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# Exactly solvable $f(R)$ inflation

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## Abstract

We show that adding a cosmological constant term to the Starobinsky model, it can be solved exactly without using the slow-roll approximation.

# 1 Introduction

To explain the homogeneity, flatness and horizon problems of the universe it has been hypothesized the existence of a rapid expansion phase of the primordial universe. Inflationary cosmology can also predict the anisotropies of the cosmic background temperature and the formation of large-scale structures. The first models of inflation were based on the coupling of one or more scalar fields ( the inflaton ) to standard gravity and correspond to a modification of the energy-momentum tensor in Einstein's equations.

However, there is another approach to explain the accelerated expansion of the universe. This corresponds to modified gravity in which the gravitational theory is modified with respect to general relativity. A subclass of such models is the gravity  $f(R)$  in which the lagrangian density is an arbitrary function of  $R$  [1]-[2]-[3]. The model  $f(R) = R + \alpha R^2$  with  $\alpha > 0$  can lead to the accelerated expansion of the universe due to the presence of the term  $\alpha R^2$ , as proposed by Starobinsky in the 1980. In all cases, the slow-roll approximation is used to extract physical information from these models. Our aim is to go beyond this approximation ( see also [4] ) and try to solve exactly the Starobinsky model. Our trick is to add a cosmological constant term to the model, which doesn't alter its main predictions and allows us to find an exact solution of the Friedmann equations.

Our work is organized as follows. First, the Friedmann equations of the modified Starobinsky model are resolved in the Jordan frame and the temporal dependence of all the physical quantities of the background is calculated exactly. Then we map this solution to the Einstein frame and the analysis of the physical quantities is repeated in full detail. Finally, for completeness we recall how to connect the Starobinsky model to the experimental data.

## 2 $f(R)$ theory

Consider a  $f(R)$  theory in the flat space-time  $FLRW$  with the metric tensor in the form  $ds^2 = -dt^2 + a^2(t) dx_i^2$ . We work in the notation  $k^2 = 8\pi G = 1$ . The background metric satisfies the following Friedmann equations (in the absence of matter and radiation)

$$\begin{aligned} 3FH^2 &= \frac{FR - f}{2} - 3H\dot{F} \\ -2F\dot{H} &= \ddot{F} - H\dot{F} \end{aligned} \tag{2.1}$$

where we introduced the notation

$$F(R) = \frac{df(R)}{dR} \quad (2.2)$$

It is well known that the  $f(R)$  gravity can be expressed as a Brans-Dicke theory in the Jordan frame. In fact the action of the model  $f(R)$  can be represented as follows:

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \varphi R - U(\varphi) \right] \quad (2.3)$$

where

$$\varphi = F(R) \quad U(\varphi) = \frac{FR - f}{2} \quad (2.4)$$

The auxiliary field  $\varphi$  is coupled in a non-minimal way to gravity (Jordan frame). The Friedmann equations can be rewritten as:

$$\begin{aligned} \ddot{\varphi} + 3H\dot{\varphi} + \frac{2}{3}(\varphi U_\varphi - 2U) &= 0 \\ 3 \left( H + \frac{\dot{\varphi}}{2\varphi} \right)^2 &= \frac{3}{4} \frac{\dot{\varphi}^2}{\varphi^2} + \frac{U}{\varphi} \end{aligned} \quad (2.5)$$

where  $U_\varphi = \frac{dU}{d\varphi}$ . The amount  $U(\varphi)$  represents an energy density, but can not be connected to an effective force. The effective potential is instead

$$U_{eff} = \int \frac{2}{3}(\varphi U_\varphi - 2U) d\varphi \quad F_{eff} = \frac{dU_{eff}}{d\varphi} \quad (2.6)$$

In the Jordan frame we try to solve exactly the Starobinsky model:

$$f(R) = R + \frac{R^2}{6M^2} \quad (2.7)$$

Our contribution is to note that by adding a cosmological constant term to (2.7) we obtain a model which can be solved exactly:

$$f(R) = R + \frac{R^2}{6M^2} + \frac{2M^2}{3} \quad (2.8)$$

The exact solution of the model (2.8) that we want to analyze here is of the type:

$$H(t) = \frac{M}{2} \left( \sqrt{\varphi(t)} - \frac{1}{3\sqrt{\varphi(t)}} \right) \quad (2.9)$$

where the time dependency of  $\varphi(t)$  is set to be:

$$\dot{\varphi}(t) = -\frac{2}{3}M\sqrt{\varphi(t)} \quad \ddot{\varphi}(t) = \frac{2}{9}M^2 \quad (2.10)$$

In turn, the time  $t$  can be rewritten in terms of the  $\varphi$  variable (in inflation the  $\varphi$  field is homogeneous):

$$t = -\frac{3}{M}\sqrt{\varphi(t)} \quad (2.11)$$

This solution allows to calculate the following quantities:

$$\begin{aligned} U(\varphi) &= \frac{1}{2}[RF(R) - f(R)] = \frac{3}{4}M^2(\varphi - 1)^2 - \frac{M^2}{3} \\ R &= 6(\dot{H} + 2H^2) = 3M^2(\varphi - 1) \end{aligned} \quad (2.12)$$

from which we can verify that the Friedmann equations (2.5) are solved exactly.

So we have an inflation model  $f(R)$  in which we can go beyond the slow-roll approximation and check if this is a good approximation. The solution for  $H(t)$  can be re-expressed as follows:

$$H(t) = \frac{d}{dt} \ln a(t) = -\frac{3}{4}\dot{\varphi}(t) + \frac{1}{4}(\dot{\ln \varphi}) \quad (2.13)$$

from which we can integrate for  $a(t)$

$$a(t) = k \varphi(t)^{\frac{1}{4}} e^{-\frac{3}{4}\varphi(t)} \quad (2.14)$$

Once we know  $a(t)$  we can determine the conformal time

$$\eta(t) = \int \frac{dt}{a(t)} = \frac{1}{k} \int_{t_i}^t dt \varphi(t)^{-\frac{1}{4}} e^{\frac{3}{4}\varphi(t)} \quad (2.15)$$

which is an incomplete gamma function.

The number of  $N$  e-foldings between the time  $t_i$  in which inflation is triggered and the time  $t_f$  in which it ends is defined by:

$$N = \int_{t_i}^{t_f} H(t) dt = \frac{1}{4}(\ln(\varphi_f) - \ln(\varphi_i)) + \frac{3}{4}(\varphi_i - \varphi_f) \quad (2.16)$$

which typically is in the order of 60 – 70. We note that the logarithmic term gives a small correction (1 %) to the main term that is obtained from the slow-roll approximation, proving that this is a good approximation.

Let's take a closer look at the solution for  $a(t)$ . Since  $\varphi(t) = \frac{M^2 t^2}{9}$  we obtain:

$$a(t) = k (-t)^{\frac{1}{2}} e^{-\frac{M^2}{12} t^2} \quad (2.17)$$

For  $t$  negative  $a(t)$  can be an increasing function of  $t$ . Calculating the derivative of  $a(t)$  we obtain that

$$H(t) = \frac{1}{2t} - \frac{M^2}{6} t \quad (2.18)$$

The condition  $\dot{a}(t) = 0$  is reached for  $t = -\frac{\sqrt{3}}{M}$ , but this time is never reached, because inflation has validity only for  $\ddot{a}(t) > 0$ :

$$\ddot{a}(t) > 0 \quad \rightarrow \quad t < t_f = -\frac{\sqrt{6 + 3\sqrt{5}}}{M} \quad (2.19)$$

The time  $t_i$  in which inflation is triggered is related to the number of  $N$  e-foldings

$$N = \int_{t_i}^{t_f} H(t) dt = \frac{1}{2} \ln(t_f - t_i) - \frac{M^2}{12} (t_f^2 - t_i^2) \quad (2.20)$$

The logarithmic term is negligible therefore we obtain:

$$t_i^2 = t_f^2 + \frac{12N}{M^2} \quad \rightarrow \quad t_i = -\frac{\sqrt{12N + 6 + 3\sqrt{5}}}{M} \quad (2.21)$$

We can also link the time  $t$  to the  $H$  function by reversing the relation (2.18)

$$t = -\frac{3H}{M^2} \left[ 1 + \sqrt{1 + \frac{M^2}{3H^2}} \right] \quad (2.22)$$

For the initial time  $t_i$  we know that the field  $H_i^2 \gg M^2$  for which we can approximate

$$t_i \sim -\frac{6H_i}{M^2} \sim -\frac{\sqrt{12N}}{M} \quad \rightarrow \quad H_i \sim \sqrt{\frac{N}{3}} M \quad (2.23)$$

while the time  $t_f$  is tied to the  $H_f$  field

$$H_f \sim \frac{M}{\sqrt{6}} \quad (2.24)$$

which is the value for  $H_f$  that is obtained from the slow-roll approximation.

Finally, we can link the number of  $N$  e-foldings to the  $H_i$  field

$$N \sim \frac{M^2}{12} (t_i^2 - t_f^2) \sim \frac{3H_i}{M^2} \sim \frac{1}{2\epsilon_1(t_i)} \quad (2.25)$$

where we introduced the slow-roll parameter

$$\epsilon_1 = -\frac{\dot{H}}{H^2} \quad \epsilon_1(t_i) \sim \frac{M^2}{6H_i^2} \quad (2.26)$$

Later it will be useful to introduce other slow-roll parameters

$$\epsilon_1 = -\frac{\dot{H}}{H^2} \quad \epsilon_2 = 0 \quad \epsilon_3 = \frac{\dot{F}}{2HF} \quad \epsilon_4 = \frac{\ddot{F}}{H\dot{F}} \quad (2.27)$$

The  $\epsilon_2$  parameter is null for a theory of pure  $f(R)$  inflation ( see [1] ). The following exact identity is valid:

$$\epsilon_1 = -\epsilon_3 (1 - \epsilon_4) \quad (2.28)$$

In the model (2.8)

$$\epsilon_3 = \epsilon_4 = \frac{2}{1 - \frac{M^2}{3}t^2} < 0 \quad \epsilon_1 = \frac{2(1 + \frac{M^2}{3}t^2)}{(1 - \frac{M^2}{3}t^2)^2} > 0 \quad (2.29)$$

Note that for  $t \sim t_i$  the parameters  $\epsilon_1, \epsilon_3, \epsilon_4$  are all much less than 1,

$$\epsilon_1(t_i) \sim -\epsilon_3(t_i) \sim \frac{M^2}{6H_i^2} \ll 1 \quad (2.30)$$

while for  $t = t_f$  the same parameters are of the order of the unity.

### 3 Einstein frame

In the Jordan frame the  $\varphi$  field is coupled not minimally to gravity. To pass to the Einstein frame in which the gravitational part of the action regains its canonical form, we must apply

the following transformation:

$$\tilde{g}_{\mu\nu} = \varphi g_{\mu\nu} \quad d\tau = \sqrt{\varphi(t)} dt \quad \tilde{a}(\tau) = \sqrt{\varphi(t)} a(t) \quad (3.1)$$

which leads to an action of the form:

$$S = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{2} \tilde{R} - \frac{3}{4} \left( \frac{\tilde{\nabla}\varphi}{\varphi} \right)^2 - \frac{U(\varphi)}{\varphi^2} \right] \quad (3.2)$$

where  $\tilde{\nabla}$  is the derivative with respect to  $\tilde{x}^\mu$  ( $\tau = \tilde{t}$ ). To obtain the canonical kinetic term for  $\varphi$  we must introduce the scalar field  $\phi$  of the Einstein frame

$$\phi = \sqrt{\frac{3}{2}} \ln \varphi \quad (3.3)$$

The canonical action in terms of  $\tilde{g}_{\mu\nu}$  and  $\phi$  is then

$$S = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{2} \tilde{R} - \frac{1}{2} (\tilde{\nabla}\phi)^2 - V(\phi) \right] \quad (3.4)$$

where the potential  $V(\phi)$  is calculable as

$$V(\phi) = \frac{(FR - f)}{2F^2} = \frac{U(\varphi)}{\varphi^2} \Big|_{\varphi=\varphi(\phi)} \quad (3.5)$$

The equations of motion are:

$$\frac{d^2\phi}{d\tau^2} + 3 \tilde{H} \frac{d\phi}{d\tau} + \frac{\partial V(\phi)}{\partial \phi} = 0 \quad \tilde{H}^2 = \frac{1}{3} \left[ \frac{1}{2} \left( \frac{d\phi}{d\tau} \right)^2 + V(\phi) \right] \quad (3.6)$$

Based on the mapping (3.1) we obtain the following identities:

$$\frac{d}{d\tau} = \frac{1}{\sqrt{\varphi(t)}} \frac{d}{dt} = -\frac{2M}{3} \frac{d}{d\varphi} = -\sqrt{\frac{2}{3}} M e^{-\sqrt{\frac{2}{3}}\phi} \frac{d}{d\phi} \quad (3.7)$$

from which we derive  $\tilde{H} = \frac{1}{a} \frac{d\tilde{a}}{d\tau}$ :

$$\tilde{H} = \frac{1}{\sqrt{F}} \left( H + \frac{\dot{F}}{2F} \right) = \frac{M}{2} \left( 1 - e^{-\sqrt{\frac{2}{3}}\phi} \right) \quad (3.8)$$

Its derivative with respect to  $\tau$  is

$$\frac{d\tilde{H}}{d\tau} = -\frac{M^2}{3} e^{-2\sqrt{\frac{2}{3}}\phi} \quad (3.9)$$

since

$$\frac{d\phi}{d\tau} = -\sqrt{\frac{2}{3}}M e^{-\sqrt{\frac{2}{3}}\phi} \quad (3.10)$$

The curvature tensor  $\tilde{R}$  turns out to be:

$$\tilde{R} = 6 \left( \frac{d\tilde{H}}{d\tau} + 2\tilde{H}^2 \right) = M^2 ( 3 - 6 e^{-\sqrt{\frac{2}{3}}\phi} + e^{-2\sqrt{\frac{2}{3}}\phi} ) \quad (3.11)$$

The potential of the solvable model is of the type:

$$V(\phi) = M^2 \left[ \frac{3}{4} - \frac{3}{2} e^{-\sqrt{\frac{2}{3}}\phi} + \frac{5}{12} e^{-2\sqrt{\frac{2}{3}}\phi} \right] \quad (3.12)$$

from which we can verify that the Friedmann equations are solved exactly. The time variable  $\tau$  is bound to the  $\phi$  field from the relation (valid only for  $\tau$  negative)

$$\tau = -\frac{3}{2M} e^{\sqrt{\frac{2}{3}}\phi} \quad (3.13)$$

from which we get:

$$\tilde{H}(\tau) = \frac{M}{2} + \frac{3}{4\tau} \quad (3.14)$$

By integrating with respect to  $\tau$  we finally obtain the temporal evolution of  $\tilde{a}(\tau)$ :

$$\tilde{a}(\tau) = k (-\tau)^{\frac{3}{4}} e^{\frac{M}{2}\tau} \quad (3.15)$$

For  $\tau$  negative the function  $\tilde{a}(\tau)$  can be an increasing function of  $\tau$ . For  $\tau = -\frac{3}{2M}$  we have  $\dot{\tilde{a}}(\tau) = 0$ , but this value is never reached. We must indeed impose that:

$$\ddot{\tilde{a}}(\tau) > 0 \quad \tau < \tau_f = -\frac{3}{2M} - \frac{\sqrt{3}}{M} \quad (3.16)$$

Let us calculate the slow-roll approximation variables:

$$\begin{aligned}
\epsilon(\tau) &= -\frac{\dot{\tilde{H}}(\tau)}{\tilde{H}^2(\tau)} = \frac{3}{(\tau M + \frac{3}{2})^2} & \epsilon(\tau_f) &= 1 \\
\eta(\tau) &= \epsilon(\tau) - \frac{\dot{\epsilon}(\tau)}{2\epsilon(\tau)\tilde{H}(\tau)} = \frac{2}{(\tau M + \frac{3}{2})} & |\eta(\tau_f)| &= \frac{2}{\sqrt{3}} \sim 1
\end{aligned} \tag{3.17}$$

Finally, the value of  $\phi_f$  at the end of the inflation is:

$$e^{\sqrt{\frac{2}{3}}\phi_f} = 1 + \frac{2}{\sqrt{3}} \quad \phi_f \sim 1 \tag{3.18}$$

Let's calculate  $\tau_i$ , that is the time when inflation starts. The number of e-foldings is

$$\begin{aligned}
N &= \int_{\tau_i}^{\tau_f} \tilde{H}(\tau) d\tau = \frac{M}{2}(\tau_f - \tau_i) + \frac{1}{2}\sqrt{\frac{3}{2}}(\phi_f - \phi_i) = \\
&= \frac{3}{4} \left( e^{\sqrt{\frac{2}{3}}\phi_i} - e^{\sqrt{\frac{2}{3}}\phi_f} \right) + \frac{1}{2}\sqrt{\frac{3}{2}}(\phi_f - \phi_i)
\end{aligned} \tag{3.19}$$

The linear contribution in  $(\phi_f - \phi_i)$  is negligible compared to the exponential of  $\phi_i$ , which is what we get from the slow-roll approximation. In first approximation

$$\frac{M}{2}(\tau_f - \tau_i) = N \quad \rightarrow \quad \tau_i = \tau_f - \frac{2N}{M} \tag{3.20}$$

We can calculate  $\epsilon(\tau_i)$  at the time of the inflation trigger:

$$\epsilon(\tau_i) = \frac{3}{(\sqrt{3} + 2N)^2} \ll 1 \quad N \sim 60 \tag{3.21}$$

## 4 Perturbation equations

For the sake of completeness, let us recall how the Starobinsky model is linked to the observables. To do this it is necessary to study a perturbed metric with respect to the *FLRW* flat background:

$$\begin{aligned}
ds^2 &= - (1 + 2\alpha)dt^2 - 2a(t)(\partial_i\beta - S_i)dt dx^i + \\
&+ a^2(t)(\delta_{ij} + 2\psi\delta_{ij} + 2\partial_i\partial_j\gamma + 2\partial_i F_j + h_{ij})dx^i dx^j
\end{aligned} \tag{4.1}$$

where  $\alpha, \beta, \psi, \gamma$  are scalar perturbations,  $S_i, F_i$  are vector perturbations and  $h_{ij}$  are tensor perturbations. In general, vector perturbations are not important in cosmology.

Varying the action of gravity  $f(R)$  with respect to  $g_{\mu\nu}$  we obtain the field equations

$$F R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F + g_{\mu\nu} \square F = 0 \quad (4.2)$$

The following perturbed quantities are then defined:

$$\chi = a(t) (\beta + a(t) \dot{\gamma}) \quad A = 3 (H \alpha - \dot{\psi}) - \frac{\Delta}{a^2(t)} \chi \quad (4.3)$$

By analyzing the field equations in the gauge condition  $\delta F = 0$ , the scalar perturbations  $\alpha$  and  $A$  can be expressed in terms of  $\psi$  ( the curvature perturbation that we will later call  $R$  ). At the end  $R$  satisfies the following simple equation in the Fourier space:

$$\ddot{R} + \frac{(a^3 \dot{Q}_s)}{a^3 Q_s} \dot{R} + \frac{k^2}{a^2} R = 0 \quad (4.4)$$

where  $k$  is a comoving wavenumber and

$$Q_s = \frac{3\dot{F}^2}{2F [H + \frac{\dot{F}}{2F}]^2} \quad (4.5)$$

Introducing the variables  $z_s = a\sqrt{Q_s}$  e  $u = z_s R$  we obtain

$$u'' + \left( k^2 - \frac{z_s''}{z_s} \right) u = 0 \quad (4.6)$$

where a prime represents a derivative with respect to the conformal time  $\eta = \int a^{-1}(t)dt$ . To derive the spectrum of the curvature perturbation generated during inflation, the following variables are introduced

$$\epsilon_1 = -\frac{\dot{H}}{H^2} \quad \epsilon_2 = 0 \quad \epsilon_3 = \frac{\dot{F}}{2HF} \quad \epsilon_4 = \frac{\ddot{F}}{H\dot{F}} \quad (4.7)$$

Equation (4.6) is very difficult to solve analytically. In practice we can only solve in the limit  $\dot{\epsilon}_i = 0$  ( $\epsilon_i$  slowly varying parameters ). For  $\epsilon_1$  constant, the conformal time can be approximated as follows:

$$\eta = -\frac{1}{(1 - \epsilon_1)aH} \quad (4.8)$$

and

$$\frac{z_s''}{z_s} = \frac{\nu_R^2 - \frac{1}{4}}{\eta^2} \quad \nu_R^2 = \frac{1}{4} + \frac{(1 + \epsilon_1 - \epsilon_3 + \epsilon_4)(2 - \epsilon_3 + \epsilon_4)}{(1 - \epsilon_1)^2} \quad (4.9)$$

The solution can be expressed as a linear combination of Hankel functions  $H_{\nu_R}^{(i)}(k|\eta|)$ . The power spectrum of the curvature perturbation is then defined as

$$P_R = \frac{4\pi k^3}{(2\pi)^3} |R|^2 \quad (4.10)$$

which can be calculated using the solution in terms of Hankel functions  $H_{\nu_R}^{(i)}(k|\eta|)$ .

The spectrum must be evaluated when crossing the Hubble radius  $k = aH$  and it can be estimated as:

$$P_R \sim \frac{1}{Q_s} \left(\frac{H}{2\pi}\right)^2 \sim \frac{1}{3\pi F} \left(\frac{H}{m_{pl}}\right)^2 \frac{1}{\epsilon_1^2} \quad (4.11)$$

We can define the spectral index of  $R$

$$n_R - 1 = \frac{d \ln P_R}{d \ln k} \Big|_{k=aH} = 3 - 2 \nu_R \sim -4 \epsilon_1 + 2 \epsilon_3 - 2 \epsilon_4 \quad (4.12)$$

For the tensor perturbations we obtain analogous equations that lead to the following amplitude:

$$P_T \sim \frac{16}{\pi} \left(\frac{H}{m_{pl}}\right)^2 \frac{1}{F} \quad (4.13)$$

The tensor-to-scalar ratio is then defined as

$$r = \frac{P_T}{P_R} \sim \frac{64\pi}{m_{pl}^2} \frac{Q_s}{F} \sim 48 \epsilon_1^2 \quad (4.14)$$

In the Starobinsky model we can approximate

$$F \sim \frac{4H^2}{M^2} \quad (4.15)$$

from which we obtain

$$P_R \sim \frac{1}{12\pi} \left( \frac{M}{m_{pl}} \right)^2 \frac{1}{\epsilon_1^2} \quad P_T \sim \frac{4}{\pi} \left( \frac{M}{m_{pl}} \right)^2 \quad (4.16)$$

$\epsilon_1^2$  must be evaluated at the time  $t_k$  of the Hubble radius crossing ( $k = aH$ ) and can be linked to the number of e-foldings from  $t = t_k$  to  $t_f$  (the end of inflation)

$$N_k \sim \frac{1}{2\epsilon_1(t_k)} \quad (4.17)$$

Then the amplitude of the curvature perturbation is given by

$$P_R \sim \frac{N_k^2}{3\pi} \left( \frac{M}{m_{pl}} \right)^2 \quad (4.18)$$

Comparing with the experimental data ( $N_k \sim 55$ ) the mass  $M$  is constrained to be

$$M \sim 3 \times 10^{-6} m_{pl} \quad (4.19)$$

The spectral index of  $R$  is reduced to

$$n_R - 1 \sim -4 \epsilon_1 \sim -\frac{2}{N_k} \sim -3.6 \times 10^{-2} \quad (4.20)$$

Finally, the tensor-to-scalar ratio can be estimated as

$$r \sim \frac{12}{N_k^2} \sim 4.0 \times 10^{-3} \quad (4.21)$$

These are the observables of the Starobinsky model that are related to the experiment [5].

## 5 Conclusion

In this article we have studied how to solve the Starobinsky model by adding an appropriate cosmological constant to the model. Once we know the solution of the Friedmann equations, we can go back to the exact parameterization of the background metric  $a(t)$  from which all the properties of the solvable model derive. The solution has been first studied in the Jordan frame and then generalized to the Einstein frame where a solvable model is obtained for a scalar field coupled to standard gravity. We compared the exact solution with the model

statements obtained by the slow-roll approximation, finding a substantial agreement. For the sake of completeness we have then recalled how the Starobinsky model is linked to the experimental observations made with the Planck satellite [5]. We hope that the proposed framework will contribute to the clarification of the theory of inflation.

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