Chaotic dynamics in a periodically perturbed Liénard system *

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Abstract

We prove the existence of infinitely many periodic solutions, as well as the presence of chaotic dynamics, for a periodically perturbed planar Liénard system of the form $\dot{x} = y - F(x) + p(\omega t),\ \dot{y} = -g(x)$. We consider the case in which the perturbing term is not necessarily small. Such a result is achieved by a topological method, that is by proving the presence of a horseshoe structure.

1 Introduction

In this paper we study the presence of chaotic dynamics for time periodic perturbations of the Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1.1}$$

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where $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Usually, equation (1.1) can be written as the equivalent system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x) \end{cases}$$

in the phase-plane. Another classical approach consists in the study of (1.1) as the first order system in the *Liénard plane*:

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \tag{1.2}$$

by setting

$$F(x) := \int_0^x f(s)\, ds. \tag{1.3}$$

Observe that with the additional assumption that g is locally Lipschitz continuous, in virtue of the smoothness of $F(x)$ inherited by its integral form, the uniqueness for the solutions of the initial value problems associated with (1.2) is ensured. Therefore, these regularity hypotheses will be assumed throughout this work.

In a recent paper [13], Messias and Alves Gouveia have investigated the case

$$\begin{cases} \dot{x} = y - ax^2 + \varepsilon p(\omega t) \\ \dot{y} = -x \end{cases} \tag{1.4}$$

where a is a real constant, ε is a small real parameter and the perturbation function p is a $2\pi/\omega$-periodic differentiable function. In [13], using the Poincaré compactification of the plane in polar coordinates, the authors prove the existence of a homoclinic loop connecting a saddle point at infinity for the associated autonomous system

$$\begin{cases} \dot{x} = y - ax^2 \\ \dot{y} = -x. \end{cases} \tag{1.5}$$

This, in turns, allows to enter in a setting where the Melnikov method can be applied. To be more specific, we recall that in [13, Theorem 1] the existence of a transversal intersection between the stable and the unstable manifolds at the saddle point at infinity is proved for $\varepsilon \neq 0$ sufficiently small and $p(\cdot)$ an even smooth function.

In our work we consider a different approach based on the theory of *topological horseshoes*. Under the name of topological horseshoes, one usually means those techniques in which the geometric properties of the Smale's horseshoe are preserved under less restrictive assumptions (see, for instance the introduction in [10] and the references quoted therein). Our aim is to study a broad class of periodic perturbations of the Liénard system (1.2) of the form

$$\begin{cases} \dot{x} = y - F(x) - E(t) \\ \dot{y} = -g(x) \end{cases} \tag{1.6}$$

with F even and g odd. The main features of our result are the following:

- we do not assume that the perturbation $E(t)$ is small;
- we deal both with the case when the associated autonomous system (1.2) has either a global center or a local center surrounded by a homoclinic loop connecting a saddle point at infinity.

We restrict our analysis to a T-periodic stepwise forcing term E of the form

$$E(t) = \begin{cases} E_1, & \text{for } 0 \le t < \tau_1 \\ E_2, & \text{for } \tau_1 \le t < \tau_1 + \tau_2 \end{cases} \tag{1.7}$$

where

$$E_1 \neq E_2$$

and τ_1, τ_2 are positive constants with

$$\tau_1 + \tau_2 = T.$$

In this case, the nonautonomous Liénard system (1.6) appears as a periodic *switched system*, in the sense that the associated dynamics is the superposition of the two autonomous systems

$$\begin{cases} \dot{x} = y - F(x) - E_1 \\ \dot{y} = -g(x) \end{cases} \tag{1.8}$$

and

$$\begin{cases} \dot{x} = y - F(x) - E_2 \\ \dot{y} = -g(x) \end{cases} \tag{1.9}$$

which switch one to the other in a T-periodic manner.

Instead of studying the switched system made by (1.8)-(1.9), one could equivalently consider equation (1.1) perturbed by a T-periodic Dirac comb forcing term of the form

$$e = -A \sum_{n \in \mathbb{Z}} \left(\delta_{\tau_1 + nT} - \delta_{nT} \right), \quad A := E_2 - E_1$$

where, as usual, δ_a is the Dirac delta distribution concentrated at the point $a \in \mathbb{R}$. Such kind of forcing term arises in the study of discrete time signal analysis, where abrupt variation of the parameters is modelled by pulse trains described by Dirac combs (see, for instance the very recent paper [3] and the references quoted therein, as well as [2] for a Liénard equation perturbed by impulses).

Coming back to system (1.6) with $E(t)$ as (1.7), our main result can be presented as follows:

Theorem 1.1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ with f continuous and g locally Lipschitz continuous and satisfying $g(x)x > 0$ for all $x \neq 0$. Suppose that f is odd (and so F is even) and g is odd and, moreover, assume that the corresponding system* (1.2) *has a center which is not isochronous. Then system* (1.6) *has infinitely many periodic solutions as well as chaotic-like dynamics for any stepwise forcing term, provided that the switching times τ_1, τ_2 are sufficiently large. Moreover, the result is robust with respect to small perturbations, namely, once we have fixed τ_1 and τ_2, there exists $\delta > 0$ such that the result holds for each T-periodic forcing term $P(t)$ with $\int_0^T |P(t) - E(t)|\, dt < \delta$.*

Here, the result obtained has to be read in the light of Definition 3.1 in Section 3 and the precise features of the obtained chaotic dynamics are those of Theorem 3.1. We anticipate that we actually get the following property: *for each given positive integers m_1, m_2 there are τ_1^* and τ_2^* such that for any $\tau_1 > \tau_1^*$ and $\tau_2 > \tau_2^*$ the Poincaré map associated with system* (1.6) *presents a full dynamics on $4m_1m_2$ symbols on a compact invariant set.* An explanation of the dynamical features of the chaotic solutions is provided in Section 3.2 immediately after the conclusion of the proof of Theorem 1.1.

The plan of the paper is as follows.

In Section 2 we present the phase-portrait of the associated Liénard autonomous system, with a list of sufficient conditions ensuring that there is a region, around the origin, filled by closed orbits. We observe that this property holds, even if the system is not Hamiltonian, in virtue of the symmetry conditions on F and g which guarantee that the trajectories have a mirror symmetry with respect to the y-axis.

In the first part of Section 3 we introduce the method for producing chaotic dynamics using a linked twist map approach. The theory of linked twist maps is a powerful approach to reveal chaotic dynamics obtained by switching between two different maps which satisfy a twist condition on linked annuli (see [4] and the interesting surveys [1, 26] where the theory is applied to fluid dynamics). More recently this framework has been extended to topological annuli in [12] thus allowing more general configurations. In the second part of Section 3 we give new applications to the specific case of system (1.6).

The last section will be devoted to some remarks and examples showing the broad range of applicability of Theorem 1.1.

2 Phase-portrait of the autonomous system

Throughout this section we consider the autonomous system

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \tag{2.1}$$

with the standard regularity and symmetry assumptions, namely

$f : \mathbb{R} \to \mathbb{R}$ is continuous and odd, with $F(x) = \int_0^x f(s)\,ds$;

$g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and odd, with $g(x)x > 0$ for $x \neq 0$.

System (2.1) is equivalent to

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x) \end{cases} \tag{2.2}$$

in the phase-plane, via the nonlinear transformation $(x, y) \mapsto (x, y - F(x))$. Therefore a closed trajectory in one system is closed also in the other one and viceversa. For this reason we can exploit some results obtained in the literature from one or the other system. It is well known that the hypotheses on f and

g imply a mirror symmetry of the trajectories with respect to the y-axis and hence, any trajectory departing at the time $t = 0$ at a point $P_0 = (0, y_0)$ (for $y_0 > 0$) on the y-axis and hitting again the first time the y-axis at a point $P_1 = (0, y_1)$ (for $y_1 < 0$) is closed. For this reason, without loss of generality, we can say that the origin is a center when this situation occurs.

Indeed, it may happen that for some special mutual behavior of $F(x)$ and $g(x)$ near the origin, the condition of being a center may fail and homoclinic loops at the origin may appear, as the so-called "petal" or "figure eight". However, this will not influence our analysis, because it will be sufficient to consider the case in which there is a closed trajectory surrounding the origin and then an external annulus filled by periodic orbits. Clearly, such annulus can fill the rest of the plane. In any case, for sake of completeness, we briefly discuss the whole situation.

Sharp conditions for having a center date back from the classical results of Filippov [5] further developed by Opial [14] and, more recently in the Eighties by Hara and Yoneyama [9]. More precisely, if $F(x)$ has a definite sign in a right hand neighborhood of 0, for instance $F(x) > 0$ for $0 < x \leq a$, then the condition

$$\exists\, \alpha > \frac{1}{4} \;:\; \frac{1}{F(x)} \int_{0^+}^{x} \frac{g(\xi)}{F(\xi)}\, d\xi \geq \alpha, \quad \forall\, 0 < x \leq a \tag{2.3}$$

implies that both the systems have a (local) center at the origin (see [14, p. 71], [9, p. 178]). In order to produce a homoclinic loop at the origin it is sufficient to violate condition (2.3) by assuming

$$\frac{1}{F(x)} \int_{0^+}^{x} \frac{g(\xi)}{F(\xi)}\, d\xi \leq \frac{1}{4}, \quad \forall\, 0 < x \leq a \tag{2.4}$$

(see, again [14, p. 73], [9, p. 183]). In this light, we can consider as an example the equation

$$\ddot{x} + (x^3 + 4x)\dot{x} + x^3 + x^7 = 0. \tag{2.5}$$

The phase-portrait of such equation in both the phase-plane and the Liénard plane has the following feature (see Figure 1 for the phase-plane). Condition (2.4) holds near the origin and therefore trajectories departing at a point $(0, y_0)$ with $y_0 > 0$ sufficiently small tend to the origin and a homoclinic loop at the origin is produced. On the other hand, being the leading coefficient of $g(x)$ large enough, any trajectory intersecting the negative y-axis is actually closed, as we will check below. Therefore we can say that for equation (2.5) both systems (2.1) and (2.2) have the so-called *property* (H) (analyzed in [23]), namely they possess a closed trajectory γ surrounding the origin and containing in its interior all the critical points of the system and such that each trajectory outside γ is closed.

Hence the main problem is to detect if we are in presence of a local or a global center. This problem has been widely investigated in the literature, in particular for nonlinearities with a polynomial growth. A detailed discussion of this kind of problems with a comprehensive list of references may be found in [24]. In particular, if we suppose that $f(x) = p_{2m+1}(x)$ is an odd polynomial of degree $2m + 1$ and $g(x) = q_{2n+1}(x)$ is an odd polynomial of degree $2n + 1$, one can prove that, if $n \geq 2(m + 1)$, then in both systems (2.1) and (2.2) there is a global center, with the exception of the above mentioned remark. Indeed,

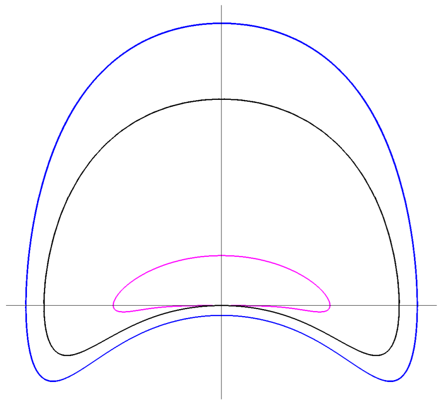

Figure 1: Phase-portrait for the equation (2.5) in the phase-plane. The aspect ratio is not the real one, for sake of simplicity.

according to [22] a global center (outside a compact neighborhood of the origin) can occur also for $n = 2m + 1$, depending on the leading coefficients.

The second possibility to be investigated is when the center is only local and a separatrix appear. Here the situation is more delicate because, a priori, we cannot exclude the possibility of blow-up phenomena, as, for instance for the equation

$$\ddot{x} + 3x\dot{x} + x(x^2 + 1) = 0,$$

which has $x(t) = -\tan(t)$ as unbounded solution run in finite time (see [24, p. 395]). Indeed, as observed in [24], in the phase plane, such equation produces the portrait of a center (at the origin) which is unbounded in the x-direction and for $y > 0$, while it is bounded below by a trajectory ($y = -x^2 - 1$) lying on the negative y-plane. A phase-plane inspection shows the existence of a separatrix between the trajectory $y = -x^2 - 1$ and the isocline $y = -(x^2 + 1)/3$. For our purposes it is crucial to control the nonexistence of blow-up phenomena for the separatrix. With this respect, a sufficient condition is

$$\limsup_{x \to +\infty} \frac{g(x)}{|f(x)|} < +\infty, \tag{2.6}$$

according to [22].

Finally we observe that there are other ways for producing a center, without assuming the natural mirror symmetry inherited by the oddness conditions on f and g. Following once again Opial [14] and Hara and Yoneyama [9] we call a center of type (S) the one obtained by the mirror symmetry as above. According to [9] (and following Filippov [5] and Sansone and Conti [21]), we call a center of type (F) provided that

$$F(G^{-1}(-x)) = F(G^{-1}(x)), \quad \forall x > 0, \tag{2.7}$$

where

$$G(x) := \int_0^x g(\xi)\, d\xi.$$

The type (F) is a generalization of the type (S). If the system (2.1) is of type (F), then the orbits have "deformed mirror symmetry" with respect to the y-axis.

3 Proof of the main result

3.1 General framework

The result that we are going to present is actually a generalization of what has been obtained in [12]. Indeed, in [12] a theorem about the presence of full symbolic dynamics for the Poincaré map on two topologically linked annuli was proved and applications were given to planar Hamiltonian systems. Therein a result in the framework of linked twist maps theory was produced under two crucial assumptions: a linked annuli hypothesis and a twist maps condition.

Our new contribution here consists in improving the linked annuli hypothesis. Moreover, our result is suited for applications to planar systems which are not necessarily Hamiltonian.

We start with some basic definitions borrowed from [17, 18]. The notation is the same as in [12] for sake of comparison. Given any continuous curve $\gamma : [a; b] \to \mathbb{R}^2$ we will use the symbol γ to denote also the image set $\gamma([a; b])$ of the curve. An *arc* is the homeomorphic image of a compact interval. An *oriented rectangle* is a pair $\widehat{R} = (R, R^-)$ where $R \subset \mathbb{R}^2$ is a bounded planar region homeomorphic to the unit square $[0; 1] \times [0; 1]$ and $R^- := R_{\mathrm{left}} \cup R_{\mathrm{right}}$ with R_{left}, R_{right} two disjoint arcs in ∂R.

Let $\phi : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map, let $\widehat{R}_1, \widehat{R}_2$ be two oriented rectangles and $H \subseteq R_1 \cap D$ be a compact set. We say that the pair (H, ϕ) *stretches* R_1 to R_2 *along the paths* and write $(H, \phi) : \widehat{R}_1 \Bumpeq\!\!\!\!\longrightarrow \widehat{R}_2$ if for every continuous curve $\gamma : [a; b] \to R_1$ with $\gamma(a) \in R_{\mathrm{left}}$ and $\gamma(b) \in R_{\mathrm{right}}$ (or viceversa) there exists a subinterval $[a_1; b_1] \subseteq [a; b]$ such that $\gamma(t) \in H$ and $\phi(\gamma(t)) \in R_2$ for all $t \in [a_1; b_1]$ and, moreover, $\phi(\gamma(a_1)), \phi(\gamma(b_1))$ belong to different components of R_2^-. Sometimes the reference to the compact set H will be omitted in order to simplify the notation. If $m \geq 1$ is an integer, we write $\phi : \widehat{R}_1 \overset{m}{\Bumpeq\!\!\!\!\longrightarrow} \widehat{R}_2$ and say that ϕ *stretches* $\widehat{R}_1$ *to* $\widehat{R}_2$ *m times* if there are m pairwise disjoint compact sets $H_1, \ldots, H_m \subseteq R_1 \cap D$ such that $(H_i, \phi) : \widehat{R}_1 \Bumpeq\!\!\!\!\longrightarrow \widehat{R}_2$ for $i = 1, \ldots, m$.

The property of stretching along the paths (SAP) is preserved by composition of maps and, in particular, it can be applied to the iterates of a given map ϕ. When $(H, \phi) : \widehat{R} \Bumpeq\!\!\!\!\longrightarrow \widehat{R}$, we know that there is at least a fixed point for ϕ in H (see [17, Theorem 3.9]). Moreover, if $\phi : \widehat{R} \overset{m}{\Bumpeq\!\!\!\!\longrightarrow} \widehat{R}$ for some $m \geq 2$, then ϕ induces a chaotic dynamics on m symbols according to the following definition.

Definition 3.1. There are $m \geq 2$ pairwise disjoint sets $H_1, \ldots, H_m \subset D$ such that for each two-sided sequence $(s_i)_{i \in \mathbb{Z}}$, where each s_i ranges in $\{1, \ldots, m\}$, there exists a corresponding sequence $(w_i)_{i \in \mathbb{Z}}$ of points in D with

$$w_i \in H_{s_i} \qquad \text{and} \qquad w_{i+1} = \phi(w_i) \quad \forall i \in \mathbb{Z}. \tag{3.1}$$

Moreover, whenever $(s_i)_{i \in \mathbb{Z}}$ is a k-periodic sequence, then there exists a k-periodic sequence $(w_i)_{i \in \mathbb{Z}}$ satisfying (3.1).

If ϕ is one-to-one (as in the case of the Poincaré map), a consequence of the above definition is that there exists a compact invariant set $\Lambda \subset R \cap D$

such that $\phi|_\Lambda$ is semi-conjugate to the two-sided Bernoulli shift on m symbols. Furthermore Λ contains as a dense subset the periodic points of ϕ and the counterimage (by the semiconiugacy) of any periodic sequence $(s_i)_{i\in\mathbb{Z}}$ of symbols contains a periodic point of ϕ having the same period of $(s_i)_{i\in\mathbb{Z}}$. The semiconiugacy with the Bernoulli shift is a typical feature associated to chaotic dynamics and, in particular, it implies the positivity of the topological entropy. We refer to [12, 17] for more details.

In the present article we apply this method via the following lemma taken from [19, Theorem 3.2].

Lemma 3.1. *Let m_1, m_2 be positive integers.*

1. *If $\phi_1 : \widehat{R}_1 \stackrel{m_1}{\Bumpeq\!\!\!\!\longrightarrow} \widehat{R}_2$ and $\phi_2 : \widehat{R}_2 \stackrel{m_2}{\Bumpeq\!\!\!\!\longrightarrow} \widehat{R}_3$, then $\phi_2 \circ \phi_1 : \widehat{R}_1 \stackrel{m_1 m_2}{\Bumpeq\!\!\!\!\longrightarrow} \widehat{R}_3$.*

2. *If*

$$\begin{aligned} &\phi_1 : \widehat{R}_1 \stackrel{m_1}{\Bumpeq\!\!\!\!\longrightarrow} \widehat{R}'_2, \qquad &&\phi_1 : \widehat{R}_1 \stackrel{m_1}{\Bumpeq\!\!\!\!\longrightarrow} \widehat{R}''_2, \\ &\phi_2 : \widehat{R}'_2 \stackrel{m_2}{\Bumpeq\!\!\!\!\longrightarrow} \widehat{R}_1, \qquad &&\phi_2 : \widehat{R}''_2 \stackrel{m_2}{\Bumpeq\!\!\!\!\longrightarrow} \widehat{R}_1, \end{aligned}$$

then $\phi_2 \circ \phi_1$ induces chaotic dynamics on $2m_1m_2$ symbols in R_1.

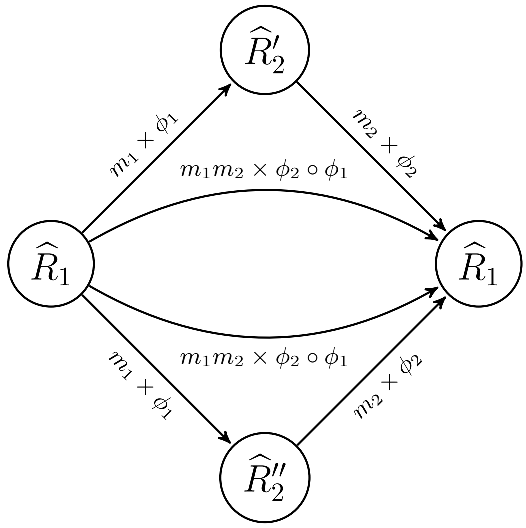

Figure 2: Graph associated with the situation described in Lemma 3.1.

Next we introduce the specific concepts which will be useful for the present paper.

If γ is a (closed) Jordan curve in $\mathbb{R}^2$ we let $\mathcal{I}(\gamma)$ and $\mathcal{E}(\gamma)$ be respectively the bounded open connected component and the unbounded one of $\mathbb{R}^2 \setminus \gamma$. Let γ_1, γ_2 be two Jordan curves in $\mathbb{R}^2$ such that $\gamma_1 \subset \mathcal{I}(\gamma_2)$, then the set $A := \overline{\mathcal{I}(\gamma_2)} \setminus \mathcal{I}(\gamma_1)$ is a (closed) topological annulus and we set

$$\mathcal{I}(A) := \mathcal{I}(\gamma_1), \quad \mathcal{E}(A) := \mathcal{E}(\gamma_2), \quad \partial^i A := \gamma_1 \quad \text{and} \quad \partial^e A := \gamma_2.$$

A *ray* in A is any simple continuous curve $\gamma : [a;b] \to A$ such that either $\gamma(a) \in \partial^i A$ and $\gamma(b) \in \partial^e A$ or, viceversa, $\gamma(a) \in \partial^e A$ and $\gamma(b) \in \partial^i A$. The following lemma shows that w.l.o.g. it is always possible to consider rays whose endpoints are the only points of theirs lying on the boundary of A.

Lemma 3.2. *Let A be a topological annulus and $\gamma : [a;b] \to \mathbb{R}^2$ be a continuous path such that $\gamma(a) \in \overline{\mathcal{I}(A)}$ and $\gamma(b) \in \overline{\mathcal{E}(A)}$. Then, there is $[c;d] \subseteq [a;b]$ such that $\gamma(c) \in \partial^i A$, $\gamma(d) \in \partial^e A$ and $\gamma(s)$ lies in the interior of A for all $s \in]c;d[$. If γ is also simple then $\gamma|_{[c;d]}$ is a ray in A.*

A similar statement holds whenever $\gamma(a) \in \overline{\mathcal{E}(A)}$ and $\gamma(b) \in \overline{\mathcal{I}(A)}$.

Proof. In the case that $\gamma(a) \in \overline{\mathcal{I}(A)}$ and $\gamma(b) \in \overline{\mathcal{E}(A)}$, it's enough to define:

$$c := \sup\{s \in [a;b] : \gamma(s) \in \partial^i A\} \quad \text{and} \quad d := \inf\{s \in [c;b] : \gamma(s) \in \partial^e A\}.$$

Similarly one can deal with the other situation. □

Two disjoint rays γ, η in an annulus A divide it into two sets which are homeomorphic to a rectangle and can be oriented by choosing the two rays as "vertical sides". We describe here the procedure. In order to fix ideas, we assume w.l.o.g. that:

- the two rays are parametrized so that $\gamma, \eta : [0;1] \to A$ with $\gamma(0), \eta(0) \in \partial^i A$ and $\gamma(1), \eta(1) \in \partial^e A$;
- $\partial^i A$ and $\partial^e A$ are parametrized by two Jordan curves $\delta^i, \delta^e : [0;1] \to \mathbb{R}^2$ with $\delta^i(0) = \delta^i(1) = \gamma(0)$, $\delta^e(0) = \delta^e(1) = \gamma(1)$, $\delta^i(1/2) = \eta(0)$ and $\delta^e(1/2) = \eta(1)$; moreover δ^i and δ^e are taken in the same homotopic class of the fundamental group of $\mathbb{R}^2 \setminus \{O\}$, with $O \in \mathcal{I}(A)$ (so that they run around O with the same orientation).

Then, the two topological rectangles we are interested in have boundaries given by the following Jordan curves:

$$\gamma * \delta^e|_{[0;1/2]} * \eta^{-1} * (\delta^i|_{[0;1/2]})^{-1} \quad \text{and} \quad \gamma * (\delta^e|_{[1/2;1]})^{-1} * \eta^{-1} * \delta^i|_{[1/2;1]}$$

where $*$ stands for the concatenation of curves and the exponent -1 is used to reverse the orientation of a curve. If R is any one of the two rectangles, then we define its vertical sides $R_{\text{left}} = \gamma$ and $R_{\text{right}} = \eta$ and we consider the corresponding oriented rectangle:

$$\widehat{R} := (R, R_{\text{left}} \cup R_{\text{right}}) = (R, \gamma \cup \eta). \tag{3.2}$$

Definition 3.2. Two annuli $A_1, A_2 \subset \mathbb{R}^2$ are *topologically linked* if there exist two rays γ_1 and γ_2 of A_1 and A_2, respectively, such that $\gamma_1 \subset \mathcal{I}(A_2)$ and $\gamma_2 \subset \mathcal{I}(A_1)$.

Proposition 3.1. *If the annuli A_1, A_2 are topologically linked, then:*

1. *$\mathcal{I}(A_1) \cap \mathcal{I}(A_2) \neq \emptyset$;*
2. *$\mathcal{I}(A_1) \cap \mathcal{E}(A_2) \neq \emptyset \neq \mathcal{E}(A_1) \cap \mathcal{I}(A_2)$;*
3. *there exist two rays η_1 and η_2 of A_1 and A_2, respectively, such that $\eta_1 \subset \mathcal{E}(A_2)$ and $\eta_2 \subset \mathcal{E}(A_1)$.*

Proof. Let $\gamma_1 : [0;1] \to A_1$ and $\gamma_2 : [0;1] \to A_2$ be two rays such that $\gamma_1 \subset \mathcal{I}(A_2)$ and $\gamma_2 \subset \mathcal{I}(A_1)$ and assume w.l.o.g. that $\gamma_1(0) \in \partial^i A_1$. We choose $r > 0$ such that $r < \text{dist}(\gamma_1(0), \partial^e A_1)$ and $r < \text{dist}(\gamma_1(0), \partial^i A_2)$. This choice grants that $B(\gamma_1(0), r) \subset \mathcal{I}(A_2)$ and $B(\gamma_1(0), r) \cap \overline{\mathcal{E}(A_1)} = \emptyset$, thus in particular one has that:

$$\mathcal{I}(A_1) \cap \mathcal{I}(A_2) \supset \mathcal{I}(A_1) \cap B(\gamma_1(0), r) \neq \emptyset.$$

Since $\gamma_1(1) \in \partial^e A_1 \cap \mathcal{I}(A_2)$, we can choose $r > 0$ such that $r < \text{dist}(\gamma_1(1), \partial^i A_2)$ and $r < \text{dist}(\gamma_1(1), \partial^i A_1)$ and deduce that $B(\gamma_1(1), r) \subset \mathcal{I}(A_2)$ and $B(\gamma_1(1), r) \cap \overline{\mathcal{I}(A_1)} = \emptyset$ and, therefore:

$$\mathcal{E}(A_1) \cap \mathcal{I}(A_2) \supset \mathcal{E}(A_1) \cap B(\gamma_1(1), r) \neq \emptyset.$$

Similarly one can show that $\mathcal{I}(A_1) \cap \mathcal{E}(A_2) \neq \emptyset$.

Now we can choose two points $p \in \mathcal{E}(A_1) \cap \mathcal{I}(A_2)$ and $q \in \mathcal{E}(A_1) \cap \mathcal{E}(A_2)$ (which is clearly non-empty). Since $\mathcal{E}(A_1)$ is open and connected, it is also path-connected and there is a continuous simple curve $\eta : [a;b] \to \mathcal{E}(A_1)$ such that $\eta(a) = p$ and $\eta(b) = q$. Since $\eta(a) \in \mathcal{I}(A_2)$ and $\eta(b) \in \mathcal{E}(A_2)$ then by Lemma 3.2 there is an interval $[c;d] \subset [a;b]$ such that $\eta_2 := \eta|_{[c;d]}$ is a ray of A_2. One can argue in a similar way to show the existence of η_1. $\square$

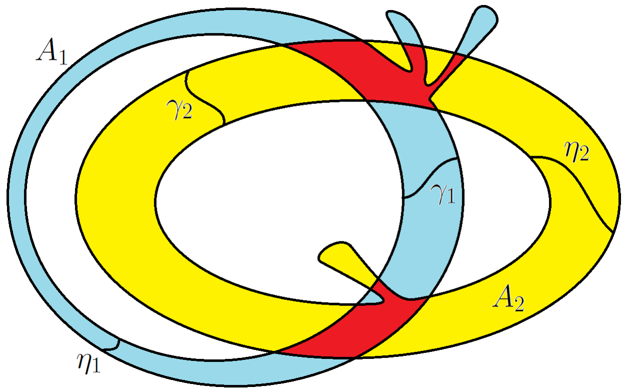

Figure 3: Example of topologically linked annuli with the curves γ_i, η_i which divide them into rectangles. Observe that the two components of $A_1 \cap A_2$ (red/darker regions) do not give rise to oriented rectangles as required in the paper [12] and, therefore, these two annuli do not fit into the frame of [12, Definition 3.1, Theorem 3.1].

When two annuli A_1, A_2 are topologically linked we will use the curves γ_i (given by Definition 3.2) and η_i (given by Lemma 3.1), for $i = 1, 2$, to determine two oriented rectangles in each annulus according to (3.2). Next we show how a twist condition like the one in [12] implies that each one of the oriented rectangle in an annulus is "stretched" across any oriented rectangle of the other annulus a suitable number of times.

Let

$$\Pi_i = \Pi_i(\theta, \alpha) : \widetilde{A}_i := \mathbb{R} \times [a_i; b_i] \to A_i \qquad i = 1, 2$$

be a covering projection for the annulus A_i where the variables θ, α generalize the angle and radius of the classical polar coordinates. Without loss of generality we assume the following properties:

- $\Pi_i(\theta+1,\alpha)=\Pi_i(\theta,\alpha)$ for all $\theta\in\mathbb{R}$ and all $\alpha\in[a_1;b_i]$;
- $[a_i;b_i]\ni\alpha\mapsto\Pi_i(k,\alpha)$ parametrizes γ_i and $[a_i;b_i]\ni\alpha\mapsto\Pi_i(k+1/2,\alpha)$ parametrizes η_i for each $k\in\mathbb{Z}$.

In particular, for each $k\in\mathbb{Z}$ the covering map Π_i homeomorphically transforms the rectangles $[k;k+1/2]\times[a_i;b_i]$ and $[k+1/2;k+1]\times[a_i;b_i]$ into the two topological rectangles in which A_i is divided by the linking condition. The rectangles in the covering space inherit the orientation of the topological rectangles in an obvious way through Π_i.

Let $\phi_i:A_i\to A_i$, for $i=1,2$, be continuous maps which admit liftings $\widetilde{\phi}_i:\widetilde{A}_i\to\widetilde{A}_i$ such that

$$\Pi_i\circ\widetilde{\phi}_i=\phi_i\circ\Pi_i\quad\text{and}\quad\widetilde{\phi}_i(\theta,\alpha)=(\theta+g_i(\theta,\alpha),R_i(\theta,\alpha))$$

where $g_i,R_i:\widetilde{A}_i\to\mathbb{R}$ are continuous functions which are 1-periodic w.r.t. their first variable. We assume what follows:

(BI) *Boundary Invariance:* $R_i(\theta,a_i)=a_i$ and $R_i(\theta,b_i)=b_i$ for all $\theta\in\mathbb{R}$.

(TC) *Twist Condition:* there exists $k_i,j_i\in\mathbb{Z}$ such that $j_i\le k_i$ and either

$$\max_{\theta\in[0;1]}g_i(\theta,a_i)\le-1+j_i\quad\text{and}\quad\min_{\theta\in[0;1]}g_i(\theta,b_i)\ge1+k_i$$

or

$$\min_{\theta\in[0;1]}g_i(\theta,a_i)\ge1+k_i\quad\text{and}\quad\max_{\theta\in[0;1]}g_i(\theta,b_i)\le-1+j_i.$$

In this framework the following result holds, which provides an extension of [12, Theorem 3.1].

Theorem 3.1. *Let $A_1,A_2\subset\mathbb{R}^2$ be topologically linked annuli and $\phi_i:A_i\to A_i$, for $i=1,2$, be two continuous maps that satisfy (BI) and (TC). For $i=1,2$ let $\widehat{R}_i$ be anyone of the two oriented rectangles determined by the linking condition in A_i. Then:*

$$\phi_1:\widehat{R}_2\overset{m_1}{\Longleftrightarrow}\widehat{R}_1\qquad\textit{and}\qquad\phi_2:\widehat{R}_1\overset{m_2}{\Longleftrightarrow}\widehat{R}_2\qquad\textit{with } m_i:=k_i-j_i+1.$$

In particular $\phi_2\circ\phi_1$ have at least four fixed points and induces chaos on $4m_1m_2$ symbols.

Proof. We have only to show that

Claim: if ϕ_1 satisfies the first twist condition in (TC), then $\phi_1:\widehat{R}_2\overset{m_1}{\Longleftrightarrow}\widehat{R}_1$ where $R_1=\Pi_1([0;1/2]\times[a_1;b_1])$ and R_2 is anyone of the two rectangles in A_2.

All the other situations can be checked in the same manner. Once this step is achieved, we get the conclusion of the proof by Lemma 3.1.

In order to prove our claim we argue as follows. Thanks to the properties of liftings, without loss of generality we can assume that actually

$$\max_{\theta\in[0;1]}g_1(\theta,a_1)\le-1\quad\text{and}\quad\min_{\theta\in[0;1]}g_1(\theta,b_1)\ge m_1\tag{3.3}$$

and we will exploit the fact that the this twist condition implies that, roughly speaking, $\widetilde{\phi}_1$ maps a lifted copy of R_1 across m_1 components of $\Pi_1^{-1}(R_1)$. More precisely, let us consider the compact set

$$\widetilde{K}_1 := \Pi_1^{-1}(A_1 \cap R_2) \cap [0;1] \times [a_1;b_1] \neq \emptyset$$

and observe that $\widetilde{K}_1 \cap \{0, 1/2, 1\} \times [a_1;b_1] = \emptyset$ since $R_2 \cap (\gamma_1 \cup \eta_1) = \emptyset$ by construction. In particular we have that $\widetilde{K}_1 \subset]0;1[\times [a_1;b_1]$ and that $\Pi_1|_{\widetilde{K}_1}$ is a homeomorphism onto $A_1 \cap R_2$. Let us consider the following pairwise disjoint rectangles in the covering space $\widetilde{A}_1$:

$$\widetilde{S}_\ell := \left[\ell - 1; \ell - \frac{1}{2}\right] \times [a_1;b_1] = (\ell - 1, 0) + \left[0; \frac{1}{2}\right] \times [a_1;b_1] \qquad \text{for } \ell \in \mathbb{Z}$$

and observe that $\Pi_1(\widetilde{S}_\ell) = R_1$ for all $\ell \in \mathbb{Z}$. Let

$$\widetilde{H}_\ell := \left\{(\theta, \alpha) \in \widetilde{K}_1 : \widetilde{\phi}_1(\theta, \alpha) \in \widetilde{S}_\ell\right\} = \widetilde{K}_1 \cap \widetilde{\phi}_1^{-1}\left(\widetilde{S}_\ell\right) \quad \text{for } \ell = 1, \dots, m_1$$

which are m_1 pairwise disjoint compact subsets of $\widetilde{K}_1$. Thus, the sets:

$$H_\ell := \Pi_1(\widetilde{H}_\ell) \qquad \ell = 1, \dots, m_1$$

are pairwise disjoint compact subsets of $A_1 \cap R_2$. We'll show that they are not empy and that

$$(H_\ell, \phi_1) : \widehat{R}_2 \Longleftrightarrow\!\!\!\!\!\rightarrow \widehat{R}_1 \qquad \text{for all } \ell = 1, \dots, m_1. \tag{3.4}$$

Let $\delta : [0;1] \to R_2$ be a continuous curve such that $\delta(0) \in \gamma_2$ and $\delta(1) \in \eta_2$. By Lemma 3.2 there exists $[s_0;t_0] \subset [0;1]$ such that $\delta_1 := \delta|_{[s_0;t_0]}$ is a continuous curve in $A_1 \cap R_2$ such that $\delta_1(s_0) \in \partial^i A_1$ and $\delta_1(t_0) \in \partial^e A_1$. Moreover, δ_1 crosses neither γ_1 nor η_1 by construction. As a consequence δ_1 admits a (unique) continuous lifting $\widetilde{\delta}_1 : [s_0;t_0] \to \widetilde{K}_1$. We can be a little more precise if we write $\widetilde{\delta}_1(t) = (\theta_1(t), r_1(t))$ since we have now that $r_1(s_0) = a_1$, $r_1(t_0) = b_1$ and either $\theta_1([s_0;t_0]) \subset]0;1/2[$ or $\theta_1([s_0;t_0]) \subset]1/2;1[$. Let's check the first component of $\widetilde{\phi}_1$ at the endpoints of $\widetilde{\delta}_1$ in the case that (3.3) holds:

$$\theta_1(s_0) + g_1(\theta_1(s_0), a_1) < 0 \qquad \text{and} \qquad \theta_1(t_0) + g_1(\theta_1(t_0), b_1) > m_1$$

which implies that the projection of $\widetilde{\phi}_1(\widetilde{\delta}_1)$ on the θ-axis strictly contains the interval $[0;m_1]$ and, roughly speaking, means that the curve $\phi_1(\delta_1)$ makes at least m_1 complete turns in A_1 around $\mathcal{I}(A_1)$ and crosses the rectangle R_1 at least the same number of times. More precisely, it is possible to determine numbers $s_k, t_k \in [s_0;t_0]$ for $k = 1, \dots, m_1$ such that

$$s_1 < t_1 < s_2 < t_2 < \cdots < s_{m_1} < t_{m_1}$$

and

$$\begin{cases} \theta_1(s_\ell) + g_1(\widetilde{\delta}_1(s_\ell)) = \ell - 1, \\ \theta_1(t_\ell) + g_1(\widetilde{\delta}_1(t_\ell)) = \ell - 1/2 \\ \ell - 1 < \theta_1(t) + g_1(\widetilde{\delta}_1(t)) < \ell - 1/2 \quad \forall t \in]s_\ell; t_\ell[\end{cases} \qquad \text{for all } \ell = 1, \dots, m_1$$

which means that

$$\begin{cases} \phi_1(\delta(s_\ell)) \in \gamma_1, \\ \phi_1(\delta(t_\ell)) \in \eta_1 \\ \phi_1(\delta(t)) \in R_1 \quad \forall t \in [s_\ell; t_\ell] \\ \delta(t) \in H_\ell \quad \forall t \in [s_\ell; t_\ell] \end{cases} \qquad \text{for all } \ell = 1, \dots, m_1$$

and (3.4) follows. □

Remark 3.1. The main difference between our Theorem 3.1 and [12, Theorem 3.1] lies in the kind of linkage which is required to be satisfied by the two annuli $\mathcal{A}_1$ and $\mathcal{A}_2$. In the paper [12] the two annuli are required to be *linked through a topological rectangle* R, which means that there should be a connected component of $\mathcal{A}_1 \cap \mathcal{A}_2$ which is homeomorphic to a rectangle and whose boundary is the concatenation of four arcs, two on $\partial\mathcal{A}_1$ and the remaining two on $\partial\mathcal{A}_2$. In particular, each arc of $\partial\mathcal{A}_1$ has to be followed by an arc of $\partial\mathcal{A}_2$ and viceversa, which, roughly speaking, means that the two annuli should intersect somewhere in a neat, transversal way. For example, the two annuli depicted in Figure 3 are not linked through a topological rectangles according to [12, Definition 3.1]. Now, the paper [12] deals with planar Hamiltonian systems and the annuli which are considered there have closed level lines of the Hamiltonian as boundaries. Using the kind of arguments employed in [6, Section 3] it is possible to construct Hamiltonians whose level lines are not starshaped and show a behavior like that in Figure 3.

We observe that it might be possible to prove Theorem 1.1 also by using in a suitable way [12, Theorem 3.1], but our Theorem 3.1 allows us to swiftly prove our main result without worrying about the way in which the orbits of the shifted systems intersect each other. We just have to check that our new notion of linkage in Definition 3.2 is satisfied and this turns out to be quite easy (see next Subsection 3.2). This greatly simplifies the treatment in all the applications.

3.2 Proof of Theorem 1.1

We are now in position to apply our geometric approach for producing the desired chaotic dynamics, being systems (1.8) and (1.9) merely a translation of system (2.1).

Just to fix the ideas, let us suppose $E_1 < E_2$ (the other case being completely symmetric). As previously observed, with the special choice of $E(t)$ as in (1.7), system (1.6) switches periodically between (1.8) and (1.9). Therefore, it is convenient analyze the auxiliary system

$$(S_E) \qquad \begin{cases} \dot{x} = y - F(x) - E \\ \dot{y} = -g(x) \end{cases}$$

where the constant E is treated as a parameter. As a result we obtain that the phase-portraits of all the systems (S_E) are just translations of (S_0) (which is actually (2.1)) in the vertical direction with the origin $(0,0)$ shifted at the point $(0,E)$. For this reason, without loss of generality, we can suppose, from now on that

$$E_1 = 0 \quad \text{and} \quad E_2 > 0.$$

By the assumption of non-isochronicity, which plays a crucial role, we take two closed orbits of (S_0) having different period. Let us call these trajectories Γ_0' and Γ_0'' with Γ_0' internal to Γ_0''. In the same light, we call σ' and σ'' the corresponding periods. It will be not restrictive to suppose

$$\sigma' < \sigma'',$$

as the other situation can be treated similarly.

Our configuration determines four intersection points of the above considered orbits with the y-axis: $(0, a), (0, b), (0, c), (0, d)$ with

$$a < b < 0 < c < d.$$

Notice that we can always select the two closed orbits Γ' and Γ'' in such a way that

$$\max\{b - a, d - c\} < c - b. \tag{3.5}$$

Indeed, as a consequence of the non-isochronicity and the continuity of the period function with respect to the initial data, one can always select two orbits with different periods which are sufficiently close to each other. We also denote by $\mathcal{W}_0$ the closed annular region bounded by Γ_0' and Γ_0''.

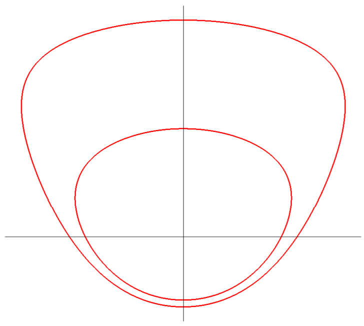

Figure 4: Phase-portraits in the Liénard plane of (S_0). For this example we have taken $f(x) = \frac{3}{5}x|x|$, $f(x) = \frac{1}{5}|x^3|$ and $g(x) = x$. The aspect ratio is not the real one, for sake of simplicity.

After these preliminary positions, we can start the proof of Theorem 1.1.

The compact annular region $\mathcal{W}_0$ is invariant for the dynamical system associated with (S_0). Hence, for every $\tau > 0$ its Poincaré map $\Phi_0^{0,\tau} : \mathcal{W}_0 \to \mathcal{W}_0$ is a well defined homeomorphism. Using the fact that $\sigma' < \sigma''$ we get that $\Phi_0^{0,\tau}$ may determine an arbitrarily large twist between Γ_0' and Γ_0'' provided that $\tau > 0$ is large enough.

Next, we take an arbitrary E_2 satisfying the condition

$$(I) \qquad \max\{b - a, d - c\} < E_2 < c - b$$

and consider the closed curves Γ_{E_2}' and Γ_{E_2}'' as well as the annular region $\mathcal{W}_{E_2}$ corresponding to system (S_A). Observe that σ' and σ'' are also the fundamental periods of Γ_{E_2}' and Γ_{E_2}'', respectively. Similarly as above, the compact annular region $\mathcal{W}_{E_2}$ is invariant for the dynamical system associated with (S_{E_2}).

Hence, for every $\tau > 0$ its Poincaré map $\Phi_{E_2}^{0,\tau} : \mathcal{W}_{E_2} \to \mathcal{W}_{E_2}$ is a well defined homeomorphism. Actually, we have that $\Phi_{E_2}^{0,\tau} = \Pi \circ \Phi_0^{0,\tau} \circ \Pi^{-1}$, where $\Pi : (x, y) \mapsto (x, y + E_2)$.

Figure 5 below shows the obtained overlapping. Condition (I) guarantees that $\mathcal{W}_0$ and $\mathcal{W}_{E_2}$ are two linked annuli.

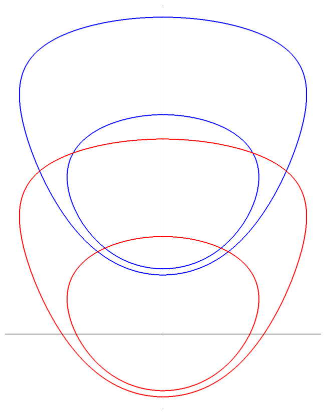

Figure 5: Phase-portraits in the Liénard plane showing the overlapping of two different systems (S_E). For this example we have taken $f(x) = \frac{3}{5}x|x|$, $F(x) = \frac{1}{5}|x^3|$ and $g(x) = x$. The overlapping is obtained by shifting (S_0) to (S_{E_2}) for $E_2 = 2.5$. The orbits Γ_0' and Γ_0'' of (S_0) are determined by choosing $c = 2$ and $d = 4$. The highlighted bold segments correspond to the arcs γ_i and η_i. The aspect ratio is not the real one, for sake of clarity.

Clearly, the Poincaré map $\Phi^{0,T}$ associated with the non-autonomous system (1.6) can be represented as

$$\Phi^{0,T} = \Phi_{E_2}^{0,\tau_2} \circ \Phi_0^{0,\tau_1},$$

wherever defined. In this manner, provided that τ_1 and τ_2 are sufficiently large, we can apply Theorem 3.1 and, therefore, the proof is complete.

The dynamics obtained for equation (1.6) by Theorem 3.1 are as follows. There are globally defined solutions that make a suitably prescribed number of turns around the corresponding equilibrium in each interval in which $E(t)$ remains constant, more precisely:

- $\ell + i_k$ turns in each interval $[kT, kT + \tau_1]$, $k \in \mathbb{Z}$, with $i_k \in \{1, \ldots, m_1\}$;
- $\ell + j_k$ turns in each interval $[kT + \tau_1, kT + T]$, $k \in \mathbb{Z}$, with $j_k \in \{1, \ldots, m_2\}$.

Here, the number $\ell \in \mathbb{N}$ depends on σ', τ_1 and τ_2, while the sequences $\{i_k\}_{k\in\mathbb{Z}}$ and $\{j_k\}_{k\in\mathbb{Z}}$ can be arbitrarily chosen a priori within the specified ranges. Moreover, it is possible to chose in which half of the respective annulus the solution should stop at the times $kT + \tau_1$ and kT in the Liénard plane (recall that our annuli are divided into two topological rectangles by the y-axis, see figure 5).

Remark 3.2. We point out that for simplicity of exposition we assumed in Theorem 1.1 that system (1.2) has a non-iscocronous center. In fact, as we see from its proof, we just need to assume that system (1.2) possesses two closed orbits of different periods.

4 Remarks and examples

In this section we present different examples in order to put in evidence the applicability of our result.

As a a first case of system in the Liénard plane, we consider the equation

$$\begin{cases} \dot{x} = y - ax^2 + \varepsilon p(\omega t) \\ \dot{y} = -x \end{cases} \tag{4.1}$$

where $a > 0$ is a fixed constant, $\varepsilon > 0$ is a real parameter and $p(\theta)$ is a given non-constant periodic function whose period is, for instance, 2π. In [13, Theorem 1], using Melnikov theory the authors proved the existence of chaotic dynamics for an *even* and smooth function $p(\theta)$ and ε sufficiently small. It is also observed that when $p(\theta)$ is odd, the stable and unstable manifolds associated to the saddle point at infinity may not intersect and therefore the method does not apply. Conversely, our result applies with no restriction on ε or symmetry conditions on the forcing term. On the other hand, we require ω small (so that the period will be large) and $p(\theta)$ close to a stepwise function.

One could observe that the Melnikov method requires to compute some integrals which require the knowledge of an analytic expression for the homoclinic orbit, which, in general is an hard task to accomplish. Our approach works well regardless the presence of an homoclinic loop at infinity. In this connection, consider for example the system

$$\begin{cases} \dot{x} = y - ax^{2m+2} + \varepsilon p(\omega t) \\ \dot{y} = -x^{2n+1} \end{cases} \tag{4.2}$$

If $n \geq 2(m+1)$, then the associated autonomous system has the property (H) from [23], as recalled in Section 2. Moreover (see [11]) the period tends to zero as the orbits are enlarged to infinity. Therefore, we can apply our result, for instance, to system

$$\begin{cases} \dot{x} = y - ax^2 + \varepsilon p(\omega t) \\ \dot{y} = -x^5 \end{cases} \tag{4.3}$$

where there is no homoclinic loop at infinity.

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