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Quasi-Monotonicity Formulas for Classical Obstacle Problems with Sobolev Coefficients and Applications

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Abstract
We establish Weiss' and Monneau's type quasi-monotonicity formulas for quadratic energies having matrix of coefficients in a Sobolev space with summability exponent larger than the space dimension and provide an application to the corresponding free boundary analysis for the related classical obstacle problems.

Keywords Classical obstacle problem · Free boundary · Monotonicity formulas

Mathematics Subject Classification 35R35 · 49N60

1 Introduction

The structure of free boundaries for classical obstacle problems has been described first by Caffarelli for quadratic energies having suitably regular matrix fields, and it is the resume of his long-term program on the subject (cf., for instance, [1–4] and the books [5–7] for more details and references also on related problems). Similar results for smooth nonlinear operators can then be obtained via a freezing argument.

In the last years, such a topic has been investigated in the case in which the quadratic energy involved has matrix of coefficients either Lipschitz continuous (cf. [8]) or belonging to a fractional Sobolev space (cf. [9]), with parameters suitably related. Let us also mention that obstacle problems for nondegenerate nonlinear variational

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energies have been studied in [10] through a linearization argument and the quoted results in the Lipschitz quadratic case.

The papers [8,9] follow the variational approach to free boundary analysis developed remarkably by Weiss [11] and by Monneau [12] that is based on (quasi-) monotonicity formulas. The extensions of Weiss' and Monneau's monotonicity formulas, obtained in [8,9], hinge upon a generalization of the Rellich and Nečas' inequality due to Payne and Weinberger (cf. [13]). On a technical side, they involve the differentiation of the matrix field.

The aim of this short note is to extend the range of validity of Weiss' and Monneau's type quasi-monotonicity formulas to classical obstacle problems, involving quadratic forms having matrix of coefficients in a Sobolev space with summability exponent lager than the space dimension.

The main difference contained in the present note, with respect to the existing literature, concerns the (quasi-)monotone quantity itself. Indeed, rather than considering the natural quadratic energy associated with the obstacle problem under study, we establish quasi-monotonicity for a related constant coefficient quadratic form. The latter result is obtained thanks to a freezing argument inspired by some computations in a paper by Monneau (cf. [12, Section 6]) in combination with the well-known quadratic lower bound on the growth of solutions from free boundary points (see Sects. 4, 5 for more details). Such an insight, though elementary, has been overlooked in the literature and enables us to obtain Weiss' and Monneau's quasi-monotonicity formulas under mild assumptions (cf. (H1) and (H3) below, the latter having no role, if the obstacle function is null), since the matrix field is not differentiated along the derivation process of the quasi-monotonicity formulas. We stress again that the mentioned quasi-monotonicity formulas are instrumental to pursue the variational approach for the analysis of the corresponding free boundaries in classical obstacle problems.

To conclude this introduction, we briefly resume the structure of the paper: Weiss' and Monneau's quasi-monotonicity formulas, the main results of the paper, together with their application to free boundaries, are stated in Sect. 2. Several preliminaries for the classical obstacle problem under study are collected in Sect. 3. The mentioned generalizations of Weiss' and Monneau's quasi-monotonicity formulas are established in Sects. 4 and 5, respectively. Finally, Sect. 6 contains the proof of the quoted applications to the free boundary stratification for quadratic problems.

2 Statement of the Main Results

In this section, we state Weiss' and Monneau's type quasi-monotonicity formulas for the quadratic problem and their application to the free boundary analysis.

We start off introducing the variational problem related to free boundaries together with the necessary notations and assumptions in the next section.

2.1 Free Boundary Analysis: Statement

We consider the functional $\mathcal{E} : W^{1,2}(\Omega) \to \mathbb{R}$ given by

$$\mathcal{E}(v) := \int_\Omega \big(\langle \mathbb{A}(x)\nabla v(x), \nabla v(x)\rangle + 2h(x)v(x)\big)\,\mathrm{d}x, \tag{1}$$

and study regularity issues related to its unique minimizer w on the set

$$\mathcal{K}_{\psi,g} := \left\{v \in W^{1,2}(\Omega) : v \geq \psi \ \ \mathcal{L}^n\text{-a.e. on } \Omega,\ \mathrm{Tr}(v) = g \ \text{ on } \partial\Omega\right\}.$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz open set, $n \geq 2$, $\psi \in C^{1,1}_{loc}(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$ are such that $\psi \leq g$ $\mathcal{H}^{n-1}$-a.e on $\partial\Omega$, $\mathbb{A} : \Omega \to \mathbb{R}^{n\times n}$ is a matrix-valued field and $f : \Omega \to \mathbb{R}$ is a function satisfying:

(H1) $\mathbb{A} \in W^{1,p}(\Omega; \mathbb{R}^{n\times n})$ with $p > n$;
(H2) $\mathbb{A}(x) = \big(a_{ij}(x)\big)_{i,j=1,\dots,n}$ is symmetric, continuous and coercive that is $a_{ij}(x) = a_{ji}(x)$ for all $x \in \Omega$ and for all $i,\ j \in \{1,\dots,n\}$, and for some $\Lambda \geq 1$

$$\Lambda^{-1}|\xi|^2 \leq \langle \mathbb{A}(x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \tag{2}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$;
(H3) $f := h - \mathrm{div}\,(\mathbb{A}\nabla\psi) > c_0$ $\mathcal{L}^n$-a.e. on Ω, for some $c_0 > 0$, and f is Dini continuous, namely

$$\int_0^1 \frac{\omega_f(t)}{t}\,\mathrm{d}t < \infty, \tag{3}$$

where $\omega_f(t) := \sup_{x,y\in\Omega,\ |x-y|\leq t} |f(x) - f(y)|$.

In some instances in place of (H3), we will require the stronger condition

(H4) $f > c_0$ $\mathcal{L}^n$-a.e. on Ω, for some $c_0 > 0$, and f is double Dini continuous, that is

$$\int_0^1 \frac{\omega_f(r)}{r}\,|\log r|^a\,\mathrm{d}r < \infty, \tag{4}$$

for some $a \geq 1$.

Note that for the zero obstacle problem, i.e. $\psi = 0$, assumptions (H3) and (H4) involve only the lower-order term h in the integrand and not the matrix field of coefficients $\mathbb{A}$. Moreover, the positivity condition on f corresponds to the concavity assumption on the obstacle function in the case of the Laplacian. Elementary examples show that it is needed to enforce free boundary regularity.

Given the assumptions introduced above, we provide a full free boundary stratification result.

Theorem 2.1 *Assume* (H1)–(H4) *to hold, and let w be the (unique) minimizer of $\mathcal{E}$ in* (1) *on $\mathcal{K}_{\psi,g}$.*

Then, w is $W^{2,p}_{loc} \cap C^{1,1-{}^{n}\!/_{p}}_{loc}(\Omega)$, and the free boundary can be decomposed as $\partial\{w = \psi\} \cap \Omega = \mathrm{Reg}(w) \cup \mathrm{Sing}(w)$, where $\mathrm{Reg}(w)$ and $\mathrm{Sing}(w)$ are called its regular and singular part, respectively. Moreover, $\mathrm{Reg}(w) \cap \mathrm{Sing}(w) = \emptyset$ and

(i) *if $a > 2$ in (H4), then $\mathrm{Reg}(w)$ is relatively open in $\partial\{w = \psi\}$ and, for every point $x_0 \in \mathrm{Reg}(w)$, there exist a radius $r = r(x_0) > 0$ such that $\partial\{w = \psi\} \cap B_r(x_0)$ is a C^1 $(n-1)$-dimensional manifold with normal vector absolutely continuous. In particular, if f is Hölder continuous, there exists $r = r(x_0) > 0$ such that $\partial\{w = \psi\} \cap B_r(x_0)$ is a $C^{1,\beta}$ $(n-1)$-dimensional manifold for some exponent $\beta \in]0, 1[$.*

(ii) *if $a \geq 1$ in (H4), then $\mathrm{Sing}(w) = \cup_{k=0}^{n-1} S_k$, with S_k contained in the union of at most countably many submanifolds of dimension k and class C^1.*

Remark 2.1 Very recently, the fine structure of the set of the so-called singular points in the case of the Dirichlet energy has been unveiled in the papers by Colombo, Spolaor and Velichkov [14] and Figalli and Serra [15] by means of a logarithmic epiperimetric inequality and new monotonicity formulas, respectively.

2.2 Quasi-Monotonicity Formulas: Statements

Theorem 2.1 is a consequence of Weiss' and Monneau's quasi-monotonicity type formulas that will be stated in this section (cf. Sect. 6 for the proofs). With this aim, we introduce first some notation.

We first reduce ourselves to the zero obstacle problem. Let w be the unique minimizer of $\mathcal{E}$ in (1) over $\mathcal{K}_{\psi,g}$, and define $u := w - \psi$. Then, u is the unique minimizer of

$$\mathscr{E}(v) := \int_{\Omega} \big(\langle \mathbb{A}(x)\nabla v(x), \nabla v(x)\rangle + 2f(x)v(x)\big)\,\mathrm{d}x, \tag{5}$$

over

$$\mathbb{K}_{\psi,g} := \left\{v \in W^{1,2}(\Omega) : \ v \geq 0 \ \ \mathcal{L}^n\text{-a.e. on } \Omega,\ \mathrm{Tr}(v) = g - \psi \ \text{ on } \partial\Omega\right\},$$

where $f = h - \mathrm{div}\,(\mathbb{A}\nabla\psi)$. Clearly, $\partial\{w = \psi\} \cap \Omega = \partial\{u = 0\} \cap \Omega =: \Gamma_u$; therefore, we shall establish all the results in Theorem 2.1 for u (notice that assumptions (H3) and (H4) are formulated exactly in terms of f).

Let $x_0 \in \Gamma_u$ be any point of the free boundary; then, the affine change of variables

$$x \mapsto x_0 + f^{-1/2}(x_0)\mathbb{A}^{1/2}(x_0)x =: x_0 + \mathbb{L}(x_0)\,x$$

leads to

$$\mathscr{E}(u) = f^{1-\frac{n}{2}}(x_0)\det(\mathbb{A}^{1/2}(x_0))\,\mathscr{E}_{\mathbb{L}(x_0)}(u_{\mathbb{L}(x_0)}), \tag{6}$$

where $\Omega_{\mathbb{L}(x_0)} := \mathbb{L}^{-1}(x_0)\,(\Omega - x_0)$, and we have set

$$\mathscr{E}_{\mathbb{L}(x_0)}(v) := \int_{\Omega_{\mathbb{L}(x_0)}} \left(\langle \mathbb{C}_{x_0}\nabla v, \nabla v\rangle + 2\frac{f_{\mathbb{L}(x_0)}}{f(x_0)}\,v\right)\mathrm{d}x, \tag{7}$$

with

$$\begin{aligned} u_{\mathbb{L}(x_0)}(x) &:= u\big(x_0 + \mathbb{L}(x_0)x\big), \\ f_{\mathbb{L}(x_0)}(x) &:= f\big(x_0 + \mathbb{L}(x_0)x\big), \\ \mathbb{C}_{x_0}(x) &:= \mathbb{A}^{-1/2}(x_0)\mathbb{A}(x_0 + \mathbb{L}(x_0)x)\mathbb{A}^{-1/2}(x_0). \end{aligned} \tag{8}$$

Note that $f_{\mathbb{L}(x_0)}(\underline{0}) = f(x_0)$ and $\mathbb{C}_{x_0}(\underline{0}) = \mathrm{Id}$. Moreover, the free boundary is transformed under this map into

$$\Gamma_{u_{\mathbb{L}(x_0)}} := \mathbb{L}^{-1}(x_0)(\Gamma_u - x_0),$$

and the energy $\mathscr{E}$ in (5) is minimized by u, if and only if $\mathscr{E}_{\mathbb{L}(x_0)}$ in (7) is minimized by the function $u_{\mathbb{L}(x_0)}$ in (8).

In addition, writing the Euler–Lagrange equation for $u_{\mathbb{L}(x_0)}$ in nondivergence form we get $\mathcal{L}^n$-a.e. on $\Omega_{\mathbb{L}(x_0)}$:

$$c_{ij}(x)\,\frac{\partial^2 u_{\mathbb{L}(x_0)}}{\partial x_i \partial x_j} + \operatorname{div}\mathbb{C}^i_{x_0}(x)\,\frac{\partial u_{\mathbb{L}(x_0)}}{\partial x_i} = \frac{f_{\mathbb{L}(x_0)}(x)}{f(x_0)}\chi_{\{u_{\mathbb{L}(x_0)}>0\}}\,,$$

(using Einstein's convention) with $\mathbb{C}_{x_0} = (c_{ij})_{i,j=1,\dots,n}$. Moreover, we may further rewrite the latter equation $\mathcal{L}^n$-a.e. on $\Omega_{\mathbb{L}(x_0)}$ as

$$\begin{aligned} \Delta u_{\mathbb{L}(x_0)} = 1 + \Big(&\frac{f_{\mathbb{L}(x_0)}(x)}{f(x_0)}\chi_{\{u_{\mathbb{L}(x_0)}>0\}} - 1 \\ &- \big(c_{ij}(x) - \delta_{ij}\big)\frac{\partial^2 u_{\mathbb{L}(x_0)}}{\partial x_i \partial x_j} - \operatorname{div}\mathbb{C}^i_{x_0}(x)\,\frac{\partial u_{\mathbb{L}(x_0)}}{\partial x_i}\Big) =: 1 + f_{x_0}(x)\,. \end{aligned} \tag{9}$$

Consider next the Weiss' type boundary adjusted energy

$$\Phi_u(x_0, r) := \frac{1}{r^{n+2}}\int_{B_r}\big(|\nabla u_{\mathbb{L}(x_0)}|^2 + 2\,u_{\mathbb{L}(x_0)}\big)\,\mathrm{d}x - \frac{2}{r^{n+3}}\int_{\partial B_r} u^2_{\mathbb{L}(x_0)}\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{10}$$

for $x_0 \in \Gamma_u$. We claim its quasi-monotonicity.

Theorem 2.2 (Weiss' quasi-monotonicity formula) *Under assumptions* (H1)–(H3)*, for every compact set $K \subset \Omega$, there exists a positive constant $C = C(n, p, \Lambda, c_0, K, \|f\|_{L^\infty}, \|\mathbb{A}\|_{W^{1,p}}) > 0$ such that for all $x_0 \in K \cap \Gamma_u$*

$$\frac{d}{dr}\Big(\Phi_u(x_0, r) + C\int_0^r \frac{\omega(t)}{t}\,dt\Big) \geq \frac{2}{r^{n+4}}\int_{\partial B_r}(\langle\nabla u_{\mathbb{L}(x_0)}, x\rangle - 2u_{\mathbb{L}(x_0)})^2 d\mathcal{H}^{n-1}, \tag{11}$$

for $\mathcal{L}^1$-a.e. $r \in]0, \frac{1}{2}\mathrm{dist}(K, \partial\Omega)[$, where $\omega(r) := \omega_f(r) + r^{1-\frac{n}{p}}$.

In particular, $\Phi_u(x_0,\cdot)$ has finite right limit $\Phi_u(x_0,0^+)$ in zero, and for all $r \in]0,\frac{1}{2}\mathrm{dist}(K,\partial\Omega)[$,

$$\Phi_u(x_0,r)-\Phi_u(x_0,0^+) \geq -C\int_0^r \frac{\omega(t)}{t}\,dt. \tag{12}$$

We recall that Weiss' original monotonicity formula for the Dirichlet energy provides an explicit expression for the derivative of $\Phi_u(x_0,\cdot)$. Namely, formula (11) is actually an equality for u, rather than for $u_{\mathbb{L}(x_0)}$, and ω is null.

The second quasi-monotonicity formula we deal with holds for a distinguished subset of points of the free boundary, that of singular points $\mathrm{Sing}(u)$. Namely, we assume that $x_0 \in \Gamma_u$ satisfies

$$\Phi_u(x_0,0^+) = \Phi_v(\underline{0},1) \tag{13}$$

for some 2-homogeneous solution v of

$$\Delta v = 1 \quad \text{on } \mathbb{R}^n. \tag{14}$$

Note that, by 2-homogeneity, elementary calculations lead to

$$\Phi_v(\underline{0},r) = \Phi_v(\underline{0},1) = \int_{B_1} v\,\mathrm{d}y, \tag{15}$$

for all $r>0$.

Theorem 2.3 (Monneau's quasi-monotonicity formula) *Under hypotheses* (H1), (H2), (H4) *with $a=1$, if $K\subset\Omega$ is a compact set and* (15) *holds for $x_0\in K\cap\Gamma_u$, then a constant $C=C(n,p,\Lambda,c_0,K,\|f\|_{L^\infty},\|\mathbb{A}\|_{W^{1,p}})>0$ exists such that the function*

$$\Big]0,\frac{1}{2}\mathrm{dist}(K,\partial\Omega)\Big[\ni r \longmapsto \frac{1}{r^{n+3}}\int_{\partial B_r}(u_{\mathbb{L}(x_0)}-v)^2\,dx + C\int_0^r\frac{dt}{t}\int_0^t\frac{\omega(s)}{s}\,ds \tag{16}$$

is nondecreasing, where v is any 2-homogeneus polynomial solution of (14)*, and ω is the modulus of continuity provided by Theorem* 2.2*.*

3 Preliminaries on the Classical Obstacle Problem

Throughout the section, we use the notation introduced in Sect. 2 and adopt Einstein' summation convention.

The next result has been established by Ural'tseva (cf. for instance [16, Theorem 2.1]) for general variational inequalities with a penalization method. Our argument instead follows the approach in [8], inspired by the ideas of Weiss for the Laplacian in [11]. Let us briefly sketch our arguments. Consider the minimizer u of the energy $\mathscr{E}$ introduced in (5). It turns out that u satisfies a PDE both in the distributional sense and $\mathcal{L}^n$-a.e. on Ω; elliptic regularity then applies to establish the smoothness of u itself.

Proposition 3.1 *Let u be the minimum of $\mathscr{E}$ on $\mathbb{K}_{\psi,g}$. Then,*

$$\operatorname{div}(\mathbb{A}\nabla u) = f\chi_{\{u>0\}} \tag{17}$$

$\mathcal{L}^n$-a.e. on Ω and in $\mathcal{D}'(\Omega)$. Moreover, $u \in W^{2,p}_{loc} \cap C^{1,1-\frac{n}{p}}_{loc}(\Omega)$.

Proof For the validity of (17), we refer to [10, Proposition 3.2], where the result is proved in the broader context of variational inequalities (see also [8, Proposition 2.2]).

From this, by taking into account that $\mathbb{A} \in C^{0,1-n/p}_{loc}(\Omega, \mathbb{R}^{n\times n})$ in view of Morrey embedding theorem, Schauder estimates yield $u \in C^{1,1-n/p}_{loc}(\Omega)$ (cf. [17, Theorem 3.13]).

Next consider the equation

$$a_{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} = f\chi_{\{u>0\}} - \operatorname{div}\mathbb{A}^j \frac{\partial u}{\partial x_j} =: \varphi, \tag{18}$$

where $\mathbb{A}^j$ denotes the j-column of $\mathbb{A}$. Being $\nabla u \in L^\infty_{loc}(\Omega, \mathbb{R}^n)$ and being $\operatorname{div}\mathbb{A}^j \in L^p(\Omega)$ for all $j \in \{1, \ldots, n\}$, then $\varphi \in L^p_{loc}(\Omega)$. [18, Corollary 9.18] implies the uniqueness of a solution $v \in W^{2,p}_{loc}(\Omega)$ to (18). By taking into account the identity $\operatorname{Tr}(\mathbb{A}\nabla^2 v) = \operatorname{div}(\mathbb{A}\nabla v) - \operatorname{div}\mathbb{A}^j \frac{\partial v}{\partial x_j}$, (18) rewrites as

$$\operatorname{div}(\mathbb{A}\nabla v) - \operatorname{div}\mathbb{A}^j \frac{\partial v}{\partial x_j} = \varphi;$$

then we have that u and v are two solutions. Then, by [19, Theorem 1.I] and (17) we deduce that $u = v$. □

We recall next the standard notations for the coincidence set and for the corresponding free boundary

$$\Lambda_u := \{x \in \Omega : u(x) = 0\}, \qquad \Gamma_u := \partial\Lambda_u \cap \Omega.$$

For any point $x_0 \in \Gamma_u$, we introduce the family of rescaled functions

$$u_{x_0,r}(x) := \frac{u(x_0 + rx)}{r^2}$$

for $x \in \frac{1}{r}(\Omega - \{x_0\})$. The existence of $C^{1,\gamma}$-limits as $r \downarrow 0$ of the latter family is standard by noting that the rescaled functions satisfy an appropriate PDE and then uniform $W^{2,p}$ estimates.

Proposition 3.2 ([9, Proposition 4.1]) *Let u be the unique minimizer of $\mathscr{E}$ over $\mathbb{K}_{\psi,g}$, and $K \subset \Omega$ a compact set. Then, for every $x_0 \in K \cap \Gamma_u$, for every $R > 0$ there exists a constant $C = C(n, p, \Lambda, R, K, \|f\|_{L^\infty}, \|\mathbb{A}\|_{W^{1,p}}) > 0$ such that, for every $r \in]0, \frac{1}{4R}\operatorname{dist}(K, \partial\Omega)[$*

$$\|u_{x_0,r}\|_{W^{2,p}(B_R)} \le C. \tag{19}$$

In particular, $(u_{x_0,r})_r$ *is equibounded in* $C^{1,\gamma}_{\mathrm{loc}}$ *for* $\gamma \in]0, 1 - {}^{n}/_{p}]$.

Then, up to extracting a subsequence, the rescaled functions have limits in the $C^{1,\gamma}$ topology. The functions arising in this process are called *blowup limits*.

Corollary 3.1 (Existence of blowups) *Let* u *be the unique minimizer of* $\mathscr{E}$ *over* $\mathbb{K}_{\psi,g}$*, and let* $x_0 \in \Gamma_u$*. Then, for every sequence* $r_k \downarrow 0$ *there exists a subsequence* $(r_{k_j})_j \subset (r_k)_k$ *such that the rescaled functions* $(u_{x_0,r_{k_j}})_j$ *converge in* $C^{1,\gamma}_{\mathrm{loc}}$, $\gamma \in]0, 1 - {}^{n}/_{p}[$*, to some function belonging to* $C^{1,1-{}^{n}/_{p}}_{\mathrm{loc}}$.

Elementary growth conditions of the solution from free boundary points are easily deduced from Proposition 3.2 and the condition $p > n$. In turn, such properties will be crucial in the derivation of the quasi-monotonicity formulas.

Proposition 3.3 *Let* u *be the unique minimizer of* $\mathscr{E}$ *over* $\mathbb{K}_{\psi,g}$*. Then, for all compact sets* $K \subset \Omega$ *a constant* $C = C(n, p, \Lambda, K, \|f\|_{L^\infty}, \|\mathbb{A}\|_{W^{1,p}}) > 0$ *exists, such that for all points* $x_0 \in \Gamma_u \cap K$*, and for all* $r \in \left]0, \frac{1}{2}\mathrm{dist}(K, \partial\Omega)\right[$ *it holds*

$$\|u\|_{L^\infty(B_r(x_0))} \le C\, r^2, \qquad \|\nabla u\|_{L^\infty(B_r(x_0),\mathbb{R}^n)} \le C\, r. \tag{20}$$

and

$$\|\nabla^2 u\|_{L^p(B_r(x_0),\mathbb{R}^{n\times n})} \le C\, r^{{}^{n}/_{p}}. \tag{21}$$

Finally, we recall the fundamental quadratic detachment property from free boundary points that entails nontriviality of blowup limits. It has been established by Blank and Hao [20, Theorem 3.9] under the sole boundedness and measurability assumptions on the matrix field $\mathbb{A}$, hypotheses clearly weaker than (H1).

Lemma 3.1 ([20, Theorem 3.9]) *There exists a positive constant* ϑ*, with* $\vartheta = \vartheta(n, \Lambda, c_0, \|f\|_{L^\infty})$*, such that for every* $x_0 \in \Gamma_u$ *and* $r \in]0, \frac{1}{2}\mathrm{dist}(x_0, \partial\Omega)[$*, it holds*

$$\sup_{x\in\partial B_r(x_0)} u(x) \ge \vartheta\, r^2.$$

4 Weiss' Quasi-Monotonicity Formula: Proof of Theorem 2.2

In this section, we prove the quasi-monotonicity of the Weiss' energy $\Phi_u(x_0, \cdot)$ defined in (10). The proof is based on equality (9) and Proposition 3.3.

Proof of Theorem 2.2 We analyse separately the volume and the boundary terms appearing in the definition of the Weiss energy in (10). For the sake of notational simplicity we write u_{x_0} in place of $u_{\mathbb{L}(x_0)}$. In what follows, with C we denote a constant $C = C(n, p, \Lambda, c_0, K, \|f\|_{L^\infty}, \|\mathbb{A}\|_{W^{1,p}}) > 0$ that may vary from line to line.

We start off with the bulk term. The Coarea formula implies for $\mathcal{L}^1$-a.e. $r \in]0, \mathrm{dist}(K, \partial\Omega)[$

$$\begin{aligned} &\frac{d}{\mathrm{d}r}\Big(\frac{1}{r^{n+2}}\int_{B_r}\Big(|\nabla u_{x_0}|^2 + 2\,u_{x_0}\Big)\,\mathrm{d}x\Big) \\ &\quad = -\frac{n+2}{r^{n+3}}\int_{B_r}\Big(|\nabla u_{x_0}|^2 + 2\,u_{x_0}\Big)\,\mathrm{d}x + \frac{1}{r^{n+2}}\int_{\partial B_r}\Big(|\nabla u_{x_0}|^2 + 2\,u_{x_0}\Big)\,\mathrm{d}x. \end{aligned} \tag{22}$$

We use the divergence theorem together with the following identities

$$\begin{aligned} |\nabla u_{x_0}|^2 &= \frac{1}{2}\mathrm{div}\,(\nabla(u_{x_0}^2)) - u_{x_0}\,\Delta u_{x_0}\,, \\ \mathrm{div}\,\Big(|\nabla u_{x_0}|^2\,\frac{x}{r}\Big) &= \frac{n-2}{r}|\nabla u_{x_0}|^2 - 2\Delta u_{x_0}\Big\langle\nabla u_{x_0}, \frac{x}{r}\Big\rangle + 2\,\mathrm{div}\,\Big(\Big\langle\nabla u_{x_0}, \frac{x}{r}\Big\rangle\nabla u_{x_0}\Big)\,, \\ \mathrm{div}\,\Big(u_{x_0}\,\frac{x}{r}\Big) &= u_{x_0}\,\frac{n}{r} + \Big\langle\nabla u_{x_0}, \frac{x}{r}\Big\rangle, \end{aligned}$$

to deal with the first, third and fourth addend in (22), respectively. Hence, we can rewrite the right-hand side of equality (22) as follows

$$\begin{aligned} &\frac{d}{\mathrm{d}r}\Big(\frac{1}{r^{n+2}}\int_{B_r}\Big(|\nabla u_{x_0}|^2 + 2\,u_{x_0}\Big)\,\mathrm{d}x\Big) \\ &\quad = \frac{2}{r^{n+2}}\int_{B_r}(\Delta u_{x_0} - 1)\Big(2\,\frac{u_{x_0}}{r} - \Big\langle\nabla u_{x_0}, \frac{x}{r}\Big\rangle\Big)\,\mathrm{d}x \\ &\qquad + \frac{2}{r^{n+2}}\int_{\partial B_r}\Big\langle\nabla u_{x_0}, \frac{x}{r}\Big\rangle^2\,\mathrm{d}\mathcal{H}^{n-1} - \frac{4}{r^{n+2}}\int_{\partial B_r}\frac{u_{x_0}}{r}\,\Big\langle\nabla u_{x_0}, \frac{x}{r}\Big\rangle\mathrm{d}\mathcal{H}^{n-1}\,. \end{aligned} \tag{23}$$

We consider next the boundary term in the expression of Φ_u. By scaling and a direct calculation, we get

$$\begin{aligned} \frac{d}{\mathrm{d}r}\Big(\frac{2}{r^{n+3}}\int_{\partial B_r}u_{x_0}^2\,\mathrm{d}\mathcal{H}^{n-1}\Big) &\overset{x=ry}{=} 2\int_{\partial B_1}\frac{d}{\mathrm{d}r}\Big(\frac{u_{x_0}(ry)}{r^2}\Big)^2\,\mathrm{d}\mathcal{H}^{n-1} \\ &= 4\int_{\partial B_1}\frac{u_{x_0}(ry)}{r^4}\Big(\big\langle\nabla u_{x_0}(ry), y\big\rangle - 2\frac{u_{x_0}(ry)}{r}\Big)\,\mathrm{d}\mathcal{H}^{n-1} \\ &\overset{x=ry}{=} \frac{4}{r^{n+2}}\int_{\partial B_r}\frac{u_{x_0}}{r}\,\Big\langle\nabla u_{x_0}, \frac{x}{r}\Big\rangle\,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad - \frac{8}{r^{n+2}}\int_{\partial B_r}\frac{u_{x_0}^2}{r^2}\,\mathrm{d}\mathcal{H}^{n-1}\,. \end{aligned} \tag{24}$$

Then, by combining together Eqs. (23) and (24) and recalling Eq. (9), we obtain

$$\begin{aligned}\Phi_u'(x_0,r) &= \frac{2}{r^{n+2}}\int_{B_r} f_{x_0}\left(2\,\frac{u_{x_0}}{r}-\langle\nabla u_{x_0},\frac{x}{r}\rangle\right)\mathrm{d}x\\ &\quad+\frac{2}{r^{n+2}}\int_{\partial B_r}\left(\left\langle\nabla u_{x_0},\frac{x}{r}\right\rangle-2\,\frac{u_{x_0}}{r}\right)^2\mathrm{d}\mathcal{H}^{n-1}\\ &=\frac{2}{r^{n+2}}\int_{B_r\setminus\Lambda_{u_{x_0}}} f_{x_0}\left(2\,\frac{u_{x_0}}{r}-\left\langle\nabla u_{x_0},\frac{x}{r}\right\rangle\right)\mathrm{d}x\\ &\quad+\frac{2}{r^{n+2}}\int_{\partial B_r}\left(\left\langle\nabla u_{x_0},\frac{x}{r}\right\rangle-2\,\frac{u_{x_0}}{r}\right)^2\mathrm{d}\mathcal{H}^{n-1},\end{aligned}$$

where in the last equality we used the unilateral obstacle condition to deduce that $\Lambda_{u_{x_0}}\subseteq\{\nabla u_{x_0}=\underline{0}\}$. Therefore, by the growth of u and ∇u from x_0 in (20) we obtain

$$\Phi_u'(x_0,r)\geq-\frac{C}{r^{n+1}}\int_{B_r\setminus\Lambda_{u_{x_0}}}|f_{x_0}|\,\mathrm{d}x+\frac{2}{r^{n+2}}\int_{\partial B_r}\left(\left\langle\nabla u_{x_0},\frac{x}{r}\right\rangle-2\,\frac{u_{x_0}}{r}\right)^2\mathrm{d}\mathcal{H}^{n-1}. \tag{25}$$

Next note that by (H1), (H3), and by the very definition of f_{x_0} in (9) it follows that

$$\frac{1}{r^{n+1}}\int_{B_r\setminus\Lambda_{u_{x_0}}}|f_{x_0}|\,\mathrm{d}x\leq\frac{\omega_f(r)}{c_0\,r}+\frac{C}{r^{n\left(1+\frac{1}{p}\right)}}\int_{B_r}|\nabla^2u_{x_0}|\,\mathrm{d}x+\frac{C}{r^n}\int_{B_r}|\mathrm{div}\,\mathbb{C}_{x_0}|\,\mathrm{d}x\,. \tag{26}$$

By (21) we estimate the second addend on the right-hand side of the last inequality as follows

$$\frac{1}{r^{n\left(1+\frac{1}{p}\right)}}\int_{B_r}|\nabla^2u_{x_0}|\,\mathrm{d}x\leq\frac{C}{r^{n\left(1+\frac{1}{p}\right)}}\|\nabla^2u_{x_0}\|_{L^p(B_r,\mathbb{R}^{n\times n})}(\omega_nr^n)^{1-\frac{1}{p}}\leq C\,r^{-\frac{n}{p}}\,, \tag{27}$$

and by Hölder inequality we get for the third addend

$$\frac{1}{r^n}\int_{B_r}|\mathrm{div}\,\mathbb{C}_{x_0}|\,\mathrm{d}x\leq\frac{1}{r^n}\|\mathrm{div}\,\mathbb{C}_{x_0}\|_{L^p(B_r,\mathbb{R}^n)}\,(\omega_nr^n)^{1-\frac{1}{p}}\leq C\,r^{-\frac{n}{p}}\,. \tag{28}$$

Therefore, we conclude from (25)–(28)

$$\Phi_u'(x_0,r)\geq-C\,\frac{\omega(r)}{r}+\frac{2}{r^{n+2}}\int_{\partial B_r}\left(\left\langle\nabla u_{x_0},\frac{x}{r}\right\rangle-2\,\frac{u_{x_0}}{r}\right)^2\mathrm{d}\mathcal{H}^{n-1},$$

where $\omega(r):=\omega_f(r)+r^{1-\frac{n}{p}}$. □

Remark 4.1 Recalling that f is Dini continuous by (H3), the modulus of continuity ω provided by Theorem 2.2 is in turn Dini continuous as $p>n$.

Remark 4.2 More generally, the argument in Theorem 2.2 works for solutions to second-order elliptic PDEs in nondivergence form of the type

$$a_{ij}(x)\,u_{ij} + b_i(x)\,u_i + c(x)\,u = f(x)\chi_{\{u>0\}}\,,$$

the only difference with the statement of Theorem 2.2 being that in this framework $\omega(r) := \omega_f(r) + r^{1-\frac{n}{p}} + r^2 \sup_{B_r} c$ (cf. [12, Appendix]).

5 Monneau's Quasi-Monotonicity Formula: Proof of Theorem 2.3

In this section, we prove Monneau's quasi-monotonicity formula for the L^2 distance on the boundary of $u_{\mathbb{L}(x_0)}$ from any 2-homogeneous solution to Eq. (14). As for Theorem 2.2, the proof of Theorem 2.3 uses equality (9) and Proposition 3.3.

Proof of Theorem 2.3 For the sake of notational simplicity, we write u_{x_0} rather than $u_{\mathbb{L}(x_0)}$ (as in the proof of Theorem 2.2).

Set $w := u_{x_0} - v$, and then arguing as in (24) and by applying the divergence theorem, we get

$$\begin{aligned} \frac{d}{\mathrm{d}r}\left(\frac{1}{r^{n+3}}\int_{\partial B_r} w^2\,\mathrm{d}\mathcal{H}^{n-1}\right) &= \frac{2}{r^{n+3}}\int_{\partial B_r} w\left(\left\langle \nabla w, \frac{x}{r}\right\rangle - 2\frac{w}{r}\right)\mathrm{d}\mathcal{H}^{n-1} \\ &= \frac{2}{r^{n+3}}\int_{B_r} \mathrm{div}\,(w\nabla w)\,\mathrm{d}x - \frac{4}{r^{n+4}}\int_{\partial B_r} w^2\,\mathrm{d}\mathcal{H}^{n-1} \\ &= \frac{2}{r^{n+3}}\int_{B_r} w\Delta w\,\mathrm{d}x + \frac{2}{r^{n+3}}\int_{B_r} |\nabla w|^2\,\mathrm{d}x \\ &\quad - \frac{4}{r^{n+4}}\int_{\partial B_r} w^2\,\mathrm{d}\mathcal{H}^{n-1}\,. \end{aligned} \tag{29}$$

For what the first term on the right-hand side of (29) is concerned, recall that $u \in W^{2,p}_{loc}(\Omega)$; thus, by locality of the weak derivatives we have that $\mathcal{L}^n\big(\{\nabla u_{x_0} = \underline{0}\} \setminus \{\nabla^2 u_{x_0} = \underline{0}\}\big) = 0$. Being $\Lambda_{u_{x_0}} \subseteq \{\nabla u_{x_0} = \underline{0}\}$, we conclude that $\Delta u_{x_0} = 0$ $\mathcal{L}^n$-a.e. in $\Lambda_{u_{x_0}}$, and therefore, in view of (9) we infer

$$w\Delta w = (u_{x_0} - v)(\Delta u_{x_0} - 1) = \begin{cases} (u_{x_0} - v)\, f_{x_0} & \mathcal{L}^n\text{-a.e. } \Omega \setminus \Lambda_{u_{x_0}} \\ v & \mathcal{L}^n\text{-a.e. } \Lambda_{u_{x_0}}\,. \end{cases}$$

Instead, estimating the second and third terms on the right-hand side of (29) thanks to (14) yields

$$\begin{aligned} &\frac{1}{r^{n+3}}\int_{B_r} |\nabla w|^2\,\mathrm{d}x - \frac{2}{r^{n+4}}\int_{\partial B_r} w^2\,\mathrm{d}\mathcal{H}^{n-1} \\ &= \frac{1}{r^{n+3}}\int_{B_r} \left(|\nabla u_{x_0}|^2 + |\nabla v|^2\right)\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
& - \frac{2}{r^{n+3}} \int_{B_r} \operatorname{div}\left(u_{x_0} \nabla v\right) \mathrm{d}x + \frac{2}{r^{n+3}} \int_{B_r} u_{x_0} \, \mathrm{d}x - \frac{2}{r^{n+4}} \int_{\partial B_r} w^2 \, \mathrm{d}\mathcal{H}^{n-1} \\
& \overset{(15)}{=} \frac{1}{r}\left(\Phi_{u_{x_0}}(x_0, r) - \Phi_v(x_0, r)\right) - \frac{2}{r^{n+4}} \int_{\partial B_r} u_{x_0} \left(\left\langle \nabla v, \frac{x}{r} \right\rangle - 2v\right) \mathrm{d}x \\
& \overset{(13)}{=} \frac{1}{r}\left(\Phi_{u_{x_0}}(x_0, r) - \Phi_{u_{x_0}}(x_0, 0^+)\right).
\end{aligned}
$$

Then, (29) rewrites as

$$
\begin{aligned}
\frac{d}{\mathrm{d}r}\left(\frac{1}{r^{n+3}} \int_{\partial B_r} w^2 \, \mathrm{d}\mathcal{H}^{n-1}\right) = {} & \frac{2}{r}\left(\Phi_u(x_0, r) - \Phi_u(x_0, 0^+)\right) \\
& + \frac{2}{r^{n+3}} \int_{B_r \setminus \Lambda_{u_{x_0}}} (u_{x_0} - v) f_{x_0} \, \mathrm{d}x \\
& + \frac{2}{r^{n+3}} \int_{B_r \cap \Lambda_{u_{x_0}}} v \, \mathrm{d}x.
\end{aligned}
$$

Inequality (12) in Theorem 2.2, the growth of the solution u from free boundary points in (20), the 2-homogeneity and positivity of v yield the conclusion (cf. (26)–(28)):

$$
\begin{aligned}
& \frac{d}{\mathrm{d}r}\left(\frac{1}{r^{n+3}} \int_{\partial B_r} w^2 \, \mathrm{d}\mathcal{H}^{n-1}\right) \\
& \geq -\frac{C}{r} \int_0^r \frac{\omega(t)}{t} \, \mathrm{d}t - \frac{C}{r^{n+1}} \int_{B_r \setminus \Lambda_{u_{x_0}}} |f_{x_0}| \, \mathrm{d}x = -\frac{C}{r} \int_0^r \frac{\omega(t)}{t} \, \mathrm{d}t
\end{aligned}
$$

for some $C = C(n, p, \Lambda, c_0, K, \|f\|_{L^\infty}, \|\mathbb{A}\|_{W^{1,p}}) > 0$. □

6 Free Boundary Analysis: Proof of Theorem 2.1

Weiss' and Monneau's quasi-monotonicity formulas proved in Sects. 4 and 5, respectively, are important tools to deduce regularity of free boundaries for classical obstacle problems for variational energies, both in the quadratic and in the nonlinear setting (see [7–12]).

In this section, we improve upon [8, Theorems 4.12 and 4.14] in the quadratic case weakening the regularity of the coefficients of the relevant energies. This is possible thanks to the above-mentioned new quasi-monotonicity formulas.

In the ensuing proof, we will highlight only the substantial changes, since the arguments are essentially those given in [8,9]. In particular, we remark again that in the quadratic case the main differences concern the quasi-monotonicity formulas established for the quantity Φ_u rather than for the natural candidate related to $\mathscr{E}$.

We follow the variational approach by Weiss [11] and by Monneau [12] for the free boundary analysis in Theorem 2.1.

Proof of Theorem 2.1 First, recall that we may establish the conclusions for the function $u = w - \psi$ introduced in Sect. 3. Given this, the only minor change to be done to the arguments in [8, Section 4] is related to the freezing of the energy, where the regularity of the coefficients plays a substantial role. More precisely, in the current framework for all $v \in W^{1,2}(B_1)$ we have

$$\left| \int_{B_1} \big(\mathbb{A}(rx)\nabla v, \nabla v\rangle + 2f(rx)v \big)\, \mathrm{d}x - \int_{B_1} \big(|\nabla v|^2 + 2v \big) \mathrm{d}x \right| \leq (r^{1-\frac{n}{p}} + \omega_f(r)) \int_{B_1} \big(|\nabla v|^2 + 2v \big) \mathrm{d}x.$$

We then describe shortly the route to the conclusion. To begin with, recall that the quasi-monotonicity formulas established in [8, Section 3] are to be substituted by those in Sects. 4 and 5. Then, the 2-homogeneity of blowup limits in [8, Proposition 4.2] now follows from Theorem 2.2. The quadratic growth of solutions from free boundary points contained in [8, Lemma 4.3] that implies nondegeneracy of blowup limits is contained in Lemma 3.1. The classification of blowup limits is performed exactly as in [8, Proposition 4.5]. The conclusions of [8, Lemma 4.8], a result instrumental for the uniqueness of blowup limits at regular points, can be obtained with essentially no difference. The proofs of [8, Propositions 4.10, 4.11, Theorems 4.12, 4.14] remain unchanged. The theses then follow at once. □

7 Conclusions

We have established quasi-monotonicity formulas of Weiss' and Monneau's type for quadratic energies having matrix of coefficients in $W^{1,p}$, $p > n$, and we have given an application to the corresponding free boundary analysis for the related classical obstacle problem.

As pointed out in Sect. 1, concerning the quasi-monotonicity formulas the main difference with the existing literature is related to the monotone quantity itself. Indeed, rather than considering the natural quadratic energy $\mathscr{E}$ associated with the obstacle problem under study, we may consider the classical Dirichlet energy thanks to a normalization. In doing this, we have been inspired by Monneau [12, Section 6]. The advantage of this formulation is that the matrix field $\mathbb{A}$ is not differentiated in deriving the quasi-monotonicity formulas contrary to [8,9]. Our additional insight is elementary, but crucial: we further exploit the quadratic growth of solutions from free boundary points in Proposition 3.3 to establish quasi-monotonicity. In view of all of this, we are able to weaken the required regularity assumptions on the matrix field $\mathbb{A}$ (cf. (H1)).

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