CONTROL IN THE SPACES OF ENSEMBLES OF POINTS*

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Abstract. We study the controlled dynamics of the *ensembles of points* of a Riemannian manifold M. Parameterized ensemble of points of M is the image of a continuous map $\gamma : \Theta \to M$, where Θ is a compact set of parameters. The dynamics of ensembles is defined by the action $\gamma(\theta) \mapsto P_t(\gamma(\theta))$ of the semigroup of diffeomorphisms $P_t : M \to M$, $t \in \mathbb{R}$, generated by the controlled equation $\dot{x} = f(x, u(t))$ on M. Therefore any control system on M defines a control system on (generally infinite-dimensional) space $\mathcal{E}_\Theta(M)$ of the ensembles of points. We wish to establish criteria of controllability for such control systems. As in our previous work ([1]) we seek to adapt the Lie-algebraic approach of geometric control theory to the infinite-dimensional setting. We study the case of finite ensembles and prove genericity of exact controllability property for them. We also find sufficient approximate controllability criterion for continual ensembles and prove a result on motion planning in the space of flows on M. We discuss the relation of the obtained controllability criteria to various versions of Rashevsky-Chow theorem for finite- and infinite-dimensional manifolds.

Key words. infinite-dimensional control systems, nonlinear control, controllability, Lie-algebraic methods

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1. Introduction and problem setting.

Let M be C^∞-smooth n-dimensional ($n \geq 2$) connected Riemannian manifold, with $d(\cdot, \cdot)$, being the Riemannian distance. Let $\mathcal{E}_\Theta(M)$ be the space of continuous maps $\gamma : \Theta \to M$, where Θ is a compact Lebesgue measure set. We call the elements of $\mathcal{E}_\Theta(M)$ *ensembles of points* or, for brevity, *ensembles*. The space $\mathcal{E}_\Theta(M)$ is infinite-dimensional, whenever Θ is an infinite set (see Section 2).

In the control-theoretic setting one looks at the action on $\mathcal{E}_\Theta(M)$ of the group of diffeomorphisms of M, which are generated by the vector fields from the family $\{f^u | \ u \in U\} \subset \text{Vect } M$. Alternatively we can consider the action of the flows, defined by the controlled equations

$$\dot{x} = f(x, u(t)), \ u(t) \in U, \tag{1.1}$$

where $u(t)$ are admissible, for example, piecewise-constant, or piecewise-continuous, or boundary measurable controls, with their values in a set U, which is a subset of a Euclidean space.

The flow $P_t^{u(\cdot)}$ ($P_0 = Id$), generated by control system (1.1) and a given admissible control $u(t) = (u_1(t), \dots, u_r(t))$, acts on $\gamma(\theta) \in \mathcal{E}_\Theta(M)$ according to the formula

$$\hat{P}_t^{u(\cdot)} : \gamma(\theta) \mapsto P_t^{u(\cdot)}(\gamma(\theta)), \theta \in \Theta.$$

Thus control system (1.1) gives rise to a control system in the space of ensembles $\mathcal{E}_\Theta(M)$. We set the controllability problem for the action of control system (1.1) on $\mathcal{E}_\Theta(M)$.

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Definition 1.1. *Ensemble* $\alpha(\cdot) \in \mathcal{E}_\Theta(M)$ *can be steered in time-*T *to ensemble* $\omega(\cdot) \in \mathcal{E}_\Theta(M)$ *by control system (1.1), if there exists a control* $\bar{u} \in L_\infty([0,T],U)$ *such that for the flow* $P_t^{\bar{u}(\cdot)}$*, generated by the equation* $\dot{x} = f(x(t), \bar{u}(t))$*, there holds*

$$P_T^{\bar{u}(\cdot)}(\alpha(\theta)) = \omega(\theta).$$

Definition 1.2. *The time-*T *attainable set from* $\alpha(\cdot) \in \mathcal{E}_\Theta(M)$ *for control system (1.1) in the space of ensembles* $\mathcal{E}_\Theta(M)$ *is*

$$\mathcal{A}_T(\alpha(\cdot)) = \{P_T^{u(\cdot)}(\alpha(\theta)) \mid u(\cdot) \in L_\infty([0,T],U)\} \subset \mathcal{E}_\Theta(M).$$

Definition 1.3. *Control system (1.1) is globally exactly controllable in time* T *in the space* $\mathcal{E}_\Theta(M)$ *from* $\alpha(\theta) \in \mathcal{E}_\Theta(M)$*, if* $\mathcal{A}_T(\alpha(\theta)) = \mathcal{E}_\Theta(M)$*. Control system (1.1) is time-*T *globally exactly controllable if it is globally exactly controllable in time-*T *from each* $\alpha(\theta) \in \mathcal{E}_\Theta(M)$ *.*

Remark 1.4. If $\Theta = \{\theta\}$ is a singleton, then the time-T attainable sets $\mathcal{A}_T(\alpha_\theta)$ coincide with the standard attainable sets of system (1.1) from the point $\alpha_\theta \in M$. The notions of global and global approximate controllability coincide with the standard notions for control system (1.1) on M.

If Θ is an infinite set, it is hard to achieve *exact* ensemble controllability for system (1.1). Instead we will study C^0- or L_p-approximate controllability property.

Definition 1.5. *Ensemble* $\alpha(\cdot) \in \mathcal{E}_\Theta(M)$ *is* C^0*-approximately steerable in time-*T *to ensemble* $\omega(\cdot) \in \mathcal{E}_\Theta(M)$ *by control system (1.1), if for each* $\varepsilon > 0$ *there exists* $\bar{u}(\cdot)$ *such that*

$$\sup_{\theta \in \Theta} d\left(\omega(\theta), P_T^{\bar{u}(\cdot)}(\alpha(\theta))\right) \leq \varepsilon. \tag{1.2}$$

Ensemble $\alpha(\cdot) \in \mathcal{E}_\Theta(M)$ *is* L_p*-approximately steerable in time-*T *to ensemble* $\omega(\cdot) \in \mathcal{E}_\Theta(M)$ *by control system (1.1), if for each* $\varepsilon > 0$ *there exists* $\bar{u}(\cdot)$ *such that*

$$\int_\Theta \left(d\left(\omega(\theta), P_T^{\bar{u}(\cdot)}(\alpha(\theta))\right)\right)^p d\theta \leq \varepsilon^p. \ .$$

Definition 1.6. *Control system (1.1) is time-*T *globally approximately controllable from* $\alpha(\cdot) \in \mathcal{E}_\Theta(M)$ *if* $\mathcal{A}_T(\alpha)$ *is dense in* $\mathcal{E}_\Theta(M)$ *in the respective metric. The system is time-*T *globally approximately controllable if it is time-*T *globally approximately controllable from each* $\alpha(\cdot) \in \mathcal{E}_\Theta(M)$*.*

It is known that the attainable sets and the controllability properties of control system (1.1) on M can be characterized via properties of the Lie brackets of the vector fields $f^u(x),\ u \in U$. In particular case for a symmetric control-linear system

$$\dot{x} = \sum_{j=1}^{s} f_j(x) u_j(t) \tag{1.3}$$

global controllability property *for singletons* is guaranteed by the *bracket generating condition*: for each point $x \in M$ the evaluations at x of the iterated Lie brackets $[f_{j_1}, [\ldots [f_{j_{N-1}}, f_{j_N}] \ldots]$ span the tangent space $T_x M$.

We are going to establish controllability criteria for control system (1.3) acting in the space of ensembles $\mathcal{E}_\Theta(M)$. The criteria for finite and continual ensembles are provided in Sections 3 and 4. As far as controlled dynamics in the space of ensembles is defined by action of the flows, generated by controlled system (1.3), it is important to analyze whether and how the controllability criterion could be "lifted" to the group of diffeomorphisms or the semigroup of flows. This is done in Section 5, where Theorem 5.1 provides a result on a Lie extension of the action of system (1.3) in the group of diffeomorphisms. In Section 6 we discuss the relation of the established controllability criterion for continual ensembles of points to various versions of Rashevsky-Chow theorem in finite and infinite dimensions. It turns out that the latter typically are not applicable to ensemble controllability.

The proofs of the main results are provided in Sections 7-9.

By now there are numerous publications on simultaneous control of ensembles of control systems

$$\dot{x} = f(x, u, \theta), \; x \in M, \; u \in U, \theta \in \Theta \tag{1.4}$$

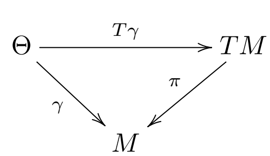
by a unique control. This direction of study has been initiated by S. Li and N. Khaneja ([13, 14]) for the case of quantum ensembles. Few other publication which took on the subject are [6, 7, 9], where readers can find more bibliographic references. In our previous publication [1] we considered the ensembles of systems (1.4), and formulated Lie algebraic controllability criteria for ensembles of systems.

In the present publication we consider ensembles of points controlled by virtue of a *single system* and *single open loop control.* This choice distinguishes the problem setting not only from the previous one, but also from the control problems, in which both the state space and the set of control parameters are infinite-dimensional. Examples of the latter kind appear in [2] and are common in the literature on mass transportation. Another range of publications operates with ensembles, named shapes, and with the group of the diffeomorphisms acting on them. An exposition of the topic and further references can be found for example in [5, 17, 18].

2. Banach manifold of ensembles. As we said ensembles of points in M are the images of continuous maps $\gamma : \Theta \to M$; the set of parameters Θ is assumed to be compact. At some moments we assume additionally the maps γ to be injective. The set of ensembles is denoted by $\mathcal{E}_\Theta(M)$.

Whenever the set of parameters Θ is finite, then the ensemble is called finite and the set of ensembles $\mathcal{E}_\Theta(M)$ is a finite-dimensional manifold.

Define for any ensemble $\gamma(\theta) \in \mathcal{E}_\Theta(M)$ a tangent space $T_\gamma \mathcal{E}_\Theta(M)$, consisting of the continuous maps $T\gamma : \Theta \to TM$, for which the diagram



is commutative. Representing an element of the tangent bundle TM as a pair $(x, \xi), \; x \in M, \xi \in T_x M$, we note that

$$T\gamma(\theta) = (\gamma(\theta), \xi(\theta)), \; \xi(\theta) \in T_{\gamma(\theta)} M, \; \theta \in \Theta.$$

If $M = \mathbb{R}^n$, then $T_\gamma \mathcal{E}_\Theta(M)$ can be identified with the set of continuous maps $C^0(\Theta, \mathbb{R}^n)$.

One can define a vector field on $\mathcal{E}_\Theta(M)$ as a section of the tangent bundle $T\mathcal{E}_\Theta(M)$.

The flow e^{tf}, generated by a time-independent vector field $f \in \mathrm{Vect}(M)$, and acting onto an ensemble $\gamma(\theta)$, defines a lift of f to the vector field

$$F \in \mathrm{Vect}\left(\mathcal{E}_\Theta(M)\right): \ F(\gamma(\cdot)) = \left.\frac{d}{dt}\right|_{t=0} e^{tf}\left(\gamma(\cdot)\right) = f(\gamma(\cdot)).$$

The same holds for time-dependent vector fields f_t.

The Lie brackets of the lifted vector fields are the lifts of the Lie brackets of the vector fields: $[F_1, F_2]|_{\gamma(\cdot)} = [f_1, f_2](\gamma(\cdot))$.

One can provide $T_{\gamma(\cdot)}\mathcal{E}_\Theta(M)$ with different metrics. Of interest for us are those obtained by the restrictions of the metrics $C^0(\Theta, TM)$, and $L_p(\Theta, TM)$ onto $T\mathcal{E}_\Theta(M)$.

3. Genericity of the controllability property for finite ensembles of points. Let $\Theta = \{1, \ldots, N\}$. A finite ensemble $\gamma : \Theta \mapsto M$ is an N-ple of points $\gamma = (\gamma_1, \ldots, \gamma_N) \in M^N$. In this Section we assume γ to be injective, so that the points γ_j are pairwise distinct. Let $\Delta^N \subset M^N$ be the set of N-ples $(x_1, \ldots, x_N) \in M^N$ with (at least) two coinciding components: $x_i = x_j$, for some $i \neq j$. Then the space of ensembles $\mathcal{E}_N(M)$ is identified with the complement of Δ^N : $\mathcal{E}_N(M) = M^N \setminus \Delta^N = M^{(N)}$.

For each $\gamma \in M^{(N)}$ the tangent space $T_\gamma M^{(N)}$ is isomorphic to

$$\bigotimes_{j=1}^{N} T_{\gamma_j}M = T_{\gamma_1}M \times \cdots \times T_{\gamma_N}M.$$

For a vector field $X \in \mathrm{Vect}M$ consider its N-fold, defined on $M^{(N)}$ as

$$X^N(x_1, \ldots, x_N) = (X(x_1), \ldots, X(x_N)).$$

For $X, Y \in \mathrm{Vect}\ M$, and $N \geq 1$ we define the Lie bracket of the N-folds X^N, Y^N on $M^{(N)}$ "componentwise": $[X^N, Y^N] = [X, Y]^N$, where $[X, Y]$ is the Lie bracket of X, Y on M. The same holds for the iterated Lie brackets.

Given the vector fields $f_1, \ldots f_s$ on M their N-folds $f_1^N, \ldots, f_s^N$ form a bracket generating system on $M^{(N)}$, if the evaluations of their iterated Lie brackets at each $\gamma \in M^{(N)}$, span the tangent space $T_\gamma M^{(N)} = \bigotimes_{j=1}^N T_{\gamma_j}M$. Evidently for $N > 1$ the property is strictly stronger, than the bracket generating property for $f_1, \ldots, f_s$ on M. We provide some comments below in Section 6.

The following result is a corollary of classical Rashevsky-Chow theorem (see Proposition 6.1).

Proposition 3.1 (global controllability criterion for system (1.3) in the space of finite point ensembles). *If the N-folds $f_1^N, \ldots, f_s^N$ are bracket generating at each point of $M^{(N)}$, then $\forall T > 0$ the system (1.3) is time-T globally exactly controllable in the space of finite ensembles $(\gamma_1, \ldots, \gamma_N) \in M^{(N)}$.*

Proposition 3.1 relates global controllability of system (1.3) for N-point ensembles to the bracket generating property on $M^{(N)}$ for the N-folds of the vector fields $f_1, \ldots, f_s$. The following result states that the bracket generating property for N-folds is generic.

Theorem 3.2. *For any $N \geq 1$ and sufficiently large ℓ, there is a set of s-ples of vector fields $(f_1, \ldots, f_s)$, which is residual in $\mathrm{Vect}M^{\otimes s}$ in Whitney C^ℓ-topology, such that for any $(f_1, \ldots, f_s)$ from this set the N-folds $(f_1^N, \ldots, f_s^N)$ are bracket generating at each point of $M^{(N)} = M^N \setminus \Delta^N$.*

Note that the notion of genericity in the theorem allows for (small) perturbations of the f_i, but not of $f_i^N = (f_i, \ldots, f_i)$ directly. Therefore the theorem extends the classical result by C.Lobry's ([15]) on genericity of the property of controllability for singletons (see also Theorem 3.1 of our previous work [1] on the genericity of controllability property for ensembles of control systems).

Proof of theorem 3.2 (for $s = 2$) is provided in Section 7.

4. Criterion of approximate steering for continual ensembles of points.

To formulate criterion for approximate steering of continual ensembles of points we impose the following assumption for control system (1.3).

ASSUMPTION 4.1 (boundedness in x). *The C^∞-smooth vector fields $f_j(x) \in$ Vect M, $j = 1, \ldots, s$, which define system* (1.3)*, are bounded on M together with their covariant derivatives of each order.*

The boundedness of f_j and of their covariant derivatives on M implies completeness of the vector fields f_j and of their Lie brackets of any order. Completeness of a vector field means that the trajectory of the vector field with arbitrary initial data can be extended to each compact subinterval of the time axis.

This assumption is rather natural. It holds for compact manifolds M. For a non-compact M it obviously holds for vector fields with compact supports. Other examples are vector fields on $\mathbb{R}^n$, whose components are trigonometric polynomials in x, or polynomial (in x) vector fields, multiplied by functions rapidly decaying at infinity (e.g. by e^{-x^2}).

Consider a couple of initial and target ensembles of points $\alpha(\theta), \omega(\theta) \in \mathcal{E}_\Theta(M)$, which we assume to be diffeotopic,[1] i.e. satisfying the relation $R_T(\alpha(\cdot)) = \omega(\cdot)$, where $t \to R_t,\ t \in [0, T],\ R_0 = \mathrm{Id}$, is a flow on M, defined by a time-dependent vector field $Y_t(x)$, with $Y_t(x), D_x Y_t(x)$ continuous.

Note that the (reference) flow R_t is a priori unrelated to control system (1.3). Denote by $\gamma_t(\theta)$ the image of $\alpha(\theta)$ under the diffeotopy

$$\gamma_t(\theta) = R_t(\alpha(\theta)),\ \gamma_0(\theta) = \alpha(\theta),\ \gamma_T(\theta) = \omega(\theta).$$

We introduce standard notation for the seminorms in the space of vector fields on M: for a compact $K \subset M$

$$\|X\|_{r,K} = \sup_{x \in K} \left(\sum_{0 \le |\beta| \le r} \left| D^\beta X(x) \right| \right)$$

and

$$\|X\|_r = \sup_{x \in M} \left(\sum_{0 \le |\beta| \le r} \left| D^\beta X(x) \right| \right).$$

Let Lie$\{f\}$ be the Lie algebra, generated by the vector fields $f_1, \ldots, f_s$. Put for $\lambda > 0$ and a compact $K \subset M$:

$$\mathrm{Lie}^\lambda_{1,K}\{f\} = \left\{ X(x) \in \mathrm{Lie}\{f\} \mid \|X\|_{1,K} < \lambda \right\},$$

and

$$\mathrm{Lie}^\lambda_1\{f\} = \left\{ X(x) \in \mathrm{Lie}\{f\} \mid \|X\|_1 < \lambda \right\}.$$

[1] We can assume instead an existence, for each $\varepsilon > 0$, of an ensemble $\omega_\varepsilon(\cdot)$, which is ε-close to $\omega(\cdot)$ in $C^0(\Theta)$-metric and diffeotopic to $\alpha(\cdot)$.

The following *bracket approximating condition along a diffeotopy* is the key part of the criterion for steering continual ensembles of points. In Section 6 we discuss the reason for the choice of this particular form of condition.

DEFINITION 4.2 (Lie bracket C^0-approximating condition along a diffeotopy). *Let the diffeotopy $\gamma_t = R_t(\alpha(\cdot))$, $t \in [0,T]$, generated by the vector field $Y_t(x)$, join $\alpha(\cdot)$ and $\omega(\cdot)$. System (1.3) satisfies Lie bracket C^0-approximating condition along γ_t, if there exist $\lambda > 0$ and a compact neighborhood $\mathcal{O}_\Gamma$ of the set $\Gamma = \{\gamma_t(\theta) | \ \theta \in \Theta, \ t \in [0,T]\}$ such that*

$$\forall t \in [0,T]: \ \inf \left\{ \sup_{\theta\in\Theta} |Y_t(\gamma_t(\theta)) - X(\gamma_t(\theta))| \ \middle| \ X \in Lie^{\lambda}_{1,\mathcal{O}_\Gamma}\{f\} \right\} = 0. \tag{4.1}$$

THEOREM 4.3 (approximate steering criterion for ensembles of points). *Let $\alpha(\theta), \omega(\theta)$ be two ensembles of points, joined by a diffeotopy $\gamma_t(\theta)$, $t \in [0,T]$. If control system (1.3) satisfies the Lie bracket C^0-approximating condition along the diffeotopy, then $\alpha(\cdot)$ can be steered C^0-approximately to $\omega(\cdot)$ by system (1.3) in time T.*

4.1. Approximate controllability for continual ensembles: basic example. We provide an example of application of Theorem 4.3. Consider the system in $\mathbb{R}^2$ with two controls:

$$\dot{x_1} = u, \ \dot{x_2} = \varphi(x_1)v, \ (u,v) \in \mathbb{R}^2. \tag{4.2}$$

It is a particular case of the control-linear system (1.3):

$$\dot{x} = f_1(x)u + f_2(x)v, \ f_1 = \partial/\partial x_1, \ f_2 = \varphi(x_1)\partial/\partial x_2. \tag{4.3}$$

We assume $\varphi(x_1)$ to be C^∞-smooth. In our example $\varphi(x_1) = e^{-x_1^2}$.

Choose the initial ensemble

$$\alpha(\theta) = (\theta, 0), \ \theta \in \Theta = [0,1]. \tag{4.4}$$

If one takes for example $u = 0$ in (4.2), then x_1 remains fixed, and by (4.2),(4.4)

$$x_2(T;\theta) = m_{v(\cdot)}\varphi(\theta),$$

where $m_{v(\cdot)} = \int_0^T v(t)dt \ \in \mathbb{R}$. Therefore for vanishing $u(\cdot)$ the set of "attainable profiles" for $x_2(T;\theta)$ is very limited.

To illustrate Theorem 4.3 we fix target ensemble $\omega(\theta) = (\theta, \theta)$ and choose a diffeotopy

$$\gamma_t(\theta) = (\theta, t\theta), \ t \in [0,1], \tag{4.5}$$

which joins $\alpha(\theta)$ and $\omega(\theta)$. The diffeotopy is generated by the (time-independent) vector field $Y(x) = Y(x_1, x_2) = x_1\partial/\partial x_2$. Evaluation of the vector field Y along the diffeotopy (4.5), equals

$$\forall t \in [0,1]: \ Y(\gamma_t(\theta)) = Y(\theta, t\theta) = \theta\partial/\partial x_2.$$

The Lie algebra, generated by f_1, f_2, is spanned in the treated case by the vector fields f_1 and the vector fields

$$\mathrm{ad}^k f_1 f_2 = \varphi^{(k)}(x_1)\partial/\partial x_2, \ k = 0, 1, 2, \dots \tag{4.6}$$

and is infinite-dimensional for our choice of $\varphi(\cdot)$.

The evaluations $f_1(\gamma_t(\theta))$ and $\mathrm{ad}^k f_1 f_2(\gamma_t(\theta))$ equal

$$f_1(\gamma_t(\theta)) = \partial/\partial x_1,\ \left(\mathrm{ad}^k f_1 f_2\right)(\gamma_t(\theta)) = \varphi^{(k)}(\theta)\partial/\partial x_2,\ k = 0, 1, 2 \dots .$$

The successive derivatives of $\varphi(x) = e^{-x^2}$ are

$$\varphi^{(m)}(x) = (-1)^m H_m(x) e^{-x^2},\ m = 0, 1, \dots, \tag{4.7}$$

where $H_m(x)$ are Hermite polynomials. Recall that $H_m(x)$ form an orthogonal complete system for $L_2(-\infty, +\infty)$ with the weight e^{-x^2}.

Let $\mathcal{H}$ be (infinite-dimensional) linear space generated by functions (4.7). Generic element of $\mathrm{Lie}\{f_1, f_2\}$ can be represented as

$$a\frac{\partial}{\partial x_1} + h(x_1)\frac{\partial}{\partial x_2},\ a \in \mathbb{R},\ h \in \mathcal{H},$$

and its evaluation at $\gamma_t(\theta)$ equals

$$a\frac{\partial}{\partial x_1} + h(\theta)\frac{\partial}{\partial x_2},\ a \in \mathbb{R},\ h \in \mathcal{H}.$$

The C^0 bracket approximating condition along $\gamma_t(\theta)$ amounts to the approximability in $C^0[0,1]$ of the function $Y_2(\theta) = \theta$ by the functions from a bounded equi-Lipschitzian subset of $\mathcal{H}$.

To establish approximability for chosen example we use the following technical lemmae.

Lemma 4.4. *There exists $\lambda > 0$ such that*

$$\inf\left\{ \sup_{\theta\in[0,1]} |\theta - h(\theta)| \;\middle|\; h(\cdot) \in \mathcal{H},\ \sup_{\theta\in[0,1]} (|h(\theta)| + |h'(\theta)|) < \lambda \right\} = 0.$$

Proof. The lemma is a corollary of the following standard facts, which concern the expansions with respect to the Hermite system.

Lemma 4.5. *Let $g(x)$ be a smooth function with compact support in $(-\infty, +\infty)$ and*

$$g(x) \simeq \sum_{m\geq 0} g_m H_m(x) \tag{4.8}$$

be its expansion with respect to Hermite system. Then:

(i) expansion (4.8) converges to $g(x)$ uniformly on any compact interval;

(ii) the expansion $\sum_{m\geq 1} g_m H'_m(x)$ converges to $g'(x)$ uniformly on any compact interval.

Proof. For (i) see e.g. [16, §8]. Statement (ii) follows easily from (i), given the relation $H'_m(x) = 2mH_{m-1}(x)$ for the Hermite polynomials. Indeed

$$\sum_{m\geq 1} g_m H'_m(x) = \sum_{m\geq 1} 2m g_m H_{m-1}(x) = \sum_{m\geq 0} 2(m+1) g_{m+1} H_m(x),$$

and it rests to verify that the coefficients of the expansion of $g'(x)$ with respect to Hermite system are precisely $2(m+1)g_{m+1}$. This in its turn follows by direct computation by the formulae

$$H'_m(x) = 2xH_m(x) - H_{m+1}(x),\ \int_{-\infty}^{+\infty} (H_m(x))^2 e^{-x^2} dx = 2^m m! \sqrt{\pi}.$$

Now in order to prove Lemma 4.4 we take a C^∞ smooth function $g(\theta)$ with compact support on $(-\infty,+\infty)$, whose restriction to $[0,1]$ coincides with the function $y(\theta)=\theta e^{\theta^2}$. By Lemma 4.5 (i) the expansion $g(\theta)\simeq\sum_m g_m H_m(\theta)$ converges uniformly on $[0,1]$ to θe^{θ^2}, and hence the series $\sum_m g_m H_m(\theta)e^{-\theta^2}$ converges to θ uniformly on $[0,1]$.

Differentiating $\sum_m c_m H_m(\theta)e^{-\theta^2}$ termwise in θ we get

$$\sum_m c_m H_m'(\theta)e^{-\theta^2}-\sum_m c_m H_m(\theta)2\theta e^{-\theta^2}.$$

By Lemma 4.5 (i) and (ii) the series $\sum_{m\geq 1} c_m H_m'$ and $\sum_{m\geq 0} c_m H_m(\theta)$ converge uniformly on $[0,1]$ to bounded functions; the partial sums of these series are equibounded and therefore partial sums of the series $\sum_m g_m H_m(\theta)e^{-\theta^2}$ are equi-Lipschitzian, what concludes the proof of Lemma 4.4. ∎

5. Lie extensions and approximate controllability for flows. The proof of Theorem 4.3, provided in Section 9, is based on an infinite-dimensional version of the method of Lie extensions ([11, 1, 4]).

According to this method one starts with establishing the property of C^0-approximate steering by means of an extended control fed into an extended (in comparison with (1.3)) control system

$$\frac{dx(t)}{dt}=\sum_{\beta\in B}X^\beta(x)v_\beta(t), \tag{5.1}$$

where $X^\beta(x)$ are the iterated Lie brackets

$$X^\beta(x)=[f_{\beta_1},[f_{\beta_2},[\ldots,f_{\beta_N}]\ldots]](x) \tag{5.2}$$

of the vector fields $f_1,\ldots,f_s$ (we assume by default, that the vector fields $f_j(x)$ are included into the family $\{X^\beta(x),\ \beta\in B\}$.) In (5.1)-(5.2) the multiindices $\beta=(\beta_1,\ldots,\beta_N)$ belong to a finite subset $B\subset\bigcup_{N\geq 1}\{1,\ldots,s\}^N$, and $(v_\beta(t))_{\beta\in B}$ is a (high-dimensional) *extended control*.

After the first step one has to prove that the action of the flow, generated by extended system (5.1) on $\mathcal{E}_\Theta(M)$, can be approximated by the action of the flow of system (1.3), driven by a low-dimensional control $u(\cdot)=(u_1(\cdot),\ldots,u_s(\cdot))$. The latter step is the core of the method of Lie extensions.

To prove the approximation result we formulate an approximate controllability criterion for flows on M, or, the same an approximate path controllability criterion in the (infinite-dimensional) group of diffeomorphisms. The result has implications for the action of the control system on ensemble of points with arbitrary Θ (see Corollary 5.2); in particular the implication for singletons gives classical Rashevsky-Chow type controllability result.

The respective formulation is given by

THEOREM 5.1. *Let $P_t^{v(\cdot)}$ be a flow on M, generated by extended control system (5.1) and an extended control $v(t)=(v_\beta(t))_{\beta\in B}$, $t\in[0,T]$. For each $\varepsilon>0$, $r\geq 0$ and compact $K\subset M$ there exists an appropriate control $u(t)=(u_1(t),\ldots,u_s(t))$ such that the flow $P_t^{u(\cdot)}$, generated by control system (1.3) and the control $u(\cdot)$, satisfies:*

$$\|P_t^{v(\cdot)}-P_t^{u(\cdot)}\|_{r,K}<\varepsilon,\ \forall t\in[0,T].$$

An obvious application of this theorem to the case of ensembles provides the following

COROLLARY 5.2. *If the ensemble $\alpha(\theta)$ can be steered approximately to the ensemble $\omega(\theta)$ in time T by an extended system (5.1), then the same can be accomplished by the original control system (1.3).*

Indeed let $v(\cdot)$ be an extended control for extended system (5.1), such that for the corresponding flow $P_t^{v(\cdot)}$ we get $\sup_{\theta\in\Theta} d\left(\omega(\theta), P_T^{v(\cdot)}(\alpha(\theta))\right) < \varepsilon/2$. By theorem 5.1 there exists a control $u(\cdot)$ for system (1.3) such that

$$\sup_{\theta\in\Theta} d\left(P_T^{v(\cdot)}(\alpha(\theta)), P_T^{u(\cdot)}(\alpha(\theta))\right) < \varepsilon/2$$

and hence

$$\sup_{\theta\in\Theta} d\left(\omega(\theta), P_T^{u(\cdot)}(\alpha(\theta))\right) < \varepsilon.$$

6. Theorem 4.3 and Rashevsky-Chow theorem(s): discussion of the formulations. The formulations of the results, provided in the two previous sections, show similarity to the formulations of Rashevsky-Chow theorem on finite-dimensional and infinite-dimensional manifolds. In this Section we survey these formulations and establish their relation to Theorem 4.3.

6.1. Lie rank/bracket generating controllability criteria. Classical Rashevsky-Chow theorem provides a sufficient (and necessary in the real analytic case) criterion for global exact controllability of system (1.3) for singletons (= single-point ensembles) on a connected finite-dimensional manifold M in terms of *bracket generating property*. This property holds for control system (1.3) at $x \in M$ if the evaluations of the iterated Lie brackets (5.2) of the vector fields $f_1, \ldots, f_r$ at x span the respective tangent space T_xM.

PROPOSITION 6.1 (Rashevsky-Chow theorem in finite dimension, [4],[11]). *Let for control system (1.3) the bracket generating property hold at each point of M. Then $\forall x_\alpha, x_\omega \in M$, $\forall T > 0$ the point x_α can be connected with x_ω by an admissible trajectory $x(t)$, $t \in [0, T]$ of system (1.3), i.e. system (1.3) is globally controllable in any time T. If the manifold M and the vector fields $f_1, \ldots, f_s$ are real analytic then the bracket generating property is necessary and sufficient for global controllability of system (1.3).*

The bracket generating property for $f_1, \ldots, f_s$ is by no means sufficient for controllability of ensembles, even finite ones. For example if this property holds but the Lie algebra $\mathrm{Lie}\{f\}$, correspondent to the system (1.3) is finite-dimensional, then the N-fold of system (1.3) can not possess bracket generating property on $M^{(N)}$ (see Section 3), if $N \dim\ M > \dim\ \mathrm{Lie}\{f\}$. Hence if $\dim \mathrm{Lie}\{f\} < +\infty$, then exact controllability in the space of N-point ensembles, with N sufficiently large, is not achievable.

Regarding continual ensembles, they form, as we said, an infinite-dimensional Banach manifold $\mathcal{E}_\Theta(M)$ (see Sections 2 and 4) and control system (1.3) admits a lift to a control system on $\mathcal{E}_\Theta(M)$.

One can think of application of infinite-dimensional Rashevsky-Chow theorem ([8],[12]) to the lifted system.

PROPOSITION 6.2 (infinite-dimensional analogue of Rashevsky-Chow theorem). *Consider a control system $\dot{y} = \sum_{j=1}^{s} F_j(y)u_j(t)$, defined on Banach manifold $\mathcal{E}$. If* *the condition*

$$\overline{Lie\{F_1, F_2, \ldots, F_m\}(y)} = T_y\mathcal{E}, \forall y \in \mathcal{E} \tag{6.1}$$

holds, then this system is globally approximately controllable, i.e. for each starting point $\tilde{y}$ the set of points, attainable from $\tilde{y}$ (by virtue of the system) is dense in $\mathcal{E}$.

Seeking to apply this result to the case of ensembles $\mathcal{E} = \mathcal{E}_\Theta(M)$ one meets two difficulties.

First, verification of the (approximate) bracket generating property (6.1) has to be done for *each* $\gamma(\cdot) \in \mathcal{E}_\Theta(M)$ and this results in a vast set of conditions, "indexed" by the elements of the functional space $\mathcal{E}_\Theta(M)$.

This difficulty can be overcome by passing to a pathwise version of Rashevsky-Chow theorem, which in the case of singletons is close to its classical formulation.

Proposition 6.3. *Let M be a finite-dimensional manifold, $x_\alpha, x_\omega \in M$. If bracket generating property holds at each point of a continuous path $\gamma(\cdot)$, joining x_α and x_ω, then x_α and x_ω can be joined by an admissible trajectory of* (1.3).

This result can be deduced directly from Proposition 6.1. Indeed if the bracket generating property holds along the path $\gamma(\cdot)$, then it also holds at each point of a connected open neighborhood $\mathcal{O}$ of the path $\gamma(\cdot)$ in M. Applying Rashevsky-Chow theorem to the restriction of the control system (1.3) to $\mathcal{O}$ we get the needed steering result.

In the case of continual ensembles it turns out though - and this is the second difficulty - that for the vector fields F, which are lifts to $\mathcal{E}_\Theta(M)$ of the vector fields $f \in \mathrm{Vect}\ M$, the (approximate) bracket generating property (6.1) can not hold at each $\gamma \in \mathcal{E}_\Theta(M)$ and may cease to hold even C^0-locally. Thus the argument just provided fails: condition (6.1) may hold along the path $p(\cdot)$ and cease to hold in a neighborhood of the path.

For example the space $\mathcal{E} = \mathcal{E}_\Theta(\mathbb{R}^n)$ of ensembles of points in $\mathbb{R}^n$, parameterized by a compact Θ, is isomorphic to the Banach space $C^0(\Theta, \mathbb{R}^n)$. Its tangent spaces are all isomorphic to $C^0(\Theta, \mathbb{R}^n)$. If Θ is not finite ($\sharp\Theta = \infty$) then in any C^0-neighborhood of an ensemble $\hat{\gamma}(\cdot) \in C^0(\Theta, \mathbb{R}^n)$ one can find an ensemble $\gamma(\cdot) \in C^0(\Theta, \mathbb{R}^n)$, which is constant on an open subset of Θ. Then $\{Y(\gamma(\theta)) | Y \in \mathrm{Vect}M\}$ is not dense in $T_\gamma\mathcal{E} = T_\gamma C^0(\Theta, \mathbb{R}^n)$ and hence condition (6.1) can not hold at $\gamma(\cdot)$. There may certainly occur other types of singularities.

The same remains true if the topology, in which the target is approximated (and hence the topology of $\mathcal{E}$) is weakened.

We end up with two remarks concerning the formulation of Theorem 4.3.

The criterion for approximate steering, provided by the Theorem has meaningful analogue also in the case of singletons.

Proposition 6.4 (bracket approximating property and approximate steering for singletons). *Let $x_\alpha, x_\omega \in M$ and $\gamma(t),\ t \in [0, T]$ be a continuously differentiable path, which joins x_α and x_ω. If the Lie bracket approximating property holds at each point $\gamma(t),\ t \in [0, T]$, then x_α can be approximately steered to x_ω by an admissible trajectory of* (1.3).

Recall that the Lie bracket approximating condition includes the assumption of Lipschitz equicontinuity of the approximating vector fields from $\mathrm{Lie}\{f\}$. The following example illustrates importance of this assumption.

Consider a control system (1.3) in $\mathbb{R}^2 = \{(x_1, x_2)\}$, such that the orbits of (1.3) are the lower and the upper open half-planes of $\mathbb{R}^2$ together with the straight-line

$x_2 = 0$. An example of such system is

$$\dot{x}_1 = u_1,\ \dot{x}_2 = x_2 u_2,\ (u_1, u_2) \in \mathbb{R}^2.$$

The points $x_\alpha = (-1, -1)$ and $x_\omega = (1, 1)$ belonging to different orbits, can not be steered approximately one to another. On the other side if we join these points by the curve $\gamma(t) = (t, t^3),\ t \in [-1, 1]$, then it is immediate to check, that $\dot{\gamma}(t) \in \mathrm{Lie}\{f\}(\gamma(t))$ for each t, but the condition of Lipshitz equicontinuity is not fulfilled. There are curves $\gamma^\delta(\cdot)$ arbitrarily close to $\gamma(\cdot)$ in C^0 metric, which intersect the line $x_2 = 0$ transversally and hence do not satisfy the condition $\dot{\gamma}^\delta(t) \in \mathrm{Lie}\{f\}(\gamma^\delta(t))$.

7. Proof of Theorem 3.2.

Proof. We provide a proof for couples of vector fields ($s = 2$); general case is treated similarly. It suffices to establish for fixed N existence of a residual subset $\mathcal{G} \subset \mathrm{Vect}M \times \mathrm{Vect}M$ such that for each couple $(X, Y) \in \mathcal{G}$ the couple of N-folds of the vector fields (X^N, Y^N) is bracket generating on $M^{(N)}$. Let $\dim M = n$.

The proof is based on application of J.Mather's multi-jet transversality theorem ([10]).

Consider the couples of vector fields (X, Y) on M as C^k-smooth sections of the fibre bundle $\pi : TM \times_M TM \to M$. Consider the set $J_k(TM \times_M TM)$ of k-jets of the couples of vector fields and the projection π_k of $J_k(TM \times_M TM)$ to M. One can define in obvious way for $N \geq 1$ the projection $\pi_k^N : J_k(TM \times_M TM)^N \to M^N$ and introduce the set $J_k^{(N)}(TM \times_M TM)^N = (\pi_k^N)^{-1}(M^{(N)})$, which is N-fold k-jet (or multi-jet) bundle for the couples of vector fields.

In other words N-fold of a vector field $X \in \mathrm{Vect}M$ is a vector field $\underbrace{(X, \ldots, X)}_{N} \in \mathrm{Vect}M^{(N)}$. For a couple $(X, Y) \in \mathrm{Vect}M \times \mathrm{Vect}M$ of vector fields the multi-jet $J_k^{(N)}(X, Y) : M^{(N)} \to J_k^{(N)}(\mathrm{Vect}M \times \mathrm{Vect}M)$ can be represented as

$$\forall (x_1, \ldots, x_N) \in M^{(N)} :$$
$$J_k^{(N)}(X, Y)(x_1, \ldots x_N) = (J_k(X, Y)(x_1), \ldots, J_k(X, Y)(x_N)) .$$

Proposition 7.1 (multi-jet transversality theorem for the couples of vector fields). *Let S be a submanifold of the space of k-multijets (N fold k-jets) $J_k^{(N)}(TM \times_M TM)^N$. Then for sufficiently large ℓ the set of the couples of the vector fields*

$$T_S = \{(X, Y) \in \mathrm{Vect}M \times \mathrm{Vect}M |\ J_k^N(X, Y) \pitchfork S\}$$

is a residual subset of $\mathrm{Vect}M \times \mathrm{Vect}M$ *in Whitney C^ℓ-topology ($\pitchfork$ stands for transversality of a map to a manifold).*

Coming back to the proof of Theorem 3.2, note that the set $\mathcal{R}$ of the couples (X, Y) of vector fields, such that at each $x \in M$ either $X(x) \neq 0$, or $Y(x) \neq 0$, is open and dense in $\mathrm{Vect}M \times \mathrm{Vect}M$. We will seek $\mathcal{G}$ as a subset of $\mathcal{R}$.

For each couple $(X, Y) \in \mathcal{R}$, and each point $\bar{x} = (x_1, \ldots, x_N) \in M^{(N)}$ we introduce the two $nN \times 2nN$-matrices:

$$V(\bar{x}) = \begin{pmatrix} Y(x_1) & \mathrm{ad}XY(x_1) & \cdots & \mathrm{ad}^{2nN-1}XY(x_1) \\ \vdots & \vdots & \vdots & \vdots \\ Y(x_N) & \mathrm{ad}XY(x_N) & \cdots & \mathrm{ad}^{2nN-1}XY(x_N) \end{pmatrix},$$

$$W(\bar{x}) = \begin{pmatrix} X(x_1) & \mathrm{ad}^2YX(x_1) & \cdots & \mathrm{ad}^{2nN}YX(x_1) \\ \vdots & \vdots & \vdots & \vdots \\ X(x_N) & \mathrm{ad}^2YX(x_N) & \cdots & \mathrm{ad}^{2nN}YX(x_N) \end{pmatrix}.$$

(Note that $W(\bar{x})$ lacks the column constituted by $\mathrm{ad}YX(x_j)$ which coincides, up to a sign, with the second column in $V(\bar{x})$).

For $(X,Y) \in \mathcal{R}$, $\bar{x} = (x_1, \ldots, x_N) \in M^{(N)}$ and each x_i, $i = 1, \ldots N$, at least one of the vectors $X(x_i), Y(x_i)$ is non null. We can choose local coordinates ξ_{ij}, $i = 1, \ldots N;\ j = 1, \ldots n$ in a neighborhood $U = U_1 \times \cdots \times U_N$ of $\bar{x} = (x_1, \ldots, x_N)$ in $M^{(N)}$ in such a way that in each U_i, $i = 1, \ldots, N$ either X or Y becomes the non null constant vector field: $X = \partial/\partial\xi_{i1}$ or $Y = \partial/\partial\xi_{i1}$. Then for each $i = 1, \ldots, N$, either $\mathrm{ad}^k XY|_{x_i}$ or $\mathrm{ad}^k YX|_{x_i}$ equal respectively to $\left.\frac{\partial^k Y}{\partial\xi_{i1}^k}\right|_{x_i}$ or $\left.\frac{\partial^k X}{\partial\xi_{i1}^k}\right|_{x_i}$.

We call *significant* those elements of the $(Nn \times 2Nn)$-matrices $V(\bar{x})$, $W(\bar{x})$ and of the corresponding $(Nn \times 4Nn)$-matrix $(V(\bar{x})|W(\bar{x}))$, which are the components of $\frac{\partial^k Y}{\partial\xi_{i1}^k}$ and of $\frac{\partial^k X}{\partial\xi_{i1}^k}$. For each $j = 1, \ldots, Nn$ either j-th row of $V(\bar{x})$ or j-th row of $W(\bar{x})$ consists of significant elements. The elements of these matrices are polynomials in the components of the multi-jets $J^{2nN}X(\bar{x}), J^{2nN}Y(\bar{x})$. Significant elements are polynomials of degree 1, distinct significant elements correspond to different polynomials, nonsignificant elements correspond to polynomials of degrees > 1. Elements of different rows of the matrices differ.

If $(X,Y) \in \mathcal{R}$ and (X^N, Y^N) lacks the bracket generating property at some $\bar{x} = (x_1, \ldots, x_N)$, then the rank r of the $(Nn \times 4Nn)$-matrix $(V|W)(\bar{x})$ is incomplete: $r < nN$.

The (stratified) manifold of $(Nn \times 4Nn)$-matrices of rank $r < nN$ is (locally) defined by rational relations, which express elements of some $(Nn - r) \times (4Nn - r)$ minor via other elements of the matrix.

As long as $4Nn - r \geq 3Nn + 1$, then each row of the minor contains $\sigma \geq 3Nn + 1 - 2Nn > Nn$ significant elements. The corresponding relations express σ distinct components of $2N$-th multi-jet of (X,Y) via other components of the multi-jet. Hence $2N$-multi-jets of the couples (X,Y), for which (X^N, Y^N) lack bracket generating property, must belong to an algebraic manifold S of codimension $\sigma > Nn$ in $J_k^N(TM \times_M TM)$.

Consider the set T_S of the couples $(X,Y) \in \mathcal{R} \subset \mathrm{Vect}M \times \mathrm{Vect}M$, for which $J_{2nN}^N(X,Y) : M^{(N)} \to J_{2nN}^N(\mathrm{Vect}M \times \mathrm{Vect}M)$ is transversal to S. According to the multijet transversality theorem (Proposition 7.1) T_S is residual in $\mathrm{Vect}M \times \mathrm{Vect}M$ in Whitney C^ℓ-topology for sufficiently large ℓ. As far as

$$\dim M^{(N)} = Nn < \sigma = \mathrm{codim}\ S$$

the transversality can take place only if, for each $\bar{x} \in M^{(N)}$, $J_{2nN}^N(X,Y)|_{\bar{x}} \notin S$. Hence for each couple (X,Y) from the residual subset T_S, the couples of N-folds (X^N, Y^N) are bracket generating at each point of $M^{(N)}$. ∎

8. Proof of Theorem 5.1.

8.1. Variational formula. We start with nonlinear version of 'variation of constants' formula, which will be employed in the next subsection.

Let $f_t(x)$ be a time-dependent and $g(x)$ a time-independent vector fields on M. We assume both vector fields to be C^∞-smooth and Lipschitz on M. Let $\overrightarrow{\exp}\int_0^t f_\tau d\tau$ denote the flow generated by the time-dependent vector field f_t (see [3, 4] for the notation), and e^{tg} stays for the flow, generated by the time-independent vector field g.

LEMMA 8.1 ([4]). *Let $f_\tau(x), g(x)$ be C^∞-smooth in x, f_τ integrable in τ. Let $U(t)$ be a Lipschitzian function on $[0,T]$, $U(0)=0$. The flow*

$$P_t = \overrightarrow{\exp}\int_0^t \left(f_\tau(x)+g(x)\dot{U}(\tau)\right)d\tau$$

generated by the differential equation

$$\dot{x} = f_t(x)+g(x)\dot{U}(t), \tag{8.1}$$

can be represented as a composition of flows

$$\overrightarrow{\exp}\int_0^t \left(f_\tau(x)+g(x)\dot{U}(\tau)\right)d\tau = \overrightarrow{\exp}\int_0^t \left(e^{-U(\tau)g}\right)_* f_\tau d\tau \circ e^{U(t)g}. \tag{8.2}$$

At the right-hand side of (8.2) $\left(e^{-U(\tau)g}\right)_*$ is the differential of the diffeomorphism $e^{-U(t)g}=\left(e^{U(t)g}\right)^{-1}$, where $e^{U(t)g}$ is the evaluation at time-instant $U(t)$ of the flow, generated by the time-independent vector field $g(x)$.

We omit at this point the questions of completeness of the vector fields involved into (8.1),(8.2), assuming that the formula (8.2) is valid, whenever the flows, involved in it, exist on the specified intervals.

For each vector field $Z \in \mathrm{Vect}\, M$ the operator ad_Z, acts on the space of vector fields: $\mathrm{ad}_Z Z_1 = [Z, Z_1]$ - the Lie bracket of Z and Z_1. The operator exponential $e^{U\mathrm{ad}Z}$ is defined formally: $e^{U\mathrm{ad}_Z}=\sum_{j=0}^\infty \frac{U^j(\mathrm{ad}_Z)^j}{j!}$. For C^∞-smooth vector fields Z, Z_1 the expansion is known (see [3],[4]) to provide asymptotic representation for $\left(e^{-U(\tau)g}\right)_*$: for each $s \geq 0$ and a compact $K \subset M$ there exists a compact neighborhood K' of K and $c>0$ such that

$$\left\|\left(\left(e^{-U(\tau)g}\right)_* - I - \sum_{j=1}^{N-1}\frac{(U(\tau))^j}{j!}\mathrm{ad}_g^j\right)Z_1\right\|_{s,K} \leq$$

$$\leq c e^{c|U(\tau)|\|g\|_{s+1,K'}}\frac{\left(|U(\tau)|\|g\|_{s+N,K'}\right)^N}{N!}\|Z_1\|_{s+N,K'}$$

(see [3] for the details). We employ the asymptotic formulae for $N=1,2$ and small magnitude of U:

$$\left\|\left(\left(e^{-U(\tau)g}\right)_* - I\right)Z_1\right\|_{s,K} = O(|U(\tau)|)\|Z_1\|_{s+1,K'}, \tag{8.3}$$

$$\left\|\left(\left(e^{-U(\tau)g}\right)_* - I - U(\tau)\mathrm{ad}_g\right)Z_1\right\|_{s,K} = o(|U(\tau)|)\|Z_1\|_{s+2,K'}, \tag{8.4}$$

as $|U| \to 0$.

We introduce at this point fast-oscillating controls by choosing 1-periodic Lipschitz function $V(t)$ with $V(0)=0$, the scaling parameters $\beta>\alpha>0$ and defining for $\varepsilon>0$: $V(t;\alpha,\beta,\varepsilon)=\varepsilon^{\alpha}V\left(t/\varepsilon^{\beta}\right)$. We introduce controls

$$u_{\varepsilon}(t)=\frac{dV(t;\alpha,\beta,\varepsilon)}{dt}=\varepsilon^{\alpha-\beta}\dot{V}\left(t/\varepsilon^{\beta}\right),$$

which are high-gain and fast-oscillating for small $\varepsilon>0$.

For a more general control

$$u_{\varepsilon}(t)=w(t)\varepsilon^{\alpha-\beta}\dot{V}\left(t/\varepsilon^{\beta}\right), \tag{8.5}$$

where $w(\cdot)$ is a Lipschitz function, the primitive of $u_{\varepsilon}(t)$ equals

$$U_{\varepsilon}(t)=\varepsilon^{\alpha}\left(w(t)V\left(t/\varepsilon^{\beta}\right)-\int_{0}^{t}V\left(\tau/\varepsilon^{\beta}\right)\dot{w}(\tau)d\tau\right)=\varepsilon^{\alpha}\hat{U}_{\varepsilon}(t), \tag{8.6}$$

and $\hat{U}_{\varepsilon}(t)=O(1)$ as $\varepsilon\to+0$ uniformly for t in a compact interval.

Substituting $U(t)=U_{\varepsilon}(t)$, defined by (8.6), into (8.2) we get

$$\overrightarrow{\exp}\int_{0}^{t}\left(f_{\tau}(x)+g(x)\varepsilon^{\alpha-\beta}w(\tau)\dot{V}\left(\frac{\tau}{\varepsilon^{\beta}}\right)\right)d\tau= \tag{8.7}$$

$$\overrightarrow{\exp}\int_{0}^{t}\left(e^{-\varepsilon^{\alpha}\hat{U}_{\varepsilon}(\tau)g}\right)_{*}f_{\tau}d\tau\circ e^{\varepsilon^{\alpha}\hat{U}_{\varepsilon}(t)g}.$$

Expanding the exponentials at the right-hand side of the equality according to formula (8.3) we get for the control $u_{\varepsilon}(t)$, defined by (8.5):

$$\overrightarrow{\exp}\int_{0}^{t}\left(f_{\tau}(x)+g(x)u_{\varepsilon}(\tau)\right)d\tau=$$

$$\overrightarrow{\exp}\int_{0}^{t}\left(f_{\tau}(x)+O(\varepsilon^{\alpha})\right)d\tau\circ\left(I+O(\varepsilon^{\alpha})\right). \tag{8.8}$$

By classic theorems on continuous dependence of trajectories on the right-hand side we conclude that the flow $\overrightarrow{\exp}\int_{0}^{t}\left(f_{\tau}(x)+g(x)u_{\varepsilon}(\tau)\right)d\tau$ with $u_{\varepsilon}(t)$, defined by (8.5), tends to $\overrightarrow{\exp}\int_{0}^{t}f_{\tau}(x)d\tau$, as $\varepsilon\to0$, uniformly in t on compact intervals. Therefore the effect of the fast-oscillating control (8.5) tends to zero as $\varepsilon\to0$ with respect to any of the seminorms $\|\cdot\|_{r,K}$:

$$\left\|\overrightarrow{\exp}\int_{0}^{t}\left(f_{\tau}(x)+g(x)u_{\varepsilon}(\tau)\right)d\tau-\overrightarrow{\exp}\int_{0}^{t}f_{\tau}(x)d\tau\right\|_{r,K}\rightrightarrows0$$

for all $r\geq0$, compact K and uniformly for $t\in[0,T]$.

8.2. Lie extension for flows. Coming back to the proof of Theorem 5.1 we first note that its conclusion can be arrived at by induction, with the step of induction, represented by the following

LEMMA 8.2. *Theorem 5.1 is valid for the controlled system*

$$\frac{d}{dt}x(t)=\sum_{j=1}^{k}X^{j}(x)u_{j}(t)+X(x)u(t)+Y(x)v(t),$$

and its Lie extension

$$\frac{d}{dt}x(t) = \sum_{j=1}^{k} X^j(x)u^e_j(t) + X(x)u^e(t) + Y(x)v^e(t) + [X,Y](x)w^e(t).$$

The proof, provided below, shows that one can leave out, without loss of generality, the summed addends $\sum_{j=1}^{k} X^k(x)u_k(t)$, $\sum_{j=1}^{k} X^k(x)u^e_k(t)$ at the right-hand side of the systems. It suffices to prove the result for the 2-input system

$$\frac{d}{dt}x(t) = X(x)u(t) + Y(x)v(t), \tag{8.9}$$

and its 3-input Lie extension

$$\frac{d}{dt}x(t) = X(x)u^e(t) + Y(x)v^e(t) + [X,Y](x)w^e(t). \tag{8.10}$$

One can assume, without loss of generality, $w^e(t)$ to be smooth, as far as smooth functions are dense in L_1-metric in the space of bounded measurable functions. Hence by classical results on continuous dependence with respect to right-hand sides, the flows, generated by measurable controls, can be approximated by flows, generated by smooth controls.

To construct the controls $u(t), v(t)$ from $u^e(t), v^e(t), w^e(t)$ we take

$$u(t) = u_\varepsilon(t) = u^e(t) + \varepsilon\dot{U}_\varepsilon(t),\ v(t) = v_\varepsilon(t) = v^e(t) + \varepsilon^{-1}\hat{v}_\varepsilon(t), \tag{8.11}$$

where ε is the parameter of approximation and the functions $U_\varepsilon(t)$ and $\hat{v}_\varepsilon(t)$ will be specified in a moment.

Feeding controls (8.11) into system (8.9) we get

$$\frac{d}{dt}x(t) = \underbrace{X(x)u^e(t) + Y(x)\left(v^e(t) + \varepsilon^{-1}\hat{v}_\varepsilon(t)\right)}_{f_t} + \underbrace{X(x)}_{g}\varepsilon\dot{U}_\varepsilon(t). \tag{8.12}$$

Applying formula (8.2) to the flow, generated by (8.12), we represent it as a composition

$$\overrightarrow{\exp}\int_0^t X(x)u^e(t) + \left(e^{-\varepsilon U_\varepsilon(t)X}\right)_* Y(x)\left(v^e(t) + \varepsilon^{-1}\hat{v}_\varepsilon(t)\right)dt\ \circ \\ e^{\varepsilon U_\varepsilon(t)X(x)}. \tag{8.13}$$

We wish the latter flow to approximate (for sufficiently small $\varepsilon > 0$) the flow, generated by (8.10). To achieve this we choose the functions

$$U_\varepsilon(t) = 2\sin(t/\varepsilon^2)w^e(t),\ \hat{v}_\varepsilon(t) = \sin(t/\varepsilon^2). \tag{8.14}$$

Approximating the operator exponential $e^{\varepsilon U_\varepsilon(t)\mathrm{ad}_X}$ by formula (8.4) we transform (8.13) into

$$\overrightarrow{\exp}\int_0^t (X(x)u^e(t){+}Y(x)v^e(t) + [X,Y](x)U_\varepsilon(t)\hat{v}_\varepsilon(t){+} \\ Y(x)\varepsilon^{-1}\hat{v}_\varepsilon(t){+}O(\varepsilon))dt \circ (I + O(\varepsilon)), \tag{8.15}$$

where all $O(\varepsilon)$ are uniform in $t \in [0,T]$.

From (8.14)

$$U_\varepsilon(t)\hat{v}_\varepsilon(t) = w^e(t) - w^e(t)\cos(2t/\varepsilon^2),$$

and (8.15) takes form

$$\overrightarrow{\exp}\int_0^t \big(X(x)u^e(t) + Y(x)v^e(t) + [X,Y](x)w^e(t) + Y(x)\varepsilon^{-1}\sin(t/\varepsilon^2) - [X,Y](x)w^e(t)\cos(2t/\varepsilon^2) + O(\varepsilon)\big)\,dt \circ (I + O(\varepsilon)). \tag{8.16}$$

Processing fast oscillating terms $Y(x)\varepsilon^{-1}\sin(t/\varepsilon^2)$, $[X,Y]w^e(t)\cos(2t/\varepsilon^2)$ according to formula (8.7) we bring the flow (8.16) to the form

$$\overrightarrow{\exp}\int_0^t (X(x)u^e(\tau) + Y(x)v^e(\tau) + [X,Y](x)w^e(\tau) + O(\varepsilon))\,d\tau \circ (I + O(\varepsilon)),$$

wherefrom one concludes for $u_\varepsilon(t), v_\varepsilon(t)$, defined by formulae (8.11)-(8.14), the convergence of the flows: for each $r \geq 0$ and compact K

$$\left\| \overrightarrow{\exp}\int_0^t (X(x)u^e(\tau) + Y(x)v^e(\tau) + [X,Y](x)w^e(\tau))\,d\tau - \overrightarrow{\exp}\int_0^t (X(x)u_\varepsilon(\tau) + Y(x)v_\varepsilon(\tau))\,d\tau \right\|_{r,K} = O(\varepsilon)$$

as $\varepsilon \to 0$.

9. Proof of Theorem 4.3.

Proposition 9.1. *Under the assumptions of Theorem 4.3, for each $\varepsilon > 0$ there exists a finite set B (depending on ε) of the multiindices $\beta = (\beta_1, \dots, \beta_N)$ and an extended differential equation (5.1) together with an extended control $(v_\beta(t))_{\beta \in B}$, $t \in [0,T]$ such that the flow, generated by (5.1) and the control steers, in time T, the initial ensemble $\alpha(\theta)$ to the ensemble $x(T;\theta)$, for which $\sup_{\theta\in\Theta} d\,(x(T;\theta), \omega(\theta)) < \varepsilon$.*

Consider the diffeotopy $\gamma_t(\theta) = P_t(\alpha(\theta))$, along which Lie bracket C^0-approximating condition holds. Let Γ be its image and $Y_t(x)$ be the time-dependent vector field, which generates the diffeotopy. We start with the following technical Lemma.

Lemma 9.2. *Let assumptions of Theorem 4.3 hold. Then there exists $\lambda > 0$ and compact neighborhood $W_\Gamma \supset \Gamma$, such that for each $\varepsilon > 0$ there exists a finite set of multi-indices B together with continuous functions $(v_\beta(t))$, $\beta \in B$ such that $X_t(x) = \sum_{\beta\in B} v_\beta(t)X^\beta(x)$ satisfies:*

$$\|X_t(x)\|_{1,W_\Gamma} < \lambda, \ \|Y_t(\gamma_t(\theta)) - X_t(\gamma_t(\theta))\|_{C^0(\Theta)} < \varepsilon. \tag{9.1}$$

Proof of Lemma 9.2. According to the Lie bracket C^0-approximating assumption along the diffeotopy there exists $\lambda > 0$ and for each $t \in [0,T]$ and each $\varepsilon > 0$ a finite set B_t of multi-indices and the coefficients $c_\beta(t)$, $\beta \in B_t$, such that

$$\left\| \sum_{\beta\in B_t} c_\beta(t)X^\beta(x) \right\|_{1,W_\Gamma} < \lambda,$$

$$\left\| Y_t(\gamma_t(\theta)) - \sum_{\beta\in B_t} c_\beta(t)X^\beta(\gamma_t(\theta)) \right\|_{C^0(\Theta)} < \varepsilon. \tag{9.2}$$

As far as $Y_t(\gamma_t(\theta))$ and $X^\beta(\gamma_t(\theta))$ vary continuously with t, the estimate

$$\left\| Y_\tau(\gamma_\tau(\theta)) - \sum_{\beta \in B_t} c_\beta(t) X^\beta(\gamma_\tau(\theta)) \right\|_{C^0(\Theta)} < \varepsilon$$

is valid for $\tau \in \mathcal{O}_t$ - a neighborhood of t. The family $\mathcal{O}_t$ $(t \in [0, T])$ defines an open covering of $[0, T]$, from which we choose finite subcovering $\mathcal{O}_i = \mathcal{O}_{t_i},\ i = 1, \ldots, N$. Putting $B_i = B_{t_i},\ i = 1, \ldots, N$ we define $c_{i\beta} = c_\beta(t_i),\ \forall i = 1, \ldots, N,\ \forall \beta \in B_i$; put $B = \bigcup_{i=1}^N B_i$.

Choose a smooth partition of unity $\{\mu_i(t)\}$ subject to the covering $\{\mathcal{O}_i\}$. Put for each $\beta \in B,\ v_\beta(t) = \sum_{i=1}^N \mu_i(t) c_{i\beta}$; it is immediate to see that $v_\beta(t)$ are continuous. For

$$X_t(x) = \sum_{\beta \in B} v_\beta(t) X^\beta(x) \tag{9.3}$$

we conclude

$$\forall \theta \in \Theta: \ \|Y_t(\gamma_t(\theta)) - X_t(\gamma_t(\theta))\| =$$
$$\left\| \sum_{i=1}^N \mu_i(t) Y_t(\gamma_t(\theta)) - \sum_{i=1}^N \sum_{\beta \in B_i} \mu_i(t) c_{i\beta} X^\beta(\gamma_t(\theta)) \right\| \leq$$
$$\sum_{i=1}^N \mu_i(t) \left\| Y_t(\gamma_t(\theta)) - \sum_{\beta \in B_i} c_{i\beta} X^\beta(\gamma_t(\theta)) \right\| \leq \varepsilon \sum_{i=1}^N \mu_i(t) = \varepsilon.$$

The first of the estimates (9.1) is proved similarly.

Coming back to the proof of Proposition 9.1 we consider the evolution of the ensemble $\alpha(\theta)$ under the action of the flow generated by the vector field X_t, defined by (9.3). We estimate

$$\|x(t;\theta) - \gamma_t(\theta)\| = \left\| \int_0^t \left(X_\tau(x(\tau;\theta), v(\tau)) - Y_\tau(\gamma_\tau(\theta)) \right) d\tau \right\| \leq$$
$$\int_0^t \|X_\tau(x(\tau;\theta)) - X_\tau(\gamma_\tau(\theta))\| \, d\tau + \int_0^t \|X_\tau(\gamma_\tau(\theta)) - Y_\tau(\gamma_\tau(\theta))\| \, d\tau.$$

By virtue of (9.2) we obtain (whenever $x(t;\theta) \in W_\Gamma$):

$$\|x(t;\theta) - \gamma_t(\theta)\| \leq \lambda \int_0^t \|x(\tau;\theta) - \gamma_\tau(\theta)\| \, d\tau + \varepsilon t,$$

and by Gronwall lemma

$$\|x(t;\theta) - \gamma_t(\theta)\| \leq \varepsilon \frac{\left(e^{\lambda t} - 1\right)}{\lambda}. \tag{9.4}$$

We should take ε sufficiently small, so that (9.4) guarantees that $x(t;\theta)$ does not leave the neighborhood W_Γ, defined by Lemma 9.2. Then

$$\|x(T;\theta) - \omega(\theta)\| \leq \varepsilon \frac{\left(e^{\lambda T} - 1\right)}{\lambda}$$

and the claim of Proposition 9.1 follows.

Theorem 4.3 follows readily from Propositions 9.1 and Corollary 5.2.

10. Conclusions. Lie algebraic/geometric approach is well adapted to studying ensemble controllability and the controllability criteria obtained are formulated in Lie rank, or Lie span, form. Up to our judgement the study is not reducible to an application of abstract versions of Rashevsky-Chow theorem on a Banach manifold.

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