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Hadamard products of varieties

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A chi mi ha aiutato
e fatto compagnia.

Contents

Introduction

The notion of Hadamard product has been introduced about a century ago for matrices, in the study of combinatorial generating functions. Besides having various applications in both statistics and physics ([L2, LN, LNP]), the Hadamard product of matrices is an interesting aspect of linear algebra and it is widely used in matrix analysis ([HM, L1, LT]). In recent papers [CMS, CTY] the authors define the Hadamard product between projective varieties $X, Y \subset \mathbb{P}^n$, denoted $X \star Y$, as the closure of the image of the rational map

$$X \times Y \dashrightarrow \mathbb{P}^n, \quad ([a_0 : \cdots : a_n], [b_0 : \cdots : b_n]) \mapsto [a_0 b_0 : a_1 b_1 : \ldots : a_n b_n].$$

This is used to describe the algebraic variety associated to the restricted Boltzmann machine, which is the undirected graphical model for binary random variables specified by the bipartite graph $K_{r,n}$.

For decades, the Hadamard product did not inspire too much investigators in projective geometry. The reason is probably due to the (immediate) observation that the product is not invariant under a charge of coordinates. Indeed, the result of the product $X \star Y$ strongly depends on the position of the two varieties with respect to the coordinate hyperplanes. Yet, recently, it turned out that applications of the Hadamard product to computer vision ([KZ, KOLKHZ]), tropical geometry ([BCK, FOW, MS]) and algebraic statistics ([CMS, CTY]) suggested that a projective analysis of its properties could be relevant in applied algebraic geometry.

The study of the properties of the Hadamard product of varieties is at the beginning. In this thesis we collect our work on this topic ([BCFL1, BCFL2, CCFL, BJC]). It could be viewed as a continuation of [BCK] which is the first paper that addresses the Hadamard product of varieties from a mathematical point of view.

In [BCK] the authors studied the Hadamard product between linear spaces when the linear spaces are generic. In Chapter 2 we still study the case of both linear spaces and points in $\mathbb{P}^2$ and $\mathbb{P}^3$, but dropping the hypothesis that these varieties be generic. The condition to be not necessarly generic means that the points can have many zero coordinates and that the linear spaces can intersect the coordinate hyperplanes in dimension greater than the expected one for the generic case. This fact forces us to study all possible pathological behaviours that can happen in the Hadamard product of such varieties. For the case of $\mathbb{P}^2$, we give a complete classification of all possible cases of the Hadamard product between a point and a line. We also study all possible cases of incidence of $Q \star L$ and $Q' \star L$ for two distinct points Q, Q' and a line L. We prove that, under suitable conditions, the Hadamard product of a finite set of collinear points X and another finite set of collinear points X' is a complete intersection. This is never the case in $\mathbb{P}^3$. However, if X and X' are general enough, we can prove that $X \star X'$ is a grid on a quadric and we are able to compute its Hilbert function when $|X| = |X'|$.

In Chapter 3 first we study the Hilbert function of some Hadamard products of sets of points or lines, both in $\mathbb{P}^3$ and in $\mathbb{P}^n$ for any n. Then we study the Hadamard product of two generic linear subspaces on $\mathbb{P}^N$, L_r and L_s, of dimensions r and s respectively, with $N = (r+1)(s+1) - 1$ and we show that $L_r \star L_s$ is projectively equivalent to the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^N$ and, as a corollary, we get that the Hilbert function of $L_r \star L_s$ is the product of the Hilbert functions of L_r and L_s.

One of the most important open questions is to find the dimension and the degree of the Hadamard product of varieties. The first result for this problem is in [BCK]. In that paper the authors give the dimension and the degree of the Hadamard product of a finite number of generic linear subspaces when the ambient space is big enough. In [FOW] the authors give an expected formula for the dimension and the degree of the Hadamard product of generic varieties. From Chapter 3 one can deduce that these formulas are true for the Hadamard product of a finite set of generic collinear points and another finite set of generic collinear points or a finite set of collinear points and a generic line. In Chapter 4 we extend the result of [BCK] and we prove that the expected formulas hold for the Hadamard product of generic degenerate subvarieties if the ambient space is large enough. Besides dealing with not necessary linear subvarieties, the ambient space could be smaller than in [BCK], thus extending the result of [BCK] and partially answering [FOW, Question 1.1]. In particular we prove that, if X and Y are two degenerate subvarieties of $\mathbb{P}^n$ contained in generic linear subspaces, then the Hadamard product $X \star Y$ and the product variety $X \times Y$ are projectively equivalent as subvarieties of $\mathbb{P}^n$, if n is bigger than a given N. As a consequence we obtain that the dimension of $X \star Y$ is the sum of the dimensions, the degree is the product of the degrees multiplied by a binomial coefficient depending on the dimensions and the Hilbert function is the product of the Hilbert functions. Then we extend these results to a finite number of degenerate subvarieties of $\mathbb{P}^n$. We also prove that, if the subvarieties are smooth, then their Hadamard product is non-singular. If n is (slightly) smaller than N we still have the dimension and the degree formulas, but the formula for the Hilbert function no longer holds. In this situation singularities may arise even when the varieties are smooth: on one hand we give a numerical sufficient condition for smoothness, on the other hand we give a sufficient numerical condition for the Hadamard product to be singular and, in this case, we give a lower bound for the dimension of the singular locus.

In [BCK] and in [CCGV] the authors want also to connect the Hadamard product to star configurations, defining a Hadamard star configuration. In Chapter 5 we want to extend those results and find some sufficient conditions, using star configurations apolar to a given form, to make a generic star configuration be a Hadamard star configuration.

Now we describe more precisely what we do in each chapter.

In Chapter 1 we give some preliminary notions on Hilbert functions and Hadamard products.

In Chapter 2, following [BCFL1], we address the Hadamard product of not necessarily generic linear varieties.

In $\mathbb{P}^2$ we obtain a complete description of the possible outcomes. In particular, in the case of a finite set of collinear points X and another finite set of collinear points X' disjoint from X, we get conditions for $X \star X'$ to be either a collinear finite set of points or a grid of $|X||X'|$ points.

In $\mathbb{P}^3$, under suitable conditions (which we prove to be generic), we show that $X \star X'$ consists of $|X||X'|$ points on the two different rulings of a non-degenerate quadric and we compute its Hilbert function in the case $|X| = |X'|$.

In Chapter 3, following [BCFL2], we address the Hadamard product of not necessarily generic linear varieties, looking in particular at its Hilbert function. We find that the Hilbert function of the Hadamard product $X \star Y$ of two varieties, with $\dim(X), \dim(Y) \leq 1$, is the product of the Hilbert functions of the original varieties X and Y. Moreover, the same result is obtained for generic linear varieties X and Y as a consequence of our showing that their Hadamard product is projectively equivalent to a Segre embedding.

In Chapter 4, following [CCFL], we continue the investigation on the connections between the Hadamard product and the Veronese-Segre variety, considering $X_1, \ldots, X_\ell$ degenerate subvarieties of $\mathbb{P}^n$ contained in generic linear subspaces of dimensions $h_1, \ldots, h_\ell$, and setting $N = (h_1 + 1) \cdots (h_\ell + 1) - 1$.

When $n \geq N$ we prove that the Hadamard product $X_1 \star \cdots \star X_\ell$ and the product variety $X_1 \times \cdots \times X_\ell$ are projectively equivalent as subvarieties of $\mathbb{P}^n$. In particular, we determine the dimension, the degree and the Hilbert function of $X_1 \star \cdots \star X_\ell$ in terms of the same invariants of $X_1, \ldots, X_\ell$. Moreover, if the varieties X_i are smooth, their Hadamard product is smooth as well.

When $n < N$ we restrict our attention to generic parameterized varieties. We show that the dimension and the degree formulas still hold, but the Hilbert function is no longer the product of the Hilbert functions. In this case the Hadamard product may be singular even if the varieties X_i are smooth. We give numerical conditions both for smoothness and for the existence of singular points.

In Chapter 5, following [BJC], we introduce a star configuration more general than the Hadamard star configurations defined in [CCGV]. Any star configuration constructed by our approach is called a weak Hadamard star configuration. We classify weak Hadamard star configurations and Hadamard star configurations. Finally we connect the Hadamard star configurations to the star configurations apolar to a given form.

Chapter 1

Preliminaries

In this chapter we introduce the definitions and the notations that we will use throughout.

1.1 Hilbert functions

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. We denote by $\mathbb{P}^n$ the projective space over $\mathbb{K}$ of dimension n.

Let R be the graded ring $\mathbb{K}[x_0, \ldots, x_n]$ and I a homogeneous ideal of R, then R_t/I_t is a vector space over $\mathbb{K}$ for all $t \in \mathbb{N}$. We denote the dimension of this vector space with $HF(R/I, t)$, and we call it the *Hilbert function* of R/I. The Hilbert function of a projective variety X is a combinatorial tool that, from a geometric point of view, studies, for each degree t, the hypersurfaces of degree t passing through X. When X is a finite set of points we can regard it as observed experimental data and use its Hilbert function to determine the number of equations needed to describe the given data. The subject of Hilbert function of projective varieties is widely studied in literature.

Let X be a subvariety of $\mathbb{P}^n$, the *Hilbert function of* X is $HF_X(t) = HF(R/I_X, t)$, where I_X is the homogeneous ideal associated to X. For all $t \gg 0$ we have that $HF_X(t) = HP_X(t)$, where HP_X is a polynomial, called the *Hilbert polynomial of* X.

If X is a set of points (simple or fat points), the Hilbert polynomial of X is constant. The smallest degree t where the Hilbert function becomes constant is known as the *regularity index*. More generally, let X be a projective variety, then the *regularity index of* X is:

$$\tau_X = \min_{\bar{t}} \{HF_X(t) = HP_X(t), \forall t \geq \bar{t}\}.$$

The regularity index is very difficult to compute, so one classical problem is to give an upper bound of τ_X (for set of points see [S, CTV, T, FaL, CFL, NT]).

Another type of regularity known in the literature is the Castelnuovo-Mumford regularity which can be defined by using a minimal free resolution of the ideal I_X of the variety X:

$$0 \to F_d \to \cdots \to F_0 \to I_X \to 0$$

with $F_i = \oplus R(-\alpha_{i,j})^{\beta_{i,j}}$, then the *Castelnuovo-Mumford regularity* of I_X is defined to be

$$reg(I_X) = \max\{\alpha_{i,j} - i\}.$$

We similarly define $reg(R/I_X)$ starting from a minimal free resolution of R/I_X:

$$0 \to F_d \to \cdots \to F_0 \to R \to R/I_X \to 0.$$

When X is a set of points (simple or fat points) we have that $reg(I_X) = \tau_X + 1$.

1.2 Hadamard products of varieties

In this section we will introduce the Hadamard product of varieties.

The Hadamard product of matrices is well known in linear algebra. Let A, B be two matrices of the same size, The Hadamard product $C = A \star B$ is the matrix whose entries are the products of the entries of A and B, i.e. $c_{ij} = a_{ij}b_{ij}$. This product has nice properties in matrix analysis ([HM, L1, LT, KT]) and has applications in both statistics and physics ([L2, LN, LNP]).

Recently, in the papers [CMS, CTY], the authors define a *Hadamard product* between projective varieties. Let X and Y be two projective subvarieties of $\mathbb{P}^n$. Then consider the usual Segre product

$$X \times Y \to \mathbb{P}^N$$

$$([a_0 : \cdots : a_n], [b_0 : \cdots : b_n]) \mapsto [a_0b_0 : a_0b_1 : \cdots : a_nb_n]$$

and denote with z_{ij} the coordinates of $\mathbb{P}^N$. Let $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$ be the projection from the linear space defined by the equations $z_{ii} = 0$ for all $i = 0, \ldots, n$. The Hadamard product of X and Y is $X \star Y = \overline{\pi(X \times Y)}$. This is equivalent to

$$X \star Y = \overline{\{P \star Q | P \in X, Q \in Y\}}$$

where $P \star Q$ is the Hadamard product of matrices (vectors) between the correspondent vectors. If $P \star Q$ is equal to $[0 : \cdots : 0]$, it does not define a point in $\mathbb{P}^n$ and so it can happen that $X \star Y$ is the empty set also when X and Y are not empty.

Note that, by good properties of the projection π, $dim(X \star Y) \leq dim(X \times Y) = dim(X) + dim(Y)$ and, if X and Y are irreducible, then $X \times Y$ is irreducible and so is $X \star Y$.

If Y is a point $[b_0 : \cdots : b_n]$, with all not zero coordinates, consider the projective map

$$\sigma : \mathbb{P}^n \to \mathbb{P}^n$$

$$[a_0 : \cdots : a_n] \mapsto [a_0 b_0 : a_1 b_1 : \cdots : a_n b_n].$$

It is clear that $\sigma(X) = X \star Y$, thus $X \star Y$ is projectively equivalent to X, and so $dim(X \star Y) = dim(X) + dim(Y)$. Note that if X and Y are generic "small" linear subspaces of dimensions greater than or equal to 1, $X \star Y$ is not linear ([BCK, Theorem 6.8]]).

Another more algebraic point of view to see $X \star Y$ is to view I_X as an ideal of $\mathbb{K}[x_0, \ldots, x_n]$, I_Y as an ideal of $\mathbb{K}[y_0, \ldots, y_n]$: now $I_{X \star Y}$ is the ideal obtained by the elimination of the $2n + 2$ variables $x_0, ..., x_n, y_0, \ldots, y_n$ from the ideal

$$I_X + I_Y + < z_0 - x_0 y_0, \ldots, z_n - x_n y_n > \subseteq \mathbb{K}[x_0, \ldots, x_n, y_0, \ldots, y_n, z_0, \ldots, z_n].$$

Given a positive integer r and a subvariety X of $\mathbb{P}^n$, the *r-th Hadamard power of* X is $X^{\star r} = X \star X^{\star r-1}$, where $X^{\star 0} = [1 : \cdots : 1]$. Note that $dim(X^{\star r}) \leq r \cdot dim(X)$.

For a finite set of points it is useful to define another type of power, the r-th square-free Hadamard power. Let X be a set of points of $\mathbb{P}^n$, then the *r-th square-free Hadamard power* is

$$X^{\underline{\star} r} = \{P_1 \star \cdots \star P_r | P_i \in X \text{ and } P_i \neq P_j, \forall i \neq j\}.$$

If X consists of generic enough points in a generic line ℓ of $\mathbb{P}^n$ then $X^{\underline{\star} r}$ is a star configuration in $\ell^{\star r}$ ([BCK, Theorem 4.7]). In [CCGV] the authors define sets which are useful to define a special star configuration connected to the Hadamard product. They say that a set $\mathcal{L} = \{L_1, \ldots, L_r\}$ of linear forms in $\mathbb{K}[x_0, \ldots, x_n]$ is a *Hadamard set* if there exist a linear form L and $P_1, \ldots, P_r$ points of $\mathbb{P}^n$ such that $V(L_i) = P_i \star V(L)$. They say that $\mathcal{L}$ is a *strong Hadamard set* if we also have that $P_i \in V(L)$. Furthermore in [CCGV], the authors define the *Hadamard star configuration*, a special star configuration, where the set of linear forms is a strong Hadamard set. In [BJC] we define a *weak Hadamard star configuration*, when the set of linear forms is only a Hadamard set.

Chapter 2

Hadamard products of linear varieties

This chapter is inspired by the paper [BCFL1] in collaboration with C. Bocci, G. Fatabbi, A. Lorenzini.

The goal of this chapter is to be a natural continuation of the paper [BCK]. Here we still study the case of both linear spaces and zero-dimensional schemes in $\mathbb{P}^2$ and $\mathbb{P}^3$, dropping the hypothesis that these varieties be generic. The condition to be not generic means that the points can have many zero coordinates and that the linear spaces can intersect the coordinate hyperplanes in dimension greater than the expected one for the generic case. This fact forces us to study all possible pathological behaviors that then can happen in the Hadamard product of such varieties. For the case of $\mathbb{P}^2$, Theorem 2.2.4 gives a complete classification of all possible cases of the Hadamard product between a point and a line, while Theorem 2.2.6 studies all possible cases of incidence of $Q \star L$ and $Q' \star L$ for two distinct points Q, Q' and a line L. These results lead to Theorem 2.2.13 where we prove that, under suitable conditions, the Hadamard product of two sets of collinear points X and Y is a complete intersection. Turning to the case of $\mathbb{P}^3$, we notice that the Hadamard product of two sets of collinear points X and Y is not, in general, a complete intersection in $\mathbb{P}^3$. However if X and Y are general enough, we can prove that $X \star Y$ is a grid on a quadric (Theorem 2.3.2) and we are able to compute its Hilbert function when $|X| = |Y|$ (Theorem 2.3.8). In this case we also prove that $X \star Y$ is never a complete intersection in $\mathbb{P}^3$ (assuming $|X| = |Y| > 1$).

2.1 General results in $\mathbb{P}^n$

As in [BCK], $H_i \subset \mathbb{P}^n$ denotes the hyperplane defined by $x_i = 0$ and

$$\Delta_i = \bigcup_{0 \leq j_1 < \cdots < j_{n-i} \leq n} H_{j_1} \cap \cdots \cap H_{j_{n-i}}.$$

Recall that Δ_i can be viewed as the $i-$dimensional variety of points having at most $i+1$ non-zero coordinate, equivalently at least $n-i$ zero coordinates.

We set Δ_{-1} to be the set $\{(0,\dots,0)\}$ and we write $P \star Q \in \Delta_{-1}$ if it is not defined.

It easily follows from [BCK, Lemma 3.2] that:

Theorem 2.1.1.

(1) *Let P, Q, A be points of $\mathbb{P}^n$ with $A \notin \Delta_{n-1}$, then $P \star A = Q \star A$ if and only if $P = Q$.*

(2) *Let $H \subset \mathbb{P}^n$ be a hyperplane defined by $a_0 x_0 + \cdots + a_n x_n = 0$ and such that $H \cap \Delta_0 = \emptyset$ and let P, Q be points not in Δ_{n-1} with $P = [p_0 : \cdots : p_n]$. Then $P \star H : \{\frac{a_0}{p_0} x_0 + \cdots + \frac{a_n}{p_n} x_n = 0\}$ and $P \star H = Q \star H$ if and only if $P = Q$.*

(3) *Let $P \notin \Delta_{n-1}$, let H, K be two hyperplanes such that $H \cap \Delta_0 = \emptyset = K \cap \Delta_0$. Then $P \star H = P \star K$ if and only if $H = K$.*

Theorem 2.1.2. *Let $P \in \mathbb{P}^n \setminus \Delta_{n-1}$ and let H,K be two hyperplanes such that $H \cap \Delta_0 = \emptyset = K \cap \Delta_0$. Then $P \star (H \cap K) = (P \star H) \cap (P \star K)$.*

Proof. We may assume $H \neq K$, for otherwise the result is trivial.

For all $Q \in H \cap K$, we have $P \star Q \in P \star (H \cap K)$ and $P \star Q \in (P \star H) \cap (P \star K)$, hence $P \star (H \cap K) \subseteq (P \star H) \cap (P \star K)$, being the right-hand side a closed set. To see the other inclusion, by (3) of Theorem 2.1.1, we have $P \star H \neq P \star K$, then $(P \star H) \cap (P \star K)$ is a linear subspace of dimension $n-2$. Since $P \notin \Delta_{n-1}$, it follows from [BCK, Lemma 3.1] that $P \star (H \cap K)$ is a linear subspace of dimension $n-2$. Therefore $P \star (H \cap K)$ is the intersection of the hyperplanes $P \star H$ and $P \star K$. □

Corollary 2.1.3. *Let $P \in \mathbb{P}^3 \setminus \Delta_2$ and let H, K be two planes such that $L = H \cap K$ and $H \cap \Delta_0 = \emptyset = K \cap \Delta_0$. Then $P \star L$ is the intersection of the two planes $P \star H$ and $P \star K$.*

Now we look at the products of two hyperplanes H and K.

Remark 2.1.4. If H and K are coordinate hyperplanes respectively defined by $x_i = 0$ and $x_j = 0$, then $H \star K$ is the hyperplane defined by $x_i = 0$, when $i = j$ and the linear subspace defined by $x_i = x_j = 0$, when $i \neq j$.

Theorem 2.1.5. *Let H, K be the hyperplanes of $\mathbb{P}^n$ defined by $a_i x_i + a_j x_j = 0$ and $b_i x_i + b_j x_j = 0$ respectively, with $i \neq j$ in $\{0, \ldots, n\}$ and either $a_i a_j \neq 0$ or $b_i b_j \neq 0$. Then $H \star K$ is the hyperplane defined by $a_i b_i x_i - a_j b_j x_j = 0$.*

Proof. For simplicity of notation we may assume $i = 0$ and $j = 1$, the other case being similar.

We distinguish the following two cases:

1. If $a_1 = 0$ and $b_0 b_1 \neq 0$, then $H : \{x_0 = 0\}$. Let $P = [0 : p_1 : p_2 : \cdots : p_n] \in H$ and let $Q \in K$, then $Q = [-\frac{b_1}{b_0} q_1 : q_1 : q_2 : \cdots : q_n]$, and so $P \star Q = [0 : p_1 q_1 : \cdots : p_n q_n]$, i.e. $H \star K : \{x_0 = 0\}$.
2. If $a_0 a_1 \neq 0$ and $b_0 b_1 \neq 0$, then $P = [-\frac{a_1}{a_0} p_1 : p_1 : p_2 \cdots : p_n] \in H$ and $Q = [-\frac{b_1}{b_0} q_1 : q_1 : q_2 : \cdots : q_n] \in K$, thus $P \star Q = [\frac{a_1 b_1}{a_0 b_0} p_1 q_1 : p_1 q_1 : p_2 q_2 : \cdots : p_n q_n]$. We claim that $H \star K$ is the hyperplane $L : \{a_0 b_0 x_0 - a_1 b_1 x_1 = 0\}$. It is obvious that $H \star K \subseteq L$. To see the other inclusion let $S = [s_0 : \cdots : s_n] \in L$ and let $P = [-\frac{a_1}{a_0} : 1 : 1 : \cdots : 1] \in H$ and consider $Q = [-\frac{a_0 s_0}{a_1} : s_1 : s_2 : \cdots : s_n]$. Clearly $P \star Q = S$ and $b_0 \frac{-(a_0 s_0)}{a_1} + b_1 s_1 = 0$, i.e. $Q \in K$.

□

Corollary 2.1.6. *Let $H \subset \mathbb{P}^n$ be the hyperplane defined by $a_i x_i + a_j x_j = 0$, with $i \neq j$ in $\{0, \ldots, n\}$ and $a_i a_j \neq 0$. Then $H \star H$ is the hyperplane defined by $a_i^2 x_i - a_j^2 x_j = 0$.*

Proof. It follows immediately from Theorem 2.1.5 with $H = K$. □

Remark 2.1.7. If $Q \in H_i$, for some $i \in \{0, \ldots, n\}$, then, for all $X \subseteq \mathbb{P}^n$, we have $Q \star X \subseteq H_i$, hence $H_i \star X \subseteq H_i$.

Example 2.1.8. Let H and K be the planes in $\mathbb{P}^3$ of equations respectively

$$H : 3x_1 - 2x_3 = 0 \quad K : -7x_1 + 4x_3 = 0.$$

Using the procedure, in `Singular`, described in Appendix A we get

```
ring r=0,(x(0..3)),dp;
ideal H=3*x(1)-2*x(3);
ideal K=-7*x(1)+4*x(3);
HPr(H,K,3);
_[1]=21*x(1)-8*x(3)
```

and, in particular,

```
HPr(H,H,3);
_[1]=9*x(1)-4*x(3)
```

2.2 Sets of collinear points in $\mathbb{P}^2$

Now we focus on $n = 2$ and on sets of at least two collinear points.

The following corollary is an application of the results of the previous section.

Corollary 2.2.1. *Let X, Y be two sets of collinear points in $\mathbb{P}^2$ such that $X \cup Y$ is contained in a line L such that $L \cap \Delta_0 \neq \emptyset$. Then $X \star Y$ are collinear points contained in the line $L \star L$.*

We need the following technical lemma, whose proof follows easily from the definitions:

Lemma 2.2.2. *Let L be a line in $\mathbb{P}^2$ such that $L \cap \Delta_0 = \emptyset$. Then*

(1) $|L \cap \Delta_1| = 3$ *and* $|L \cap \Delta_1 \cap H_i| = 1$, *for all* $i \in \{0, 1, 2\}$.

(2) *For all $P, Q \in L \cap \Delta_1$, we have that $P \star Q \notin \Delta_0$ if and only if $P = Q$.*

Theorem 2.2.3. *Let $X, Y \subseteq \mathbb{P}^2$ be two sets of points with $|X|, |Y| \geq 3$, $|X \cup Y| \geq 4$ and $X \cup Y \subseteq L$, where L is a line. Then the points of $X \star Y$ are not collinear if and only if $L \cap \Delta_0 = \emptyset$.*

Proof. The necessary part follows from Corollary 2.2.1.

To prove the sufficient part, first we note that, by (1) of Lemma 2.2.2, $|\Delta_1 \cap L| = 3$, and so $X \cup Y \nsubseteq \Delta_1$, thus there exists at least one point of X or Y not in Δ_1.

Suppose that there exists a unique point $P_1 \in X$ such that $P_1 \notin \Delta_1$, hence $X \setminus \{P_1\} \subseteq L \cap \Delta_1$.

Then either there exists $Q_1 \in Y$ with $Q_1 \neq P_1$ such that $Q_1 \notin \Delta_1$ or $Y \setminus \{P_1\} \subseteq L \cap \Delta_1$.

In the first case for each $P \in X$ and for each $Q \in Y$, with $P \neq P_1$ and $Q \neq Q_1$, we have that $P \star Q_1 \neq P_1 \star Q_1 \neq P_1 \star Q$, by (1) of Theorem 2.1.1.

Since $P_1 \star L \neq Q_1 \star L$ by (2) of Theorem 2.1.1, the points $P \star Q_1, P_1 \star Q_1, P_1 \star Q$ cannot be collinear.

In the second case we can show that the points of $X \star (Y \setminus \{P_1\})$ are not collinear. In fact, for every $Q, Q' \in Y \setminus \{P_1\}$ with $Q \neq Q'$ and for each $P \in X \setminus \{P_1\}$, we have $Q \star L \subseteq H_i$ and $Q' \star L \subseteq H_j$, with $i \neq j$, since $L \cap \Delta_0 = \emptyset$. Then $P_1 \star Q, P \star Q \in H_i$ and $P_1 \star Q', P \star Q' \in H_j$. Now, we have either $P \neq Q$ or $P \neq Q'$, whence, by (2) of Lemma 2.2.2, $P \star Q \in \Delta_0$ or $P \star Q' \in \Delta_0$. Thus, either $P_1 \star Q \neq P \star Q \neq P_1 \star Q'$

or $P_1 \star Q \neq P \star Q' \neq P_1 \star Q'$, for $P_1 \star Q, P_1 \star Q' \notin \Delta_0$. On the the other hand, $P_1 \star Q \neq P_1 \star Q'$ by (2) of Theorem 2.1.1. Since $P_1 \star Q, P_1 \star Q' \notin H_i \cap H_j \subset \Delta_0$, the points $P_1 \star Q, P \star Q, P_1 \star Q', P \star Q'$ cannot be collinear.

Now suppose that there exist at least $P_1, P_2 \in X$ and $Q_1, Q_2 \in Y$, all distinct, such that either $P_1, P_2 \notin \Delta_1$ or $Q_1, Q_2 \notin \Delta_1$.

We may assume that $P_1, P_2 \notin \Delta_1$, the other case being similar.

For every $Q, Q' \in Y$, with $Q \neq Q'$ we have that $P_1 \star Q \neq P_1 \star Q'$ and $P_2 \star Q \neq P_2 \star Q'$, by (1) of Theorem 2.1.1. Since $P_1 \star L \neq P_2 \star L$, by (1) of Theorem 2.1.1, the points $P_1 \star Q, P_1 \star Q', P_2 \star Q, P_2 \star Q'$ cannot be all collinear. □

By [BCK, Lemma 3.1] we have that if $L \subset \mathbb{P}^n$ is a linear subspace of dimension m and P is a point, then $P \star L$ is either empty or it is a linear subspace of dimension at most m. If $P \notin \Delta_{n-1}$, then $\dim(P \star L) = m$. In the following theorem we give a description of what occurs in the plane in some cases. We will use these technical results later.

Theorem 2.2.4. *Let L be the line in $\mathbb{P}^2$ defined by $a_0x_0 + a_1x_1 + a_2x_2 = 0$, set $A = [a_0 : a_1 : a_2] \in \mathbb{P}^2$ and let $Q = [q_0 : q_1 : q_2]$ be any point. Then:*

1. *if $Q \notin \Delta_1$, then $Q \star L$ is the line defined by $\frac{a_0}{q_0}x_0 + \frac{a_1}{q_1}x_1 + \frac{a_2}{q_2}x_2 = 0$;*
2. *if $Q \in \Delta_1 \setminus \Delta_0$ (and so $Q \in H_j$, for some $j \in \{0, 1, 2\}$), $A \in \Delta_i \setminus \Delta_{i-1}$, ($i \in \{0, 1, 2\}$) and $Q \star A \in \Delta_{i-1}$, then $Q \star L = H_j$;*
3. *if $Q \in \Delta_1 \setminus \Delta_0$ (and so $Q \in H_j$ for some $j \in \{0, 1, 2\}$) and we are not in the hypothesis of (2), then $Q \star L$ is the point of the intersection of the line H_j with the line defined by $a_kq_lx_k + a_lq_kx_l = 0$, with $k, l \neq j$;*
4. *if $Q \in \Delta_0$ and $Q \neq A$, then $Q \star L = Q$;*
5. *if $Q \in \Delta_0$ and $Q = A$, then $Q \star L$ is not defined.*

Proof.

1. If $A \notin \Delta_1$ the result follows immediately from (2) of Theorem 2.1.1.

   Now we suppose $A \in \Delta_1 \setminus \Delta_0$, say $A \in H_2$, then $L : a_0x_0 + a_1x_1 = 0$. Let $P \in L$, then $P = [-\frac{a_1}{a_0}p_1 : p_1 : p_2]$ and $P \star Q = [-\frac{a_1}{a_0}q_0p_1 : q_1p_1 : q_1p_2]$, whence $Q \star L$ is contained in the line $a_0q_1x_0 + a_1q_0x_1 = 0$. To see the other inclusion consider $S = [-\frac{a_1q_0}{a_0q_1}s_1 : s_1 : s_2]$. Then we have that $S = Q \star P$, where $P = [-\frac{a_1}{a_0q_1}s_1 : \frac{s_1}{q_1} : \frac{s_2}{q_2}]$. Since $a_0(-\frac{a_1}{a_0q_1}s_1) + a_1(\frac{s_1}{q_1}) = 0$, then $P \in L$.

   The proof is similar if we suppose $a_0 = 0$ or $a_1 = 0$.

   Finally suppose $A \in \Delta_0$, so that $L = H_i$, for some $i \in \{0, 1, 2\}$. In this case it is easy to see that $Q \star L$ is H_i.

(2), (3) Since $Q \in \Delta_1 \setminus \Delta_0$, without loss of generality, we may assume that $Q = [0 : q_1 : q_2]$. First consider the case $i = 2$, i.e. $A \notin \Delta_1$. In this case necessarily $Q \star A \in \Delta_1$. Let $P \in L$, then $P = [-\frac{a_1p_1+a_2p_2}{a_0} : p_1 : p_2]$ and $P \star Q = [0 : p_1q_1 : p_2q_2]$, i.e $Q \star L = H_0$, because $q_1q_2 \neq 0$.

Now suppose $i = 1$, i.e. $A \in \Delta_1 \setminus \Delta_0$. In this case we need to distinguish whether $Q \star A \in \Delta_0$ or not.

If $Q \star A \in \Delta_0$, then $a_0 \neq 0$ and we may assume $a_2 = 0$, i.e. $L : \{a_0x_0 + a_1x_1 = 0\}$.

Let $P \in L$, then $P = [-\frac{a_1p_1}{a_0} : p_1 : p_2]$ and $P \star Q = [0 : p_1q_1 : p_2q_2]$, i.e $Q \star L = H_0$, because $q_1q_2 \neq 0$.

If $Q \star A \in \Delta_1 \setminus \Delta_0$, then we may assume $q_0 = a_0 = 0$, i.e. $L : \{a_1x_1 + a_2x_2 = 0\}$. Let $P \in L$, then $P = [p_0 : -\frac{a_2}{a_1}p_2 : p_2]$ and $Q \star P = [0 : -\frac{a_2}{a_1}p_2q_1 : p_2q_2] = [0 : -\frac{a_2}{a_1}q_1 : q_2]$, whence $Q \star L = [0 : -\frac{a_2}{a_1}q_1 : q_2] = H_0 \cap \{a_1q_2x_1 + a_2q_2x_2 = 0\}$.

Now suppose $i = 0$, i.e. $A \in \Delta_0$, then $Q \star A$ can be defined or not.

If $Q \star A \in \Delta_{-1}$, then $L : \{x_0 = 0\} = H_0$ and $Q \star L = H_0$.

If $Q \star A \notin \Delta_{-1}$, then we may assume $L : \{x_1 = 0\} = H_1$. In this case $Q \star L = \{[0 : 0 : 1]\} = H_0 \cap \{x_1 = 0\}$.

(4), (5) They follow immediately from the definition of the Hadamard product.

□

Corollary 2.2.5. *Let X be a set of collinear points in $\mathbb{P}^2$ and let Q be a point. Then $Q \star X$ is contained in a line.*

Theorem 2.2.6. *Let L be a line in $\mathbb{P}^2$ defined by $a_0x_0 + a_1x_1 + a_2x_2 = 0$, set $A = [a_0 : a_1 : a_2]$ and let $Q = [q_0 : q_1 : q_2], Q' = [q'_0 : q'_1 : q'_2]$ be two distinct points. Suppose $A, Q, Q' \notin \Delta_0$, then:*

(1) *$Q \star L$ and $Q' \star L$ are lines when:*

  (a) *$A \notin \Delta_1$ and either $Q \notin \Delta_1$ or $Q' \notin \Delta_1$ or both. The two lines are distinct;*

  (b) *$A \notin \Delta_1, Q, Q' \in \Delta_1$. The two lines are distinct if and only if $Q \star Q' \in \Delta_0$;*

  (c) *$A \in \Delta_1$, $Q, Q' \notin \Delta_1$. The two lines are distinct if and only if we have that* $\det \begin{pmatrix} q_j & q_k \\ q'_j & q'_k \end{pmatrix} \neq 0$ *where $j, k \neq i$ and $A \in H_i$;*

  (d) *$A \in \Delta_1$, $Q' \notin \Delta_1$ and $Q \star A \in \Delta_0$. The two lines are distinct;*

(e) $A \in \Delta_1$, $Q \star A \in \Delta_0$ *and* $Q' \star A \in \Delta_0$. *The two lines are distinct if and only if* $Q \star Q' \in \Delta_0$.

(2) $Q \star L$ *is a point and* $Q' \star L$ *is a line when:*

(a) $A, Q \in \Delta_1$, $Q' \notin \Delta_1$ *and* $Q \star A \notin \Delta_0$. *The point* $Q \star L$ *belongs to the line* $Q' \star L$ *if and only if* $det \begin{pmatrix} q_j & q_k \\ q'_j & q'_k \end{pmatrix} = 0$, *where* $j, k \neq i$ *and* $A \in H_i$;

(b) $A, Q \in \Delta_1$, $Q \star A \notin \Delta_0$ *and* $Q' \star A \in \Delta_0$. *The point* $Q \star L$ *does not belong to the line* $Q' \star L$.

(3) $Q \star L$ *and* $Q' \star L$ *are two distinct points when* $A, Q, Q' \in \Delta_1$, $Q \star A \notin \Delta_0 \not\ni Q' \star A$.

Proof.

(1)-(a) If $Q, Q' \notin \Delta_1$ then $Q \star L$ and $Q' \star L$ are two distinct lines by (2) of Theorem 2.1.1.

If $Q \in \Delta_1$ and $Q' \notin \Delta_1$, then, by (2) of Theorem 2.2.4, $Q \star L = H_i$, for some $i \in \{0, 1, 2\}$, which is a line distinct from $Q' \star L$.

(1)-(b) Since $Q, Q' \in \Delta_1 \setminus \Delta_0$, then, by (2) of Theorem 2.2.4, $Q \star L = H_i$ and $Q' \star L = H_j$, for some $i, j \in \{0, 1, 2\}$. Clearly, these lines are distinct if and only if $Q \star Q' \in \Delta_0$.

In the rest of the proof we have $A \in \Delta_1 \setminus \Delta_0$ and so, without loss of generality, we may assume $A = [0 : a_1 : a_2] \in H_0$, with $a_1 a_2 \neq 0$, whence $L : \{a_1 x_1 + a_2 x_2 = 0\}$.

(1)-(c) Since $Q, Q' \notin \Delta_1$, then, by (1) of Theorem 2.2.4, $Q \star L : \{a_1 q_2 x_1 + a_2 q_1 x_2 = 0\}$ and $Q' \star L : \{a_1 q'_2 x_1 + a_2 q'_1 x_2 = 0\}$ are lines. Clearly, these lines are distinct if and only if $det \begin{pmatrix} q_1 & q_2 \\ q'_1 & q'_2 \end{pmatrix} \neq 0$.

(1)-(d) Since $Q' \notin \Delta_1$, then $Q' \star L : \{a_1 q'_2 x_1 + a_2 q'_1 x_2 = 0\}$, by (1) of Theorem 2.2.4. Since $Q \star A \in \Delta_0$, then necessarily $Q \in \Delta_1$, and so, by (2) of Theorem 2.2.4, $Q \star L = H_i$, for some $i \in \{1, 2\}$. These two lines are distinct because $Q' \notin \Delta_1$.

(1)-(e) Since $Q \star A, Q' \star A \in \Delta_0$, then necessarily $Q, Q' \in \Delta_1$, and so by (2) of Theorem 2.2.4, $Q \star L = H_i$ and $Q' \star L = H_j$, for some $i \in \{1, 2\}$. Clearly, these lines are distinct lines if and only if $Q \star Q' \in \Delta_0$.

(2)-(a) Since $Q \star A \notin \Delta_0$, then, by (3) of Theorem 2.2.4, $Q \star L = [0 : -a_2q_1 : a_1q_2]$, which belongs to the line $Q' \star L : \{a_1q'_2x_1 + a_2q'_1x_2 = 0\}$ if and only if $det\begin{pmatrix} q_1 & q_2 \\ q'_1 & q'_2 \end{pmatrix} = 0$.

(2)-(b) Since $Q \star A \notin \Delta_0$ and $Q' \star A \in \Delta_0$, then, by (2), (3) of Theorem 2.2.4, $Q \star L = [0 : -a_2q_1 : a_1q_2]$ and $Q' \star L = H_i$ for some $i \in \{1, 2\}$. Clearly, the point does not belong to the line.

(3) Since $Q \star A, Q' \star A \notin \Delta_0$ and $A, Q, Q' \in \Delta_1$, then, by (3) of Theorem 2.2.4, $Q \star L = [0 : -a_2q_1 : a_1q_2]$, and $Q' \star L = [0 : -a_2q'_1 : a_1q'_2]$. These two points are distinct because $Q \neq Q'$ and $Q, Q' \in \Delta_1$ imply $det\begin{pmatrix} q_1 & q_2 \\ q'_1 & q'_2 \end{pmatrix} \neq 0$.

□

Notice that Theorem 2.2.6 gives an exhaustive description of all the possible cases.

Remark 2.2.7. Let L be a line in $\mathbb{P}^2$ such that $L \cap \Delta_0 = \emptyset$, and let $P, Q \in L$, with $P \neq Q$. Then the minors of order two of the matrix $\begin{pmatrix} P \\ Q \end{pmatrix}$ are all not zero.

Theorem 2.2.8. *Let L be a line in $\mathbb{P}^2$ defined by $a_0x_0 + a_1x_1 + a_2x_2 = 0$ and let $A = [a_0 : a_1 : a_2]$. Let L' be a line in $\mathbb{P}^2$ defined by $a'_0x_0 + a'_1x_1 + a'_2x_2 = 0$ and such that $[a'_0 : a'_1 : a'_2] \notin \Delta_1$. Let $X' \subseteq L'$ be a set of m' collinear points and suppose $L' \cap \Delta_1 \subseteq X'$. Then:*

(1) *If $A \notin \Delta_1$, then $X' \star L$ is a set of m' distinct lines.*

(2) *If $A \in \Delta_1 \setminus \Delta_0$, then $X' \star L$ is a set of $m' - 1$ distinct lines and a point which does not belong to any line of $X' \star L$.*

(3) *If $A \in \Delta_0$, then $X' \star L = L$.*

Proof.

(1) Since $A \notin \Delta_1$, then $\{P' \star L | P' \in X' \setminus \Delta_1\}$ is a set of $m' - 3$ distinct lines. In fact, by (1) of Lemma 2.2.2, $|L \cap \Delta_1| = 3$, and by (2) of Theorem 2.1.1, each $P' \star L$ is a line and these lines are all distinct.

By (2) of Theorem 2.2.4, if $P' \in X' \cap \Delta_1$, then $P' \star L = H_i$ (for some $i \in \{0, 1, 2\}$) and they are distinct by (1)-(b) of Theorem 2.2.6;

(2) Since $A \in \Delta_1 \setminus \Delta_0$, then $\{P' \star L | P' \in X' \setminus \Delta_1\}$ is a set of $m' - 3$ distinct lines. In fact, by (1) of Theorem 2.2.4, each $P' \star L$ is a line and it is easy to prove that they are all distinct by using Remark 2.2.7.

Since $L' \cap \Delta_1 \subseteq X'$, by (1) of Lemma 2.2.2 there exists only one point $P'_1 \in X' \cap \Delta_1$ such that $P'_1 \star A \notin \Delta_0$. Thus, by (3) of Theorem 2.2.4, $P'_1 \star L$ is a point which does not belong to any of the lines of $\{P' \star L | P' \in X' \setminus \Delta_1\}$ by (2)-(a) of Theorem 2.2.6 in view of Remark 2.2.7. For the remaining two points $P'_2, P'_3 \in X' \setminus \Delta_1$ we have $P'_2 \star L = H_i$ and $P'_3 \star L = H_j$, for some $i, j \in \{0, 1, 2\}$, with $i \neq j$.

(3) Since $A \in \Delta_0$, then $L = H_i$, for some $i \in \{0, 1, 2\}$. By Remark 2.1.7 $X' \star L \subseteq L$. On the other hand, from $L' \cap \Delta_1 \subseteq X'$, $[a'_0 : a'_1 : a'_2] \notin \Delta_1$ and (1) of Lemma 2.2.2, it follows that there exists a point $P' \in (X' \cap \Delta_1) \setminus \Delta_0$, such that $P' \star A \in \Delta_{-1}$. The conclusion follows from (2) of Theorem 2.2.4.

□

Corollary 2.2.9. *Let L be a line in $\mathbb{P}^2$ defined by $a_0x_0 + a_1x_1 + a_2x_2 = 0$ and let $A = [a_0 : a_1 : a_2]$. Let L' be a line in $\mathbb{P}^2$ defined by $a'_0x_0 + a'_1x_1 + a'_2x_2 = 0$ and such that $[a'_0 : a'_1 : a'_2] \notin \Delta_1$. Let $X' \subseteq L'$ be a set of m' collinear points. Then:*

1. *If $A \notin \Delta_1$, then $X' \star L$ is a set of m' distinct lines;*
2. *If $A \in \Delta_1 \setminus \Delta_0$, then $X' \star L$ is either a set of m' distinct lines or a set of $m' - 1$ distinct lines and a point which does not belong to any line of $X' \star L$;*
3. *If $A \in \Delta_0$ and $m' \geq 3$, then $X' \star L = L$.*

Proof. The only difference with the previous theorem is that we no longer have the hypothesis $L' \cap \Delta_1 \subseteq X'$, and so the existence of $P' \in X' \cap \Delta_1$ such that $P' \star A \notin \Delta_0$ is not granted, therefore we can obtain m' lines and no extra point.

As for (3), if there exists $P' \in X' \setminus \Delta_1$, we are done by (1) of Theorem 2.2.4. If every $P' \in X'$ is in Δ_1, then, in view of (1) of Lemma 2.2.2, we have $r = 3$ and $L' \cap \Delta_1 = X'$. □

Example 2.2.10. Let $L' \subset \mathbb{P}^2$ be the line of equation $2x_0 - 3x_1 + 132x_2$ and let $X' \subset L'$ be the following set of five points (randomly chosen in L' by `Singular`)

$$X' = \{[27 : 238 : 5], [12 : 96 : 2], [15 : 142 : 3], [21 : 234 : 5], [33 : 242 : 5]\}.$$

After setting $X' = Y$, we get that the ideal I of Y is generated by I[1] and I[2], where:

```
I[1]=2*x(0)-3*x(1)+132*x(2)
I[2]=375*x(1)^5-89300*x(1)^4*x(2)+8505840*x(1)^3*x(2)^2+
-405077872*x(1)^2*x(2)^3+9645291984*x(1)*x(2)^4-
-91862394624*x(2)^5
```

As L consider the line $2x_0 - 3x_1 - 11x_2$; clearly we are in the case $A \notin \Delta_1$. Computing the Hadamard product $X' \star L$, in `Singular` we get

```
ideal J=2*x(0)-3*x(1)-11*x(2);
ideal YL=HPr(I,J,2);
degree(YL);
// dimension (proj.)  = 1
// degree (proj.)   = 5
genus(YL);
-4
```

which tell us that $X' \star L$ is the union of five lines. In particular, looking at the primary decomposition of the ideal `YL` we recover the five lines

$$\begin{aligned}
&16x_0 - 3x_1 - 528x_2 = 0\\
&284x_0 - 45x_1 - 7810x_2 = 0\\
&2380x_0 - 405x_1 - 70686x_2 = 0\\
&260x_0 - 35x_1 - 6006x_2 = 0\\
&220x_0 - 45x_1 - 7986x_2 = 0.
\end{aligned}$$

Lemma 2.2.11. *Let L be the line defined by $a_0x_0+a_1x_1+a_2x_2 = 0$ and L' defined by $a'_0x_0 + a'_1x_1 + a'_2x_2 = 0$, let $A = [a_0 : a_1 : a_2]$ and $A' = [a'_0 : a'_1 : a'_2]$ with $A, A' \notin \Delta_1$. Let $P_1, P_2 \in L \setminus \Delta_1$ and $P'_1, P'_2 \in L' \setminus \Delta_1$ with $\{P_1, P_2\} \cap \{P'_1, P'_2\} = \emptyset$. If $P_1 \star P'_1 = P_2 \star P'_2$, then either $P_1 = P_2$ and $P'_1 = P'_2$ or $P_i \star A = P'_j \star A'$ for $i, j \in \{1, 2\}$ with $i \neq j$.*

Proof. If $P_1 = P_2$, then because $P_1, P_2 \notin \Delta_1$ and $P_1 \star P'_1 = P_2 \star P'_2$, then $P'_1 = P'_2$, by (1) of Theorem 2.1.1.

Suppose $P_1 = [p_{10} : p_{11} : p_{12}] \neq P_2 = [p_{20} : p_{21} : p_{22}]$ and $P'_1 = [p'_{10} : p'_{11} : p'_{12}] \neq P'_2 = [p'_{20} : p'_{21} : p'_{22}]$. Since $P'_1 \neq P'_2$, we have $P_1 \star P'_2 \neq P_1 \star P'_1 = P_2 \star P'_2$. Through $P_1 \star P'_2$ and $P_1 \star P'_1$ there is only the line $P_1 \star L'$ defined by $\frac{a'_0}{p_{10}}x_0 + \frac{a'_1}{p_{11}}x_1 + \frac{a'_2}{p_{12}}x_2 = 0$, and through $P_1 \star P'_2$ and $P_2 \star P'_2$ there is only the line $P'_2 \star L$ defined by $\frac{a_0}{p'_{20}}x_0 + \frac{a_1}{p'_{21}}x_1 + \frac{a_2}{p'_{22}}x_2 = 0$. Since $P_1 \star P'_1 = P_2 \star P'_2$, these two lines must coincide, i.e. $\frac{a'_i}{p_{1i}} = \alpha\frac{a_i}{p'_{2i}}$, for $i \in \{0, 1, 2\}$, which gives $P_1 \star A = P'_2 \star A'$. □

Definition 2.2.12. We say that a set of ab distinct points of $\mathbb{P}^n$ is a *grid* if there exist a set of a distinct lines and a set of b distinct lines intersecting exactly in the given points.

Observe that, when $n = 2$, such a grid is a complete intersection of type (a, b).

Theorem 2.2.13. *Let X be a set of m collinear points of $\mathbb{P}^2$, with $X \subseteq L \setminus \Delta_1$ and L defined by $a_0x_0 + a_1x_1 + a_2x_2 = 0$. Let X' be a set of m' collinear points, with $X' \subseteq L' \setminus \Delta_1$ and L' defined by $a'_0x_0 + a'_1x_1 + a'_2x_2 = 0$. Set $A = [a_0 : a_1 : a_2]$ and $A' = [a'_0 : a'_1 : a'_2]$. If $A, A' \notin \Delta_1$ and $X \cap X' = \emptyset$, then, $X \star X'$ is the mm' element grid $(X \star L') \cap (X' \star L)$ if and only if $P \star A \neq P' \star A'$, for all $P \in X$ and all $P' \in X'$.*

Proof. If $(X \star L') \cap (X' \star L)$ is a grid with mm' elements, then the lines $\{P \star L' \mid P \in X\}$ and $\{P' \star L \mid P' \in X'\}$ are all distinct. With the same reasoning of Lemma 2.2.11, we can prove that $P \star A \neq P' \star A'$ for all $P \in X$ and $P' \in X'$.

Conversely, since $P \star A \neq P' \star A'$, then $\{P \star L' \mid P \in X\}$ and $\{P' \star L \mid P' \in X'\}$ are two families of distinct lines by Corollary 2.2.9. Moreover, since $P \star A \neq P' \star A'$, for all $P \in X$ and $P' \in X'$, as in the proof of Lemma 2.2.11, we obtain that also $\{P \star L', P' \star L \mid P \in X, P' \in X'\}$ is a family of distinct lines. On the other hand it is easy to check that $X \star X' = (X \star L') \cap (X' \star L)$. Now suppose $X \star X'$ has fewer than mm' elements, then there exist $P_1, P_2 \in X$ and $P'_1, P'_2 \in X'$ with $P_1 \neq P_2$ and $P'_1 \neq P'_2$ such that $P_1 \star P'_1 = P_2 \star P'_2$. By Lemma 2.2.11 this forces $P_1 \star A = P'_2 \star A'$ against the hypothesis. ☐

Corollary 2.2.14. *Let X, Y be two disjoint sets of points of $\mathbb{P}^2$ both contained in the same line L. Suppose $X \cap \Delta_1 = \emptyset = Y \cap \Delta_1$ and $L \cap \Delta_0 = \emptyset$. If $|X| = m$ and $|Y| = m'$, then $X \star Y$ is the mm' element grid $(X \star L) \cap (Y \star L)$.*

Proof. Let L be defined by $a_0x_0 + a_1x_1 + a_2x_2 = 0$ and let $A = [a_0 : a_1 : a_2]$, then, for all $P \in X$ and all $P' \in Y$, we have $P \star A \neq P' \star A$ by (1) of Theorem 2.1.1. Now the conclusion follows from Theorem 2.2.13. ☐

Corollary 2.2.15. *Let L, L' be two generic distinct lines in $\mathbb{P}^2$. There is a generic choice of a finite set of points $X \subseteq L$ for which it is possible to find a generic choice of a finite set of points $X' \subseteq L'$ such that $X \star X'$ is the grid $(X \star L') \cap (X' \star L)$.*

Proof. Let L, L' be defined by $a_0x_0 + a_1x_1 + a_2x_2 = 0$ and $a'_0x_0 + a'_1x_1 + a'_2x_2 = 0$ respectively and let $A = [a_0 : a_1 : a_2]$ and $A' = [a'_0 : a'_1 : a'_2]$. We may assume $L \cap \Delta_0 = \emptyset = L' \cap \Delta_0$, whence $A, A' \notin \Delta_1$. Let $P, Q \in L \setminus \Delta_1$ and $P' \in L' \setminus \Delta_1$ be distinct points. By (1) of Theorem 2.1.1 $P \star A \neq Q \star A$, then either $P \star A \neq P' \star A'$ or $Q \star A \neq P' \star A'$. Suppose $P \star A \neq P' \star A'$ and consider the 2×3 matrix $M(\lambda, \mu) = \begin{pmatrix} A \\ A' \end{pmatrix} \star \begin{pmatrix} \lambda P + \mu Q \\ P' \end{pmatrix}$ with $[\lambda : \mu] \in \mathbb{P}^1$. Then $M(1, 0)$ has a non-zero 2×2 minor. The corresponding minor in $M(\lambda, \mu)$ is a non-zero linear form $F(\lambda, \mu)$. Let $P_0 \in L$ be the point corresponding to the zero locus of F. Thus the set $L \setminus \{P_0\}$ is a non empty open subset U of L. Moreover, if $R \in U$ then $R \star A \neq P' \star A'$.

Now consider a finite set of points $X \subseteq U \cap (L \setminus \Delta_1)$. For any point $R \in X$, by the same reasoning as before, we find a non empty open subset U'_R of L' such that $R \star A \neq R' \star A'$ for any point $R' \in U'_R$. Set $U' = \bigcap_{R \in X} U'_R$. If $X' \subseteq U' \cap (L' \setminus \Delta_1)$ is a finite set of points then $P \star A \neq P' \star A'$ for all $P \in X$ and $P' \in X'$.

Now the claim follows from Theorem 2.2.13. □

Remark 2.2.16. Since grids in $\mathbb{P}^2$ are complete intersections, their Hilbert functions and even their resolutions are known.

Example 2.2.17. Let L and L' be respectively the lines $3x_0 + x_1 - 30x_2 = 0$ and $67x_0 - 6x_1 - 110x_2$ (randomly chosen by `Singular`). Consider the sets of points (still randomly chosen by `Singular`), which satisfy the hypotheses of Theorem 2.2.13,

$$\begin{aligned} &X = \{[6:12:1], [22:54:4], [29:63:5]\} \subset L \\ &X' = Y = \{[22:154:5], [28:221:5], [34:288:5], [18, 146, 3]\} \subset L' \end{aligned}$$

Using the procedure of Appendix A we compute the ideal I of $X \star X'$ and then its Hilbert function

```
ideal I=HPr(X,Y,2);

HF(2,I,0)=1;
HF(2,I,1)=3;
HF(2,I,2)=6;
HF(2,I,3)=9;
HF(2,I,4)=11;
HF(2,I,5)=12;
HF(2,I,6)=12;
```

that is $HF_{X \star X'}(t) = 12$ for $t \geq 5$.

As expected, $X \star X'$ is a complete intersection.

2.3 Sets of collinear points in $\mathbb{P}^3$

We keep assuming that the sets of points under consideration have cardinalities at least two.

Lemma 2.3.1. *Let L be a line in $\mathbb{P}^3$ such that $L \cap \Delta_0 = \emptyset$ and let H be a generic plane trough L. Then $H \cap \Delta_0 = \emptyset$. Equivalently, if A is the point corresponding to H in the dual space, then $A \notin \Delta_2$.*

Proof. It is immediate since the set of planes through L which contain some coordinate point is finite. $\square$

Theorem 2.3.2. *Let L, L' be two lines in $\mathbb{P}^3$, $L = H \cap K$, $L' = H' \cap K'$, let A, B, A' and B' be the points which correspond to H, K, H', K' in the dual space and suppose $A, B, A', B' \notin \Delta_2$. Let $X \subseteq L$ and $X' \subseteq L'$ be two finite sets of points such that $X \cap \Delta_2 = \emptyset = X' \cap \Delta_2$ and $X \cap X' = \emptyset$. Suppose rank* $\begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} P \\ P \\ P' \\ P' \end{pmatrix} > 2$ *for all $P \in X \subset L$ and $P' \in X' \subset L'$, then, $X \star X' = (X \star L') \cap (X' \star L)$ and $|X \star X'| = |X||X'|$.*

Proof. Let $P = [p_0 : \cdots : p_3] \in X$ and $P' = [p'_0 : \cdots : p'_3] \in X'$. First we show that $P \star L'$ and $P' \star L$ are distinct lines. They are lines by [BCK, Lemma 3.1]. On the other hand, by Corollary 2.1.3, we have that $P \star L' = (P \star H') \cap (P \star K')$, and $P' \star L = (P' \star H) \cap (P' \star K)$. If we had $P \star L' = P' \star L$, then, after denoting $\frac{1}{P} = [\frac{1}{p_0} : \cdots : \frac{1}{p_3}]$ and $\frac{1}{P'} = [\frac{1}{p'_0} : \cdots : \frac{1}{p'_3}]$, we would have that *rank* $\begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} \frac{1}{P'} \\ \frac{1}{P'} \\ \frac{1}{P} \\ \frac{1}{P} \end{pmatrix} = 2.$

But a straightforward computation shows that

$$\mathit{rank} \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} \frac{1}{P'} \\ \frac{1}{P'} \\ \frac{1}{P} \\ \frac{1}{P} \end{pmatrix} = \mathit{rank} \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} P \\ P \\ P' \\ P' \end{pmatrix},$$

in contradiction with the hypothesis.

Now let $P_1, P_2 \in X$, we shall show that if $P_1 \star L' = P_2 \star L'$ then $P_1 = P_2$. In fact, let $P' \in X'$. Then, we just showed that $P_1 \star L'$ and $P' \star L$ are distinct lines, hence $(P_1 \star L') \cap (P' \star L)$ is the point $P_1 \star P'$. Similarly, $P_2 \star P' = (P_2 \star L') \cap (P' \star L)$. Therefore $P_1 \star P' = P_2 \star P'$, hence $P_1 = P_2$, by (1) of Theorem 2.1.1.

In a similar way we can prove that, for any $P'_1, P'_2 \in X'$, if $P'_1 \star L = P'_2 \star L$, then $P'_1 = P'_2$. Finally, we prove that for any $P_1, P_2 \in X$ and for any $P'_1, P'_2 \in X'$, $P_1 \star P'_1 \neq P_2 \star P'_2$ provided $P_1 \neq P_2$ and $P'_1 \neq P'_2$. Assume, by contradiction, that $P_1 \star P'_1 = P_2 \star P'_2$ but then we would have $P_1 \star L' = P'_2 \star L$.

$\square$

Remark 2.3.3. Observe that, under all hypotheses of Theorem 2.3.2, the hypothesis $rank \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} P \\ P \\ P' \\ P' \end{pmatrix} > 2$ forces $rank \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} P \\ P \\ P' \\ P' \end{pmatrix} = 3$, since $rank \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} P \\ P \\ P' \\ P' \end{pmatrix} = 4$ would imply that $P' \star L$ and $P \star L'$ are disjoint, while they meet in $P \star P'$.

Lemma 2.3.4. *Assume the hypotheses as in Theorem 2.3.2. If there exist $P, Q \in X$ with $P \neq Q$ such that $(P \star L') \cap (Q \star L') \neq \emptyset$, then $X \star L'$ and $X' \star L$ are contained in the same plane. Similarly if there exist $P', Q' \in X'$ with $P' \neq Q'$ such that $(P' \star L) \cap (Q' \star L) \neq \emptyset$, then $X \star L'$ and $X' \star L$ are contained in the same plane.*

Proof. We only prove the first statement, the other being similar.

Since $(P \star L') \cap (Q \star L') \neq \emptyset$, then they determine a plane π. Now, let P' be any point of X' and consider the line $P' \star L$. By the proof of Theorem 2.3.2, one has

$$(P \star L') \cap (P' \star L) = P \star P' \text{ and } (Q \star L') \cap (P' \star L) = Q \star P'$$

hence $P \star P', Q \star P'$ are distinct points of π and thus also the line $P' \star L$ lies in π, hence $X' \star L$ is contained in the plane π. Now let Q' be any other point of X', then $(P' \star L) \cap (Q' \star L) \neq \emptyset$ and, from what we have proved, $P' \star L$ and $Q' \star L$ both lie in π. With the same reasoning we have that $X \star L'$ is contained in the plane determined by $P' \star L$ and $Q' \star L$, which is π. □

Corollary 2.3.5. *Let L, L' be two generic lines in $\mathbb{P}^3$. There is a generic choice of a finite set of points $X \subseteq L$ for which it is possible a generic choice of a finite set of points $X' \subseteq L'$ such that:*

(1) *$X \star X' = (X \star L') \cap (X' \star L)$ and $|X \star X'| = |X||X'|$.*

(2) *$L \star L'$ is an irreducible and non-degenerate quadric, and $X \star L'$ and $X' \star L$ are lines of the two different rulings.*

Proof.

(1) We may assume that $L \cap \Delta_1 = \emptyset = L' \cap \Delta_1$, so that $L \cap \Delta_2$ and $L' \cap \Delta_2$ are finite. By Lemma 2.3.1 we can write $L = H \cap K$ and $L' = H' \cap K'$, with $A, B, A', B' \notin \Delta_2$, where A, B, A' and B' are the points which correspond to H, K, H', K' in the dual space.

If $rank \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} P \\ P \\ P' \\ P' \end{pmatrix} = 2$, for all $P \in L \setminus \Delta_2$ and $P' \in L' \setminus \Delta_2$, then $P \star L' = P' \star L$ is a line, say L'', for all $P \in L \setminus \Delta_2$ and $P' \in L' \setminus \Delta_2$. Thus

$$L \star L' = \overline{\bigcup_{P \in L} \{P \star L'\}} = \overline{(\bigcup_{P \in L \setminus \Delta_2} \{P \star L'\}) \bigcup (\bigcup_{P \in L \cap \Delta_2} \{P \star L'\})} =$$

$$= \overline{L'' \bigcup \left(\bigcup_{P \in L \cap \Delta_2} \{P \star L'\} \right)},$$

which is a union of a line and a finite number of linear spaces of dimension less than or equal to 1. This contradicts [BCK, Theorem 6.8] in view of [BCK, Remark 6.9].

Hence there exist $P \in L \setminus \Delta_2$ and $P' \in L' \setminus \Delta_2$ such that

$$rank \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} P \\ P \\ P' \\ P' \end{pmatrix} = 3.$$

Consider a point $Q \in L$ and the 4×4 matrix

$$M(\lambda, \mu) = \begin{pmatrix} A \\ B \\ A' \\ B' \end{pmatrix} \star \begin{pmatrix} \lambda P + \mu Q \\ \lambda P + \mu Q \\ P' \\ P' \end{pmatrix}$$

with $[\lambda : \mu] \in \mathbb{P}^1$. Now we get the conclusion by mimicking the proof of Corollary 2.2.15 and by applying Theorem 2.3.2.

(2) Since L and L' are generic, by [BCK, Theorem 6.8], $L \star L'$ is a quadric, in fact, an irreducible one, as noticed right after Remark 2.5 of [BCK]. Since the quadric is irreducible, then

$$(P' \star L) \cap (Q' \star L) = \emptyset \quad \forall P', Q' \in X'$$

and similarly

$$(P \star L') \cap (Q \star L') = \emptyset \quad \forall P, Q \in X.$$

In fact, suppose $P' \star L$ and $Q' \star L$ intersect in a point. Then, by Lemma 2.3.4, the lines $P' \star L$, $Q' \star L$ and $P \star L'$ are all distinct and lie in the same plane. But then $L \star L'$ would be reducible.

On the other hand $P \star L'$ and $P' \star L$ intersect in $P \star P'$ for all $P \in X$ and for all $P' \in X'$.

Therefore $L \star L'$ is also non-degenerate.

□

Remark 2.3.6. If both $|X|$ and $|X'|$ are strictly greater than 2, then we have at least three skew lines each with at least three points of $X \star X'$ and this is enough to prove that $L \star L'$ is the unique quadric through $X \star X'$. It would be interesting to understand the geometry of $X \star X'$ on such a quadric.

Example 2.3.7. In this example we compute the ideal of $X \star X'$ and its Hilbert function, where X and X' are two sets of collinear points satisfying the hypotheses of Corollary 2.3.5.

Let H, K be the planes defined by $x_0 - x_1 + x_2 + 2x_3 = 0$ and $x_0 + 2x_1 - x_2 + x_3 = 0$ and let H', K' be the planes defined by $x_0 + 2x_1 - 2x_2 + x_3 = 0$ and $2x_0 + 2x_1 + x_2 - 4x_3 = 0$. Let $L = H \cap K$ and $L' = H' \cap K'$. Choose $X \subset L$ and $X' \subset L'$ where

$$X = \{[-2 : 1 : 1 : 1], [-1 : -1 : -2 : 1], [-2 : 3 : 4 : 1]\}$$

and

$$X' = \{[-1 : 2 : 2 : 1], [11 : -8 : -2 : 1], [-7 : 7 : 4 : 1]\}.$$

By computing the ideal of $X \star X'$ with `Singular`, we obtain

```
ideal I=HPr(X,X’,3)

I[1]=-3/31x(0)^2+41/62x(0)x(1)-15/31x(1)^2-169/186x(0)x(2)+
+59/62x(1)x(2)-21/62x(2)^2-59/62x(0)x(3)+
+845/186x(1)x(3)-287/124x(2)x(3)-105/62x(3)^2
I[2]=-3/31x(0)x(1)^2+18/31x(1)^3+10/31x(0)x(1)x(2)-
-87/31x(1)^2x(2)-25/93x(0)x(2)^2+140/31x(1)x(2)^2-
-75/31x(2)^3-24/31x(0)x(1)x(3)+137/31x(1)^2x(3)+
+35/31x(0)x(2)x(3)-2261/186x(1)x(2)x(3)+
+505/62x(2)^2x(3)+51/31x(0)x(3)^2-362/31x(1)x(3)^2+
+1443/62x(2)x(3)^2--873/31x(3)^3
I[3]=6/31x(0)x(1)x(2)-36/31x(1)^2x(2)-10/31x(0)x(2)^2+
+114/31x(1)x(2)^2-90/31x(2)^3+42/31x(0)x(2)x(3)-
```

```
-238/31x(1)x(2)x(3)+303/31x(2)^2x(3)-1152/155x(0)x(3)^2+
+192/31x(1)x(3)^2+3762/155x(2)x(3)^2-1728/31x(3)^3
I[4]=6/31x(0)x(2)^2-36/31x(1)x(2)^2+54/31x(2)^3+
+3456/775x(0)x(1)x(3)-576/155x(1)^2x(3)-
-1584/155x(0)x(2)x(3)+7944/775x(1)x(2)x(3)-
-1476/155x(2)^2x(3)+3456/775x(0)x(3)^2+
+4608/155x(1)x(3)^2-28296/775x(2)x(3)^2+
+5184/155x(3)^3
I[5]=-6/5x(1)^3+6x(1)^2x(2)-10x(1)x(2)^2+50/9x(2)^3+
+336/155x(0)x(1)x(3)-1954/155x(1)^2x(3)-
-112/31x(0)x(2)x(3)+15742/465x(1)x(2)x(3)-
-5972/279x(2)^2x(3)-428/155x(0)x(3)^2+
+13652/465x(1)x(3)^280218/1395x(2)x(3)^2+
+91204/1395x(3)^3
I[6]=-x(1)^2x(2)+10/3x(1)x(2)^2-25/9x(2)^3+
+56/31x(0)x(2)x(3)-884/93x(1)x(2)x(3)+
+2986/279x(2)^2x(3)-3956/465x(0)x(3)^2+
+392/31x(1)x(3)^2+28958/1395x(2)x(3)^2-
-16252/279x(3)^3
I[7]=-x(1)x(2)^2+5/3x(2)^3+3956/775x(0)x(1)x(3)-
-1176/155x(1)^2x(3)-5102/465x(0)x(2)x(3)+
+15944/775x(1)x(2)x(3)-6703/465x(2)^2x(3)+
+3956/775x(0)x(3)^2+12724/465x(1)x(3)^2-
-88388/2325x(2)x(3)^2+16252/465x(3)^3
I[8]=-1/4x(2)^3-1278/775x(0)x(1)x(3)-252/155x(1)^2x(3)+
+687/155x(0)x(2)x(3)+3228/775x(1)x(2)x(3)-
-172/155x(2)^2x(3)-4518/775x(0)x(3)^2-234/155x(1)x(3)^2
+988/775x(2)x(3)^2-422/155x(3)^3
```

whose Hilbert function is given by

```
HF(3,I,0)=1
HF(3,I,1)=4
HF(3,I,2)=9
HF(3,I,3)=9
```

that is $HF_{X\star X'}(t) = 9$ for $t \geq 2$.

The example above shows that the finite set $X\star X'$ in Corollary 2.3.5, in general, is not a complete intersection. However we are able to compute its Hilbert function in the case $|X| = |X'|$ and this allows us to prove that $X \star X'$ is never a complete intersection as long as $m > 1$ (obviously it is for $m = 1$).

Theorem 2.3.8. *Assume the hypotheses as in Corollary 2.3.5. Also suppose that* $|X| = |X'| = m$. *Then* $\tau_{X\star X'} = m - 1$ *and* $HF_{X\star X'} = HF_X HF_{X'}$.

Proof. For $m = 2$ the four points of $X \star X'$ cannot be coplanar since they belong to the two skew lines of $X \star L'$ each containing two points. Thus $HF_{X\star X'}(t) = 4$, for all $t \geq 1$.

For $m \geq 3$, if F is a form of $I_{X\star X'}$ of degree t with $2 \leq t < m$, then we have $X \star L' \subseteq V(F) \supseteq X' \star L$. In fact, $|(P \star L') \cap V(F)| \geq m > t$ for all $P \in X$ so that $X\star L \subseteq V(F)$. Similarly $X'\star L \subseteq V(F)$. Hence $(X\star L')\cup(X'\star L) \subseteq V(F)\cap(L\star L')$. Therefore, since $X \star L' \cup X' \star L$ is a set of $2m$ distinct lines, by intersection theory, we have that the quadric $L \star L'$ (which is the unique quadric through $X \star X'$ by Remark 2.3.6) is a fixed component of $I_{X\star X'}$ in each degree $2 \leq t < m$. Then, for $0 \leq t < m$ we have

$$HF_{X\star X'}(t) = \binom{t+3}{3} - \binom{t+1}{3} = (t+1)^2$$

which equals $HF_X(t)HF_{X'}(t)$ since $HF_X(t) = HF_{X'}(t) = t + 1$ for $t < m$.

In particular, $HF_{X\star X'}(m-1) = m^2 = |X||X'|$, hence $\tau_{X\star X'} = m - 1$ and, for all $t \geq m - 1$, $HF_{X\star X'}(t) = m^2 = HF_X(t)HF_{X'}(t)$. □

Obviously Theorem 2.3.8 works also for $m = 1$.

Remark 2.3.9. If X is a finite set of projective points we set

$$h_X = (HF_X(0), \dots, HF_X(\tau_X)).$$

With this notation we can rephrase Theorem 2.3.8 as

$$h_{X\star X'} = h_X \star h_{X'}.$$

The following example shows that we may still have $HF_{X\star X'} = HF_X HF_{X'}$ even when $|X| \neq |X'|$. In the next chapter (Theorem 3.1.3) we prove that this is always the case.

Example 2.3.10. Let H, K be the planes defined by $11x_1 - 14x_2 - 2x_3$ and $22x_0 - 25x_2 - 13x_3$ and let H', K' be the planes defined by $21x_1 - 2x_2 - 11x_3$ and $7x_0 - 6x_2 + 2x_3$ Let $L = H \cap K$ and $L' = H' \cap K'$. Choose $X \subset L$ and $X' \subset L'$ where

$$X = \{[4:4:3:1], [7:4:2:8], [11:8:5:9]\}$$

and

$$X' = \{[2:3:4:5], [6:4:9:6], [18:17:30:27], [94:76:149:118]\}.$$

Let I, J, K be respectively the ideals of X, X' and $X \star X'$. By `Singular` we obtain

```
HF(3,I,0)=1        HF(3,J,0)=1        HF(3,K,0)=1
HF(3,I,1)=2        HF(3,J,1)=2        HF(3,K,1)=4
HF(3,I,2)=3        HF(3,J,2)=3        HF(3,K,2)=9
HF(3,I,3)=3        HF(3,J,3)=4        HF(3,K,3)=12
HF(3,I,4)=3        HF(3,J,4)=4        HF(3,K,4)=12
```

Corollary 2.3.11. *Assume the hypotheses as in Corollary 2.3.5 and $|X| = |X'| = m \geq 2$. Then $X \star X'$ is not a complete intersection.*

Proof. First assume $m = 2$. Then $dim_{\mathbb{K}} \left(I_{X\star X'}\right)_t = \begin{cases} 0 & t = 0,1 \\ 6 & t = 2 \end{cases}$. Thus a minimal system of generators of $I_{X\star X'}$ contains at least six quadrics and so $X \star X'$ cannot be complete intersection.

Now assume $m \geq 3$. From Remark 2.3.6 we know that

$$dim_{\mathbb{K}} \left(I_{X\star X'}\right)_t = \begin{cases} 0 & t = 0,1 \\ 1 & t = 2 \end{cases},$$

and so $dim_{\mathbb{K}} \left(I_{X\star X'}\right)_t \geq \binom{t+1}{3}$, $\forall t \geq 2$. As in the proof of Theorem 2.3.8 we have that the quadric $L \star L'$ is a fixed component of $I_{X\star X'}$ in each degree $2 \leq t < m$, and so we need $\binom{m+3}{3} - m^2 - \binom{m+1}{3} = 2m + 1$ generators of degree m. Thus a minimal system of generators of $I_{X\star X'}$ consists of $2m + 2 > 3$ forms and so $X \star X'$ cannot be complete intersection. □

If we drop some of the assumptions of Corollary 2.3.5 several behaviours may occur, as the following examples show.

Example 2.3.12. Let H, K be the planes defined by $x_1 - x_3$ and $14x_0 - 27x_2 + 10x_3$ and let H', K' be the planes defined by $9x_1 + 5x_2 - 11x_3$ and $x_0 - x_2$. Let $L = H \cap K$ and $L' = H' \cap K'$. Note that this time $L \cap \Delta_1 \neq \emptyset$ and $L' \cap \Delta_1 \neq \emptyset$. Choose $X \subset L$ and $X' \subset L'$ where

$$X = \left\{ \begin{array}{l} [1:4:2:4], [8:5:6:5], [37:40:34:40], \\ [9:9:8:9], [65:98:70:98] \end{array} \right\}$$

and

$$X' = \left\{ \begin{array}{l} [2:5:2:5], [3:2:3:3], [24:27, 24, 33], \\ [13:16:13:19], [130:127:130:163] \end{array} \right\}.$$

Let I, J, K be respectively the ideals of X, X' and $X \star X'$. By `Singular` we obtain

```
HF(3,I,0)=1        HF(3,J,0)=1        HF(3,K,0)=1
HF(3,I,1)=2        HF(3,J,1)=2        HF(3,K,1)=3
HF(3,I,2)=3        HF(3,J,2)=3        HF(3,K,2)=6
HF(3,I,3)=4        HF(3,J,3)=4        HF(3,K,3)=10
HF(3,I,4)=5        HF(3,J,4)=5        HF(3,K,4)=15
HF(3,I,5)=5        HF(3,J,5)=5        HF(3,K,4)=19
HF(3,I,6)=5        HF(3,J,6)=5        HF(3,K,6)=22
HF(3,I,7)=5        HF(3,J,7)=5        HF(3,K,7)=24
HF(3,I,8)=5        HF(3,J,8)=5        HF(3,K,8)=25
```

Notice that, in this case, the Hilbert function of $X \star X'$ is not the product of the Hilbert functions of X and X'.

As a matter of fact, looking at the ideal of $X \star X'$, we can notice that the first generator is

```
K[1]=14*x(0)-18*x(1)-27*x(2)+22*x(3)
```

that is, $X \star X'$ is a planar set of points. Moreover the first difference of its Hilbert function is $(1, 2, 3, 4, 5, 4, 3, 2, 1)$ showing that $X \star X'$ is a complete intersection.

The following two examples show that $L \star L'$ can be a quadric (necessarily irreducible) also under the condition that $L \cap \Delta_1 \neq \emptyset$ or $L \cap \Delta_1 \neq \emptyset \neq L' \cap \Delta_1$. In both examples $X \star X'$ is not a complete intersection.

Example 2.3.13. In this example we compute the ideal of $L \star L'$ and the ideal of $X \star X'$ with its Hilbert function, where X and X' are two sets of collinear points satisfying the hypotheses of Theorem 2.3.2, $L \cap \Delta_1 \neq \emptyset$ and $L' \cap \Delta_1 = \emptyset$.

Let H, K be the planes defined by $x_0 + 2x_1 + x_2 + x_3 = 0$ and $x_0 + x_1 + x_2 - 3x_3 = 0$ and let H', K' be the planes defined by $x_0 + 2x_1 - 2x_2 + x_3 = 0$ and $2x_0 + 2x_1 + x_2 - 4x_3 = 0$. Let $L = H \cap K$ and $L' = H' \cap K'$. Choose $X \subset L$ and $X' \subset L'$ where

$$X = \{[4 : -4 : 3 : 1], [6 : -4 : 1 : 1], [5 : -4 : 2 : 1]\}$$

and

$$X' = \{[-1 : 2 : 2 : 1], [11 : -8 : -2 : 1], [-7 : 7 : 4 : 1]\}.$$

By computing the ideal of $L \star L'$ and the ideal of $X \star X'$ with `Singular`, we obtain

```
ideal J=HPr(L,L',3)
```

```
J[1]=1/5xy-21/50y^2-3/5yz-12/5xw+77/25yw-14/5zw+588/25w^2
```

```
ideal I=HPr(X,X',3)
I[1]=1/5x(0)x(1)-21/50x(1)^2-3/5x(1)x(2)-12/5x(0)x(3)+
+77/25x(1)x(3)-14/5x(2)x(3)+588/25x(3)^2
I[2]=1/5x(0)^3-9261/5000x(1)^3-9/5x(0)^2x(2)+
+27/5x(0)x(2)^2-27/5x(2)^3-15x(0)^2x(3)+
+25137/625x(1)^2x(3)+27x(0)x(2)x(3)+
+54x(2)^2x(3)+370x(0)x(3)^2-350763/1250x(1)x(3)^2-
-165x(2)x(3)^2-1389774/625x(3)^3
I[3]=-x(0)^2x(2)+441/100x(1)^2x(2)+6x(0)x(2)^2-9x(2)^3+
+40x(0)x(2)x(3)-1071/25x(1)x(2)x(3)+90x(2)^2x(3)-
-15x(1)x(3)^2-14274/25x(2)x(3)^2+180x(3)^3
I[4]=-x(0)x(2)^2+21/10x(1)x(2)^2+3x(2)^3-3/10x(1)^2x(3)-
-11/2x(1)x(2)x(3)-101/5x(2)^2x(3)+36/5x(1)x(3)^2+
+66x(2)x(3)^2-216/5x(3)^3
I[5]=1/10x(1)^3+2/5x(1)^2x(3)-464/5x(1)x(3)^2-3584/5x(3)^3
I[6]=-x(1)^2x(2)-16x(1)x(2)x(3)+320x(0)x(3)^2-672x(1)x(3)^2-
-224x(2)x(3)^2-3136x(3)^3
I[7]=-x(1)x(2)^2+32/5x(0)^2x(3)-3528/125x(1)^2x(3)-
-152/5x(0)x(2)x(3)-84/5x(1)x(2)x(3)+28/5x(2)^2x(3)-
-448x(0)x(3)^2+84672/125x(1)x(3)^2-392/5x(2)x(3)^2+
+471968/125x(3)^3
I[8]=-25/16x(2)^3-6x(0)^2x(3)+5367/200x(1)^2x(3)+
+57/2x(0)x(2)x(3)+181/8x(1)x(2)x(3)+6x(2)^2x(3)+
+365x(0)x(3)^2-53229/100x(1)x(3)^2+349/4x(2)x(3)^2-
-76294/25x(3)^3
```

whose Hilbert function is given by

```
HF(3,I,0)=1
HF(3,I,1)=4
HF(3,I,2)=9
HF(3,I,3)=9
```

that is $HF_{X\star X'}(t) = 9$ for $t \geq 2$.

Example 2.3.14. In this example we compute the ideal of $L \star L'$ and the ideal of $X \star X'$ with its Hilbert function, where X and X' are two sets of collinear points satisfying the hypotheses of Theorem 2.3.2 and $L \cap \Delta_1 \neq \emptyset \neq L' \cap \Delta_1$.

Let H, K be the planes defined by $x_0+2x_1+x_2+x_3=0$ and $x_0+x_1+x_2-3x_3=0$ and let H', K' be the planes defined by $x_0+x_1-2x_2+x_3=0$ and $x_0+x_1+x_2-4x_3=0$. Let $L=H\cap K$ and $L'=H'\cap K'$. Choose $X\subset L$ and $X'\subset L'$ where

$$X=\{[4:-4:3:1],[6:-4:1:1],[5:-4:2:1]\}$$

and

$$X'=\{[1:-1:\frac{5}{3}:1],[2:-2:\frac{5}{3}:1],[3:-3:\frac{5}{3}:1]\}.$$

By computing the ideal of $L\star L'$ and the ideal of $X\star X'$ with `Singular`, we obtain

```
ideal J=HPr(L,L',3);
J[1]=-3/5x(1)x(2)-4x(0)x(3)+7x(1)x(3)-28/5x(2)x(3)+196/3x(3)^2

ideal I=HPr(x(0),x(0)',3);
I[1]=-3/5x(1)x(2)-4x(0)x(3)+7x(1)x(3)
I[2]=-3/5x(2)^3+6x(2)^2x(3)-55/3x(2)x(3)^2+50/3x(3)^3
I[3]=-x(0)x(2)^2-5/3x(0)x(2)x(3)-50x(0)x(3)^2+250/3x(1)x(3)^2
I[4]=-x(0)^2x(2)-40/3x(0)^2x(3)+185/6x(0)x(1)x(3)-
-25/2x(1)^2x(3)
I[5]=1/4x(1)^3-6x(1)^2x(3)+44x(1)x(3)^2-96x(3)^3
I[6]=-x(0)x(1)^2+24x(0)x(1)x(3)-176x(0)x(3)^2-288/5x(2)x(3)^2+
+672x(3)^3
I[7]=-x(0)^2x(1)+24x(0)^2x(3)+132/5x(0)x(2)x(3)+
+216/25x(2)^2x(3)-308x(0)x(3)^2-1008/5x(2)x(3)^2+
+1176x(3)^3
I[8]=-x(0)^3+90x(0)^2x(3)-111x(0)x(1)x(3)+45x(1)^2x(3)+
+99x(0)x(2)x(3)+162/5x(2)^2x(3)-341x(0)x(3)^2-
-330x(1)x(3)^2-2448/5x(2)x(3)^2+2022x(3)^3
```

whose Hilbert function is given by

```
HF(3,I,0)=1
HF(3,I,1)=4
HF(3,I,2)=9
HF(3,I,3)=9
```

that is $HF_{X\star X'}(t)=9$ for $t\geq 2$.

Chapter 3

The Hilbert function of some Hadamard products

This chapter is inspired by the paper [BCFL2] in collaboration with C. Bocci, G. Fatabbi, A. Lorenzini.

We first study the Hilbert function of some Hadamard products of sets of points or lines, both in $\mathbb{P}^3$ and in $\mathbb{P}^n$ for any n. Then we study the Hadamard product of two generic linear spaces L_r and L_s, of dimensions respectively r and s, in $\mathbb{P}^n$ with $n = (r+1)(s+1) - 1$, showing that $L_r \star L_s$ is projectively equivalent to the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^n$. As a corollary we get that the Hilbert function of $L_r \star L_s$ is the product of the Hilbert functions of L_r and L_s.

3.1 Points and lines in $\mathbb{P}^3$

Let L and L' be lines in $\mathbb{P}^3$. Let X be a finite set of points on L and let X' be a finite set of points on L'. In this section we address the case $|X'| > |X|$.

In view of computing the Hilbert function of $X \star X'$ in the case $|X| \neq |X'|$ we first prove some general results.

Theorem 3.1.1. *Let $L_1, \dots, L_m$ be lines of a ruling of an irreducible and non-degenerate quadric Q in $\mathbb{P}^3$ with $m \geq 3$. If $Z = L_1 \cup \cdots \cup L_m$, then $\tau_Z = m - 1$ and $HF_Z(t) = (t+1)^2$ for $0 \leq t \leq m - 1$.*

Proof. First assume $t < m$. Consider Z' the union of m distinct lines of the other ruling. It is obvious that $I_{Z \cup Z'} \subseteq I_Z \subseteq I_{Z \cap Z'}$. By Bezout's Theorem applied twice, we have that $(I_{Z \cap Z'})_t = (I_{Z \cup Z'})_t = (I_Q)_t$.

Thus, for $t < m$ we have

$$HF_Z(t) = HF_Q(t) = \binom{t+3}{3} - \binom{t-2+3}{3} = (t+1)^2.$$

From [E, Theorem 4.2-2.], we obtain

$$\tau_Z \leq reg(R/I_Z) + pd(R/I_Z) - 3 = reg(I_Z) + pd(I_Z) - 3,$$

since $reg(R/I_Z) = reg(I_Z) - 1$ and $pd(R/I_Z) = pd(I_Z) + 1$. Furthermore in [DS, Theorem 2.1] it is proved that $reg(I_Z) \leq m$ and in [P, Corollary 1.9-2)] that $pd(I_Z) \leq 2$, so that $\tau_Z \leq m - 1$.

But $HF_Z(m-2) \neq HP_Z(m-2)$ and so $\tau_Z = m - 1$. □

Corollary 3.1.2. *Let Z and Z' be two sets of, respectively, m and m' distinct lines of the two different rulings of an irreducible and non-degenerate quadric Q in $\mathbb{P}^3$ with $m' \geq m \geq 3$. Then $\tau_{Z \cap Z'} = m' - 1$, $HF_{Z \cap Z'}(t) = (t+1)^2$, for $0 \leq t \leq m-1$, and $HF_{Z \cap Z'}(t) = m(t+1)$, for $m \leq t \leq m' - 1$.*

Proof. First assume $t < m'$. Then we have $HF_{Z \cap Z'}(t) = HF_Z(t)$ since, by Bezout's Theorem, $(I_{Z \cap Z'})_t = (I_Z)_t$. Thus, by Theorem 3.1.1, we have $HF_{Z \cap Z'}(t) = (t+1)^2$ for $0 \leq t \leq m-1$ and $HF_{Z \cap Z'}(t) = m(t+1)$ for $m \leq t \leq m'-1$. Finally observe that $HF_{Z \cap Z'}(m'-1) = mm' = |Z \cap Z'|$, i.e. $\tau_{Z \cap Z'} = m' - 1$. □

Now, we are able to extend Theorem 2.3.8 to the case $|X| \neq |X'|$.

Theorem 3.1.3. *Assume the hypotheses of Corollary 2.3.5. Further set $|X| = m$ and $|X'| = m'$ and assume $m' > m \geq 2$. Then $\tau_{X \star X'} = m' - 1$ and $HF_{X \star X'} = HF_X HF_{X'}$.*

Proof. First assume $m = 2$. Then, by Corollary 2.3.5, $X \star L'$ is the union of two skew lines. If $0 < t < m'$ then, by Bezout's Theorem, it follows that $(I_{X \star X'})_t = (I_{X \star L'})_t$ and so

$$HF_{X \star X'}(t) = HF_{X \star L'}(t) = 2(t+1) = HF_X(t) HF_{X'}(t).$$

Therefore

$$HF_{X \star X'}(m'-1) = 2m' = HP_{X \star X'}$$

and $\tau_{X \star X'} = m' - 1$ since $HF_{X \star X'}(m'-2) \neq HP_{X \star X'}$.

Now assume $m > 2$ and set $Z = X \star L'$ and $Z' = X' \star L$, whence $Z \cap Z' = X \star X'$. By Corollary 2.3.5 we can apply Corollary 3.1.2 to obtain

$$HF_{X \star X'}(t) = \begin{cases} (t+1)^2 & t < m, \\ m(t+1) & m \leq t < m' \\ mm' & t \geq m' - 1 \end{cases}.$$

On the other hand $(t+1)^2 = HF_X(t)HF_{X'}(t)$ for $t < m$, since $HF_X(t) = HF_{X'}(t) = t+1$ for $t < m$; $m(t+1) = HF_X(t)HF_{X'}(t)$ for $m \leq t < m'$, since $HF_X(t) = m$ and $HF_{X'}(t) = t+1$ for $m \leq t < m'$; and $mm' = HP_X HP_{X'}$. $\square$

Theorem 3.1.4. *Let L, L' be two generic lines in $\mathbb{P}^3$. Then there is a generic choice of a finite set of points $X \subseteq L$ such that $X \star L'$ are lines of a ruling of the quadric $L \star L'$, $\tau_{X \star L'} = |X| - 1$ and $HF_{X \star L'} = HF_X HF_{L'}$.*

Proof. By Corollary 2.3.5 there exist two finite sets of points $X \subseteq L$ and $X' \subseteq L'$ such that $X \star L'$ and $X' \star L$ are lines of the two different rulings of the irreducible and non-degenerate quadric $L \star L'$. Choose $|X| = |X'| = m$.

If $m = 2$ the result is obvious, so assume $m > 2$. Then, by Theorem 3.1.1, we have that, for $0 \leq t \leq m-1$, $HF_{X \star L'}(t) = (t+1)^2$ which equals $HF_X(t)HF_{L'}(t)$ since $HF_X(t) = HF_{L'}(t) = t+1$, while, for $t \geq m-1$, $HF_{X \star L'}(t) = m(t+1)$ which equals $HF_X(t)HF_{L'}(t)$ since $HF_X(t) = m$ and $HF_{L'}(t) = t+1$. It follows that $\tau_{X \star L'} = m-1$.

$\square$

Theorem 3.1.5. *Let L, L' be two generic lines in $\mathbb{P}^3$. Then $HF_{L \star L'} = HF_L HF_{L'}$.*

Proof. Since L and L' are generic, by [BCK, Theorem 6.8], $L \star L'$ is a quadric. In fact, $L \star L'$ is an irreducible quadric, as noticed right after Remark 2.5 of [BCK]. Thus we have

$$HF_{L \star L'}(t) = \binom{t+3}{3} - \binom{t-2+3}{3} = (t+1)^2$$

which equals $HF_L(t)HF_{L'}(t)$ since $HF_L(t) = HF_{L'}(t) = t+1$, for every $t \geq 0$. $\square$

Example 3.1.6. Let L and L', respectively, be the lines in $\mathbb{P}^3$ of equations

$$L : \begin{cases} 18x_1 + 3x_2 + 22x_3 = 0 \\ 10x_0 + 3x_1 + 3x_2 + 2x_3 = 0 \end{cases}$$

and

$$L' : \begin{cases} 18x_1 + 29x_2 - 9x_3 = 0 \\ 7x_0 + 5x_1 + 10x_2 + x_3 = 0 \end{cases}$$

and let I and J be respectively their ideals. Using the procedure `HPr`, in `Singular`, described in Appendix A, we compute the ideal K of $L \star L'$ as

```
ideal K=HPr(I,J,3);
```

which is, as expected, a quadric surface of equation

```
> K;
K[1]=68904*x(0)^2+104976*x(0)*x(1)+2430*x(1)^2-37758*x(0)*x(2)-
-8595*x(1)*x(2)+2465*x(2)^2+75636*x(0)*x(3)+11718*x(1)*x(3)-
-33048*x(2)*x(3)+6732*x(3)^2
```

Computing the Hilbert function of I, J and K we get

```
HF(3,I,0)=1        HF(3,J,0)=1        HF(3,K,0)=1
HF(3,I,1)=2        HF(3,J,1)=2        HF(3,K,1)=4
HF(3,I,2)=3        HF(3,J,2)=3        HF(3,K,2)=9
HF(3,I,3)=4        HF(3,J,3)=4        HF(3,K,3)=16
HF(3,I,4)=5        HF(3,J,4)=5        HF(3,K,4)=25
HF(3,I,5)=6        HF(3,J,5)=6        HF(3,K,5)=36
HF(3,I,6)=7        HF(3,J,6)=7        HF(3,K,6)=49
HF(3,I,7)=8        HF(3,J,7)=8        HF(3,K,7)=64
HF(3,I,8)=9        HF(3,J,8)=9        HF(3,K,8)=81
     ⋮                  ⋮                  ⋮
```

Example 3.1.7. Even when the lines L and L' are coplanar, $L \star L'$ might still be a quadric. For example, consider the lines $L = \overline{P_1P_2}$ and $L' = \overline{P_1P_3}$ where

$$P_1 = [1:1:1:1]\,,\ P_2 = \left[3 : \frac{3}{2} : 5 : \frac{7}{2}\right],\ P_3 = \left[\frac{3}{2} : 3 : \frac{4}{3} : \frac{7}{5}\right].$$

Hence the lines have equations

$$L : \begin{cases} 3x_1 + 4x_2 - 7x_3 = 0 \\ 7x_0 - 4x_1 - 3x_2 = 0 \end{cases}$$

and

$$L' : \begin{cases} x_1 + 24x_2 - 25x_3 = 0 \\ 10x_0 - x_1 - 9x_2 = 0 \end{cases} \quad .$$

The computations in `Singular`

```
> ring R=0,(x(0..3)),dp;
> ideal J1=3*x(1)+4*x(2)-7*x(3),7*x(0)-4*x(1)-3*x(2);
> ideal J2=x(1)+24*x(2)-25*x(3), 10*x(0)-x(1)-9*x(2);
> ideal K=HPr(J1,J2,3);
> K;
K[1]=1120*x(0)^2-68*x(0)*x(1)+x(1)^2+1056*x(0)*x(2)-
-30*x(1)*x(2)+216*x(2)^2-3500*x(0)*x(3)+110*x(1)*x(3)-
-1530*x(2)*x(3)+2625*x(3)^2
```

show, indeed, that $L \star L'$ is still a quadric. As a consequence

$$HF_{L\star L'}(t) = (t+1)^2 = HF_L(t)HF_{L'}(t),\ \forall t.$$

However, there are cases in which $L \star L'$ is a plane as the following example shows.

Example 3.1.8. Let $A_i = [\alpha_i : \beta_i]$, for $i = 0, \dots, 3$, be four distinct points of $\mathbb{P}^1 \setminus \Delta_0$ and consider the points

$$P_1 = [1:1:1:1]$$

$$P_2 = \left[\frac{\alpha_0+\beta_0}{\alpha_0} : \frac{\alpha_1+\beta_1}{\alpha_1} : \frac{\alpha_2+\beta_2}{\alpha_2} : \frac{\alpha_3+\beta_3}{\alpha_3}\right].$$

$$P_3 = \left[\frac{\alpha_0+\beta_0}{\beta_0} : \frac{\alpha_1+\beta_1}{\beta_1} : \frac{\alpha_2+\beta_2}{\beta_2} : \frac{\alpha_3+\beta_3}{\beta_3}\right],$$

In [BC], it is shown that for the lines $L = \overline{P_1P_2}$ and $L' = \overline{P_1P_3}$, $L \star L'$ is a plane. For example, consider

$$A_0 = [1:2],\ A_1 = [2:1],\ A_2 = [1:3],\ A_3 = [2:5].$$

Thus the points are

$$P_1 = [1:1:1:1]\,,\ P_2 = \left[3:\frac{3}{2}:4:\frac{7}{2}\right],\ P_3 = \left[\frac{3}{2}:3:\frac{4}{3}:\frac{7}{5}\right].$$

Hence the lines have equations

$$L : \begin{cases} x_1 + 4x_2 - 5x_3 = 0 \\ 5x_0 - 2x_1 - 3x_2 = 0 \end{cases}$$

and

$$L' : \begin{cases} x_1 + 24x_2 - 25x_3 = 0 \\ 10x_0 - x_1 - 9x_2 = 0 \end{cases}.$$

Notice that, with respect to the previous example, only the third coordinate of P_2 is changed.

The computations in `Singular`

```
> ring R=0,(x(0..3)),dp;
> ideal I1=x(1)+4*x(2)-5*x(3),5*x(0)-2*x(1)-3*x(2);
```

```
> ideal I2=x(1)+24*x(2)-25*x(3), 10*x(0)-x(1)-9*x(2);
> ideal K=HPr(I1,I2,3);
> K;
K[1]=40*x(0)-x(1)+36*x(2)-75*x(3)
```

show, indeed, that $L \star L'$ is a plane.

Notice that, in this case, we have that

$$HF_{L\star L'}(t) = \binom{t+2}{2} \neq (t+1)^2 = HF_L(t)HF_{L'}(t), \forall t \geq 1.$$

3.2 Points and lines in $\mathbb{P}^n$

In order to both extend and give a more constructive version of Corollary 2.3.5 in $\mathbb{P}^n$ (Theorem 3.2.4), we need the following two lemmas, the first of which extends Theorem 2.1.1-(1).

Lemma 3.2.1. *Let L be a line of $\mathbb{P}^n$ such that $|L \cap \Delta_{n-1}| = n+1$, and let P be a point in $\Delta_{n-1} \setminus \Delta_{n-2}$. For every two points $Q, R \in L$ we have $P \star Q = P \star R$ if and only if $Q = R$.*

Proof. Since $|L \cap \Delta_{n-1}| = n+1$, write $L \cap \Delta_{n-1} = \{P_0, \dots P_n\}$ with $P_i \in H_i \setminus (\bigcup_{j\neq i} H_j)$. We may assume $P = [0 : p_1 \cdots : p_n]$ with $p_i \neq 0$ for all $i = 1, \dots, n$. The statement is obvious if $Q, R \notin H_0$.

In the other cases, let $Q = [q_0 : \cdots : q_n]$ and $R = [r_0 : \cdots : r_n]$. If $P \star Q = P \star R$, then we have $[0 : p_1q_1 : \cdots : p_nq_n] = [0 : p_1r_1 : \cdots : p_nr_n]$, and so $R = [r_0 : q_1 : \cdots : q_n]$. If $Q \neq R$, then we have $[1 : 0 : \cdots : 0] \in L \cap H_i = \{P_i\}$, for all $i = 1, \dots, n$, in contradiction with $P_i \in H_i \setminus (\bigcap_{j\neq i} H_j)$. □

Remark 3.2.2. If $P \notin \Delta_{n-1}$ and L' is any line, then the set $\{P \star P' \mid P' \in L'\}$ is projectively equivalent to the line L', and therefore it is closed. Thus $P \star L' = \{P \star P' \mid P' \in L'\}$. Actually, there is no need for L' to be a line: with the same reasoning we obtain $P \star Y = \{P \star Q \mid Q \in Y\}$, where $P \notin \Delta_{n-1}$ and Y is a closed subset of $\mathbb{P}^n$.

Lemma 3.2.3. *Let L, L' be two generic lines in $\mathbb{P}^n$. Then, for all $P, Q \in L$ with $P \neq Q$, we have that $(P \star L') \cap (Q \star L') = \emptyset$. Similarly, $(L \star P') \cap (L \star Q') = \emptyset$, for all $P', Q' \in L'$ with $P' \neq Q'$.*

Proof. We will prove only the first assertion, the other one being similar.

Since L and L' are generic, we may assume that they are skew and that they intersect Δ_{n-1} in exactly $n+1$ points and don't intersect Δ_{n-2}.

First we observe that if $P, Q \notin \Delta_{n-1}$ then both $P \star L'$ and $Q \star L'$ are lines [BCK, Lemma 3.1]. Even when $P \in \Delta_{n-1}$, then $P \star L'$ is a linear space of dimension less than or equal to 1, but by Lemma 3.2.1 $P \star L'$ contains at least 2 points and so it is a line. Similarly for $Q \star L'$.

Now assume, by contradiction, that there exist two distinct points P, Q in L such that $(P \star L') \cap (Q \star L') \neq \emptyset$. We claim that either $P \star L'$ and $Q \star L'$ are distinct or that there exist $P', Q' \in L'$ such that $L \star P'$ and $L \star Q'$ are distinct with non-empty intersection. In fact, if $P \star L' = Q \star L'$ and $P \in H_i$, for some i, then also $P \star L' \subseteq H_i$, whence $Q \star L' \subseteq H_i$. Now pick $P' \in L' \setminus \Delta_{n-1}$. Then $Q \star P' \in H_i$ yields $Q \in H_i$. Since L intersects H_i in a single point, necessarily $P = Q$. But $P \neq Q$ by hypothesis, and so $P, Q \notin \Delta_{n-1}$. Hence, by Remark 3.2.2, $P \star L' = \{P \star Q' \mid Q' \in L'\}$. Thus we have that for each $P' \in L'$, there exists $Q' \in L'$ such that $P \star P' = Q \star Q'$, and so $(L \star P') \cap (L \star Q') \neq \emptyset$. Since $L \star L'$ is an irreducible variety of dimension 2, we may assume that $L \star P' \neq L \star Q'$ and this proves the claim.

Therefore, by exchanging the roles if necessary, we may assume that $P \star L'$ and $Q \star L'$ generate a unique plane π.

Now, $\forall P' \in L'$ we have

$$\{P \star P', Q \star P'\} \subseteq (L \star P') \cap [(P \star L') \cup (Q \star L')] .$$

Either by Theorem 2.1.1-(1) or by Lemma 3.2.1, we have $P \star P' \neq Q \star P'$ and so $L \star P' \subseteq \pi$. Thus

$$L \star L' \subseteq \overline{\bigcup_{P' \in L'} (L \star P')} \subseteq \pi.$$

But this is impossible since $L \star L'$ is an irreducible variety of dimension 2 and degree 2. ☐

Theorem 3.2.4. *Let L, L' be two generic distinct lines in $\mathbb{P}^n$ and let X and X' be two finite sets of points with $X \subseteq L$ and $X' \subseteq L'$. Then:*

(1) *$X \star X' = (X \star L') \cap (X' \star L)$ and $|X \star X'| = |X||X'|$.*

(2) *$L \star L' \subset \mathbb{P}^3 \subset \mathbb{P}^n$, hence, as a subvariety of $\mathbb{P}^3$, $L \star L'$ is an irreducible and non-degenerate quadric, and $X \star L'$ and $X' \star L$ are lines of the two different rulings.*

(3) *The rulings of the quadric $L \star L'$ are given by $\{P \star L' | P \in L\}$ and $\{L \star P' | P' \in L'\}$.*

Proof.

(1) The first equality is clear, since it follows from Lemma 3.2.3 that $P \star L' \neq L \star P'$ for all $P \in X$ and for all $P' \in X'$. In order to prove the second equality assume, by contradiction, that there exist $P, Q \in X$ and $P', Q' \in X'$ such that $P \star P' = Q \star Q'$. Now, $P \star P' \in L \star P'$ and $Q \star Q' \in L \star Q'$, and so $(L \star P') \cap (L \star Q') \neq \emptyset$. By Lemma 3.2.3 this means $P' = Q'$, which in turn implies $P = Q$ either by Theorem 2.1.1-(1) or by Lemma 3.2.1.

(2) We already observed that $L \star L'$ is an irreducible variety of dimension 2 and degree 2. Now we prove that it is contained in $\mathbb{P}^3$. Let P, Q be two distinct points of X. By Lemma 3.2.3, $P \star L'$ and $Q \star L'$ generate a linear subspace $\Lambda \simeq \mathbb{P}^3$. Now, for all $P' \in L'$, we have $L \star P' \subset \Lambda$ and so

$$L \star L' = \overline{\{L \star P' | P' \in L'\}} \subset \Lambda.$$

The last part of the statement and the fact that $L \star L'$ is non degenerate follow easily from Lemma 3.2.3.

(3) We prove that a ruling is given by $\{L \star P' | P' \in L'\}$, the other proof being similar. Fix $P \in L \setminus \Delta_{n-1}$ and consider $P \star L'$. By Remark 3.2.2, $P \star L' = \{P \star Q' \mid Q' \in L'\}$. Now if L'' is any line of the other ruling, then $L'' \cap (P \star L') \neq \emptyset$ and thus there is $P' \in L'$ such that $L'' \cap (P \star L') = \{P \star P'\} = (P \star L') \cap (L \star P')$. Since $L \star L'$ is not degenerate, this forces $L'' = L \star P'$.

□

Remark 3.2.5. Part (3) of Theorem 3.2.4 yields a first extension of Remark 3.2.2 also to points of $\Delta_{n-1} \setminus \Delta_{n-2}$, as long as we have L, L' generic with $P \in L$. But we also obtain $L \star L' = \{P \star P' | P \in L, P' \in L'\}$ for generic lines L and L', which is another extension of Remark 3.2.2.

Example 3.2.6. Let L be a generic line in $\mathbb{P}^n$ and let $P = [p_0 : \cdots : p_n]$, $P' = [p'_0 : \cdots : p'_n]$ be two distinct points not in Δ_{n-1} with $P \in L$ and $P' \notin L$. Denote $\frac{P'}{P} = \left[\frac{p'_0}{p_0} : \cdots : \frac{p'_n}{p_n}\right]$. Pick any $Q' \in L \star \frac{P'}{P}$ and consider the line, L', joining P' and Q'. By Remark 3.2.2, we have that $L \star \frac{P'}{P} = \left\{Q \star \frac{P'}{P} | Q \in L\right\}$, and so there is $Q \in L$ such that $Q' = Q \star \frac{P'}{P}$. But then $Q \star P' = P \star Q'$, so that $(L \star P') \cap (L \star Q') \neq \emptyset$. As in the proof of Lemma 3.2.3 we may assume $L \star P' \neq L \star Q'$ and therefore $L \star L'$ is contained in a plane π (in fact $L \star L' = \pi$).

In general $X \star X'$ is not a complete intersection in $\mathbb{P}^n$ (Corollary 2.3.11). But we shall see that it is a *complete intersection on the quadric* $L \star L'$, i.e. its ideal in the coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ is generated by two bi-homogeneous polynomials of degree $(m, 0)$ and $(0, m')$, respectively ([GMR, Remark 1.3]).

Corollary 3.2.7. *Let* L, L' *be two generic distinct lines in* $\mathbb{P}^n$ *and let* Y *be a finite set of points on* $L \star L'$*. Then* Y *is a complete intersection on the quadric* $L \star L'$ *if and only if* $Y = X \star X'$ *for suitable finite sets of points* $X \subset L$ *and* $X' \subset L'$*.*

Proof. The "only if " part follows from (1) and (2) of Theorem 3.2.4. The other implication follows from (3) of the same theorem. □

Now we want to extend Theorem 3.1.3 and Theorem 2.3.8 to $\mathbb{P}^n$ (Theorem 3.2.9), but to do so we need a preliminary result which generalizes [FL, Corollary 3.3].

Proposition 3.2.8. *Let* V *be a reduced (not necessarily irreducible) subvariety of* $\mathbb{P}^n$ *which is contained in* $\mathbb{P}^r$ *for some* $r < n$*. Then*

$$HF_{V,\mathbb{P}^n} = HF_{V,\mathbb{P}^r}.$$

Proof. First assume $r = n - 1$. The Hilbert function of V in $\mathbb{P}^n$ can be recovered from the Hilbert function of V in $\mathbb{P}^{n-1}$ by using [FHL, Corollary 3.2], which works throughout without the assumption that $L_1, \ldots, L_r$ (of [FHL]) are linear subvarieties. After taking $W = V \subseteq \mathbb{P}^n$, $Y = V \subseteq \mathbb{P}^r$, $k = 0$, $l' = 1$ and $l_0 = 0$ (whence $\lambda = 0$) in [FHL, Corollary 3.2], we obtain

$$dim_{\mathbb{K}} \left(I_{V,\mathbb{P}^n}\right)_t = \begin{cases} dim_{\mathbb{K}} \left(I_{V,\mathbb{P}^{n-1}}\right)_0 = 0 & t = 0 \\ dim_{\mathbb{K}} \left(I_{V,\mathbb{P}^{n-1}}\right)_t + \binom{t-1+n}{n} & t > 0. \end{cases}$$

Therefore, for all $t > 0$, $HF_{V,\mathbb{P}^n}(t) = \binom{t+n}{n} - dim_{\mathbb{K}} \left(I_{V,\mathbb{P}^{n-1}}\right)_t - \binom{t-1+n}{n} = \binom{t+n-1}{n-1} - dim_{\mathbb{K}} \left(I_{V,\mathbb{P}^{n-1}}\right)_t = HF_{V,\mathbb{P}^{n-1}}(t)$. Since for $t = 0$ the equality is obvious, we are done for the case $r = n - 1$.

The result follows by iterating the above procedure. □

Theorem 3.2.9. *Assume the hypotheses of Theorem 3.2.4. Set* $|X| = m$ *and* $|X'| = m'$ *and assume* $m' \geq m \geq 2$*. Then* $\tau_{X\star X'} = m' - 1$ *and* $HF_{X\star X'} = HF_X HF_{X'}$*.*

Proof. By (2) of Theorem 3.2.4, $X \star X' \subset L \star L' \subset \mathbb{P}^3$ and hence, by Proposition 3.2.8, we may compute its Hilbert function in $\mathbb{P}^3$. Now Theorem 3.1.3 and Theorem 2.3.8 apply throughout. □

The Hilbert function of the Hadamard product of two subsets, X and X', of collinear points depends on the genericity of the lines in which they are contained. Two behaviours may occur: either $HP_{X\star X'} \neq HP_X HP_{X'}$ (hence, a fortiori, $HF_{X\star X'} \neq HF_X HF_{X'}$) or $HP_{X\star X'} = HP_X HP_{X'}$ but $HF_{X\star X'} \neq HF_X HF_{X'}$, as the following examples show.

Example 3.2.10. We use the construction introduced in Example 3.2.6: L is a generic line, $P \in L$ and $P' \notin L$, $Q' \in L \star \frac{P'}{P}$ and $Q \in L$ such that $Q' = Q \star \frac{P'}{P}$, so that $Q \star P' = P \star Q'$. Set $X = \{P, Q\} \subseteq L$ and $X' = \{P', Q'\} \subseteq L'$. Then $|X \star X'| = 3$ and so $HP_{X \star X'} \neq HP_X HP_{X'}$.

Example 3.2.11. Let $A_i = [\alpha_i : \beta_i]$, for $i = 0, \dots, n$, be distinct points of $\mathbb{P}^1 \setminus \Delta_0$ and consider the two sets of collinear points

$$P_h = \left[\frac{\alpha_0 + h\beta_0}{\alpha_0} : \frac{\alpha_1 + h\beta_1}{\alpha_1} : \cdots : \frac{\alpha_n + h\beta_n}{\alpha_n}\right], \tag{3.1}$$

$$P'_j = \left[\frac{j\alpha_0 + \beta_0}{\beta_0} : \frac{j\alpha_1 + \beta_1}{\beta_1} : \cdots : \frac{j\alpha_n + \beta_n}{\beta_n}\right], \tag{3.2}$$

with, h, j non-negative integers.

Let L and L' be respectively the lines through the P_h's and the P'_j's. In [BC], it is shown that:

1) $P_k \star P'_l = P_r \star P'_s$ if and only if $k = r$ and $l = s$;

2) $P_0 \star P'_0 = [1 : 1 : \cdots : 1] = P_0 = P'_0$;

3) $L \star L'$ is a plane.

Consider the sets $X = \{P_1, P_2\}$ and $X' = \{P'_1, P'_2\}$. By 1) we have $HP_{X \star X'} = HP_X HP_{X'}$, while, by 3), we know that $X \star X'$ is contained in a plane and hence $HF_{X \star X'}(1) = 3 \neq 2 \cdot 2 = HF_X(1) HF_{X'}(1)$.

Also Theorem 3.1.4 can be extended to $\mathbb{P}^n$ as follows:

Theorem 3.2.12. *Let L, L' be two generic lines in $\mathbb{P}^n$ and let X be a finite set of points with $X \subseteq L$. Then $\tau_{X \star L'} = |X| - 1$ and $HF_{X \star L'} = HF_X HF_{L'}$.*

Proof. Let $X' \subset L'$ with $|X'| = |X| = m$. By Theorem 3.2.4 $X \star X' \subset X \star L' \subset L \star L' \subset \mathbb{P}^3$ and so, by Proposition 3.2.8, we may compute the Hilbert function of $X \star X'$ in $\mathbb{P}^3$. Now we can use Theorem 3.1.1 and proceed as in the proof of Theorem 3.1.4. □

Now we extend Theorem 3.1.5 to $\mathbb{P}^n$.

Theorem 3.2.13. *Let L, L' be two generic lines in $\mathbb{P}^n$. Then $HF_{L \star L'} = HF_L HF_{L'}$.*

Proof. By (2) of Theorem 3.2.4, $L \star L' \subset \mathbb{P}^3$, and hence, by Proposition 3.2.8, we can apply Theorem 3.1.5. □

Remark 3.2.14. Examples 3.1.8 and 3.2.6 show that Theorem 3.2.13 may fail if we drop the assumption that L and L' are generic. In fact, in those cases we have that $L \star L'$ is a plane and so

$$HF_{L\star L'}(t) = \binom{t+2}{2} \neq (t+1)^2 = HF_L(t)HF_{L'}(t), \forall t \geq 1.$$

3.3 Hadamard product of two generic linear spaces

In this section we extend Theorem 3.1.5 and Theorem 3.2.13.

We shall use the notation $\mathcal{M}_{h\times k}(\mathbb{K})$ for the set of $h \times k$ matrices with entries in $\mathbb{K}$ and I_j for the $j \times j$ identity matrix.

Theorem 3.3.1. *Let L_r and L_s be two generic linear spaces, of dimensions r and s, respectively, in $\mathbb{P}^N$ with $N = (r+1)(s+1) - 1$. Then $L_r \star L_s$ is projectively equivalent to the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ in $\mathbb{P}^N$.*

Proof. Assume that L_r and L_s have parametric equations given respectively by

$$L_r : \begin{cases} x_0 = f_0(y_0, \dots, y_r) \\ x_1 = f_1(y_0, \dots, y_r) \\ \quad\vdots \\ x_N = f_N(y_0, \dots, y_r) \end{cases} \text{and } L_s : \begin{cases} x_0 = g_0(z_0, \dots, z_s) \\ x_1 = g_1(z_0, \dots, z_s) \\ \quad\vdots \\ x_N = g_N(z_0, \dots, z_s) \end{cases},$$

where $f_i(y_0, \dots, y_r) = a_{i0}y_0 + a_{i1}y_1 + \cdots + a_{ir}y_r$ and $g_i(z_0, \dots, z_s) = b_{i0}z_0 + b_{i1}z_1 + \cdots + b_{is}z_s$, for $i = 0, \dots, N$. By definition of the Hadamard product, $L_r \star L_s$ is the closure of the set Σ with parametric equations

$$\Sigma : \begin{cases} x_0 = f_0(y_0, \dots, y_r) \cdot g_0(z_0, \dots, z_s) \\ x_1 = f_1(y_0, \dots, y_r) \cdot g_1(z_0, \dots, z_s) \\ \quad\vdots \\ x_N = f_N(y_0, \dots, y_r) \cdot g_N(z_0, \dots, z_s) \end{cases},$$

or more explicitly

$$\Sigma : \begin{cases} x_0 = (a_{00}y_0 + a_{01}y_1 + \cdots + a_{0r}y_r)(b_{00}z_0 + b_{01}z_1 + \cdots + b_{0s}z_s) \\ x_1 = (a_{10}y_0 + a_{11}y_1 + \cdots + a_{1r}y_r)(b_{10}z_0 + b_{11}z_1 + \cdots + b_{1s}z_s) \\ \quad\vdots \\ x_N = (a_{N0}y_0 + a_{N1}y_1 + \cdots + a_{Nr}y_r)(b_{N0}z_0 + b_{N1}z_1 + \cdots + b_{Ns}z_s) \end{cases}.$$

Consider the matrix M of size $(N+1) \times (N+1)$ defined as

$$M = \begin{pmatrix} a_{00}b_{00} & \dots & a_{00}b_{0s} & a_{01}b_{00} & \dots & a_{0r}b_{0s} \\ a_{10}b_{10} & \dots & a_{10}b_{1s} & a_{11}b_{10} & \dots & a_{1r}b_{1s} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N0}b_{N0} & \dots & a_{N0}b_{Ns} & a_{N1}b_{N0} & \dots & a_{Nr}b_{Ns} \end{pmatrix}.$$

Notice that M can be expressed as the Khatri-Rao product (developed by single rows)

$$M = A \otimes_{\mathrm{KR}} B$$

where

$$A = \begin{pmatrix} a_{00} & a_{01} & \dots & a_{0r} \\ a_{10} & a_{11} & \dots & a_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N0} & a_{N1} & \dots & a_{Nr} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{00} & b_{01} & \dots & b_{0s} \\ b_{10} & b_{11} & \dots & b_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N0} & b_{N1} & \dots & b_{Ns} \end{pmatrix}.$$

Equivalently M can be expressed as the Hadamard product

$$M = \tilde{A} \star \tilde{B}$$

where $\tilde{A}$ and $\tilde{B}$ are matrices of size $(N+1) \times (N+1)$ defined as

$$\tilde{A} = \left(\underbrace{\begin{matrix} a_{00} & \dots & a_{00} \\ a_{10} & \dots & a_{10} \\ \vdots & \vdots & \vdots \\ a_{N0} & \dots & a_{N0} \end{matrix}}_{s+1 \text{ times}} \; \underbrace{\begin{matrix} a_{01} & \dots & a_{01} \\ a_{11} & \dots & a_{11} \\ \vdots & \vdots & \vdots \\ a_{N1} & \dots & a_{N1} \end{matrix}}_{s+1 \text{ times}} \; \begin{matrix} \dots \\ \dots \\ \ddots \\ \end{matrix} \; \underbrace{\begin{matrix} a_{0r} & \dots & a_{0r} \\ a_{1r} & \dots & a_{1r} \\ \vdots & \vdots & \vdots \\ a_{Nr} & \dots & a_{Nr} \end{matrix}}_{s+1 \text{ times}} \right)$$

$$\tilde{B} = \left(\underbrace{B \quad B \quad \dots \quad B}_{r+1 \text{ times}} \right).$$

By considering each f_i as a point of $\mathbb{P}^r$ (by duality), we can view L_r as a point of $\underbrace{\mathbb{P}^r \times \cdots \times \mathbb{P}^r}_{N \text{ times}}$.

Similarly L_s can be viewed as a point of $\underbrace{\mathbb{P}^s \times \cdots \times \mathbb{P}^s}_{N \text{ times}}$.

Now, since $\det(M)$ is multi-homogeneous, the locus where $\det M \neq 0$ is an open subset of $\underbrace{(\mathbb{P}^r \times \cdots \times \mathbb{P}^r)}_{N \text{ times}} \times \underbrace{(\mathbb{P}^s \times \cdots \times \mathbb{P}^s)}_{N \text{ times}}$.

To see that such an open set is non-empty choose

$$A^i = \begin{pmatrix} 0 & \dots & 0 & \overset{i}{1} & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \mathcal{M}_{(s+1)\times(r+1)}(\mathbb{K}),\ \forall i = 1, \dots, r+1,$$

$$A = \begin{pmatrix} A^1 \\ \vdots \\ A^i \\ \vdots \\ A^{r+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} I_{s+1} \\ \vdots \\ I_{s+1} \\ \vdots \\ I_{s+1} \end{pmatrix} \in \mathcal{M}_{(r+1)(s+1)\times(s+1)}(\mathbb{K}).$$

Then $M = \tilde{A} \star \tilde{B} = I_{N+1}$.

Thus we can generically choose the f_i's and the g_i's so that $\det(M) \neq 0$.

Observe that $rk(\tilde{A}) = rk(A)$ and $rk(\tilde{B}) = rk(B)$ and recall that $rk(\tilde{A} \star \tilde{B}) \leq rk(\tilde{A})rk(\tilde{B})$ ([M, Theorem 4.5]). Therefore when $\det(M) \neq 0$ we have $rk(A) = r+1$ and $rk(B) = s+1$. Thus we can generically choose L_r and L_s so that M gives a projective isomorphism.

If the Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$ is defined as

$$\begin{array}{ccc} \mathbb{P}^r \times \mathbb{P}^s & \longrightarrow & \mathbb{P}^N \\ ([y_0 : \cdots : y_r], [z_0 : \cdots : z_s]) & \mapsto & [y_0z_0 : y_0z_1 : \cdots : y_0z_s : y_1z_0 : \cdots : y_rz_s], \end{array}$$

and its image is denoted by S, then the map $\mathbb{P}^N \xrightarrow{M} \mathbb{P}^N$ sends each point $P \in S$ to a point $M(P) \in \Sigma \subseteq L_r \star L_s$. Since $dim(S) = r + s = dim(L_r \star L_s)$, they are projectively equivalent. □

Remark 3.3.2. In the proof of Theorem 3.3.1 the genericity of the linear subspaces L_r and L_s is only used to say that the matrix M has non-zero determinant.

Example 3.3.3. Let L_r be the line in $\mathbb{P}^5$ of equations

$$L_r : \begin{cases} 5x_3 + 17x_4 - 8x_5 = 0 \\ x_2 + 2x_4 - x_5 = 0 \\ 5x_1 + 11x_4 - 4x_5 = 0 \\ 5x_0 + 7x_4 - 3x_5 = 0 \end{cases}$$

and let L_s be the plane in $\mathbb{P}^5$ of equations

$$L_s : \begin{cases} 105x_2 - 109x_3 - 42x_4 + 13x_5 = 0 \\ 6x_1 + 2x_3 - 3x_4 + x_5 = 0 \\ 7x_0 + 4x_3 + x_5 = 0 \end{cases} .$$

Using `Singular` we can check that $L_r \star L_s$ is equivalent to the Segre embedding S of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$ via the projectivity M defined in the proof of Theorem 3.3.1.

The parametric equations of L_r and L_s are respectively

$$L_r : \begin{cases} x_0 = y_0 + y_1 \\ x_1 = y_0 + 2y_1 \\ x_2 = 2y_0 + y_1 \\ x_3 = 3y_0 + 2y_1 \\ x_4 = y_0 - 2y_1 \\ x_5 = 4y_0 - 3y_1 \end{cases}$$

and

$$L_s : \begin{cases} x_0 = z_0 + z_1 - z_2 \\ x_1 = z_0 + 2z_1 + 3z_2 \\ x_2 = 2z_0 - z_1 + 5z_2 \\ x_3 = z_0 - 2z_1 + 2z_2 \\ x_4 = -z_0 + 3z_1 + 7z_2 \\ x_5 = -11z_0 + z_1 - z_2 \end{cases} .$$

Hence

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \\ 3 & 2 \\ 1 & -2 \\ 4 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 3 \\ 2 & -1 & 5 \\ 1 & -2 & 2 \\ -1 & 3 & 7 \\ -11 & 1 & -1 \end{pmatrix}$$

and, following the proof of Theorem 3.3.1, one has

$$\tilde{A} \star \tilde{B} = M = \begin{pmatrix} 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 2 & 4 & 6 \\ 4 & -2 & 10 & 2 & -1 & 5 \\ 3 & -6 & 6 & 2 & -4 & 4 \\ -1 & 3 & 7 & 2 & -6 & -14 \\ -44 & 4 & -4 & 33 & -3 & 3 \end{pmatrix} .$$

Let `Lrs` and `Segre` be respectively the ideals of $L_r \star L_s$ and of S:

```
ring R=0,(x(0..5)),dp;
ideal Lr=5*x(3)+17*x(4)-8*x(5), x(2)+2*x(4)-x(5),
5*x(1)+11*x(4)-4*x(5),5*x(0)+7*x(4)-3*x(5);
ideal Ls=105*x(2)-109*x(3)-42*x(4)+13*x(5),
6*x(1)+2*x(3)-3*x(4)+x(5),7*x(0)+4*x(3)+x(5);
ideal Lrs=HPr(Lr,Ls,5);
ideal Segre=x(2)*x(4)-x(1)*x(5),x(2)*x(3)-x(0)*x(5),
x(1)*x(3)-x(0)*x(4);
```

The matrix M defines the following map between rings

$$\begin{array}{ccc}
\mathbb{K}[x_0,x_1,x_2,x_3,x_4,x_5] & \to & \mathbb{K}[x_0,x_1,x_2,x_3,x_4,x_5] \\
x_0 & \mapsto & x_0+x_1-x_2+x_3+x_4-x_5 \\
x_1 & \mapsto & x_0+2x_1+3x_2+2x_3+4x_4+6x_5 \\
x_2 & \mapsto & 4x_0-2x_1+10x_2+2x_3-x_4+5x_5 \\
x_3 & \mapsto & 3x_0-6x_1+6x_2+2x_3-4x_4+4x_5 \\
x_4 & \mapsto & -x_0+3x_1+7x_2+2x_3-6x_4-14x_5 \\
x_5 & \mapsto & -44x_0+4x_1-4x_2+33x_3-3x_4+3x_5
\end{array}$$

which, in `Singular`, is expressed as

```
map f= R,x(0)+x(1)-x(2)+x(3)+x(4)-x(5),
x(0)+2*x(1)+3*x(2)+2*x(3)+4*x(4)+6*x(5),
4*x(0)-2*x(1)+10*x(2)+2*x(3)-x(4)+5*x(5),
3*x(0)-6*x(1)+6*x(2)+2*x(3)-4*x(4)+4*x(5),
-x(0)+3*x(1)+7*x(2)+2*x(3)-6*x(4)-14*x(5),
-44*x(0)+4*x(1)-4*x(2)+33*x(3)-3*x(4)+3*x(5);
```

To prove the statement it is sufficient to show that the image of the ideal `Lrs`, under f, is the ideal `Segre`, i.e.

`(f(Lrs):Segre)=(1)` and `(Segre:f(Lrs))=(1)`.

This can be easily verified in `Singular`:

```
> quotient(f(Lrs),Segre);
_[1]=1
> quotient(Segre,f(Lrs));
_[1]=1
```

Remark 3.3.4. It follows from the proof of Theorem 3.3.1 that if L_r and L_s are two generic linear spaces, of dimensions respectively r and s, in $\mathbb{P}^N$ with $N=(r+1)(s+1)-1$, then

$$L_r \star L_s = \{P \star Q | P \in L_r, Q \in L_s\}.$$

This is an extension of Remark 3.2.5.

Corollary 3.3.5. *If L_r and L_s are two generic linear spaces, of dimensions respectively r and s, in $\mathbb{P}^N$ with $N=(r+1)(s+1)-1$, then $HF_{L_r \star L_s} = HF_{L_r} HF_{L_s}$.*

Example 3.3.6. Consider again the linear spaces L_r and L_s of Example 3.3.3. Computing, in `Singular`, the Hilbert functions of L_r, L_s and $L_r \star L_s$ we get

```
HF(3,Lr,0)=1        HF(3,Ls,0)=1        HF(3,Lrs,0)=1
HF(3,Lr,1)=2        HF(3,Ls,1)=3        HF(3,Lrs,1)=6
HF(3,Lr,2)=3        HF(3,Ls,2)=6        HF(3,Lrs,2)=18
HF(3,Lr,3)=4        HF(3,Ls,3)=10       HF(3,Lrs,3)=40
HF(3,Lr,4)=5        HF(3,Ls,4)=15       HF(3,Lrs,4)=75
HF(3,Lr,5)=6        HF(3,Ls,5)=21       HF(3,Lrs,5)=126
HF(3,Lr,6)=7        HF(3,Ls,6)=28       HF(3,Lrs,6)=196
HF(3,Lr,7)=8        HF(3,Ls,7)=36       HF(3,Lrs,7)=288
HF(3,Lr,8)=9        HF(3,Ls,8)=45       HF(3,Lrs,8)=405

      ⋮                   ⋮                    ⋮
```

Remark 3.3.7. The Hadamard product of two generic linear spaces L_r and L_s, of dimensions respectively r and s, can be defined also in $\mathbb{P}^m$ with $\max\{r, s\} < m < (r+1)(s+1)-1$. However, according to [BCK, Remark 6.7], in this case $L_r \star L_s$ is not identifiable i.e. the secant variety does not have the expected dimension, since the dimension of the linear span of $L_r \star L_s$ is strictly less than $(r+1)(s+1)-1$. Notice that, in this case, Corollary 3.3.5 is not true. Consider, for example, the Hadamard product of a line L_r and a plane L_s in $\mathbb{P}^4$ of equations respectively

$$L_r : \begin{cases} 8x_2 - 5x_3 - x_4 = 0 \\ 2x_1 - x_3 + x_4 = 0 \\ 8x_0 - 3x_3 + x_4 = 0 \end{cases}$$

and

$$L_s : \begin{cases} 26x_1 - 35x_2 + 45x_3 + x_4 = 0 \\ 13x_0 - 15x_2 + 23x_3 + 6x_4 = 0 \end{cases} .$$

The Hadamard product $L_r \star L_s$ is a variety of dimension 3 and degree 3, as in the case of Example 3.3.3. However, computations in `Singular` show that the Hilbert function of $L_r \star L_s$ is not the product of the Hilbert functions of L_r and L_s:

```
ring R=0,(x(0..4)),dp;
ideal Lr=8*x(2)-5*x(3)-x(4),2*x(1)-x(3)+x(4),
8*x(0)-3*x(3)+x(4);
ideal Ls=26*x(1)-35*x(2)+45*x(3)+x(4),
13*x(0)-15*x(2)+23*x(3)+6*x(4)
ideal Lrs=HPr(Lr,Ls,4);
```

```
HF(3,Lr,0)=1        HF(3,Ls,0)=1        HF(3,Lrs,0)=1
```

```
HF(3,Lr,1)=2         HF(3,Ls,1)=3         HF(3,Lrs,1)=5
HF(3,Lr,2)=3         HF(3,Ls,2)=6         HF(3,Lrs,2)=15
HF(3,Lr,3)=4         HF(3,Ls,3)=10        HF(3,Lrs,3)=36
HF(3,Lr,4)=5         HF(3,Ls,4)=15        HF(3,Lrs,4)=65
HF(3,Lr,5)=6         HF(3,Ls,5)=21        HF(3,Lrs,5)=111

     ⋮                    ⋮                     ⋮
```

Chapter 4

Hadamard product of degenerate subvarieties

This chapter is inspired by the paper [CCFL] in collaboration with E. Carlini, G. Fatabbi, A. Lorenzini.

Here we want to estimate the dimension and the degree of Hadamard product of degenerate subvarieties, partially answering [FOW, Question 1.1]. In Section 4.1 we prove that, if X and Y are two degenerate subvarieties of $\mathbb{P}^n$ contained in generic linear subspaces of dimension h and k respectively, with $n \geq (h+1)(k+1)-1$, then the Hadamard product $X \star Y$ and the product variety $X \times Y$ are projectively equivalent as subvarieties of $\mathbb{P}^n$. As a consequence we obtain that the dimension of $X \star Y$ is the sum of the dimensions, the degree is the product of the degrees multiplied by a binomial coefficient depending on the dimensions and the Hilbert function is the product of the Hilbert functions.

Then we extend these results to the case of ℓ degenerate subvarieties $X_1, \ldots, X_\ell$ of $\mathbb{P}^n$ contained in ℓ generic linear subspaces $L_1, \ldots, L_\ell$ of dimensions $h_1, \ldots, h_\ell$ respectively, with $n \geq (h_1+1)\cdots(h_\ell+1)-1$, thus obtaining analogous formulas for the dimension, the degree and the Hilbert function of their Hadamard product. These degree and dimension formulas generalize the ones in [BCK, Theorem 6.8] which are only given for linear spaces. We also prove that, if the varieties X_i are smooth, then their Hadamard product is non-singular.

In Section 4.2 we consider two generic parameterized subvarieties of $\mathbb{P}^n$ of dimension r, s and degree d_X, d_Y respectively, with $N-(r+s) \leq n \leq N-1$ where $N = \binom{r+d_X}{d_X}\binom{s+d_Y}{d_Y}-1$. In this case the formula for the Hilbert function no longer holds, but we still have the dimension and degree formulas. We also extend these results to a finite number of subvarieties. In this situation singularities may arise even when the varieties are smooth: on one hand we give a numerical sufficient condition for smoothness, on the other hand we give a sufficient numerical condition for the Hadamard product to be singular and, in this case, we give a lower bound for the dimension of the singular locus.

We conclude with some explicit examples in Section 4.3. These examples show the role of the genericity assumption and how singularities can arise.

4.1 Large ambient space

In this section we consider the Hadamard product of subvarieties contained in generic linear subspaces and in particular the case in which the ambient space has dimension large enough in a very precise sense. We note that Theorem 3.3.1 considered this situation for the product of generic linear spaces.

Theorem 4.1.1. *Let L_h and L_k be generic linear subspaces of $\mathbb{P}^n$ of dimensions h and k respectively, with $n \geq N = (h+1)(k+1)-1$. Let X and Y be two subvarieties of $\mathbb{P}^n$ contained in L_h and L_k respectively. Then the Hadamard product $X \star Y$ and the product variety $X \times Y$ are projectively equivalent as subvarieties of $\mathbb{P}^n$.*

Proof. First we show that $L_h \star L_k$ is projectively equivalent to the product variety $L_h \times L_k$ as subvarieties of $\mathbb{P}^n$.

Following Theorem 3.3.1, set $\Sigma = \{P \star Q | P \in L_h, Q \in L_k\}$ and assume that L_h and L_k have parametric equations given respectively by

$$L_h : \begin{cases} x_0 = f_0(y_0, \dots, y_h) \\ x_1 = f_1(y_0, \dots, y_h) \\ \quad\vdots \\ x_n = f_n(y_0, \dots, y_h) \end{cases} \qquad L_k : \begin{cases} x_0 = g_0(z_0, \dots, z_k) \\ x_1 = g_1(z_0, \dots, z_k) \\ \quad\vdots \\ x_n = g_n(z_0, \dots, z_k) \end{cases} ,$$

where $f_i(y_0, \dots, y_h) = a_{i0}y_0 + a_{i1}y_1 + \cdots + a_{ih}y_h$ and $g_i(z_0, \dots, z_k) = b_{i0}z_0 + b_{i1}z_1 + \cdots + b_{ik}z_k$, for $i = 0, \dots, n$. Consider the matrix of size $(n+1) \times (N+1)$, with $N = (h+1)(k+1)-1$, defined as

$$M' = \begin{pmatrix} a_{00}b_{00} & \dots & a_{00}b_{0k} & a_{01}b_{00} & \dots & a_{0h}b_{0k} \\ a_{10}b_{10} & \dots & a_{10}b_{1k} & a_{11}b_{10} & \dots & a_{1h}b_{1k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n0}b_{n0} & \dots & a_{n0}b_{nk} & a_{n1}b_{n0} & \dots & a_{nh}b_{nk} \end{pmatrix}.$$

The genericity of L_h and L_k gives that the matrix M' has maximal rank $N+1$.

For $n > N$, we can complete the matrix M' to a matrix M of size $n+1 \times n+1$ with $det(M) \neq 0$, so that M gives a projective isomorphism.

Let σ be the Segre embedding of $\mathbb{P}^h \times \mathbb{P}^k$ in $\mathbb{P}^n$ defined by

$$\begin{array}{ccc} \mathbb{P}^h \times \mathbb{P}^k & \longrightarrow & \mathbb{P}^n \\ ([y_0 : \cdots : y_h], [z_0 : \cdots : z_k]) & \mapsto & [y_0z_0 : y_0z_1 : \cdots : y_0z_k : y_1z_0 : \cdots : y_hz_k : 0 : \cdots : 0], \end{array}$$

then the map $\mathbb{P}^n \overset{M}{\to} \mathbb{P}^n$ sends each point $P \in \sigma(\mathbb{P}^h \times \mathbb{P}^k)$ to a point $M(P) \in \Sigma \subseteq L_h \star L_k$. Since $h+k = dim(\sigma(\mathbb{P}^h \times \mathbb{P}^k)) \leq dim(L_h \star L_k) \leq h+k$, they are projectively equivalent. Thus we have that $L_h \star L_k = \Sigma$.

A direct computation shows that

$$(M \circ \sigma)([y_0 : \cdots : y_h], [z_0 : \cdots : z_k]) =$$

$$[f_0(y_0, \dots, y_h)g_0(z_0, \dots, z_k) : \cdots : f_n(y_0, \dots, y_h)g_n(z_0, \dots, z_k)].$$

In other words, if $P = P(y_0, \dots, y_h)$ and $Q = Q(z_0, \dots, z_k)$, then

$$(M \circ \sigma)([y_0 : \cdots : y_h], [z_0 : \cdots : z_k]) = P \star Q.$$

By abuse of notation, we denote by X and Y also the corresponding subvarieties of $\mathbb{P}^h$ and $\mathbb{P}^k$ respectively.

We just proved that $M(\sigma(X \times Y)) \subseteq \{P \star Q | P \in X, Q \in Y\} \subseteq X \star Y$. Since $dim(X) + dim(Y) = dim(M(\sigma(X \times Y)) \leq dim(X \star Y) \leq dim(X) + dim(Y)$, they are projectively equivalent, as we wished. □

Remark 4.1.2. Note that Theorem 4.1.1 generalizes Theorem 3.3.1 in two directions: we consider not only the product of linear spaces, but also the product of degenerate varieties, and we also consider ambient spaces of larger dimension.

Remark 4.1.3. It the proof of the Theorem above we also proved that

$$X \star Y = \overline{\{P \star Q | P \in X, Q \in Y\}} = \{P \star Q | P \in X, Q \in Y\}.$$

Theorem 4.1.1 easily yields the following Corollary which gives some useful formulas about invariants of the Hadamard product of two varieties.

Corollary 4.1.4. *Let L_h and L_k be generic linear subspaces of $\mathbb{P}^n$ of dimensions h and k respectively, with $n \geq N = (h+1)(k+1) - 1$. Let X be a subvariety of $\mathbb{P}^n$ contained in L_h of dimension r and degree d_X, and let Y be a subvariety of $\mathbb{P}^n$ contained in L_k of dimension s and degree d_Y. Then*

i) $dim(X \star Y) = r + s = dim(X) + dim(Y)$

ii) $deg(X \star Y) = \binom{r+s}{s} d_X d_Y = \binom{r+s}{s} deg(X) deg(Y)$

iii) $HF_{X\star Y} = HF_X HF_Y$

Remark 4.1.5. In Theorem 4.1.1 and Corollary 4.1.4 it is not necessary, but it is convenient, to assume that h and k are the minimal dimensions of linear subspaces containing X and Y respectively.

Remark 4.1.6. In the proof of Theorem 4.1.1 the genericity of the linear subspaces L_h and L_k is only used to say that the matrix M' has maximal rank $N+1$, and so we can characterize a closed set C of

$$\underbrace{\left(\mathbb{P}^h \times \cdots \times \mathbb{P}^h\right)}_{n+1 \text{ times}} \times \underbrace{\left(\mathbb{P}^k \times \cdots \times \mathbb{P}^k\right)}_{n+1 \text{ times}}$$

as the zero locus of the maximal minors of M' which are multi-homogeneous polynomials of the multi-graded ring

$$\mathbb{K}[a_{00}, \ldots, a_{0h}, \ldots, a_{n0}, \ldots, a_{nh}, b_{00}, \ldots, b_{0k}, \ldots, b_{n0}, \ldots, b_{nk}].$$

The complement of C is an open subset which can be proved to be non-empty as in Theorem 3.3.1, and each point of this open subset gives a parameterization of two linear subspaces of $\mathbb{P}^n$ of dimensions h and k respectively, for which Theorem 4.1.1 and Corollary 4.1.4 hold.

In Example 4.3.1 we shall see a matrix M' that does not have maximal ranks and $X \star Y$ is neither projectively equivalent nor isomorphic to the product variety $X \times Y$. In fact, in Example 4.3.1, $\operatorname{Sing}(X \star Y) \neq \emptyset$, even if X and Y are smooth. Note that, the dimension and degree formulas still hold, but the Hilbert function formula does not hold.

Now we extend Theorem 4.1.1 and Corollary 4.1.4 to a finite number of subvarieties.

Theorem 4.1.7. *Let ℓ be a positive integer and let $L_1, \ldots, L_\ell$ be generic linear subspaces of $\mathbb{P}^n$ of dimensions $h_1, \ldots, h_\ell$, with $n \geq (h_1+1)\cdots(h_\ell+1)-1$. For each $i = 1, \ldots, \ell$, let X_i be a subvariety of $\mathbb{P}^n$ contained in L_i. Then the Hadamard product $X_1 \star \cdots \star X_\ell$ and the product variety $X_1 \times \cdots \times X_\ell$ are projectively equivalent as subvarieties of $\mathbb{P}^n$.*

Proof. We proceed by induction on ℓ, the case $\ell = 2$ being given in Theorem 4.1.1, but, for simplicity of notation, we only prove the case $\ell = 3$.

Assume that L_1, L_2 and L_3 have parametric equations given respectively by

$$L_1 : \begin{cases} x_0 = f_0(y_0, \dots, y_{h_1}) \\ x_1 = f_1(y_0, \dots, y_{h_1}) \\ \quad\vdots \\ x_n = f_n(y_0, \dots, y_{h_1}) \end{cases} \quad L_2 : \begin{cases} x_0 = g_0(z_0, \dots, z_{h_2}) \\ x_1 = g_1(z_0, \dots, z_{h_2}) \\ \quad\vdots \\ x_n = g_n(z_0, \dots, z_{h_2}) \end{cases} \quad L_3 : \begin{cases} x_0 = \ell_0(t_0, \dots, t_{h_3}) \\ x_1 = \ell_1(t_0, \dots, t_{h_3}) \\ \quad\vdots \\ x_n = \ell_n(t_0, \dots, t_{h_3}) \end{cases},$$

where $f_i(y_0, \dots, y_{h_1}) = a_{i0}y_0 + a_{i1}y_1 + \dots + a_{ih_1}y_{h_1}$, $g_i(z_0, \dots, z_{h_2}) = b_{i0}z_0 + b_{i1}z_1 + \dots + b_{ih_2}z_{h_2}$ and $\ell_i(t_0, \dots, t_{h_3}) = c_{i0}z_0 + c_{i1}z_1 + \dots + c_{ih_3}z_{h_3}$, for $i = 0, \dots, n$.

Apply Theorem 4.1.1 to get that $X_1 \star X_2$ is projectively equivalent to the product variety $X_1 \times X_2$. Set $N = (h_1+1)(h_2+1)-1$ and observe that $L_1 \star L_2$ is contained in a unique linear subspace L projectively equivalent to the $\mathbb{P}^N$ defined by the equations $\{x_{N+1} = 0, \dots, x_n = 0\}$, by using the projective morphism induced by M of Theorem 4.1.1. Thus a parameterization of L is given by $M(u_0 \ \dots \ u_N \ 0 \ \dots \ 0)^T$.

In order to apply Theorem 4.1.1 again, we consider the new matrix M' associated to the parameterizations of L and L_3 defined above. By construction we have that M' is the matrix

$$\begin{pmatrix} M'_{00} & \dots & M'_{0h_2} & M'_{10} & \dots & M'_{1h_2} & \dots & M'_{h_10} & \dots & M'_{h_1h_2} \end{pmatrix}$$

where, for all $i = 0, \dots, h_1$ and for all $j = 0, \dots, h_2$,

$$M'_{ij} = \begin{pmatrix} a_{0i}b_{0j}c_{00} & \dots & a_{0i}b_{0j}c_{0h_3} \\ a_{1i}b_{1j}c_{10} & \dots & a_{1i}b_{1j}c_{1h_3} \\ \vdots & \vdots & \vdots \\ a_{ni}b_{nj}c_{n0} & \dots & a_{ni}b_{nj}c_{nh_3} \end{pmatrix}.$$

By the genericity of L_1, L_2 and L_3, M' has maximal rank, and so we can apply Theorem 4.1.1 again, to obtain that $(X_1 \star X_2) \star X_3$ is projectively equivalent to $(X_1 \star X_2) \times X_3$ which in turn is projectively equivalent to $(X_1 \times X_2) \times X_3$. $\square$

Remark 4.1.8. An immediate consequence of Theorem 4.1.7 is that if $X_1, \dots, X_\ell$ are non-singular, then also $X_1 \star \dots \star X_\ell$ is non-singular.

Theorem 4.1.7 yields the following Corollary which extends the dimension and the degree formulas of [BCK, Theorem 6.8] beyond linear spaces.

Corollary 4.1.9. *Let ℓ be a positive integer and let $L_1, \dots, L_\ell$ be generic linear subspaces of $\mathbb{P}^n$ of dimensions $h_1, \dots, h_\ell$, with $n \geq (h_1 + 1) \cdots (h_\ell + 1) - 1$. For each $i = 1, \dots, \ell$, let X_i be a subvariety of $\mathbb{P}^n$ contained in L_i of dimension r_i and degree d_i. Then* *i)* $dim(X_1 \star \cdots \star X_\ell) = r_1 + \cdots + r_\ell = \sum_{i=1}^{\ell} dim(X_i)$.

ii) $deg(X_1 \star \cdots \star X_\ell) = \binom{r_1+\cdots+r_\ell}{r_1,\ldots,r_\ell} \prod_{i=1}^{\ell} d_i = \binom{r_1+\cdots+r_\ell}{r_1,\ldots,r_\ell} \prod_{i=1}^{\ell} deg(X_i)$, *where* $\binom{r_1+\cdots+r_\ell}{r_1,\ldots,r_\ell} = \frac{(r_1+\cdots+r_\ell)!}{r_1!\cdots r_\ell!}$.

iii) $HF_{X_1\star\cdots\star X_\ell} = \prod_{i=1}^{\ell} HF_{X_i}$.

Remark 4.1.10. As before, in Theorem 4.1.7 and Corollary 4.1.9, it is convenient to assume that h_i is the minimal dimension of a linear subspace containing X_i, for all $i = 1, \ldots, \ell$.

Before ending this section we introduce the notion of *generic parameterized subvariety*, which allows us to extend our results to the case of a small ambient space. Given a subvariety $Z \subset \mathbb{P}^n$ of dimension r and degree d with a parametric representation, it is clear that Z is contained in a linear subspace of dimension $\binom{r+d}{d} - 1$. Thus Z is degenerate as soon as $\binom{r+d}{d} - 1 < n$. Moreover, we say that Z is a *generic parameterized subvariety*, if the $n+1$ degree d polynomials in $r+1$ variables defining it have coefficients which can be generically chosen.

Remark 4.1.11. Let Z be a generic parameterized subvariety of $\mathbb{P}^n$ of dimension r and degree d. If $n \geq \binom{r+d}{d}$ we had just seen that Z is degenerate. Moreover Z is non-singular, in fact it is projectively equivalent to the d-uple Veronese embedding of $\mathbb{P}^r$.

Corollary 4.1.12. *Let ℓ be a positive integer. For $i = 1, \ldots, \ell$, let r_i, d_i, be positive integers and let $n \geq \binom{r_1+d_1}{d_1} \cdots \binom{r_\ell+d_\ell}{d_\ell} - 1$. For $i = 1, \ldots, \ell$, let X_i be a generic parameterized subvariety of $\mathbb{P}^n$ of dimension r_i and degree d_i. Then the Hadamard product $X_1 \star \cdots \star X_\ell$ and the product variety $X_1 \times \cdots \times X_\ell$ are projectively equivalent as subvarieties of $\mathbb{P}^n$.*

Proof. For $i = 1, \ldots, \ell$, assume that X_i has parametric equations given by

$$X_i : \begin{cases} x_0 = f_{i0}(y_{i0}, \ldots, y_{ir_i}) \\ x_1 = f_{i1}(y_{i0}, \ldots, y_{ir_i}) \\ \qquad \vdots \\ x_n = f_{in}(y_{i0}, \ldots, y_{ir_i}) \end{cases}$$

where $f_{ij}(y_{i0}, \ldots, y_{ir_i}) \in \mathbb{K}[y_{i0}, \ldots, y_{ir_i}]_{d_i}$, for $j = 0, \ldots, n$.

Since $dim_{\mathbb{K}}\left(\mathbb{K}[y_{i0},\dots,y_{ir_i}]_{d_i}\right)=\binom{r_i+d_i}{d_i}$, then X_i is contained in a linear subspace L_i of dimension $\binom{r_i+d_i}{d_i}-1$.

Therefore, by Theorem 4.1.7, we have that $X_1\star\cdots\star X_\ell$ and $X_1\times\cdots\times X_\ell$ are projectively equivalent as subvarieties of $\mathbb{P}^n$. $\square$

Corollary 4.1.12 easily yields the following Corollary.

Corollary 4.1.13. *Let ℓ be a positive integer. For $i=1,\dots,\ell$, let r_i, d_i, be positive integers and let $n\geq\binom{r_1+d_1}{d_1}\cdots\binom{r_\ell+d_\ell}{d_\ell}-1$. For $i=1,\dots,\ell$, let X_i be a generic parameterized subvariety of $\mathbb{P}^n$ of dimension r_i and degree d_i. Then:*

i) $dim(X_1\star\cdots\star X_\ell)=r_1+\cdots+r_\ell=\sum_{i=1}^{\ell}dim(X_i)$

ii) $deg(X_1\star\cdots\star X_\ell)=\binom{r_1+\cdots+r_\ell}{r_1,\dots,r_\ell}\prod_{i=1}^{\ell}d_i=\binom{r_1+\cdots+r_\ell}{r_1,\dots,r_\ell}\prod_{i=1}^{\ell}deg(X_i).$

iii) $HF_{X_1\star\cdots\star X_\ell}=\prod_{i=1}^{\ell}HF_{X_i}$

iv) $X_1\star\cdots\star X_\ell$ *is non-singular.*

4.2 Small ambient space

Now we consider ℓ generic parameterized subvarieties of $\mathbb{P}^n$ of dimension r_i and degree d_i, respectively. Let $N=\binom{r_1+d_1}{d_1}\cdots\binom{r_\ell+d_\ell}{d_\ell}-1$.

In the previous section, for $n\geq N$, we determined the dimension, the degree and the Hilbert function of the Hadamard product in terms of the same invariants of the factors.

Now we consider the case $N-(r+s)\leq n\leq N-1$. We will see that the dimension and the degree formulas still hold, even if the relation on the Hilbert functions fails. Moreover, the Hadamard product can be a singular variety, even if the factors are smooth.

In order to study Hadamard products in a small ambient space we use Segre-Veronese varieties ([CGG]), thus we briefly recall some basic notation about them.

Let ℓ be a positive integer. Let $r_1,\dots,r_\ell,d_1,\dots,d_\ell$ be positive integers and set $N=\binom{r_1+d_1}{d_1}\cdots\binom{r_\ell+d_\ell}{d_\ell}-1$. We denote by S the image in $\mathbb{P}^N$ of a Segre-Veronese embedding of type $(d_1,\dots,d_\ell)$ from $\mathbb{P}^{r_1}\times\cdots\times\mathbb{P}^{r_\ell}$ to $\mathbb{P}^N$.

Theorem 4.2.1. *Let r,s,d_X,d_Y be positive integers, let $N=\binom{r+d_X}{d_X}\binom{s+d_Y}{d_Y}-1$ and $N-(r+s)\leq n\leq N-1$. Let X and Y be two generic parameterized subvarieties of $\mathbb{P}^n$ of dimensions r, s and degrees d_X, d_Y, respectively. If $n>r+s$, then:*

i) $dim(X \star Y) = r + s = dim(X) + dim(Y)$

ii) $deg(X \star Y) = \binom{r+s}{s} d_X d_Y = \binom{r+s}{s} deg(X) deg(Y)$.

Proof. Consider the Segre-Veronese embedding of type (d_X, d_Y) from $\mathbb{P}^r \times \mathbb{P}^s$ to $\mathbb{P}^N$ and let S be its image.

Assume that X and Y have parametric equations given respectively by

$$X : \begin{cases} x_0 = f_0(y_0, \dots, y_r) \\ x_1 = f_1(y_0, \dots, y_r) \\ \vdots \\ x_n = f_n(y_0, \dots, y_r) \end{cases} \qquad Y : \begin{cases} x_0 = g_0(z_0, \dots, z_s) \\ x_1 = g_1(z_0, \dots, z_s) \\ \vdots \\ x_n = g_n(z_0, \dots, z_s) \end{cases}$$

where $f_i(y_0, \dots, y_r) \in \mathbb{K}[y_0, \dots, y_r]_{d_X}$ and $g_i(z_0, \dots, z_s) \in \mathbb{K}[z_0, \dots, z_s]_{d_Y}$, for $i = 0, \dots, n$.

Observe that, for each $i = 0, \dots, n$, the form $f_i g_i$ has bi-degree (d_X, d_Y) in $\mathbb{K}[y_0, \dots, y_r, z_0, \dots, z_s]$. Since $\mathbb{K}[y_0, \dots, y_r, z_0, \dots, z_s]_{(d_X, d_Y)}$ has dimension $N + 1$, then, for each $i = 0, \dots, n$, $f_i g_i$ defines a point P_i of $\mathbb{P}^N$ belonging to S.

Since X and Y are generic parameterized subvarieties, the projective space generated by the points $P_0, \dots, P_n$ is a linear subspace of $\mathbb{P}^N$ of dimension n.

Consider the $(n+1) \times (N+1)$ matrix M' whose rows are the coordinates of the points $P_0, \dots, P_n$. Again since X and Y are generic parameterized subvarieties, M' has maximum rank, hence it defines a projection π from $\mathbb{P}^N$ to $\mathbb{P}^n$ whose center we call Λ. Note that $dim(\Lambda) = N - n - 1$ and Λ can be seen as the dual of the linear span of the points $P_0, \dots, P_n$.

Now, any pair of generic choices of the polynomials f_i's and g_j's determines $n+1$ points of $\mathbb{P}^N$ (belonging to S) which generate a linear subspace of $\mathbb{P}^N$ of dimension n. Conversely, any $n+1$ points of S can be obtained from parameterizations (with suitable coefficients) of two subvarieties of $\mathbb{P}^n$ of the given dimensions and degrees.

On the other hand, for any generic linear subspace L of $\mathbb{P}^N$ of dimension n, defined by $N-n$ generic hyperplanes $H_1, \dots, H_{N-n}$, we shall consider $S_i = S \cap H_1 \cap \dots \cap H_i$. Since $n \geq N-(r+s)$, we have that $dim(S_i) \geq 2$ for all $i = 1, \dots, N-n-2$ and $dim(S_{N-n-1}) \geq 1$. Therefore by [H, Proposition 18.10], S_{N-n} contains at least $n + 1$ points which generate L. Thus we may assume that the linear subspaces of $\mathbb{P}^N$ of dimension n generated by $n + 1$ points of S are generic, and so Λ is generic as well.

For $n \geq r + s = dim(S)$, since Λ is generic, we have $dim(\pi(S)) = dim(S) = r + s$. Since $n > r + s$, we also have $\pi(S) \neq \mathbb{P}^n$, and so the projection $\pi_{|S}$ is a birational map. Hence $deg(\pi(S)) = deg(S) = \binom{r+s}{s} d_X d_Y$.

Set $\Sigma = \{P \star Q | P \in X, Q \in Y\}$. It is easy to see that $\pi(S) \subseteq \Sigma \subseteq X \star Y$.

Since $r+s = dim(\pi(S)) \leq dim(X \star Y) \leq r+s$, we have that $\pi(S) = X \star Y$, and so $dim(X \star Y) = dim(\pi(S)) = r+s$ and $deg(X \star Y) = deg(\pi(S)) = \binom{r+s}{s} d_X d_Y$.
□

Remark 4.2.2. In the proof of Theorem 4.2.1 we also proved that

$$X \star Y = \overline{\{P \star Q | P \in X, Q \in Y\}} = \{P \star Q | P \in X, Q \in Y\}.$$

In order to make Theorem 4.2.1 more effective, we can find explicit numerical conditions on X and Y so that $n \geq N-(r+s)$ yields $n > r+s$.

Lemma 4.2.3. *Using the notation of Theorem 4.2.1, we have that: if* (d_X, d_Y, r, s) *is in the following table, then* $N-(r+s) > r+s$.

d_X	d_Y	r	s
≥ 2	$\forall$	$\forall$	$\forall$
$\forall$	≥ 2	$\forall$	$\forall$
1	1	≥ 3	≥ 2
1	1	≥ 2	≥ 3

Remark 4.2.4. Notice that, in the hypotheses of Theorem 4.2.1, we have that $HF_{X\star Y} \neq HF_X HF_Y$. In fact, since X is not contained in a linear subspace of dimension less than $\binom{r+d_X}{d_X}-1$ and similarly Y, we have

$$HF_X(1) = HF_{\mathbb{P}^{\binom{r+d_X}{d_X}-1}}(1) = \binom{r+d_X}{d_X}$$

and

$$HF_Y(1) = HF_{\mathbb{P}^{\binom{s+d_Y}{d_Y}-1}}(1) = \binom{s+d_Y}{d_Y}$$

and so

$$HF_X(1)HF_Y(1) = \binom{r+d_X}{d_X}\binom{s+d_Y}{d_Y} > N \geq HF_{X\star Y}(1).$$

Remark 4.2.5. In Remark 4.1.6 we saw that being M' of maximum rank is sufficient to have the formulas for the dimension, the degree and the Hilbert function, when $n \geq N$. When $n < N$, besides the failure of the Hilbert function formula (Remark 4.2.4), M' of maximum rank does not grant the degree formula, as Example 4.3.2 shows.

Using a similar technique to that contained in the proof of Theorem 4.2.1, we can extend this Theorem to a finite number of subvarieties.

Theorem 4.2.6. *Let ℓ be a positive integer. For $i = 1, \dots, \ell$, let r_i, d_i, be positive integers, let $N = \binom{r_1+d_1}{d_1} \cdots \binom{r_\ell+d_\ell}{d_\ell} - 1$ and $N - (r_1 + \cdots + r_\ell) \leq n \leq N - 1$. For $i = 1, \dots, \ell$, let X_i be a generic parameterized subvariety of $\mathbb{P}^n$ of dimension r_i and degree d_i. If $n > \sum_i r_i$, then:*

i) $dim(X_1 \star \cdots \star X_\ell) = r_1 + \cdots + r_\ell = \sum_{i=1}^{\ell} dim(X_i)$

ii) $deg(X_1 \star \cdots \star X_\ell) = \binom{r_1+\cdots+r_\ell}{r_1,\dots,r_\ell} \prod_{i=1}^{\ell} d_i = \binom{r_1+\cdots+r_\ell}{r_1,\dots,r_\ell} \prod_{i=1}^{\ell} deg(X_i)$.

Now we provide a numerical condition for the Hadamard product to be smooth and we give an estimate on how big the singular locus is when singularities occur. In order to do this we will use the variety of secant lines to a variety S that we denote by $\sigma_2(S)$.

Notice that, for n in our range, when using generic parameterized subvarieties of $\mathbb{P}^n$, we are sure that we are dealing with smooth varieties, as the following Lemma shows.

Lemma 4.2.7. *Let r, s, d_X, d_Y be positive integers, let $N = \binom{r+d_X}{d_X}\binom{s+d_Y}{d_Y} - 1$ and $N - (r + s) \leq n \leq N - 1$. Let X and Y be two generic parameterized subvarieties of $\mathbb{P}^n$ of dimensions r, s and degrees d_X, d_Y, respectively. Then X and Y are non-singular.*

Proof. We only prove that X is non-singular (similarly for Y).

The case $d_X = 1$ is clear, hence we may assume $d_X > 1$.

By Remark 4.1.11, it is enough to show that $\binom{r+d_X}{d_X} \leq N - (r + s)$. Clearly it is enough to consider only the case $d_Y = 1$ and $s = 1$. In this case the inequality holds, in fact, for $d_X > 1$, we have

$$\binom{r + d_X}{d_X} \leq 2\binom{r + d_X}{d_X} - 1 - (r + 1).$$

□

Proposition 4.2.8. *Let r, s, d_X, d_Y be positive integers, let $N = \binom{r+d_X}{d_X}\binom{s+d_Y}{d_Y} - 1$ and $N - (r + s) \leq n \leq N - 1$. Let X and Y be two generic parameterized subvarieties of $\mathbb{P}^n$ of dimensions r, s and degrees d_X, d_Y, respectively.*

i) If $n \geq dim(\sigma_2(S))$, then $X \star Y$ is smooth.

ii) If $r + s < n < dim(\sigma_2(S))$, then $dim(Sing(X \star Y)) \geq 2r + 2s - n$,

where S is the Segre-Veronese embedding of type (d_X, d_Y) of $\mathbb{P}^r \times \mathbb{P}^s$.

Proof. We use the same notations as in the proof of Theorem 4.2.1 and we let $\sigma_2 = \sigma_2(S)$.
i) If $n \geq dim(\sigma_2)$, since Λ is generic, we have that $\Lambda \cap \sigma_2 = \emptyset$, then $\pi(S) = X \star Y$ is smooth.
ii) Define the incidence correspondence $\Theta \subseteq \mathrm{Sing}(X \star Y) \times (\Lambda \cap \sigma_2)$ where

$$\Theta = \{(Q, P) : Q = \pi(\langle \Lambda, r_P \rangle), r_P \text{ is a tangent or secant line to } S \text{ through } P\}.$$

We consider the projection maps $p_1 : \Theta \to \mathrm{Sing}(X \star Y)$ and $p_2 : \Theta \to \Lambda \cap \sigma_2$. First we prove that p_1 has a finite fiber over a point $Q \in \mathrm{Sing}(X \star Y)$. Since Λ is a hyperplane in $\pi^{-1}(Q)$, and $\Lambda \cap S = \emptyset$, then $\pi^{-1}(Q) \cap S$ contains only a finite number of points and thus a finite number of secant, or tangent, lines to S; by the genericity of Λ each of these lines contains a finite number of points of $\Lambda \cap \sigma_2$. Hence, $p_1^{-1}(Q)$ is finite. Now we consider the fiber of p_2 over $P \in \Lambda \cap \sigma_2$, i.e. the family of secant and tangent lines to S through P, which has dimension at least $2r + 2s + 1 - dim(\sigma_2)$. Since $dim(\Lambda \cap \sigma_2) = dim(\Lambda) + dim(\sigma_2) - N$, we conclude that

$$dim(\mathrm{Sing}(X \star Y)) = dim(\Theta) \geq dim(\Lambda) + 2r + 2s + 1 - N = 2r + 2s - n.$$

□

Remark 4.2.9. If X and Y are not generic enough, it can happen that the dimension of $Sing(X \star Y)$ is smaller than $2r + 2s - n$, as Example 4.3.2 shows.

Also note that the bound of Proposition 4.2.8-*ii)* can be sharp, as Example 4.3.3 shows.

Remark 4.2.10. If $(d_X, d_Y) = (1, 1)$, then $\sigma_2(S)$ can be identified with the variety of $r \times s$ matrices of rank 1 or 2 and so $dim(\sigma_2(S)) = 2r + 2s - 1$.

If $(d_X, d_Y) \neq (1, 1)$, by [AB, Theorem 4.2], we have that

$$dim(\sigma_2(S)) = \min\{N, 2r + 2s + 1\},$$

and it is easy to check that $dim(\sigma_2(S)) = 2r + 2s + 1$.

Remark 4.2.11. In the case $(d_X, d_Y) = (1, 1)$, Proposition 4.2.8 yields that $X \star Y$ is either smooth or $dim(\mathrm{Sing}(X \star Y)) \geq 2r + 2s - n > 2r + 2s - dim(\sigma_2(S)) = 1$. Thus, if $X \star Y$ is not smooth, it is singular at least along a surface.

The following conditions show that the hypotheses of Proposition 4.2.8 hold in a large number of cases.

Lemma 4.2.12. *Using the notations of Proposition 4.2.8, we have that:*

i) If (d_X, d_Y, r, s) *is in the following table, then* $N - (r+s) \geq dim(\sigma_2(S))$.

d_X	d_Y	r	s
≥ 2	≥ 2	$\forall$	$\forall$
≥ 3	1	$\forall$	$\forall$
2	1	≥ 2	$\forall$
1	≥ 3	$\forall$	$\forall$
1	2	$\forall$	≥ 2
1	1	3	≥ 5
1	1	4	≥ 4
1	1	5	≥ 3

ii) If (d_X, d_Y, r, s, n) *is in the following table, then* $r + s < N - (r+s) \leq n \leq dim(\sigma_2(S))$.

d_X	d_Y	r	s	n
2	1	1	$\forall$	$2s+1 \leq n \leq 2s+2$
1	2	$\forall$	1	$2r+1 \leq n \leq 2r+2$
1	1	2	≥ 3	$2s \leq n \leq 2s+2$
1	1	≥ 3	2	$2r \leq n \leq 2r+2$
1	1	3	3	$9 \leq n \leq 10$
1	1	3	4	$n = 12$
1	1	4	3	$n = 12$

Remark 4.2.13. It is easy to check that in the cases of Lemma 4.2.12-*ii)* the lower bound on $dim(\text{Sing}(X \star Y))$ does not depend on r and s.

Remark 4.2.14. Let S be the Segre-Veronese variety with $\ell > 2$. By [AB, Theorem 4.2], S does not have a defective secant line variety, and thus

$$dim(\sigma_2(S)) = \min\{N, 2\sum r_i + 1\},$$

and it is easy to check that $dim(\sigma_2(S)) = 2\sum r_i + 1$.

Notice that Lemma 4.2.7 easily extends to a finite number of varieties. Moreover by using Remark 4.2.14, Proposition 4.2.8 can be extended to a finite number of varieties.

Proposition 4.2.15. *Let $\ell > 2$. For $i = 1, \ldots, \ell$, let r_i, d_i, be positive integers, let $N = \binom{r_1+d_1}{d_1} \cdots \binom{r_\ell+d_\ell}{d_\ell} - 1$ and $N - (r_1 + \cdots + r_\ell) \leq n \leq N - 1$. For $i = 1, \ldots, \ell$, let X_i be a generic parameterized non-singular subvariety of $\mathbb{P}^n$ of dimension r_i and degree d_i.*

i) if $n \geq 2\sum r_i + 1$, then $X_1 \star \cdots \star X_\ell$ is smooth;

ii) if $\sum r_i < n < 2\sum r_i + 1$, then $dim(Sing(X_1 \star \cdots \star X_\ell)) \geq 2\sum r_i - n$.

4.3 Some examples

Here we collect some examples to show the role of the genericity assumption in our results.

In Example 4.3.1 we have $n \geq N$, but X and Y are not generic enough to have the matrix M' of maximal rank (see Theorem 4.1.1 and Corollary 4.1.4). Also, the varieties X and Y are both non-singular, but $\mathrm{Sing}(X \star Y) \neq \emptyset$, and so $X \star Y$ is neither projectively equivalent nor isomorphic to the product variety $X \times Y$.

In Example 4.3.2 we have $n < N$, X and Y are generic enough to have the matrix M' of maximal rank, but, X and Y are not generic enough to give a generic center of projection Λ (see Theorem 4.2.1 and Proposition 4.2.8). Also, the degree formula and the lower bound on the dimension of the singular locus do not hold.

In Example 4.3.3 the dimension of the singular locus is equal to the lower bound.

Finally we give an example (Example 4.3.4) which is not computable but can be directly deduced from our results.

Example 4.3.1. Let X be the line of $\mathbb{P}^5$ of equations $\{x_0 - x_1 = 0, x_0 - x_2 = 0, x_3 - x_5 = 0, x_0 + x_3 - x_4 = 0\}$ and let Y be the conic of $\mathbb{P}^5$ of equations $\{x_0 - 2x_3 + 3x_5 = 0, x_1 + x_4 - x_5 = 0, x_2 + 2x_3 - 3x_4 = 0, x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + 5x_0x_1 + 8x_0x_1 - 2x_2x_5 + 10x_0x_4 = 0\}$. Here $h = 1$ and $k = 2$ and so $N = (h+1)(k+1) - 1 = 5$.

We use `CoCoA` ([CoCoA]), following the procedure given in Section 4.1 of Chapter 2 and denoting by IH the ideal of the Hadamard product $X \star Y$:

```
Use RR::=QQ[a,b,c,d,e,f,g,h,i,l,m,n,x,y,z,t,w,s];
L:=Ideal(a-b,a-c,d-f,a+d-e);
C:=Ideal(g-2l+3n,h-n+m,i+2l-3m,
g^2+h^2+i^2+l^2+m^2+n^2+5gh+8lm-2in+10gm);
H:=L+C+Ideal(x-ag,y-bh,z-ci,t-dl,w-em,s-fn);
SH:=Saturation(H,Ideal(xyztws));
IH:=Elim(a..n,SH);
```

```
Use TT::=QQ[x,y,z,t,w,s];
%Redefine IH in the ring TT:
IH:= Ideal(x + 3y + z,
y^2 + 1/4yz - 9/16z^2 - 653/88yt + 9/11zt - 585/176t^2 + 7yw +
35/8zw - 195/88tw - 65/16w^2 + 8/11ys - 621/176zs - 65/88ts +
1495/176ws + 65/44s^2,
z^2t - 62/11zt^2 + 161/11t^3 - 22/3ztw + 322/33t^2w +
161/9tw^2 + 8/9yzs + 2/9z^2s - 320/99yts + 497/99zts -
974/99t^2s - 8/3yws + 14/3zws - 4567/99tws - 16w^2s +
184/33ys^2 + 226/99zs^2 - 932/99ts^2 + 368/11ws^2 +
64/11s^3, yzt + 40/33zt^2 - 161/33t^3 - 161/27ytw + ztw -
322/99t^2w - 161/27tw^2 + 1/2z^2s + 701/99yts - 460/297zts +
3703/594t^2s - 121/27zws + 5152/297tws + 161/18w^2s -
160/99ys^2 + 1543/594zs^2 + 1127/297ts^2 - 3703/198ws^2 -
322/99s^3,
yt^2 + 1/3zt^2 - 2/3t^3 - 22/27ytw - 4/9t^2w - 22/27tw^2 +
2/9yts + 2/27zts + 23/27t^2s - 11/27zws + 64/27tws + 11/9w^2s
- 4/9ys^2 - 4/27zs^2 + 14/27ts^2 - 23/9ws^2 - 4/9s^3);
Hilbert(TT/IH);
H(0) = 1
H(t) = 2t^2 + 3t   for t >= 1

JF:=Jacobian(Gens(IH));

Sing:=Saturation(Ideal(Minors(3,JF))+IH,Ideal(x,y,z,t,w,s));

Hilbert(TT/Sing);
H(0) = 1
H(t) = 5   for t >= 1
```

Here we see that the Hadamard product has dimension $2 = r + s = dim(X) + dim(Y)$ and degree $4 = \binom{r+s}{r} deg(X) deg(Y)$ as expected, but $HF_{X\star Y} \neq HF_X HF_Y$. Also, the singular locus has dimension 0 and degree 5.

In this case the matrix M' of the proof of Theorem 4.1.1 does not have maximum rank. In fact, first we write the parameterizations of $L_1 = X$ and of the plane L_2 containing Y:

$$L_1 : \begin{cases} x_0 = y_1 \\ x_1 = y_1 \\ x_2 = y_1 \\ x_3 = y_0 \\ x_4 = y_0 + y_1 \\ x_5 = y_0 \end{cases} \qquad L_2 : \begin{cases} x_0 = 2z_0 - 3z_2 \\ x_1 = -z_1 + z_2 \\ x_2 = -2z_0 + 3z_1 \\ x_3 = z_0 \\ x_4 = z_1 \\ x_5 = z_2 \end{cases}$$

and then we obtain

$$M' = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 & 3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

whose determinant equals 0.

Example 4.3.2. Let X be the line of $\mathbb{P}^4$ of equations $\{x_0 - x_1 = 0, x_0 - x_2 = 0, x_3 - 2x_4 = 0\}$ and let Y be the conic $\mathbb{P}^4$ of equations $\{x_0 - x_3 = 0, x_1 - x_4 = 0, x_1^2 - x_0x_2 = 0\}$.

We again use `CoCoA`:

```
Use RR::=QQ[a,b,c,d,e,g,h,i,l,m,x,y,z,t,w];
L:=Ideal(a-b,a-c,d-2e);
C:=Ideal(g-l,h-m,h^2-gi);
H:=L+C+Ideal(x-ag,y-bh,z-ci,t-dl,w-em);
SH:=Saturation(H,Ideal(xyztw));
IH:=Elim([a,b,c,d,e,g,h,i,l,m],SH);
Use TT::=QQ[x,y,z,t,w];
%Redefine IH in the ring TT:
IH:= Ideal(y^2 - xz, yt - 2xw, zt - 2yw);

Hilbert(TT/IH);
H(t) = 3/2t^2 + 5/2t + 1   for t >= 0

JF:=Jacobian(Gens(IH));

Sing:=Saturation(Ideal(Minors(2,JF))+IH,Ideal(x,y,z,t,w));
```

```
Hilbert(TT/Sing);
H(t) = 0   for t >= 0
```

In this case $X \star Y$ has dimension $2 = r + s = deg(X) + deg(Y)$ but it has degree $3 < \binom{r+s}{s} dim(X) dim(Y)$.

Surprisingly enough $X \star Y$ does not have singularities and $dim(Sing(X \star Y)) < 2r + 2s - n = 0$ (see Proposition 4.2.8). Moreover M' has maximum rank. In fact, writing the parameterization of X and Y

$$X : \begin{cases} x_0 = y_0 - y_1 \\ x_1 = y_0 - y_1 \\ x_2 = y_0 - y_1 \\ x_3 = y_0 \\ x_4 = 2y_0 \end{cases} \qquad Y : \begin{cases} x_0 = z_0^2 \\ x_1 = z_0 z_1 \\ x_2 = z_1^2 \\ x_3 = z_0^2 \\ x_4 = z_0 z_1 \end{cases}$$

we obtain

$$M' = \begin{pmatrix} -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Also observe that, in this case, Λ is the point $[0 : 0 : -2 : 0 : 0 : 1]$ and so it belongs to the Segre-Veronese variety S and this is why our genericity hypothesis on X and Y is not satisfied.

Example 4.3.3. Let X be the line of $\mathbb{P}^3$ of equations $\{x_0 + x_1 + x_2 + 2x_3 = x_0 - x_1 + 4x_2 - x_3 = 0\}$ and let Y be the conic of $\mathbb{P}^3$ of equations $\{x_0 + 2x_1 + 3x_2 + x_3 = x_0^2 + 2x_0x_2 + 2x_0x_3 + x_12 + 2x_1x_2 - 2x_1x_3 + x_2^2 + 2x_2x_3 + x_3^2 = 0\}$. Here $r = s = 1$, $d_X = 1$ and $d_Y = 2$, so 3 is the minimum possible value for n, moreover we are in the case *ii)* of Proposition 4.2.8.

Once more we use `CoCoA`:

```
Use RR::=QQ[a,b,c,d,e,f,g,h,x,y,z,t];
IX:=Ideal(a+b+c+2d,a-b+4c-d);
IY:=Ideal(e+2f+3g+h,e^2+2eg+2eh+f^2+2fg-2fh+g^2+2gh+h^2);
H:=IX+IY+Ideal(x-ae,y-bf,z-cg,t-dh);
SH:=Saturation(H,Ideal(xyzt));
IH:=Elim(a..h,SH);
Use TT::=QQ[x,y,z,t];
```

```
%Redefine IH in the ring TT:
IH:= Ideal(x^3y - 26/3x^2y^2 + 79/9xy^3 - 5/24y^4 - 26x^2yz +
413/6xy^2z- 155/36y^3z - 15/2x^2z^2 + 2611/20xyz^2 -
569/24y^2z^2 + 45xz^3 - 159/4yz^3 - 27/2z^4 - 32/5x^2yt +
227/15xy^2t - 203/20y^3t + 803/5xyzt - 11099/60y^2zt + 63xz^2t -
14613/20yz^2t - 297z^3t + 113/20xyt^2 - 363/40y^2t^2-
741/20yzt^2 - 243/2z^2t^2 - 27/20yt^3);

Hilbert(TT/IH);
H(0) = 1
H(t) = 2t^2 + 2   for t >= 1

JF:=Jacobian(Gens(IH));
Sing:=Saturation(Ideal(JF[1]),Ideal(x,y,z,t));
Hilbert(TT/Sing);
H(0) = 1
H(1) = 4
H(2) = 10
H(t) = 3t + 5   for t >= 3
```

In this case $X \star Y$ is a singular quartic surface and the singular locus is of dimension $1 = 2r + 2s - n$.

Example 4.3.4. Let k be a positive integer. Let $\mathcal{C}$ be a generic conic of $\mathbb{P}^{2k+1}$. Let L be a generic linear subspace of $\mathbb{P}^{2k+1}$ of dimension k. In view of Lemma 4.2.12, we can use Theorem 4.2.1 and Proposition 4.2.8 to obtain $dim(\mathcal{C} \star L) = k + 1$, $deg(\mathcal{C} \star L) = \binom{k+1}{k} \cdot 2 \cdot 1 = 2(k + 1)$ and $dim(Sing(\mathcal{C} \star L)) \geq 2 + 2k - (2k + 1) = 1$.

Chapter 5

Weak Hadamard star configurations and apolarity

This chapter is inspired by the paper [BJC] written in collaboration with I. Bahmani Jafarloo.

In Section 5.1, we prove some general properties of the standard Cremona transformation that can be useful to characterize weak Hadamard star configurations and Hadamard star configurations.

Section 5.2 is the heart of the chapter. In this section we prove that when we start from a finite number of points and a hyperplane, we have a sufficient and necessary condition to have that the Hadamard product gives a weak Hadamard star configuration or a Hadamard star configuration. At the end of this section we give a sufficient condition to have that the star configuration given by [BCK] is a Hadamard star configuration. Also in [CCGV] it was given a sufficient condition: we prove that if this sufficient condition holds also our condition holds.

In Section 5.3 we connect Hadamard star configurations and star configurations apolar to a form.

5.1 Properties of Standard Cremona transformation

In this section we study some properties of the Standard Cremona transformation.

Notation 5.1.1. We denote by $\sigma : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ the Standard Cremona transformation

$$\sigma\left([p_0 : \ldots : p_n]\right) = \left[\frac{1}{p_0} : \ldots : \frac{1}{p_n}\right].$$

Definition 5.1.2. Let $r \geq n+1$ and let $P_1, \dots, P_r$ points in $\mathbb{P}^n$. We say that $P_1, \dots, P_r$ are in *general position* if there are no $n+1$ points on a hyperplane.

Lemma 5.1.3. *Let $P_1, \cdots, P_r \in \mathbb{P}^n \setminus \Delta_{n-1}$ with $P_i = [p_0(i) : \dots : p_n(i)]$. Then $\sigma(P_1), \dots, \sigma(P_r)$ are in general position if and only if the matrix M has all non-zero maximal minors, where*

$$M = \begin{pmatrix} \dfrac{1}{p_0(1)} & \cdots & \dfrac{1}{p_n(1)} \\ \vdots & & \vdots \\ \dfrac{1}{p_0(r)} & \cdots & \dfrac{1}{p_n(r)} \end{pmatrix}.$$

Proof. Observe that

$$M = \begin{pmatrix} \dfrac{1}{p_0(1)} & \cdots & \dfrac{1}{p_n(1)} \\ \vdots & & \vdots \\ \dfrac{1}{p_0(r)} & \cdots & \dfrac{1}{p_n(r)} \end{pmatrix} = \begin{pmatrix} \sigma(P_1) \\ \vdots \\ \sigma(P_r) \end{pmatrix}$$

and so it is clear that $\sigma(P_1), \dots, \sigma(P_r)$ are in general position if and only if the matrix M has all non-zero maximal minors. $\square$

Lemma 5.1.4. *Let $P_1, \dots, P_r$ be generic points in $\mathbb{P}^n$. Then $\sigma(P_1), \dots, \sigma(P_r)$ are in general position.*

Proof. In order to prove that $\sigma(P_1), \dots, \sigma(P_r)$ are in general position, it suffices to show that any maximal minor of the matrix M in Lemma 5.1.3 is non-zero. We set, for instance $\det\begin{pmatrix}\sigma(P_1) & \cdots & \sigma(P_{n+1})\end{pmatrix}^T = \lambda_{1,\dots,n+1}$. Note that $F = \lambda_{1,\dots,n+1} p_0(1) \cdots p_n(1) \cdots p_0(n+1) \cdots p_n(n+1)$ can be viewed as a multi-homogeneous polynomial in $\mathbb{K}[p_0(1), \dots, p_n(1), \dots, p_0(r), \dots, p_n(r)]$ of multi-degree $(n, \dots, n, 0, \dots, 0)$. Therefore, it defines a closed subset C_1 in $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$. Obviously, the polynomial F is non-zero and thus C_1 is a proper subset. If we change the minor, we have another proper closed subset. Let C be the union of this finite number of proper closed subsets. Clearly C is a proper closed subset. By the genericity of $P_1, \dots, P_r$ we can suppose that they are not in C. We conclude that $F(P_1, \dots, P_r) = F(P_1, \dots, P_{n+1}) \neq 0$. It follows that $\lambda_{1,\dots,n+1} = \det\begin{pmatrix}\sigma(P_1) & \cdots & \sigma(P_{n+1})\end{pmatrix}^T \neq 0$. Hence, all maximal minors of the matrix M are non-zero. $\square$

Remark 5.1.5. If the points $P_1, \dots, P_r$ are in general position, then this does not guarantee that $\sigma(P_1), \dots, \sigma(P_r)$ are in general position. In fact, if we consider a generic hyperplane H, we have that $\sigma(H)$ is a hypersurface of degree $d > 1$ and so there exist $P_1, \dots, P_{n+1}$ in $\sigma(H)$ which are in general position. Clearly, $\sigma(\sigma(H)) = H$, and this allows us to conclude that $\sigma(P_1), \dots, \sigma(P_{n+1})$ are not in general position.

We believe the following result is known, but we could not find it in the literature so we state and prove it for further reference.

Lemma 5.1.6. *Let $F \in \mathbb{K}[x_0, \dots, x_n]$ and $n \geq 2$. If $V(F)$ is a reduced irreducible hypersurface of degree $d > 1$, then for all $k \in \mathbb{N}$ there exist $P_1, \dots, P_k \in V(F)$ such that $P_1, \dots, P_k$ are in general position.*

Proof. We proceed by induction on k. If $k = n + 1$, then $n + 1$ points are in general position if and only if they generate $\mathbb{P}^n$. It is clear that F must be a non-degenerate hypersurface, and so there are $n+1$ points on F such that they generate $\mathbb{P}^n$. Now assume $k > n + 1$. By induction there exist $P_1, \dots, P_{k-1}$ points on $V(F)$ in general position. Now we can consider all the hyperplanes $L_1, \dots, L_t$ generated by any n of these points. Since $V(F)$ is irreducible, $V(F)$ in not contained in any hyperplane and so $L_i \cap V(F)$ has dimension $n - 2$ for all i. Thus $V(F)$ can not be contained in $L_1 \cup \cdots \cup L_t$. Therefore, there exists a point P_k on F such that $P_k \in V(F) \setminus (L_1 \cup \cdots \cup L_t)$. Thus the points $P_1, \dots, P_k$ are in general position. □

Lemma 5.1.7. *Let $H \subset \mathbb{P}^n$ be a generic hyperplane and let $P_1, \dots, P_r$ be generic points in H. Then $\sigma(P_1), \dots, \sigma(P_r)$ are in general position.*

Proof. Since H is a generic hyperplane, we can assume that H has equation $a_0x_0 + \cdots + a_nx_n = 0$, with $a_i \neq 0$ for all i. So we have that $P = [p_0 : \cdots : p_n] \in H$ if and only if $p_n = -(a_0p_0 + \cdots + a_{n-1}p_{n-1})/a_n$.

Hence the last coordinate $p_n(i)$ of the point P_i can be seen as a linear combination of $p_0(i), \dots, p_{n-1}(i)$ for $i = 1, \dots, r$. Therefore, $P_1, \dots, P_r$ can be viewed as points of $\mathbb{P}^{n-1}$. If in the proof of Lemma 5.1.3 we replace all $p_n(i)$ with the linear combinations of $p_0(i), \dots, p_{n-1}(i)$, then the polynomial F which we defined in Lemma 5.1.4 is also in this case a multi-homogeneous polynomial in the ring $\mathbb{K}[p_0(1), \dots, p_{n-1}(1), \dots, p_0(r), \dots, p_{n-1}(r)]$. Therefore, similarly to Lemma 5.1.4, we define an open subset $\mathcal{A}$ where the lemma holds. Now we prove that the open set $\mathcal{A}$ is not empty. Since $\sigma(H)$ is an irreducible hypersurface of degree $d > 1$, by Lemma 5.1.6, we can find r points $\{Q_1, \dots, Q_r\}$ in $\sigma(H)$ which are in general position. Let $P_1, \dots, P_r$ be points on H such that $\sigma(P_1) = Q_1, \dots, \sigma(P_r) = Q_r$. It can be deduced that $\mathcal{A}$ is non-empty since we found that $P_1, \dots, P_r$ are in H and $\sigma(P_1), \dots, \sigma(P_r)$ are in general position. □

Remark 5.1.8. Note that general position on H is not generic enough. If the points $P_1, \dots, P_r$ are in general position on H, then it is not guaranteed that

$\sigma(P_1), \dots, \sigma(P_r)$ are in general position. Consider the plane $H \subset \mathbb{P}^3$ of equation $x_0 + 2x_1 + 3x_3 - x_4 = 0$. Then we have that the points $P_1 = [1 : 2 : 3 : 14]$, $P_2 = [1 : 1 : 1 : 6]$, $P_3 = [-1 : 2 : -2 : -3]$ and $P_4 = [-1 : -2 : 190/33 : 135/11]$ are in general position on H. A computation with `Macaulay2` [GS] shows that $\sigma(P_1)$, $\sigma(P_2)$, $\sigma(P_3)$ and $\sigma(P_4)$ are not in general position in $\mathbb{P}^3$.

```
Macaulay2, version 1.11
i1 : --pick 4 points on the plane H
P1={1,2,3,14}; P2={1,1,1,6};
P3={-1,2,-2,-3}; P4={-1,-2,190/33,135/11};
i5 : --we construct the matrix of the points as follows
M=matrix{P1,P2,P3,P4};
i6 :--we compute the determinant of the matrix M
det M
o6 = 0
o6 : QQ
i7 : --it verifies that they are on a plane
--we obtaine the 3 minors of the matrix M
--to see that there is no 3 collinear points
numgens minors(3,M)== binomial(4,3)*binomial(4,3)
o7 = true
i8 : -- by the standard cremona transformation we have
SigmaP1={1,1/2,1/3,1/14};SigmaP2={1,1,1,1/6};
i10 : SigmaP3={-1,1/2,-1/2,-1/3};SigmaP4={-1,-1/2,33/190,11/135};
i12 : N=matrix{SigmaP1,SigmaP2,SigmaP3,SigmaP4};
i13 : det(N)
o13 = 0
o13 : QQ
```

5.2 Weak Hadamard star configuration

Our goal in this section is to find a necessary and sufficient condition for a generally linear set of linear forms to be a weak Hadamard star configuration. We want to give a characterization of Hadamard sets producing weak Hadamard star configurations or producing Hadamard star configurations.

Recall that a set $\mathcal{L} = \{L_1, \dots, L_r\}$ of linear forms in $R = \mathbb{K}[x_0, \dots, x_n]$ is a set of *generally linear* forms if $r \geq n+1$ and if for any choice of $n+1$ distinct element in $\mathcal{L}$ they are linearly independent. Using sets of generally linear forms we can define a *codimension c star configuration*, or simply a *star configuration* as follows:

$$\mathbb{X}_c(\mathcal{L}) = \bigcup_{1 \leq i_1 < \ldots < i_c \leq r} V(L_{i_1}, \ldots, L_{i_c}).$$

Let $H_1, \ldots, H_r$ be hyperplanes of $\mathbb{P}^n$. We denote by

$$\mathbb{X}_c(H_1, \ldots, H_r) = \bigcup_{1 \leq i_1 < \ldots < i_c \leq r} H_{i_1} \cap \ldots \cap H_{i_c}.$$

A set $\mathcal{L} = \{L_1, \ldots, L_r\}$ of linear forms in R is a *Hadamard set* if there exists a linear form L and $P_1, \ldots, P_r$ points of $\mathbb{P}^n$ such that $V(L_i) = P_i \star V(L)$ for all $i = 1, \ldots, r$. Moreover $\mathcal{L}$ is a *strong Hadamard set* if $P_i \in V(L)$ for all $i = 1, \ldots, r$.

Definition 5.2.1. A star configuration $\mathbb{X}_c(\mathcal{L})$ is a *weak Hadamard star configuration* if $\mathcal{L}$ is a Hadamard set. Moreover a star configuration $\mathbb{X}_c(\mathcal{L})$ is a *Hadamard star configuration* if $\mathcal{L}$ is a strong Hadamard set.

Definition 5.2.2. Let L be a linear form. The *support* of L is the set of variables appearing in L with non-zero coefficients.

Proposition 5.2.3. *Let $\mathcal{L} = \{L_1, \ldots, L_r\}$ be a generally linear set of linear forms in R. The set $\mathbb{X}_c(\mathcal{L})$ is a weak Hadamard star configuration if and only if $L_i \cap \Delta_0 = \emptyset$ for all $i = 1, \ldots, r$.*

Proof. By [CCGV, Remark 2.10], we know that $\mathcal{L}$ is a Hadamard set if and only if $L_1, \ldots, L_r$ have the same support.

If $L_i \cap \Delta_0 = \emptyset$ for all $i = 1, \ldots, r$, it is clear that $L_1, \ldots, L_r$ have the same support (all coefficient are different from zero) and so $\mathbb{X}_c(\mathcal{L})$ is a weak Hadamard star configuration. Conversely, if $\mathbb{X}_c(\mathcal{L})$ is a weak Hadamard star configuration, then $\mathcal{L}$ is a Hadamard set and so $L_1, \ldots, L_r$ have the same support. By contradiction suppose that there exists $i \in \{1, \ldots, n\}$ such that $L_i \cap \Delta_0 \neq \emptyset$. Then there exists $j \in \{0, \ldots, n\}$ such that $L_i \in \mathbb{K}[x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n]$. Hence, since $L_1, \ldots, L_r$ have the same support, for all $i = 1, \ldots, r$, we have that $L_i \in \mathbb{K}[x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n]$ and so $\mathcal{L}$ is not generally linear. □

Theorem 5.2.4. *Let $H \subset \mathbb{P}^n$ be a hyperplane such that $H \cap \Delta_0 = \emptyset$. Consider $P_1, \ldots, P_r \in \mathbb{P}^n \setminus \Delta_{n-1}$ and set $H_i = P_i \star H$ for all $i = 1, \ldots, r$. Then, $\mathbb{X}_c(H_1, \ldots, H_r)$ is a weak Hadamard star configuration if and only if the points $\sigma(P_1), \ldots, \sigma(P_r)$ are in general position in $\mathbb{P}^n$.*

Proof. Assume that H is defined by $a_0x_0 + \cdots + a_nx_n = 0$ with $a_i \neq 0$ for all $i = 0, \ldots, n$. For all $j = 1, \ldots, r$, we define $L_j = \dfrac{a_0x_0}{p_0(j)} + \cdots + \dfrac{a_nx_n}{p_n(j)}$. Let

$\mathcal{L} = \{L_1, \dots, L_r\}$. Since $V(L_j) = P_j \star H$, then $\mathbb{X}_c(H_1, \dots, H_r) = \mathbb{X}_c(\mathcal{L})$. What remains to prove is: $\mathcal{L}$ is generally linear if and only if $\sigma(P_1), \dots, \sigma(P_r)$ are in general position. First suppose that $\mathcal{L}$ is not generally linear, i.e., there exist $n+1$ forms in $\mathcal{L}$ which are linearly dependent, say $L_1, \dots, L_{n+1}$. Therefore, there exists $(\lambda_1, \dots, \lambda_{n+1}) \neq (0, \dots, 0)$ such that $\sum_{j=1}^{n+1} \lambda_j L_j = 0$, i.e.

$$0 = \sum_{j=1}^{n+1} \lambda_j \left(\frac{a_0 x_0}{p_0(j)} + \cdots + \frac{a_n x_n}{p_n(j)} \right) = \sum_{i=0}^{n} \left(\frac{\lambda_1}{p_i(1)} + \cdots + \frac{\lambda_{n+1}}{p_i(n+1)} \right) a_i x_i.$$

Hence, we get the following system (since $a_i \neq 0$ for all $i = 0, \dots, n$):

$$\begin{cases} \dfrac{\lambda_1}{p_0(1)} + \cdots + \dfrac{\lambda_{n+1}}{p_0(n+1)} = 0 \\ \qquad\qquad \vdots \\ \dfrac{\lambda_1}{p_n(1)} + \cdots + \dfrac{\lambda_{n+1}}{p_n(n+1)} = 0 \end{cases}.$$

The system above has a not-trivial solutions since $(\lambda_1, \dots, \lambda_{n+1}) \neq (0, \dots, 0)$. It follows that

$$\det \begin{pmatrix} \dfrac{1}{p_0(1)} & \cdots & \dfrac{1}{p_n(1)} \\ \vdots & & \vdots \\ \dfrac{1}{p_0(n+1)} & \cdots & \dfrac{1}{p_n(n+1)} \end{pmatrix} = 0.$$

By Lemma 5.1.3, we have that $\sigma(P_1), \dots, \sigma(P_r)$ are not in general position.

For the converse, suppose that $\sigma(P_1), \dots, \sigma(P_r)$ are not in general position and we will prove that $\mathcal{L}$ is not generally linear. Since $\sigma(P_1), \dots, \sigma(P_r)$ are not in general position, $n+1$ points of these are in a hyperplane. Without loss of generality we may suppose that $\det(\sigma(P_1) \ \dots \ \sigma(P_{n+1})) = 0$. Therefore, considering the same system as in the first part of the proof, it follows that there exists $(\lambda_1, \dots, \lambda_{n+1}) \neq (0, \dots, 0)$ such that $\sum_{i=1}^{n+1} \lambda_i L_i = 0$. Therefore the elements in $\mathcal{L}$ are not linearly independent. $\square$

Remark 5.2.5. Note that if $\sigma(P_1), \dots, \sigma(P_r)$ are in general position, any 3 of these points are not collinear, and so any 3 points of $\left\{ \left[\frac{a_0}{p_0(i)} : \cdots : \frac{a_n}{p_n(i)} \right] \right\}$ are not collinear. Thus, as in [CCGV, Theorem 4.3], there is no rational normal curve containing the coordinates points and the points P_i, P_j, and P_k for all possible choices of $1 \leq i < j < k \leq r$.

Corollary 5.2.6. *Let* $P_1 \dots, P_r$ *be generic points in* $\mathbb{P}^n$. *Let* H *be a hyperplane such that* $H \cap \Delta_0 = \emptyset$ *and set* $H_i = P_i \star H$ *for all* $i = 1, \dots, r$. *Then,* $\mathbb{X}_c(H_1, \dots, H_r)$ *is a weak Hadamard star configuration.*

Proof. It follows from Lemma 5.1.4 and the Theorem 5.2.4. □

Remark 5.2.7. It is not sufficient that the points $P_1 \dots, P_r$ be only in general position to conclude that $\mathbb{X}_c(H_1, \dots, H_r)$ is a weak Hadamard star configuration. In fact, from Remark 5.1.5 there exist points $P_1 \dots, P_r$ in general position in $\mathbb{P}^n$ such that $\sigma(P_1), \dots, \sigma(P_r)$ are not in general position. Hence, $\mathbb{X}_c(H_1, \dots, H_r)$ is not a weak Hadamard star configuration.

Corollary 5.2.8. *Let* $H \subset \mathbb{P}^n$ *be a hyperplane such that* $H \cap \Delta_0 = \emptyset$. *Consider* $P_1, \dots, P_r \in H \setminus \Delta_{n-1}$ *and let* $H_i = P_i \star H$ *for all* $i = 1, \dots, r$. *Then* $\mathbb{X}_c(H_1, \dots, H_r)$ *is a Hadamard star configuration if and only if* $\sigma(P_1), \dots, \sigma(P_r)$ *are in general position in* $\mathbb{P}^n$.

Proof. From Theorem 5.2.4, we have that $\mathbb{X}_c(H_1, \dots, H_r)$ is a weak Hadamard star configuration if and only if $\sigma(P_1), \dots, \sigma(P_r)$ are in general position in $\mathbb{P}^n$. But in this case $P_i \in H$ for all $i = 1, \dots, r$, and so a weak Hadamard star configuration is a Hadamard star configuration. □

Corollary 5.2.9. *Let* $H \subset \mathbb{P}^n$ *be a hyperplane such that* $H \cap \Delta_0 = \emptyset$. *Consider generic points* $P_1, \dots, P_r$ *in* H *and let* $H_i = P_i \star H$ *for all* $i = 1, \dots, r$. *Then,* $\mathbb{X}_c(H_1, \dots, H_r)$ *is a Hadamard star configuration.*

Proof. It can be deduced from Lemma 5.1.7 and the above corollary. □

Remark 5.2.10. Also in this case we see that the points $P_1 \dots, P_r$ only being in general position in H is not sufficient to grant that $\mathbb{X}_c(H_1, \dots, H_r)$ is a Hadamard star configuration. In fact, Remark 5.1.8 shows that there exist $P_1 \dots, P_r$ points in general position in H such that $\sigma(P_1), \dots, \sigma(P_r)$ are not in general position in $\mathbb{P}^n$, and so $\mathbb{X}_c(H_1, \dots, H_r)$ is not a Hadamard star configuration.

Recall that, if $\mathbb{X}$ is a finite set of points in $\mathbb{P}^n$, then the r-th square-free Hadamard product of $\mathbb{X}$ is

$$\mathbb{X}^{\underline{\star} r} = \{P_1 \star \dots \star P_r | P_i \in \mathbb{X} \text{ and } P_i \neq P_j\}.$$

Theorem 5.2.11. *Let* ℓ *be a line in* $\mathbb{P}^n$ *such that* $\ell \cap \Delta_{n-2} = \emptyset$, *and let* $\mathbb{X} \subseteq \ell$ *be a set of* $m > n$ *points with* $\mathbb{X} \cap \Delta_{n-1} = \emptyset$. *Then* $\mathbb{X}^{\underline{\star} n}$ *is weak Hadamard star configuration.*

Proof. From [BCK, Theorem 4.7], we have that $\mathbb{X}^{\underline{\star} n}$ is a star configuration defined by the set of hyperplanes $\{P \star \ell^{\star(n-1)} | P \in \mathbb{X}\}$. Hence, by definition, $\mathbb{X}^{\underline{\star} n}$ is a weak Hadamard star configuration. □

Now we want to extend [CCGV, Theorem 2.17] and give a sufficient condition to make the weak Hadamard star configuration of the Theorem above be a Hadamard star configuration.

Theorem 5.2.12. *Let ℓ be a line in $\mathbb{P}^n$ such that $\ell \cap \Delta_{n-2} = \emptyset$, and let $\mathbb{X} \subseteq \ell$ be a set of $m > n$ points such that $\mathbb{X} \cap \Delta_{n-1} = \emptyset$. If there exist distinct points $P = [p_0, \dots, p_n]$ and $Q = [q_0, \dots, q_n]$ on ℓ such that*

$$\det \begin{pmatrix} p_0^{n-1} & \cdots & p_n^{n-1} \\ p_0^{n-2}q_0 & \cdots & p_n^{n-2}q_n \\ \vdots & & \vdots \\ q_0^{n-1} & \cdots & q_n^{n-1} \\ p_0 & \cdots & p_n \end{pmatrix} = \det \begin{pmatrix} p_0^{n-1} & \cdots & p_n^{n-1} \\ p_0^{n-2}q_0 & \cdots & p_n^{n-2}q_n \\ \vdots & & \vdots \\ q_0^{n-1} & \cdots & q_n^{n-1} \\ q_0 & \cdots & q_n \end{pmatrix} = 0, \qquad (\clubsuit)$$

then $\mathbb{X}^{\star n}$ is a Hadamard star configuration.

Proof. From Theorem 5.2.12, we know that $\mathbb{X}^{\star n}$ is a weak Hadamard star configuration. By [BCK, Corollary 3.7], $\ell^{\star(n-1)}$ is defined by the following equation:

$$\det \begin{pmatrix} p_0^{n-1} & p_1^{n-1} & \cdots & p_n^{n-1} \\ p_0^{n-2}q_0 & p_1^{n-2}q_1 & \cdots & p_n^{n-2}q_n \\ \vdots & \vdots & & \vdots \\ q_0^{n-1} & q_1^{n-1} & \cdots & q_n^{n-1} \\ x_0 & x_1 & \cdots & x_n \end{pmatrix} = 0.$$

By the hypothesis on P and Q we have that P and Q are in $\ell^{\star(n-1)}$, and so $\mathbb{X} \subseteq \ell \subseteq \ell^{\star(n-1)}$. In Theorem 5.2.11, we say that the set of hyperplanes of $\mathbb{X}^{\star n}$ is $\{P \star \ell^{\star(n-1)} | P \in \mathbb{X}\}$, and so, by the definition, $\mathbb{X}^{\star n}$ is a Hadamard star configuration. ☐

Remark 5.2.13. ($\clubsuit$) is a numerical sufficient condition to have that $\mathbb{X}^{\star n}$ be a Hadamard star configuration. More geometrically ($\clubsuit$) means that P, Q are in the linear subspace generated by $P^{\star(n-1)}, P^{\star(n-2)} \star Q, \dots, P \star Q^{\star(n-2)}, Q^{\star(n-1)}$.

If $[1 : \cdots : 1] \in \ell$, then the above theorem holds for all $Q \in \ell$. In fact one can verify that ($\clubsuit$) holds. When $n = 2$ the above theorem always holds (see [CCGV, Theorem 2.17]).

5.3 Apolar Hadamard star configuration

In this section, we study the existence of Hadamard star configurations apolar to a homogeneous polynomial. In [BJ] the author has shown for which triples

(d, r, n) a star configuration $\mathbb{X}(L_1, \ldots, L_r) := \mathbb{X}_n(L_1, \ldots, L_r) \subset \mathbb{P}^n$ is apolar to a given form $F \in R_d$. We recall some basic facts.

Let us consider the standard graded polynomial ring $T = \mathbb{K}[y_0, \ldots, y_n]$. We make T act on R via differentiation, i.e. we think of $y_j = \partial/\partial x_j$. For any form F of degree d in R, we define the ideal $F^{\perp} \subseteq T$, called the *perp ideal* or the ideal of the *inverse system* of F, as follows:

$$F^{\perp} = \{\partial \in T : \partial F = 0\}.$$

Lemma 5.3.1 (Apolarity Lemma). *A homogeneous degree d form $F \in R$ can be written as*

$$F = \sum_{i=1}^{s} L_i^d, \ \textit{with} \ \ L_i \in R_1 \ \textit{pairwise linearly independent}$$

if and only if there exists $I \subseteq F^{\perp}$ such that I is the ideal of a set of s distinct points in $\mathbb{P}(R_1)$.

Definition 5.3.2. We say that a set of points $\mathbb{X}$ of $\mathbb{P}^n$ is *apolar to a form F* if the ideal of the set of points is such that $I_{\mathbb{X}} \subset F^{\perp}$. We say that $\mathbb{X}$ is an *apolar Hadamard star configuration for F* if the set $\mathbb{X}$ is a Hadamard star configuration.

Remark 5.3.3. A linear form can be seen as a point of $\mathbb{P}(R_1)$ and a set $\mathcal{L}$ of generally linear forms can be viewed as a set of points in general position on $\mathbb{P}(R_1)$. In this case Proposition 5.2.3 says that if $\mathcal{L} \cap \Delta_{n-1} = \emptyset$, then $\mathcal{L}$ is a weak Hadamard star configuration. Using this point of view it is clear that a generic star configuration is a weak Hadamard star configuration.

Remark 5.3.4. Let F be a generic form of degree $d \geq 2$ in $n + 1$ variables. If $r < d + n$, there is no Hadamard star configuration $\mathbb{X}(L_1, \ldots, L_r)$ apolar to F with some exceptions (see, [BJ, Lemma 3.1, 3.2, Corollory 4.9, Theorem 5.3, 5.4 and Lemma 5.2]).

Theorem 5.3.5. *Let F be a form of degree $d \geq 2$ in $n + 1$ variables. If $r \geq d + n$, then there exists a Hadamard star configuration $\mathbb{X}(\mathcal{L})$ apolar to F, where $\mathcal{L} = \{H_1, \ldots, H_r\}$ is a set of hyperplanes in $\mathbb{P}^n$.*

Proof. By Corollary 5.2.9 we have that there exists a Hadamard star configuration $\mathbb{X}_n(\mathcal{L})$. The desired result follows from by [BJ, Lemma 3.1, 3.2 and Corollory 4.10] and [BJ, Remark 5.1] for $n = 2$ and $n \geq 3$, respectively. $\square$

Example 5.3.6. Let $F = \frac{1}{5}x_0^2 + x_0x_1 + 3x_1^2 + \frac{7}{9}x_0z_2 + \frac{5}{4}x_1x_2 + \frac{5}{4}x_2^2$ be a generic ternary quadratic form and $\mathcal{L} = \{L_1, L_1, L_3, L_4\}$ where $L_1 = (13/4)y_0 + (1/2)y_1 + (1/3)y_2$,

$L_2 = -(13/15)y_0 + (1/3)y_1 + (1/6)y_2$, $L_3 = (1/7)y_0 + (1/7)y_1 + (1/5)y_2$, $L_4 = y_0 + (1/3)y_1 + (1/4)y_2$. It is clear that $\mathbb{X}(\mathcal{L})$ is a weak Hadamard star configuration and its defining ideal is generated by forms of degree 3. By an easy computation we can find the perp ideal of F:

$$F^{\perp} = (2y_1y_2 - y_2^2, 45y_0y_2 - 14y_2^2, 5y_1^2 - 12y_2^2, 5y_0y_1 - 2y_2^2, 25y_0^2 - 4y_2^2).$$

Since $(\mathbb{K}[y_0, y_1, y_2])_3 \subseteq (F^{\perp})_3$, then we conclude that $I_{\mathbb{X}(\mathcal{L})} \subset F^{\perp}$. By [CCGV, Theorem 3.1], this weak Hadamard star configuration is a Hadamard star configuration since

$$\operatorname{rk}\begin{pmatrix} 4/13 & 2 & 3 \\ -15/13 & 3 & 6 \\ 7 & 7 & 5 \\ 1 & 3 & 4 \end{pmatrix} = 2.$$

Remark 5.3.7. A weak Hadamard star configuration apolar to a generic form is not necessarily a Hadamard star configuration, as the following example shows.

Example 5.3.8. The four linear forms $L_1 = y_0 + 3y_1 - 2y_2$, $L_2 = -3y_0 + 5y_1 + y_2$, $L_3 = -(1/2)y_0 + (1/4)y_1 + 7y_2$, $L_4 = 4y_0 + 3y_1 + y_2$ gives an apolar weak Hadamard star configuration $\mathbb{X}(L_1, \dots, L_4)$ for the form F in the previous example which is not a Hadamard star configuration since

$$\operatorname{rk}\begin{pmatrix} 1 & 1/3 & -1/2 \\ -1/3 & 1/5 & 1 \\ -2 & 4 & 1/7 \\ 1/4 & 1/3 & 1 \end{pmatrix} \neq 2.$$

In the next example we give a special form M such that, for $r = 4$, all weak Hadamard star configurations apolar to M are Hadamard star configurations apolar to M.

Example 5.3.9. Let $M = x_0x_1x_2 \in (\mathbb{K}[x_0, x_1, x_2])_3$ be a ternary monomial form. If $r < 4$ there are no star configurations apolar to M and so there are neither weak Hadamard star configurations nor Hadamard star configurations (see, [BJ, Remark 6.3]). If $r > 4$, by Theorem 5.3.5 we have that there exists a Hadamard star configuration apolar to M. So the only case we need to check is $r = 4$. We are interested in finding an apolar Hadamard star configuration $\mathbb{X}(L_1, \dots, L_4)$ and we proceed as follows:
pick a generic Hadamard star configuration $\mathbb{X}(L_1, \dots, L_4)$ where

$$L_1 = a_1y_0 + b_1y_1 + c_1y_2,\ L_2 = a_2y_0 + b_2y_1 + c_2y_2,$$

$$L_3 = a_3y_0 + b_3y_1 + c_3y_2,\ L_4 = a_4y_0 + b_4y_1 + c_4y_2$$

are four general linear forms in $(\mathbb{K}[x_0, x_1, x_2])_1$ with all a_i, b_i, c_i different from zero. Since $c_i \neq 0$ we can suppose that $c_1 = c_2 = c_3 = c_4 = 1$. By direct computation, we have that $\mathbb{X}(L_1, \dots, L_4)$ is apolar to M, i.e. $I_{\mathbb{X}(L_1,\dots,L_4)} \subseteq M^\perp = (y_0^2, y_1^2, y_2^2)$, if and only if

$$\begin{aligned}
b_3a_4 + b_2a_4 + a_3b_4 + a_2b_4 + b_2a_3 + a_2b_3 &= 0,\\
b_3a_4 + b_1a_4 + a_3b_4 + a_1b_4 + b_1a_3 + a_1b_3 &= 0,\\
b_2a_4 + b_1a_4 + a_2b_4 + a_1b_4 + b_1a_2 + a_1b_2 &= 0,\\
b_2a_3 + b_1a_3 + a_2b_3 + a_1b_3 + b_1a_2 + a_1b_2 &= 0.
\end{aligned} \tag{$\bigstar$}$$

A possible solution for ($\bigstar$) with $a_i \neq 0$ and $b_i \neq 0$ gives a weak Hadamard star configuration $\mathbb{X}(L_1, \dots, L_4)$ apolar to M if the four linear forms $L_1, \dots, L_4$ are generally linear. By [CCGV, Theorem 3.1], the star configuration $\mathbb{X}(L_1, \dots, L_4)$ is a Hadamard star configuration if and only if

$$\operatorname{rk}\begin{pmatrix} \frac{1}{a_1} & \frac{1}{b_1} & 1\\ \frac{1}{a_2} & \frac{1}{b_2} & 1\\ \frac{1}{a_3} & \frac{1}{b_3} & 1\\ \frac{1}{a_4} & \frac{1}{b_4} & 1 \end{pmatrix} = 2.$$

Therefore, $\mathbb{X}(L_1, \dots, L_4)$ is a Hadamard star configuration if and only if the determinants of all the 3×3 minors of the above matrix are zeros. So we have,

$$\begin{aligned}
\tfrac{1}{a_1b_2} + \tfrac{1}{a_3b_1} + \tfrac{1}{a_2b_3} - \tfrac{1}{a_3b_2} - \tfrac{1}{a_2b_1} - \tfrac{1}{a_1b_3} &= 0,\\
\tfrac{1}{a_1b_2} + \tfrac{1}{a_4b_1} + \tfrac{1}{a_2b_4} - \tfrac{1}{a_4b_2} - \tfrac{1}{a_2b_1} - \tfrac{1}{a_1b_4} &= 0,\\
\tfrac{1}{a_1b_3} + \tfrac{1}{a_4b_1} + \tfrac{1}{a_3b_4} - \tfrac{1}{a_4b_3} - \tfrac{1}{a_3b_1} - \tfrac{1}{a_1b_4} &= 0,\\
\tfrac{1}{a_2b_3} + \tfrac{1}{a_4b_2} + \tfrac{1}{a_3b_4} - \tfrac{1}{a_4b_3} - \tfrac{1}{a_3b_2} - \tfrac{1}{a_2b_4} &= 0.
\end{aligned} \tag{$\blacklozenge$}$$

Since a_i and b_i are not zero, ($\blacklozenge$) is equivalent to

$$\begin{aligned}
a_1a_2a_3b_1b_2b_3\left(\tfrac{1}{a_1b_2} + \tfrac{1}{a_3b_1} + \tfrac{1}{a_2b_3} - \tfrac{1}{a_3b_2} - \tfrac{1}{a_2b_1} - \tfrac{1}{a_1b_3}\right) &= 0\\
a_1a_2a_4b_1b_2b_4\left(\tfrac{1}{a_1b_2} + \tfrac{1}{a_4b_1} + \tfrac{1}{a_2b_4} - \tfrac{1}{a_4b_2} - \tfrac{1}{a_2b_1} - \tfrac{1}{a_1b_4}\right) &= 0\\
a_1a_3a_4b_1b_3b_4\left(\tfrac{1}{a_1b_3} + \tfrac{1}{a_4b_1} + \tfrac{1}{a_3b_4} - \tfrac{1}{a_4b_3} - \tfrac{1}{a_3b_1} - \tfrac{1}{a_1b_4}\right) &= 0\\
a_2a_3a_4b_2b_3b_4\left(\tfrac{1}{a_2b_3} + \tfrac{1}{a_4b_2} + \tfrac{1}{a_3b_4} - \tfrac{1}{a_4b_3} - \tfrac{1}{a_3b_2} - \tfrac{1}{a_2b_4}\right) &= 0
\end{aligned} \tag{$\blacklozenge\blacklozenge$}$$

By using `Macaulay2` one can check that the equations ($\blacklozenge\blacklozenge$) can be written as combinations (with coefficients in $\mathbb{Q}[a_1, \dots, a_4, b_1, \dots, b_4]$) of the equations ($\bigstar$), thus a solution of ($\bigstar$) is a solution of ($\blacklozenge\blacklozenge$) i.e. it is a solution of ($\blacklozenge$). Therefore we conclude that a solution for (★) gives a Hadamard star configuration $\mathbb{X}(L_1,\ldots,L_4)$ apolar to M if the four linear forms $L_1,\ldots,L_4$ are generally linear. Thus a weak Hadamard star configuration $\mathbb{X}(L_1,\ldots,L_4)$ apolar to M is a Hadamard star configuration as well. For example, a possible solution of (★) is the following:

$$a_1 = \frac{\sqrt{2641}+119}{4(\sqrt{2641}+47)},\ a_2 = \frac{2(-\sqrt{2641}-59)}{\sqrt{2641}+47},\ a_3 = 2,\ a_4 = 1,$$

$$b_1 = \frac{3(-\sqrt{2641}-39)}{16},\ b_2 = 5,\ b_3 = 9,\ b_4 = \frac{\sqrt{2641}+11}{4}.$$

One can check that those coefficients define a set of four linearly independent lines $L_1,\ldots,L_4$. Hence there exists a Hadamard star configuration $\mathbb{X}(L_1,\ldots,L_4)$ apolar to M.

Remark 5.3.10. The previous example gives us a sufficient condition to grant that a star configuration $\mathbb{X}(L_1,\ldots,L_4)$ be a Hadamard star configuration: if $\mathbb{X}(L_1,\ldots,L_4)$ is a star configuration apolar to $x_0x_1x_2$ and $L_i \cap \Delta_0 = \emptyset$, then $\mathbb{X}(L_1,\ldots,L_4)$ is a Hadamard star configuration. This condition is not necessary, in fact the Hadamard star configuration of Example 5.3.6 is not apolar to $x_0x_1x_2$.

Remark 5.3.11. The author in [BJ, Conjecture 1] suggests that any generic ternary form of degree d has an apolar star configuration $\mathbb{X}(L_1,\ldots,L_{d+1})$ with $d \geq 5$. He also has a complete proof for the case of quadratic and cubic forms. It is interesting to understand when the star configuration $\mathbb{X}(L_1,\ldots,L_{d+1})$ is a Hadamard star configuration or simply a weak Hadamard star configuration.

Theorem 5.3.12. *For any ternary cubic of rank five there exists an apolar weak Hadamard star configuration* $\mathbb{X}(L_1,\ldots,L_4)$.

Proof. By [BJ, Proposition 6.8] and Proposition 5.2.3, the proof is done. □

Appendix A

Computing Hadamard products with `Singular`

Given the ideals I and J of the varieties X and Y in $\mathbb{P}^n$, respectively, the computation of the ideal of $X \star Y$ may be achieved with a saturation and elimination as follows:

- Let $I(y_{1i})$ be the ideal of X in $\mathbb{C}[y_{10}, \dots, y_{1n}]$ and let $I(y_{2i})$ be the ideal of Y in $\mathbb{C}[y_{20}, \dots, y_{2n}]$.
- Work in the ring $\mathbb{C}[y_{10}, \dots, y_{1n}, y_{20}, \dots, y_{2n}, x_0, \dots, x_n]$.
- See $I(y_{1i})$ and $I(y_{2i})$ as ideals of $\mathbb{C}[y_{10}, \dots, y_{1n}, y_{20}, \dots, y_{2n}, x_0, \dots, x_n]$.
- Form the ideal $I(y_{1i}) + J(y_{2i}) + \langle x_0 - y_{10}y_{20}, \dots, x_n - y_{1n}y_{2n}\rangle$.
- Saturate with respect to the product $x_0 \cdots x_n$.
- Eliminate the $2n+2$ variables $y_{i0}, \dots, y_{in}$.

For completeness, we show here a procedure in `Singular` which performs the previous steps and that we used to compute the examples in the chapter.

```
LIB "ncalg.lib";
LIB "poly.lib";
LIB "rootsmr.lib";
LIB "elim.lib";


proc HPr(ideal I1, ideal I2, int n) /* where n+1 is the number of variables */
{
```

```
ring RH=0,(y(1..2)(0..n),x(0..n)),dp;
int i;
ideal T1;
ideal T2;
poly elle1;
poly elle2;
poly elle3=1;
map f1;
map f2;

T1=y(1)(0);
for (i=1; i<=n; i=i+1)
{
elle1=y(1)(i);
T1=T1+elle1;
}
f1=r,T1;
ideal H1=f1(I1);


T2=y(2)(0);
for (i=1; i<=n; i=i+1)
{
elle2=y(2)(i);
T2=T2+elle2;
}
f2=r,T2;
ideal H2=f2(I2);

int j;
ideal H=0;
for (j=0; j<=n; j=j+1)
{
H=H+ideal(x(j)-y(1)(j)*y(2)(j));
elle3=elle3*x(j);
}

H=H+H1+H2;
```

```
ideal Ksat=elle3;
ideal HH=sat(H,Ksat)[1];

ideal HHH=elim(H,1..2*(n+1));

setring r;
ideal HFin=imap(RH,HHH);

return(HFin);
}
```

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