LOCAL BOUNDEDNESS OF VECTORIAL MINIMIZERS OF NON-CONVEX FUNCTIONALS
LIMITATEZZA LOCALE DI MINIMI VETTORIALI DI FUNZIONALI NON CONVESSI

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Abstract. We prove a local boundedness result for local minimizers of a class of non-convex functionals, under special structure assumptions on the energy density. The proof follows the lines of that in [10], where a similar result is proved under slightly stronger assumptions on the energy density.

Sunto. Dimostriamo un risultato di limitatezza locale per minimi locali di una classe di funzionali non convessi, con particolari ipotesi di struttura sulla densità di energia. La dimostrazione procede come quella in [10], dove un risultato simile è dimostrato con ipotesi leggermente più forti sulla densità di energia.

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1. Introduction

In this paper we consider a class of variational integrals

$$\mathcal{F}(u,\Omega) = \int_\Omega f(x,Du)\,dx \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded set, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued map (i.e., $m > 1$) and Du is the $m \times n$ Jacobian matrix of its partial derivatives

$$u \equiv \left(u^1, u^2, \dots, u^m\right), \qquad Du = \left(\frac{\partial u^\alpha}{\partial x_i}\right)_{i=1,2,\dots,n}^{\alpha=1,2,\dots,m}.$$

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The integrand $f : \Omega \times \mathbb{R}^{m\times n} \to [0,\infty)$ is a Carathéodory function, $\xi \mapsto f(x,\xi)$ has a polynomial growth and is possibly non-convex with respect to ξ. The local boundedness of local minimizers will be considered.

1.1. **"Weak convexity".** The lack of convexity of the energy density f is relevant in the vectorial case ($m > 1$). Indeed the convexity condition is a reasonable assumption in the scalar case ($m = 1$), but not in the vectorial case. Two justifications of this fact are the following.

– The direct method in calculus of variations to prove the existence of a minimizer of $\mathcal{F}$ in a suitable class X (usually a subspace of a Sobolev space, e.g. $W_0^{1,p}(\Omega)$) relies on the Weierstrass Theorem and require two properties:

(a) the existence of a convergent minimizing sequence of $\mathcal{F}$ in X

(b) the sequential lower semicontinuity of the functional $\mathcal{F}$.

It is well known that in the scalar case (i.e. $m = 1$) the sequential lower semicontinuity of the functional $\mathcal{F}$ is essentially equivalent to the convexity of f with respect to gradient variable, see [19]. In the vectorial case, this equivalence does not hold true anymore: the convexity of f with respect to gradient variable implies (b), but the reverse implication does not hold in general. As it was proved by Morrey 1952 [28], Meyers 1965 [27] and, in full generality, by Acerbi-Fusco 1984 [1] (see also Marcellini 1985 [26]) the sequential lower semicontinuity of the functional $\mathcal{F}$ is equivalent to the *quasi-convexity* of $z \mapsto f(x,z)$. The quasi-convexity is a weaker property than the convexity and it says that any linear function furnishes the absolute minimum among all Lipschitzian functions coinciding with it on the boundary. Precisely, its definition is the following:

a Carathéodory function $f : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$, $f(x,\cdot)$ locally integrable, is quasi-convex *(in Morrey's sense) if*

$$\mathcal{L}^n(\Omega) f(x,\xi) \le \int_\Omega f\left(x, \xi + D\varphi(y)\right) dy, \tag{2}$$

for every $\xi \in \mathbb{R}^{m\times n}$, $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$, and for $\mathcal{L}^n$ a.e. $x \in \Omega$.

– The second motivation to study non-convex integrands f comes by the applications to nonlinear elasticity. In a model in nonlinear elasticity, the stored energy of an elastic body occupying in a reference configuration the bounded domain $\Omega \subset \mathbb{R}^3$ is expressed by the functional (1), where $u : \Omega \to \mathbb{R}^3$ is the displacement. J. Ball in 1977 pointed out in [2] that the convexity of f with respect to Du conflicts, for instance, with the natural requirement that the elastic energy is frame-indifferent. In this setting the convexity is usually replaced by the polyconvexity assumption, a weaker assumption than the convexity. In the general case:

a Carathéodory function $f : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is polyconvex if there exists a function $g : \Omega \times \mathbb{R}^{\tau} \to \mathbb{R}$, with $g(x, \cdot)$ convex for $\mathcal{L}^n$ a.e. $x \in \Omega$, such that

$$f(x,\xi) = g\big(x, T(\xi)\big)\,. \tag{3}$$

In the last item we have adopted the standard notation

$$T(\xi) = \big(\xi, \mathrm{adj}_2\xi, \ldots, \mathrm{adj}_i\xi, \ldots, \mathrm{adj}_{m\wedge n}\xi\big)\,.$$

where $\mathrm{adj}_i\xi$ is the adjugate matrix of order $i \in \{2, \ldots, m \wedge n\}$ of the matrix $\xi \in \mathbb{R}^{m\times n}$, that is the $\binom{m}{i} \times \binom{n}{i}$ matrix of all minors of order i of ξ. Thus $T(\xi)$ is a vector in $\mathbb{R}^{\tau}$, with

$$\tau = \sum_{i=1}^{m\wedge n} \binom{m}{i}\binom{n}{i}.$$

If $n = m = 3$, as we will consider in this note, a polyconvex function f is in the form

$$f(x,\xi) := g(x, \xi, \mathrm{adj}_2\,\xi, \det\xi), \qquad g(x, \cdot, \cdot, \cdot) \text{ convex.}$$

Notice that polyconvexity is implied by the convexity, but it is easy to find examples of polyconvex, non-convex functions: for instance $f(\xi) = \det\xi$ or $f(\xi) = |\det\xi|$.

We recall here also the definition of *rank-one convexity*, another way to be "weakly convex":

a Carathéodory function $f : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is rank-one convex *if for all $\lambda \in [0, 1]$ and for all ξ, $\eta \in \mathbb{R}^{m\times n}$ with* $\mathrm{rank}(\xi - \eta) \le 1$

$$f(x, \lambda\xi + (1-\lambda)\eta) \le \lambda f(x,\xi) + (1-\lambda) f(x,\eta) \tag{4}$$

for $\mathcal{L}^n$ a.e. $x \in \Omega$.

It can be proved that

$$f \text{ convex } \Longrightarrow f \text{ polyconvex } \Longrightarrow f \text{ quasiconvex } \Longrightarrow f \text{ rank-one convex.}$$

As it immediately follows by the definition of rank-one convexity, all these implications are equivalences if $m = 1$. On the other hand, none of the previous implications can be reversed except for some particular cases. We refer to the monograph by Dacorogna [11] for more properties and results on these "weak" convexity properties.

1.2. **Partial regularity vs Everywhere regularity.** The celebrated regularity result for elliptic equations proved by De Giorgi [13] in 1957 states that, given a second order linear elliptic equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} a_{ij}\left(x\right)\, u_{x_j} \right) = 0,$$

with essentially bounded measurable coefficients a_{ij} satisfying

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n$$

for some $\nu > 0$, then the $W^{1,2}$-weak solutions are Hölder continuous. The method used by De Giorgi is a very powerful one and it was a breakthrough in the study of regularity. The proof of the fundamental result of De Giorgi relies on a sophisticated application of the maximum principle. The first step is the proof of Caccioppoli-type inequalities for the weak solution u on its super-(sub-)level sets, that is estimates of integrals of Du with integrals of u on level sets of the form:

$$\int_{A_{k,\rho}} |Du|^p \, dx \leq \frac{c}{(R-\rho)^p} \int_{A_{k,R}} (u-k)^p \, dx, \qquad 0 < \rho < R$$

where $A_{k,s} = \{x \in B_s \,:\, u(x) \geq k\}$.

This step of course requires u to be a scalar valued function and it is based on the truncation of the solution. As many counterexamples show, starting by the famous example by De Giorgi 1968 [14], weak solutions to nonlinear elliptic systems or vector valued minimizers of integrals may be irregular, even unbounded. Therefore, motivated by these counterexamples, we find in the mathematical literature two directions of research in regularity of generalized solutions of elliptic systems or of vectorial minimizers of integrals:

- *partial regularity*: i.e., regularity in an open set $\Omega_0 \subseteq \Omega$, $\operatorname{meas}(\Omega \setminus \Omega_0) = 0$,
- *regularity in the interior of* Ω, under suitable structure assumptions and/or assuming *apriori* some regularity (e.g. the local boundedness).

Roughly, a typical condition that forces the regularity is the dependence of the operator/functional on the modulus of the gradient, see the pioneering result by Uhlenbeck [31].

For the polyconvex case, only few everywhere regularity results are available; we mention here those by Fusco and Hutchinson in [16], where the everywhere continuity is proved and Fuchs and Seregin [15] and [7] where Hölder continuity is discussed. Global pointwise bounds are in [20], [12], [24], [25], [22], [23]. Interesting results are contained in [30], [3], [4], [5], [6]; see also [21] and [18].

1.3. **Our result.** We consider $\Omega \subseteq \mathbb{R}^3$ open set, a function $f : \Omega \times \mathbb{R}^{3\times 3} \to [0, +\infty)$, and the functional

$$\mathcal{F}(u) := \int_\Omega f(x, Du(x))\, dx$$

where $u : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$,

$$Du := \begin{pmatrix} Du^1 \\ Du^2 \\ Du^3 \end{pmatrix} = \begin{pmatrix} u^1_{x_1} & u^1_{x_2} & u^1_{x_3} \\ u^2_{x_1} & u^2_{x_2} & u^2_{x_3} \\ u^3_{x_1} & u^3_{x_2} & u^3_{x_3} \end{pmatrix}.$$

We assume that there exist Carathéodory functions $F_\alpha : \Omega \times \mathbb{R}^3 \to [0, +\infty)$, $G_\alpha : \Omega \times \mathbb{R}^3 \to [0, +\infty)$, $\alpha \in \{1, 2, 3\}$, and $H : \Omega \times \mathbb{R} \to [0, +\infty)$, such that

$$\lambda \to G_\alpha(x, \lambda), \qquad \text{and} \qquad t \to H(x, t) \text{ are convex,}$$

with

$$f(x, \xi) := \sum_{\alpha=1}^{3} \{F_\alpha(x, \xi^\alpha) + G_\alpha(x, (\operatorname{adj}_2 \xi)^\alpha)\} + H(x, \det \xi). \tag{5}$$

Here

$$\xi = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \xi_3^1 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 \\ \xi_1^3 & \xi_2^3 & \xi_3^3 \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix}, \qquad \xi^\alpha \in \mathbb{R}^3 \text{ for } \alpha \in \{1,2,3\}.$$

and $\operatorname{adj}_2 \xi \in \mathbb{R}^{3\times 3}$ denotes the adjugate matrix of order 2 whose components are

$$(\operatorname{adj}_2 \xi)_{\gamma i} = (-1)^{\gamma+i} \det \begin{pmatrix} \xi_k^\alpha & \xi_l^\alpha \\ \xi_k^\beta & \xi_l^\beta \end{pmatrix} \qquad \gamma, i \in \{1,2,3\},$$

where $\alpha, \beta \in \{1,2,3\} \setminus \{\gamma\}$, $\alpha < \beta$, and $k, l \in \{1,2,3\} \setminus \{i\}$, $k < l$. Moreover,

$$(\operatorname{adj}_2 \xi)^\alpha = ((\operatorname{adj}_2 \xi)_{\alpha 1}, (\operatorname{adj}_2 \xi)_{\alpha 2}, (\operatorname{adj}_2 \xi)_{\alpha 3}).$$

A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^3)$ is a local minimizer of I if $f(Du) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\mathcal{F}(u, \operatorname{supp} \varphi) \le \mathcal{F}(u + \varphi, \operatorname{supp} \varphi),$$

for all $\varphi \in W^{1,1}(\Omega, \mathbb{R}^3)$ with $\operatorname{supp} \varphi \subset\subset \Omega$.

To have regular local minimizers some growth conditions have to be considered. We assume that there exist exponents $1 < p \le 3$, $1 < q$, $1 \le r$, constants $k_1, k_3 > 0$, $k_2 \ge 0$ and functions $a, b, c : \Omega \to [0, +\infty)$ such that, for all $\alpha \in \{1,2,3\}$,

$$k_1 |\lambda|^p - k_2 \le F_\alpha(x, \lambda) \le k_3 \left(|\lambda|^p + 1\right) + a(x) \qquad \forall \lambda \in \mathbb{R}^3 \tag{6}$$

$$k_1 |\lambda|^q - k_2 \le G_\alpha(x, \lambda) \le k_3 \left(|\lambda|^q + 1\right) + b(x) \qquad \forall \lambda \in \mathbb{R}^3 \tag{7}$$

$$0 \le H(x, t) \le k_3 \left(|t|^r + 1\right) + c(x) \qquad \forall t \in \mathbb{R} \tag{8}$$

where $a, b, c \in L^\sigma(\Omega)$, $\sigma > 1$.

Our main result is the following.

Theorem 1.1. *Let f satisfy* (5) *and growth conditions* (6), (7), (8), *with* $1 \le r < q < p \le 3$. *Assume*

$$\frac{p}{p^*} < \min\left\{1 - \frac{qp^*}{p(p^* - q)}, 1 - \frac{rp^*}{q(p^* - r)}, 1 - \frac{1}{\sigma}\right\}, \tag{9}$$

where $p^* = \frac{3p}{3-p}$, *if* $p < 3$, *and, if* $p = 3$, *then* p^* *is any* $\nu > 3$.

Then all the local minimizers $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^3)$ *of* I *are locally bounded.*

Remark 1.1. *If $\sigma = \infty$ then $\frac{1}{\sigma}$ must be read as 0. Moreover, we remark that if $p = 3$, then p^* can be chosen large enough so that (9) is implied by the assumptions $1 \leq r < q < p$ and $\sigma > 1$.*

Remark 1.2. *As an application of Theorem 1.1, let us consider the functional (1) with*

$$f(Du) := \sum_{\alpha=1}^{3} \left(|Du^\alpha|^{14/5} + |\operatorname{adj}_2 Du^\alpha|^2 \right) + |\det Du|^{3/2}.$$

By Theorem 1.1 every local minimizer $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ of I is locally bounded. Note that the existence of a minimizer of I in $\bar{u} + W_0^{1,\frac{14}{5}}(\Omega)$, with $\bar{u} \in W^{1,\frac{14}{5}}(\Omega)$, comes from Remark 8.32 in [11].

An analogous result has been proved in [10], but assuming that $\lambda \mapsto F_\alpha(x, \lambda)$ are convex functions for every α. Here we improve that result, by removing this convexity hypothesis, by using the same argument used in the recent [9], based on the hole-filling method. We remark that a result, similar to the one in [10], was proved in [8] for functionals satisfying different structure assumptions, see the Appendix.

In [9] the Hölder continuity has been proved for minimizers of functionals with special classes of rank-one convex or polyconvex functions.

The main novelty of the result in [10] is the strategy used to obtain the regularity result. Indeed we prove the local boundedness of vector valued minimizers $u = (u^1, u^2, u^3)$ by employing De Giorgi's iteration method, used until now only in the scalar case. Indeed, we first show that each component u^α satisfies a Caccioppoli's inequality (see Proposition 4.1); then we apply De Giorgi's procedure, separately, to each u^α.

Our paper is organized as follows. In the next section we present some preliminary results used to prove Theorem 1.1. In Section 3 we provide a sketch of the proof of Theorem 1.1. In the last Section, we give the proof of the Caccioppoli-type inequalities.

2. Technical results

The following lemma finds an important application in the hole-filling method. The proof can be found for example in [19, Lemma 6.1] .

Lemma 2.1. *Let $h : [r, R_0] \to \mathbb{R}$ be a non-negative bounded function and $0 < \vartheta < 1$, $A, B \geq 0$ and $\beta > 0$. Assume that*

$$h(s) \leq \vartheta h(t) + \frac{A}{(t-s)^\beta} + B,$$

for all $r \leq s < t \leq R_0$. Then

$$h(r) \leq \frac{cA}{(R_0 - r)^\beta} + cB,$$

where $c = c(\vartheta, \beta) > 0$.

Given a vector $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$ we write $|v| := \sqrt{\sum_{i=1}^n v_i^2}$. Analogously, given a matrix $A = (a_{ij})$, $i, j \in \{1, \cdots, n\}$, A^i is its i-th row and $|A| := \sqrt{\sum_{i,j=1}^n a_{ij}^2}$.

Lemma 2.2 (Lemma 4.1 in [10]). *Consider the matrices $A, B \in \mathbb{R}^{3\times 3}$*

$$A = \begin{pmatrix} A^1 \\ B^2 \\ B^3 \end{pmatrix}, \qquad B = \begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix}.$$

Then the following estimates hold:

(a) $|A| \leq |A^1| + |B^2| + |B^3|$,
(b) $|\det A| \leq |A^1| |\left(\operatorname{adj}_2 B\right)^1|$,
(c) $|(\operatorname{adj}_2 A)_{2j}| \leq |A^1||B^3|$ *and* $|(\operatorname{adj}_2 A)_{3j}| \leq |A^1||B^2|$, *for all $j \in \{1, 2, 3\}$.*

3. Sketch of the proof of Theorem 1.1

We now provide a sketch of the proof of Theorem 1.1. For a local minimizer $u = (u^1, u^2, u^3)$ we will prove that each component is locally bounded. In the following we consider the first component u^1. We can argue similarly for the other components u^2, u^3.

STEP 1. Caccioppoli inequality for u^1. We use the minimality condition with a suitable test function; such a test function and the particular structure (5) of the density f guarantee a Caccioppoli inequality for u^1 on every superlevel set $\{u^1 > k\}$. More precisely, fixed $x_0 \in \Omega$ and a ball $B_{R_0}(x_0) \subset\subset \Omega$ (we will not write the center x_0 if no confusion may arise) we have that there exists $c > 0$ such that for all $s, t > 0$, $s < t \le R_0$,

$$\int_{\{u^1>k\}\cap B_s} |Du^1|^p \, dx \le c \int_{\{u^1>k\}\cap B_t} \left(\frac{u^1 - k}{t - s} \right)^{p^*} dx + c|\{u^1 > k\} \cap B_t|^{\vartheta} \tag{10}$$

with a suitable $\vartheta > 0$. The Caccioppoli inequality (10) permits to apply the classical methods to get the regularity in the scalar case. Observe that on the right hand side of (10) we do not get the same exponent p as in the left hand side, but the larger p^*; it still allows us to prove the local boundedness of u^1, see also [17] and [29].

STEP 2. Decay of the "excess" on superlevel sets. For a suitable radius $R < R_0$ and a suitable level d, we define a sequence ρ_h of radii starting from R and decreasing to $\frac{R}{2}$, another sequence k_h of levels starting from $\frac{d}{2}$ and increasing to d. We define the "excess" on the superlevel set as follows

$$J_h := \int_{\{u^1>k_h\}\cap B_{\rho_h}} \left(u^1 - k_h\right)^{p^*} dx. \tag{11}$$

Note that J_h is a decreasing sequence. Using Sobolev inequality and Caccioppoli estimate (10) we are able to show that

$$J_{h+1} \le c Q^h J_h^{\vartheta p^*/p} \tag{12}$$

for some constants $c, Q > 1$.

STEP 3. Iteration. In the right hand side of (12) there is competition between the increasing Q^h and the decreasing $J_h^{\vartheta p^*/p}$; if $\vartheta p^*/p > 1$ and the initial value J_0 is small, then

$$J_h \le Q^{\frac{-h}{\vartheta p^*/p - 1}} J_0, \tag{13}$$

so that

$$\lim_{h \to +\infty} J_h = 0, \tag{14}$$

which implies

$$u^1 \le d \qquad \text{a. e. in } B_{R/2}. \tag{15}$$

Since assumption (9) guarantees $\vartheta p^*/p > 1$ we get (15). Lower bounds for u^1 can be obtained by showing that $-u$ is a minimizer for a similar functional.

4. Proof of the first step: the Caccioppoli inequalities

Once the Caccioppoli inequalities are proved, the remaining part (Steps 2-3 in the previous section) of the proof of Theorem 1.1 is quite standard: for them, we refer to [10]. Here we limit ourselves to prove the Caccioppoli-type inequalities.

The particular structure (5) of the density f guarantees a Caccioppoli inequality for any component u^α of u on every superlevel set $\{u^\alpha > k\}$. In the next proposition we state this result in the case of the first component u^1.

Proposition 4.1. *Let f be as in* (5)*, satisfying the growth conditions* (6)*,* (7)*,* (8)*, with*

$$q < \frac{p^* p}{p^* + p} \qquad \text{and} \qquad r < \frac{p^* q}{p^* + q}. \tag{16}$$

Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$ be a local minimizer of I.

Let $B_{R_0}(x_0) \subset\subset \Omega$, $|B_{R_0}| < 1$, $R_0 < 1$, and, fixed $k \in \mathbb{R}$, denote

$$A^1_{k,\tau} := \{x \in B_\tau(x_0) \,:\, u^1(x) > k\} \qquad 0 < \tau \le R_0.$$

Then there exists $c > 0$, independent of k, such that for every $0 < \rho < R \le R_0$.

$$\begin{aligned}\int_{A^1_{k,\rho}} |Du^1|^p\,dx \le c\int_{A^1_{k,R}} \left(\frac{u^1-k}{r-\rho}\right)^{p^*} dx \\ + c\left\{1 + \left(\int_{B_R} (|Du^2| + |Du^3|)^p\,dx\right)^{\frac{qp^*}{(p^*-q)p}} + \left(\int_{B_R} \left|(\mathrm{adj}_2\, Du)^1\right|^q\,dx\right)^{\frac{rp^*}{(p^*-r)q}}\right\} |A^1_{k,R}|^\vartheta\end{aligned} \tag{17}$$

where $\vartheta := \min\{1 - \frac{qp^*}{p(p^*-q)}, 1 - \frac{rp^*}{q(p^*-r)}, 1 - \frac{1}{\sigma}\}$.

Proof. For the sake of simplicity we will give a proof assuming that the integrand function f is independent on x, and, consequently, that a, b, c in (6), (7), (8) are equal to 0.

Let $B_{R_0}(x_0) \subset\subset \Omega$, $|B_{R_0}| < 1$, $R_0 < 1$. Let ρ, s, t, R be such that $\rho \le s < t \le R \le R_0$. Consider a cut-off function $\eta \in C^\infty_0(B_t)$ satisfying the following assumptions:

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ in } B_s(x_0), \quad |D\eta| \le \frac{2}{t-s}. \tag{18}$$

Fixed $k \in \mathbb{R}$, define $w \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$,

$$w^1 := \max(u^1 - k, 0), \quad w^2 := 0, \quad w^3 := 0,$$

and, for $\mu \geq p^*$,

$$\varphi := -\eta^\mu w,$$

that has support in B_t.

By the minimality of u, $f(Du) \in L^1(B_t)$. Let us prove that $f(Du + D\varphi) \in L^1(B_t)$.

First, we notice that a.e. x in $\Omega \setminus (\{\eta > 0\} \cap \{u^1 > k\})$ we have $\varphi = 0$, thus

$$f(Du + D\varphi) = f(Du) \qquad \text{a.e. in } \Omega \setminus (\{\eta > 0\} \cap \{u^1 > k\}). \tag{19}$$

Moreover, for a.e. x in $\{\eta > 0\} \cap \{u^1 > k\}$

$$Du + D\varphi = \begin{pmatrix} (1-\eta^\mu)Du^1 + \mu\eta^{\mu-1}(k-u^1)D\eta \\ Du^2 \\ Du^3 \end{pmatrix}. \tag{20}$$

therefore

$$F_2((Du + D\varphi)^2) = F_2(Du^2), \quad F_3((Du + D\varphi)^3) = F_3(Du^3) \qquad \text{a.e. in } \Omega. \tag{21}$$

Let us define the polyconvex function G,

$$G(\xi) := \sum_{\alpha=1}^{3} G_\alpha(\operatorname{adj}_2 \xi) + H(\det \xi), \tag{22}$$

so

$$f(\xi) = \sum_{\alpha=1}^{3} F_\alpha(\xi^\alpha) + G(\xi).$$

By (19), (21), and using $F_\alpha, G \geq 0$, $\alpha \in \{1,2,3\}$, we have

$$\begin{aligned}
\int_{B_t} f(Du + D\varphi)\,dx &\leq \int_{B_t \setminus (\{\eta>0\} \cap \{u^1>k\})} f(Du)\,dx + \int_{B_t} F_1((Du + D\varphi)^1)\,dx \\
&+ \int_{B_t \cap \{\eta>0\} \cap \{u^1>k\}} \sum_{\alpha=2}^{3} F_\alpha(Du^\alpha)\,dx + \int_{B_t \cap \{\eta>0\} \cap \{u^1>k\}} G(Du + D\varphi)\,dx \\
&\leq 3\int_{B_t} f(Du)\,dx + \int_{B_t} F_1((Du + D\varphi)^1)\,dx + \int_{B_t \cap \{\eta>0\} \cap \{u^1>k\}} G(Du + D\varphi)\,dx
\end{aligned} \tag{23}$$

The integral $\int_{B_t} F_1((Du+D\varphi)^1)\,dx$ is finite. Indeed, by (20), the right inequality in (6) and the convexity of $t \mapsto |t|^p$,

$$\begin{aligned}
&\int_{B_t} F_1((Du+D\varphi)^1)\,dx \le \int_{B_t} k_3(|(Du+D\varphi)^1|^p+1)\,dx \\
&\le k_3 \int_{B_t} \Big((1-\eta^\mu)|Du^1|^p + |p(k-u^1)D\eta|^p + 1)\Big)\,dx \\
&\le c\int_{B_t} |Du^1|^p\,dx + c\int_{B_t} \Big(\Big(\frac{u^1-k}{t-s}\Big)^p + 1\Big)\,dx
\end{aligned} \tag{24}$$

with $c = c(k_3, p)$. Since $u \in W^{1,p}(\Omega;\mathbb{R}^3)$ the right hand side is finite.

It remains to establish that $\int_{B_t\cap\{\eta>0\}\cap\{u^1>k\}} G(Du+D\varphi)\,dx$ is finite. We begin noting that

$$Du + D\varphi = (1-\eta^\mu)Du + \eta^\mu A, \qquad \text{a.e. in } B_t \cap \{\eta>0\}\cap\{u^1>k\}, \tag{25}$$

where

$$A := \begin{pmatrix} \mu\eta^{-1}(k-u^1)D\eta \\ Du^2 \\ Du^3 \end{pmatrix}. \tag{26}$$

Since $Du - A$ is a rank-one matrix and $\xi \mapsto \det\xi$ and $\xi \mapsto \operatorname{adj}_2(\xi)$ are quasi-affine functions, see [11], then

$$\operatorname{adj}_2(Du+D\varphi) = (1-\eta^\mu)\operatorname{adj}_2 Du + \eta^\mu \operatorname{adj}_2 A,$$

and

$$\det(Du+D\varphi) = (1-\eta^\mu)\det Du + \eta^\mu \det A.$$

Therefore, by the polyconvexity of G,

$$G(Du+D\varphi) \le (1-\eta^\mu)G(Du) + \eta^\mu G(A) \qquad \text{a.e. in } \{\eta>0\}\cap\{u^1>k\}, \tag{27}$$

so that

$$\begin{aligned}
\int_{B_t\cap\{\eta>0\}\cap\{u^1>k\}} G(Du+D\varphi)\,dx &\le \int_{B_t\cap\{\eta>0\}\cap\{u^1>k\}} \Big((1-\eta^\mu)G(Du) + \eta^\mu G(A)\Big)\,dx \\
&\le \int_{B_t} f(Du)\,dx + \int_{A^1_{k,t}} \eta^\mu G(A)\,dx,
\end{aligned}$$

where in the last inequality we used $F_\alpha \geq 0$, $\alpha \in \{1,2,3\}$. The first integral at right hand side if finite. Let us consider the last one. Taking into account (26), we obtain

$$G_1((\operatorname{adj}_2 A)^1) = G_1((\operatorname{adj}_2 Du)^1), \tag{28}$$

therefore, since $G_1 \leq f$,

$$\begin{aligned}\int_{A^1_{k,t}} \eta^\mu G(A)\,dx &= \int_{A^1_{k,t}} \eta^\mu \left(G_1((\operatorname{adj}_2 Du)^1) + \sum_{\alpha=2}^{3} G_\alpha((\operatorname{adj}_2 A)^\alpha) + H(\det A)\right) dx \\ &\leq \int_{B_t} f(Du)\,dx + \int_{A^1_{k,t}} \eta^\mu \left(\sum_{\alpha=2}^{3} G_\alpha((\operatorname{adj}_2 A)^\alpha) + H(\det A)\right) dx. \qquad (29)\end{aligned}$$

By (7) and Lemma 2.2-(c)

$$\begin{aligned}\eta^\mu \sum_{\alpha=2}^{3} G_\alpha((\operatorname{adj}_2 A)^\alpha) &\leq \eta^\mu k_3 \sum_{\alpha=2}^{3} (|(\operatorname{adj}_2 A)^\alpha|^q + 1) \\ &\leq c\eta^\mu + c\mu^q \eta^{\mu-q} \left(\frac{u^1-k}{t-s}\right)^q (|Du^2| + |Du^3|)^q.\end{aligned}$$

The first inequality in (16) implies $q < p^*$. Using the Young inequality with exponents $\frac{p^*}{q}$ and $\frac{p^*}{p^*-q}$ we get that, a.e. in $A^1_{k,t}$,

$$c\mu^q \eta^{\mu-q} \left(\frac{u^1-k}{t-s}\right)^q (|Du^2| + |Du^3|)^q \leq c\left(\frac{u^1-k}{t-s}\right)^{p^*} + c\,(|Du^2| + |Du^3|)^{\frac{qp^*}{p^*-q}}.$$

We have so proved that

$$\int_{A^1_{k,t}} \eta^\mu \sum_{\alpha=2}^{3} G_\alpha((\operatorname{adj}_2 A)^\alpha)\,dx \leq c \int_{A^1_{k,t}} \left\{ 1 + \left(\frac{u^1-k}{t-s}\right)^{p^*} + (|Du^2| + |Du^3|)^{\frac{qp^*}{p^*-q}} \right\} dx.$$

By the first condition in (16), $\frac{qp^*}{p^*-q} < p$, therefore, by Hölder inequality,

$$\begin{aligned}\int_{A^1_{k,t}} \eta^\mu \sum_{\alpha=2}^{3} G_\alpha((\operatorname{adj}_2 A)^\alpha)\,dx &\leq c \int_{A^1_{k,t}} \left\{ 1 + \left(\frac{u^1-k}{t-s}\right)^{p^*} \right\} dx \\ &+ c\left(\int_{B_t} (|Du^2| + |Du^3|)^p\,dx\right)^{\frac{qp^*}{(p^*-q)p}} |A^1_{k,t}|^{1-\frac{qp^*}{p(p^*-q)}}. \qquad (30)\end{aligned}$$

Let us now prove that $\int_{A^1_{k,t}} \eta^\mu H(\det A)$ is finite. By (8) and computing $\det A$ with respect to the first row (see Lemma 2.2-(b))

$$\eta^\mu H(\det A) \leq c\eta^\mu + c\mu^r \eta^{\mu-r} \left(\frac{u^1-k}{t-s}\right)^r \left|(\operatorname{adj}_2 Du)^1\right|^r.$$

Notice that, by (16), $r < p^*$. By the Young inequality with exponents $\frac{p^*}{r}$ and $\frac{p^*}{p^*-r}$ we get

$$c\mu^r \eta^{\mu-r} \left(\frac{u^1-k}{t-s} \right)^r \left|(\mathrm{adj}_2\, Du)^1\right|^r \le c \left(\frac{u^1-k}{t-s} \right)^{p^*} + c\left|(\mathrm{adj}_2\, Du)^1\right|^{\frac{rp^*}{p^*-r}}.$$

Therefore

$$\int_{A^1_{k,t}} \eta^\mu H(\det A)\, dx \le c \int_{A^1_{k,t}} \left\{ 1 + \left(\frac{u^1-k}{t-s} \right)^{p^*} + \left|(\mathrm{adj}_2\, Du)^1\right|^{\frac{rp^*}{p^*-r}} \right\} dx. \tag{31}$$

Taking into account that the second inequality in (16) is equivalent to $\frac{rp^*}{p^*-r} < q$, by Hölder inequality we obtain

$$\int_{A^1_{k,t}} \left|(\mathrm{adj}_2\, Du)^1\right|^{\frac{rp^*}{p^*-r}}\, dx \le \left(\int_{B_t} \left|(\mathrm{adj}_2\, Du)^1\right|^q\, dx \right)^{\frac{rp^*}{(p^*-r)q}} |A^1_{k,t}|^{1-\frac{rp^*}{q(p^*-r)}}$$

that, together with (31), implies

$$\begin{aligned} \int_{A^1_{k,t}} \eta^\mu H(\det A)\, dx &\le c \int_{A^1_{k,t}} \left\{ 1 + \left(\frac{u^1-k}{t-s} \right)^{p^*} \right\} dx \\ &+ c \left(\int_{B_t} \left|(\mathrm{adj}_2\, Du)^1\right|^q\, dx \right)^{\frac{rp^*}{(p^*-r)q}} |A^1_{k,t}|^{1-\frac{rp^*}{q(p^*-r)}}. \end{aligned} \tag{32}$$

Collecting (30) and (32) we get

$$\begin{aligned} \int_{A^1_{k,t}\cap\{\eta>0\}} \eta^\mu \left\{ \sum_{\alpha=2}^3 G_\alpha((\mathrm{adj}_2\, A)^\alpha) + H(\det A) \right\} dx \le c \int_{A^1_{k,t}} \left\{ 1 + \left(\frac{u^1-k}{t-s} \right)^{p^*} \right\} dx \\ + c \left\{ \left(\int_{B_t} (|Du^2|+|Du^3|)^p\, dx \right)^{\frac{qp^*}{(p^*-q)p}} + \left(\int_{B_t} \left|(\mathrm{adj}_2\, Du)^1\right|^q\, dx \right)^{\frac{rp^*}{(p^*-r)q}} \right\} |A^1_{k,t}|^\vartheta, \end{aligned} \tag{33}$$

where

$$\vartheta := \min\left\{ 1 - \frac{qp^*}{p(p^*-q)}, 1 - \frac{rp^*}{q(p^*-r)} \right\}.$$

This inequality and (29) imply

$$\begin{aligned} \int_{A^1_{k,t}} \eta^\mu G(A)\, dx &\le \int_{B_t} f(Du)\, dx + c \int_{A^1_{k,t}} \left\{ 1 + \left(\frac{u^1-k}{t-s} \right)^{p^*} \right\} dx \\ &+ c \left\{ \left(\int_{B_t} (|Du^2|+|Du^3|)^p\, dx \right)^{\frac{qp^*}{(p^*-q)p}} + \left(\int_{B_t} \left|(\mathrm{adj}_2\, Du)^1\right|^q\, dx \right)^{\frac{rp^*}{(p^*-r)q}} \right\} |A^1_{k,t}|^\vartheta, \end{aligned} \tag{34}$$

and the right hand side is finite, by the left inequalities in (6), (7) and the assumption $f(Du) \in L^1(B_t)$. We have so proved that $\int_{B_t \cap \{\eta>0\} \cap \{u^1>k\}} G(Du + D\varphi)\,dx$ is finite. This fact, together with the inequalities (23) and (24), gives $f(Du + D\varphi) \in L^1(B_t)$.

We can now turn to the proof of the Caccioppoli-type inequality (17). By definition of local minimality of u, (19), (21) and (27) we have

$$\int_{A^1_{k,t} \cap \{\eta>0\}} f(Du)\,dx = \int_{A^1_{k,t} \cap \{\eta>0\}} F_1(Du^1)\,dx + \int_{A^1_{k,t} \cap \{\eta>0\}} \left(\sum_{\alpha=2}^{3} F_\alpha(Du^\alpha) + G(Du) \right) dx$$

$$\leq \int_{A^1_{k,t} \cap \{\eta>0\}} F_1((Du + D\varphi)^1)\,dx + \int_{A^1_{k,t} \cap \{\eta>0\}} \left(\sum_{\alpha=2}^{3} F_\alpha(Du^\alpha) + (1-\eta^\mu)G(Du) + \eta^\mu G(A) \right) dx.$$

The inequality above implies

$$\int_{A^1_{k,t} \cap \{\eta>0\}} F_1(Du^1)\,dx + \int_{A^1_{k,t} \cap \{\eta>0\}} \eta^\mu G(Du)\,dx$$
$$\leq \int_{A^1_{k,t} \cap \{\eta>0\}} F_1((Du + D\varphi)^1)\,dx + \int_{A^1_{k,t} \cap \{\eta>0\}} \eta^\mu G(A)\,dx.$$

Taking into account the definition of G, see (22), and (28), we obtain

$$\int_{A^1_{k,t} \cap \{\eta>0\}} F_1(Du^1)\,dx + \int_{A^1_{k,t} \cap \{\eta>0\}} \eta^\mu \left\{ \sum_{\alpha=2}^{3} G_\alpha((\mathrm{adj}_2\, Du)^\alpha) + H(\det Du) \right\} dx$$
$$\leq \int_{A^1_{k,t} \cap \{\eta>0\}} F_1((Du + D\varphi)^1)\,dx + \int_{A^1_{k,t} \cap \{\eta>0\}} \eta^\mu \left\{ \sum_{\alpha=2}^{3} G_\alpha((\mathrm{adj}_2\, A)^\alpha) + H(\det A) \right\} dx.$$

Since

$$\int_{A^1_{k,t} \cap \{\eta>0\}} \eta^\mu \left\{ \sum_{\alpha=2}^{3} G_\alpha((\mathrm{adj}_2\, Du)^\alpha) + H(\det Du) \right\} dx \geq 0,$$

we obtain

$$\int_{A^1_{k,t} \cap \{\eta>0\}} F_1(Du^1)\,dx \leq \int_{A^1_{k,t} \cap \{\eta>0\}} F_1((Du + D\varphi)^1)\,dx$$
$$+ \int_{A^1_{k,t} \cap \{\eta>0\}} \eta^\mu \left\{ \sum_{\alpha=2}^{3} G_\alpha((\mathrm{adj}_2\, A)^\alpha) + H(\det A) \right\} dx. \tag{35}$$

By the left inequality in (6),

$$\int_{A^1_{k,s}} (k_1|Du^1|^p - k_2)\,dx \leq \int_{A^1_{k,t} \cap \{\eta>0\}} (k_1|Du^1|^p - k_2)\,dx \leq \int_{A^1_{k,t} \cap \{\eta>0\}} F_1(Du^1)\,dx. \tag{36}$$

By the right inequality in (6), (25), the convexity of $t \mapsto |t|^p$ and (18), we get

$$\int_{A^1_{k,t}\cap\{\eta>0\}} F_1((Du+D\varphi)^1)\,dx \le \int_{A^1_{k,t}\cap\{\eta>0\}} k_3(|(Du+D\varphi)^1|^p+1)\,dx$$

$$\le k_3 \int_{A^1_{k,t}\cap\{\eta>0\}} \big((1-\eta^\mu)|Du^1|^p + |p(k-u^1)D\eta|^p + 1)\big)\,dx$$

$$\le c\int_{(A^1_{k,t}\setminus A^1_{k,s})\cap\{\eta>0\}} |Du^1|^p\,dx + c\int_{A^1_{k,t}\cap\{\eta>0\}} \Big(\Big(\frac{u^1-k}{t-s}\Big)^p + 1\Big)\,dx$$

$$\le c\int_{A^1_{k,t}\setminus A^1_{k,s}} |Du^1|^p\,dx + c\int_{A^1_{k,t}} \Big(\Big(\frac{u^1-k}{t-s}\Big)^{p^*} + 1\Big)\,dx \tag{37}$$

with $c = c(k_3, p)$. Therefore, (35), (36), (37) and (33) imply

$$k_1\int_{A^1_{k,s}} |Du^1|^p\,dx \le c\int_{A^1_{k,t}\setminus A^1_{k,s}} |Du^1|^p\,dx + c\int_{A^1_{k,t}} \left\{1 + \left(\frac{u^1-k}{t-s}\right)^{p^*}\right\}dx$$

$$+\, c\left\{\left(\int_{B_R} (|Du^2|+|Du^3|)^p\,dx\right)^{\frac{qp^*}{(p^*-q)p}} + \left(\int_{B_R} \left|(\mathrm{adj}_2\, Du)^1\right|^q\,dx\right)^{\frac{rp^*}{(p^*-r)q}}\right\}|A^1_{k,R}|^\vartheta \tag{38}$$

with c also depending on k_2. By hole-filling, i.e. adding

$$c\int_{A^1_{k,s}} |Du^1|^p\,dx$$

to both sides of (38) we obtain

$$\int_{A^1_{k,s}} |Du^1|^p\,dx \le \frac{c}{k_1+c}\int_{A^1_{k,t}} |Du^1|^p\,dx + \frac{c}{k_1+c}\int_{A^1_{k,t}} \left\{1 + \left(\frac{u^1-k}{t-s}\right)^{p^*}\right\}dx$$

$$+\,\frac{c}{k_1+c}\left\{\left(\int_{B_R} (|Du^2|+|Du^3|)^p\,dx\right)^{\frac{qp^*}{(p^*-q)p}} + \left(\int_{B_R} \left|(\mathrm{adj}_2\, Du)^1\right|^q\,dx\right)^{\frac{rp^*}{(p^*-r)q}}\right\}|A^1_{k,R}|^\vartheta.$$

Using Lemma 2.1 we obtain (17).

□

To conclude the proof, two more steps are necessary: a measure of the decay of the "excess" on superlevel sets and an iteration asrgument. Since they are quite standard, we refer to [10] for their description.

Remark 4.1. *We proved Theorem 1.1 by assuming that the integrand function is indipendent of x. In the general case, f depending on x and satisfying the general growth* *conditions* (6)-(8), *with* a, b, c *belonging to* L^σ, $\sigma > 1$, *the proof goes in a similar way, with the additional condition* $1 - \frac{1}{\sigma} > \frac{p}{p^*}$.

5. Appendix

The functionals studied in [8] are $\int_\Omega f(x, Du)\, dx$, with

$$f(x, \xi) := \tilde{F}(x, |\xi|^2) + \tilde{G}(x, |\operatorname{adj}_2 \xi|^2) + \tilde{H}(x, \det \xi), \tag{39}$$

where $\tilde{F} : \Omega \times [0, +\infty) \to [0, +\infty)$, $\tilde{G} : \Omega \times [0, +\infty) \to [0, +\infty)$ and $\tilde{H} : \Omega \times \mathbb{R} \to [0, +\infty)$ satisfy growth properties and some more assumptions. We exhibit here an example of an energy density of type (5) (considered in [10] and in the present paper), that cannot be expressed as in (39).

Lemma 5.1. *We assume that* $\tilde{F}, \tilde{G} : [0, +\infty) \mapsto [0, +\infty)$ *and* $\tilde{H} : \mathbb{R} \mapsto [0, +\infty)$*; let* $p, q, r \in (0, +\infty)$ *with* $p \neq 2$*. Then, it is false that*

$$\sum_{\alpha=1}^{3} |\xi^\alpha|^p + \sum_{\alpha=1}^{3} |(\mathrm{adj}_2 \xi)^\alpha|^{\mathrm{q}} + |\det \xi|^{\mathrm{r}} = \tilde{\mathrm{F}}(|\xi|^2) + \tilde{\mathrm{G}}(|\mathrm{adj}_2 \xi|^2) + \tilde{\mathrm{H}}(\det \xi) \tag{40}$$

for every $\xi \in \mathbb{R}^{3\times 3}$.

Proof. We argue by contradiction: if (40) holds true, then we can use (40) with

$$\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{41}$$

and we get

$$\mathrm{adj}_2 \xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{42}$$

with $\det \xi = 0$, so that

$$0 = \tilde{F}(0) + \tilde{G}(0) + \tilde{H}(0); \tag{43}$$

we keep in mind that $\tilde{F}, \tilde{G}, \tilde{H} \geq 0$ and we get

$$\tilde{F}(0) = \tilde{G}(0) = \tilde{H}(0) = 0. \tag{44}$$

Now we use (40) with

$$\xi = \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{45}$$

and we get

$$\mathrm{adj}_2 \xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{46}$$

with $\det \xi = 0$, so that

$$|t|^p = \tilde{F}(t^2) + \tilde{G}(0) + \tilde{H}(0); \tag{47}$$

we keep in mind (44) and we get

$$\tilde{F}(t^2) = |t|^p = (t^2)^{p/2}, \tag{48}$$

for every $t \in \mathbb{R}$. Now we take

$$\xi = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{49}$$

and we get

$$\mathrm{adj}_2 \xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{50}$$

with $\det \xi = 0$, $|\xi^1| = 1 = |\xi^2| = |\xi^3|$, $|\xi|^2 = 3$ and (40) implies

$$3 = 3^{p/2}; \tag{51}$$

such an equality is a contradiction, since $p \neq 2$. This ends the proof. $\square$

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