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PREFACE

Calculus of Variations in Mechanics and Related Fields

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With *Calculus of Variations*—we know—we indicate a body of mathematical techniques aiming at determining existence and properties of functions on which certain classes of functionals attain optimal values, according to some prescribed criteria, and pertinent results. Our common belief that nature minimizes energy at constant entropy and maximizes entropy at constant energy—a belief not falsified so far by common experiments—is at roots of the continuous interest in calculus of variations, besides aesthetic evaluation of the theory per se. Companion motivation for the attention on such a topic is also the technological interest for designing objects with some optimal property—e.g., shape, strength, conductivity—under some constraints, or to get optimal control of processes, e.g., in mechanics or economy.

Problems requiring recourse to calculus of variations techniques to be tackled emerge in several sectors, even in social sciences, but above all in mechanics (be it classical, quantum, or relativistic), condensed matter physics, chemistry, and else. The motion of a three-dimensional rigid body can be viewed as a geodetic curve (the one with minimal length) over the special orthogonal group, while perfect fluids move along geodetic paths over the special group of diffeomorphisms. We can also aim at controlling optimally the motion of certain systems, be them multi-rigid-bodies with flexible mutual constraints or continua suffering distributed strain, as, e.g., rods are. We may ask to find the minimal energy of an atom, a molecule, a thin film, or we may tackle optimality questions connected with chemical reactions, or we aim at printing and connecting microstructures, in order to obtain an artifact with some optimized properties, what we call a metamaterial. Also, we may be interested in optimal transportation of mass, charges, and their like.

Energy minimization characterizes equilibrium configurations. When coupled with appropriate monotonicity conditions mimicking irreversible behavior, such a minimization procedure may allow us to describe classes of (rate-independent) dissipative processes, such as plastic flows, damage, some phase transitions, or nucleation of fractures, by adapting Ennio De Giorgi's idea of minimizing movements. The idea is to partition the time interval into finitely many sub-intervals, presuming to go from

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the state at instant t_k to the one at t_{k+1} by minimizing some functional onto an appropriate function class. For example, in fracture processes, a crack path can be viewed as coinciding with the jump set of special bounded variation functions (SBV) or the support of varifolds. (SBV is a class of functions with derivative a measure having absolutely continuous component with respect to the Lebesgue volume n-dimensional measure, and a nonsingular part concentrated over a $\mathcal{H}^{n-1}$-measurable set, with $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure. Varifolds are vector-valued measures admitting a generalized notion of curvature.) The two possibilities are not equivalent: when we choose just deformations in SBV, imposing that they are one-to-one and preserve almost everywhere the orientation of volumes outside the jump set, we are thinking only to open fractures, while when we make use of varifolds, we may describe fractures which have a portion of the margins in contact, although no material bonds intervene between them. This and other examples of phenomena suggesting recourse to a variational view to be described open often challenging problems. Tackling them drives the evolution of this rich sector of mathematical analysis, possibly indicating connections with other sectors, as it happened in the analysis of parabolic partial differential equations, now connected with calculus of variations in the optimal transportation theory.

The early days of the analytical minimization theory of functionals dates back to Leonhard Euler and Giuseppe Luigi Lagrange in XVIII Century (Euler himself introduced the syntagma *Calculus of Variations* in 1766), although we have not to forget previous work by Pierre de Fermat—who believed that "nature operates by means and ways that are 'easiest and fastest'"—Jacob Bernoulli, Isaac Newton himself and that optimal problems interested even ancient Greeks (Hero and Pappus of Alessandria in the first Century), although they did not have even rough tools of infinitesimal calculus. With his 1946 monograph *Lectures on Calculus of Variations*, Gilbert Ames Bliss presented first the matter as a body of mathematics finding end in itself, not as a mere adjunct of mechanics.

Already a beginner in mathematical analysis knows the way of finding minima, maxima, and stationary point of a differentiable real function on $[a, b] \subset \mathbb{R}$ through the evaluation of zeros for the first derivative and the analysis of second derivative, when available. The classical approach to calculus of variations has been properly the extension of that standard method to functions defined on functional spaces rather than just on $\mathbb{R}^n$. The derivative becomes a variation obtained through appropriate test functions. The Euler–Lagrange equations determine necessary conditions for a function u to be an extremal for a functional $\mathcal{F}(u)$ given, for example, by

$$\mathcal{F}(u) := \int_\Omega F(x, u(x), Du(x))\, \mathrm{d}\mu(x),$$

with F a density assumed to be differentiable with respect to its entries, and μ a volume measure over Ω, a smooth open set in $\mathbb{R}^n$. When F is convex and the pertinent Euler–Lagrange equations admit unique solution u, we are sure that $\mathcal{F}$ attains its minimum value over u. Otherwise, once proving existence of solutions for the Euler–Lagrange equations, we should evaluate the second variation of $\mathcal{F}$ over them. When Ω is an interval in $\mathbb{R}$, the pertinent Euler–Lagrange equations are ordinary with boundary data, and not always we find conditions assuring existence of their solutions. Beyond one-dimensional ground space, the Euler–Lagrange equations are partial, with pertinent difficulties.

Around the end of XIX Century, Bernhard Riemann suggested to reverse the view along a path already used (in a sense implicitly) by Carl Friedrich Gauss and William Thompson Lord (1^{st} Baron) Kelvin. If we are able to find a minimum for $\mathcal{F} : X \to \mathbb{R} \cup \{\infty\}$ by looking just at minimizing sequences, we have, in turn, a solution of the pertinent Euler–Lagrange equations in some sense, i.e., depending on the regularity showed by the minimum. This is what we call *direct method* in calculus of variations, explored by many scholars, starting from David Hilbert, who included related questions in his path-opening 1900 address to the International Congress of Mathematicians in Paris. Such an approach emerges once again by what we do on functions on $\mathbb{R}$. In fact, to prove that a continuous real function defined on a compact set $K \in \mathbb{R}^n$ attains its minimum value, first we take a minimizing sequence $\left\{x_j\right\}$ such that $f(x_j) \to \inf_{x\in K} f(x)$ as $j \to \infty$. In K there exists a converging subsequence $\{x_j\}$ and the continuity of f implies $\lim_{j\to\infty} f(x_j) = f(x)$. Although working on minimizing sequences is the idea, its version in infinite-dimensional space cannot be reached straight away. In fact, consider, for example, $\mathcal{F}$ to be such that, for $u \in L^2(\Omega, \mathrm{d}\mu)$, whenever $\|u_j \to u\|_{L^2}$ as $j \to +\infty$, $\mathcal{F}(u_j) \to \mathcal{F}(u)$, i.e., $\mathcal{F}$ is strongly continuous. If we look at the unit ball $K := \left\{u \in L^2(\Omega, \mathrm{d}\mu) \ : \ \|u\|_{L^2} \leq 1\right\}$ as a putative set for finding the minimum of $\mathcal{F}$, although K is closed and bounded, we do not necessarily find a convergent subsequence $\left\{u_j\right\}$ in K. If we look at weak convergence, we find that every sequence in K has a weakly convergent subsequence. However, $\mathcal{F}$ is not necessarily weakly continuous. In other words, the more we relax the notion of convergence, the less likely $\mathcal{F}$ is continuous on the pertinent sequences. Things may be adjusted when $\mathcal{F}$ is such that $\liminf_{j\to\infty} \mathcal{F}(u_j) \geq \mathcal{F}(u)$ as $u_j \rightharpoonup u$, and in this case we say that $\mathcal{F}$ is weakly lower semicontinuous—remarkably, Leonida Tonelli established first in 1920 necessary and sufficient conditions of lower semicontinuity for a functional defined on a one-dimensional space. Thus, if $\mathcal{F}$ is lower semicontinuous, for $\left\{u_j\right\}$ a minimizing sequence in the sense that $\mathcal{F}(u_j) \to \inf\{\mathcal{F}(u) \ : \ u \in C\} =: \gamma$, there exists a subsequence $\left\{u_j\right\}$ such that $u_j \rightharpoonup u$, so that $\gamma = \lim_{j\to\infty} \mathcal{F}(u_j) \geq \mathcal{F}(u) \geq \gamma$, i.e., $\mathcal{F}(u) = \gamma$. Instead of thinking in sequential terms (convergence of sequences being not necessarily weak), we can speak of lower semicontinuity for a functional $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$, with X a topological space, if for any $t \in \mathbb{R}$ the set $\mathcal{F}_t := \{u \in X \ : \ \mathcal{F}(u) > t\}$ is open. A functional lower semicontinuous in this topological sense is also so in sequential terms. The opposite is true if every point of X admits a countable fundamental system of neighborhoods.

In analyzing a functional class in terms of the direct method, a key point is to have at disposal a lower semicontinuity result. In the academic year 1968/1969, in the notes of a course in Rome at the "Istituto Nazionale di Alta Matematica", never published, Ennio De Giorgi presented the first proof of the (sequential) lower semicontinuity of the functional $\mathcal{F}(u, v) = \int_\Omega F(x, u(x), v(x))\, \mathrm{d}x$, with respect to strong convergence of u and weak convergence of v, under assumption that the density $F(x, s, \xi)$ is jointly continuous with respect to the three variables entering it and convex in ξ. Such a result and the technique used in reaching it opened the way to a rich crop of lower semicontinuity results (e.g., the L^p norm is a lower semicontinuous functional). Among them, we mention the case in which $\mathcal{F}(u) = \frac{1}{2}\int_\Omega |Du|^2 \, \mathrm{d}x$, and u is a map taking values on a differentiable manifold not embedded in a linear space, provided that the manifold is Riemannian and complete, because it plays a nontrivial role in determining equilibrium configurations in the general model-building framework of the mechanics of complex materials, where descriptors of the material microstructure are manifold-valued maps.

We also owe to De Giorgi—among several things—the notion of Γ-convergence, which is fundamental in evaluating dimensional reductions (e.g., from thick material layers to thin films) or the passage from a discrete (atomic-scale) representation of matter to a continuum view, a fundamental step in justifying from an atomistic viewpoint the continuum representation of condensed matter behavior. The idea of Γ-convergence is to have a sequence of functionals and a companion sequence of minima. A question is to find conditions assuring that the limiting function is a minimum for the limiting functional. Consider, in fact, functionals $\mathcal{F}_\varepsilon : X_\varepsilon \to \mathbb{R} \cup \{\infty\}$ and a sequence $\{\min \mathcal{F}_\varepsilon(u_\varepsilon) \; : \; u_\varepsilon \in X_\varepsilon\}$, which we assume to be equi-coercive, i.e., there exists a pre-compact minimizing sequence such that $\mathcal{F}_\varepsilon(u_\varepsilon) \leq \inf \mathcal{F}_\varepsilon + o(1)$, also $u_\varepsilon \to u_0$, as $\varepsilon \to 0$, with u_0 solution to $\{\min \mathcal{F}_0(u_0) \; : \; u_0 \in X_0\}$. We call $\mathcal{F}_0$ the Γ-limit of $\mathcal{F}_\varepsilon$ when two conditions are satisfied. The first is that for every $u \in X_0$ and every $u_\varepsilon \to u$ we have $\mathcal{F}(u) \leq \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon)$. The second condition is the existence of a sequence $\bar{u}_\varepsilon \to u_0$ for every $u_0 \in X_0$ such that $\inf \mathcal{F}_0 \geq \limsup_{\varepsilon \to 0} \inf \mathcal{F}_\varepsilon$. Γ-convergence and equi-coerciveness imply convergence of minimum problems.

Again the choice of convergence is crucial: a weaker convergence, with many converging sequences, makes equi-coerciveness easier to fulfill, but at the same time makes the $\liminf$ inequality more difficult to hold. Often, an appropriate choice is strong convergence in L^p spaces. Connected with the selection of convergence is the companion choice of energy scaling to assure equi-coerciveness.

Analytical problems in calculus of variations are manifold and faceted. They exceed largely the brief incomplete sketch above. Also, besides purely analytical questions, when we look at the world around us with the aim of interpreting it qualitatively and quantitatively—we repeat—we meet recurrently phenomena offering themselves as a playground for calculus of variations or suggesting further analytical problems in a fruitful mathematical field.

The collection of papers presented here offers a partial, although variegate, view on problems and techniques in calculus of variations connected with mechanics and related fields. Topics range from shape and compliance optimization to identification of material properties, optimal control, plastic flows, gradient polyconvexity, optimal potentials, constrained and obstacle problems, quasi-monotonicity, waves, numerical techniques and simulations. The papers in this collection offer results which could be source of further work. Also they are examples of how foundational knowledge and command of appropriate mathematical techniques may address us toward applications going out of the rut and indicating, as such, possible new scientific and technological paths.