varphi(x)$, $\xi = Du(x) + (1+t)D\varphi(x)$, we have $2\xi_0 - \xi = Du(x) + (t-1)D\varphi(x)$ and

$$\begin{aligned} f(x, Du + tD\varphi) - f(x, Du + (t-1)D\varphi) \quad &\leq \quad \sum_{i=1}^n \sum_{\alpha=1}^m \frac{\partial f}{\partial \xi_i^\alpha}(x, Du + tD\varphi)\varphi^\alpha_{x_i} \\ &\leq \quad f(x, Du + (1+t)D\varphi) - f(x, Du + tD\varphi). \end{aligned}$$

Therefore, since f is non-negative,

$$\Big| \sum_{i=1}^n \sum_{\alpha=1}^m \frac{\partial f}{\partial \xi_i^\alpha}(x, Du + tD\varphi)\varphi^\alpha_{x_i} \Big| \leq f(x, Du + (1+t)D\varphi) + f(x, Du + (t-1)D\varphi).$$

Using again the convexity and Lemma 3.3 we get

$$\begin{aligned} f(x, Du+(1+t)D\varphi) &\leq tf(x, Du+2D\varphi)+(1-t)f(x, Du+D\varphi) \\ &\leq f(x, Du+2D\varphi)+f(x, Du+D\varphi) \leq 2^{\gamma}\left(f(x, Du+D\varphi)+f(x, D\varphi)\right) \\ &\leq 2^{2\gamma}\left(f(x, Du)+f(x, D\varphi)\right) \end{aligned}$$

and, similarly,

$$\begin{aligned} f(x, Du+(t-1)D\varphi) &\leq tf(x, Du)+(1-t)f(x, Du-D\varphi) \\ &\leq f(x, Du)+f(x, Du-D\varphi) \\ &\leq 2^{\gamma}\left(f(x, Du)+f(x, -D\varphi)\right). \end{aligned}$$

We have so proved that

$$\Big|\sum_{i=1}^{n}\sum_{\alpha=1}^{m}\frac{\partial f}{\partial \xi_i^{\alpha}}(x, Du+tD\varphi)\varphi_{x_i}^{\alpha}\Big| \leq 2^{3\gamma}\left(f(x, Du)+f(x, D\varphi)+f(x, -D\varphi)\right) =: H(x).$$

Since $u, \varphi, -\varphi \in W^{1,\mathcal{F}}$, we conclude. □

4. Proof of the main results

We first state a lemma useful for the proof of Theorem 2.1. In the statement, the functions λ, μ, and the exponents p, q, r, are the same of the statement of Theorem 2.1. Moreover, $x_0 \in \Omega$ and $0 < R_0 \leq 1$ are such that $B_{R_0} := B_{R_0}(x_0) \Subset \Omega$. Fixed $R \in (0, R_0]$, the function $\eta \in C_c^{\infty}(B_R(x_0))$ denotes a cut-off function satisfying

$$0 \leq \eta \leq 1, \qquad \eta \equiv 1 \text{ in } B_{\rho}(x_0), \qquad |D\eta| \leq \frac{2}{R-\rho}, \tag{4.1}$$

where $0 < \rho < R$.

Lemma 4.1. *Let $u \in W^{1,\frac{pr}{r+1}}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ be a local minimizer of (2.1). Then for every $\beta \geq 0$*

$$\int_{B_R} \lambda(x)(|u|+1)^{p\beta}|Du|^p\ \eta^q\, dx \leq \frac{c_1}{(R-\rho)^q}\int_{B_R}\mu(x)(|u|+1)^{q+p\beta}\, dx \tag{4.2}$$

for some c_1 depending on n, m, p, q, γ, but independent of β, u, R and ρ.

Proof. We begin using Proposition 3.4, with a suitable test function.

Let us approximate the identity function id : $\mathbb{R}_+ \to \mathbb{R}_+$ with an increasing sequence of C^1 functions $h_k : \mathbb{R}_+ \to \mathbb{R}_+$, with the following properties:

$$h_k(t) = 0 \quad \forall t \in [0, \frac{1}{k}], \qquad h_k(t) = k \quad \forall t \in [k+1, +\infty], \qquad 0 \leq h_k'(t) \leq 2 \text{ in } [0, \infty). \tag{4.3}$$

Fix $k \in \mathbb{N}$ and $\beta \geq 0$. Let $\Phi_k^{(\beta)} : \mathbb{R}_+ \to \mathbb{R}_+$ be the increasing function defined as follows

$$\Phi_k^{(\beta)}(t) := h_k(t^{p\beta}). \tag{4.4}$$

Define $\varphi_k^{(\beta)} : B_R(x_0) \to \mathbb{R}^m$,

$$\varphi_k^{(\beta)}(x) := \Phi_k^{(\beta)}(|u(x)|)u(x)[\eta(x)]^q. \tag{4.5}$$

We have that φ_k is in $W^{1,f}(B_R(x_0); \mathbb{R}^m)$, $\operatorname{supp}\varphi \Subset B_R(x_0)$ (see the proof of Lemma 5.1 in [8]).

Let us consider the Euler's equation (3.2) with test function $\varphi_k^{(\beta)}$. From now on, we write φ_k and Φ_k in place of $\varphi_k^{(\beta)}$ and $\Phi_k^{(\beta)}$, respectively. We obtain

$$\begin{aligned} I_1 + I_2 :=& \sum_{j=1}^{n}\sum_{\alpha=1}^{m}\int_{B_R} \frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\, u^\alpha_{x_j}\, \Phi_k(|u|)\,\eta^q\,dx \\ &+\sum_{j=1}^{n}\sum_{\alpha,\beta=1}^{m}\int_{B_R} \frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\, u^\alpha \frac{u^\beta}{|u|}\, u^\beta_{x_j}\, \Phi_k'(|u|)\,\eta^q\,dx \\ &\le q\left|\sum_{j=1}^{n}\sum_{\alpha=1}^{m}\int_{B_R} \frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\,\Phi_k(|u|)\,u^\alpha\,\eta^{q-1}\eta_{x_j}\,dx\right| =: I_3. \end{aligned} \tag{4.6}$$

Now, we separately estimate I_1, I_2, I_3.

Estimate of I_1

As far as I_1 is concerned, we use that $f(x,\cdot)$ is convex. Thus,

$$I_1 \ge \int_{B_R} \left(f(x,Du) - f(x,0)\right)\,\Phi_k(|u|)\,\eta^q\,dx.$$

Using (2.2), we get

$$I_1 \ge \int_{B_R} f(x,Du)\,\Phi_k(|u|)\,\eta^q\,dx - \int_{B_R}\mu\Phi_k(|u|)\,\eta^q\,dx. \tag{4.7}$$

Estimate of I_2

We claim that $I_2 \ge 0$. Indeed, by (2.4), $\frac{\partial f}{\partial \xi_j^\alpha}(x,Du) = g_t(x,|Du|)\frac{u^\alpha_{x_j}}{|Du|}$. Therefore

$$\sum_{j=1}^{n}\sum_{\alpha,\beta=1}^{m}\frac{\partial f}{\partial \xi_j^\alpha}(x,Du)\,u^\alpha u^\beta\, u^\beta_{x_j} = \sum_{j=1}^{n} g_t(x,|Du|)\frac{\left(\sum_{\alpha=1}^m u^\alpha\, u^\alpha_{x_j}\right)^2}{|Du|} \ge 0.$$

Thus, by the monotonicity of Φ_k we have

$$I_2 = \int_{B_R}\sum_{j=1}^{n} g_t(x,|Du|)\frac{\left(\sum_{\alpha=1}^m u^\alpha\, u^\alpha_{x_j}\right)^2}{|Du|\,|u|}\,\Phi_k'(|u|)\,\eta^q\,dx \ge 0. \tag{4.8}$$

Estimate of I_3

By (2.4) and (4.1) we have

$$I_3 \le \frac{2mq}{R-\rho}\int_{A_R^-\cup A_R^+} g_t(x,|Du|)|u|\Phi_k(|u|)\,\eta^{q-1}\,dx, \tag{4.9}$$

where

$$A_R^- := B_R \cap \left\{\eta \ne 0, |Du| \le \frac{2mqL|u|}{\eta(R-\rho)}\right\}$$

and

$$A_R^+ := B_R \cap \left\{\eta \ne 0, |Du| > \frac{2mqL|u|}{\eta(R-\rho)}\right\}$$

with $L>0$ to be chosen later.

By (2.2), (2.4) and the assumption $x \in A_R^-$ the following inequality follows:

$$g_t(x,|Du|)\frac{2mq|u|}{\eta(R-\rho)} \le \frac{1}{L} g_t\left(x, \frac{2mqL|u|}{\eta(R-\rho)}\right)\frac{2mqL|u|}{\eta(R-\rho)},$$

and, by Lemma 3.3 (iii), (2.4) and (2.2),

$$\frac{1}{L} g_t\left(x, \frac{2mqL|u|}{\eta(R-\rho)}\right) \frac{2mqL|u|}{\eta(R-\rho)} \le \frac{\gamma}{L} g\left(x, \frac{2mqL|u|}{\eta(R-\rho)}\right) \le \frac{\gamma}{L}\mu(x)\left\{\left(\frac{2mqL|u|}{\eta(R-\rho)}\right)^q + 1\right\}.$$

Therefore

$$\begin{aligned} &\frac{2mq}{R-\rho}\int_{A_R^-} g_t(x,|Du|)|u|\Phi_k(|u|)\,\eta^{q-1}\,dx \\ &\le \gamma\left(\frac{2mq}{R-\rho}\right)^q \int_{B_R}\mu(x)L^{q-1}|u|^q\Phi_k(|u|)\,dx + \frac{\gamma}{L}\int_{B_R}\mu(x)\Phi_k(|u|)\,\eta^q\,dx. \end{aligned} \tag{4.10}$$

Let us now deal with A_R^+. For a.e. $x \in A_R^+$, by Lemma 3.3 (iii), (2.4) and (2.2) it follows

$$g_t(x,|Du|)\frac{2mq|u|}{\eta(R-\rho)} \le \frac{1}{L} g_t(x,|Du|)|Du| \le \frac{\gamma}{L} f(x,Du),$$

thus

$$\frac{2mq}{R-\rho}\int_{A_R^+} g_t(x,|Du|)|u|\Phi_k(|u|)\,\eta^{q-1}\,dx \le \frac{\gamma}{L}\int_{B_R} f(x,Du)\Phi_k(|u|)\eta^q\,dx. \tag{4.11}$$

By (4.9), (4.10) and (4.11) we obtain

$$\begin{aligned} I_3 &\le \gamma\left(\frac{2mq}{R-\rho}\right)^q \int_{B_R}\mu(x)L^{q-1}|u|^q\Phi_k(|u|)\,dx \\ &+ \frac{\gamma}{L}\int_{B_R} f(x,Du)\Phi_k(|u|)\eta^q\,dx + \frac{\gamma}{L}\int_{B_R}\mu(x)\Phi_k(|u|)\,\eta^q\,dx. \end{aligned} \tag{4.12}$$

By (4.7), (4.8), (4.10) and (4.12) we get

$$\begin{aligned} \left(1-\frac{\gamma}{L}\right)\int_{B_R} f(x,Du)\,\Phi_k(|u|)\,\eta^q\,dx &\le \gamma\left(\frac{2mq}{R-\rho}\right)^q \int_{B_R}\mu(x)L^{q-1}|u|^q\Phi_k(|u|)\,dx \\ &+ \left(1+\frac{\gamma}{L}\right)\int_{B_R}\mu(x)\Phi_k(|u|)\,\eta^q\,dx. \end{aligned}$$

Choosing $L = 2\gamma$ and using (2.2), we get

$$\begin{aligned} \int_{B_R} f(x,Du)\,\Phi_k(|u|)\,\eta^q\,dx &\le (2\gamma)^q\left(\frac{2mq}{R-\rho}\right)^q \int_{B_R}\mu(x)|u|^q\Phi_k(|u|)\,dx \\ &+ 3\int_{B_R}\mu(x)\Phi_k(|u|)\,dx. \end{aligned} \tag{4.13}$$

Inequalities (2.2) and (4.13) imply

$$\int_{B_R}\lambda(x)|Du|^p\ \Phi_k(u)\ \eta^q\,dx \le \frac{c_0}{(R-\rho)^q}\int_{B_R}\mu(x)\left\{|u|^q+1\right\}\Phi_k(|u|)\,dx,$$

where we also used $R_0 \le 1$. We recall that $\Phi_k = \Phi_k^{(\beta)}$ and we explicitly notice that c_0 is independent of β, ρ and R. Using the monotone convergence theorem we let k go to $+\infty$ and by the definition of Φ we obtain

$$\begin{aligned} &\int_{B_R}\lambda(x)|u|^{p\beta}|Du|^p\ \eta^q\,dx \le \frac{c_0}{(R-\rho)^q}\int_{B_R}\mu(x)\left\{|u|^{q+p\beta}+|u|^{p\beta}\right\}\,dx \\ &\le \frac{2c_0}{(R-\rho)^q}\int_{B_R}\mu(x)(|u|+1)^{q+p\beta}\,dx\,. \end{aligned}$$

In particular, if $\beta = 0$:

$$\int_{B_R} \lambda(x)|Du|^p \ \eta^q \, dx \le \frac{2c_0}{(R-\rho)^q} \int_{B_R} \mu(x)(|u|+1)^q \, dx \, .$$

Thus, (4.2) follows. □

Proof of Theorem 2.1. Let $u \in W^{1,\frac{pr}{r+1}}_{\rm loc}(\Omega;\mathbb{R}^m)$ be a local minimizer of (2.1). Consider $x_0 \in \Omega$ and $0 < R_0 \le 1$, such that $B_{R_0} := B_{R_0}(x_0) \Subset \Omega$. Fix also $0 < \rho < R \le R_0$ and consider a cut-off function η satisfying (4.1). We split the proof into two steps.

Step 1. First we prove that, if $\delta \ge 1$ and $|u|^\delta \in W^{1,\sigma}(B_R)$, then

$$\begin{aligned}\|(|u|+1)^\delta\|_{L^{\sigma^*}(B_\rho)} \le & c\frac{q+\delta}{(R-\rho)^{q/p}} \|\lambda^{-1}\|^{\frac{1}{p}}_{L^r(B_{R_0})} \|\mu\|^{\frac{1}{p}}_{L^s(B_{R_0})} \times \\ & \times \||u|+1\|^{\frac{q-p}{p}}_{L^{qs'}(B_{R_0})} \|(|u|+1)^\delta\|_{L^{qs'}(B_R)}. \end{aligned} \tag{4.14}$$

To prove the above inequality, we notice that for any $\beta := \delta - 1 \ge 0$ we have that

$$\begin{aligned}\int_\Omega \left| D((|u|+1)^{\beta+1}\eta^q)\right|^p \lambda(x)\, dx \le \int_\Omega \left(q\eta^{q-1}|D\eta|\right)^p (|u|+1)^{(\beta+1)p}\lambda(x)\, dx \\ + (\beta+1)^p \int_\Omega \lambda(x)\eta^{qp}(|u|+1)^{p\beta}|Du|^p \, dx. =: J_1 + J_2. \end{aligned} \tag{4.15}$$

To estimate J_1 we observe that

$$J_1 \le c\frac{q^p}{(R-\rho)^p} \int_{B_R} (|u|+1)^{(\beta+1)p}\lambda(x)\, dx \le c\frac{q^p}{(R-\rho)^p} \int_{B_R} (|u|+1)^{q+\beta p}\lambda(x)\, dx. \tag{4.16}$$

Since $\eta^{qp} \le \eta^q$, we can estimate J_2 using Lemma 4.1. Thus,

$$J_2 \le c\frac{(\beta+1)^p}{(R-\rho)^q} \int_{B_R} \mu(x)\,(|u|+1)^{q+p\beta} \ dx. \tag{4.17}$$

By Lemma 3.2 applied to $v = (|u|+1)^{\beta+1}\eta^q$ the inequality (3.1) holds, that is

$$\left\{\int_\Omega \left((|u|+1)^{\beta+1}\eta^q\right)^{\sigma^*} \ dx\right\}^{\frac{p}{\sigma^*}} \le c\|\lambda^{-1}\|_{L^r(B_{R_0})} \int_\Omega \lambda(x) \left| D((|u|+1)^{\beta+1}\eta^q)\right|^p \ dx. \tag{4.18}$$

Collecting this inequality, (4.15), (4.16) and (4.17), we get

$$\begin{aligned}\left\{\int_\Omega \left((|u|+1)^{\beta+1}\eta^q\right)^{\sigma^*} \ dx\right\}^{\frac{p}{\sigma^*}} \le & c\|\lambda^{-1}\|_{L^r(B_{R_0})} \frac{q^p}{(R-\rho)^p} \int_{B_R} \lambda(x)(|u|+1)^{q+\beta p}\, dx \\ & + c\|\lambda^{-1}\|_{L^r(B_{R_0})} \frac{(\beta+1)^p}{(R-\rho)^q} \int_{B_R} \mu(x)\,(|u|+1)^{q+p\beta} \ dx. \end{aligned}$$

By (2.2) $\lambda \le 2\mu$ a.e., and $R_0 \le 1$, so we get

$$\left\{\int_\Omega \left((|u|+1)^{\beta+1}\eta^q\right)^{\sigma^*} \ dx\right\}^{\frac{p}{\sigma^*}} \le c\|\lambda^{-1}\|_{L^r(B_{R_0})} \frac{q^p + (\beta+1)^p}{(R-\rho)^q} \int_{B_R} \mu(x)(|u|+1)^{q+\beta p}\, dx. \tag{4.19}$$

By Hölder inequality and $\mu \in L^s_{\rm loc}(\Omega)$ we obtain

$$\int_{B_R} \mu(x)(|u|+1)^{q+\beta p}\, dx \le \|\mu\|_{L^s(B_{R_0})} \left(\int_{B_R} (|u|+1)^{(q+\beta p)s'}\, dx\right)^{\frac{1}{s'}}. \tag{4.20}$$

If $p = q$, (4.19) and (4.20) give

$$\|(|u|+1)^{\beta+1}\|_{L^{\sigma^*}(B_\rho)} \le c\frac{q+\beta}{R-\rho}\|\lambda^{-1}\|^{\frac{1}{p}}_{L^r(B_{R_0})}\|\mu\|^{\frac{1}{p}}_{L^s(B_{R_0})}\|(|u|+1)^{\beta+1}\|_{L^{ps'}(B_R)},$$

that is (4.14) holds for $q = p$ and $\delta = \beta + 1$.

If $p < q$, let us apply Hölder inequality in the last integral in (4.20):

$$\begin{aligned} \left(\int_{B_R}(|u|+1)^{(q+\beta p)s'}\,dx\right)^{\frac{1}{s'}} &= \left(\int_{B_R}(|u|+1)^{(q-p)s'}(|u|+1)^{(\beta+1)ps'}\,dx\right)^{\frac{1}{s'}} \\ \le \ & \left(\int_{B_R}(|u|+1)^{qs'}\,dx\right)^{\frac{q-p}{qs'}}\left(\int_{B_R}(|u|+1)^{(\beta+1)qs'}\,dx\right)^{\frac{p}{qs'}}. \end{aligned} \tag{4.21}$$

Then, (4.19), (4.20) and (4.21) give

$$\begin{aligned} \|(|u|+1)^{\beta+1}\|_{L^{\sigma^*}(B_\rho)} \le & c\frac{q+\beta+1}{(R-\rho)^{q/p}}\|\lambda^{-1}\|^{\frac{1}{p}}_{L^r(B_{R_0})}\|\mu\|^{\frac{1}{p}}_{L^s(B_{R_0})}\times \\ & \times \||u|+1\|^{\frac{q-p}{p}}_{L^{qs'}(B_{R_0})}\|(|u|+1)^{\beta+1}\|_{L^{qs'}(B_R)}. \end{aligned}$$

Since $\delta = \beta + 1$, we get that (4.14) holds also for $p < q$.

Step 2. Let us define $G(x) := \max\{1, |u(x)|\}$. Now, we prove the boundedness of G, and then of u, using the Moser's iteration technique. The inequality (4.14) implies that for any $\delta \ge 1$,

$$\|G^\delta\|_{L^{\sigma^*}(B_\rho)} \le c\frac{q+\delta}{(R-\rho)^{q/p}}\|\lambda^{-1}\|^{\frac{1}{p}}_{L^r(B_{R_0})}\|\mu\|^{\frac{1}{p}}_{L^s(B_{R_0})}\|G\|^{\frac{q-p}{p}}_{L^{qs'}(B_{R_0})}\|G^\delta\|_{L^{qs'}(B_R)}. \tag{4.22}$$

Now, we prove the boundedness of G, and then of u, using the Moser's iteration technique. For all $h \in \mathbb{N}$ define $\delta_h = \left(\frac{\sigma^*}{qs'}\right)^{h-1}$, $R_h = R_0/2 + R_0/2^h$ and $\rho_h = R_{h+1}$. Notice that the choice of δ_h has been done in such a way that $\delta_1 = 1$ and $\delta_h\sigma^* = \delta_{h+1}qs'$. By (4.22), replacing δ, R and ρ with δ_h, R_h and ρ_h, respectively, we have that $G \in L^{\delta_h qs'}(B_{R_h})$ implies $G \in L^{\delta_{h+1}qs'}(B_{R_{h+1}})$. Precisely,

$$\begin{aligned} \|G\|\|_{L^{\delta_{h+1}s'q}(B_{R_{h+1}})} &= \|G\|_{L^{\delta_h\sigma^*}(B_{R_{h+1}})} = \|G^{\delta_h}\|^{\frac{1}{\delta_h}}_{L^{\sigma^*}(B_{R_{h+1}})} \\ \le \ & \left\{c\tfrac{q+\delta_h}{(R_h-\rho_h)^{q/p}}\|\lambda^{-1}\|^{\frac{1}{p}}_{L^r(B_{R_0})}\|\mu\|^{\frac{1}{p}}_{L^s(B_{R_0})}\|G\|^{\frac{q-p}{p}}_{L^{qs'}(B_{R_0})}\right\}^{\frac{1}{\delta_h}}\|G\|_{L^{\delta_h qs'}(B_{R_h})} \end{aligned} \tag{4.23}$$

holds true for every h. For instance, if $h = 1$, we get

$$\|G\|_{L^{\sigma^*}(B_{3R_0/4})} \le c\frac{q+1}{R_0^{q/p}}\|\lambda^{-1}\|^{\frac{1}{p}}_{L^r(B_{R_0})}\|\mu\|^{\frac{1}{p}}_{L^s(B_{R_0})}\|G\|^{\frac{q-p}{p}}_{L^{qs'}(B_{R_0})}\|G\|_{L^{qs'}(B_{R_0})}.$$

Notice that the right hand side is finite, because $u \in W^{1,\sigma}$, so $u \in L^{\sigma^*}$ and, by (2.8), $qs' < \sigma^*$. Since

$$\sum_{h=1}^{\infty}\frac{1}{\delta_h} = \sum_{i=0}^{\infty}\left(\frac{qs'}{\sigma^*}\right)^i = \frac{\sigma^*}{\sigma^*-qs'},$$

an iterated use of (4.23) implies the existence of a constant c such that

$$\|G\|_{L^\infty(B_{R_0/2}(x_0))} \le c\left(\frac{\|\lambda^{-1}\|_{L^r(B_{R_0})}\,\|\mu\|_{L^s(B_{R_0})}}{R_0^q}\right)^{\frac{\sigma^*}{p(\sigma^*-qs')}}\|G\|^{\frac{\sigma^*(q-p)}{p(\sigma^*-qs')}}_{L^{qs'}(B_{R_0})}\|G\|_{L^{qs'}(B_{R_0}(x_0))}.$$

The inequality above implies that u is in $L^\infty(B_{R_0/2}(x_0);\mathbb{R}^m)$ and the estimate (2.7). $\square$

References

[1] P. Baroni, M. Colombo, G. Mingione: *Harnack inequalities for double phase functionals,* Nonlinear Analysis 121 (2015) 206-222.

[2] M. Carozza, J. Kristensen, A. Passarelli di Napoli: *Regularity of minimizers of autonomous convex variational integrals,* Ann. Sc. Norm. Super. Pisa Cl. Sci. 13 (2014) 1065-1089.

[3] M. Colombo, G. Mingione: *Regularity for double phase variational problems,* Arch. Rat. Mech. Anal. 215 (2015) 443-496.

[4] M. Colombo, G. Mingione: *Bounded minimisers of double phase variational integrals,* Arch. Rat. Mech. Anal. 218 (2015) 219-273.

[5] D. Cruz-Uribe, P. Di Gironimo, C. Sbordone: *On the continuity of solutions to degenerate elliptic equations,* J. Differential Equations 250 (2011) 2671-2686.

[6] G. Cupini, P. Marcellini, E. Mascolo: *Regularity under sharp anisotropic general growth conditions,* Discrete Contin. Dyn. Syst. Ser. B 11 (2009) 66-86.

[7] G. Cupini, P. Marcellini, E. Mascolo: *Local boundedness of solutions to quasilinear elliptic systems,* Manuscripta Math. 137 (2012) 287-315.

[8] G. Cupini, P. Marcellini, E. Mascolo: *Local boundedness of solutions to some anisotropic elliptic systems,* Recent trends in nonlinear partial differential equations. II. Stationary problems 169-186, Contemp. Math. 595, Amer. Math. Soc., Providence, RI, 2013.

[9] G. Cupini, P. Marcellini, E. Mascolo: *Existence and regularity for elliptic equations under $p,q-$growth,* Adv. Differential Equations 19 (2014) 693-724.

[10] G. Cupini, P. Marcellini, E. Mascolo: *Regularity of minimizers under limit growth conditions,* Nonlinear Anal. 153 (2017) 294-310.

[11] A. Dall'Aglio, E. Mascolo: *L^∞ estimates for a class of nonlinear elliptic systems with nonstandard growth,* Atti Sem. Mat. Fis. Univ. Modena 50 (2002) 65-83.

[12] E. De Giorgi: *Un esempio di estremali discontinue per un problema variazionale di tipo ellittico,* Boll. Un. Mat. Ital.(4) 1 (1968) 135-137.

[13] M. Eleuteri, P. Marcellini, E. Mascolo: *Lipschitz estimates for systems with ellipticity conditions at infinity,* Ann. Mat. Pura e Appl. (4) 195 (2016) 1575-1603.

[14] M. Eleuteri, P. Marcellini, E. Mascolo: *Lipschitz continuity for energy integrals with variable exponents,* Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 27 (2016) 61-87.

[15] M. Eleuteri, P. Marcellini, E. Mascolo: *Regularity for scalar integrals without structure conditions,* Adv. Calc. Var. (2018), in press, DOI: https://doi.org/10.1515/acv-2017-0037.

[16] N. Fusco, C. Sbordone: *Some remarks on the regularity of minima of anisotropic integrals,* Comm. Partial Differential Equations 18 (1993) 153-167.

[17] F. Giannetti, A. Passarelli di Napoli: *Higher differentiability of minimizers of variational integrals with variable exponents,* Math. Z. 280 (2015) 873-892.

[18] E. Giusti, M. Miranda: *Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni,* Boll. Un. Mat. Ital. 2 (1968) 1-8.

[19] F. Leonetti, P. V. Petricca: *Bounds for vector valued minimizers of some relaxed functionals,* Complex Var. Elliptic Equ. 58 (2013) 221-230.

[20] P. Marcellini, C. Sbordone: *An approach to the asymptotic behaviour of elliptic-parabolic operators,* J. Math. Pures Appl. 56 (1977) 157-182.

[21] P. Marcellini, C. Sbordone: *Homogenization of nonuniformly elliptic operators,* Applicable Anal. 8 (1978/79) 101-113.

[22] G. Mingione:*Regularity of minima: an invitation to the dark side of the calculus of variations,* Appl. Math. 51 (2006) 355-426.

[23] V. Rădulescu, Q. Zhang: *Double phase anisotropic variational problems and combined effects of reaction and absorption terms,* J. Math. Pures Appl. (2018), to appear.

[24] V. Sveràk, X. Yan: *A singular minimizer of a smooth strongly convex functional in three dimensions,* Calc. Var. Partial Differential Equations 10 (2000) 213-221.

[25] N.S. Trudinger: *On the regularity of generalized solutions of linear, nonuniformly elliptic equations,* Arch. Rational Mech. Anal. 42 (1971) 50-62.

[26] V.V. Zhikov: *Averaging of functionals of the calculus of variations and elasticity theory,* Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986) 675-710.

E-mail address: giovanni.cupini@unibo.it
E-mail address: marcellini@math.unifi.it
E-mail address: mascolo@math.unifi.it

Giovanni Cupini: Dipartimento di Matematica, Università di Bologna, Piazza di Porta S.Donato 5, 40126 - Bologna, Italy

Paolo Marcellini and Elvira Mascolo: Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Viale Morgagni 67/A, 50134 - Firenze, Italy