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POLYNOMIALS AND THE EXPONENT OF MATRIX MULTIPLICATION

LUCA CHIANTINI, JONATHAN D. HAUENSTEIN, CHRISTIAN IKENMEYER, J.M. LANDSBERG, AND GIORGIO OTTAVIANI

Abstract. The *exponent of matrix multiplication* is the smallest constant ω such that two $n \times n$ matrices may be multiplied by performing $O(n^{\omega+\epsilon})$ arithmetic operations for every $\epsilon > 0$. Determining the constant ω is a central question in both computer science and mathematics. Strassen [39] showed that ω is also governed by the tensor rank of the matrix multiplication tensor. We define certain symmetric tensors, i.e., cubic polynomials, and our main result is that their symmetric rank also grows with the same exponent ω, so that ω can be computed in the symmetric setting, where it may be easier to determine. In particular, we study the symmetrized matrix multiplication tensor $sM_{\langle n\rangle}$ defined on an $n\times n$ matrix A by $sM_{\langle n\rangle}(A) = \mathrm{trace}(A^3)$. The use of polynomials enables the introduction of additional techniques from algebraic geometry in the study of the matrix multiplication exponent ω.

1. Introduction

The *exponent of matrix multiplication* is the smallest constant ω such that two $n \times n$ matrices may be multiplied by performing $O(n^{\omega+\epsilon})$ arithmetic operations for every $\epsilon > 0$. It is a central open problem to estimate ω since it governs the complexity of many basic algorithms in linear algebra. The current state of the art [15, 30, 38, 41] is

$$2 \leq \omega < 2.374.$$

A *tensor* $t \in \mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$ has *(tensor) rank* r if r is the minimum such that there exists $u_i, v_i, w_i \in \mathbb{C}^N$ with $t = \sum_{i=1}^r u_i \otimes v_i \otimes w_i$. In this case, we write $\mathbf{R}(t) = r$. Let $V = \mathbb{C}^n$ and $\mathbb{C}^N = \mathrm{End}(V) = \mathrm{Mat}_n$ be the vector space of $n \times n$ matrices over $\mathbb{C}$. The matrix multiplication tensor $M_{\langle n\rangle} \in \mathrm{Mat}_n^\vee \otimes \mathrm{Mat}_n^\vee \otimes \mathrm{Mat}_n^\vee$ is

$$M_{\langle n\rangle}(A, B, C) = \mathrm{trace}(ABC), \tag{1.1}$$

where $\mathrm{Mat}_n^\vee$ is the vector space dual to Mat_n.

Strassen [39] showed that $\omega = \liminf[\log_n(\mathbf{R}(M_{\langle n\rangle}))]$. If the tensor t can be expressed as a limit of tensors of rank s (but not a limit of tensors of rank at most $s-1$), then t has *border rank* s, denoted $\underline{\mathbf{R}}(t) = s$. This is equivalent to t being in the Zariski closure of the set of tensors of rank s but not in the Zariski closure of the set of tensors of rank at most $s-1$, see, e.g., [31, Thm. 2.33]. This was rediscovered in complexity theory in [3]. Bini [7] showed that $\omega = \liminf[\log_n(\underline{\mathbf{R}}(M_{\langle n\rangle}))]$.

The determination of the fundamental constant ω is a central question. In 1981, Schönhage [36] showed the exponent ω could be bounded using disjoint sums of matrix multiplication tensors. Then, in 1987, Strassen [40] proposed using tensors other than $M_{\langle n\rangle}$ which are easier to analyze due to their combinatorial properties

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to prove upper bounds on ω. These other tensors are then degenerated to disjoint matrix multiplication tensors. The main goal of this paper is to open a different path to bounding ω by introducing polynomials that are closely related to matrix multiplication.

We expect these polynomials are easier to work with in two ways. First, we want to take advantage of the vast literature in algebraic geometry regarding the geometry of cubic hypersurfaces. Second, we want to exploit recent numerical computational techniques. The difficulty of the usual matrix multiplication tensor is the sheer size of the problem, even for relatively small n. Despite considerable effort, no 4×4 decompositions, other than the standard rank 64 decomposition and the rank 49 decomposition obtained by squaring Strassen's 2×2 decomposition, have appeared in the literature. With our approach, the polynomials are defined on much smaller spaces thereby allowing one to perform more computational experiments and produce additional data for forming conjectures.

Let $\mathrm{Sym}^3\mathbb{C}^N \subset (\mathbb{C}^N)^{\otimes 3}$ and $\Lambda^3\mathbb{C}^N \subset (\mathbb{C}^N)^{\otimes 3}$ respectively denote the space of symmetric and skew-symmetric tensors. Tensors in $\mathrm{Sym}^3\mathbb{C}^N$ may be viewed as homogeneous cubic polynomials in N variables. While the matrix multiplication tensor $M_{\langle n\rangle}$ is neither symmetric nor skew-symmetric, it is $\mathbb{Z}_3$-invariant where $\mathbb{Z}_3$ denotes the cyclic group on three elements permuting the factors since $\mathrm{trace}(ABC) = \mathrm{trace}(BCA)$. The space of $\mathbb{Z}_3$-invariant tensors in $(\mathbb{C}^N)^{\otimes 3}$ is

$$[(\mathbb{C}^N)^{\otimes 3}]^{\mathbb{Z}_3} = \mathrm{Sym}^3\mathbb{C}^N \oplus \Lambda^3\mathbb{C}^N.$$

Thus, respectively define the symmetrized and skew-symmetrized part of the matrix multiplication tensor, namely

$$sM_{\langle n\rangle}(A,B,C) := \frac{1}{2}[\mathrm{trace}(ABC) + \mathrm{trace}(BAC)] \tag{1.2}$$

$$\Lambda M_{\langle n\rangle}(A,B,C) := \frac{1}{2}[\mathrm{trace}(ABC) - \mathrm{trace}(BAC)] \tag{1.3}$$

so that

$$M_{\langle n\rangle} = sM_{\langle n\rangle} + \Lambda M_{\langle n\rangle}. \tag{1.4}$$

The $\mathbb{Z}_3$-invariance implies $sM_{\langle n\rangle} \in \mathrm{Sym}^3\mathbb{C}^N$ and $\Lambda M_{\langle n\rangle} \in \Lambda^3\mathbb{C}^N$.

The tensor $M_{\langle n\rangle}$ is the structure tensor for the algebra Mat_n. Similarly, the skew-symmetrized matrix multiplication tensor $\Lambda M_{\langle n\rangle}$ is (if one ignores the $\frac{1}{2}$) the structure tensor for the Lie algebra $\mathfrak{gl}(V)$. The symmetrized matrix multiplication tensor $sM_{\langle n\rangle}$ is the structure tensor for Mat_n considered as a Jordan algebra, i.e., with the multiplication $A\circ B = \frac{1}{2}(AB+BA)$. In particular, considered as a cubic polynomial on Mat_n,

$$sM_{\langle n\rangle}(A) = \mathrm{trace}(A^3).$$

We further define the following cubic polynomials (symmetric tensors):

- $sM^S_{\langle n\rangle}$: restriction of $sM_{\langle n\rangle}$ to symmetric matrices Sym^2V,
- $sM^{S,0}_{\langle n\rangle}$: restriction of $sM^S_{\langle n\rangle}$ to traceless symmetric matrices, and
- $sM^Z_{\langle n\rangle}$: restriction of $sM^S_{\langle n\rangle}$ to symmetric matrices with zeros on diagonal.

In order to have an invariant definition of $sM^S_{\langle n\rangle}$ and $sM^{S,0}_{\langle n\rangle}$, one needs an identification of V with V^*. Two natural ways of obtaining this identification are via a nondegenerate symmetric quadratic form or, when $\dim V$ is even, a skew-symmetric form. We will often use the former, which reduces the symmetry group from the general linear group to the orthogonal group. We do not know of a nice invariant definition for the polynomial $sM^Z_{\langle n\rangle}$.

For a homogeneous degree d polynomial P, the *symmetric* or *Waring rank* $\mathbf{R}_s(P)$ is the smallest r such that $P=\sum_{j=1}^r \ell_j^d$, where ℓ_j are linear forms. The *symmetric border rank* $\underline{\mathbf{R}}_s(P)$ is the smallest r such that P is a limit of polynomials of symmetric rank at most r. Note that

$$\mathbf{R}(P)\le \mathbf{R}_s(P) \quad \text{and} \quad \underline{\mathbf{R}}(P)\le \underline{\mathbf{R}}_s(P). \tag{1.5}$$

We notice that there are several general cases where equality holds in both of these relations. We refer to [8, 14] for a discussion.

Our main result is that one can compute the exponent ω of matrix multiplication using these polynomials even when considering symmetric rank and border rank.

Theorem 1.1. *Let $\omega=\liminf[\log_n(\mathbf{R}(M_{\langle n\rangle}))]$ be the exponent of matrix multiplication. Then $\omega=\liminf[\log_n F(G_n)]$, where*
G_n is one of the families of symmetric tensors defined above:

1. *Symmetrized matrix multiplication tensor $sM_{\langle n\rangle}(A)=\mathrm{trace}(A^3)$,*
2. *$sM^S_{\langle n\rangle}$: restriction of $sM_{\langle n\rangle}$ to symmetric matrices A,*
3. *$sM^{S,0}_{\langle n\rangle}$: restriction of $sM^S_{\langle n\rangle}$ to traceless symmetric matrices A,*
4. *$sM^Z_{\langle n\rangle}$: restriction of $sM^S_{\langle n\rangle}$ to symmetric matrices A with zeros on diagonal,*

and F is one of the following functions on cubic polynomials/symmetric tensors:

(a) *tensor rank,*
(b) *tensor border rank*
(c) *symmetric (Waring) rank*
(d) *symmetric (Waring) border rank.*

Explicitly we have the following chain of equalities

$$\begin{aligned}
\omega &= \liminf_n \left[\log_n \mathbf{R}(sM_{\langle n\rangle})\right] = \liminf_n \left[\log_n \underline{\mathbf{R}}(sM_{\langle n\rangle})\right] && (1.6)\\
&= \liminf_n \left[\log_n \mathbf{R}_s(sM_{\langle n\rangle})\right] = \liminf_n \left[\log_n \underline{\mathbf{R}}_s(sM_{\langle n\rangle})\right]\\
&= \liminf_n \left[\log_n \mathbf{R}(sM^S_{\langle n\rangle})\right] = \liminf_n \left[\log_n \underline{\mathbf{R}}(sM^S_{\langle n\rangle})\right] && (1.7)\\
&= \liminf_n \left[\log_n \mathbf{R}_s(sM^S_{\langle n\rangle})\right] = \liminf_n \left[\log_n \underline{\mathbf{R}}_s(sM^S_{\langle n\rangle})\right]\\
&= \liminf_n \left[\log_n \mathbf{R}(sM^{S,0}_{\langle n\rangle})\right] = \liminf_n \left[\log_n \underline{\mathbf{R}}(sM^{S,0}_{\langle n\rangle})\right] && (1.8)\\
&= \liminf_n \left[\log_n \mathbf{R}_s(sM^{S,0}_{\langle n\rangle})\right] = \liminf_n \left[\log_n \underline{\mathbf{R}}_s(sM^{S,0}_{\langle n\rangle})\right]\\
&= \liminf_n \left[\log_n \mathbf{R}(sM^Z_{\langle n\rangle})\right] = \liminf_n \left[\log_n \underline{\mathbf{R}}(sM^Z_{\langle n\rangle})\right] && (1.9)\\
&= \liminf_n \left[\log_n \mathbf{R}_s(sM^Z_{\langle n\rangle})\right] = \liminf_n \left[\log_n \underline{\mathbf{R}}_s(sM^Z_{\langle n\rangle})\right].
\end{aligned}$$

Proofs are given in §2 for (1.6), §3 for (1.7) and (1.8), and §4 for (1.9).

1.1. **Explicit ranks and border ranks.** For any $t\in\mathbb{C}^N\otimes\mathbb{C}^N\otimes\mathbb{C}^N$, the symmetrization of t is $\mathcal{S}(t):=\frac{1}{6}\sum_{\pi\in\mathfrak{S}_3}\pi(t)\in \mathrm{Sym}^3\mathbb{C}^N$. In particular, $\mathcal{S}(t)=t$ if and only if $t\in\mathrm{Sym}^3\mathbb{C}^N$. The following provides bounds relating t and $\mathcal{S}(t)$.

Lemma 1.2. *For $t\in\mathbb{C}^N\otimes\mathbb{C}^N\otimes\mathbb{C}^N$, $\mathbf{R}_s(\mathcal{S}(t))\le 4\mathbf{R}(t)$ and $\underline{\mathbf{R}}_s(\mathcal{S}(t))\le 4\underline{\mathbf{R}}(t)$.*

Proof. If $t = \sum_{i=1}^r u_i \otimes v_i \otimes w_i$ with $u_i, v_i, w_i, \in \mathbb{C}^N$, then $\mathcal{S}(t) = \sum_{i=1}^r (u_i v_i w_i)$. Since $\mathbf{R}_s(xyz) = 4$ (see, e.g., [25, §10.4]), this immediately yields that $\mathbf{R}_s(\mathcal{S}(t)) \leq 4\mathbf{R}(t)$. In the same way, if t is a limit of tensors of the form $\sum_{i=1}^r u_i \otimes v_i \otimes w_i$, this yields $\underline{\mathbf{R}}_s(\mathcal{S}(t)) \leq 4\underline{\mathbf{R}}(t)$. □

In particular, $\mathbf{R}(sM_{\langle n\rangle}) \leq 2\mathbf{R}(M_{\langle n\rangle}) < 2n^3$ (as $sM_{\langle n\rangle}$ is the sum of two matrix multiplications, by (1.2)) so that $\mathbf{R}_s(sM_{\langle n\rangle}) \leq 8\mathbf{R}(M_{\langle n\rangle}) < 8n^3$ and similarly for all its degenerations.

The following summarizes some results about small cases.

Theorem 1.3.

(1) $\mathbf{R}_s(sM_{\langle 2\rangle}) = 6$ *and* $\underline{\mathbf{R}}_s(sM_{\langle 2\rangle}) = 5$ *([37, IV, §97], [28, Prop. 7.2]).*
(2) $\underline{\mathbf{R}}_s(sM_{\langle 3\rangle}) \geq 14$.
(3) $\mathbf{R}_s(sM^S_{\langle 2\rangle}) = \underline{\mathbf{R}}_s(sM^S_{\langle 2\rangle}) = 4$ *([37, IV §96] or [28, §8]).*
(4) $\underline{\mathbf{R}}_s(sM^S_{\langle 3\rangle}) = 10$.
(5) $sM^{S,0}_{\langle 2\rangle} = 0$ *while* $\underline{\mathbf{R}}_s(sM^{S,0}_{\langle 3\rangle}) = \mathbf{R}_s(sM^{S,0}_{\langle 3\rangle}) = 8$.
(6) $\underline{\mathbf{R}}_s(sM^{S,0}_{\langle 4\rangle}) \geq 14$.
(7) $sM^Z_{\langle 2\rangle} = 0$ *while* $\underline{\mathbf{R}}_s(sM^Z_{\langle n\rangle}) = \mathbf{R}_s(sM^Z_{\langle n\rangle}) = 2^{n-1}$ *for* $n = 3, 4, 5$, *with* $\mathbf{R}_s(sM^Z_{\langle 6\rangle}) \leq 30$, $\mathbf{R}_s(sM^Z_{\langle 7\rangle}) \leq 48$, *and* $\mathbf{R}_s(sM^Z_{\langle 8\rangle}) \leq 64$.

The cases (1) and (3) are discussed respectively in §2.1 and §3.1. The case (4) is proved in §3.2 with a tableau evaluation. The cases (2), (5), (6) are proved with the technique of Young flattenings introduced in [27] which has already been used in the case of general tensors in [26]. In particular, Proposition 2.6 below considers (2) with the other cases following analogously. The case (7) is proved by exhibiting explicit decompositions in Theorems 4.2, 4.3, and 4.4.

Since one of our goals is to simplify the problem in order to further exploit numerical computations, we experiment with numerical tools and probabilistic methods via `Bertini` [6]. We believe the computations could likely be converted to rigorous proofs, e.g., by showing that an overdetermined system has a solution nearby the given numerical approximation [2]. We write **Theorem*** when we mean the result of a numerical computation.

Theorem* 1.4. $\mathbf{R}_s(sM_{\langle 3\rangle}) \leq 18$.

We show this in Theorem* 2.7 with data regarding this and other computations available at `http://dx.doi.org/10.7274/R0VT1Q1J`.

Remark 1.5. *Very recently in [5] it was shown* $\mathbf{R}_s(sM_{\langle 3\rangle}) \leq 18$ *with an exact decomposition.*

Notation and conventions. The group of invertible linear maps $\mathbb{C}^N \to \mathbb{C}^N$ is denoted GL_N and the permutation group on d elements by $\mathfrak{S}_d$. For $u, v, w \in \mathbb{C}^N$, we have $u \otimes v \otimes w \in (\mathbb{C}^N)^{\otimes 3}$ and $uvw \in \mathrm{Sym}^3\mathbb{C}^N$. The space Mat_n is canonically self-dual. Given a matrix L, when we consider $L \in \mathrm{Mat}_n^\vee$, we write $L^3 \in \mathrm{Sym}^3(\mathrm{Mat}_n^\vee)$ for the cubic polynomial function which sends the matrix A to $\left[\mathrm{trace}(L^T A)\right]^3$, where L^T is the transpose of L. Note that L^3 is a function and *not* the cube of the matrix L. In particular, $sM_{\langle n\rangle} = \sum_{i=1}^k L_i^3$ means that

$$\mathrm{trace}(A^3) = \sum_{i=1}^k \left[\mathrm{trace}(L_i^T A)\right]^3.$$

For a partition π of d, $S_\pi\mathbb{C}^N$ denotes the corresponding GL_N-module and $[\pi]$ the corresponding $\mathfrak{S}_d$-module. In particular $S_{(d)}\mathbb{C}^N = \mathrm{Sym}^d\mathbb{C}^N$ and $S_{(1^d)}\mathbb{C}^N = \Lambda^d\mathbb{C}^N$.

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2. The polynomial $sM_{\langle n\rangle}$

We start with the first statement from Theorem 1.1.

Proof of (1.6). Lemma 1.2 and (1.5) imply

$$4\mathbf{R}(M_{\langle n\rangle}) \geq \mathbf{R}_s(sM_{\langle n\rangle}) \geq \mathbf{R}(sM_{\langle n\rangle})$$

so that

$$\omega \geq \liminf_n \left[\log_n \mathbf{R}_s(sM_{\langle n\rangle})\right] \geq \liminf_n \left[\log_n \mathbf{R}(sM_{\langle n\rangle})\right].$$

For $n\times n$ matrices A, B, C consider the $3n\times 3n$ matrix $X = \begin{pmatrix} 0 & 0 & A \\ C & 0 & 0 \\ 0 & B & 0\end{pmatrix}$. Then, $X^3 = \begin{pmatrix} ABC & 0 & 0 \\ 0 & CAB & 0 \\ 0 & 0 & BCA\end{pmatrix}$ and $\mathrm{trace}(X^3) = 3\,\mathrm{trace}(ABC)$. This shows that $\mathbf{R}(M_{\langle n\rangle}) \leq \mathbf{R}(sM_{\langle 3n\rangle})$ yielding the inequality $\omega \leq \liminf_n \left[\log_n \mathbf{R}(sM_{\langle n\rangle})\right]$. The border rank statement follows similarly by taking limits. □

As a GL_N-module via the Cauchy formula, $\mathrm{Sym}^3(\mathrm{End}(V)) = \mathrm{Sym}^3(V\otimes V^*)$ decomposes as

$$\mathrm{Sym}^3(\mathrm{End}(V)) = \mathrm{Sym}^3 V\otimes \mathrm{Sym}^3 V^* \oplus S_{21}V\otimes S_{21}V^* \oplus \Lambda^3 V\otimes \Lambda^3 V^* \tag{2.1}$$

$$= \mathrm{End}(\mathrm{Sym}^3 V)\oplus \mathrm{End}(S_{21}V)\oplus \mathrm{End}(\Lambda^3 V). \tag{2.2}$$

The tensor $M_{\langle n\rangle}\in \mathrm{Mat}_n^\vee\otimes\mathrm{Mat}_n^\vee\otimes\mathrm{Mat}_n^\vee = \mathrm{End}(V^{\otimes 3})$ corresponds to the identity endomorphism. Since $V^{\otimes 3} = \mathrm{Sym}^3 V\oplus (S_{21}V)^{\oplus 2}\oplus\Lambda^3 V$, it follows that $\mathrm{End}(V^{\otimes 3})$, as a $GL(V)$-module, contains the submodule

$$\mathrm{End}(\mathrm{Sym}^3 V)\oplus \left(\mathrm{End}(S_{21}V)^2\right)\oplus \mathrm{End}(\Lambda^3 V).$$

The projection of $sM_{\langle n\rangle}$ onto each of the three summands in (2.2) is the identity endomorphism (the last summand requires $n\geq 3$ to be nonzero). In particular, all three projections are nonzero when $n\geq 3$.

For $n\geq 2$, the following shows that in any symmetric rank decomposition of $sM_{\langle n\rangle}$, it is impossible to have all summands corresponding to matrices L_i of rank one. Moreover, for $n\geq 3$, at least one summand corresponds to a matrix having rank at least 3. We note that this statement is in contrast to tensor decompositions of $M_{\langle n\rangle}$ where there do exist decompositions constructed from rank one matrices, e.g., the standard decomposition.

Theorem 2.1. *Suppose that $sM_{\langle n\rangle} = \sum_{i=1}^k L_i^3$ is a symmetric rank decomposition. If $n=2$, there exists i with $\mathrm{rank}(L_i)=2$. Moreover, if $n\geq 3$, $\max_i \mathrm{rank}(L_i)\geq 3$.*

Proof. Any summand L_i^3 with rank $L_i = 1$ is of the form $L_i = v_i \otimes \omega_i \in V \otimes V^\vee$ and induces an element of rank one that takes $a \otimes b \otimes c$ to $\omega_i(a)\omega_i(b)\omega_i(c)v_i^3$ which vanishes outside $\mathrm{Sym}^3 V$. This element lies in $\mathrm{End}(\mathrm{Sym}^3 V)$ in the decomposition (2.2). Hence, any sum of these elements lies in this subspace and thus projects to zero in the second and third factors in (2.2).

Similarly, any summand of rank two only gives rise to a term appearing in $\mathrm{Sym}^3 V \otimes \mathrm{Sym}^3 V^* \oplus S_{21} V \otimes S_{21} V^*$ because one needs three independent vectors for a term in $\Lambda^3 V \otimes \Lambda^3 V^*$. $\square$

This following provides a slight improvement over the naïve bound of $8n^3$.

Proposition 2.2 (A modest upper bound)**.** $\mathbf{R}_s(sM_{\langle n\rangle}) \leq 8\binom{n}{3} + 4\binom{n}{2} + n.$

Proof. Every monomial appearing in $sM_{\langle n\rangle}$ has the form $a_{ij}a_{jk}a_{ki}$. This bound arises from considering the symmetric ranks of each of these monomials. There are $2\binom{n}{3}$ monomials corresponding to distinct cardinality 3 sets $\{i,j,k\} \subset \{1,\ldots,n\}$ and each monomial has symmetric rank 4. There are $2\binom{n}{2}$ monomials corresponding to distinct cardinality 2 sets $\{i,j\} \subset \{1,\ldots,n\}$ and they group together in $\binom{n}{2}$ pairs as $a_{ij}a_{ji}(a_{ii} + a_{jj})$ with each such term having symmetric rank four. Finally, there are n monomials of the form a_{ii}^3 for $i = 1,\ldots,n$. $\square$

The following considers algebraic geometric aspects of $sM_{\langle n\rangle}$.

Proposition 2.3. *(i) The singular locus of* $\{sM_{\langle n\rangle} = 0\} \subset \mathbb{P}\mathrm{Mat}_n$ *is*

$$\{[A] \in \mathbb{P}\mathrm{Mat}_n \mid A^2 = 0\}.$$

(ii) The polynomial $sM_{\langle 2\rangle}$ *is reducible, while* $sM_{\langle n\rangle}$ *is irreducible for* $n \geq 3$.

Proof. The first derivatives of $\mathrm{tr}(A^3)$ vanish if and only if the first polar $\mathrm{tr}(X \cdot A^2)$ vanishes for every matrix X. Since the map $(A,B) \mapsto \mathrm{tr}(AB^t)$ is a nondegenerate pairing, this proves (i).

Alternatively, note that the (i,j) entry of A^2 coincides, up to scalar multiple, with the partial derivative $\dfrac{\partial\left(sM_{\langle n\rangle}\right)}{\partial a_{j,i}}$. In order to prove (ii), we estimate the dimension of the singular locus computed in (i). If A belongs to the singular locus of $\{sM_{\langle n\rangle} = 0\}$, we know $\ker(A) \subseteq \mathrm{im}(A)$ so that $\mathrm{rank}(A) \leq n/2$. It follows that the singular locus of $\{sM_{\langle n\rangle} = 0\}$ has codimension ≥ 3 for $n \geq 3$ showing that $sM_{\langle n\rangle}$ must be irreducible. If not, the singular locus contains the intersection of any two irreducible components, having codimension ≤ 2. The $n = 2$ case follows from (2.3) below. $\square$

2.1. **Decomposition of $sM_{\langle 2\rangle}$.** The reducibility of $sM_{\langle 2\rangle}$ is as follows:

$$\begin{aligned} sM_{\langle 2\rangle} &= a_{0,0}^3 + 3a_{0,0}a_{0,1}a_{1,0} + 3a_{0,1}a_{1,0}a_{1,1} + a_{1,1}^3 \\ &= \underbrace{\mathrm{trace}(A)}_{\text{non tg hyperp.}} \cdot \underbrace{\left(\mathrm{trace}^2(A) - 3\det(A)\right)}_{\text{smooth quadric}}. \end{aligned} \tag{2.3}$$

In particular, for this classically studied polynomial, its zero set is the union of a smooth quadric and a non-tangent hyperplane. A general cubic surface has a unique Waring decomposition as a sum of 5 summands by the Sylvester Pentahedral

Theorem [32, Theor. 3.9]. Hence, every $f \in \mathrm{Sym}^3\mathbb{C}^4$ has $\underline{\mathbf{R}}_s(f) \leq 5$. However, $\mathbf{R}_s(sM_{\langle 2\rangle}) = 6$ (see [37, IV, §97]) with a minimal Waring decomposition given by

$$sM_{\langle 2\rangle} = \sum_{i=1}^{6} L_i^3 \tag{2.4}$$

where

$$L_1 = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix},\ L_2 = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix},\ L_3 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ L_4 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$L_5 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } L_6 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark 2.4 (A remark on 5 summands)**.** *For the decomposition presented in* (2.4), $\mathrm{rank}(L_i) = 2$ *for* $i = 1, 2$ *while* $\mathrm{rank}(L_i) = 1$ *for* $i = 3, 4, 5, 6$ *in agreement with Theorem 2.1. Since* $sM_{\langle 2\rangle}$ *is* GL_2*-invariant for the conjugate action which takes* A *to* $G^{-1}AG$ *for every* $G \in GL_2$*, the matrices* L_i *can be replaced in* (2.4) *with* $G^{-1}L_iG$ *for any* $G \in GL_2$*.*

Consider a family $f_{2,\epsilon}$ *which has a Waring decomposition given by five matrices* $L_{i,\epsilon}$ *for* $\epsilon \neq 0$ *and* $f_{2,0} = sM_{\langle 2\rangle}$*. In all the examples we have found, the five matrices* $L_{i,\epsilon}$ *converge as* $\epsilon \to 0$ *to the identity matrix that is indeed a fixed point for the conjugate action.*

The following Remark provides a geometric description for decompositions of $sM_{\langle 2\rangle}$ using six terms.

Remark 2.5. *Identify the projective space of* 2×2 *matrices with* $\mathbb{P}^3$*. Let* $\mathcal{Q}$ *be the quadric of matrices of rank* 1 *and let* ℓ *denote the line spanned by the identity* I *and the skew-symmetric point* Λ*.*

For a choice of 3 *points* Q_1, Q_2, Q_3 *in the intersection of* $\mathcal{Q}$ *with the plane of traceless matrices, let* $A_1, A_2, B_1, B_2, C_1, C_2$ *denote the 6 points of intersection of the two rulings of* $\mathcal{Q}$ *passing through each* Q_i*. These points, together with* I*, determine a minimal decomposition of the general tensor* $M_{\langle 2\rangle}$*, as explained in [12].*

A decomposition of $sM_{\langle 2\rangle}$ *is determined as follows: let* Q_3 *be the intersection of the lines* (B_1C_1) *and* (B_2C_2)*. Then the six points* $L_1 \ldots L_6$ *are obtained by taking* $L_6 = A_2$*,* $L_5 = A_1$*,* $L_4 =$ *the intersection of* (B_1, C_1) *with the plane* π *of symmetric matrices,* $L_3 =$ *the intersection of* (B_2, C_2) *with* π*,* $L_2 =$ *the intersection of the line* (Q_3A_2) *with* ℓ *(they meet), and* $L_1 =$ *the intersection of the line* (Q_3, A_1) *with* ℓ*.*

For instance, starting with $Q_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$*,* $Q_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$*, and* $Q_3 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$*, we obtain the six points* $L_1, \ldots, L_6$ *of the decomposition* (2.4) *described above.*

We ask if an analogous geometric description could provide small decompositions of $sM_{\langle n\rangle}$ for $n \geq 3$.

2.2. **Case of** $sM_{\langle 3\rangle}$**.** The polynomial $sM_{\langle 3\rangle}$ is irreducible by Proposition 2.3 with the following lower bound on border rank.

Proposition 2.6. $\underline{\mathbf{R}}_s(sM_{\langle 3\rangle}) \geq 14$.

Proof. Let $W = \mathbb{C}^9$. For any $\phi \in \mathrm{Sym}^3 W$ we have the linear map

$$A_\phi \colon W^\vee \otimes \Lambda^4 W \to \Lambda^5 W \otimes W$$

which is defined by contracting the elements of the source with ϕ and then projecting to the target. This projection is well-defined because the map

$$S^2W \otimes \Lambda^4 W \to \Lambda^5 W \otimes W$$

is a $GL(W)$-module map and the image of the projection is the unique copy of $S_{21111}W \subset \Lambda^5 W \otimes W$. This map was denoted as $\mathrm{YF}_{3,8}(\phi)$ in [27, Eq. (2)]. Using [18], direct computation shows that $\operatorname{rank} A_{w^3} = 70$ for a nonzero $w \in W$ and $\operatorname{rank} A_{sM_{\langle 3\rangle}} = 950$. By linearity $\underline{\mathbf{R}}_s(sM_{\langle 3\rangle}) \geq \lceil \frac{950}{70} \rceil = 14$. □

The following provides information on the rank.

Theorem* 2.7. $\mathbf{R}_s(sM_{\langle 3\rangle}) \leq 18$ *with a Waring decomposition of* $sM_{\langle 3\rangle}$ *with* 18 *summands found numerically by* `Bertini` *[6] with all* 18 *summations having rank* 3.

Proof. After numerically approximating a decomposition with `Bertini` [6], applying the isosingular local dimension test [22] suggested that there is at least one 9-dimensional family of decompositions. We used the extra 9 degrees of freedom to set 9 entries to 0, 1, or -1 producing a polynomial system which has an isolated nonsingular root with an approximation given in Appendix A and electronically available at `http://dx.doi.org/10.7274/R0VT1Q1J`. □

Decompositions with 18 summands were highly structured leading to the following.

Conjecture 2.8. $\mathbf{R}_s(sM_{\langle 3\rangle}) = 18$.

In our experiments, we were unable to compute a decomposition of $sM_{\langle 3\rangle}$ using 18 summands with real matrices.

3. The polynomials $sM^S_{\langle n\rangle}$ and $sM^{S,0}_{\langle n\rangle}$

We start with statements from Theorem 1.1.

Proof of (1.7) *and* (1.8). The following two inequalities are trivial since $sM^S_{\langle n\rangle}$ is a specialization of $sM_{\langle n\rangle}$:

$$\mathbf{R}_s(sM^S_{\langle n\rangle}) \leq \mathbf{R}_s(sM_{\langle n\rangle}) \quad \text{and} \quad \mathbf{R}(sM^S_{\langle n\rangle}) \leq \mathbf{R}(sM_{\langle n\rangle}).$$

For $n \times n$ matrices A, B, C consider the $3n \times 3n$ symmetric matrix

$$X = \begin{pmatrix} 0 & C^T & A \\ C & 0 & B^T \\ A^T & B & 0 \end{pmatrix}.$$

We have $\operatorname{trace}(X^3) = 6 \operatorname{trace}(ABC)$ since

$$X^3 = \begin{pmatrix} ABC + C^T B^T A^T & * & * \\ * & CAB + B^T A^T C^T & * \\ * & * & BCA + A^T C^T B^T \end{pmatrix}.$$

It immediately follows $\mathbf{R}(M_{\langle n\rangle}) \leq \mathbf{R}(sM^S_{\langle 3n\rangle})$. Hence, (1.7) follows by a similar argument as in the proof of (1.6).

Since X is traceless, the same argument also proves (1.8). □

3.1. **Decomposition of $sM^S_{\langle 2\rangle}$.** As in the general case (2.3), $sM^S_{\langle 2\rangle}$ is a reducible polynomial while $sM^S_{\langle n\rangle}$ is irreducible for $n \geq 3$ (the same argument as in Proposition 2.3 works). In fact,

$$sM^S_{\langle 2\rangle}\begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix} = (a_0 + a_2)(a_0^2 + 3a_1^2 - a_0a_2 + a_2^2),$$

which corresponds to the union of a smooth conic with a secant (not tangent) line. Moreover, it was known classically that $\underline{\mathbf{R}}_s(sM^S_{\langle 2\rangle}) = \mathbf{R}_s(sM^S_{\langle 2\rangle}) = 4$, which is the generic rank in $\mathbb{P}(\mathrm{Sym}^3\mathbb{C}^3)$ with a minimal Waring decomposition given by

$$6 \cdot sM^S_{\langle 2\rangle} = L_1^3 + L_2^3 - 2L_3^3 - 2L_4^3 \tag{3.1}$$

where

$$L_1 = \begin{pmatrix} 2 & \frac{\sqrt{-1}}{2} \\ \frac{\sqrt{-1}}{2} & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & -\frac{\sqrt{-1}}{2} \\ -\frac{\sqrt{-1}}{2} & 2 \end{pmatrix}, L_3 = \begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, L_4 = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix}.$$

We note that L_1 and L_2 are similar as well as L_3 and L_4 and all have rank 2.

3.2. **Case of $sM^S_{\langle 3\rangle}$.** We consider $sM^S_{\langle 3\rangle}$ as a cubic polynomial on $\mathbb{C}^6$. Since the generic rank in $\mathbb{P}(\mathrm{Sym}^3\mathbb{C}^6)$ is 10 (see [4]), we have $\underline{\mathbf{R}}_s(sM^S_{\langle 3\rangle}) \leq 10$. To show that equality holds, consider the degree 10 invariant in $\mathrm{Sym}^{10}(\mathrm{Sym}^3\mathbb{C}^6)$ corresponding to the following Young diagram (see, e.g., [34, §3.9], for the symbolic notation of invariants):

$$T_{10} = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 3 & 3 & 4 \\ \hline 4 & 4 & 5 & 5 & 5 \\ \hline 6 & 6 & 7 & 6 & 7 \\ \hline 7 & 8 & 8 & 9 & 8 \\ \hline 9 & 10 & 9 & 10 & 10 \\ \hline \end{array}. \tag{3.2}$$

This invariant is generalized in [9, Prop. 3.25].

Lemma 3.1. *The polynomial T_{10} defined by* (3.2) *forms a basis of the SL_6-invariant space* $(\mathrm{Sym}^{10}(\mathrm{Sym}^3\mathbb{C}^6))^{SL_6}$ *and is in the ideal of $\sigma_9(\nu_3(\mathbb{P}^5))$. Moreover, $T_{10}(sM^S_{\langle 3\rangle}) \neq 0$ showing that $\underline{\mathbf{R}}_s(sM^S_{\langle 3\rangle}) > 9$.*

Proof. A plethysm calculation, e.g., using `Schur` [10], shows that

$$\dim(\mathrm{Sym}^{10}(\mathrm{Sym}^3\mathbb{C}^6))^{SL_6} = 1.$$

We explicitly evaluated $T_{10}(sM^S_{\langle 3\rangle})$ using the same algorithm as in [1] and [11] which phrases the evaluation as a tensor contraction and ignores summands that contribute zero to the result. The result was that $T_{10}(sM^S_{\langle 3\rangle}) \neq 0$.

We now consider evaluating T_{10} on all cubics of the form $f = \sum_{i=1}^9 \ell_i^3$. The expression $T_{10}(f)$ splits as the sum of several terms of the form $T_{10}(\ell_{i_1}^3, \ldots, \ell_{i_{10}}^3)$ where, in each of these summands, there is a repetition of some ℓ_i. We claim that every $T_{10}(\ell_{i_1}^3, \ldots, \ell_{i_{10}}^3)$ vanishes due to this repetition. Indeed, each pair (i, j) with $1 \leq i < j \leq 10$ appears in at least one column of (3.2). In other words, for any $g\colon \{1, \ldots, 10\} \to \{1, \ldots, 9\}$, the tableau evaluation $g(T_{10})$ has a repetition in at least one column and thus vanishes. This approach is the main tool used in [1]. $\square$

Since the polynomial T_{10} vanishes on $\sigma_9(\nu_3(\mathbb{P}^5))$, Lemma 3.1 immediately yields that $\underline{\mathbf{R}}_s(sM^S_{\langle 3\rangle}) = 10$, because $\sigma_{10}(v_3(\mathbb{P}^5))$ equals $\mathbb{P}\mathrm{Sym}^3\mathbb{C}^6$. The following considers decompositions with 10 summands.

Proposition* 3.2. $\mathbf{R}_s(sM^S_{\langle 3\rangle}) = 10$.

Proof. Consider decompositions consisting of 3 symmetric matrices each of the form

$$\begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & * & 0 \\ * & * & * \\ 0 & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix}, \tag{3.3}$$

each of which is clearly rank deficient, and one symmetric matrix of the form

$$\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix} \tag{3.4}$$

which is clearly traceless. Upon substituting these forms, which have a total of $3 \cdot 10 = 30$ unknowns, into the $\binom{5+3}{3} = 56$ equations which describe the decompositions of $sM^S_{\langle 3\rangle}$, there are 28 equations which vanish identically leaving 28 polynomial equations in 30 affine variables. The isosingular local dimension test [22] in `Bertini` [6] suggests that this system has at least one 3-dimensional solution component which we utilize the 3 extra degrees of freedom to make one entry either ± 1 in one of each of the three types of matrices in (3.3). The resulting system has an isolated solution which we present one here to 4 significant digits:

$$\begin{pmatrix} 0.1755 & 2.16 & -1 \\ 2.16 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0.6889 & -0.4607 & -0.8745 \\ -0.4607 & 0 & 0 \\ -0.8745 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0.874 & 0.1991 & 0.5836 \\ 0.1991 & 0 & 0 \\ 0.5836 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & -0.7877 & 0 \\ -0.7877 & 0.5269 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1.431 & 0 \\ 1.431 & 0.326 & 0.9555 \\ 0 & 0.9555 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0.076 & 0 \\ 0.076 & 0.9356 & -0.4331 \\ 0 & -0.4331 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0.2278 \\ 1 & 0.2278 & 0.6677 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -0.6362 \\ 0 & 0 & 0.4255 \\ -0.6362 & 0.4255 & 0.8077 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -0.09825 \\ 0 & 0 & -1.21 \\ -0.09825 & -1.21 & 0.5599 \end{pmatrix},$$
$$\begin{pmatrix} 0 & -2.317 & 0.8998 \\ -2.317 & 0 & -0.4797 \\ 0.8998 & -0.4797 & 0 \end{pmatrix}.$$

The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the first 9 summands satisfy

$$\lambda_3 = (\lambda_1 + \lambda_2)(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2) - 1 = 0$$

while the eigenvalues of the traceless matrix satisfy

$$\lambda_1 + \lambda_2 + \lambda_3 = (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3) + 2 = 0.$$

□

The variety $\sigma_9(\nu_3(\mathbb{P}^5))$ has codimension 2 as expected. The following describes generators of its ideal.

Theorem* 3.3. *The variety $\sigma_9(\nu_3(\mathbb{P}^5))$ has codimension 2 and degree 280. It is the complete intersection of the solution set of T_{10} and a hypersurface of degree 28.*

Proof. It is easy to computationally verify that the variety $X := \sigma_9(\nu_3(\mathbb{P}^5)) \subset \mathbb{P}^{55}$ has the expected dimension of 53, e.g., via [20, Lemma 3]. This also follows from the Alexander-Hirschowitz Theorem [4]. We used the approach in [19, §2] with `Bertini` [6] to compute a so-called pseudowitness set [20] for X yielding $\deg X = 280$. With this pseudowitness set, [16, 17] shows that X is arithmetically

Cohen-Macaulay and arithmetically Gorenstein. In particular, the Hilbert function of the finite set $X \cap \mathcal{L}$ where $\mathcal{L} \subset \mathbb{P}^{55}$ is a general linear space of dimension 2 is

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 65, 75, 85, 95, 105, 115, 125, 135, 145, 155, 165, 175, 185, 195, 205, 215, 225, 235, 244, 252, 259, 265, 270, 274, 277, 279, 280, 280.$$

Thus, the ideal of $X \cap \mathcal{L}$ is minimally generated by a degree 10 polynomial (corresponding to T_{10}) and a polynomial of degree 28. The same holds for X, i.e., X is a complete intersection defined by the vanishing of T_{10} and a polynomial of degree 28, since X is arithmetically Cohen-Macaulay. The Hilbert series of X is

$$\frac{\begin{gathered}1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + 7t^6 + 8t^7 + 9t^8 + 10t^9(1 + t + t^2 + \cdots + t^{18}) \\ + 9t^{28} + 8t^{29} + 7t^{30} + 6t^{31} + 5t^{32} + 4t^{33} + 3t^{34} + 2t^{35} + t^{36}\end{gathered}}{(1-t)^{54}}$$

□

Remark 3.4. *The generic polynomial in $\sigma_9(\nu_3(\mathbb{P}^5))$ has exactly two Waring decompositions, which is the last subgeneric $\sigma_k(\nu_d(\mathbb{P}^n))$ whose generic member has a non-unique Waring decomposition [13, Thm. 1.1]. The equations of $\sigma_9(\nu_3(\mathbb{P}^5))$ are already discussed in some papers. The invariant T_{10} corresponds to the Iliev-Ranestad divisor D_{IR} introduced in [23] and studied by Ranestad and Voisin in [35, §2] and by Jelisiejew in [24, Prop. 2]. Indeed it is observed in [24, Remark 28] that D_{IR} is the SL_6-invariant of smallest degree on $\mathrm{Sym}^3\mathbb{C}^6$. Jelisiejew poses the interesting question if the degree 28 divisor of Theorem* 3.3 is (up to multiples of T_{10}) the divisor D_{V-ap} of cubic fourfolds apolar to a Veronese surface [35]. At present, as far as we know, the question is still unsolved.*

We close with the traceless 3×3 case $sM^{S,0}_{\langle 3 \rangle}$ where we take $a_5 = -(a_0 + a_3)$.

Proposition 3.5. $\underline{\mathbf{R}}_s(sM^{S,0}_{\langle 3 \rangle}) = \mathbf{R}_s(sM^{S,0}_{\langle 3 \rangle}) = 8$

Proof. Although $\sigma_7(\nu_3(\mathbb{P}^4)) \subset \mathbb{P}^{34}$ is expected to fill the ambient space, it is defective: it is a hypersurface of degree 15 defined by the cubic root of the determinant of a 45×45 matrix, e.g., see [1,33]. This 45×45 matrix evaluated at $sM^{S,0}_{\langle 3 \rangle}$ has full rank showing that $\underline{\mathbf{R}}_s(sM^{S,0}_{\langle 3 \rangle}) > 7$. Since 8 is the generic rank, $\underline{\mathbf{R}}_s(sM^{S,0}_{\langle 3 \rangle}) = 8$.

To show the existence of a decomposition using 8 summands, we need to solve a system of $\binom{4+3}{3} = 35$ polynomials in 40 affine variables. By including the determinant of the matrices corresponding to the first 5 summands, we produce a square system with 40 polynomials in 40 variables. We prove the existence of a solution via α-theory using `alphaCertified` [21] starting with the following approximation:

$$\begin{pmatrix} 0.2533609 - 0.3253227i & 0.3900781 - 0.4785431i & 0.2123864 - 0.4078949i \\ 0.3900781 - 0.4785431i & 2.017554 - 0.09428536i & 0.2851566 + 1.393898i \\ 0.2123864 - 0.4078949i & 0.2851566 + 1.393898i & -2.2709149 + 0.41960806i \end{pmatrix},$$
$$\begin{pmatrix} 0.04310556 + 0.1656553i & 0.1274312 + 0.3981205i & -0.4116999 - 0.01601833i \\ 0.1274312 + 0.3981205i & -1.934801 + 0.2834521i & -0.4336855 - 1.882461i \\ -0.4116999 - 0.01601833i & -0.4336855 - 1.882461i & 1.89169544 - 0.4491074i \end{pmatrix},$$
$$\begin{pmatrix} -0.3785169 + 0.4133459i & 0.250335 - 1.167081i & 0.2879607 - 0.03026006i \\ 0.250335 - 1.167081i & 0.3783613 - 0.2548455i & -0.2320977 - 0.1928574i \\ 0.2879607 - 0.03026006i & -0.2320977 - 0.1928574i & 0.00015556 - 0.1585004i \end{pmatrix},$$
$$\begin{pmatrix} 0.3088717 + 0.01475953i & -0.1783686 - 0.3906188i & 0.2316816 + 0.3868949i \\ -0.1783686 - 0.3906188i & 2.09271 - 0.2084929i & 0.4260931 + 1.812166i \\ 0.2316816 + 0.3868949i & 0.4260931 + 1.812166i & -2.4015817 + 0.19373337i \end{pmatrix},$$
$$\begin{pmatrix} 0.1338705 - 0.583525i & 0.8378364 - 1.288966i & 0.9626108 + 0.2184638i \\ 0.8378364 - 1.288966i & 0.3123518 - 0.5020508i & 0.9180592 + 0.8550508i \\ 0.9626108 + 0.2184638i & 0.9180592 + 0.8550508i & -0.4462223 + 1.0855758i \end{pmatrix},$$
$$\begin{pmatrix} 0.1972212 - 0.8229062i & -0.1848761 - 0.6480481i & 0.1550351 - 0.03293642i \\ -0.1848761 - 0.6480481i & 1.063376 - 1.941261i & 1.409034 + 0.4606352i \\ 0.1550351 - 0.03293642i & 1.409034 + 0.4606352i & -1.2605972 + 2.7641672i \end{pmatrix},$$
$$\begin{pmatrix} 0.8841674 - 0.3162212i & 1.394277 + 0.2241248i & 0.1553027 + 0.6211393i \\ 1.394277 + 0.2241248i & 0.7687433 - 0.1439827i & -0.1014701 + 1.35298i \\ 0.1553027 + 0.6211393i & -0.1014701 + 1.35298i & -1.6529107 + 0.4602039i \end{pmatrix},$$
$$\begin{pmatrix} 0.1094107 + 0.06367402i & -0.3608902 + 1.394814i & -0.7766249 - 0.2283304i \\ -0.3608902 + 1.394814i & -0.5622866 + 0.5667869i & -0.01617272 + 0.05988273i \\ -0.7766249 - 0.2283304i & -0.01617272 + 0.05988273i & 0.4528759 - 0.63046092i \end{pmatrix}$$

where $i = \sqrt{-1}$. □

4. The polynomial $sM^Z_{\langle n\rangle}$

Let Z_n be the space of symmetric matrices with zeros on the diagonal. The cubic $sM^Z_{\langle n\rangle}(A)$ is a polynomial in $\binom{n}{2}$ indeterminates, its naïve expression has $\binom{n}{3}$ terms:

$$sM^Z_{\langle n\rangle}(A) = \sum_{1\le i<j<k\le n} a_{ij}a_{jk}a_{ik} \tag{4.1}$$

The proof of (1.9) is similar to the others and thus omitted.

Since $sM^Z_{\langle 2\rangle}$ is identically zero, we take $n \ge 3$. Let P_n denote the finite set of 2^{n-1} vectors of the form $v = (1, \pm1, \ldots, \pm1)^T \in \mathbb{Z}^n$. In Theorem 4.1, we use P_n to construct a decomposition of $sM^Z_{\langle n\rangle}$. Although such a decomposition is not minimal for $n \ge 6$ (see Proposition 4.3), a modification of it constructs the decomposition (4.3) for $n = 8$ which we expect is minimal (see Remark 4.5).

For each $v \in P_n$, $vv^T - I_n$ belongs to Z_n with eigenvalues $\{-1, \ldots, -1, n-1\}$ and off-diagonal elements ±1. Here I_n denotes the $n \times n$ identity matrix.

Theorem 4.1. *For $n \ge 3$, we have the decomposition with 2^{n-1} summands:*

$$2^{n+2} sM^Z_{\langle n\rangle}(A) = \sum_{v\in P_n} \left(\mathrm{trace}[(vv^T - I_n)\cdot A]\right)^3. \tag{4.2}$$

Proof. If $v = (v_1, \ldots, v_n)^T \in P_n$, then the (i,j)-entry of $(vv^T - I_n)$ is $v_iv_j - \delta_{ij}$. The monomials appearing in $\left(\mathrm{trace}[(vv^T - I_n)\cdot A]\right)^3 = \left(\sum_{i<j} a_{ij}v_iv_j\right)^3$ divide into three groups

(1) $a_{ij}^3 v_iv_j$ for $i < j$,
(2) $a_{ij}^2 a_{pq}v_pv_q$ for $i < j$ and $p < q$,
(3) $a_{ij}a_{pq}a_{rs}v_iv_jv_pv_qv_rv_s$.

Summing over P_n, the monomials of the first group cancel each other because, for any fixed value $v_i \in \{-1, +1\}$, the vectors $v \in P_n$ having this fixed value divide into two subsets of equal size, having respectively $v_j = -1$ or $v_j = 1$. This argument includes the case $i = 0$, when $v_0 = 1$.

For the same reason the monomials of the second group cancel each other.

In the third group, all monomials when $\#\{i, j, p, q, r, s\} \geq 4$ cancel each other because there is an index which appear only once, and the above argument shows that the sum over this index makes zero. If $\#\{i, j, p, q, r, s\} = 3$ and the monomial is not in the first or second group, then each index appears exactly twice and we get exactly all the summands which appear in (4.1).

Since these cover all cases, the right-hand side of (4.2) sums up to a scalar multiple of the left-hand side. □

Proposition 4.2. *The decompositions in (4.2) are minimal for $n = 3, 4, 5$. In particular, $\underline{\mathbf{R}}_s(sM^Z_{\langle n\rangle}) = \mathbf{R}_s(sM^Z_{\langle n\rangle}) = 2^{n-1}$ for $n = 3, 4, 5$.*

Proof. We compute the Koszul flattening $\mathrm{YF}_{3,n^2-1}(sM^Z_{\langle n\rangle})$ as in [27, (2)] where r_n is its rank. Let $q_n = \operatorname{rank} \mathrm{YF}_{3,n^2-1}(\ell^3) = \binom{m_n}{\lfloor m_n/2\rfloor}$ for any linear form ℓ, where $m_n = n(n-1)/2 - 1$. With this setup, [27, Prop. 4.1.1] and Theorem 4.1 provide

$$\left\lceil \frac{r_n}{q_n} \right\rceil \leq \underline{\mathbf{R}}_s(sM^Z_{\langle n\rangle}) \leq \mathbf{R}_s(sM^Z_{\langle n\rangle}) \leq 2^{n-1}.$$

The result follows immediately from the following table:

n	r_n	q_n	$\left\lceil \frac{r_n}{q_n} \right\rceil$	2^{n-1}
3	8	2	4	4
4	72	10	8	8
5	1920	126	16	16

□

For comparison, the known lower bounds on the border rank of $M_{\langle n\rangle}$ when $n = 3, 4, 5$ are $15, 29, 47$, respectively, with the general lower bound from [29] of $\underline{\mathbf{R}}(M_{\langle n\rangle}) \geq 2n^2 - \log_2(n) - 1$.

For $n = 6, 7$, the decomposition (4.2) has 32 and 64 summands, respectively. The following shows that such decompositions are not minimal and we expect that this holds for any $n \geq 6$.

Proposition 4.3. $\mathbf{R}_s(sM^Z_{\langle 6\rangle}) \leq 30$ *and* $\mathbf{R}_s(sM^Z_{\langle 7\rangle}) \leq 48$.

Proof. See Appendix A for a decomposition of $32sM^Z_{\langle 6\rangle}$ with 30 summands having integer coefficients and a decomposition of $160sM^Z_{\langle 7\rangle}$ with 48 summands having coefficients in $\mathbb{Q}[\sqrt{5}]$. □

When $n = 8$, the following provides a decomposition using 64 summands.

Proposition 4.4. *Let P_8^+ be the subset of P_8 consisting of v such that $+1$ appears an even number of times, so $\#P_8^+ = 64$. Then,*

$$2^9 sM^Z_{\langle 8\rangle}(A) = \sum_{v \in P_8^+} \left(\operatorname{trace}[(vv^T - I_n) \cdot A]\right)^3 \tag{4.3}$$

Proof. This is easy to verify by direct computation. □

Remark 4.5. *We expect the decomposition* (4.3) *with* 64 *summands is minimal. In the* $\binom{8}{2} = 28$ *indeterminants of the matrix* A*, consider the polynomial* $f(A; \ell_1, \ldots, \ell_{64}) = \sum_{i=1}^{64} \left(\mathrm{trace}[(\ell_i^T A])\right)^3$*. Evaluated at* $\ell_i = (v_i v_i^T - I_n)$ *where* $v_i \in P_8^+$*, the polynomial* f *has maximal rank of* $1792 = 28 \cdot 64$*.*

Proposition 4.4 suggests one should look for strategic (not yet known) subsets $P_n^? \subset P_n$ analogous to $P_8^+ \subset P_8$ to produce minimal decompositions. For $n = 9$ and 10, we can use P_9^+ and P_{10}^+ to obtain (again by direct computation) $\mathbf{R}_s(sM^Z_{\langle 9\rangle}) \leq 128$ and $\mathbf{R}_s(sM^Z_{\langle 10\rangle}) \leq 256$, but both seem to not be sharp.

References

[1] A. Abdesselam, C. Ikenmeyer, G. Royle, *16,051 formulas for Ottaviani's invariant of cubic threefolds*, Journal of Algebra **447** (2016), 649–663

[2] T.A. Akoglu, J.D. Hauenstein, A. Szanto, *Certifying solutions to overdetermined and singular polynomial systems over Q*, J. Symb. Comput., **84** (2018), 147–171.

[3] A. Alder, *Grenzrang und Grenzkomplexität aus algebraischer und topologischer Sicht*, PhD thesis, Universität Zürich, 1984.

[4] J. Alexander, A. Hirschowitz, *Polynomial interpolation in several variables*, J. Algebraic Geom. **4** (1995), no. 2, 201–222.

[5] A. Conner, *A rank* 18 *Waring decomposition of* $sM_{\langle 3\rangle}$ *with* 432 *symmetries*, arXiv:1711.05796

[6] D.J. Bates, J.D. Hauenstein, A.J. Sommese, C.W. Wampler. *Bertini: Software for numerical algebraic geometry*. Available at `bertini.nd.edu`.

[7] D. Bini, *Relations between exact and approximate bilinear algorithms. Applications*, Calcolo **17** (1980), 87–97.

[8] J. Buczyński, A. Ginensky, J. M. Landsberg, *Determinantal equations for secant varieties and the Eisenbud-Koh-Stillman conjecture*, J. Lond. Math. Soc. (2), **88** (2013), 1–24.

[9] P. Bürgisser, C. Ikenmeyer, *Fundamental invariants of orbit closures*, Journal of Algebra, **477** (2017), 390–434.

[10] F. Butelle, R. King, F. Toumazet, G. Wybourne, *Schur Group Theory Software*, `http://schur.sourceforge.net`

[11] M. Cheung, C. Ikenmeyer, S. Mkrtchyan, *Symmetrizing tableaux and the 5th case of the Foulkes conjecture*, Journal of Symbolic Computation **80** (2017), no. 3, 833–843

[12] L. Chiantini, C. Ikenmeyer, J.M.. Landsberg, G. Ottaviani, *The geometry of rank decompositions of matrix multiplication I:* 2×2 *matrices*, Experimental Mathematics, `https://doi.org/10.1080/10586458.2017.1403981` , arXiv:1610.08364

[13] L. Chiantini, G. Ottaviani, N. Vannieuwenhoven, *On generic identifiability of symmetric tensors of subgeneric rank*, Trans. Amer. Math. Soc. **369** (2017), 4021–4042.

[14] P. Comon, G.H. Golub, L-H. Lim, B. Mourrain, *Symmetric tensors and symmetric tensor rank*, SIAM J. Matrix Anal. Appl. **30** (2008), no. 3, 1254–1279.

[15] D. Coppersmith, S. Winograd, *Matrix multiplication via arithmetic progressions*, J. Symb. Comput. **9** (1990), 251–280.

[16] N.S. Daleo, J.D. Hauenstein, *Numerically deciding the arithmetically Cohen-Macaulayness of a projective scheme*, J. Symb. Comput. **72** (2016), 128–146.

[17] N.S. Daleo, J.D. Hauenstein, *Numerically testing generically reduced projective schemes for the arithmetic Gorenstein property*, LNCS **9582** (2016), 137–142.

[18] D. Grayson, M. Stillman, Macaulay 2: *a software system for research in algebraic geometry.* Available at `http://www.math.uiuc.edu/Macaulay2`.

[19] J.D. Hauenstein, C. Ikenmeyer, J.M. Landsberg, *Equations for lower bounds on border rank*, Exp. Math. **22** (2013), 372–383.

[20] J.D. Hauenstein, A.J. Sommese, *Witness sets of projections*, Appl. Math. Comput. **217** (2010), 3349–3354.

[21] J.D. Hauenstein and F. Sottile, *Algorithm 921: alphaCertified: Certifying solutions to polynomial systems*, ACM Trans. Math. Softw. **38** (2012), 28.

[22] J.D. Hauenstein and C.W. Wampler, *Isosingular sets and deflation*, Found. Comput. Math. **13** (2013), 371–403.

[23] A. Iliev, K. Ranestad, *K3 surfaces of genus 8 and varieties of sums of powers of cubic fourfolds*, Trans. Amer. Math. Soc. **353** (2001), no. 4, 1455–1468.

[24] J. Jelisiejew, *VSP's of cubic fourfolds and the Gorenstein locus of the Hilbert scheme of 14 points on A^6*, arXiv:1611.04345

[25] J.M. Landsberg, *Tensors: Geometry and applications*, Graduate Studies in Mathematics, vol. 128, AMS, Providence, Rhode Island, 2012.

[26] J.M. Landsberg, G. Ottaviani, *New lower bounds for the border rank of matrix multiplication*, Theory of Computing, **11** (2015), 285–298.

[27] J.M. Landsberg, G. Ottaviani, *Equations for secant varieties of Veronese and other varieties*, Annali di Matematica Pura e Applicata, 192 (2013), 569–606.

[28] J.M. Landsberg, Z. Teitler, *On the ranks and border ranks of symmetric tensors*, Found. Comput. Math. **10** (2010), no. 3, 339–366

[29] J.M. Landsberg, M. Michalek, *A $2n^2 - log(n) - 1$ lower bound for the border rank of matrix multiplication*, International Mathematics Research Notices, `http://dx.doi.org/10.1093/imrn/rnx025`, arXiv:1608.07486,

[30] F. Le Gall, *Powers of tensors and fast matrix multiplication*, Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation (ISSAC 2014), pp. 296-303, 2014.

[31] D. Mumford, Complex projective varieties, Springer-Verlag, Berlin, 1995.

[32] L. Oeding and G. Ottaviani. *Eigenvectors of tensors and algorithms for Waring decomposition.* J. Symb. Comput. **54** (2013), 9–35.

[33] G. Ottaviani, *An invariant regarding Waring's problem for cubic polynomials*, Nagoya Math. J., **193** (2009), 95–110.

[34] G. Ottaviani, *Five Lectures on projective Invariants*, lecture notes for Trento school, September 2012, Rendiconti del Seminario Matematico Univ. Politec. Torino, **71**, 1 (2013), 119–194.

[35] K. Ranestad, C. Voisin, *Variety of power sums and divisors in the moduli space of cubic fourfolds*, Documenta Mathematica, **22**, (2017), 455–504. arXiv:1309.1899.

[36] A. Schönhage, *Partial and total matrix multiplication*, SIAM J. Comput., **10** (1981), 434–455.

[37] B. Segre, *The Non-singular Cubic Surfaces* , Oxford University Press, Oxford, 1942

[38] A.J. Stothers, *On the complexity of matrix multiplication*, PhD thesis, University of Edinburgh, 2010

[39] V. Strassen, *Rank and optimal computation of generic tensors*, Linear Algebra Appl., **52/53**, (1983), 645–685.

[40] V. Strassen, *Relative bilinear complexity and matrix multiplication*, J. Reine Angew. Math., **375/376**, (1987), 406–443.

[41] V. Williams, *Multiplying matrices faster than Coppersmith-Winograd*, Proceedings 44th ACM Symposium on Theory of Computing (STOC 12) , pages 887–898, 2012.

Appendix A. Decompositions

The following 18 matrices of rank 3 form a numerical approximation of a decomposition of $sM_{\langle 3\rangle}$ where $i = \sqrt{-1}$:

$$\begin{pmatrix} -0.13-0.311i & 0.499-0.51i & -0.464-0.387i \\ -1.4-2.08i & 2.46-0.687i & -1.56+0.414i \\ -0.141-0.542i & 0.44-0.374i & -0.783-0.0408i \end{pmatrix},$$
$$\begin{pmatrix} 0.568+1.31i & -0.592+0.375i & 1 \\ 0.0129-0.785i & 0.598+0.73i & -1.48+0.0928i \\ 0.943-0.486i & 0.407+0.64i & -0.55-0.572i \end{pmatrix},$$
$$\begin{pmatrix} -0.557-0.103i & 0.169-0.756i & 0.198+0.804i \\ 0.815-1.25i & 1 & -1 \\ 1.23+0.517i & -0.491+0.197i & 0.516-1.16i \end{pmatrix},$$
$$\begin{pmatrix} -0.649-0.377i & 0.787+0.21i & 0 \\ 1.26-1.57i & 1.2+2.02i & -0.712-1.79i \\ -0.314+0.107i & 0.602+0.0423i & 0.0664-0.178i \end{pmatrix},$$
$$\begin{pmatrix} 0.714+0.0554i & 0.283-0.0242i & -0.0436-1.28i \\ 0.491+2.16i & -0.449-0.276i & 2.3-1.63i \\ 0.685+1.21i & -0.692-0.311i & 0.695-1.04i \end{pmatrix},$$
$$\begin{pmatrix} -1.34-0.753i & -0.344-0.339i & -0.0879+1.74i \\ 0.00563-2.43i & -0.0178-0.303i & -1.91+1.15i \\ -0.148-0.755i & 0.106+0.39i & -0.312+0.239i \end{pmatrix},$$

$$\begin{pmatrix} -1.42-0.99i & 0.779-0.573i & -1.33+0.129i \\ -1.36-0.599i & 0.496-0.462i & -0.474-0.937i \\ -1.5-6.93\cdot 10^{-5}i & 0.209-0.712i & -0.747+0.635i \end{pmatrix},$$
$$\begin{pmatrix} 0.918+0.932i & -0.867-0.478i & 1+0.368i \\ -1.01+0.753i & 0.132-1.7i & -0.195+2.31i \\ 1 & -0.659+0.171i & 0.493-0.273i \end{pmatrix},$$
$$\begin{pmatrix} 1.22+1.7i & -0.408+0.328i & 0.739-1.56i \\ -0.731+2.33i & -0.0271+0.0704i & 1.36-0.778i \\ 0.137+0.165i & 0.395-0.0697i & -0.575-0.311i \end{pmatrix},$$
$$\begin{pmatrix} 1.5+0.508i & 0.406+0.256i & 0.672-0.572i \\ 0.665-0.0681i & 1.5+0.508i & -0.529+0.896i \\ 0.224+0.652i & 0.000481-0.584i & 1.02+0.317i \end{pmatrix},$$
$$\begin{pmatrix} 0.0701-0.426i & 0.128+0.459i & 0.0339+0.782i \\ -0.144-1.56i & 0.409+0.605i & -1.92+1.94i \\ -1.12-0.862i & 0.596-0.152i & -0.351+1.68i \end{pmatrix},$$
$$\begin{pmatrix} -1.25+1.2i & -0.0161-1.08i & 0.231+1.64i \\ -0.909+0.146i & 0.96-0.305i & -1.86+0.264i \\ -0.134+1.44i & -0.622-0.495i & 0.903+0.567i \end{pmatrix},$$
$$\begin{pmatrix} -1 & -0.09-0.772i & -1.17+0.839i \\ -1.95+0.717i & -0.092-1.51i & -0.511+0.706i \\ -0.905+0.354i & -0.208-0.553i & -0.484+1.31i \end{pmatrix},$$
$$\begin{pmatrix} -1.44-1.13i & 0.414-0.261i & -0.828+1.13i \\ 1 & -0.751+0.559i & 0.0548-1.41i \\ 0.0382+0.716i & -0.481+0.194i & 0.519-0.242i \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0.0696+0.285i & 0.537+0.0341i \\ 0.321+0.252i & 0 & -0.612+0.169i \\ -0.178+0.381i & 0.248-0.256i & -0.126-0.284i \end{pmatrix},$$
$$\begin{pmatrix} 0.261-0.0237i & -0.258+0.311i & -0.338-0.577i \\ -1.21-0.256i & -0.221-0.63i & -0.347+0.516i \\ -0.129-1.47i & 0.789+0.348i & -1.71-0.163i \end{pmatrix},$$
$$\begin{pmatrix} -0.294+1.33i & -0.471-0.0831i & 0.353-1.03i \\ -0.411+1.21i & -0.534-0.0483i & 1.34-1.84i \\ 1.27+0.0632i & -0.0116+0.723i & -0.748-1.48i \end{pmatrix},$$
$$\begin{pmatrix} -1.47+1.27i & -0.701-0.533i & 1.66+0.192i \\ -1.88+0.176i & -0.351-0.513i & 1.59+0.427i \\ 0.568+0.624i & -0.306+0.631i & 0.153-1.58i \end{pmatrix}.$$

A decomposition of $32sM^Z_{\langle 6\rangle}$ using 30 summands is:

$$\begin{aligned}
&(a_2-2a_3+a_4-a_6-a_8+2a_9-a_{10}-a_{12}-a_{13}-a_{15})^3+\\
&(2a_1+a_2-2a_3-a_4+a_6-2a_7-a_8-a_{10}-2a_{11}-a_{12}+a_{13}+a_{15})^3+\\
&(2a_3-a_2-a_4-a_6+2a_7-a_8-a_{10}+2a_{11}-a_{12}-a_{13}+2a_{14}-a_{15})^3+\\
&(a_2-2a_1+2a_3-a_4-a_6+a_8-2a_9+a_{10}+a_{12}-a_{13}+2a_{14}-a_{15})^3+\\
&(2a_1+a_2-2a_3+a_4+2a_5+a_6-2a_7+a_8+2a_9-a_{10}+a_{12}-a_{13}-2a_{14}+a_{15})^3+\\
&(2a_1-a_2-a_4-a_6-a_8-a_{10}+a_{12}-a_{13}-2a_{14}+a_{15})^3+\\
&(a_2+a_4-2a_5+a_6+a_8-2a_9-a_{10}+2a_{11}-a_{12}-a_{13}+2a_{14}-a_{15})^3+\\
&(a_2+2a_3-a_4-2a_5+a_6+2a_7-a_8-2a_9+a_{10}-a_{12}-a_{13}+a_{15})^3+\\
&(2a_3-a_2+a_4-2a_5-a_6+2a_7+a_8-2a_9-a_{10}+a_{12}+a_{13}-a_{15})^3+\\
&(a_4-a_2-2a_1-2a_5+a_6+2a_7-a_8+a_{10}+a_{12}-a_{13}+2a_{14}-a_{15})^3+\\
&(2a_1+a_2+a_4+a_6+a_8+a_{10}-a_{12}+a_{13}-2a_{14}-a_{15})^3+\\
&(a_2-a_4-a_6-2a_7+a_8+2a_9+a_{10}-2a_{11}-a_{12}-a_{13}-2a_{14}+a_{15})^3+\\
&(a_2+a_4+2a_5-a_6-2a_7-a_8+a_{10}+a_{12}+a_{13}+a_{15})^3+\\
&(2a_1-a_2-2a_3+a_4-a_6-2a_7+a_8+a_{10}-2a_{11}+a_{12}-a_{13}-a_{15})^3+\\
&(2a_3-a_2-2a_1-a_4-2a_5+a_6+a_8-a_{10}+2a_{11}+a_{12}-a_{13}+a_{15})^3+\\
&(a_6-2a_3-a_4-a_2+a_8+2a_9+a_{10}+a_{12}+a_{13}+a_{15})^3+\\
&(a_6-a_2-a_4-2a_1+2a_7+a_8-2a_9+a_{10}+2a_{11}-a_{12}+a_{13}-a_{15})^3+\\
&(a_4-a_2+a_6-2a_7-a_8+2a_9-a_{10}-2a_{11}+a_{12}+a_{13}-2a_{14}-a_{15})^3+\\
&(2a_5-a_4-a_2+a_6-2a_7+a_8-a_{10}-a_{12}-a_{13}-a_{15})^3+\\
&(2a_3-a_2-2a_1+a_4+a_6-a_8-2a_9-a_{10}-a_{12}+a_{13}+2a_{14}+a_{15})^3+\\
&(2a_1+a_2-a_4+2a_5+a_6-a_8+2a_9+a_{10}-2a_{11}+a_{12}-a_{13}-a_{15})^3+\\
&(2a_1-a_2+a_4+2a_5-a_6+a_8+2a_9-a_{10}-2a_{11}-a_{12}+a_{13}+a_{15})^3+\\
&(a_2-2a_1+a_4-a_6+2a_7-a_8-2a_9-a_{10}+2a_{11}+a_{12}-a_{13}+a_{15})^3+\\
&(a_{10}-a_4-2a_5-a_6-a_8-2a_9-a_2+2a_{11}+a_{12}+a_{13}+2a_{14}+a_{15})^3+\\
&(2a_1-a_2-2a_3-a_4+2a_5-a_6-2a_7-a_8+2a_9+a_{10}-a_{12}+a_{13}-2a_{14}-a_{15})^3+\\
&(a_4-2a_3-a_2+2a_5+a_6-a_8+a_{10}-2a_{11}-a_{12}-a_{13}-2a_{14}+a_{15})^3+\\
&(a_2+2a_3+a_4+a_6+2a_7+a_8+a_{10}+2a_{11}+a_{12}+a_{13}+2a_{14}+a_{15})^3+\\
&(a_2-2a_1-a_4-2a_5-a_6+2a_7+a_8-a_{10}-a_{12}+a_{13}+2a_{14}+a_{15})^3+\\
&(a_2-2a_1+2a_3+a_4-2a_5-a_6-a_8+a_{10}+2a_{11}-a_{12}+a_{13}-a_{15})^3+\\
&(a_2-2a_3-a_4+2a_5-a_6+a_8-a_{10}-2a_{11}+a_{12}+a_{13}-2a_{14}-a_{15})^3.
\end{aligned}$$

With $\beta = \sqrt{5}/2$, a decomposition of $160sM^Z_{\langle 7\rangle}$ using 48 summands is:

$$
\begin{aligned}
&(a_3 - a_1 - 2a_4 + a_5 - a_6 - a_8 + 2a_9 - a_{10} + a_{11} - 2a_{16} + a_{17} - a_{18} - 2a_{19} + 2a_{20} - a_{21}\\
&\quad - 2a_2\beta + 2a_7\beta - 2a_{12}\beta + a_{13}\beta - 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(2a_3 - 2a_1 + a_4 + 2a_5 - 2a_6 - 2a_8 - a_9 - 2a_{10} + 2a_{11} + a_{16} + 2a_{17} - 2a_{18} + a_{19} - a_{20} - 2a_{21})^3 +\\
&(a_4 - 2a_3 - 2a_1 - 2a_5 - 2a_6 + 2a_8 - a_9 + 2a_{10} + 2a_{11} - a_{16} + 2a_{17} + 2a_{18} - a_{19} - a_{20} + 2a_{21})^3 +\\
&(a_1 - a_3 + 2a_4 + a_5 + a_6 - a_8 + 2a_9 + a_{10} + a_{11} - 2a_{16} - a_{17} - a_{18} + 2a_{19} + 2a_{20} + a_{21}\\
&\quad + 2a_2\beta + 2a_7\beta - 2a_{12}\beta + a_{13}\beta + 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(2a_1 + 2a_3 - a_4 - 2a_5 + 2a_6 + 2a_8 - a_9 - 2a_{10} + 2a_{11} - a_{16} - 2a_{17} + 2a_{18} + a_{19} - a_{20} - 2a_{21})^3 +\\
&(2a_3 - 2a_1 - a_4 - 2a_5 - 2a_6 - 2a_8 + a_9 + 2a_{10} + 2a_{11} - a_{16} - 2a_{17} - 2a_{18} + a_{19} + a_{20} + 2a_{21})^3 +\\
&(a_1 - a_3 - 2a_4 - a_5 + a_6 - a_8 - 2a_9 - a_{10} + a_{11} + 2a_{16} + a_{17} - a_{18} + 2a_{19} - 2a_{20} - a_{21}\\
&\quad + 2a_2\beta + 2a_7\beta - 2a_{12}\beta - a_{13}\beta - 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(a_1 - a_3 + 2a_4 - a_5 - a_6 - a_8 + 2a_9 - a_{10} - a_{11} - 2a_{16} + a_{17} + a_{18} - 2a_{19} - 2a_{20} + a_{21}\\
&\quad + 2a_2\beta + 2a_7\beta - 2a_{12}\beta + a_{13}\beta - 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(a_1 + a_3 - 2a_4 - a_5 - a_6 + a_8 - 2a_9 - a_{10} - a_{11} - 2a_{16} - a_{17} - a_{18} + 2a_{19} + 2a_{20} + a_{21}\\
&\quad + 2a_2\beta + 2a_7\beta + 2a_{12}\beta - a_{13}\beta - 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(2a_4 - a_3 - a_1 + a_5 - a_6 + a_8 - 2a_9 - a_{10} + a_{11} - 2a_{16} - a_{17} + a_{18} + 2a_{19} - 2a_{20} - a_{21}\\
&\quad + 2a_2\beta - 2a_7\beta - 2a_{12}\beta + a_{13}\beta + 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(a_1 + a_3 - 2a_4 + a_5 + a_6 + a_8 - 2a_9 + a_{10} + a_{11} - 2a_{16} + a_{17} + a_{18} - 2a_{19} - 2a_{20} + a_{21}\\
&\quad + 2a_2\beta + 2a_7\beta + 2a_{12}\beta - a_{13}\beta + 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(a_3 - a_1 + 2a_4 - a_5 - a_6 - a_8 - 2a_9 + a_{10} + a_{11} + 2a_{16} - a_{17} - a_{18} - 2a_{19} - 2a_{20} + a_{21}\\
&\quad - 2a_2\beta + 2a_7\beta - 2a_{12}\beta - a_{13}\beta + 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(a_1 + a_3 + 2a_4 + a_5 - a_6 + a_8 + 2a_9 + a_{10} - a_{11} + 2a_{16} + a_{17} - a_{18} + 2a_{19} - 2a_{20} - a_{21}\\
&\quad - 2a_2\beta - 2a_7\beta - 2a_{12}\beta - a_{13}\beta - 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(a_3 - a_1 + 2a_4 - a_5 - a_6 - a_8 - 2a_9 + a_{10} + a_{11} + 2a_{16} - a_{17} - a_{18} - 2a_{19} - 2a_{20} + a_{21}\\
&\quad + 2a_2\beta - 2a_7\beta + 2a_{12}\beta + a_{13}\beta - 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(2a_4 - a_3 - a_1 - a_5 + a_6 + a_8 - 2a_9 + a_{10} - a_{11} - 2a_{16} + a_{17} - a_{18} - 2a_{19} + 2a_{20} - a_{21}\\
&\quad - 2a_2\beta + 2a_7\beta + 2a_{12}\beta - a_{13}\beta + 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(a_1 + a_3 + 2a_4 - a_5 + a_6 + a_8 + 2a_9 - a_{10} + a_{11} + 2a_{16} - a_{17} + a_{18} - 2a_{19} + 2a_{20} - a_{21}\\
&\quad - 2a_2\beta - 2a_7\beta - 2a_{12}\beta - a_{13}\beta + 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(a_8 - a_3 - 2a_4 - a_5 - a_6 - a_1 + 2a_9 + a_{10} + a_{11} + 2a_{16} + a_{17} + a_{18} + 2a_{19} + 2a_{20} + a_{21}\\
&\quad + 2a_2\beta - 2a_7\beta - 2a_{12}\beta - a_{13}\beta - 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(2a_3 - 2a_1 + a_4 - 2a_5 + 2a_6 - 2a_8 - a_9 + 2a_{10} - 2a_{11} + a_{16} - 2a_{17} + 2a_{18} - a_{19} + a_{20} - 2a_{21})^3 +\\
&(a_3 - a_1 - 2a_4 - a_5 + a_6 - a_8 + 2a_9 + a_{10} - a_{11} - 2a_{16} - a_{17} + a_{18} + 2a_{19} - 2a_{20} - a_{21}\\
&\quad + 2a_2\beta - 2a_7\beta + 2a_{12}\beta - a_{13}\beta - 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(2a_1 - 2a_3 - a_4 + 2a_5 + 2a_6 - 2a_8 - a_9 + 2a_{10} + 2a_{11} + a_{16} - 2a_{17} - 2a_{18} - a_{19} - a_{20} + 2a_{21})^3 +\\
&(2a_4 - a_3 - a_1 - a_5 + a_6 + a_8 - 2a_9 + a_{10} - a_{11} - 2a_{16} + a_{17} - a_{18} - 2a_{19} + 2a_{20} - a_{21}\\
&\quad + 2a_2\beta - 2a_7\beta - 2a_{12}\beta + a_{13}\beta - 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(a_3 - a_1 + 2a_4 + a_5 + a_6 - a_8 - 2a_9 - a_{10} - a_{11} + 2a_{16} + a_{17} + a_{18} + 2a_{19} + 2a_{20} + a_{21}\\
&\quad - 2a_2\beta + 2a_7\beta - 2a_{12}\beta - a_{13}\beta - 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(a_3 - a_1 + 2a_4 + a_5 + a_6 - a_8 - 2a_9 - a_{10} - a_{11} + 2a_{16} + a_{17} + a_{18} + 2a_{19} + 2a_{20} + a_{21}\\
&\quad + 2a_2\beta - 2a_7\beta + 2a_{12}\beta + a_{13}\beta + 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(2a_3 - 2a_1 - a_4 + 2a_5 + 2a_6 - 2a_8 + a_9 - 2a_{10} - 2a_{11} - a_{16} + 2a_{17} + 2a_{18} - a_{19} - a_{20} + 2a_{21})^3 +\\
&(a_3 - a_1 - 2a_4 + a_5 - a_6 - a_8 + 2a_9 - a_{10} + a_{11} - 2a_{16} + a_{17} - a_{18} - 2a_{19} + 2a_{20} - a_{21}\\
&\quad + 2a_2\beta - 2a_7\beta + 2a_{12}\beta - a_{13}\beta + 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(a_1 + a_3 - 2a_4 + a_5 + a_6 + a_8 - 2a_9 + a_{10} + a_{11} - 2a_{16} + a_{17} + a_{18} - 2a_{19} - 2a_{20} + a_{21}\\
&\quad - 2a_2\beta - 2a_7\beta - 2a_{12}\beta + a_{13}\beta - 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(a_3 - a_1 - 2a_4 - a_5 + a_6 - a_8 + 2a_9 + a_{10} - a_{11} - 2a_{16} - a_{17} + a_{18} + 2a_{19} - 2a_{20} - a_{21}\\
&\quad - 2a_2\beta + 2a_7\beta - 2a_{12}\beta + a_{13}\beta + 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(2a_1 - 2a_3 + a_4 + 2a_5 - 2a_6 - 2a_8 + a_9 + 2a_{10} - 2a_{11} - a_{16} - 2a_{17} + 2a_{18} + a_{19} - a_{20} - 2a_{21})^3 +\\
&(a_1 + a_3 - 2a_4 - a_5 - a_6 + a_8 - 2a_9 - a_{10} - a_{11} - 2a_{16} - a_{17} - a_{18} + 2a_{19} + 2a_{20} + a_{21}\\
&\quad - 2a_2\beta - 2a_7\beta - 2a_{12}\beta + a_{13}\beta + 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(a_1 - a_3 + 2a_4 - a_5 - a_6 - a_8 + 2a_9 - a_{10} - a_{11} - 2a_{16} + a_{17} + a_{18} - 2a_{19} - 2a_{20} + a_{21}\\
&\quad - 2a_2\beta - 2a_7\beta + 2a_{12}\beta - a_{13}\beta + 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(2a_4 - a_3 - a_1 + a_5 - a_6 + a_8 - 2a_9 - a_{10} + a_{11} - 2a_{16} - a_{17} + a_{18} + 2a_{19} - 2a_{20} - a_{21}\\
&\quad - 2a_2\beta + 2a_7\beta + 2a_{12}\beta - a_{13}\beta - 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(a_5 - a_3 - 2a_4 - a_1 + a_6 + a_8 + 2a_9 - a_{10} - a_{11} + 2a_{16} - a_{17} - a_{18} - 2a_{19} - 2a_{20} + a_{21}\\
&\quad - 2a_2\beta + 2a_7\beta + 2a_{12}\beta + a_{13}\beta - 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(a_1 - a_3 - 2a_4 + a_5 - a_6 - a_8 - 2a_9 + a_{10} - a_{11} + 2a_{16} - a_{17} + a_{18} - 2a_{19} + 2a_{20} - a_{21}\\
&\quad - 2a_2\beta - 2a_7\beta + 2a_{12}\beta + a_{13}\beta - 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(a_1 + a_3 + 2a_4 - a_5 + a_6 + a_8 + 2a_9 - a_{10} + a_{11} + 2a_{16} - a_{17} + a_{18} - 2a_{19} + 2a_{20} - a_{21}\\
&\quad + 2a_2\beta + 2a_7\beta + 2a_{12}\beta + a_{13}\beta - 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(2a_5 - 2a_3 - a_4 - 2a_1 - 2a_6 + 2a_8 + a_9 - 2a_{10} + 2a_{11} + a_{16} - 2a_{17} + 2a_{18} - a_{19} + a_{20} - 2a_{21})^3 +\\
&(a_8 - a_3 - 2a_4 - a_5 - a_6 - a_1 + 2a_9 + a_{10} + a_{11} + 2a_{16} + a_{17} + a_{18} + 2a_{19} + 2a_{20} + a_{21}\\
&\quad - 2a_2\beta + 2a_7\beta + 2a_{12}\beta + a_{13}\beta + 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(2a_1 + 2a_3 + a_4 + 2a_5 + 2a_6 + 2a_8 + a_9 + 2a_{10} + 2a_{11} + a_{16} + 2a_{17} + 2a_{18} + a_{19} + a_{20} + 2a_{21})^3 +\\
&(a_5 - a_3 - 2a_4 - a_1 + a_6 + a_8 + 2a_9 - a_{10} - a_{11} + 2a_{16} - a_{17} - a_{18} - 2a_{19} - 2a_{20} + a_{21}\\
&\quad + 2a_2\beta - 2a_7\beta - 2a_{12}\beta - a_{13}\beta + 2a_{14}\beta + 2a_{15}\beta)^3 +\\
&(2a_1 + 2a_3 - a_4 + 2a_5 - 2a_6 + 2a_8 - a_9 + 2a_{10} - 2a_{11} - a_{16} + 2a_{17} - 2a_{18} - a_{19} + a_{20} - 2a_{21})^3 +\\
&(2a_6 - 2a_3 - a_4 - 2a_5 - 2a_1 + 2a_8 + a_9 + 2a_{10} - 2a_{11} + a_{16} + 2a_{17} - 2a_{18} + a_{19} - a_{20} - 2a_{21})^3 +\\
&(2a_1 + 2a_3 + a_4 - 2a_5 - 2a_6 + 2a_8 + a_9 - 2a_{10} - 2a_{11} + a_{16} - 2a_{17} - 2a_{18} - a_{19} - a_{20} + 2a_{21})^3 +\\
&(a_1 - a_3 + 2a_4 + a_5 + a_6 - a_8 + 2a_9 + a_{10} + a_{11} - 2a_{16} - a_{17} - a_{18} + 2a_{19} + 2a_{20} + a_{21}\\
&\quad - 2a_2\beta - 2a_7\beta + 2a_{12}\beta - a_{13}\beta - 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(2a_1 - 2a_3 - a_4 - 2a_5 - 2a_6 - 2a_8 - a_9 - 2a_{10} - 2a_{11} + a_{16} + 2a_{17} + 2a_{18} + a_{19} + a_{20} + 2a_{21})^3 +\\
&(a_4 - 2a_3 - 2a_1 + 2a_5 + 2a_6 + 2a_8 - a_9 - 2a_{10} - 2a_{11} - a_{16} - 2a_{17} - 2a_{18} + a_{19} + a_{20} + 2a_{21})^3 +\\
&(a_1 + a_3 + 2a_4 + a_5 - a_6 + a_8 + 2a_9 + a_{10} - a_{11} + 2a_{16} + a_{17} - a_{18} + 2a_{19} - 2a_{20} - a_{21}\\
&\quad + 2a_2\beta + 2a_7\beta + 2a_{12}\beta + a_{13}\beta + 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(2a_1 - 2a_3 + a_4 - 2a_5 + 2a_6 - 2a_8 + a_9 - 2a_{10} + 2a_{11} - a_{16} + 2a_{17} - 2a_{18} - a_{19} + a_{20} - 2a_{21})^3 +\\
&(a_1 - a_3 - 2a_4 - a_5 + a_6 - a_8 - 2a_9 - a_{10} + a_{11} + 2a_{16} + a_{17} - a_{18} + 2a_{19} - 2a_{20} - a_{21}\\
&\quad - 2a_2\beta - 2a_7\beta + 2a_{12}\beta + a_{13}\beta + 2a_{14}\beta - 2a_{15}\beta)^3 +\\
&(a_1 - a_3 - 2a_4 + a_5 - a_6 - a_8 - 2a_9 + a_{10} - a_{11} + 2a_{16} - a_{17} + a_{18} - 2a_{19} + 2a_{20} - a_{21}\\
&\quad + 2a_2\beta + 2a_7\beta - 2a_{12}\beta - a_{13}\beta + 2a_{14}\beta - 2a_{15}\beta)^3.
\end{aligned}
$$

Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena, Italy
E-mail address: luca.chiantini@unisi.it

Department of Applied and Computational Mathematics and Statistics, University of Notre Dame, Notre Dame, IN, USA
E-mail address: hauenstein@nd.edu

Max Planck Institute for Informatics, Saarland Informatics Campus, Germany
E-mail address: cikenmey@mpi-inf.mpg.de

Department of Mathematics, Texas A&M University, College Station, TX, USA
E-mail address: jml@math.tamu.edu

Dipartimento di Matematica e Informatica "Ulisse Dini", Università di Firenze, Italy
E-mail address: ottavian@math.unifi.it