Basic structures on derived critical loci

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To the memory of my teacher
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Abstract

We review the derived algebraic geometry of derived zero loci of sections of vector bundles, with particular emphasis on derived critical loci. In particular we some of the structures carried by derived critical loci: the homotopy Batalin-Vilkovisky structure, the action of the 2-monoid of the self-intersection of the zero section, and the derived symplectic structure of degree -1. We also show how this structure exists, more generally, on derived lagrangian intersections inside a symplectic algebraic manifold.

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Contents

1 Introduction

We quickly explore the derived algebraic geometry of derived zero loci of sections of vector bundles, with particular emphasis on derived critical loci. In particular we single out many of the derived geometric structures carried by derived critical loci: the homotopy *Batalin-Vilkovisky* structure, the *action of the* 2-*monoid* of the self-intersection of the zero section, and the *derived symplectic structure* of degree -1, and show how this structure exists, more generally, on *derived lagrangian intersections* inside a symplectic algebraic manifold. These are just applications of a small part of a much larger project investigating quantization of derived moduli spaces started in [PTVV].

These pages were motivated by a series of lectures - on the *formal* derived aspects of the same topics treated here - by Kevin Costello at DennisFest 2011, Stony Brook. The idea was to describe those global structures on a derived critical locus, that after passing to the formal completion (i.e. to the associated *formal moduli problem* in the sense of [L]) could recover Costello-Gwilliam's set-up in [CG].
The aim of these notes is more at giving a roadmap and an overview of all the constructions than at giving full proofs. These notes were written before [CPTVV] was completed, and, while the contents of Sections 3 and 4.1 are somehow new in print (though most probably known to the experts), the other parts are closer to standard constructions or contained, in a different and less elementary way, in [PTVV].

A word to differential geometers. Though we work here in the setting of derived algebraic geometry, most of what we say can be translated into derived differential geometry (where stacks can be modelled by Lie groupoids up to Morita equivalence, and derived stacks by derived Lie groupoids, etc.). The formal analog (i.e. formally at a given point) in derived differential geometry of what we describe can be found in [CG], while derived symplectic structures in differential geometry are studied in details in [PS]. A very nice and general approach to derived differential geometry is [N] that might be used as a background for extending what is described in this paper to the C^∞ setting. There are already several applications of the differential geometric side of the circle of ideas described in this paper: to Chern-Simons theory ([S1]), to Batalin-Vilkovisky ([HG]), and to Poisson-Lie structures ([S2]). A gentle introduction to some of these applications is [C].

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I wish to thank Kevin Costello, Bertrand Toën and Boris Tsygan for many interesting discussions on these notes' topics. Thanks are also due to the anonymous referee for a useful report.

Notations and related warnings. We work over a base field k of characteristic 0. The other notations are standard or recalled otherwise. Our reference for derived algebraic geometry will be [HAG-II]. In particular, $\mathbf{dSt}_k$ is the model category of derived stacks over k, as defined in [HAG-II, 2.2.3].
As Kevin Costello pointed out to me, there are different notions of Gerstenhaber and Batalin-Vilkovisky structures in the literature, so my choice of terminology might not be standard. In particular, what we call a dg Gerstenhaber bracket has degree 1, and therefore coincides with what Jacob Lurie calls a P_0 bracket. And we use the term dg-Batalin-Vilkovisky (dgBV) algebra according to the physics literature, while Costello calls it a Beilinson-Drinfel'd (BD) algebra. All the structures are explicitly defined in the text, so we hope our terminology will not cause insurmountable problems to the reader.

2 Koszul complexes and derived zero loci of sections of vector bundles

Since there are tons of complexes named after Koszul in the literature, we give here the definitions we'll use and recall some more or less well known related facts.

2.1 Affine case

Let R be a commutative k-algebra, and P a projective R-module of finite type. Let $S := \mathrm{Sym}_R(P^\vee)$ the symmetric algebra on the R-dual $P^\vee$: we will *always* disregard its internal grading. S is a commutative R-algebra. Let $\bigwedge^\bullet P^\vee$ be the exterior algebra of $P^\vee$ as an R-module. Consider the nonpositively graded S-module $S \otimes_R \bigwedge^\bullet P^\vee$ graded by $(S \otimes_R \bigwedge^\bullet P^\vee)_m := S \otimes_R \bigwedge^{-m} P^\vee$ for $m \leq 0$. This is a graded S-modules which is degreewise projective over S. One has the following further structures on $S \otimes_R \bigwedge^\bullet P^\vee$

- An obvious R-*augmentation* $S \otimes_R \bigwedge^\bullet P^\vee \to (S \otimes_R \bigwedge^\bullet P^\vee)_0 = S \to R$ coming from the canonical (*zero section*) augmentation of S.
- Since $\bigwedge^\bullet P^\vee$ is a graded commutative R-algebra, $S \otimes_R \bigwedge^\bullet P^\vee$ is a *graded commutative S-algebra.*
- There is a natural degree 1 *differential* on the graded S-module $S \otimes_R \bigwedge^\bullet P^\vee$ induced by contraction and the canonical map
$$h : R \longrightarrow \mathrm{Hom}_R(P, P) \simeq P^\vee \otimes_R P.$$
More precisely, if $h(1) = \sum_j \alpha_j \otimes x_j$, then
$$d : S \otimes_R \bigwedge^{n+1} P^\vee \longrightarrow S \otimes_R \bigwedge^{n} P^\vee$$
acts as
$$d(a \otimes (\beta_1 \wedge \ldots \wedge \beta_{n+1})) = \sum_j a \cdot \alpha_j \otimes \sum_k (-1)^k \beta_k(x_j)(\beta_1 \wedge \ldots \wedge \check{\beta}_k \wedge \ldots \wedge \beta_{n+1}).$$
This is obviously S-linear.
- Together with such a d, $S \otimes_R \bigwedge^\bullet P^\vee$ becomes a *commutative differential non-positively graded* (cdga) over S.

Definition 2.1 *The S-cdga* $\mathrm{K}(R; P) := (S \otimes_R \bigwedge^\bullet P^\vee, d)$ *is called the* fancy Koszul cdga *of the pair* (R, P).

The following result is well known (and easy to verify, or see e.g. [BGS], [MR, 2.3])

Proposition 2.2 *The cohomology of the fancy Koszul cdga* $\mathrm{K}(R; P)$ *is zero in degrees* < 0*, and* $H^0(\mathrm{K}(R; P)) \simeq R$.

We will be interested in the following translation of the previous result

Corollary 2.3 *The augmentation map* $\mathrm{K}(R; P) \longrightarrow R$ *is a cofibrant resolution of* R *in the model category* $\mathbf{cdga}_S$ *of cdga's over* S.

We now explain the relation of the fancy Koszul cdga to the usual Koszul complex associated to an element $s \in P$, and interpret this relation geometrically.
Let
$$\mathrm{K}(R, P; s) := (\bigwedge^\bullet P^\vee, d_s)$$
be the usual non-positively graded Koszul cochain complex, whose differential d_s is induced by contraction along s (see [Bou-AH]). Together with the exterior product $\mathrm{K}(R, P; s)$ is a cdga over R.
The choice of an element $s \in P$ induces a map of commutative algebras $\varphi_s : S \to R$ (corresponding to the evaluation-at-s map of R-modules $P^\vee \to R$). We will denote R with the corresponding S-algebra structure by R_s; R_s is a cdga over S (concentrated in degree 0). We can therefore form the tensor product of cdga's $R_s \otimes_S \mathrm{K}(R; P)$. This is an S-cdga, whose underlying graded S-module is
$$R_s \otimes_S \mathrm{K}(R; P) = R_s \otimes_S (S \otimes_R \bigwedge^\bullet P^\vee) \simeq \bigwedge^\bullet P^\vee,$$

where $\bigwedge^{\bullet} P^{\vee}$ is viewed as an S-module via the composite k-algebra morphism

$$S \xrightarrow{\varphi_s} R \xrightarrow{\text{can}} \bigwedge^{\bullet} P^{\vee} .$$

It is easy to verify that, under this isomorphism, the induced differential on $R_s \otimes_S \mathrm{K}(R;P)$ becomes exactly d_s. In other words, there is an isomorphism of S-cdga's (and therefore of R-cdga's, via the canonical map $R \to S$)

$$R_s \otimes_S \mathrm{K}(R;P) \simeq \mathrm{K}(R,P;s).$$

2.2 Affine derived zero loci

Let R, P, S and s be as above. Let $X := \operatorname{Spec} R$, and $\mathbb{V}(P) := \operatorname{Spec}(S)$. The canonical map of k-algebras $R \to S$, indices a map $\pi : \mathbb{V}(P) \to X$, making $\mathbb{V}(P)$ a vector bundle on X with R-module of sections P. The zero section $0 : X \to \mathbb{V}(P)$ corresponds to the natural augmentation $S \to R$, while the section s, corresponding to the element $s \in P$, induces the other augmentation we denoted as $\varphi_s : S \to S$ before. Now we con consider the homotopy pullback

$$\begin{array}{ccc} Z^h(s) & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle 0} \\ X & \xrightarrow[s]{} & \mathbb{V}(P) \end{array}$$

in the model category $\mathbf{dSt}_k$ of étale derived stacks over k ([HAG-II, 2.2.3]). The derived affine stack $Z^h(s)$ is called the *derived zero locus* of the section s. By definition of homotopy pullback in $\mathbf{dSt}_k$, we may choose any cofibrant replacement of the natural augmentation $S \to R$ in the category of S-cdga's, for example (Cor. 2.3) $\mathrm{K}(R;P) \to R$, and get

$$Z^h(s) \simeq \mathbb{R}\mathrm{Spec}(R_s \otimes_S^{\mathbb{L}} R) \simeq \mathbb{R}\mathrm{Spec}(R_s \otimes_S \mathrm{K}(R;P)) \simeq \mathbb{R}\mathrm{Spec}(\mathrm{K}(R,P;s))$$

by the computation above. In other words

Proposition 2.4 *The usual Koszul cdga* $\mathrm{K}(R,P;s)$ *is the algebra of functions on the* derived zero locus *of the section s.*

Remark 2.5 The corresponding statement in the C^{∞}-category can be found in [CG, Appendix].

2.3 General - i.e. non-affine - case

Let X be a scheme over k, E a vector bundle on X, $\mathcal{E}$ the corresponding $\mathcal{O}_X$-Module of local sections of E (so that $E \simeq \operatorname{Spec}_X(\operatorname{Sym}^{\bullet}_{\mathcal{O}_X} \mathcal{E}^{\vee})$), $s \in \Gamma(X, \mathcal{E})$ a global section of E, and

$$\mathrm{K}(X,E;s) := (\bigwedge^{\bullet} \mathcal{E}^{\vee}, d_s)$$

the usual non-positively graded Koszul cochain complex, whose differential d_s is induced by contraction along s. Together with the exterior product $\mathrm{K}(X,E;s)$ is a sheaf of $\mathcal{O}_X$-cdga's. Define the derived zero locus $Z^h(s)$ via the homotopy cartesian diagram (in $\mathbf{dSt}_k$)

$$\begin{array}{ccc} Z^h(s) & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle s} \\ X & \xrightarrow[0]{} & E. \end{array}$$

The following result is an immediate consequence of the corresponding result in the affine case

Proposition 2.6 $\mathrm{K}(X,E;s)$ *is the cdg-Algebra* $\mathcal{O}_{Z^h(s)}$ *of functions on the* derived zero locus $Z^h(s)$ *of the section s.*

3 The self-intersection 2-monoid and its actions

Let $X = \operatorname{Spec} R$ be a smooth k-scheme, and P be a finitely generated projective R-module. The derived self-intersection $0^2 := Z^h(0)$ of the zero section 0 of $\pi : \mathbb{V}(P) \to X$ has, *relative to* X, one monoid structure (by loop[1] composition), and one abelian group structure (by addition of loops, coming from the group structure of $\mathbb{V}(P)$). These structures are compatible in the sense that the functor corresponding to 0^2 factors through the category of group-like Segal monoid objects in simplicial abelian groups. This structure will be denoted by $(0^2, \circ, +)$, Note that this whole structure acts, relative to X, on the derived zero locus of *any* section of $\pi : \mathbb{V}(P) \to X$.

Let us see this structure more precisely. First of all, the derived self-intersection 0^2 of the zero section 0 of $\pi : \mathbb{V}(P) \to X$ is

$$0^2 \simeq \mathbb{R}\mathrm{Spec}(\mathrm{K}(R, P; 0)) \simeq \mathbb{R}\mathrm{Spec}(\bigwedge^{\bullet} P^{\vee}, 0).$$

Remark 3.1 Note that if $R = k$ and P is free of rank 1, then $0^2 \simeq \mathbb{R}\mathrm{Spec}(k[\varepsilon])$, where $\deg \varepsilon = -1$. In this case the functor of points of 0^2, defined on a k-cdga A is given by $0^2(A) \simeq \mathrm{Map}_*(S^1, A_*)$ (pointed maps, S^1 being the simplicial circle) where A_* is the simplicial abelian group $\mathrm{Map}_{\mathrm{cdga}}(k[x], A)$ (the group structure being induced from the coalgebra structure of $S = k[x]$). This explains the name of loops given to points of 0^2, even when the vector bundle E is not a trivial line bundle, and R is an arbitrary commutative k-algebra. On the other hand, for arbitrary R, and $P := \Omega^1_{R/k}$, then 0^2 is the derived spectrum of the dg-algebra of polyvectorfields on X - arranged in non-positive degrees and with zero differential.

Now, the derived Čech nerve of the zero-section inclusion $0 : X \hookrightarrow \mathbb{V}(P)$ is a Segal groupoid object in the model category of derived stacks *over* X, the composition law (that we will call *loop composition*) being given - as usual for derived Cech nerves - by

$$(X \times^h_{\mathbb{V}(P)} X) \times^h_X (X \times^h_{\mathbb{V}(P)} X) \xrightarrow{\sim} X \times^h_{\mathbb{V}(P)} X \times^h_{\mathbb{V}(P)} X \xrightarrow{p_{13}} X \times^h_{\mathbb{V}(P)} X\ .$$

This first stage operation can be extended to the whole Segal object, as usual, using projections and diagonal maps.
Analogously, the linear structure on $\mathbb{V}(P)$ allows one to define another composition law on 0^2 by simply pointwise adding the "loops" (this corresponds to the natural dg-coalgebra structure on $(\bigwedge^{\bullet} P^{\vee}, 0)$). More geometrically, the commutative diagram

$$\begin{tikzcd} X \times_X X \arrow[r, "="] \arrow[d, "{(0,0)}"'] & X \arrow[d, "0"] \\ \mathbb{V}(P) \times_X \mathbb{V}(P) \arrow[r, "+"] & \mathbb{V}(P) \end{tikzcd}$$

induces by pullback along $+$ a *loop addition* on 0^2.

Summing up, we get

Proposition 3.2 *There is a functor*

$$Z^h(0)^{\bullet\bullet} : \Delta^{op} \times \Gamma \longrightarrow \mathbf{dSt}_k/X$$

such that

- *it prolonges*[2] *the two previous loop operations;*

[1] For the explanation of this name, see Remark 3.1.

[2] In the sense that 0^2 is just the first stage of the Segal object $Z^h(0)^{\bullet\bullet}$.

- *it defines a group-like Segal monoid object in commutative Segal monoid objects in* $\mathbf{dSt}_k/X$*;*
- $Z^h(0)^{\bullet\bullet}$ *is well defined in the homotopy category of group-like Segal monoid objects in commutative Segal monoid objects in* $\mathbf{dSt}_k/X$*;*

Here, as customary, Γ is the category of finite pointed sets and pointed maps. Observe that the Γ argument corresponds to addition of loops while the Δ^{op} one to the composition of loops.

Remark 3.3 The results in Prop. 3.2 are contained in [P], in the case where P is the free k-module of rank 1 (i.e. the trivial line bundle $\mathbb{A}^1_X$ on $X = \operatorname{Spec} k$), and interestingly related to matrix factorizations dg-derived categories.

We will call a group-like Segal monoid object in commutative Segal monoid objects (in $\mathbf{dSt}_k/X$), a (G,Comm) 2-*monoid* (in $\mathbf{dSt}_k/X$).

Proposition 3.4 *For any section* s *of* $\mathbb{V}(P) \to X$*, there is an action of the (G,Comm)* 2-monoid *(in* $\mathbf{dSt}_k/X$*)* $Z^h(0)^{\bullet\bullet}$ *on* $Z^h(s)$ *over* X*.*

Remark 3.5 Similar results obviously hold in the non-affine case. We leave to the reader the parallel statements of such results.

4 Derived critical loci

Let M be a smooth algebraic variety over k, $f \in \mathcal{O}(M)$, and df the induced section of the cotangent bundle of M.

Definition 4.1 *The* derived critical locus $\mathrm{Crit}^h(f)$ *of* f *is the derived zero locus* $Z^h(df)$ *of* df*.*

As a corollary of Proposition 3.2, we have

Proposition 4.2 $\mathrm{K}(M, T^\vee M; df)$ *is the cdg-Algebra* $\mathcal{O}_{\mathrm{Crit}^h(f)}$ *of functions on the* derived critical locus $\mathrm{Crit}^h(f)$ *of the function* f*.*

In the next Section we will describe some of the derived geometrical structures carried by derived critical loci.

4.1 Geometric structures on derived critical loci

4.1.1 G_∞-structure

For Let $X = \operatorname{Spec} R$, we will denote by T_X the R-dual module to $\Omega^1(X) \equiv \Omega^1_{R/k}$. For definitions and basic properties of differential graded (homotopy) Gerstenhaber algebras we refer the reader to [Ma, TT, G].

Proposition 4.3 *Let* $X = \operatorname{Spec} R$ *be a* smooth *affine scheme over* k*, and* $\alpha \in \Omega^1(X)$ *a* closed 1*-form on* X*. Then, the usual Koszul cdga* $(\mathrm{K}(R, \Omega^1(X); \alpha), d_\alpha) = (\bigwedge^\bullet \mathrm{T}_X, d_\alpha)$*, together with the Schouten bracket* $[\![-,-]\!]$ *is a differential graded commutative Gerstenhaber algebra.*

Proof. The result follows from the easily verified formula

$$\alpha([X,Y]) = d_\alpha[\![X,Y]\!] = [\![d_\alpha X, Y]\!] + [\![X, d_\alpha Y]\!] = -Y(\alpha(X)) + X(\alpha(Y))$$

valid for any $X, Y \in \mathrm{T}_X$, $[-,-]$ being the usual Lie bracket of vector fields on X. □

Since

$$Z^h(\alpha) \simeq \mathbb{R}\mathrm{Spec}(\bigwedge^{\bullet} \mathrm{T}_X, d_\alpha)$$

(isomorphism in the homotopy category of $\mathbf{dSt}_k$) - i.e. $(\bigwedge^{\bullet} \mathrm{T}_X, d_\alpha)$ is a model for the cdga of functions on $Z^h(\alpha)$ - by the general $G - G_\infty$ principle ($\equiv$ "strict G on a dg-model $\Rightarrow G_\infty$ on any dg-model"), we get

Corollary 4.4 *Let $X = \mathrm{Spec}\, R$ be a* smooth *affine scheme over k, $\alpha \in \Omega^1(X)$ a* closed *1-form on X, and $Z^h(\alpha)$ the derived zero locus of α. Then the cdga $\mathcal{O}_{Z^h(\alpha)}$ of functions on $Z^h(\alpha)$ is a dg-G_∞-algebra.*

4.1.2 BV_∞-structure

If X has a volume form vol (e.g. X is symplectic or Calabi-Yau), then we may consider the differential Δ induced by the de Rham differential via the contraction-with-vol (or divergence) isomorphism

$$i_{\mathsf{vol}} : \bigwedge^{\bullet} T_X \longrightarrow \bigwedge^{n-\bullet} \Omega^1_X$$

where $n := \dim X$. Together with the bracket defined above, this gives an explicit BV structure on the dg-model $(\bigwedge^{\bullet} \mathrm{T}_X, d_\alpha)$ hence a dg-BV_∞-structure on $\mathcal{O}_{Z^h(\alpha)}$.

4.2 Derived symplectic structures

We will first define degree n derived symplectic structures on a derived stack, and then show how derived intersections of smooth lagrangians (in particular derived critical loci) carry canonical degree (-1) derived symplectic structures.

4.2.1 Generalities on derived symplectic structures

Derived symplectic structures are introduced and studied in detail in [PTVV], and in [CPTVV] they are shown to induce E_n-deformation of the derived dg-category of the underlying derived stack. We will only briefly sketch their definitions here, and show how they naturally arise on derived intersections of lagrangian submanifolds.

Definition 4.5 *Let Y be a derived algebraic stack over k, $\mathcal{L}Y := \mathbb{R}\mathrm{HOM}_{\mathbf{dSt}_k}(S^1 := B\mathbb{Z}, Y)$ its* derived loop stack*, and $\widehat{\mathcal{L}Y}$ its* derived formal loop stack *(i.e. the derived completion of $\mathcal{L}Y$ along the constant loops Y) together with the canonical action of $\mathbb{G}_m \ltimes B\mathbb{G}_a$.*

- *The* Hochschild homology *complex of Y is $\mathrm{HH}(Y) := \mathcal{O}_{\widehat{\mathcal{L}Y}}$, together with the additional internal grading by the $\mathbb{G}_m$-action; the corresponding internal degree will be called the* weight.
- *The* negative cyclic homology *complex of Y is $\mathrm{HC}^-(Y) := \mathcal{O}_{\widehat{\mathcal{L}Y}}^{B\mathbb{G}_a}$, together with the additional internal grading by the residual $\mathbb{G}_m$-action; the corresponding internal degree will be called the* weight.
- *Let $i : \mathrm{HC}^-(Y) \longrightarrow \mathrm{HH}(Y)$ be the obvious canonical (weight-preserving) map.*

Note that if $\mathbb{L}_Y$ is the cotangent complex of Y, an element $\omega \in \mathrm{HH}_m(Y)$ corresponds to a map $\omega : \mathcal{O}_Y \to \mathrm{Sym}^{\bullet}(\mathbb{L}_Y[1])[-m]$ in $\mathrm{D}(Y)$, and its weight is just its degree in $\mathrm{Sym}^{\bullet}$.
Recall from [Ill] that the derived functors $\bigwedge^m_{\mathcal{O}_Y}$ and $\mathrm{Sym}^m_{\mathcal{O}_Y}$, $m \geq 0$ are related by

$$\bigwedge^m_{\mathcal{O}_Y} P \simeq \mathrm{Sym}^m_{\mathcal{O}_Y}(P[1])[-m]$$

for any $\mathcal{O}_Y$-Module P. In particular, we have

$$\mathrm{Sym}^2_{\mathcal{O}_Y}(\mathbb{L}_Y[1]) \simeq \mathbb{L}_Y \wedge_{\mathcal{O}_Y} \mathbb{L}_Y[2].$$

Therefore there is a canonical projection

$$\mathrm{Sym}^{\bullet}(\mathbb{L}_Y[1]) \longrightarrow \mathbb{L}_Y \wedge_{\mathcal{O}_Y} \mathbb{L}_Y[2]$$

to the weight 2 part.

We are now ready to define generalized or derived symplectic structures.

Definition 4.6 *Let Y be a derived algebraic stack over k, $\mathbb{L}_Y$ its cotangent complex, and $\mathbb{T}_Y =: \mathbb{L}_Y^{\vee}$ its tangent complex. A* derived symplectic structure of degree $n \in \mathbb{Z}$ *on Y is an element $\omega \in \mathrm{HC}^{-}_{2-n}(Y)$ of weight 2 such that its weight 2 projection*

$$\overline{\omega} : \mathcal{O}_Y \longrightarrow \mathrm{Sym}^{\bullet}(\mathbb{L}_Y[1])[n-2] \longrightarrow \mathbb{L}_Y \wedge \mathbb{L}_Y[n]$$

induces, by adjunction, an isomorphism $\mathbb{T}_Y \simeq \mathbb{L}_Y[n]$ (non-degeneracy condition).

Intuitively, a derived symplectic form is therefore a non-degenerate map $\mathbb{T}_Y \wedge \mathbb{T}_Y \to \mathcal{O}_Y[n]$ which is $B\mathbb{G}_a$-equivariant (i.e. it lifts to $\mathrm{HC}^{-}_{2-n}(Y)$, and this is a datum[3]).

Remark 4.7 If Y is a (quasi-smooth) derived Deligne-Mumford stack and $\omega \in \mathrm{HC}^{-}_{3}(Y)$ is a (-1) derived symplectic structure on Y, let us consider the following construction. By definition of derived exterior and symmetric powers of complexes, we have, for any complex P on Y, a canonical isomorphism

$$P \wedge P \simeq \mathrm{Sym}^2(P[1])[-2],$$

and taking $P = \mathbb{L}_Y[-1]$ ($\mathbb{L}_Y$ being the cotangent complex of the derived stack Y), we get a canonical isomorphism

$$\mathrm{can} : \mathbb{L}_Y[-1] \wedge \mathbb{L}_Y[-1] \simeq \mathrm{Sym}^2(\mathbb{L}_Y)[-2].$$

Since our derived stack Y is endowed with a (-1) derived symplectic form, its image $\overline{\omega} \in \mathrm{HH}_3(Y)$ yields a morphism

$$\hat{\omega} : \mathbb{T}_Y \wedge \mathbb{T}_Y \longrightarrow \mathcal{O}_Y[-1]$$

such that the map

$$\varphi_{\omega} : \mathbb{T}_Y \longrightarrow \mathbb{L}_Y[-1],$$

induced by adjunction, is an isomorphism. Therefore we may consider the (-2-shifted) associated Poisson bivector, i.e. the composition

$$s_{\omega} : \mathrm{Sym}^2(\mathbb{L}_Y)[-2] \xrightarrow{\mathrm{can}^{-1}} \mathbb{L}_Y[-1] \wedge \mathbb{L}_Y[-1] \xrightarrow{\quad f_{\omega} \wedge f_{\omega} \quad} \mathbb{T}_Y \wedge \mathbb{T}_Y \xrightarrow{\overline{\omega}} \mathcal{O}_Y[-1].$$

If $j : \mathrm{t}_0(Y) \hookrightarrow Y$ denotes the closed immersion of the truncation[4], then the canonical map

$$j^* \mathbb{L}_Y \longrightarrow \mathbb{L}_{\mathrm{t}_0(Y)}$$

together with the 2-shifted restricted map

$$j^*(s_{\omega}[2]) : \mathrm{Sym}^2(j^* \mathbb{L}_Y) \longrightarrow \mathcal{O}_{\mathrm{t}_0(Y)}[1]$$

define a *symmetric perfect obstruction theory* - in the sense of [BF] - on $\mathrm{t}_0(Y)$.
Note that symmetric obstruction theories that are induced as explained above from (-1) derived symplectic structures (on some quasi-smooth derived extension Y of the given DM stack on which they are

[3] Equivariance in derived geometry is not a property but rather a datum, since the equalities testing usual equivariance are replaced by paths in the appropriate space, together with all higher coherences/compatibilities This corresponds to giving a specified lift to $\mathrm{HC}^{-}_{2-n}(Y)$.

[4] There is a truncation functor t_0 from the category of derived stacks to the category of stacks, right adjoint to the fully faithful inclusion (see [HAG-II]). On affine derived schemes, t_0 sends a cdga A to $H^0(A)$.)

defined) have indeed more "structure", i.e. their lift to derived stack corresponds to a *closed* element in $\mathrm{HH}_3(Y)$ - i.e. an element admitting a lift to $\mathrm{HC}_3^-(Y)$. It might be the case that a symmetric obstruction theory that is induced by a (-1) derived symplectic form, is étale locally isomorphic to the canonical one existing on the derived zero locus of a *closed* 1-form on a smooth scheme - instead of just an *almost-closed* 1-form, as in the case of a general symmetric obstruction theory, see [BF].

Heuristically, a derived symplectic structure of degree $n \geq 0$ (resp. $n \leq 0$) on Y induces a *quantization* of Y, i.e. a deformation of the derived dg-category $\mathrm{D}(Y)$ (resp. the periodic derived dg-category $\mathrm{D}^{\mathbb{Z}/2}(Y)$) as an E_n-monoidal dg-category (resp. as an E_{-n}-monoidal dg-category). This is the result that makes the notion of derived symplectic form geometrically interesting. We address the reader to [CPTVV] for detailed statements and proofs.

Example 4.8 We will now show that the canonical symplectic form on the cotangent bundle T^*X of a smooth variety X/k, induces on the derived zero locus of any closed 1-form on X a natural derived symplectic structure of degree -1. This is true, more generally, for the locus defined by the derived intersection of two (algebraic, smooth) Lagrangian subvarieties in a (smooth algebraic) symplectic manifold (see §4.2.2).
Let $\alpha \in$ be a closed 1-form on X. The homotopy cartesian square

$$\begin{tikzcd} Z^h(\alpha) \arrow[r, "j"] \arrow[d, "j"'] & X \arrow[d, "0"] \\ X \arrow[r, "\alpha"'] & T^*X \end{tikzcd}$$

induces the following exact triangle ($q := s \circ j$)

$$q^*\mathbb{L}_{T^*X} \longrightarrow j^*\mathbb{L}_X \oplus j^*\mathbb{L}_X \longrightarrow \mathbb{L}_{Z^h(\alpha)} .$$

But X is smooth so $\mathbb{L}_X \simeq \Omega^1_X$ and $\mathbb{L}_{T^*X} \simeq \Omega^1_{T^*X}$, and there is a canonical symplectic form $\omega_0 : \mathcal{O}_{T^*X} \to \Omega^2_{T^*X}$ on T^*X. Since 0 and α defines Lagrangian immersions of X with respect to ω_0, we have that the composition

$$q^*\mathcal{O}_{T^*X} \simeq \mathcal{O}_{Z^h(\alpha)} \xrightarrow{q^*\omega_0} q^*\Omega^2_{T^*X} \longrightarrow \wedge^2(j^*\Omega^1_X \oplus j^*\Omega^1_X)$$

is zero. By the exact triangle above, we deduce the existence of a map

$$\omega : \mathcal{O}_{Z^h(\alpha)}[1] \longrightarrow \mathbb{L}_{Z^h(\alpha)} \wedge \mathbb{L}_{Z^h(\alpha)}.$$

This gives the underlying 2-form of a derived symplectic structure of degree -1 on $Z^h(\alpha)$.

Remark 4.9 Note that $A := (\bigwedge^\bullet \mathrm{T}_X, d_\alpha)$ is not cofibrant as a cdga over k, so its $\Omega^2_{A/k}$ is not directly related to $\mathbb{L}_{Z^h(\alpha)} \wedge \mathbb{L}_{Z^h(\alpha)} \equiv \mathbb{L}_{A/k} \wedge \mathbb{L}_{A/k}$, hence we cannot exhibit an explicit map $A[1] \to \Omega^2_{A/k}$ in $D(A)$ realizing ω. It is however possible, in the affine case, to compute explicitly the element in HC_3^- corresponding to ω, at least in the case X is affine - see [PTVV].

4.2.2 Symplectic structure on derived intersections of lagrangian subvarieties

Let (Y, ω_0) a smooth algebraic symplectic variety, L_1 and L_2 smooth lagrangian subvarieties in X. We will show that the derived intersection $Z := L_1 \times^h_X L_2$ has a canonical (-1)-symplectic structure, and therefore by [CPTVV] the derived dg-category $D^{\mathbb{Z}/2}(Z)$ has a canonical deformation $D^{\mathbb{Z}/2}_{\omega_0}(Z)$ as a monoidal dg-category.

Consider the homotopy cartesian diagram

$$\begin{tikzcd} Z \arrow[r, "j_1"] \arrow[d, "j_2"'] & L_1 \arrow[d, "i_1"] \\ L_2 \arrow[r, "i_2"'] & Y \end{tikzcd}$$

By definition of homotopy cartesian diagram we have a suitably functorial homotopy $H : j_1^* i_1^* \sim j_2^* i_2^*$. By applying this to ω_0, and using that each L_i is lagrangian, we get a canonical self-homotopy h_{ω_0} of the *zero* map

$$j_1^* i_1^*(T_Y \wedge T_Y) \longrightarrow \mathcal{O}_Z.$$

By composing with the canonical derivative map $\mathbb{T}_Z \wedge \mathbb{T}_Z \to j_1^* i_1^*(T_Y \wedge T_Y)$, we get then a self-homotopy of the zero map $\mathbb{T}_Z \wedge \mathbb{T}_Z \longrightarrow \mathcal{O}_Z$, i.e. a map

$$\omega : \mathbb{T}_Z \wedge \mathbb{T}_Z \longrightarrow \mathcal{O}_Z[-1].$$

Hence ω is an element of $\mathrm{HH}_3(Z)$. We let the reader verify that such an ω lifts canonically to $\mathrm{HC}_3^-(Z)$ (i.e. it is $B\mathbb{G}_a$-equivariant, in other words "closed") and that the non-degeneracy of ω_0 implies that of ω. In other words, ω is in fact a (-1)-symplectic form on the derived intersection $Z := L_1 \times_X^h L_2$.

If $Y = T^*M$, with M a smooth algebraic variety, then any closed 1-form α on M defines a Lagrangian subvariety L_α in T^*M, therefore the derived critical locus

$$\mathrm{Crit}^h(f) \equiv Z^h(df) := L_{df} \times_{T^*M}^h L_0,$$

admits a derived (-1)-symplectic form.

Question 4.10 Are the BV_∞ structure, the (-1)-symplectic structure, and the action of the (G,Comm) 2-monoid $Z^h(0)^{\bullet\bullet}$ on a derived critical locus *compatible*, in a suitable sense ?
This should be true but even giving a rigorous definition of compatibility requires some non-trivial work.

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