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Reeder's Conjecture for Lie Algebras of type C

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"Solitude, récif, étoile
À n'importe ce qui valut
Le blanc souci de notre toile." [1]

[1] S. Mallarmé - *Salut*

Contents

Introduction

It is common to find in Science objects that appear ubiquitously in many fields and prove to be central in apparently far theories.

Restricting our focus to Mathematics, one of these remarkable objects is undoubtedly the algebra of differential forms.

We are in particular interested to study this object in an algebro-geometric framework, more precisely in the context of Lie Theory, where the properties of left and right invariance of the forms are crucial to obtain many relevant and elegant results.

The study of this kind of problems started in the first half of the twentieth century, when some mathematicians proposed to face with the concrete problem of better understanding the topological properties of spaces endowed with a transitive action of an algebraic group of transformations.

Many works were published around the fifties about this subject, most of them introducing some new brilliant ideas and powerful instruments as spectral sequences, fiber bundles and characteristic classes. (See A.Borel's paper [7] for a more precise exposition about this fascinating subject)

One of the most famous and elegant results of that period establishes a link between the cohomology of a simple compact Lie group and the invariants in exterior algebra of its complexified Lie algebra:

Theorem 0.0.1. *Let G be a compact connected Lie group and let $\mathfrak{g}$ its complexified Lie algebra. Then*

$$H^*(G) \otimes \mathbb{C} \backsimeq (\Lambda\mathfrak{g})^G$$

as graded vector spaces.

Observing that, by differentiation, considering Ad_G invariants corresponds to take invariants by $\mathrm{ad}_{\mathfrak{g}}$-action of $\mathfrak{g}$ on itself $\mathfrak{g}$, this elegant result can be restated in Lie algebras representation theory language recalling that considering the $\mathfrak{g}$ invariants in $\Lambda\mathfrak{g}$ correspond to identify the trivial representation of $\mathfrak{g}$ in the exterior algebra. In other terms, observing that adjoint action preserves the grading, the above theorem is equivalent to

$$\dim_{\mathbb{C}} H^i(G) \otimes \mathbb{C} = \dim_{\mathbb{C}} \left(\Lambda^i\mathfrak{g}\right)^{\mathfrak{g}} = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(V_0, \Lambda^i\mathfrak{g})$$

Therefore, to compute the Betti numbers of a compact Lie group is sufficient to find the dimension of the homogeneous components of $\Lambda\mathfrak{g}^{\mathfrak{g}}$. Moreover, one can ask if this invariant subspace has some algebraic structure compatible with the grading that can make the computation of the graded multiplicities more efficient.

In [17] H.Hopf proved that such a structure on $\Lambda\mathfrak{g}^{\mathfrak{g}}$ exists and is again a structure of exterior algebra:

Theorem 0.0.2 (H.Hopf)**.** *Let G be a compact Lie group and let $\mathfrak{g}$ be its complexified Lie algebra. Then*

$$\Lambda\mathfrak{g}^{\mathfrak{g}} = \Lambda(x_1, \dots, x_k)$$

where k is the rank of $\mathfrak{g}$ and $\deg x_i = d_i$ is an odd positive integer for every i.

This shows immediately that a compact simple Lie group has the same cohomology of a product of odd dimensional spheres. In the language of Poincaré polynomials it can be translated in the following way:

Corollary 0.0.3. *Let G be a compact simple Lie group. Set*

$$P_G(t) = \sum_{i=0}^{\dim G} \dim H^i(G) t^i = \sum_{i=0}^{\dim G} \dim \left(\Lambda^i \mathfrak{g}\right)^G t^i$$

then the polynomial $P_G(t)$ has the following factorization

$$P_G(t) = \prod_{i=1}^{\mathrm{rk}\mathfrak{g}} \left(1 + t^{d_i}\right).$$

The problem of computing completely the graded structure of the cohomology ring of G has been reduced to the problem of find the exponents of the variable t in the above expression.

The most revolutionary approach to this problem was announced in 1952 by Claude Chevalley at the International Mathematical Congress (see [9]) as a corollary of his famous restriction theorem.

The work of Chevalley about the determination of the exponents is based on a result by André Weil that links the generators $\{x_i\}$ to the invariant polynomials in the symmetric algebra $S(\mathfrak{g})$. More precisely for each homogeneous polynomial $p \in S(\mathfrak{g})^G$ of degree d, Weil constructed a G-invariant differential form of degree $2d-1$. Chevalley proved that $S(\mathfrak{g})^G$ is generated by $k = \mathrm{rk}\mathfrak{g}$ homogeneous polynomials $\{p_1, \dots, p_k\}$. Actually the degrees of these generators are exactly the so called exponents $\{e_1, \dots, e_n\}$ of the Lie algebra, as proved in [10]. It can be shown that $\Lambda\mathfrak{g}^{\mathfrak{g}}$ is then generated as graded algebra exactly by the differential forms constructed by Weil starting from the homogeneous polynomials $\{p_1, \dots, p_k\}$. These differential forms have consequently odd degrees, exactly of the form $2e_i - 1$.

More in detail, Chevalley's idea is to relate the G invariants $S(\mathfrak{g})^G$ to the invariants by the action of the Weyl group on the algebra symmetric $S(\mathfrak{h})$ of polynomials over a maximal toral subalgebra $\mathfrak{h}$, reducing the computation to a (simpler) problem of finite group representations.

Theorem 0.0.4 (C.Chevalley). *Let G be a complex connected semisimple Lie group. Fix a maximal torus T. Denoting with $\mathfrak{g}$ and $\mathfrak{h}$ the respective Lie algebras we have the following isomorphism*

$$S(\mathfrak{g})^G \simeq S(\mathfrak{h})^W.$$

As remarked above, taking the invariants in a G-representation V corresponds to locate copies of the trivial representation among the isotypical components of V. Some interesting questions can be posed now thinking in some sense to a generalization of these results:

Q1: What are the isotypical components appearing in $S(\mathfrak{g})$ and in $\Lambda\mathfrak{g}$?

Q2: What about their (graded) multiplicities?

For the symmetric algebra this problem has been solved starting from a result by Kostant (see [25]) proving the decomposition of $S(\mathfrak{g})$ as direct sum of the harmonic polynomials submodule with the invariants $S(\mathfrak{g})^G$. To find the irreducibles in $S(\mathfrak{g})$ and their graded multiplicities is then equivalent to identify the irreducible subrepresentations and their graded multiplicities in the harmonic polynomials. Such topic is studied in many works (see for example [20], [23] and [19]) that give a complete answer to the problem: the multiplicities are given by some special Kazhdan-Luzstig polynomials determined by the affine Weyl group of $\mathfrak{g}$.

On the other side, in spite of finite dimensionality of $\Lambda\mathfrak{g}$, determining the components of the exterior algebra seems to be quite difficult. The decomposition of exterior algebra in its irreducible components has been extensively studied by Kostant in [26]. In particular, he proves that the exterior algebra is isomorphic, as $\mathfrak{g}$ module, to the direct sum of $2^{\mathrm{rk}\mathfrak{g}}$ copies of the tensor product representation $V_\rho \otimes V_\rho$, where ρ denotes the Weyl vector. Unfortunately, this decomposition is not compatible with grading of Λ^G. In our special case, Kostant formulated the following very elegant conjecture:

Conjecture 0.0.5 (Kostant). *Let be V_ρ the irreducible $\mathfrak{g}$-representation of highest weight ρ. The irreducible representation V_μ appears in the decomposition of $V_\rho \otimes V_\rho$ if and only if $\mu \leq 2\rho$ in the dominance order.*

This conjecture, actually unsolved, has surprisingly an uniform solution only in the case of Lie algebra $\mathfrak{sl}_n$, as proved in [5] and [24].

Moreover in [12] it is proved that for simply laced cases this conjecture is implied by the saturation conjecture.

In the thesis we prove some partial results in the case of the symplectic algebra $\mathfrak{sp}_{2n}$. More closely we prove that if $\mu = (\mu_1, \ldots, \mu_n)$ is a weight smaller than 2ρ and $\mu_i \leq n - i + 1$ (i.e. the shape of μ, viewed as partition, is contained in the shape of 2ρ) then V_μ appears in the irreducible decomposition of $\Lambda\mathfrak{g}$.

Moreover, we propose an analogue of Kostant conjecture for the exterior algebra of little adjoint representation. Our conjecture is trivial in the case B_n and can be verified by inspection for F_4. In the case C_n it seems be quite complicated and linked to the Kostant conjecture for classical cases.

Coming back to the problem of studying the isotypical components of $\Lambda\mathfrak{g}$ and changing a bit our point of view, we can reformulate our "global" problem in terms of $\mathfrak{g}$ equivariant functions, aiming to study the space $\mathrm{Hom}_{\mathfrak{g}}(V_\lambda, \Lambda\mathfrak{g})$, usually called the module of covariants of type V_λ. On the other hand, the problem of determining the graded multiplicities is equivalent to find $\dim \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, \Lambda^i\mathfrak{g})$

These topics have been extensively studied in literature in some recent papers. In [26] the dimension of this module is computed for the adjoint representation and in [3] Bazlov gives an explicit formula for the graded multiplicities of the adjoint representation in the exterior algebra. Recently, in [15] it has been proved that in the case of some special representations the module $\mathrm{Hom}_{\mathfrak{g}}(V_\lambda, \Lambda\mathfrak{g})$ has a structure of free algebra over $\Lambda(x_1, \dots, x_{k-1})$ and its dimension is computed. On the other side, in 1995 Broer [8] proved that Chevalley restriction induces a graded isomorphism between module of covariants for certain representations V_λ such that λ is in the root lattice and $\lambda \not\geq 2\alpha$ for all positive roots α. These representations such that the weight λ is "near" to 0 are called "small representations".

Theorem 0.0.6 (Broer)**.** *The graded homomorphism induced by the Chevalley restriction theorem $S(\mathfrak{g})^G \simeq S(\mathfrak{h})^W$*

$$\mathrm{Hom}_{\mathfrak{g}}(V_\lambda, S(\mathfrak{g})) \to \mathrm{Hom}_W(V_\lambda^0, S(\mathfrak{h}))$$

is a graded isomorphism if and only if V_λ is small.

In the same year of Broer's paper, Mark Reeder [29] proposed a proof of the Theorem 0.0.1 based on the invariant theory. More precisely, he proved the isomorphism as graded vector spaces of the cohomology ring with the invariants for the action of the Weyl Group on the cohomology of some homogeneous spaces:

Theorem 0.0.7. *Let G be a compact Lie Group, $T \subset G$ a maximal torus and W the Weyl Group. The Weyl map $\psi : G/T \times T \to G$ induces in cohomology the following graded isomorphism:*

$$H^*(G) \simeq (H^*(G/T) \otimes H^*(T))^W .$$

Aiming at deepening the study of the exterior algebra, in the following paper [30] Reeder proves many formulae about graded multiplicities of representations appearing in $\Lambda\mathfrak{g}$ using a large amount of techniques coming from combinatorics, Lie theory and theory of symmetric functions.

More precisely, if I is a subset of the simple roots and denoting by $\delta(I)$ their sum, he gives a closed formula for graded multiplicities of representations of highest weights $2\rho - \delta(I)$ in terms of the connected component of the complementary of I in the Dynkin diagram of $\mathfrak{g}$.

In the same paper [30], inspired by Broer's work, Reeder starts a systematic study of small representation in the exterior algebra. Using suitable operators on the set of dominant weights of $\mathfrak{g}$, he proves a general recursive formula for the multiplicity of a representation V_λ in $\Lambda\mathfrak{g}$. Unfortunately, this formula involves many sign changes and it seems easy to compute neither the multiplicities in the general case, nor if they are different form zero.

Nevertheless, the sign changes disappear in the case when 2α is not a weight of V_λ, i.e. exactly in the case of small representations, for which the following elegant formula holds:

Theorem 0.0.8. *Let λ be a* small *weight and let m_λ^0 be the dimension of the zero weight space in the irreducible representation V_λ. Then*

$$\dim\mathrm{Hom}_G(V_\lambda, \Lambda\mathfrak{g}) = 2^{\mathrm{rk}\mathfrak{g}} m_\lambda^0.$$

For the sake of completeness, we specify that in general the left hand side of the above equality is smaller than the right hand side.

Furthermore Reeder looked at the problem of determining graded multiplicities of these representations, by studying firstly the simpler case of the adjoint representation and then generalizing his conjectures to the other small modules. He conjectured that for small representations in exterior algebra a similar result to the one proved by Broer holds. Such a conjecture is posed in [30] in terms of Poincaré polynomials of graded multiplicities:

Conjecture 0.0.9 (Reeder)**.** *Let us denote with $\mathcal{H}^i$ the space of W-harmonic polynomials. Consider the two polynomials*

$$P(V_\lambda, \bigwedge \mathfrak{g}, u) = \sum_{n\geq 0} \dim \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, \bigwedge^n \mathfrak{g}) u^n$$

$$P_W(V_\lambda^0, \mathcal{H}, x, y) = \sum_{n\geq 0} \dim \mathrm{Hom}_W(V_\lambda^0, \bigwedge^k \mathfrak{h} \otimes \mathcal{H}^h) x^k y^h.$$

If V_λ is a small representation of highest weight λ, then the following equality holds:

$$P(V_\lambda, \bigwedge \mathfrak{g}, q) = P_W(V_\lambda^0, \mathcal{H}, q, q^2)$$

Curiously, this conjecture was implicitly proved for the case A_n already before Reeder's paper was published in the works of Stembridge [36] for the " Lie algebra" part and by Kirillov - Pak and Molchanov for the "Weyl group" part.

Moreover in [37] many tools potentially useful for a case by case proof of the conjecture are introduced. Of crucial importance for our work are some recursive relations for the coefficients $C_\lambda(t, s)$ in the characters expansion of Macdonald kernels. Such polynomials, specialized in $t = -q$ and $s = q^2$ give exactly the Poincaré polynomials of the representations V_λ in $\Lambda\mathfrak{g}$.

In the thesis we propose a case by case proof of the conjecture for classical groups of type B and C using the recursive relations of [37] and closed formulae of [18] for the Weyl group part. As mentioned before in [37] some recursive relations are proved for the coefficients in the expansion of Macdonald Kernels in term of the characters of irreducible representations of $\mathfrak{g}$. An important remark by Stembridge assures that in such a recursion, for a small weight λ, only $C_\mu(t,s)$ associated to other small weights μ smaller then λ in the dominance ordering appear. Our strategy is then based on inductive reasoning: we compute explicitly a formula for the polynomials P_W and then substitute them in the specialized recurrence.We obtain a triangular system of equations and we solve it using combinatorial methods to reduce the coefficients of the recurrence to more a computable form. We hope to prove the remaining cases of the Conjecture in future works, using the same techniques for the more complicate case of D_n and some direct computations for the exceptional ones.

The problem of finding an uniform proof of Reeder's Conjecture is still unsolved but we have to mention that in [14] the authors proposed a possible strategy to approach this problem. They prove the existence of a graded map

$$\Phi_V : \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, \bigwedge \mathfrak{g}) \to \mathrm{Hom}_W(V_\lambda^0, \bigwedge \mathfrak{h} \otimes \mathcal{H})$$

and verified that this is an isomorphism for the adjoint representation. Furthermore, they formulated the following conjecture:

Conjecture 0.0.10 (De Concini - Papi)**.** *The map Φ_V is injective for all finite dimensional $\mathfrak{g}$ representations V.*

Recalling that, if V is a small representation, the equality

$$\mathrm{dimHom}_G(V, \Lambda\mathfrak{g}) = \mathrm{dimHom}_W(V_\lambda^0, \Lambda\mathfrak{h} \otimes \mathcal{H})$$

holds by ([30], Corollary 4.2), the injectivity would then imply that Φ_V is an isomorphism, obtaining as a consequence an uniform proof of Reeder's conjecture.

Finally, we have to point out that small representations have been recently on the focus of some papers in geometric representation theory (see [1], [31], [32] and [33]), where such representations (and their zero weight spaces) appear in connection with Satake and Springer correspondences. However, Reeder's Conjecture has been considered in this more geometrical setting only very recently in [34] by Reiner and Shepler in the context of their study of invariant derivations and differential forms in complex reflection groups.

In the first chapter of the thesis we presents the tools involved in our proof of the Reeder's Conjecture for the Lie algebras of type B and C. We study the structure of the zero weight space for these algebras and find closed formulae for the polynomials P_W using the results of [18]. Moreover we will study the polynomials $C_\mu(q,t)$ recalling the so called "minuscule" and "quasi minuscule" recurrences proved by Stembridge in [37].

The following two chapters are dedicated to our proofs of the Reeder's Conjecture. Using Stembridge's recurrences to prove the Reeder's Conjecture we have to solve an upper triangular system of linear equations with polynomials coefficients.

In the second chapter we prove the Conjecture in the case of odd orthogonal groups. Starting from the Stembridge's minuscule recurrence and using the combinatorics of weights and the action of the Weyl group we find some nice closed expressions of the coefficients. Some consequent simplifications allow us to rearrange the recurrence and reduce it to a two terms relation between the polynomials C_μ. An inductive reasoning concludes the proof in this case.

For the proof in the case C_n, contained in Chapter 3, we change our strategy and use the quasi minuscule recurrence. Fixed a weight λ the coefficients of the recurrence for C_λ are described in terms of some suitable subsets in the orbits of (λ, θ) by the action of the Weyl group, where θ denotes the highest root. We use the combinatorics of weights to express the associated coefficients in a recursive way. In the case of small weights of the form ω_{2k} the system of linear equations reduces easily to a two terms recursion that we solve using again an inductive process. For the weights $\omega_1 + \omega_{2k+1}$ the problem is more complicated: the zero weight representation V_λ^0 of the Weyl group is non irreducible (except in the case $k = 0$) and the combinatorics of the coefficients is more complex. We prove that the system of equation for $C_{\omega_1+\omega_{2k+1}}$ can be reduced to a three terms relations involving $C_{\omega_{2k}}$ and $C_{\omega_{2(k+1)}}$. This allow us to reduce the proof of the conjecture to prove a univariate polynomials identity that we have verified with the program SAGE.

Finally in the fourth Chapter we briefly expose some results about Kostant's Conjecture in the case of Symplectic groups. Berestein and Zelevinsky in [4] prove that the irreducible components appearing in $V_\mu \otimes V_\nu$ are in bijection with integral points in some polytopes described by suitable inequalities. We give an explicit construction of these integral points for the family of weights with shape contained in the shape of 2ρ. Moreover we conjecture that a statement similar to the Kostant conjecture holds for the exterior algebra of little adjoint representation: the irreducible weights appearing in Λ_{θ_s} are exactly the dominant weights smaller or equal to $2\rho_s$ where ρ_s is half the sum of the short roots. We prove finally that our conjecture is trivial for the Lie Algebras of type B by a theoretical argument and for F_4 and G_2 by direct computation. For type C it seems to be as difficult as the Kostant conjecture by the great number of weights smaller or equal to $2\rho_s$.

Chapter 1

Poincaré polynomials for graded multiplicities and Reeder's Conjecture

In this chapter we present an overview on the combinatorial aspects of representation theory of Lie algebras of type C and B we are going to use in our work.

After introducing some classical results about weights and representations, we will recall some recursive identities proved in [37] for the polynomials $P(V_\lambda, \Lambda\mathfrak{g}, q)$ and explicit closed formulae (see [1]) for $P_W(V_\lambda^0, \mathcal{H}, x, y)$.

1.1 Combinatorics of Representation for Lie Algebras of type B and C

Let $\mathfrak{g}$ be a finite dimensional simple Lie Algebra over $\mathbb{C}$ of rank n and let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. The action of $\mathfrak{h}$ on $\mathfrak{g}$ induces a root space decomposition. Let us denote by Φ the corresponding root system and by W its Weyl Group. Choose a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and denote with Φ^+ the corresponding set of positive roots and with $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots.

Finally we denote with P and Q respectively the weight and coweight lattice, i.e. the set of $\lambda \in \mathfrak{h}$ such that $\langle\lambda, \alpha\rangle := 2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ (resp. the set of $\lambda \in \mathfrak{h}$ such that $(\lambda, \alpha) \in \mathbb{Z}$) for all $\alpha \in \Phi$, and with P^+ the set of dominant weights, i.e. $\langle\lambda, \alpha\rangle \geq 0$ for all $\alpha \in \Phi^+$. We recall that the set P is spanned by the fundamental weights $\{\omega_1, \ldots, \omega_n\}$, defined by $\langle\omega_i, \alpha_j\rangle = \delta_{ij}$, as $\mathbb{Z}$- module. The *Weyl vector* is the sum of fundamental weights and is denoted by ρ.

Remark 1.1.1 (Dominance ordering). The choice of a positive set of roots Φ^+ induces an ordering on the weights: we say that a weight λ is greater than μ if and only if $\lambda - \mu$ is a sum of positive roots.

Remark 1.1.2. In the dominance order, for every root length, there exists a unique maximal element in Φ, corresponding to the only dominant root of such length. We will refer to this roots as "highest" long and short roots an we will denote them as θ and θ_s.

We can now examine more closely the root systems that we are interested in.

The root systems B_n and C_n can be explicitly constructed starting from a real euclidean space of dimension n with orthonormal basis $e_1, \dots, e_n$ with respect to a positive definite inner product $(\ ,\ )$

Remark 1.1.3 (Construction of Root System B_n). The set of roots Φ is made by the vectors of the form $\pm e_i \pm e_j$ and by the vectors $\{\pm e_1, \dots, \pm e_n\}$. We choose as simple roots the set $e_1 - e_2, \dots, e_{n-1} - e_n, e_n$. This choice implies that the set of positive roots is made by vectors of the form $e_i \pm e_j$ with $i < j$ and by $\{e_1, \dots, e_n\}$. The highest root for the induced dominance order is $\theta = e_1 + e_2$

In this realization the fundamental weights are $\omega_i = e_1 + \cdots + e_i$ if $i < n$ and $\omega_n = \frac{e_1 + \cdots + e_n}{2}$. Moreover the Weyl vector ρ is equal to $\frac{1}{2} \sum (2n - 2j + 1) e_j$.

Remark 1.1.4 (Construction of Root System C_n). We can chose as set of roots Φ the vectors of the form $\pm e_i \pm e_j$ and the vectors $\{\pm 2e_1, \dots, \pm 2e_n\}$. We choose as simple roots the set $e_1 - e_2, \dots, e_{n-1} - e_n, 2e_n$.

The set of positive roots is made by vectors of the form $e_i \pm e_j$ with $i < j$ and by $\{2e_1, \dots, 2e_n\}$. The highest root is $\theta = 2e_1$

The fundamental weights are all of the form $\omega_i = e_1 + \cdots + e_i$. The Weyl vector is $\rho = \sum (n - j + 1) e_j$

In both cases the Weyl Group is the Hyperoctaedral Group $S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$. By abuse of notation, when the context appears clear, we will denote this group with B_n.

We briefly remark some classical results and properties of representations of Lie Algebras.

Remark 1.1.5. Let V be a finite dimensional representation of $\mathfrak{g}$. The diagonalizable action of $\mathfrak{h}$ on V induce a weight spaces decomposition

$$V = \bigoplus_{\mu \in \Pi} V^\mu$$

The set Π characterizes completely the finite dimensional irreducible representation of $\mathfrak{g}$:

Theorem 1.1.6. *Let V a finite dimensional representation of $\mathfrak{g}$ over $\mathbb{C}$.*

- *The set Π is saturated, i.e. if $\mu \in \Pi$ then all its W-conjugates are in Π;*
- *V is irreducible if and only if there exists a unique dominant integral weight λ maximal between the weights on Π for the dominance order.*

We will denote the highest weight module of highest weight λ by V_λ.

Remark 1.1.7. The Weyl Group acts linearly on the zero weight space of V_λ, hence V_λ^0 is a finite dimensional representation of the finite group W.

A complete description of these representations in the classical cases can be found in [16] and [21].

1.2 Weyl Group Representations

In this section we display the closed formulae that express explicitly the polynomials P_W appearing in the "Weyl Group" part of the Reeder Conjecture. These formulae are proved in [2], where the authors are interested to some properties of special matrix-valued generating functions linked to the characters of Weyl groups.

Let χ be a character for a Weyl group W; following notation of [2] we define the rational function:

$$\tilde{\tau}(\chi; x, y) := \frac{1}{\prod_{i=1}^{n}\left(1 - x^{m_i+1}\right)} \sum_{p,q \geq 0} \operatorname{dimHom}(\chi, \mathcal{H}^q \otimes \Lambda^p \mathfrak{h}) x^q y^p \qquad (1.2.1)$$

where $\mathcal{H}^n$ denotes the space W-harmonic polynomials on $\mathfrak{h}$ of degree n, i.e. the polynomials of degree n on h that are annihilated by W-invariants costrant coefficients differential operators with zero constant term. Here the numbers $m_1, \ldots, m_n$ are the exponents of the Root System associated to W (as defined in [22]).

We recall that in our Lie theoretic context the Weyl Group acts on the Cartan subalgebra $\mathfrak{h}$ as the reflection representation. The above expression is then linked to the polynomial P_W by a simple algebraic relation:

$$\begin{aligned} \tilde{\tau}(\chi; x, y) \prod_{i=1}^{n}\left(1 - x^{m_i+1}\right) &= \\ \sum_{p,q \geq 0} \operatorname{dim} Hom(\chi, \mathcal{H}^q \otimes \Lambda^p \mathfrak{h}) x^q y^p &= \\ P_W(\chi, x, y). \end{aligned}$$

In [2] the authors obtain closed formulae for $\tilde{\tau}$ using the combinatorics of representations of Weyl Group W. In our case, we are interested to the hyperoctaedral group, i.e. the Weyl Group of type B_n. If we consider a pair (α, β) of partition such that $|\alpha| = k$, $|\beta| = h$ and $k + h = n$, we can construct a representation of the Weyl Group W of type B_n in the following way

- Each partition in the pair (α, β) defines a representation of symmetric groups S_k and S_h. We will denote these representation as π_α and π_β.
- We can view the group S_k as subgroups of Weyl groups of type B_k and consider the representations χ_α of B_k such that $\chi_\alpha \simeq \pi_\alpha$ as S_k- representation and such that $(\mathbb{Z}/2\mathbb{Z})^k$ acts on it trivially.
- Considering similarly S_h as subgroup of B_h, we will denote with χ_β the representation such that $\chi_\beta \simeq \pi_\beta$ as S_h representation tensored with the sign representation of $(\mathbb{Z}/2\mathbb{Z})^h$.
- Let us denote with $\chi_{\alpha,\beta}$ the induced representation $\operatorname{Ind}_{B_k \times B_h}^{B_n} \chi_\alpha \otimes \chi_\beta$.

It can be shown that $\chi_{\alpha,\beta}$ is irreducible and that all the irreducible representations of B_n can be constructed in this way.

Theorem 1.2.1 ([19], Proposition 3.3)**.** *Let $\chi_{\alpha,\beta}$ a representation of the Weyl Group W as described above. Then the following formula hold:*

$$\tilde{\tau}(\chi_{\alpha,\beta}; x, y) = x^{2n(\alpha)+2n(\beta)+|\beta|} \prod_{(i,j)\in\alpha} \frac{1 + yx^{2c(i,j)+1}}{1 - x^{2h(i,j)}} \prod_{(i,j)\in\beta} \frac{1 + yx^{2c(i,j)-1}}{1 - x^{2h(i,j)}}, \tag{1.2.2}$$

where $h(i,j)$, $c(i,j)$ are respectively the hook lenght of the box (i,j) and its content $i-j$ in the partition, displayed in the English way. Moreover, for a partition $\lambda = (\lambda_1, \geq \lambda_2, \ldots, \geq, \lambda_n)$, $n(\lambda)$ denotes the quantity $\sum_{i=1}^{n}(i-1)\lambda_i$.

Remark 1.2.2. We can rearrange the formula (1.2.2)as follows, showing a link with the box structure of the partitions α and β:

$$x^{2n(\alpha)+2n(\beta)+|\beta|} \prod_{(i,j)\in\alpha} \frac{1 + yx^{2c(i,j)+1}}{1 - x^{2h(i,j)}} \prod_{(i,j)\in\beta} \frac{1 + yx^{2c(i,j)-1}}{1 - x^{2h(i,j)}} =$$
$$\left(x^{2n(\alpha)} \prod_{(i,j)\in\alpha} \frac{1 + yx^{2c(i,j)+1}}{1 - x^{2h(i,j)}} \right) \cdot \left(x^{2n(\beta)+|\beta|} \prod_{(i,j)\in\beta} \frac{1 + yx^{2c(i,j)-1}}{1 - x^{2h(i,j)}} \right)$$

Combinatorially we can interpret this rewriting in the following way: In the first partition a box in the i-th row of the shape gives a contribution equal to $2(i-1)$ to the exponent of x, otherwise such a contribution for the second partition is equal to $2(i-1)+1$. We then arrive to the explicit formula we are going to use in the following chapters.

$$\tilde{\tau}(\chi_{\alpha,\beta}; x, y) = \prod_{(i,j)\in\alpha} \frac{x^{2(i-1)} + yx^{2j-1}}{1 - x^{2h(i,j)}} \prod_{(i,j)\in\beta} \frac{x^{2i-1} + yx^{2(j-1)}}{1 - x^{2h(i,j)}} \tag{1.2.3}$$

1.3 Small Representations

In the next chapters we will deal with zero weight spaces of special irreducible representations V_λ such that the maximal weight λ is in some sense "very close " to 0.

Definition 1.3.1 (Small Representations)**.** *Let λ be a dominant weight in the root lattice. We say that V_λ is* small *if $\lambda \ngeq 2\alpha$ for every dominant root α.*

The small representations for classical groups and the structure of their zero weight spaces as W representation are classified in the simply laced cases by Reeder in [31]. In [26] Kostant attributes the complete description of zero weight spaces to Chari and Pressley. Such a classification for Lie Algebras B_n and C_n is displayed in the following

Table 1.1: Zero weight spaces of small representations: Type B

Small Representation	**Zero Weight Space**
Highest weight	(α, β) description
ω_i, $i < n$, $i = 2k$	$((n-k),(k))$
ω_i, $i < n$, $i = 2k+1$	$((k),(n-k))$
$2\omega_n$, $n = 2k$	$((k),(k))$
$2\omega_n$, $n = 2k+1$	$((k),(k+1))$

tables. In the right column of the tables are reported the pair corresponding to the irreducible components of V_λ^0.

In particular in the odd orthogonal case the zero weight space representations are all irreducible. This does not happen for the symplectic group.

Table 1.2: Zero weight spaces of small representations: Type C

Small Representation	**Zero Weight Space**
Highest weight	(α, β) description
$2\omega_1$	$((n-1),(1))$
ω_{2i}	$((n-i,i),\emptyset)$
$\omega_1 + \omega_{2i+1}$, $0 < i$	$((n-i-1,i),(1)) \oplus ((n-i-1,i,1),\emptyset)$

Here the right part of the last row denotes that the zero weight space of representations $V_{\omega_1+\omega_{2i+1}}$ is not irreducible as W representations and splits as the direct sum of two W-representations indexed by partitions in the table.

1.4 Macdonald Kernels

In this section we remark some of the results due to Stembridge (see [37]) about coefficients in the expansion of Macdonald Kernels. We are in the general context of any irreducible root system Φ associated to a Lie Algebra $\mathfrak{g}$.

Definition 1.4.1 (Macdonald Kernel)**.**

$$MDK(q,t) := \prod_{i\geq 0}\left(\frac{1-q^{i+1}}{1-tq^i}\right)^n \cdot \prod_{i\geq 0}\prod_{\alpha\in\Phi}\frac{1-q^{i+1}e^\alpha}{1-tq^ie^\alpha}.$$

This expression defines a formal power series with coefficients in the ring $\mathbb{Z}\langle e^\alpha\rangle_{\alpha\in\Phi}$ and is clearly invariant for the action of the Weyl Group of Φ.

Remark 1.4.2. The Weyl characters χ_λ (i.e. the characters of irreducible representations of $\mathfrak{g}$) are a free $\mathbb{Z}$-basis of the ring $\mathbb{Z}\langle e^\alpha\rangle_{\alpha\in\Phi}$.

We can then write the coefficients of $MDK(q,t)$ in Weyl Characters - expansion obtaining:

$$MDK(q,t) = \sum_{\mu \in P^+} C_\mu(q,t)\chi_\mu$$

Actually the polynomials $C_\mu(q,t)$ are elements of $\mathbb{Z}[q,t]$ and they inherit from Macdonald Kernels some properties related to graded multiplicities in exterior and symmetric algebras (see [3] and [37]):

- $C_\mu(0,t)$ is the Poincaré polynomial of graded multiplicities of the representation V_μ in the symmetric slgebra $S\mathfrak{g}$
- The polynomial $C_\mu(-q,q^2)$ gives the graded multiplicities of the representation V_μ in the exterior algebra $\Lambda\mathfrak{g}$

We remark that the definition of polynomials C_μ can be extended to every integral weight $\mu \in \Pi$ using the following rule:

$$C_\mu(q,t) = \begin{cases} 0 & \text{if } \mu+\rho \text{ is not regular,} \\ (-1)^{l(\sigma)}C_\lambda & \text{if } \sigma(\mu+\rho) = \lambda+\rho \text{ , } \lambda \in P^+, \sigma \in W. \end{cases} \tag{1.4.1}$$

If there exists σ such that $\sigma(\lambda+\rho) = \mu+\rho$ we will say that $\lambda+\rho$ is *conjugated* to $\mu+\rho$ (or equivalently that λ is *conjugated* to μ) and we will write $\lambda+\rho \sim \mu+\rho$.

The polynomials $C_\mu(q,t)$ are the main object of [37]. In that paper Stembridge proposes some recursive relations to explicitly determine their expression. We will use two of them to prove Reeder's Conjecture for Lie Algebras of type B_n and C_n.

Before recalling Stembridge's recursive relations we have to introduce some more definition concerning some special coweights.

Definition 1.4.3. *We say that a weight (resp. coweight) ω is* minuscule *if $(\omega,\alpha^\vee) \in \{0,\pm1\}$ (resp. (ω,α)) for all positive roots and that it is* quasi minuscule *if $(\omega,\alpha^\vee) \in \{0,\pm1,\pm2\}$ (resp. (ω,α)) for all positive roots.*

Example 1.4.4. If we consider the Lie Algebra B_n as realized in 1.1.3, it is clear by direct inspection that $\omega = e_1$ is a minuscule coweight.

In our construction for the Lie Algebra C_n the coweight $\theta^\vee = e_1$ is a quasi minuscule coweight.

In our work we will use the following recursions as an effective tool to compute explicitly the polynomials C_μ.

$$\sum_{w\in W} \left(t^{-(\rho,w\omega)} - q^{(\mu,\omega)}t^{(\rho,w\omega)}\right) C_{w\mu}(q,t) = 0, \tag{1.4.2}$$

where ω is a minuscule coweight. The previous expression can be modified to obtain a simpler formula involving stabilizer W_μ of μ in Weyl Group W.

$$\sum_{i=1}^{k} C_{w_i\mu}(q,t)\left(\sum_{\psi\in O_\omega}\left(t^{-(\rho,w_i\psi)}-q^{(\mu,\omega)}t^{(\rho,w_i\psi)}\right)\right)=0. \tag{1.4.3}$$

Here $w_1,\dots,w_k$ are minimal coset representatives of W/W_μ and O_ω is the orbit $W_\mu\cdot\omega$. We are going to call this recursive relation as *minuscule recurrence.*

In the case $\mathfrak{g}$ is not simply laced, the coroot $\theta^\vee$ is a quasi minuscule coweight. Set $\omega=\theta^\vee$. In such a case Stembridge proves the following recurrence:

$$\sum_{(\lambda,\beta)}\sum_{i\geq 0}\left[f_i^\beta(q,t)-q^{(\mu,\omega)}f_i^\beta(q^{-1},t^{-1})\right]C_{\mu-i\beta}(q,t)=0. \tag{1.4.4}$$

Here the pairs (λ,β) are elements of the set $\{(w\mu,w\theta)|w\in W\ and\ w\theta\geq 0\}$ and the rational functions f_i^β are are linked to the coefficients of a special generating polynomial:

$$(1-tz)(1-qtz)\frac{((t^2z)^{(\rho,\beta^\vee)}-1)}{t^2z-1}=\sum_{i\geq 0}t^{(\rho,\beta^\vee)}f_i^\beta(q,t)z^i.$$

We will refer to the recursive relation (1.4.4) as *quasi-minuscule recurrence.*

Considering the recurrences 1.4.3 and 1.4.4 with the C_ν polynomials in their reduced form, Stembridge proves a nice characterization of the weights appearing in the recursion associated to the general dominant weight μ.

Remark 1.4.5. The dominant weights appearing in the reduced form of recurrence 1.4.3 and 1.4.4 are exactly the dominant weights smaller or equal to μ in dominance order.

As observed by Stembridge in [37], the set of recurrences associated to weights smaller or equal to μ define a triangular system, hence we can use a recursive process to obtain computable expressions for C_μ.

More precisely, our strategy is the following: we determine closed formulae for the polynomials of the "Weyl Group" part of Reeder's Conjecture(explicitly computed in [18]), after that we prove by induction that this closed formulae satisfy the Stembridge's specialized recursive relations.

Chapter 2

Reeder's Conjecture: the B_n case

In this chapter we will present a proof of the Reeder's Conjecture in the case of odd orthogonal groups. I would like to thank the professors De Concini and Papi for providing me the sketches of their computations in this case. First of all we find closed expressions for the polynomials P_W for the representations in the Table 1.3. After that we find formulae for the solutions of system of the linear equations given by recurrences 1.4.3, using combinatorical reasoning to simplify their coefficients. We finally obtain the Reeder's Conjecture proving the equality between these two polynomials.

2.1 Small Representations for Odd Orthogonal Groups

As mentioned before in this section we will find closed formulae for the polynomials $P_W(V_\lambda^0, x, y)$, where the zero weight spaces V_λ^0 are the ones listed in Table 1.3.

We recall that we are interested to the specializations $x \to q^2$ and $y \to q$ of these polynomials. We are going to use extensively the q-analogue notation, which we will now recall:

Definition 2.1.1 (q-analogue). *Let n be a natural number. The q-analogue of n is the polynomial*

$$(n)_q = \frac{q^n - 1}{q - 1} = q^{n-1} + \cdots + 1$$

Analogously to the case of natural numbers, we can define the the q-factorial and the q-binomial:

$$(n)_q! = \prod_{i=1}^{n} (i)_q$$

$$\binom{n}{k} = \frac{(n)_q!}{(k)_q!(n-k)_q!}$$

This notation allows us to obtain more compact and handy formulae for P_W.

Let us start with the case of generic fundamental weight ω_i with i even and smaller than n. Suppose $i = 2k$, then we obtain

$$P_W(\chi_{(n-k)(k)}, x, y) =$$

$$\prod_{j=1}^{n-k} \frac{1 + yx^{(2j-1)}}{1 - x^{2(n-k-j+1)}} = \prod_{l=1}^{k} \frac{y^2 + y \cdot x^{(2l-2)}}{1 - x^{2(k-l+1)}} \prod_{i=1}^{n} (1 - x^{2i}) =$$

$$= \binom{n}{k}_{x^2} \prod_{j=1}^{n-k} (1 + yx^{(2j-1)}) \prod_{l=1}^{k} (y^2 + y \cdot x^{(2l-2)}),$$

which leads us to the specialized formula

$$P_W(\chi_{(n-k)(k)}, q^2, q) = q^{2k-1}(q+1)\binom{n}{k}_{q^4} \prod_{j=1}^{n-k} (1 + q^{4j-1}) \prod_{l=1}^{k-1} (1 + q^{4l-1}). \tag{2.1.1}$$

Otherwise, if i is smaller then n but odd, supposing $i = 2k+1$, a very similar computations gives the formula

$$P_W(\chi_{(k)(n-k)}, x, y) =$$

$$\prod_{j=1}^{k} \frac{1 + yx^{(2j-1)}}{1 - x^{2(n-k-j+1)}} = \prod_{l=1}^{n-k} \frac{y^2 + y \cdot x^{(2l-2)}}{1 - x^{2(k-l+1)}} \prod_{i=1}^{n} (1 - x^{2i}) =$$

$$\binom{n}{k}_{x^2} \prod_{j=1}^{k} (1 + yx^{(2j-1)}) \prod_{l=1}^{n-k} (y^2 + y \cdot x^{(2l-2)}),$$

so that

$$P_W(\chi_{(k)(n-k)}, q^2, q) = q^{2(n-k)-1}(q+1)\binom{n}{k}_{q^4} \prod_{j=1}^{k} (1 + q^{4j-1}) \prod_{l=1}^{n-k-1} (1 + q^{4l-1}). \tag{2.1.2}$$

For $2\omega_n$, we have to distinguish two cases, depending on the parity of n. We recall that if $n = 2k$, the zero weight space $V^0_{2\omega_n}$ is isomorphic to the representation $\chi_{(k)(k)}$. Otherwise, if $n = 2k+1$, it is isomorphic to $\chi_{(k),(k+1)}$. The polynomial $P_W(\chi, q^2, q)$ has then the following form:

$$P_W(\chi_{(k)(k)}, q^2, q) = q^{2k-1}(q+1)\binom{2k}{k}_{q^4} \prod_{j=1}^{k} (1 + q^{4j-1}) \prod_{l=1}^{k-1} (1 + q^{4l-1}), \tag{2.1.3}$$

$$P_W(\chi_{(k)(k+1)}, q^2, q) = q^{2(k+1)-1}(q+1)\binom{2k+1}{k}_{q^4} \prod_{j=1}^{k} (1 + q^{4j-1}) \prod_{l=1}^{k} (1 + q^{4l-1}). \tag{2.1.4}$$

2.2 The minuscule recurrence.

Let us start considering the recurrence 1.4.3 and make the evaluation $q \to -q$ and $t \to q^2$. We recall that in this case $\rho = \frac{1}{2}\sum(2n-2j+1)e_j$, the small weights are of the form $\omega_i = e_1 + \cdots + e_i$ for $i < n$ and $2\omega_n = e_1 + \cdots + e_n$. Moreover we can choose $\omega = e_1$ as minuscule coweight, thus for all non zero small weights λ of B_n, the inner product (λ, ω) is equal to 1. If we set $\lambda = \omega_k$, Stembridge's recurrence 1.4.3 becomes consequently:

$$\sum_{i=1}^{l} C_{w_i\lambda} \sum_{j=1}^{k} (q^{-2(\rho, w_i e_j)} + q^{1+2(\rho, w_i e_j)}) = 0. \tag{2.2.1}$$

A similar results with the index of the second sum running from 1 to n holds for $\lambda = 2\omega_n$. We consider now, instead of the above recurrence, the one where all the C_μ are in reduced form. Such a recurrence is then of the following form:

$$\sum_{\mu \leq \lambda} \Gamma^\lambda_\mu(q) C_\mu(q) = 0 \tag{2.2.2}$$

for some coefficients $\Gamma^\lambda_\mu(q)$ (denoted respectively as Γ^k_i if $\lambda = \omega_k$, $\mu = \omega_i$ and as Γ^n_i if $\lambda = 2\omega_n$, $\mu = \omega_i$). We will refer to this recurrence as the *reduced* one. Our purpose is to make more explicit the coefficients of the polynomials C_μ using the combinatorics of the Weyl group.

Remark 2.2.1. The weight appearing as index of the polynomials C_μ in 2.2.2 are the ones smaller than λ in the dominance order. Then if $\lambda = \omega_k$ in the reduced reccurence appears the C_μ indexed by the weights 0 and ω_i with $i \leq k$. In the case $\lambda = 2\omega_n$ all the C_{ω_i} with $i < n$ and $C_{2\omega_n}$ appear.

Set

$$c_m = \frac{1-q^{2m}}{1-q^2} q^{-2n+1}(1 + q^{4n-2m+1}), \tag{2.2.3}$$

$$b_n = (q+1)q^{-2n+2}\frac{1-q^{4n-2}}{1-q^2}. \tag{2.2.4}$$

Proposition 2.2.2. *Set $\lambda = \omega_k$ or $\lambda = 2\omega_n$. The coefficient of C_λ in the reduced recurrence is c_k*

Proof. We will prove that $C_{w_i\lambda} = C_\lambda$ if and only if $w_i = id$. Let w be an element of the Weyl Group, first of all observe that the first $n-1$ coordinates of $w\lambda + \rho$ are all positive independently from the choice of w. Moreover the last one can be equal only to $\pm 1/2$ or to $3/2$. Then we must obtain $\lambda + \rho$ from $w\lambda + \rho$ just by the action of the Symmetric group S_n and, eventually, by the change of sign on the last coordinate. Now we have to impose the condition $w\lambda + \rho \sim \lambda + \rho$. If all the coordinates of $w\lambda + \rho$ are positive, such coordinates must be a permutation of the ones of $\lambda + \rho$, i.e. of the vector $(\frac{2n+1}{2},\dots,\frac{2n-2i+3}{2},\frac{2n-2i-1}{2},\dots\frac{1}{2})$. The only way to obtain a coordinate equal to $\frac{2n+1}{2}$ between the ones of $w\lambda+\rho$ is that $(w\lambda)_1=1$. Iterating this reasoning on all the coordinates we obtain that $(w\lambda)_j=1$ for all $j\le k$ (resp $j\le n$) and w have to act on the fist k (resp n) coordinates as an element of the symmetric group S_k (resp. S_n). Consequently it must be in the stabilizer of λ and then be in the same class of id in the quotient W/W_λ. Conversely suppose $(w\lambda+\rho)_n=-\frac{1}{2}$ and let be $e_j=-w^{-1}(e_n)$. Consider λ_j to be the weight with all coordinates equal to the ones of λ, except for the j-th that we set equal to 0. We obtain that $w\lambda+\rho$ is the conjugated to $w\lambda_j+\rho$. The latter one cannot be conjugated to $\lambda+\rho$ because we can "add 1" at most in $k-1$ (resp. $n-1$) coordinates and the the resulting weight can not be a permutation of $(\frac{2n+1}{2},\dots,\frac{2n-2i+3}{2},\frac{2n-2i-1}{2},\dots\frac{1}{2})$. Recalling that conjugation is an equivalence relation we obtain an absurd and this case cannot happen.

Now it is easy to compute explicitly the coefficient of C_λ:

$$\Gamma_\lambda^\lambda=\sum_{j=1}^{i}(q^{-2(\rho,e_j)}+q^{1+2(\rho,e_j)})=$$
$$\sum_{j=1}^{i}(q^{-(2n-2j+1)}+q^{1+(2n-2j+1)})=$$
$$q^{-2n+1}\left(\sum_{j=1}^{i}(q^{2j-2}+q^{4n-2j+1})\right)=$$
$$q^{-2n+1}\left[\sum_{t=0}^{i-1}q^{2t}+q^{4n+1}\left(\sum_{j=1}^{i}q^{-2j}\right)\right]=$$
$$q^{-2n+1}\left[\sum_{t=0}^{i-1}q^{2t}+q^{4n-2k+1}\left(\sum_{j=0}^{i-1}q^{2j}\right)\right]=$$
$$\frac{q^{2i}-1}{q^2-1}q^{-2n+1}(1+q^{4n-2i+1})$$

□

We want now to find some similar formulae for the general coefficients Γ_i^k. We will start with the case of coefficient Γ_0^λ. We obtain different formulae, depending on the parity of k (resp. n). To obtain a more compact notation we need some suitably defined univariate polynomials we are going to introduce now.

Set $J(h,k,r)=\{((j_1,\dots,j_r))\mid h<j_1<j_2-1<\dots<j_r-(r-1)\le k\}$. We can then define

$$\Gamma(h,k;r)=\sum_{\underline{j}\in J(h,k,r)}(q+1)\sum_{s=1}^{r}(q^{2(j_s-1)}+q^{-2(j_s-1)}). \qquad (2.2.5)$$

Proposition 2.2.3. *Set again $\lambda = \omega_k$ or $\lambda = 2\omega_n$. If $k = 2s$ (resp $n = 2s$) the coefficient Γ_0^λ is equal to*

$$\Gamma_0^\lambda = (-1)^s \Gamma(1, n, s). \tag{2.2.6}$$

Otherwise, if $k = 2s + 1$ (resp $n = 2s + 1$),

$$\Gamma_0^\lambda = (-1)^{s+1} \left[\binom{n-s-1}{s} b_1 + \Gamma(2, n, s) \right]. \tag{2.2.7}$$

Proof. First of all, we want to understand when the weight $w_i\lambda + \rho$ is conjugated to ρ.

Lemma 2.2.4. *Let us suppose $w_i\lambda + \rho$ is conjugated to ρ. Then*

- If k is even *then $w_i\lambda$ has all the coordinates equal to zero except for $k/2$ pairs of consecutive coordinates of the form $(-1, 1)$. Moreover $w_i\lambda + \rho$ is conjugated to ρ by a permutation σ of signature equal to $(-1)^{k/2}$.*
- If k is odd *then $w_i\lambda$ has all the coordinates equal to zero, except for a choice of $(k-1)/2$ pairs of coordinates equal to $(-1, 1)$ and for the last one that must be equal to -1. In this case $w_i\lambda + \rho$ is conjugated to ρ by a permutation σ of signature equal to $(-1)^{(k-1)/2+1}$.*

Proof. As in the Proposition 2.2.2, we observe that the coordinates of $w_i\lambda + \rho$ are all positive independently by w_i except to the last one that can be equal to $-\frac{1}{2}$. Let us start with the case where all the coordinates of $w_i\lambda + \rho \sim \rho$ are all positive. In this case $w_i\lambda + \rho$ is just the vector obtained by a permutation of the coordinates of ρ. Moreover, we have some more restriction on the possible values of each coordinate of $w_i\lambda + \rho$: the coordinates of $w_i\lambda$ can be only equal to 0 o to ± 1. Consequently the first coordinate can be equal to n or $n - 1$ only. In the first case there are no more restriction on the second coordinate: iterating the reasoning, it can be equal to $n - 2$ or $n - 1$. Otherwise, the second one is forced to be $(w_i\lambda + \rho)_2 = n$, nevertheless we cannot obtain a coordinate equal to n and $w_i\lambda + \rho$ cannot be conjugated to ρ. In other words our request forces $w_i\lambda$ to be of the form $(0, \lambda')$ or of the form $(-1, 1, \lambda'')$. Iterating the reasoning on λ' and λ'' we obtain that $w_i\lambda$ is obtained by choosing pair of consecutive indices $(k, k + 1)$ and setting $(w_i\lambda)_k = -1$ and $(w_i\lambda)_{k+1} = 1$. We have to underline that this analysis holds only if k is even, and in this case the number of chosen pair is exactly $k/2$. If k is odd a similar analysis shows that $w_i\lambda$ is obtained choosing $(k - 1)/2$ pair of consecutive indices and setting $(w_i\lambda)_k = -1$ and $(w_i\lambda)_{k+1} = 1$, moreover there is one "unpaired" coordinate that gives a contribution of ± 1 to some coordinate. This contribution cannot be equal to 1, otherwise $w_i\lambda + \rho$ is not regular. The only possible case is then that its contribution is on the n-th coordinate: in all the other cases can be checked easily that $w_i\lambda + \rho$ is not regular. The statements about the signature of σ are immediate by the above construction. □

As an immediate corollary we can compute the number of the weights that give a contribution to Γ_0^k.

Corollary 2.2.5. *Set* $Conj_0^{nk} = \{w_i \,|w_i\omega_k + \rho \sim \rho\}$*. Then*

$$|Conj_0^{nk}| = \begin{cases} \binom{n-\frac{k}{2}}{\frac{k}{2}} & \textit{if } k \textit{ is even} \\ \binom{n-\frac{k-1}{2}-1}{\frac{k-1}{2}} & \textit{if } k \textit{ is odd.} \end{cases}$$

Proof. By above lemma, the cardinality of $Conj_0^{nk}$ is equal to the number of choices of consecutive coordinates between n or $n-1$ coordinates respectively when k is even or odd. We can count the number of these possible choices in the following way: we can choose k indices from a the set $I = \{1 \dots n-k\}$ and expand our choices as a pair of consecutive indices. The assert of the lemma is now immediate. $\square$

The case $\lambda = 2\omega_n$ is exactely the same with n instead of k in the above formulae. Observe that to each element $\gamma \in Conj_0^{nk}$ is a associated a vector of indices $(j_1, \dots j_s)$ such that $\gamma_{j_h} = 1$ for $1 \leq h \leq s$. We will denote the set of these vectors as $J(n,k)$. Let us fix one of these vectors, say $v = (j_1, \dots j_s)$, and let $w_i\omega_k = \gamma_v \in Conj_0^{nk}$ be the associated weight. We want to understand which is the contribution of γ_v to Γ_0^k. Suppose first that $k = 2s$, by the above lemma we know that $w_i \cdot (W_{\omega_k} \cdot e_1) = w_i \cdot (e_1, \dots, e_k) = (-e_{j_1}, e_{j_1+1}, \dots, -e_{j_s}, e_{j_2+1})$. We obtain that γ_v contributes to Γ_0^k with a term equal to

$$\sum_{t=1}^{s} \left[q^{2(\rho,e_{j_t})} + q^{1-2(\rho,e_{j_t})} + q^{-2(\rho,e_{j_t+1})} + q^{1+2(\rho,e_{j_t+1})}\right] =$$
$$\sum_{t=1}^{s} \left[q^{2n-2j_t+1} + q^{-2n+2j_t} + q^{-2n+2j_t+1} + q^{2n-2j_t}\right] =$$
$$\sum_{t=1}^{s} (q+1)\left[q^{2(n-j_t)} + q^{-2(n-j_t)}\right].$$

Summing up on the vectors $v \in J(n,k)$ we obtain that for $k = 2s$

$$\Gamma_0^k = (-1)^s(q+1) \sum_{v \in J(n,k)} \sum_{t=1}^{s} \left(q^{2(n-j_t)} + q^{-2(n-j_t)}\right). \tag{2.2.8}$$

The case of $k = 2s+1$ is very similar, we have only to consider that the last coordinate of $w_i\omega_k$ must be equal to -1 and choose the pairs of consecutive indices between $\{1, \dots n-1\}$. To the previous expression we have to sum b_1, coming from the contribution of $(w_i\omega_k)_n = -1$

$$\Gamma_0^k = (-1)^{s+1} \sum_{v \in J(n-1,2s)} \left[(q+1)\sum_{t=1}^{s}\left(q^{2(n-j_t)} + q^{-2(n-j_t)}\right) + b_1\right] =$$
$$(-1)^{s+1}(q+1) \sum_{v \in J(n-1,2s)} \sum_{t=1}^{s} \left(q^{2(n-j_t)} + q^{-2(n-j_t)}\right) + \binom{n-k-1}{s} b_1.$$

Now we have just to prove that

$$\Gamma(1,n,s) = (q+1) \sum_{v \in J(n,2s)} \sum_{t=1}^{s} \left(q^{2(n-j_t)} + q^{-2(n-j_t)} \right) \tag{2.2.9}$$

and

$$\Gamma(2,n,s) = (q+1) \sum_{v \in J(n-1,2s)} \sum_{t=1}^{s} \left(q^{2(n-j_t)} + q^{-2(n-j_t)} \right) \tag{2.2.10}$$

We remark that $J(n,k) = J(0,n-1,s)$ in the even case and $J(n,k) = J(0,n-2,s)$ in the odd one. Now our conclusion is immediate observing that if $v = (j_1, \dots j_s) \in J(0,n-2,s)$ (resp $v \in J(0,n-1,s)$) then $v' = (n-j_s+1, \dots, n-j_1+1) \in J(2,n,s)$ (resp. $J(1,n,s)$).

□

We want now use this proposition to give formulae for generic coefficient Γ_0^k.

Proposition 2.2.6. *The coefficient Γ_i^k is equal to*

$$\binom{n-i-s}{s} c_i + \Gamma(1, n-i; s) \tag{2.2.11}$$

if $k-i = 2s$. Otherwise, if $k-i = 2s+1$ it is equal to

$$\binom{n-i-s-1}{s} (c_i + b_1) + \Gamma(2, n-i; s) \tag{2.2.12}$$

Proof. Again, we have first of all understand which weights of the form $w_i \omega_k$ give a contribution to the coefficient of C_i. Let us denote as

$$Conj_i^{n,k} = \{ w_j \omega_k \omega_k \mid w_j \omega_k + \rho \sim \omega_i + \rho \}$$

Lemma 2.2.7. *The weight $\gamma \in Conj_i^{n,k}$ gives a contribution to Γ_i^k if and only if*

$$\gamma = (1, \dots, 1, \gamma')$$

where $\gamma' \in Conj_0^{n-i,k-i}$.

Proof. Again we observe that all the coordinates of $\gamma + \rho$ are positive except for the last one that can be equal to $-\frac{1}{2}$ and then we can obtain $\omega_i + \rho$ just permuting coordinates and changing the sign to the last one. Imposing $\gamma + \rho \sim \omega_i + \rho$ it follows that the first i coordinates of γ must be equal to 1. It remains $k-i$ coordinates of γ to place in the way that the last $n-j$ coordinates of $\gamma + \rho$, rearranged by the action of S_n and eventually by the change of sign of the n-th coordinate, are equal to the last $n-j$ coordinates of ρ. Considering the immersion $B_{n-j} \to B_n$ induced by inclusion of Dynkin diagrams, this correspond to say that the restriction to B_{n-j} of γ is conjugated to 0, or, in other therms, that $\gamma' \in Conj_0^{n-i,k-i}$. □

Now the statement of proposition comes easily from the one for Γ_0^k observing that the single contribution of each vector $w_i\omega_k$ is equal to

$$c_i + (-1)^s \sum_{t=1}^{s} (q+1) \left[q^{2(n-j_t)} + q^{-2(n-j_t)}\right]$$

if $k - i = 2s$ and to

$$c_i + b_1 + (-1)^{s+1} \sum_{t=1}^{s} (q+1) \left[q^{2(n-j_t)} + q^{-2(n-j_t)}\right]$$

if $k - i = 2s + 1$. Summing up again on the set $Conj_i^{n,k}$ and using identities 2.2.9 and 2.2.10 we obtain the thesis. $\square$

Again, if $\lambda = 2\omega_n$ all the above resutls holds with n instead of k. For brevity we will denote with C_i the polynomial C_{ω_i}. We can now write the reduced recurrence in the following way :

$$C_m c_m = \sum_{i=0}^{[\frac{m-1}{2}]} (-1)^i C_{m-2i-1} \left(\binom{n-m+i}{i} (c_{m-2i-1} + b_1) + \Gamma(2, n-m+2i+1; i) \right) \tag{2.2.13}$$
$$+ \sum_{i=1}^{[\frac{m}{2}]} (-1)^{i-1} C_{m-2i} \left(\binom{n-m+i}{i} c_{m-2i} + \Gamma(1, n-m+2i; i) \right)$$

The coefficients Γ_i^k are now more explicit but less handy for a concrete computation. In the following proposition we prove an equivalent form of the recurrence with explicit and compact coefficients.

Proposition 2.2.8.

$$C_m c_m = \sum_{i=1}^{[\frac{m+1}{2}]} C_{m-2i+1} b_i + \sum_{i=1}^{[\frac{m}{2}]} C_{m-2i} b_{n-m+i+1}. \tag{2.2.14}$$

Proof. We will prove that the coefficient of $C_i, i < m$ in the right hand sides of (2.2.13) and (2.2.14) match. We start checking the statement for the coefficient of C_0.

Lemma 2.2.9. *The coefficient of C_0 in the expression 2.2.13 is equal to the coefficient of C_0 in 2.2.14.*

Proof. We want to use induction. Assume (2.2.14) holds for C_{m-h} $h \geq 1$. Then

$$C_{m-h}c_{m-h} = \sum_{i=0}^{[\frac{m-h-1}{2}]} C_{m-h-2i-1}b_{i+1} + \sum_{i=1}^{[\frac{m-h}{2}]} C_{m-h-2i}b_{n-m+h+i+1}. \qquad (2.2.15)$$

First consider the case $m = 2k+1$. If h is even, $m-h$ is odd and the coefficient of C_0 in (2.2.15) is $b_{(m-h-1)/2}$. If h is odd, $m-h$ is even the coefficient of C_0 in (2.2.15) is $b_{n+(m-h)/2+1}$. Substituting into (2.2.13), the coefficient of C_0 is

$$(-1)^k\left(\binom{n-1+k}{k}b_1+\Gamma(2,n;k)\right)+\sum_{s=0}^{k-1}(-1)^s\binom{n-m+s}{s}b_{n-k+s+1}+\sum_{r=1}^{k}(-1)^{r-1}\binom{n-m+r}{r}b_{k-r+1} \qquad (2.2.16)$$

We want to show that this coefficients equals b_{k+1}, which is the coefficient of C_0 in the right hand side of (2.2.14). This is in turn equivalent to prove the following equality.

$$\Gamma(2,n;k) = \sum_{s=0}^{k-1}(-1)^{s+k+1}\binom{n-2k-1+s}{s}b_{n-k+s+1}+\sum_{r=0}^{k-1}(-1)^{r+k}\binom{n-2k-1+r}{r}b_{k-r+1}. \qquad (2.2.17)$$

Set $\Psi = (q+1)(q^{2n-2}+q^{2-2n})$, we remark that the following relation holds

$$\Gamma(2,n;k) = \binom{n-k-2}{k-1}\Psi + \Gamma(2,n-2;k-1) + \Gamma(2,n-1;k). \qquad (2.2.18)$$

Since $\Psi + b_{n-1} = b_n$, we have by induction

$$\begin{aligned}
&\Gamma(2,n;k) = \\
&\binom{n-k-2}{k-1}\Psi+ \\
&+\sum_{s=0}^{k-2}(-1)^{s+k}\binom{n-2k-1+s}{s}b_{n-k+s}+\sum_{r=0}^{k-2}(-1)^{r+k+1}\binom{n-2k-1+r}{r}b_{k-r}+ \\
&+\sum_{s=0}^{k-1}(-1)^{s+k+1}\binom{n-2k+s-2}{s}b_{n-k+s}+\sum_{r=0}^{k-1}(-1)^{r+k}\binom{n-2k+r-2}{r}b_{k-r+1} = \\
&\binom{n-k-2}{k-1}\Psi+\binom{n-k-3}{k-1}b_{n-1}+ \\
&+\sum_{s=0}^{k-2}(-1)^{s+k}\left(\binom{n-2k-1+s}{s}-\binom{n-2k+s-2}{s}\right)b_{n-k+s} \\
&+(-1)^k b_{k+1}+\sum_{r=1}^{k-1}(-1)^{r+k}\left(\binom{n-2k+r-2}{r-1}+\binom{n-2k+r-2}{r}\right)b_{k-r+1} = (*)
\end{aligned}$$

Now using the well known identity $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$ we have:

$$(*) = \binom{n-k-2}{k-1}b_n - \binom{n-k-3}{k-2}b_{n-1} + \sum_{s=1}^{k-2}(-1)^{s+k}\binom{n-2k+s-2}{s-1}b_{n-k+s}+$$

$$+\sum_{r=0}^{k-1}(-1)^{r+k}\binom{n-2k-1+r}{r}b_{k-r+1}$$

$$\binom{n-k-2}{k-1}b_n-\binom{n-k-3}{k-2}b_{n-1}+\sum_{s=0}^{k-3}(-1)^{s+k+1}\binom{n-2k-1+s}{s}b_{n-k+s+1}+$$

$$+\sum_{r=0}^{k-1}(-1)^{r+k}\binom{n-2k-1+r}{r}b_{k-r+1}=$$

$$\sum_{s=0}^{k-1}(-1)^{s+k+1}\binom{n-2k-1+s}{s}b_{n-k+s+1}+\sum_{r=0}^{k-1}(-1)^{r+k}\binom{n-2k-1+r}{r}b_{k-r+1}.$$

Now assume $m=2k$. Proceeding as above, the equality to prove is

$$b_{n-k+1}=(-1)^{k-1}\Gamma(1,n;k)+\sum_{s=1}^{k-1}(-1)^{s-1}\binom{n-2k+s}{s}b_{n-k+s+1}+\sum_{r=0}^{k-1}(-1)^{r}\binom{n-2k+r}{r}b_{k-r} \tag{2.2.19}$$

or

$$\Gamma(1,n;k)=\sum_{s=0}^{k-1}(-1)^{k+s-1}\binom{n-2k+s}{s}b_{n-k+s+1}+\sum_{r=0}^{k-1}(-1)^{k+r}\binom{n-2k+r}{r}b_{k-r}$$

As for (2.2.18), we have

$$\Gamma(1,n;k)=\binom{n-k-1}{k-1}\Psi+\Gamma(1,n-2;k-1)+\Gamma(1,n-1;k). \tag{2.2.20}$$

Again we have by induction

$$\Gamma(1,n;k)=$$

$$\binom{n-k-1}{k-1}\Psi+\sum_{s=0}^{k-2}(-1)^{s+k}\binom{n-2k+s}{s}b_{n-k+s}+\sum_{r=0}^{k-2}(-1)^{r+k+1}\binom{n-2k+r}{r}b_{k-r-1}$$

$$+\sum_{s=0}^{k-1}(-1)^{s+k+1}\binom{n-2k+s-1}{s}b_{n-k+s}+\sum_{r=0}^{k-1}(-1)^{r+k}\binom{n-2k+r-1}{r}b_{k-r}=$$

$$\binom{n-k-1}{k-1}\Psi+\sum_{s=0}^{k-2}(-1)^{s+k}\left(\binom{n-2k+s}{s}-\binom{n-2k+s-1}{s}\right)b_{n-k+s}+$$

$$+\binom{n-k-2}{k-1}b_{n-1}+$$

$$+\sum_{r=1}^{k-1}(-1)^{r+k}\left(\binom{n-2k+r-1}{r}+\binom{n-2k+r-1}{r-1}\right)b_{k-r}+(-1)^k b_k=$$

$$\binom{n-k-1}{k-1}\Psi+\sum_{s=0}^{k-1}(-1)^{s+k}\binom{n-2k+s-1}{s-1}b_{n-k+s}$$

$$+\binom{n-k-2}{k-1}b_{n-1}+\sum_{r=1}^{k-1}(-1)^{r+k}\binom{n-2k+r}{r}b_{k-r}+(-1)^k b_k =$$

$$\binom{n-k-1}{k-1}\Psi+\sum_{s=0}^{k-3}(-1)^{s+k+1}\binom{n-2k+s}{s}b_{n-k+s+1}$$

$$+\binom{n-k-2}{k-1}b_{n-1}+\sum_{r=0}^{k-1}(-1)^{r+k}\binom{n-2k+r}{r}b_{k-r}=(*)$$

and again recalling the properties of binomial coefficients and using $\Psi + b_{n-1} = b_n$ we have

$$(*)=\binom{n-k-1}{k-1}b_n-\binom{n-k-2}{k-2}b_{n-1}+\sum_{s=0}^{k-3}(-1)^{s+k+1}\binom{n-2k+s}{s}b_{n-k+s+1}$$

$$+\sum_{r=0}^{k-1}(-1)^{r+k}\binom{n-2k+r}{r}b_{k-r}$$

$$=\sum_{s=0}^{k-1}(-1)^{k+s-1}\binom{n-2k+s}{s}b_{n-k+s+1}+\sum_{r=0}^{k-1}(-1)^{k+r}\binom{n-2k+r}{r}b_{k-r}.$$

$\square$

Now we want prove the equality for the other coefficients. We start identifing such coefficients in the equation 2.2.13 where we have substituted $C_{m-h}c_{m-h}$ with the corresponding expression 2.2.14.

$$C_m c_m = \sum_{i=0}^{[\frac{m-1}{2}]}(-1)^i\binom{n-m+i}{i}C_{m-2i-1}c_{m-2i-1}+\sum_{i=1}^{[\frac{m}{2}]}(-1)^{i-1}\binom{n-m+i}{i}C_{m-2i}c_{m-2i}+$$

$$+\sum_{i=0}^{[\frac{m-1}{2}]}(-1)^i C_{m-2i-1}\left(\binom{n-m+i}{i}b_1+\Gamma(2,n-m+2i+1;i)\right)+$$

$$+\sum_{i=1}^{[\frac{m}{2}]}(-1)^{i-1}C_{m-2i}\Gamma(1,n-m+2i;i)=$$

$$\sum_{i=0}^{[\frac{m-1}{2}]}(-1)^i\binom{n-m+i}{i}\left(\sum_{j=1}^{[\frac{m-2i}{2}]}C_{m-2i-2j}b_j+\sum_{j=1}^{[\frac{m-2i-1}{2}]}C_{m-2i-2j-1}b_{n-m+2i+j+2}\right)+$$

$$+\sum_{i=1}^{[\frac{m}{2}]}(-1)^{i-1}\binom{n-m+i}{i}\left(\sum_{j=1}^{[\frac{m-2i+1}{2}]}C_{m-2i-2j+1}b_j+\sum_{j=1}^{[\frac{m-2i}{2}]}C_{m-2i-2j}b_{n-m+2i+j+1}\right)+$$

$$+\sum_{i=0}^{[\frac{m-1}{2}]}(-1)^i C_{m-2i-1}\left(\binom{n-m+i}{i}b_1+\Gamma(2,n-m+2i+1;i)\right)+$$

$$+\sum_{i=1}^{[\frac{m}{2}]}(-1)^{i-1}C_{m-2i}\Gamma(1,n-m+2i;i).$$

Now set $m-h=2s$, then the coefficient of C_h in the above expression is

$$\sum_{i+j=s}\sum_{i=0}^{\lfloor\frac{m-1}{2}\rfloor}\sum_{j=1}^{\lfloor\frac{m-2i}{2}\rfloor}(-1)^i\binom{n-m+i}{i}b_j+$$
$$+\sum_{i+j=s}\sum_{i=1}^{\lfloor\frac{m-1}{2}\rfloor}\sum_{j=1}^{\lfloor\frac{m-2i}{2}\rfloor}(-1)^{i-1}\binom{n-m+i}{i}b_{n-h-j+1}+(-1)^{s-1}\Gamma(1,n-h,s)$$

Otherwise if $m-h=2s+1$, the coefficient of C_h is

$$\sum_{i+j=s}\sum_{i=0}^{\lfloor\frac{m-1}{2}\rfloor}\sum_{j=1}^{\lfloor\frac{m-2i-1}{2}\rfloor}(-1)^i\binom{n-m+i}{i}b_{n-h-j+1}+$$
$$+\sum_{i+j=s}\sum_{i=1}^{\lfloor\frac{m}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{m-2i-1}{2}\rfloor}(-1)^{i-1}\binom{n-m+i}{i}b_{j+1}+(-1)^s\left(\binom{n-m+s}{s}b_1+\Gamma(2,n-h;s)\right)$$

We can rearrange the indices observing that the conditions on j are redundant, obtaining that coefficients are

$$\gamma^p(n,m,h)=$$
$$\sum_{i=0}^{s-1}(-1)^i\binom{n-m+i}{i}b_{s-i}+\sum_{i=1}^{s-1}(-1)^{i-1}\binom{n-m+i}{i}b_{n-h-s+i+1}+(-1)^{s-1}\Gamma(1,n-h,s)$$

for $m-h=2s$ and

$$\gamma^d(n,m,h)=$$
$$\sum_{i=0}^{s-1}(-1)^i\binom{n-m+i}{i}b_{n-h-s+i+1}+\sum_{i=1}^{s}(-1)^{i-1}\binom{n-m+i}{i}b_{s-i+1}+$$
$$+(-1)^s\left(\binom{n-m+s}{s}b_1+\Gamma(2,n-h;s)\right)$$

for $m-h=2s+1$. Observe that both γ^p_h and γ^d_h, with s fixed, are invariant for translation:

$$\gamma^{p/d}(n,m,h)=\gamma^{p/d}(n+1,m+1,h+1)$$

On the other hand, for fixed s, the same hold for coefficients $b_{n-m+s+1}$ and b_{s+1}. We want prove

$$b_{n-m+s+1} = \gamma^p(n,k,h)$$
$$b_{s+1} = \gamma^d(n,k,h)$$

By translation invariance we can then reduce to the case $h = 0$ that we have proved in Lemma 2.2.9.

□

2.3 Proof of Reeder's Conjecture for B_n

We are now ready to use this new formulation of minuscule recurrence to prove Reeder's Conjecture.

Remark 2.3.1. The Poincaré polynomial for the graded multiplicities of trivial representation is well known: it is the Poincaré polynomial of the cohomology of $SO(2n+1)$. Moreover, V_{ω_1} in this case is the little adjoint representation an the polyonomial $P(V_{\omega_1}, \Lambda\mathfrak{g}, q)$ can be easily computed using the results in [37].

It can be checked by direct inspection that the Reeder's Conjecture holds in this two cases. We can then apply induction using (2.2.14).

Theorem 2.3.2. *Set $h = n - m/2$ if m is even and $h = (m-1)/2$ is m is odd. Then, if $h < n$,*

$$C_m = \binom{n}{h}_{q^4} \prod_{j=1}^{h}(1+q^{4j-1}) \prod_{r=1}^{n-h-1}(1+q^{4r-1})(q^{2(n-h)} + q^{2(n-h)-1}). \tag{2.3.1}$$

Proof. We start with the even case: $m = 2k$; first we write (2.2.14):

$$C_m c_m = \sum_{i=1}^{k} C_{2(k-i)+1} b_i + \sum_{i=1}^{k} C_{2(k-i)} b_{n-2k+i+1} = \sum_{j=0}^{k-1} C_{2j+1} b_{k-j} + \sum_{j=0}^{k-1} C_{2j} b_{n-k-j+1}. \tag{2.3.2}$$

Set, for $h > 0$,

$$S_h = \begin{cases} \frac{1}{1-q^2}\binom{n}{h}_{q^4} \prod_{j=1}^{n-h-1}(1+q^{4j-1}) \prod_{r=1}^{h-1}(1+q^{4r-1})(q+1)^2 & \text{for } h > 0 \\ \frac{1}{1-q^2}\prod_{j=1}^{n-1}(1+q^{4j-1})(q+1) & \text{for } h = 0 \end{cases}$$

Recalling (2.2.4), and using by induction 2.3.1 we have, for $h > 0$

$$\begin{aligned} C_{2h+1} b_{k-h} + C_{2h} b_{n-k-h+1} = S_h \Big[&(1+q^{4h-1}) q^{2(n-k)+1} (1 - q^{4(k-h)-2}) \\ &+ (1+q^{4(n-h)-1}) q^{4h+2k-2n-1} (1 - q^{4(n-k-h+1)-2}) \Big] \end{aligned} \tag{2.3.3}$$

$$= S_h(q^{4h} - q^{4(n-h)})(q^{2n-2k} + q^{2k-2n-1})$$

If $h = 0$ we have

$$\begin{aligned} C_1 b_k + C_0 b_{n-k+1} &= S_0\Big[q^{2n-2k+1}(q+1)(1-q^{4k-2}) + (1+q^{4n-1})q^{2k-2n}(1-q^{4n-4k+2})\Big] \\ &= S_0(q - q^{4n+1})(q^{2n-2k} + q^{2k-2n-1}). \end{aligned}$$

Hence, if we set $P_h = S_h(q^{4h} - q^{4(n-h)}),\ h > 0,\quad P_0 = S_0(q - q^{4n+1})$, we can rewrite (2.3.2) as

$$C_m c_m = \sum_{j=0}^{k-1} C_{2j+1} b_{k-j} + \sum_{j=0}^{k-1} C_{2j} b_{n-k-j+1} = (\sum_{h=0}^{k-1} P_h)(q^{2n-2k} + q^{2k-2n-1}). \tag{2.3.4}$$

Now, by induction,

$$C_{2k-2} c_{2k-2} = (\sum_{h=0}^{k-2} P_h)(q^{2n-2k+2} + q^{2k-2n-3})$$

and substituting into (2.3.4) we obtain

$$\sum_{j=0}^{k-1} C_{2j+1} b_{k-j} + \sum_{j=0}^{k-1} C_{2j} b_{n-k-j+1} = (P_{k-1} + \frac{C_{2k-2} c_{2k-2}}{q^{2n-2k+2} + q^{2k-2n-3}})(q^{2n-2k} + q^{2k-2n-1}).$$

Now

$$c_{2k-2} = \frac{1-q^{4k-4}}{1-q^2} q^{-2n+1}(1 + q^{4n-4k+5}) = \frac{1-q^{4k-4}}{1-q^2} q^{4-2k}(q^{2n-2k+2} + q^{2k-2n-3}),$$

whence

$$\frac{c_{2k-2}}{q^{2n-2k+2} + q^{2k-2n-3}} = \frac{1-q^{4k-4}}{1-q^2} q^{4-2k},$$

and in turn

$$\sum_{j=0}^{k-1} C_{2j+1} b_{k-j} + \sum_{j=0}^{k-1} C_{2j} b_{n-k-j+1} = (P_{k-1} + C_{2k-2} \frac{1-q^{4k-4}}{1-q^2} q^{4-2k})(q^{2n-2k} + q^{2k-2n-1}).$$

Now observe that

$$c_{2k} = \frac{1-q^{4k}}{1-q^2} q^{-2n+1}(1 + q^{4n-4k+1}) = \frac{1-q^{4k}}{1-q^2} q^{2-2k}(q^{2k-2n-1} + q^{2n-2k})$$

so we are reduced to prove that

$$C_{2k}(1-q^{4k}) q^{2-2k} = (1-q^2) P_{k-1} + C_{2k-2}(1-q^{4k-4}) q^{4-2k}. \tag{2.3.5}$$

Divide both sides of (2.3.5) by $\frac{1}{1-q^2}\binom{n}{k-1}_{q^4}\prod_{j=1}^{n-k}(1+q^{4j-1})\prod_{r=1}^{k-2}(1+q^{4r-1})(q+1)$. We get, for the r.h.s.

$$q(1+q^{4n-4k+3})(1-q^{4k-4})+(q+1)(q^{4k-4}-q^{4n-4k+4})=q+q^{4k-4}-q^{4n}-q^{5-4k+4n}=$$
$$q(1+q^{4k-5})(1-q^{4n-4k+4})$$

and for the l.h.s

$$q(1+q^{4k-5})(1-q^{4n-4k+4})$$

So equality (2.3.5) holds and the proof is completed in the even case.

Let us pass to the odd case, $m=2k+1$. After a suitable changes of variables we have

$$C_{2k+1}c_{2k+1}=\sum_{j=0}^{k}C_{2j}b_{k-j+1}+\sum_{j=0}^{k-1}C_{2j+1}b_{n-k-j}$$

Let us look at

$$C_{2h+1}b_{n-k-h}+C_{2h}b_{k-h+1}$$

From Formula 2.3.3 we immediately deduce that for $h>0$,

$$\begin{aligned}C_{2h+1}b_{n-k-h}+C_{2h}b_{k-h+1}&=S_h\Big[(1+q^{4h-1})q^{2k+1}(1-q^{4(n-k-h)-2})\\&+(1+q^{4(n-h)-1})q^{4h-2k-1}(1-q^{4(k-h+1)-2})\Big]\\&=S_h(q^{4h}-q^{4(n-h)})(q^{2k}+q^{-2k-1})\end{aligned}$$

For $h=0$,

$$\begin{aligned}C_1b_{n-k}+C_0b_{k+1}&=S_0\Big[q^{2k+1}(q+1)(1-q^{4(n-k)-2})+(1+q^{4n-1})q^{2k}(1-q^{4k)+2})\Big]\\&=S_0(q-q^{4n+1})(q^{2k}+q^{-2k-1}).\end{aligned}$$

Now we know that

$$C_{2k}c_{2k}=(\sum_{h=0}^{k-1}P_h)(q^{2n-2k}+q^{2k-2n-1})$$

and also

$$\frac{c_{2k}}{q^{2n-2k}+q^{2k-2n-1}}=\frac{1-q^{4k}}{1-q^2}q^{2-2k},$$

Thus we deduce that

$$C_{2k+1}c_{2k+1}=C_{2k}b_1+C_{2k}\frac{1-q^{4k}}{1-q^2}q^{2-2k}(q^{2k}+q^{-2k-1})$$

Multiplying by $1-q^2$ we get

$$C_{2k+1}(1-q^{4k+2})q^{-2n+1}(1+q^{4n-4k-1})=C_{2k}((q+1)(1-q^2)+(1-q^{4k})q^{2-2k}(q^{2k}+q^{-2k-1}))=$$

$$C_{2k}(q^{4k}+q)(q^{-4k}-q^2)$$

It this then a simple computation that if we assume the formula for C_{2k+1}, we get that

$$C_{2k+1}c_{2k+1}(1-q^2) = C_{2k}(q^{4k}+q)(q^{-4k}-q^2).$$

□

Note that (2.3.1) can be rewritten as

$$C_{2h+1} = \binom{n}{h}_{q^4} \prod_{j=1}^{h}(1+q^{4j-1}) \prod_{r=1}^{n-h-1}(1+q^{4r-1})q^{2(n-h)-1}(q+1), \tag{2.3.6}$$

$$C_{2k} = \binom{n}{k}_{q^4} \prod_{j=1}^{n-k}(1+q^{4j-1}) \prod_{r=1}^{k-1}(1+q^{4r-1})q^{2k-1}(q+1). \tag{2.3.7}$$

Making a direct comparison with the formulae 2.1.1, 2.1.2, 2.1.3 and 2.1.4 we obtain immediately that the Reeder's conjecture is verified for odd orthogonal groups.

Chapter 3

Reeder's Conjecture for Symplectic Groups

The aim of this chapter is to present a complete proof of Reeder's Conjecture for Lie Groups of type C. First of all we will list the small representations for C_n and then we analyze the structure of their zero weights space as representation of W and find some close formulae for polynomials P_W. After that we deal with quasi minuscule recurrence, using some symmetries of the set of weights involved in Stembridge's formulae to reduce the associated triangular system. The non-irreducibility of zero weight representation V_λ^0 if $\lambda = \omega_1 + \omega_{2k+1}$, leads us to subdivide the proof in two different cases with different combinatorical aspect.

3.1 Zero weight spaces in Small Representations

In this section we summarize what has been stated in Chapter 1 about small representations of Lie algebras of type C and give explicit formulae for the Poincaré polynomials of graded multiplicities of zero weight representations. We recall from Table 1.3 that the small weights representations for C_n are of two different kinds:

- *Type 1* V_λ with $\lambda = \omega_{2i} = e_1 + \cdots + e_{2i}$. In this case the zero weight representation is irreducible.
- *Type 2* V_λ with $\lambda = \omega_1 + \omega_{2i+1} = 2e_1 + \cdots + e_{2i+1}$. The zero weight representation splits in two irreducible representations.

A very simple remark about the (dominant) ordering of this weights will be very useful when we will work with Stembridge's recursive formulae:

Remark 3.1.1. The weights smaller or equal to ω_{2i} in dominance ordering are 0 and the ones of the form ω_{2j} with $j \leq i$. The ordering is more complicated for weights of the form $\lambda = \omega_1 + \omega_{2i+1}$:

If $n = 2i+1$ then μ is smaller than λ if and only if $\mu = \omega_{2j}$ with $j \leq i$ or $\mu = 0$ or $\mu = \omega_1 + \omega_{2j+1}$ with $j \leq i$.

If $n > 2i+1$ to the previous list the weight $\omega_{2(i+1)}$ should be added.

We now want to compute $P_W(\chi, s, t)$ for these representations.

3.1.1 Kirillov-Pak-Molchanov Formulae

As we have remarked in the Chapter 1 we have the following description of the zero weight space as W representation in the case $\lambda = \omega_{2k}$:

$$V^0_{\omega_{2k}} \cong \pi_{((n-k,k),\emptyset)}.$$

As a consequence of this description we can recover the rational function $\tilde{\tau}(V^0_\lambda, s, t)$ as described in [18] using the formula 1.2.3.

We are interested to closed formulae for specializations in (q, q^2) of the polynomials $P_W(V^0_{\omega_{2k}}, s, t)$ (i.e. the rational functions $\tilde{\tau}(V^0_{\omega_{2k}}, s, t)$ multiplied for $\prod_{i=1}^n (1 - t^{2i})$, c.f.r. 1.2.2). We are going to denote these polynomials by the more compact notation $\mathcal{C}_{k,n}(q)$ or $\mathcal{C}_{k,n}$. Using the formula 1.2.3 we have

$$\mathcal{C}_{k,n}(q) := \tilde{\tau}(\pi_{((n-k,k);\emptyset)})(q, q^2) \cdot \prod_{i=1}^{n} (q^{4i} + 1) =$$
$$q^{4k-1}(q+1) \binom{n}{k}_{q^4} \frac{\left(q^{4(n-2k+1)} - 1\right)}{\left(q^{4(n-k+1)} - 1\right)} \prod_{i=1}^{n-k} \left(q^{4i-1} + 1\right) \prod_{i=1}^{k-1} \left(q^{4i-1} + 1\right)$$

Using this explicit formula we obtain that, fixed a value for n, the polynomials $\mathcal{C}_{k,n}(q)$ with $1 \leq k \leq \lfloor n/2 \rfloor$ are linked by the following relation:

$$\mathcal{C}_{k+1,n}(q) = \mathcal{C}_{k,n}(q) \frac{q^4 (q^{4(n-k+1)} - 1)(q^{4(n-2k-1)} - 1)(q^{4k-1} + 1)}{(q^{4(k+1)} - 1)(q^{4(n-2k+1)} - 1)(q^{4(n-k)-1} + 1)} \tag{3.1.1}$$

The formulae are more complicate when V^0_λ is not irreducible (i.e. when $\lambda = \omega_1 + \omega_{2i+1}$ and $i > 0$). In this case we have to consider the sum of two $\tilde{\tau}$ rational functions. We recall first of all that

$$V^0_{\omega_1 + \omega_{2k}} \cong \pi_{((n-k-1,k),(1))} \oplus \pi_{((n-k-1,k,1),\emptyset)}$$

We have to compute $\tilde{\tau}$ for each irreducible component using again formula 1.2.3 and then sum up the two corresponding polynomials.

We will denote the polynomials $P_W(V_{\omega_1 + \omega^0_{2k+1}, s, t})$ more compactly by $\mathcal{C}_{2|k,n}(s,t)$ (or just $\mathcal{C}_{2|k}$ if the n is fixed or the context is clear)

Differently from the previous case, the recursive relation between the polynomials $\mathcal{C}_{2|k,n}$ are less compact and more difficult to handle.

It is more convenient to highlight the relations linking $\mathcal{C}_{2|k,n}$ with polynomials $\mathcal{C}_{k,n}$

We start defining the polynomial $P_{k,n}$ as

$$P_{k,n} = (t+s)(t^{2(n-k+1)}-1)(t^{2(k+1)}-1) + (t^4+st)(t^{2(n-k)}-1)(t^{2k}-1)$$

We obtain the following formula for $\mathcal{C}_{2|k}(q)$:

$$\mathcal{C}_{2|k,n}(s,t) = \frac{\prod_{i=1}^{n-k-1}(1+st^{2i-1}) \cdot \prod_{i=1}^{k}(t^2+st^{2i-1}) \cdot \prod_{i=1}^{n}(t^{2i}-1) \cdot P_{k,n}}{H(n-k,k+1)(1-t^2)}$$

Here $H(n,k)$ is a compact notation for the polynomial $\prod_{(i,j)}(1-t^{2h(i,j)})$ where (i,j) denotes the box of coordinates (i,j) the partition (n,k) read in English notation.

We can rewrite the explicit formula for the non specialized $\mathcal{C}_{k,n}$ in the following more compact formula

$$\mathcal{C}_{k+1,n}(s,t) = \frac{\prod_{i=1}^{n-k-1}(1+st^{2i-1}) \cdot \prod_{i=1}^{k+1}(t^2+st^{2i-1}) \cdot \prod_{i=1}^{n}(1-t^{2i})}{H(n-k-1,k+1)}$$

$$\mathcal{C}_{k,n}(s,t) = \frac{\prod_{i=1}^{n-k}(1+st^{2i-1}) \cdot \prod_{i=1}^{k}(t^2+st^{2i-1}) \cdot \prod_{i=1}^{n}(1-t^{2i})}{H(n-k,k)}$$

Remark 3.1.2. The following relations between the $H(n,k)$ polynomials hold

$$H(n-k.k+1) = H(n-k-1,k+1) \cdot \frac{(t^{2(n-2k-1)}-1)(t^{2(n-k+1)}-1)}{(t^{2(n-2k)}-1)}$$

$$H(n-k,k+1) = H(n-k,k) \cdot \frac{(t^{2(n-2k+1)}-1)(t^{2(k+1)}-1)}{(t^{2(n-2k)}-1)}$$

As an immediate consequence we obtain the announced transition formulae. The first one is from $\mathcal{C}_{k+1,n}$ to $\mathcal{C}_{2|k,n}$

$$\mathcal{C}_{2|k,n} = \frac{\mathcal{C}_{k+1,n}P_{kn}}{(t^2+st^{2k+1})(1-t^2)} \cdot \frac{H(n-k-1,k+1)}{H(n-k,k+1)} = \frac{\mathcal{C}_{k+1,n}P_{kn}}{(t^2+st^{2k+1})(1-t^2)} \frac{(t^{2(n-2k)}-1)}{(t^{2(n-2k-1)}-1)(t^{2(n-k+1)}-1)} \tag{3.1.2}$$

and the second one from $\mathcal{C}_{k,n}$ to $\mathcal{C}_{2|k,n}$

$$\mathcal{C}_{2|k,n} = \frac{\mathcal{C}_{k,n}P_{kn}}{(1+st^{2(n-k)-1})(1-t^2)} \cdot \frac{H(n-k,k)}{H(n-k,k+1)} =$$

$$\frac{\mathcal{C}_{k,n}P_{kn}}{(1+st^{2(n-k)-1})(1-t^2)}\frac{(t^{2(n-2k)}-1)}{(t^{2(n-2k+1)}-1)(t^{2(k+1)}-1)}$$

In the following sections we want to deal with the specializations of polynomials $C_{k,n}$ and $C_{2|k,n}$ that satisfies Stembridge's recurrences presented in [37]. We want to prove first of all that $C_{k,n} = \mathcal{C}_{k,n}$ and then $C_{2|k,n} = \mathcal{C}_{2|k,n}$. Both proof work by induction showing that the relations between $\mathcal{C}$-polynomials as stated above are satisfied by the polynomials $C_{k,n}$ and $C_{2|k,n}$.

3.2 Case 1: Small weights of the form ω_{2k}

Before starting the proof of the Conjecture we have to point out what happens to the quasi minuscule recurrence 1.4.4 in our case. After that we have to deal with the problem of solve a triangular system of equations with coefficients in $\mathbb{C}[q,t]$. The combinatorial structure of these coefficients will help us to find a two terms recurrence that solves our problem.

3.2.1 The recurrence for the $C_\lambda(q,t)$

Set $\lambda = \omega_{2k}$. As we have seen in the Chapter 1, Stembridge proves the following recurrence:

$$\sum_{(\mu,\beta)}\sum_{i\geq 0}\left[f_i^\beta(q,t) - q^{(\lambda,\omega)}f_i^\beta(q^{-1},t^{-1})\right]C_{\mu-i\beta}(q,t) \tag{3.2.1}$$

where ω is a quasi minuscule coweight, (μ,β) are element of the set $\{(w\lambda, w\theta)|w \in W\,,\ w\theta \geq 0\}$ and the rational functions f_i^β are defined by the following generating polynomial:

$$(1-tz)(1-qtz)((\rho,\beta^\vee))_{t^2z} = \sum_{i\geq 0} t^{(\rho,\beta^\vee)}f_i^\beta(q,t)z^i.$$

In our case the weight $\lambda = \omega_{2k} = e_1 + \cdots + e_{2k}$, the root θ is equal to $2e_1$ and $\omega = \theta^\vee = e_1$ is a quasi minuscule weight. The recurrence above the can be rewritten in the following form:

$$\sum_{(\mu,2e_j)}\sum_{i\geq 0}\left[f_i^j(q,t) - qf_i^j(q^{-1},t^{-1})\right]C_{\mu-2ie_j}(q,t), \tag{3.2.2}$$

where we denote with f_i^j the rational function $f_i^{2e_j}$ defined implicitely as above.

Remark 3.2.1. The C_μ appearing in the above recursion are not in general indexed by dominant weights. They can be reduced in the form ϵC_ν whit ν integral dominant weight and $\epsilon \in \{\pm 1, 0\}$ following the rule 1.4.1. As remarked in 3.1.1, in the recurrence for ω_{2k}, appear only the $C_{\omega_{2h}}$ with $0 \leq h \leq k$.

Now, suppose to reduce in dominant form each C_μ in the equation (3.2.2); we want to investigate the coefficients of each $C_{\omega_{2h}}$ in the recursion. Set $B_j = (\rho, e_j) = n - j + 1$. We can give explicit formulae for the rational functions f_i^j as follows:

- *Case* $i = 0$. We have $f_0^j = 1/t^{B_j}$,
- *Case* $i = 1$. In this case $f_1^j = (t^2 - (q+1)t)/t^{B_j}$,
- *Case* $2 \le i \le B_j - 1$. We have

$$f_i^j = (t^{2i+4)} - (q+1)t^{2i+3} + qt^{2i+2})/t^{B_j} = \left[t^{2i-2}(t-q)(t-1)\right]/t^{B_j},$$

- *Case* $i = B_j$. We have $f_{B_j}^j = (-(q+1)t^{B_j-1} + qt^{B_j-2})$,
- *Case* $i = B_j + 1$. In this last case $f_{B_j+1}^j = qt^{B_j}$.

So we obtain explicit expressions for the coefficients $F_i^j = f_i^\beta(q,t) - qf_i^\beta(q^{-1},t^{-1})$:

$$\begin{aligned}
F_0^j(q,t) &= \frac{1}{t^{B_j}} - qt^{B_j}, \\
F_1^j(q,t) &= \frac{t}{t^{B_j}}(q - qt - t) - t^{B_j}\frac{(t-(q+1))}{t^2}, \\
F_i^j(q,t) &= (t-q)(t-1)\left(\frac{t^{2i-2}}{t^{B_j}} - \frac{t^{B_j}}{t^{2i}}\right), \\
F_{B_j}^j(q,t) &= \frac{t^{2(B_j-1)}}{t^{B_j}}(q - qt - t) - \frac{t^{B_j}}{t^{2(B_j-1)}}\frac{t(t-(q+1))}{t^2}, \\
F_{B_j+1}^j(q,t) &= qt^{B_j} - \frac{1}{t^{B_j}}.
\end{aligned}$$

Remark 3.2.2. If we set $A(i,j) = n - i - j + 2$ we have $F_i^j = -F_{A(i,j)}^j$. In particular if $A(i,j) = i$ we obtain $F_i^j = 0$.

Now we want to investigate the coefficients of the generic polynomial C_μ in the recurrence. Let us fix n and set as above $\lambda = \omega_{2k}$. Set

$$\Gamma_\nu^{i,j} = \{(w\lambda, \epsilon)\ w \in W,\ \epsilon \in \{\pm 1\} \mid we_1 = e_j\ ,\ C_{w(\lambda - 2ie_j)} = \epsilon C_\nu\}$$

Lemma 3.2.3. *We have a bijection between* $\Gamma_\nu^{i,j}$ *and* $\Gamma_\nu^{A(i,j),j}$.
This bijection sends a pair $(w\lambda, \epsilon)$ *to a pair of the form* $(w'\lambda, -\epsilon)$.

Proof. Let $(\mu_1, \ldots, \mu_n)$ be the coordinates of $w\lambda$. By definition of $\Gamma_\nu^{i,j}$ we have $\mu_j = 1 - 2i$.
We define

$$\Psi(\mu) = \begin{cases} \mu_h & \text{if } h \neq j \\ 1 - 2(n - j - i + 2) & \text{if } h = j \end{cases}$$

It follows immediately from the definition that $\Psi(\mu)+\rho$ differs from $\mu+\rho$ just for the j-th coordinate which is equal to

$$n-j+1-2n+2j+2i-3=-n+j+2i-2=-(n-j+1)-(1-2i)=-(\mu+\rho)_j$$

Actually $(\Psi(\mu),-\epsilon)$ is an element of $\Gamma_\nu^{A_i,j}$ and we can complete the proof observing that the map Ψ is an involution. $\square$

Substituting the polynomials C_μ in their reduced form, we can rewrite 3.2.2 as

$$\sum_{i=0}^{k}\Lambda_i^{k,n}C_{\omega_{2i}}$$

for some polyonomial coefficients $\Lambda_i^{k,n}$. We will use Lemma 3.2.3 and Remark 3.2.2 to give a first simplification in the explicit computation of these coefficient.

Let J be the set of pairs (i,j) such that $1\leq j\leq n$ and $i\leq B_j+1$ and let I be the subset of pairs such that $i\neq n-j-i+2$.

By the remark 3.2.2 and by the definition of f_i^j, those are exactly the pairs (i,j) that can give a non zero contribution to a coefficient in the recurrence.

We can partition I in two subset I' and I'' in the following way

$$I'=\{(i,j)\mid i<n-j-i+2\},$$

$$I''=\{(i,j)\mid i>n-j-i+2\}.$$

The generic coefficient $\Lambda_\nu^{k,n}$ of the polynomial C_ν can then be written in the following form

$$\Lambda_\nu^{k,n}=\sum_{(i,j)\in I'}F_i^j|\Gamma_\nu^{i,j}|-\sum_{(i,j)\in I''}F_i^j|\Gamma_\nu^{i,j}|.$$

This expression by Lemma 3.2.3 can be written as

$$\Lambda_\nu^{k,n}=\sum_{(i,j)\in I'}F_i^j|\Gamma_\nu^{i,j}|-F_A^j|\Gamma_\nu^{A,j}|=\sum_{(i,j)\in I'}\left(F_i^j-F_A^j\right)|\Gamma_\nu^{i,j}|=2\sum_{(i,j)\in I'}F_i^j|\Gamma_\nu^{i,j}|.$$

Without loss of generality, we can then consider only the contributions to $\Lambda_\nu^{k,n}$ given by the pairs in I', i.e. we can suppose $i<n-i-j+2$.

This simplification implies that $2i-1<n-j+1=\rho_j$. This very simple remark leads us to the recursive relations between the coefficients $\Lambda_\nu^{k,n}$ we were looking for:

Fix i and j and suppose $w(e_1)=e_j$, one can observe that the coordinates of $w\omega_{2k}-2ie_j+\rho$ have modulus smaller or equal to 1 except for the j-th coordinate which is equal to $1-2i$. Now, let fix only i and consider $\bar{j}=max\{j\mid i<n-j-i+2\}$. We have

$$0 < n-\bar{j}+1+(1-2i) = \rho_{\bar{j}}+(w\omega_{2k}-2ie_j)_{\bar{j}} < \rho_j+(w\omega_{2k}-2ie_j)_j = n-j+1+(1-2i) \ \ \forall j < \bar{j}$$

This implies that, for the pairs in I', the coordinates of $w\omega_{2k} - 2ie_j$ are all non negative and the reduced form of $C_{w\omega_{2k}-2ie_j}$ in our case can be computed just by the action of the symmetric group on $w\omega_2 k - 2ie_j + \rho$.

Let us start the explicit computation of the coefficients $\Lambda_h^{k,n}$ with a general remark.

Remark 3.2.4. Set $\lambda = \omega_{2k}$ and suppose $w(e_1) = e_j$. If $w\lambda - 2ie_j$ is conjugated to ω_{2h}, we can write $w\lambda - 2ie_j + \rho$ in the form $\big((\omega_{2h} + \rho)_{\sigma(1)}, \ldots, (\omega_{2h} + \rho)_{\sigma(n)}\big)$, i.e. a coordinate permutation of the vector $(n+1, \ldots, n-2h+2, n-2h, \ldots, 1)$.

If follows from our assumption on i and j that $(w\lambda - 2ie_j + \rho)_t > \rho_t$ only if $t \neq j$ and $(w\lambda)_t > 0$. Then, to obtain a contribution to the coefficient of $C_{\omega_{2h}}$, the first $2h$ coordinates of $w\lambda - 2ie_j$ must be equal to 1.

As an immediate consequence of this observation we obtain that the only contributions to the coefficient of $C_{\omega_{2k}}$ in the recursion (3.2.2) for $\lambda = \omega_{2k}$ come from the case $i = 0$. More precisely, $w(\omega_{2k} - 2ie_1) + \rho$ must be equal to $\omega_{2k} + \rho$ and this implies $i = 0$ as stated above, but also force w to be a permutation of the first k coordinates. On the other side, if w is a permutation of the first k coordinates and $i = 0$, we have $w(\omega_{2k} - 2ie_1) + \rho = \omega_{2k} - 2ie_{w(1)} + \rho = \omega_{2k} + \rho$.

This immediately implies that the pair $\{(\mu, e_j)\}$ appearing in the quasi minuscule recurrence (3.2.2) and giving contribution to $\Lambda_k^{k,n}$ are all of the form (ω_{2k}, e_j) with $1 \leq j \leq 2k$ and that

$$\Lambda_k^{k,n} = \sum_{i=1}^{2k} F_0^{i,n} = \sum_{i=1}^{2k} \left(\frac{1}{t^{n-i+1}} - qt^{n-i+1} \right) = \frac{(t^{2k-1} - qt^{2n})(t^{2k} - 1)}{t^{n+2k-1}(t-1)} \tag{3.2.3}$$

If we analyze the coefficient $\Lambda_h^{k,n}$, it is harder to find an equivalent explicit closed formula but we can use Remark 3.2.4 to obtain relations between the $\Lambda_h^{k,n}$ coefficients . First of all, we need to prove an useful Lemma.

Lemma 3.2.5. *Consider a dominant weight λ of the form $\lambda = \omega_{2k}$. Let w be an element of the Weyl group W. Then:*

1. *$w\lambda + \rho$ is conjugated to ρ if and only if the $2k$ non zero coordinates of $w\lambda$ are pair of consecutive coordinates $\big((w\lambda)_{(j)}, (w\lambda)_{(j)+1}\big)$ of the form $(-1, 1)$.*

2. *The number of weights $w\lambda$ such that $w\lambda + \rho \sim \rho$ is equal to $\binom{n-k}{k}$.*

3. *If $w\lambda$ is conjugated to 0, then there exists a permutation $\sigma \in S_n$ of length $l(\sigma) = k$ such that $\sigma(w\lambda + \rho) = \rho$.*

Proof. 1. First observe that $(w\lambda)_1$ can not be equal to 1, because in this case the 1-st coordinate is equal to $n+1$ and $w\lambda+\rho$ is not conjugated to ρ. If $(w\lambda)_j = 1$ we have $(w\lambda+\rho)_j = n-j+2$, but in $w\lambda+\rho$ must appear a coordinate equal to $n-j+1$. This can be obtained or adding 1 to the $j+1$-th component or adding -1 to the $j-1$-th component. In the first case the fact that $w\lambda+\rho$ must be regular forces to have $(w\lambda)_h = 1\ \forall h \geq i$ and $w\lambda+\rho$ cannot be conjugated to ρ. The second case is exactly the thesis.

2. We have to count the number of placements of k pairs of coordinates in a vector of length n. This is equivalent to choose k coordinates in a vector of length $n-k$, doubling the chosen ones. The number of this choices is exactly equal to $\binom{n-k}{k}$.

3. If $\{((w\lambda)_{h_1}, (w\lambda)_{h_1+1}), \ldots, ((w\lambda)_{h_k}, (w\lambda)_{h_k+1})\}$ are the pairs of coordinates as in (1), to prove (3) we can consider the permutation $\sigma = \prod_{i=1}^{k}(h_i, h_i+1)$.

□

Example 3.2.6. Consider the case of $\omega_2 \in C_4$. In this case the weights $w\omega_2$ conjugated to 0 are $(-1,1,0,0)$, $(0,-1,1,0)$ and $(0,0,-1,1)$.

As observed in 3.2.4, the weight $w(\omega_{2k} - 2ie_1) + \rho$ must be conjugated to $\omega_{2h} + \rho$ by a permutation $\sigma \in S_n$, furthermore the first $2h$ coordinates of $w(\omega_{2k})$ must be equal to 1. Now, to understand more about the pairs $(w\omega_{2k}, e_j)$ that give contribution to the coefficient $\Lambda_h^{k,n}$ in 3.2.2, we have to analyze two two cases:

- *Case 1:* $w(1) = j \leq 2h$. This case give a contribution to the coefficient only if $i = 0$ because, as just observed, $(w(\omega_{2k} - 2ie_1))_j = (\omega_{2k})_j - 2ie_j$ must be equal to 1. In this case $w\omega_{2k}$ is of the form $\omega_{2h} + w\nu$, where ν is the vector that has coordinates equal to 1 from the $2h+1$-st to the $2k$-th entry and zero elsewhere. This weight corresponds to the weight $\omega_{2(k-h)}$ when contracted to the subalgebra $C_{n-2h} \to C_n$, where the restriction is induced by inclusion of Dynkin diagrams. Furthermore $w\nu+\rho$ must be conjugated to ρ and the same holds for the projection to C_{n-2h}. Using the Lemma 3.2.5 we know that the number of different weights of the form $w\nu$ is, in this case, equal to $\binom{(n-2h)-(k-h)}{k-h} = \binom{n-h-k}{k-h}$. The contribution to $\Lambda_h^{k,n}$ is then equal to
$$(-1)^{k-h} F_0^j \binom{n-h-k}{k-h}.$$

- *Case 2:* $\sigma(1) = j > 2h$. In this second case the first $2h$ coordinates of $w\omega_{2k}$ are all equal to 1, then $w\omega_{2k} = \omega_{2h} + w\nu$, where ν is defined as above and $w\nu$, projected on C_{n-2h} is such that $(w\nu)_{>2h} + \rho_{n-h}$ is conjugated to ρ_{n-2h}. This second case then gives a contribution equal to $\Lambda_0^{k-h,n-2h}$.

Summing up what we have just observed we obtain the following recursive formula

$$\Lambda_h^{k,n} = \Lambda_0^{k-h,n-2h} + (-1)^{k-h} \sum_{i=1}^{2h} F_0^j \binom{n-h-k}{k-h} = \Lambda_0^{k-h,n-2h} + (-1)^{k-h} \Lambda_h^{h,n} \binom{n-h-k}{k-h}$$

Finally, we want produce a recursive relation for the coefficient related to the weight 0. Remark that, if $h = 0$, we can obtain the coefficient $\Lambda_0^{k,\,n}$ as the sum of the contributions obtained fixing a value of i. In other words, we have

$$\Lambda_0^{k,\,n} = \sum_{i=0}^{\lfloor n/2 \rfloor} \Lambda_{0i}^{k,\,n}.$$

We want use the Remark 3.2.4 to find a recurrence relation for the i-th contribution.

Let fix $i > 0$, we start describing the set $\Gamma_{0,\,i}^{k,\,n} = \{w(\lambda - 2ie_1) \mid w(\lambda - 2ie_1) + \rho \sim \rho\}$. Again, we have to consider some different cases:

- *Case 1:* $\mu \in \Gamma_{0,\,i}^{k,\,n}$ *and* $\mu_1 = 0$. In this case μ can be contracted to a weight $\mu' \in \Gamma_{0,\,i}^{k,\,n-1}$
- *Case 2:* $\mu \in \Gamma_{0,\,i}^{k,\,n}$ *and* $\mu_1 = -1$. The condition $w(\lambda - 2ie_1) + \rho \sim \rho$ forces $\mu_2 = 1$ and μ is of the form $(-1, 1, \mu')$, where μ' is a weight in $\Gamma_{0,\,i}^{k-1,\,n-1}$.
- *Case 3:* $\mu \in \Gamma_{0,\,i}^{k,\,n}$ *and* $\mu_1 = 1 - 2i$. First of all suppose $k = i$. In this case there is only one weight with this property, that precisely is the weight $(1 - 2i, 1, \ldots, 1)$. If $k > i$ we obtain that μ must be of the form $(1 - 2i, 1, \ldots, 1, \mu')$ where μ' is such that $\mu' + \rho_{n-2i} \sim \rho_{n-2i}$.

In the first case we obtain a contribution to $\Lambda_{0i}^{k,\,n}$ equal to $\Lambda_{0i}^{k,\,n-1}$, in the second case it is equal to $-\Lambda_{0i}^{k-1,\,n-2}$ and in the third case, using again what we have proved in the Lemma 3.2.5, the contribution is equal to $(-1)^{k-i+1} F_i^{1,\,n} \binom{(n-2i)-(k-2i)}{k-i}$. Summarizing we obtain

$$\Lambda_{0,i}^{k,\,n} = \Lambda_{0,i}^{k,\,n-1} - \Lambda_{0,i}^{k-1,\,n-2} + (-1)^{k-i+1} F_i^{1,n} \binom{n-i-k}{k-i}$$

If $i = 0$ the recursion for the coefficient must be treated differently. In fact for $\Lambda_{00}^{k,\,n}$ the case 3 discussed above does not appear. It is substituted by weights of the form $w\lambda = (-1, 1, \mu')$ where $w(1) = 2$ and $\mu' + \rho_{n-2} \sim \rho_{n-2}$. Using again Lemma 3.2.5 we obtain that the contribution of these weights is equal to $(-1)^k F_0^{2,\,n} \binom{n-k-1}{k-1}$. We can now compute a recursive expression for $\Lambda_{00}^{k,\,n}$:

$$\Lambda_{0,0}^{k,\,n} = \Lambda_{0,0}^{k,\,n-1} - \Lambda_{0,0}^{k-1,\,n-2} + (-1)^k F_0^2 \binom{n-k-1}{k-1},$$

and finally sum all the contributions obtaining

$$\Lambda_0^{k,\,n} = \Lambda_0^{k,\,n-1} - \Lambda_0^{k-1,\,n-2} + (-1)^k \binom{n-k-1}{k-1} F_0^{2,n} + \sum_{i=1}^{k} (-1)^{k-i+1} F_i^{1,n} \binom{n-i-k}{k-i}.$$

We can summarized what is proved above about in the following proposition.

Proposition 3.2.7. *Rewriting the recurrence 3.2.2 using the reduced form of the polynomials C_μ, yields the following equation:*

$$R_k \ : \ \sum_{i=0}^{k} C_{\omega_{2i}}(q,t)\Lambda_i^{k,\,n} = 0$$

where the coefficients $\Lambda_i^{k,\,n}$ satisfies the following recursive relations and closed formulae:

$$\Lambda_k^{k,\,n} = \frac{(t^{2k-1} - qt^{2n})(t^{2k} - 1)}{t^{n+2k-1}(t-1)},$$

$$\Lambda_h^{k,\,n} = \Lambda_0^{k-h,\,n-2h} + (-1)^{k-h}\Lambda_h^{h,\,n}\binom{n-h-k}{k-h},$$

$$\Lambda_0^{k,\,n} = \Lambda_0^{k,\,n-1} - \Lambda_0^{k-1,\,n-2} + (-1)^k\binom{n-k-1}{k-1}F_0^{2,n} + \sum_{i=1}^{k}(-1)^{k-i+1}F_i^{1,n}\binom{n-i-k}{k-i}.$$

We end our investigation about the coefficients of the recurrence with the computation of $\Lambda_0^{1,\,n}$.

Lemma 3.2.8.

$$\Lambda_0^{1,\,n} = -\frac{(t-q)(t^{2n-2}-1)}{t^{n-1}(t-1)}. \tag{3.2.4}$$

Proof. If $i = 0$ we are looking for weights such that $w(1,1,0,\ldots,0) + \rho \sim \rho$. By the Lemma 3.2.5 these weights are of the form $(0\ldots0,-1,1,0\ldots,0)$, then $w(1) \in \{2,\ldots,n\}$ and the contribution to $\Lambda_0^{1,\,n}$ is equal to $-\sum_{j=2}^{n} F_0^{jn}$.

By a similar argument the weights such that $w(-1,1,0,\ldots,0)+\rho \sim \rho$ are again of the form $(0\ldots0,-1,1,0\ldots,0)$ but in this case $w(1) \in \{1,\ldots,n-1\}$ and the contribution is equal to $-\sum_{j=1}^{n-1} F_1^{jn}$.

Observing that $F_0^{j,n} = F_0^{2,n-j+2}$ and $F_1^{j-1,n} = F_1^{1,n-j+2}$ we obtain

$$\Lambda_0^{1,\,n} = -\sum_{j=2}^{n}\left(F_0^{j,n} + F_1^{j-1,n}\right) = -\sum_{j=2}^{n}\left(F_0^{2,j} + F_1^{1,j}\right) = -\frac{(t-q)(t^{2n-2}-1)}{t^{n-1}(t-1)}$$

□

As a first corollary we can re-prove a known formula (see [37]) for $C_{\omega_2}(q,t)$.

Corollary 3.2.9.

$$C_{\omega_2,n}(q,t) = \frac{t(t-q)(t^{2n-2}-1)}{(1-qt^{2n-1})(t^2-1)}C_{0,n}$$

Proof. By our computation of the coefficients for the recurrence R_1 we obtain

$$R_1 \; : \; \frac{(1-qt^{2n-1})(t+1)}{t^n} C_{\omega_2,n}(q,t) - \frac{(t-q)(t^{2n-2}-1)}{t^{n-1}(t-1)} C_{0,n} = 0$$

which implies the equality stated by the corollary. □

3.2.2 Reeder's Conjecture for $\lambda = \omega_{2k}$

We are now ready to prove Reeder's conjecture. The conjecture hold for $k=1$ as a consequence of the Corollary 3.2.9. We want use an inductive argument to obtain the general proof.

Notation 3.2.10. Coherently with notation in Section 2, will denote the polynomial $C_{\omega_{2k},n}(q,t)$ with the more concise notation $C_{k,n}(q,t)$ or by $C_{k,n}$.

In particular we obtain the conjecture as a consequence of the following proposition.

Proposition 3.2.11. *Let $\{R_i\}_{i\leq k}$ be the set of recursion defined in 3.2.7. Then there exist a family of integers $\{A_i^{k,n}\}_{i\leq k}$ such that*

$$\sum A_i^{k,n} R_i = \Lambda_k^{k,n} C_{k,n}(q,t) + \Lambda_0^{1,n-2k+2} \left(C_{k-1,n}(q,t) + \cdots + C_{0,n}(q,t) \right)$$

Proof. Let us start defining the integers $A_k^{k,n}$. We will use a recursive definition setting:

$$A_h^{k,n} = \begin{cases} 0 & \text{if } h > k \\ 1 & \text{if } h = k \\ A_{i-1}^{k-1,n-2} & \text{if } k > h > 1 \\ \sum_{i=2}^{k} (-1)^i \binom{n-i-1}{i-1} A_{i-1}^{k-1,n-2} & \text{if } h = 1 \end{cases} \tag{3.2.5}$$

Aiming to prove Proposition 3.2.11, we can rearrange the expression $\sum A_i^{k,n} R_i$

$$\sum_{i=1}^{k} A_i^{k,n} R_i = \sum_{i=1}^{k} \left(A_i^{k,n} \sum_{j=0}^{i} \Lambda_j^{i,n} C_{j,n} \right) = \sum_{j=0}^{k} \left(\sum_{i=j}^{k} A_i^{k,n} \Lambda_j^{i,n} \right) C_{j,n} =$$
$$\Lambda_k^{k,n} C_{k,n} + \sum_{j=0}^{k-1} \left(\sum_{i=j}^{k} A_i^{k,n} \Lambda_j^{i,n} \right) C_{j,n}.$$

So we are reduced to prove the following identity

$$\sum_{i=h}^{k} A_i^{k,n} \Lambda_h^{i,n} = \Lambda_0^{1,n-2k+2}.$$

We need now two preliminary Lemmata

Lemma 3.2.12.

$$F_1^{1,n-2k+3} + F_0^{2,n-2k+3} + F_k^{1,n} = F_1^{1,n-2k+2} + F_0^{2,n-2k+2}.$$

Proof. First of all we remark that

$$F_1^{1,n} + F_0^{2,n} = \frac{(t-q)(t^{2n-2}+t)}{t^n}$$

By simple algebraic computations we obtain

$$F_1^{1,n-2k+3} + F_0^{2,n-2k+3} + F_k^{1,n} = (t-q)\left[\frac{(t^{2(n-2k+3)}+t)}{t^{n-2k+3}} + \frac{(t-1)(t^{2k-2}-t^{2n-2k})}{t^n}\right] =$$
$$= \frac{(t-q)(t^{2n-2k}+t^{2k-1})}{t^n} = \frac{(t-q)(t^{2n-4k+2}+t)}{t^{n-2k+2}} = F_1^{1,n-2k+2} + F_0^{2,n-2k+2}.$$

□

Lemma 3.2.13. *Let $A_1^{k,n}$ be the integers defined above, then $A_h^{k,n} = A_h^{k,n-1} + A_h^{k-1,n-1}$.*

Proof. Observe that without loss of generality it is sufficient show the thesis for $h = 1$. In fact by the definition we have

$$A_h^{k,,n} = A_1^{k-h,n-2h} = A_1^{k-h,(n-1)-2h} + A_1^{k-h-1,(n-1)-2h} = A_h^{k,n-1} + A_h^{k-1,n-1}.$$

Now we can write down the expression for $A_1^{k,n-1}$ and $A_1^{k-1,n-1}$ using the definition:

$$A_i^{k,n-1} = \sum_{i=2}^{k} (-1)^i \binom{n-i-2}{i-1} A_{i-1}^{k-1,n-3}$$
$$A_i^{k-1,n-1} = \sum_{i=2}^{k-1} (-1)^i \binom{n-i-2}{i-1} A_{i-1}^{k-2,n-3}$$

and then

$$A_i^{k,n-1} + A_i^{k-1,n-1} =$$
$$\sum_{i=2}^{k-1} (-1)^i \binom{n-i-2}{i-1} \left[A_{i-1}^{k-2,n-3} + A_{i-1}^{k-2,n-3}\right] + (-1)^k \binom{n-k-2}{k-1} A_{k-1}^{k-1,n-3} =$$
$$\sum_{i=2}^{k-1} (-1)^i \binom{n-i-2}{i-1} A_{i-1}^{k-1,n-2} + (-1)^k \binom{n-k-2}{k-1} A_{k-1}^{k-1,n-2} =$$
$$\sum_{i=2}^{k} (-1)^i \binom{n-i-2}{i-1} A_{i-1}^{k-1,n-2}$$

where for the second to last equality we used the identity $A_{k-1}^{k-1,n-3} = 1 = A_{k-1}^{k-1,n-2}$.

Now we can write

$$\sum_{i=2}^{k}(-1)^i\binom{n-i-2}{i-1}A_{i-1}^{k-1,n-2} =$$

$$\sum_{i=2}^{k}(-1)^i\left[\binom{n-i-1}{i-1}-\binom{n-i-2}{i-2}\right]A_{i-1}^{k-1,n-2} =$$

$$A_1^{k,n}-\sum_{i=2}^{k}(-1)^i\binom{n-i-2}{i-2}A_{i-1}^{k-1,n-2}.$$

To complete the proof it is enough show the identity

$$\sum_{i=2}^{k}(-1)^i\binom{n-i-2}{i-2}A_{i-1}^{k-1,n-2} = 0$$

If we translate the index i by 1 setting $i = t+1$ we obtain

$$\sum_{t=1}^{k-1}(-1)^{t+1}\binom{n-2-t-1}{t-1}A_{t}^{k-1,n-2} =$$

$$\left[A_1^{k-1,n-2}-\sum_{t=2}^{k-1}(-1)^t\binom{n-2-t-1}{t-1}A_{t-1}^{k-2,n-4}\right]$$

Now recalling the definition of $A_1^{k-1,n-2}$ we remark that

$$A_1^{k-1,n-2} = \sum_{i=2}^{k-1}(-1)^i\binom{n-2-i-1}{i-1}A_{i-1}^{k-2,n-4}$$

obtaining the thesis.

□

Now we can start the proof of Proposition 3.2.11 from the case $h = 0$. We want prove the following identity:

$$\sum_{i=1}^{k}A_i^{k,n}\Lambda_0^{i,n} = \Lambda_0^{1,n-2k+2}$$

We recall the general form for the zero coefficient from Proposition 3.2.7

$$\Lambda_0^{k,\,n} = \Lambda_0^{k,\,n-1}-\Lambda_0^{k-1,\,n-2}+(-1)^k\binom{n-k-1}{k-1}F_0^{2,n}+\sum_{i=1}^{k}(-1)^{k-i+1}F_i^{1,n}\binom{n-i-k}{k-i}$$

and use it to expand the expression $\sum_{i=1}^{k} A_i^{k,n}\Lambda_0^{i,n}$.

$$\sum_{i=1}^{k} A_i^{k,n}\Lambda_0^{i,n} =$$

$$\sum_{i=1}^{k} A_i^{k,n}\Lambda_0^{i,n-1} - \sum_{i=1}^{k} A_i^{k,n}\Lambda_0^{i-1,n-2} + \sum_{i=1}^{k}(-1)^i A_i^{k,n}\binom{n-i-1}{i-1}\left(F_0^{2,n}+F_1^{1,n}\right)+$$

$$+\sum_{i=2}^{k}\left[A_i^{k,n}\sum_{j=2}^{i}(-1)^{i-j+1}F_j^{1,n}\binom{n-i-j}{i-j}\right] =$$

$$\sum_{i=1}^{k} A_i^{k,n}\Lambda_0^{i,n-1} - \sum_{i=2}^{k} A_{i-1}^{k-1,n-2}\Lambda_0^{i-1,n-2} + \left(F_0^{2,n}+F_1^{1,n}\right)\left(\sum_{i=1}^{k}(-1)^i A_i^{k,n}\binom{n-i-1}{i-1}\right)+$$

$$+\sum_{i=2}^{k}\left[A_i^{k,n}\sum_{j=2}^{i}(-1)^{i-j+1}F_j^{1,n}\binom{n-i-j}{i-j}\right] =$$

$$\sum_{i=1}^{k} A_i^{k,n}\Lambda_0^{i,n-1} - \sum_{t=1}^{k-1} A_t^{k-1,n-2}\Lambda_0^{t,n-2} + \left(F_0^{2,n}+F_1^{1,n}\right)\left[-A_1^{k,n}+\sum_{i=2}^{k}(-1)^i A_i^{k,n}\binom{n-i-1}{i-1}\right]+$$

$$+\sum_{j=2}^{k}F_j^{1,n}\left[\sum_{i=j}^{k}A_i^{k,n}(-1)^{i-j+1}\binom{n-i-j}{i-j}\right] =$$

$$\sum_{i=1}^{k} A_i^{k,n}\Lambda_0^{i,n-1} - \Lambda_0^{1,n-2k+2} + \sum_{j=2}^{k}F_j^{1,n}\left[\sum_{t=1}^{k-j+1}A_{t+j-1}^{k,n}(-1)^t\binom{n-2j-t+1}{t-1}\right]$$

Where by inductive hypothesis we replaced $\sum_{t=1}^{k-1} A_t^{k-1,n-2}\Lambda_0^{t,n-2}$ with $\Lambda_0^{1,n-2k+2}$.

$$\sum_{i=1}^{k} A_i^{k,n}\Lambda_0^{i,n-1} - \Lambda_0^{1,n-2k+2} + \sum_{j=2}^{k}F_j^{1,n}\left[\sum_{t=1}^{k-j+1}A_{t+j-1}^{k,n}(-1)^t\binom{n-2j-t+1}{t-1}\right]$$

$$\sum_{i=1}^{k} A_i^{k,n}\Lambda_0^{i,n-1} - \Lambda_0^{1,n-2k+2} + \sum_{j=2}^{k}F_j^{1,n}\left[\sum_{t=1}^{k-j+1}A_{t}^{k-j+1,n-2j+2}(-1)^t\binom{n-2j+2-t-1}{t-1}\right] =$$

$$\sum_{i=1}^{k} A_i^{k,n}\Lambda_0^{i,n-1} - \Lambda_0^{1,n-2k+2} - F_k^{1,n}$$

Now we use Lemma 3.2.13 to expand $A_i^{k,n}$

$$\sum_{i=1}^{k} A_i^{k,n}\Lambda_0^{i,n-1} - \Lambda_0^{1,n-2k+2} - F_k^{1,n} = \sum_{i=1}^{k}\left[A_i^{k,n-1}+A_i^{k-1,n-1}\right]\Lambda_0^{i,n-1} - \Lambda_0^{1,n-2k+2} - F_k^{1,n} =$$

$$\sum_{i=1}^{k} A_i^{k,n-1}\Lambda_0^{i,n-1} + \sum_{i=1}^{k-1} A_i^{k-1,n-1}\Lambda_0^{i,n-1} - \Lambda_0^{1,n-2k+2} - F_k^{1,n} =$$
$$\Lambda_0^{1,n-2k+1} + \Lambda_0^{1,n-2k+3} - \Lambda_0^{1,n-2k+2} - F_k^{1,n} =$$
$$\Lambda_0^{1,n-2k+1} - \sum_{j=2}^{n-2k+3}\left[F_0^{2,j} + F_1^{1,j}\right] + \sum_{j=2}^{n-2k+2}\left[F_0^{2,j} + F_1^{1,j}\right] - F_k^{1,n} =$$
$$\Lambda_0^{1,n-2k+1} - F_1^{1,n-2k+3} - F_0^{2,n-2k+3} - F_k^{1,n}$$

and by Lemma 3.2.12

$$\Lambda_0^{1,n-2k+1} - F_1^{1,n-2k+3} - F_0^{2,n-2k+3} - F_k^{1,n} =$$
$$\Lambda_0^{1,n-2k+1} - \left[F_1^{1,n-2k+2} + F_0^{2,n-2k+2}\right] =$$
$$-\sum_{j=2}^{n-2k+1}\left[F_0^{2,j} + F_1^{1,j}\right] - \left[F_1^{1,n-2k+2} + F_0^{2,n-2k+2}\right] = -\sum_{j=2}^{n-2k+2}\left[F_1^{1,j} + F_0^{2,j}\right] =$$
$$\Lambda_0^{1,n-2k+2}$$

We can now finish the proof of Proposition 3.2.11 using the case $h = 0$ to prove the case $k \geq h > 0$.

$$\sum_{i=h}^{k} A_i^{k,n}\Lambda_h^{i,n} = \sum_{i=0}^{k-h} A_{h+i}^{k,n}\Lambda_h^{h+i,n} = \sum_{i=1}^{k-h} A_{h+i}^{k,n}\Lambda_h^{h+i,n} + A_h^{k,n}\Lambda_h^{h,n}$$

Recalling the recursive identity 3.4.5 in Proposition 3.2.7 we can expand $\Lambda_h^{h+i,n}$ obtaining

$$\sum_{i=1}^{k-h} A_{h+i}^{k,n}\Lambda_h^{h+i,n} + A_h^{k,n}\Lambda_h^{h,n} =$$
$$\sum_{i=1}^{k-h} A_{h+i}^{k,n}\left[\Lambda_0^{i,n-2h} + (-1)^i\binom{n-2h-i}{i}\Lambda_h^{h,n}\right] + A_h^{k,n}\Lambda_h^{h,n} =$$
$$\sum_{i=1}^{k-h} A_i^{k-h,n-2h}\Lambda_0^{i,n-2h} + \Lambda_h^{h,n}\left[A_h^{k,n} + \sum_{i=1}^{k-h}(-1)^i\binom{n-2h-i}{i}A_{h+i}^{k,n}\right] =$$
$$\Lambda_0^{1,n-2k+2} + \Lambda_h^{h,n}\left[A_h^{k,n} + \sum_{i=1}^{k-h}(-1)^i\binom{n-2h-i}{i}A_i^{k-h,n-2h}\right]$$

Replacing i with $t = i + 1$ we can rewrite the expression as

$$\Lambda_0^{1,n-2k+2} + \Lambda_h^{h,n}\left[A_h^{k,n} + \sum_{t=2}^{k-h+1}(-1)^{t-1}\binom{n-2h-t+1}{t-1}A_{t-1}^{k-h,n-2h}\right] =$$

$$\Lambda_0^{1,n-2k+2} + \Lambda_h^{h,n}\left[A_h^{k,n} - \sum_{t=2}^{k-h+1}(-1)^t\binom{n-2h+2-t-1}{t-1}A_{t-1}^{k-h,n-2h}\right] =$$

$$\Lambda_0^{1,n-2k+2} + \Lambda_h^{h,n}\left[A_1^{k-h+1,n-2h+2} - \sum_{t=2}^{k-h+1}(-1)^t\binom{n-2h+2-t-1}{t-1}A_t^{k-h+1,n-2h+2}\right] =$$

$$\Lambda_0^{1,n-2k+2}$$

where for the last equality we used the definition of $A_1^{k-h+1,n-2h+2}$. □

We are now ready to prove that Reeder's Conjecture holds for Lie Groups of type C and small weights of the form ω_{2k}.

Theorem 3.2.14. *In the same notation of Proposition 3.2.11, the following formula holds:*

$$C_{k+1n}(q,t) = \frac{(t^{2(n-2k-1)}-1)(t^{2(n-k+1)}-1)(1-qt^{2k-1})t^2}{(t^{2(n-2k+1)}-1)(t^{2(k+1)}-1)(1-qt^{2(n-k)-1})}C_{kn}(q,t). \tag{3.2.6}$$

Proof. By Proposition 3.2.11

$$\Lambda_{k+1}^{k+1,n}C_{k+1,n}(q,t) = -\Lambda_0^{1,n-2k}\left(C_{kn}(q,t) + \cdots + C_{0n}(q,t)\right)$$

Multiplying by $\Lambda_0^{1,n-2k+2}$ we obtain

$$\Lambda_{k+1}^{k+1,n}\Lambda_0^{1,n-2k+2}C_{k+1,n}(q,t) =$$

$$-\Lambda_0^{1,n-2k}\Lambda_0^{1,n-2k+2}\left(C_{kn}(q,t) + \cdots + C_{0n}(q,t)\right) =$$

$$-\Lambda_0^{1,n-2k}\Lambda_0^{1,n-2k+2}C_{kn}(q,t) - \Lambda_0^{1,n-2k}\left[\Lambda_0^{1,n-2k+2}\left(C_{k-1n}(q,t) + \cdots + C_{0n}(q,t)\right)\right]$$

Now by induction we have

$$\Lambda_k^{k,n}C_{kn}(q,t) = -\Lambda_0^{1,n-2k+2}\left(C_{k-1n}(q,t) + \cdots + C_{0n}(q,t)\right)$$

and substituting we obtain

$$-\Lambda_0^{1,n-2k}\Lambda_0^{1,n-2k+2}C_{kn}(q,t) - \Lambda_0^{1,n-2k}\left[\Lambda_0^{1,n-2k+2}\left(C_{k-1n}(q,t) + \cdots + C_{0n}(q,t)\right)\right] =$$

$$-\Lambda_0^{1,n-2k}\Lambda_0^{1,n-2k+2}C_{kn}(q,t) + \Lambda_0^{1,n-2k}\Lambda_k^{k,n}C_{kn}(q,t) =$$

$$-\Lambda_0^{1,n-2k}\left(\Lambda_0^{1,n-2k+2} - \Lambda_k^{k,n}\right)C_{kn}(q,t)$$

And then

$$C_{k+1,n}(q,t) = -\frac{\Lambda_0^{1,n-2k}\left(\Lambda_0^{1,n-2k+2} - \Lambda_k^{k,n}\right)}{\Lambda_{k+1}^{k+1,n}\Lambda_0^{1,n-2k+2}}C_{kn}(q,t)$$

We have closed expressions for all the Λ-coefficients in the formula (see Proposition 3.2.7 and equation 3.2.4). Replacing them with their closed formulas we obtain exactly the statement of Theorem. $\square$

Corollary 3.2.15.

$$C_{k+1,n}(-q,q^2) = C_{k,n}(-q,q^2)\frac{q^4(q^{4(n-k+1)}-1)(q^{4(n-2k-1)}-1)(q^{4k-1}+1)}{(q^{4(k+1)}-1)(q^{4(n-2k+1)}-1)(q^{4(n-k)-1}+1)} \tag{3.2.7}$$

Proof. This is just the specialization $(q,t) \longrightarrow (-q,q^2)$ in the formula 3.2.6. $\square$

Finally, we have proved that for fixed n, the polynomials $C_{k,n}(-q,q^2)$ and the polynomials $\mathcal{C}_{k,n}(q)$ satisfy the same transition formula from k to $k+1$.

By Corollary 3.2.9 is known that $C_{1,n}(-q,q^2) = \mathcal{C}_{1,n}(q)$ and then the polynomials must be equal for each $k > 0$. This prove the Reeder's Conjecture in our case.

3.3 Stability of the coefficients

We observed empirically that the coefficients of the polynomials $C_{k,n}$ and $C_{k,n+1}$ are the same in a range of degrees that depends by n and k.

By duality we know that polynomials $C_{k,n}$ must be reciprocal polynomials, symmetric by a central term of degree $n(2n+1)/2$.

We want now to determine the maximum degree M of q in $C_{k,n}$, smaller than $n(2n+1)/2$ and such that $C_{k,n}$ and $C_{k,n+1}$ have the same i-th coefficients for all $i < M$.

By direct computation one can obtain a transition formula from $C_{k,n}$ to $C_{k,n+1}$, similar to 3.1.1 in first section.

$$C_{2k;n+1}(q) = C_{2k;n}(q)\frac{\left(1-q^{4(n-2k+2)}\right)\left(1+q^{4(n+1-k)-1}\right)\left(1-q^{4(n+1)}\right)}{\left(1-q^{4(n-k+2)}\right)\left(1-q^{4(n-2k+1)}\right)}$$

Now we can compute explicitly M just looking at the degree of the lowest degree monomial with non zero coefficient in the polynomial $D(q) = C_{k,n+1} - C_{k,n}$.

$$D(q) = C_{k+1,n} - C_{k,n} =$$

$$C_{k,n}(q)\left[\frac{\left(1-q^{4(n-2k+2)}\right)\left(1+q^{4(n+1-k)-1}\right)\left(1-q^{4(n+1)}\right)}{\left(1-q^{4(n-k+2)}\right)\left(1-q^{4(n-2k+1)}\right)} - 1\right] =$$

$$C_{k,n}(q)\left[\frac{\left(1-q^{4(n-2k+2)}\right)\left(1+q^{4(n+1-k)-1}\right)\left(1-q^{4(n+1)}\right) - \left(1-q^{4(n-k+2)}\right)\left(1-q^{4(n-2k+1)}\right)}{\left(1-q^{4(n-k+2)}\right)\left(1-q^{4(n-2k+1)}\right)}\right]$$

The numerator is equal to

$$-q^{4(n-2k+2)} - q^{4(n+1)} + q^{4(2n-2k+3)} + q^{4(n-k)+3} - q^{4(2n-3k)+11} - q^{4(2n-k)+7} +$$

$$+q^{4(2n-3k)+15} - q^{4(2n-3k+3)} + q^{4(n-2k+1)} + q^{4(n-k+2)} = q^{4(n-2k+1)}P(q)$$

where $P(q)$ is a polynomial and $P(0) = 1$.
Now we recall that q^{4k-1} divides $C_{k,n}(q)$ and then we can write

$$D(q) = q^{4n-4k+3}\left(\frac{C_{k,n}(q)}{q^{4k-1}}P(q)\right)$$

Actually $\frac{C_{k,n}(q)}{q^{4k-1}}P(q)$ is a polynomial and its evaluation at 0 is equal to 1. Hence we obtain the following statement.

Theorem 3.3.1. *Let $C_{k,n}(q)$ and $C_{k,n+1}(q)$ be the polynomials described in section 1, then*

$$[q^i]C_{k,n}(q) = [q^i]C_{k,n+1}(q)$$

for all i smaller than $4n-4k+3$, where the symbol $[q^i]$ denote the coefficient of the i-th power of q in the polynomial. Equivalently, for all $i < 4n-4k+3$, the following equality holds:

$$\dim Hom_{Sp_{2n}}(V_{\omega_{2k}}, \Lambda^i \mathfrak{sp}_{2n}) = \dim Hom_{Sp_{2(n+1)}}(V_{\omega_{2k}}, \Lambda^i \mathfrak{sp}_{2n+2}).$$

3.4 Case 2: Small weights of the form $\omega_1 + \omega_{2k+1}$

As observed in the Section 1 of this Chapter, in this case (except when $k = 0$, which has a different zero weight structure and is treated in a different way) the zero weights space of V_λ is not irreducible. Anyway our strategy is very similar to the case ω_{2k}: we will find some recursive relations between coefficients in the recurrence and then use such relations to reduce the triangular system of equations to a three terms recurrence that we solve by induction.

Notation 3.4.1. We will denote with $C_{2|k,n}$ and C_k respectively the polynomials $C_{\omega_1+\omega_{2k+1}}$ and $C_{\omega_{2k}}$ in the reduced Stembridge's recurrence.

Differently from the previous one, in this case we will look a transition from $C_{2|k}$ to C_k and C_{k+1}: using the "natural" one from $C_{2|k}$ to $C_{2|k-1}$ has revealed to be too complicated for a general computation.

3.4.1 The recurrence for the $C_\lambda(q,t)$

Set $\lambda = \omega_1 + \omega_{2k+1}$. This weight, in e_j-notation, is equal to $2e_1 + e_2 + \cdots + e_{2k+1}$ and then the inner product $(\lambda, \omega) = (\omega_1 + \omega_{2k+1}, e_1)$ is equal to 2 for all k. The recurrence 1.4.4 can be rewritten in the following form:

$$\sum_{(\mu,2e_j)} \sum_{i\geq 0} \left[f_i^j(q,t) - q^2 f_i^j(q^{-1},t^{-1})\right] C_{\mu-2ie_j}(q,t) \tag{3.4.1}$$

Remark 3.4.2. Differently from the case ω_{2k} we have to distinguish two different sets of C_μ appearing in the recursion, depending on n.

In the case $n = 2k+1$ the weights smaller or equal to $\omega_1 + \omega_{2k+1}$ in the dominance order are the weights ω_{2j} and $\omega_1 + \omega_{2j+1}$ with $j \leq k$.

In the other case, with $n > 2k+1$, to the set above we must to add the weight $\omega_{2(k+1)}$

We are going now to make some remarks about the coefficients in the recursion that hold independently for the two cases. However prove the conjecture in the two cases needs different computations and then we will treat them separately.

Setting again with $B_j = (\rho, e_j) = n - j + 1$, we can give also in this case explicit formulae for the coefficients $F_j^i = f_i^j(q,t) - q^2 f_i^j(q^{-1}, t^{-1}$ in the non reduced recurrence:

$$F_0^j(q,t) = \frac{(1 - q^2 t^{2B_j})}{t^{B_j}},$$

$$F_1^{n,n} = q^2 - 1$$

$$F_1^j(q,t) = \frac{t}{t^{B_j}}(t - q - 1) - qt^{B_j}\frac{(q - tq - t)}{t^2},$$

$$F_i^j(q,t) = (t-q)(t-1)\left(\frac{t^{2i-2}}{t^{B_j}} - \frac{qt^{B_j}}{t^{2i}}\right),$$

$$F_{B_j}^j(q,t) = \frac{t^{2(B_j-1)}}{t^{B_j}}(q - qt - t) - \frac{qt^{B_j}}{t^{2(B_j-1)}}\frac{t(t-(q+1))}{t^2},$$

$$F_{B_j+1}^j(q,t) = \frac{q(t^{2B_j} - 1)}{t^{B_j}}.$$

Remark 3.4.3. As in the ω_{2k} case, there is a kind of "inductive stability" for the F_i^{jn}. We have, for $0 \geq h$,

$$F_i^{j+h,n+h} = F_i^{j,n}$$

Notation 3.4.4. For some reasons that will be clear in the next pages, it is convenient to denote by $\Psi_i^{j,n}$ the following expression

$$\Psi_i^{j,n} = F_i^{j,n} - F_{B(i,j)}^{j,n}$$

where $B(i,j) = n - i - j + 3$

As done in the case of the weights ω_{2k}, we want to investigate some symmetry in the weights appearing in the non reduced form of the recursion.

Let us fix n and set $\lambda = \omega_1 + \omega_{2k+1}$. Set

$$\Gamma_\nu^{i,j} = \{(w\lambda, \epsilon)\ w \in W,\ \epsilon \in \{\pm 1\} \mid we_1 = e_j\ ,\ C_{w(\lambda - 2ie_j)} = \epsilon C_\nu\}$$

Lemma 3.4.5. *We have a bijection between* $\Gamma_\nu^{i,j}$ *and* $\Gamma_\nu^{B(i,j),j}$.

This bijection sends a pair $(w\lambda, \epsilon)$ *in a pair of the form* $(w'\lambda, -\epsilon)$.

Proof. Let $(\mu_1, \dots, \mu_n)$ be the coordinates of $w\lambda$. By definition of $\Gamma_\nu^{i,j}$ we have $\mu_j = 2-2i$.
We define

$$\Psi(\mu) = \begin{cases} \mu_h & \text{if } h \neq j \\ 2 - 2(n-j-i+3) & \text{if } h = j \end{cases}$$

It follows immediately from the definition that $\Psi(\mu) + \rho$ differs from $\mu + \rho$ just for the j-th coordinate which is equal to

$$n - j + 1 - 2n + 2j + 2i - 4 = -n + j + 2i - 3 = -(n - j + 1) - (2 - 2i) = -(\mu + \rho)_j$$

Actually $(\Psi(\mu), -\epsilon)$ is an element of $\Gamma_\nu^{B(i,j),j}$ and we can complete the proof observing that the map Ψ is an involution. □

Notation 3.4.6. We will denote with $\Lambda_{2|h}^{2|k,n}$ and with $\Lambda_h^{2|k,n}$ respectively the coefficients of $C_{2|k,n}$ and C_k in the Stembridge's recurrence.

As in the previous case, we are looking for recursive expressions of the coefficient in the reduced Stembridge's recursion with the aim to simplify the computations. The following remarks are crucial for this purpose

Remark 3.4.7. We recall that the rational functions F_i^j are defined by a generating polynomial of degree $B_j + 1$, i.e. the polynomials $F_{B_j+2}^j$ are equal to 0. This behavior of the concrete expression of the coefficients breaks the symmetry in the following sense: if there is a pair (μ, ϵ) appearing in $\Gamma_\nu^{i,j}$ it gives a contribution equal to $\epsilon\ F_i^j$ to the coefficient of C_ν and its symmetric weight in $\Gamma_\nu^{B(i,j),j}$ gives a contribution $-\epsilon\ F_{B(i,j)}^j$. Altough the set $\Gamma_\nu^{0,j}$ is symmetric to $\Gamma_\nu^{B_j+2,j}$ and then the only contribution in this case is ϵF_0^j.

Remark 3.4.8. By previous remark, we can reduce to compute weights in the sets $\Gamma_\nu^{i,j}$ for $i < n-i-j+3$ with the convention that a weight in $\Gamma_\nu^{i,j}$ gives a contribution to the coefficient equal to Ψ_i^j if $i > 0$ and equal to F_0^j if $i = 0$.

This simplification implies that we can suppose $2i-2 < n-j+1 = (e_j, \rho)$. As in the case ω_{2k}, this simple observation lead us to some recurrence rations between the $\Lambda_\nu^{2|k,n}$.

Suppose $w(e_1) = e_j$. Observe again that the coordinates of $w\lambda - 2ie_j$ have modulus smaller or equal to 1 except for the j-th coordinate that, in this case, is equal to $2 - 2i$.

If we fix i and consider $\bar{j} = max\{j \,|\, i < n - j - i + 3\}$, we have

$$0 < n-\bar{j}+1+(2-2i) = \rho_{\bar{j}}+(w\lambda-2ie_j)_{\bar{j}} < \rho_j+(w\lambda-2ie_j)_j = n-j+1+(2-2i) \ \ \forall j < \bar{j}$$

This implies again that without loss of generality we can suppose the coordinates of $w\lambda - 2ie_j$ are all non negative and the the reduced form of $C_{w\lambda-2ie_j}$ can be again computed just by the action of the symmetric group on $w\lambda - 2ie_j + \rho$.

Let us start the explicit computation of the coefficients $\Lambda_h^{2|k,n}$ with a general observation

Remark 3.4.9. Similarly to what observed in the case ω_{2k}, to obtain a contribution to the coefficient of $C_{\omega_1+\omega_{2h+1}}$, the first coordinate of $w\lambda$ must be equal to 2 and the latter $2h$ ones must be equal to 1

As an immediate consequence of this remark, we obtain that the only contributions to the coefficient of $C_{\omega_1+\omega_{2k+1}}$ in the recursion 3.2.2 come from the case $i = 0$.

Recalling the Lemma 3.2.5, we can compute easily the coefficient $\Lambda^{2|k,n}_{2|h}$ in the following way:

As observed before, the only possibility to obtain a contribution to $\Lambda^{2k,n}_{2h}$ is that the vector $(w\lambda_1, \dots w\lambda_{2h+1})$ is equal to $(2, 1, \dots, 1)$, in particular $i = 0$ and $w(e_1) = e_1$. The other $n - 2h - 1$ coordinates must restrict to a weight of the form $w\omega_{2(k-h)}$ conjugated to 0 in C_{n-2h-1}. Then we obtain immediately from the Lemma 3.2.5 that

$$\Lambda^{2|k,n}_{2|h} = (-1)^{k-h} F^{1,n}_0 \binom{(n-2h-1)-(k-h)}{k-h} = (-1)^{k-h} F^{1,n}_0 \binom{n-k-h-1}{k-h}. \tag{3.4.2}$$

Now we want find a recursive expansion of the coefficients $\Lambda^{2|k,n}_{h}$ looking at the combinatoric structure of the weights in $\Gamma^{k,n}_{h} = \{w(\omega_1 + \omega_{2k+1} - 2ie_1) \mid w(\omega_1 + \omega_{2k+1} - 2ie_1) + \rho \sim \omega_{2h} + \rho\}$ We will recall that, by our assumption on i, the coordinates of $w(\omega_1+\omega_{2k+1}-2ie_1)$ must be a permutation of the coordinates of $\omega_{2h}+\rho$. Let us suppose, first of all, that $h > 1$. We have five different cases:

- *Case 1: The fist two coordinates of* $w\lambda$ *are equal to 1.* Then the weight $w\lambda$ contract to a weight of the form $w(\omega_1 + \omega_{2(k-1)+1})$ conjugated to $\omega_{2(h-1)}$ in C_{n-2}
- *Case 2; The first coordinate of* $w\lambda$ *is equal to* 1 *and the second is equal to* 0*.* To obtain the weight ω_{2h} we must have coordinates equal to n and $n-1$ (this holds only if $h > 1$) and this force the third one to be equal to 2 and the fourth to be equal to 1. The remaining $n-4$ coordinates contract to a weight $w\omega_{2(k-1)}$ in C_{n-4} conjugated to $\omega_{2(h-1)}$.
- *Case 3: The first coordinate is equal to* 1 *and the second is equal to* -1*.* Again, because $h > 1$, we must have some coordinate equal to n and $n-1$, this forces the third coordinate to be equal to 2. But now it is impossible to obtain $n-1$ adding 1 to coordinates from the third to the n-th.
- *Case 4: The first coordinate is equal to* 0*.* This forces the second one to be equal to 2 and the coordinates from the third to the $2h$-th to be equal to 1. The remaining coordinates must contract to a weight $\omega_{2(k-h+1)}$ in C_{n-2h} conjugated to 0. It is crucial in the formulation of the recursions to observe that this is equivalent to contract $(w\lambda_2, \dots, w\lambda_n)$ to a weight $w(\omega_1+\omega_{2k+1})$ in C_{n-1} conjugated to $\omega_1 + \omega_{2(h-1)+1}$.
- *Case 5: The first coordinate is equal to* -1*.* This force the second one to be equal to 2 and again is impossible to obtain a coordinate equal to n adding 1 to the remaining ones.

We can translate this enumerative analysis in the following recursive relation:

$$\Lambda_h^{2|k,n} = \Lambda_{h-1}^{2|k-1,n-2} - \Lambda_{2h-1}^{2|k,n-1} - (-1)^{k-h} F_0^{3,n} \binom{n-k-h-1}{k-h+1}$$

The case $h = 1$, i.e. $w(\omega_1 + \omega_{2k+1} - 2ie_1) + \rho \simeq \omega_2 + \rho$ is very similar to the latter

- *Case 1: The first two coordinates of $w\lambda$ are equal to 1.* Then the weight $w\lambda$ contract in C_{n-2} to a weight of the form $w'(\omega_1 + \omega_{2(k-1)+1})$ conjugated to 0
- *Case 2; The first coordinate of $w\lambda$ is equal to* 1 *and the second is equal to* 0*.* To obtain the weight ω_2 we must have a coordinate equal to n and this force the third one to be equal to 2. Now $(w\lambda_4, \dots, w\lambda_n)$ must contract to a weight of the form $w\omega_{2k-1}$ conjugated to 0. Such a weight cannot exist by a parity argument.
- *Case 3: The first coordinate is equal to* 1 *and the second is equal to* -1*.* Again we must have some coordinate equal to n and this forces the third coordinate to be equal to 2. The remaining coordinates must contract to $w\omega_{2(k-1)}$ conjugated to zero.
- *Case 4: The first coordinate is equal to* 0*.* Again this is equivalent to contract $(w\lambda_2, \dots, w\lambda_n)$ to a weight $w(\omega_1 + \omega_{2k+1})$ in C_{n-1} conjugated to the weight $2\omega_1$.
- *Case 5: The first coordinate is equal to* -1*.* This force the second one to be equal to 2 and again is impossible to obtain a coordinate equal to n adding 1 to the remaining ones.

In terms of coefficients, the prevuous analysis can be transleted in the following recurrence:

$$\Lambda_1^{2|k,n} = \Lambda_0^{2|k-1,n-2} - \Lambda_{20}^{2|k,n-1} + (-1)^k F_0^{3,n} \binom{n-k-2}{k-1}$$

The zero case is again more complicated and we need to examine it very carefully. If we denote with $\Lambda_{0,i}^{2|k,n}$ the contribution to the coefficient $\Lambda_0^{2|k,n}$ for a fixed i we have

$$\Lambda_0^{2k,n} = \sum_{i=0}^{k+1} \Lambda_{0,i}^{2k,n}.$$

We want recover again a recurrence for $\Lambda_0^{2|k,n}$ summing up recurrences for each $\Lambda_{0,i}^{2|k,n}$. Such recurrences depend from the fist coordinates of $\mu \in \Gamma_0^{k,n} = \{w(\omega_1 + \omega_{2k+1} - 2ie_1) \mid w(\omega_1 + \omega_{2k+1} - 2ie_1) + \rho \simeq \rho\}$:

- μ_1 *is equal to 0.* Then μ can be contracted to a weight μ' for C_{n-1} conjugated to 0.
- μ_1 *is equal to* -1 *and* $\mu_2 = 1$*.* Then μ is of the form $(-1, 1, \mu')$ with μ' conjugated to 0 in C_{n-2}

- *The first j coordinates are equal to -1.*. In this case $\mu+\rho$ is then of the form $(n-1,\dots,n-2j,\dots)$. We must obtain a coordinate equal to n because $\mu+\rho$ have to be conjugated to 0. This is possible only if $j=2$ and $\mu_3=2$ (i.e. $i=0$ and $we_1=e_3$). Moreover $(\mu_4,\dots,\mu_n)$ is a weight for C_{n-3} of the form $w\omega_{2(k-1)}$ conjugated to 0.

- *The first coordinate is equal to $2-2i$.* The first coordinate of $\mu+\rho$ is equal to $n+2-2i=n-(2i-1)+1$ This forces the second coordinate of μ to be equal to 1, idem for the third and so on until the $2i-1$-th. We cannot add 1 to the $2i$-th coordinate because otherwise $(\mu+\rho)_1=(\mu+\rho)_{2i}$ and $\mu+\rho$ is not regular. We obtain μ is then of the form $(2-2i,1,\dots,1,\dots)$. The latter $(n-2i+1)$ coordinates contract to a weight $w\omega_{2(k-i+1)}$ conjugated to 0 in C_{n-2i}

We can translate what stated above in the following two relations:

$$\Lambda_{0,i}^{2|k,n}=\Lambda_{0,i}^{2|k,n-1}-\Lambda_{0,i}^{2|k-1,n-2}+(-1)^{k-i+1}\Psi_i^{1,n}\binom{n-k-i}{k-i+1}$$

if $i>0$. Conversly, in the case $i=0$ we obtain

$$\Lambda_{0,0}^{2|k,n}=\Lambda_{0,0}^{2|k,n-1}-\Lambda_{0,0}^{2|k-1,n-2}+(-1)^{k-1}F_0^{3,n}\binom{n-k-2}{k-1}$$

and finally we can sum up all the contributions obtaining

$$\Lambda_0^{2|k,n}=\Lambda_0^{2|k,n-1}-\Lambda_0^{2|k-1,n-2}+(-1)^{k-1}\binom{n-k-2}{k-1}F_0^{3,n}+\sum_{i=1}^{k+1}(-1)^{k-i+1}\Psi_i^{1,n}\binom{n-k-i}{k-i+1}$$

We can summarized what is proved above about the coefficients of the recursion in the following proposition.

Proposition 3.4.10. *Rewriting the recursion 3.2.2 using the reduced form of the polynomials C_μ, in the general case $n\neq 2k+1$ we obtain the recursion*

$$R_k\ :\ \sum_{i=0}^{k}C_{\omega_1+\omega_{2i+1}}\Lambda_{2|i}^{2|k,n}+\sum_{i=0}^{k+1}C_{\omega_{2i}}\Lambda_i^{2|k,n}=0 \tag{3.4.3}$$

where the coefficients of the form $\Lambda_{2|i}^{2|k,n}$ have the following closed form

$$\Lambda_{2|h}^{2|k,n}=(-1)^{k-h}F_0^{1,n}\binom{n-k-h-1}{k-h} \tag{3.4.4}$$

and the coefficients $\Lambda_h^{2|k,n}$ satisfies the recursive relations below:

$$\Lambda_h^{2|k,n}=\Lambda_{h-1}^{2|k-1,n-2}-\Lambda_{2|h-1}^{2|k,n-1}-(-1)^{k-h}F_0^{3,n}\binom{n-k-h-1}{k-h+1} \tag{3.4.5}$$

$$\Lambda_1^{2|k,n} = \Lambda_0^{2|k-1,n-2} - \Lambda_{2|0}^{2|k,n-1} + (-1)^k F_0^{3,n} \binom{n-k-2}{k-1} \tag{3.4.6}$$

$$\Lambda_0^{2|k,\,n} = \Lambda_0^{2|k,\,n-1} - \Lambda_0^{2|k-1,\,n-2} + (-1)^{k-1} \binom{n-k-2}{k-1} F_0^{3,n} + \sum_{i=1}^{k+1} (-1)^{k-i+1} \Psi_i^{1,n} \binom{n-k-i}{k-i+1} \tag{3.4.7}$$

If $n = 2k+1$, otherwise, the polynomial C_{k+1} does not appear in the recursion:

$$R_k \;:\; \sum_{i=0}^{k} C_{\omega_1+\omega_{2i+1}} \Lambda_{2|i}^{2|k,n} + \sum_{i=0}^{k} C_{\omega_{2i}} \Lambda_i^{2|k,\,n} = 0 \tag{3.4.8}$$

and the coefficients $\Lambda_h^{2|k,\,n}$ satisfy recursive relations similar to the previous ones:

$$\Lambda_{2|h}^{2|k,\,n} = (-1)^{k-h} F_0^{1,n}, \tag{3.4.9}$$

$$\Lambda_h^{2|k,n} = \Lambda_{h-1}^{2|k-1,n-2}, \tag{3.4.10}$$

$$\Lambda_1^{2|k,n} = \Lambda_0^{2|k-1,n-2} + (-1)^k F_0^{3,n}, \tag{3.4.11}$$

$$\Lambda_0^{2|k,\,n} = -\Lambda_0^{2|k-1,\,n-2} + (-1)^{k-1} F_0^{3,n} + \sum_{i=1}^{k+1} (-1)^{k-i+1} \Psi_i^{1,n}. \tag{3.4.12}$$

Before to start dealing with the general case of the Reeder Conjecture, we need some basic steps for inductive reasoning about coefficients.

3.4.2 The "well known" case $\lambda = 2\omega_1$ revised

This case is very different from the other ones (i.e. $k > 0$). It is well known in literature and the polynomials of graded multiplicities has been previously computed, for example by Stembridge in [37] or in [15]. The most important difference concerns the W-representation $(V_{2\omega_1})^0$ which is irreducible and isomorphic to $\pi_{(n-1,1),\emptyset}$ (see [1]).

To be more precise, in Stembridge's paper appears the following formula

Theorem 3.4.11.

$$C_{2\omega_1}(q,t) = C_0(q,t) \frac{(t-q)(t^{2n}-1)}{(1-qt^{2n-1})(t-1)}.$$

As a first computation we want to show again this result. Let us examine closely the sets $\Gamma_\nu^{i,n}$ for $\nu \in \{2\omega_1,\, \omega_2,\, 0\}$. All the previous observations about the simplifications in the computation of coefficients holds. In particular for the coefficient $\Lambda_{2|0}^{2|0,n}$ we have the following formula:

$$\Lambda_{2|0}^{2|0,n} = F_0^{1,n} = \frac{(1-q^2t^{2n})}{t^n}$$

Now we want examine the elements of the set $\Gamma_{\omega_2}^{i,n}$. The weight $w(2\omega_1 - 2ie_1) + \rho$ must be conjugated to $\omega_2 + \rho = (n+1, n, n-2, \dots, 1)$. Our assumption on i (i.e. $i < n - j - i + 3$) implies that the components of $w(2\omega_1 - 2ie_1)$ must be all positive and we must have a component equal to $n+1$, then the only possible case is $i = 0$ and $w(2\omega_1) = (0, 2, 0, \dots, 0)$. Moreover $w(2\omega_1) + \rho$ is conjugated to $\omega_2 + \rho$ by the permutation (12). It follows that the coefficient $\Lambda_1^{2|0,n}$ can be expressed as:

$$\Lambda_1^{2|0,n} = -F_0^{2,n} = -\frac{(1-q^2t^{2n-2})}{t^{n-1}}.$$

Finally, we need to compute the coefficient $\Lambda_0^{2|0,n}$. If we fix a value of i, then $w(2-2i, 0, \dots, 0) + \rho$ must be conjugated to ρ. If we suppose $we_1 = e_j$, then $(w(2-2_i, 0, \dots, 0) + \rho)_j = (n-j+1) + (2-2i) = n - j - 2i + 3 = n - (j + 2i - 2) + 1$. Recall that, by our assumption on i, the coordinates of $w(2-2_i, 0, \dots, 0) + \rho$ are all positive. Then $w(2-2i, 0, \dots, 0) + \rho$ is regular and conjugates to 0 only if $i = 1$. Then we have

$$\Gamma_0^{2\omega_1,n} = \bigcup_{j=1}^{n} \{((0, \dots, 0), e_j)\}.$$

Then we can describe the coefficient $\Lambda_0^{2|0,n}$ by the following formula:

$$\Lambda_0^{2|0,n} = \sum_{i=1}^{n} \Psi_1^{1,i}.$$

Remark 3.4.12.

$$\Psi_1^{1,1} = F_1^{n,n} - F_2^{n,n} = -\frac{(t-q)(1+qt)}{t},$$

$$\Psi_1^{1,i} = F_1^{1,i} - F_{n+1}^{1,i} = \frac{(t-q)(t-1)(1-qt^{2(i-1)})}{t^i}.$$

Lemma 3.4.13.

$$\Lambda_0^{2|0,n} = -\frac{(t-q)(1+qt^{2n-1})}{t^n}.$$

Proof. By the previous remark, the identity holds for $n = 1$. The thesis is then a just an explicit computation.

□

Now we are ready to prove again the Stembridge's result. We start from the recursion for $C_{2\omega_1,n}(q,t)$.

$$R_0 \ : \ \Lambda_{2|0}^{2|0,n} C_{2\omega_1,n}(q,t) + \Lambda_1^{2|0,n} C_{\omega_2,n}(q,t) + \Lambda_0^{2|0,n} C_{0,n}(q,t) = 0.$$

We now recall the explicit formula for C_1 from the previous case:

Remark 3.4.14.

$$C_{1,n}(q,t) = C_{0,n}(q,t)\frac{t^2(t-q)(t^{2(n-1)}-1)}{t^n(1-qt^{2n-1})(t^2-1)}.$$

We can then substitute this expression in the recurrence, obtaining:

$$\Lambda_{2|0}^{2|0,n} C_{2\omega_1,n}(q,t) = -C_{0,n}(q,t)\left(\frac{t^2(t-q)(t^{2(n-1)}-1)}{t^n(1-qt^{2n-1})(t^2-1)}\Lambda_1^{2|0,n} + \Lambda_0^{2|0,n}\right).$$

Finally, using the explicit expressions for the coefficient stated above, we prove the thesis

$$\begin{aligned}
&\Lambda_{2|0}^{2|0,n} C_{2\omega_1,n}(q,t) = \\
-C_{0,n}(q,t)&\frac{(t-q)}{t^n(1-qt^{2n-1})(t^2-1)}\Big(t^2(t^{2(n-1)}-1)(1-q^2t^{2(n-1)}) + (1-q^2t^{4n-2})(t^2-1)\Big) = \\
&C_{0,n}(q,t)\frac{(t-q)}{t^n(1-qt^{2n-1})(t^2-1)}(t^{2n}-1)(1-q^2t^{2n}).
\end{aligned}$$

and then

$$\begin{aligned}
&C_{2\omega_1,n}(q,t) = \\
C_{0,n}(q,t)&\frac{(t-q)}{t^n(1-qt^{2n-1})(t^2-1)}(t^{2n}-1)(1-q^2t^{2n})\cdot\frac{t^n}{(1-q^2t^{2n})} = \\
&C_{0,n}(q,t)\frac{(t-q)(t^{2n}-1)}{(1-qt^{2n-1})(t^2-1)}.
\end{aligned}$$

3.4.3 Recurrences, Coefficients and Closed Formulae

In this section we give a proof of the Reeder's Conjecture in the case of $\lambda = \omega_1 + \omega_{2k+1}$ for the Lie Algebras of type C_n.

We have just proved the conjecture for $k = 0$ in the previous sections. We want use induction to obtain the general proof.

We are going to obtain the conjecture as a consequence of the following proposition.

Proposition 3.4.15. *Let $\{R_i\}_{i\le k}$ be the set of recursion defined as in 3.4.10, then there exist a family of integers $\{B_i^{k,n}\}_{i\le k}$ such that if $n = 2k+1$*

$$\begin{aligned}
&\Gamma^{k,2k+1} := \\
\sum B_i^{k,2k+1} R_i = \Lambda_{2|k}^{2|k,2k+1} C_{2|k} &+ \Gamma_k^{k,2k+1} C_k + \Gamma_0^{k,2k+1}\left(C_{k-1} + \cdots + C_0\right).
\end{aligned}$$

and, for generic n

$\Gamma^{k,n} :=$

$$\sum B_i^{k,n} R_i = \Lambda_{2|k}^{2|k,n} C_{2|k} + \Gamma_{k+1}^{k,n} C_{k+1} + \Gamma_k^{k,n} C_k + \Gamma_0^{k,n} \left(C_{k-1} + \cdots + C_0\right).$$

Proof. Let us start, as in the previous case, by defining the integers $B_k^{k,n}$. We will use a recursive definition, setting:

$$B_h^{k,n} = \begin{cases} 0 & \text{if } h > k \\ 1 & \text{if } h = k \\ B_{i-1}^{k-1 n-2} & \text{if } k > h > 0 \\ \sum_{i=1}^{k} (-1)^i \binom{n-i-1}{i} B_i^{k,n} & \text{if } h = 0 \end{cases} \tag{3.4.13}$$

Remark 3.4.16. The following, more general, identity holds:

$$\sum_{i=h}^{k} (-1)^{i-h} \binom{n-i-h-1}{i-h} B_i^{kn} = 0. \tag{3.4.14}$$

It can be proved making the substitution $t = i - h$.

Aiming to prove the Proposition 3.4.15 we can rearrange the expression $\sum B_i^{k,n} R_i$

$$\sum_{i=1}^{k} B_i^{k,n} R_i = \sum_{i=0}^{k} \left(B_i^{k,n} \sum_{j=0}^{i} \Lambda_{2|j}^{2|i,n} C_{2|j} \right) + \sum_{i=0}^{k} \left(B_i^{k,n} \sum_{j=0}^{i/i+1} \Lambda_j^{i,n} C_j \right) =$$

$$\sum_{j=0}^{k} \left(\sum_{i=j}^{k} B_i^{k,n} \Lambda_{2|j}^{2|i,n} \right) C_{2|j} + \sum_{j=0}^{k+1/k} \left(\sum_{i=j-1}^{k} B_i^{k,n} \Lambda_j^{2|i,n} \right) C_j =$$

$$\Lambda_{2|k}^{2|k,n} C_{2|k} + \sum_{j=0}^{k-1} \left(\sum_{i=j}^{k} B_i^{k,n} \Lambda_{2|j}^{2|i,n} \right) C_{2|j} + \sum_{j=0}^{k/k+1} \left(\sum_{i=j-1}^{k} B_i^{k,n} \Lambda_j^{2|i,n} \right) C_j$$

Where the notation $k/k+1$ simply remarks the fact that if $n = 2k+1$ the polynomial C_{k+1} is not involved in the recursive formula. Let us denote with $\Gamma_{2|h}^{k,n}$ the coefficient

$$\Gamma_{2|h}^{k,n} = \sum_{i=h}^{k} B_i^{k,n} \Lambda_{2|h}^{2|i,n}.$$

Lemma 3.4.17. *In the above notation, for all* $0 \leq h < k$, *we have* $\Gamma_{2|h}^{k,n} = 0$

Proof. We recall that, by above combinatorical computation, the following explicit expression for the single coefficient $\Lambda_{2|h}^{2|i,n}$ holds:

$$\Lambda_{2|h}^{2|i,\, n} = (-1)^{i-h} F_0^{1,n} \binom{n-i-h-1}{i-h}$$

Substituting this formula in the expression for the coefficient $\Gamma_{2|h}^{k,n}$ we obtain

$$\Gamma_{2|h}^{k,n} = \sum_{i=h}^{k} B_i^{k,n} \Lambda_{2|h}^{2|i,n} = \sum_{i=h}^{k} B_i^{k,n} (-1)^{i-h} F_0^{1,n} \binom{n-i-h-1}{i-h} =$$
$$F_0^{1,n} \left(\sum_{i=h}^{k} (-1)^{i-h} B_i^{k,n} \binom{n-i-h-1}{i-h} \right).$$

By equation 3.4.14 we obtain what we were looking for.

□

To prove the Proposition 3.4.15 it is enough to show that, for all $h < k$ we have

$$\sum_{i=h}^{k} B_i^{k,n} \Lambda_h^{2|i,n} = \Gamma_h^{k,n} = \Gamma_0^{k,n} = \sum_{i=0}^{k} B_i^{k,n} \Lambda_0^{2|i,n}$$

We need now two preliminary Lemmata

Lemma 3.4.18.

$$B_h^{k+1,n+1} = B_h^{k+1,n} + B_h^{k,n}.$$

Proof. Without loss of generality it is sufficient show the thesis for $h = 0$, in fact by the definition we have

$$B_h^{k,n} = B_0^{k-h,n-2h} = B_0^{k-h,(n-1)-2h} + B_0^{k-h-1,(n-1)-2h} = B_h^{k,n-1} + B_h^{k-1,n-1}$$

Now we can write down the expression for $B_0^{k,n-1}$ and $B_0^{k-1,n-1}$ using the definition:

$$B_0^{k,n} = \sum_{i=1}^{k} (-1)^{i+1} \binom{n-i-1}{i} B_i^{k,n}$$

$$B_0^{k+1,n} = \sum_{i=1}^{k+1} (-1)^{i+1} \binom{n-i-i}{i} B_i^{k+1,n}$$

and then

$$B_0^{k,n} + B_0^{k+1,n} =$$
$$\sum_{i=1}^{k} (-1)^{i+1} \binom{n-i-1}{i} \left[B_i^{k,n} + B_i^{k+1,n} \right] + (-1)^{k+2} \binom{n-k-2}{k+1} B_{k+1}^{k+1,n} =$$
$$\sum_{i=1}^{k} (-1)^{i+1} \binom{n-i-1}{i} \left[B_0^{k-i,n-2i} + B_0^{k-i+1,n-2i+1} \right] + (-1)^{k+2} \binom{n-k-2}{k+1} B_{k+1}^{k+1,n} =^*$$

$$\sum_{i=1}^{k}(-1)^{i+1}\binom{n-i-1}{i}B_0^{k-i+1,n-2i+1}+(-1)^{k+2}\binom{n-k-2}{k+1}B_{k+1}^{k+1,n}=$$

$$\sum_{i=1}^{k}(-1)^{i+1}\binom{n-i-1}{i}B_i^{k+1,n+1}+(-1)^{k+2}\binom{n-k-2}{k+1}B_{k+1}^{k+1,n}=$$

$$\sum_{i=1}^{k}(-1)^{i+1}\binom{n-i-1}{i}B_i^{k+1,n+1}$$

where for the marked equality we used the equation 3.4.14 and for the last one we used, by definition, $B_{k+1}^{k+1,n+1}=B_{k+1}^{k+1,n}$. Now we will use the well know identity between binomial coefficients

$$\binom{n-i-1}{i}=\binom{n-i}{i}-\binom{n-i-1}{i-1}$$

to expand the sum and obtain the thesis

$$\sum_{i=1}^{k}(-1)^{i+1}\binom{n-i-1}{i}B_i^{k+1,n+1}=$$

$$\sum_{i=1}^{k}(-1)^{i+1}\left[\binom{n-i}{i}-\binom{n-i-1}{i-1}\right]B_i^{k+1,n+1}=^*$$

$$\sum_{i=1}^{k}(-1)^{i+1}\binom{n-i}{i}B_i^{k+1,n+1}-\sum_{t=0}^{k}(-1)^{t+2}+\binom{(n+1)-t-1}{t}B_{t+1}^{k+1,n+1}=^D$$

$$B_0^{k+1,n+1}-\sum_{t=0}^{k}(-1)^{t}+\binom{(n+1)-t-1}{t}B_t^{k,n-1}=^*B_0^{k+1,n+1}$$

where the equalities marked by a $*$ are obtained using again equation 3.4.14 and the D-marked one uses the definition of $B_i^{k,n}$. ☐

Lemma 3.4.19. *For $h<k-1$ the following identity holds*

$$\sum_{i=h}^{k}(-1)^{i-h}\binom{n-i-h-2}{i-h}B_i^{k,n}=0. \tag{3.4.15}$$

Proof. First of all we observe that without loss of generality we can suppose $h=0$. In fact, setting $t=i-h$, we obtain

$$\sum_{i=h}^{k}(-1)^{i-h}\binom{n-i-h-2}{i-h}B_i^{k,n}=\sum_{t=0}^{k-h}(-1)^{t}\binom{n-2h-t-2}{t}B_{t+h}^{k,n}=$$

$$\sum_{t=0}^{k-h}(-1)^t\binom{n-2h-t-2}{t}B_t^{k-h,n-2h}.$$

Now, recalling the definition of $B_h^{k,n}$, we have the following identity

$$\sum_{i=0}^{k}(-1)^i\binom{n-i-1}{i}B_i^{k,n}=0$$

Now we can subtract the expression 3.4.15 with $h=0$ obtaining

$$\sum_{i=0}^{k}(-1)^i\binom{n-i-1}{i}B_i^{k,n}-\sum_{i=0}^{k}(-1)^i\binom{n-i-2}{i}B_i^{k,n}=$$
$$\sum_{i=0}^{k}(-1)^i\left[\binom{n-i-1}{i}-\binom{n-i-2}{i}\right]B_i^{k,n}=$$
$$\sum_{i=1}^{k}(-1)^i\binom{n-i-2}{i-1}B_i^{k,n}=-\sum_{i=1}^{k}(-1)^i\binom{n-i-2}{i-1}B_i^{k,n}.$$

If $k>1$ (we are using here the hypothesis $h<k-1$) the latter expression is equal to zero by equation 3.4.14. This implies immediately our thesis.

□

Now we can start the proof of Proposition 3.4.15 showing an iterative formula to compute the coefficients $\Gamma_h^{k,n}$. Let us start from the case $k-1\geq h>1$. We have:

$$\Gamma_h^{k,n}=\sum_{i=h-1}^{k}B_i^{k,n}\Lambda_h^{2|i,n}$$

Using Proposition 3.4.10 we can first of all expand the coefficient $\Lambda_h^{2|in}$ for $n\neq 2k+1$

$$\Gamma_h^{k,n}=\sum_{i=h-1}^{k}B_i^{k,n}\Lambda_h^{2|i,n}=$$
$$\sum_{i=h-1}^{k}B_i^{k,n}\Lambda_{h-1}^{2|i-1,n-2}-\sum_{i=h-1}^{k}B_i^{k,n}\Lambda_{2|h-1}^{2|i,n-1}-\left[\sum_{i=h-1}^{k}(-1)^{i-h+1}B_i^{k,n}\binom{n-i-h-1}{i-h+1}\right]F_0^{3,n}=^{Def}$$
$$\sum_{i=h-1}^{k}B_{i-1}^{k-1,n-2}\Lambda_{h-1}^{2|i-1,n-2}-\sum_{i=h-1}^{k}B_i^{k,n}\left[(-1)^{i-h+1}\binom{n-i-h-1}{i-h+1}\right]F_0^{1,n-1}+$$
$$-\left[\sum_{i=h-1}^{k}(-1)^{i-h+1}B_i^{k,n}\binom{n-i-h-1}{i-h+1}\right]F_0^{3,n}=^{t\to i-1}$$

$$\sum_{t=h-2}^{k-1} B_t^{k-1,n-2}\Lambda_{h-1}^{2|t,n-2} - \left[\sum_{i=h-1}^{k}(-1)^{i-h+1}B_i^{k,n}\binom{n-i-h-1}{i-h+1}\right]F_0^{1,n-1}+$$
$$-\left[\sum_{i=h-1}^{k}(-1)^{i-h+1}B_i^{k,n}\binom{n-i-h-1}{i-h+1}\right]F_0^{3,n} =^{Equation\ 3.4.14}$$
$$\sum_{t=h-2}^{k-1} B_t^{k-1,n-2}\Lambda_{h-1}^{2|t,n-2} = \Gamma_{h-1}^{k-1,n-2}.$$

The result is exactly the same if $n = 2k+1$, but in this case the much simpler recursions lead to simpler computations:

$$\Gamma_h^{k,n} = \sum_{i=h-1}^{k} B_i^{k,n}\Lambda_h^{2|i,n} = \sum_{i=h-1}^{k} B_i^{k,n}\Lambda_{h-1}^{2|i-1,n-2} =^{t\to i-1}$$
$$\sum_{t=h-2}^{k-1} B_t^{k-1,n-2}\Lambda_{h-1}^{2|t,n-2} = \sum_{t=h-2}^{k-1} B_t^{k-1,n-2}\Lambda_{h-1}^{2|t,n-2} = \Gamma_{h-1}^{k-1,n-2}.$$

We want prove a similar formula for the coefficient $\Gamma_1^{k,n}$

$$\Gamma_1^{k,n} = \sum_{i=0}^{k} B_i^{k,n}\Lambda_1^{2|i,n} =$$
$$\sum_{i=1}^{k} B_i^{k,n}\Lambda_1^{2|i-1,n-2} - \sum_{i=0}^{k} B_i^{k,n}\Lambda_{2|0}^{2|i,n-1} + \left[\sum_{i=0}^{k}(-1)^{i-2}B_i^{k,n}\binom{n-i-2}{i-1}\right]F_0^{3,n} =^{Def}$$
$$\sum_{i=1}^{k} B_i^{k,n}\Lambda_0^{2|i-1,n-2} - \left[\sum_{i=0}^{k}(-1)^{i}B_i^{k,n}\binom{n-i-1}{i}\right]F_0^{1,n-1}+$$
$$+\left[\sum_{i=0}^{k}(-1)^{i-2}B_i^{k,n}\binom{n-i-2}{i-1}\right]F_0^{3,n} =^{t\to i-1}$$
$$\sum_{t=0}^{k} B_{t+1}^{k,n}\Lambda_0^{2|t,n-2} - \left[\sum_{i=0}^{k}(-1)^{i}B_i^{k,n}\binom{n-i-1}{i}\right]F_0^{1,n-1}+$$
$$+\left[\sum_{i=0}^{k}(-1)^{i-2}B_i^{k,n}\binom{n-i-2}{i-1}\right]F_0^{3,n} =^{Def}$$
$$\sum_{t=0}^{k-1} B_t^{k-1,n-2}\Lambda_0^{2|t,n-2} = \Gamma_0^{k-1,n-2}.$$

The case $n = 2k+1$ is more complicated and needs a preliminary Lemma

Lemma 3.4.20. $B_j^{k,2,k} = 0$ *for all* $j < k$.

Proof. First of all observe that if $n = 2k$, for all i we have

$$\binom{n-k-j-1}{k-j} = \binom{2k-k-j-1}{k-j} = \binom{k-j-1}{k-j} = 0.$$

Let suppose now $j = k-1$, then by definition

$$0 = -B^{k,n}_{k-1} + B^{k,n}_{k}\binom{n-k-(k-1)-1}{1} = -B^{k,n}_{k-1}.$$

Iterating the argument we can prove the statement for a general j: let's suppose $B^{k,2k}_i = 0$ for all $i > j$, then by definition

$$0 = \sum_{s=j}^{k}(-1)^{s-j}\binom{2k-s-j-1}{s-j}B^{k,2k}_j = (-1)^{s-j}B^{k,n} + \binom{k-j-1}{k-j}B^{k,2k}_k = (-1)^{s-j}B^{k,n}.$$

$\square$

Now we are ready to expand the expression for $\Gamma^{k,2k+1}_1$:

$$\Gamma^{k,n}_1 = \sum_{i=0}^{k} B^{k,n}_i \Lambda^{2|i,n}_1 =$$

$$\sum_{i=1}^{k} B^{k,n}_i \Lambda^{2|i-1,n-2}_1 - \sum_{i=0}^{k-1} B^{k,n}_i \Lambda^{2|i,n-1}_{2|0} + \left[\sum_{i=0}^{k}(-1)^{i-2}B^{k,n}_i\binom{n-i-2}{i-1}\right]F^{3,n}_0 =^{Def}$$

$$\sum_{i=1}^{k} B^{k,n}_i \Lambda^{2|i-1,n-2}_0 - \sum_{i=0}^{k-1} B^{k,n}_i \Lambda^{2|i,n-1}_{2|0} =^{t\to i-1}$$

$$\sum_{t=0}^{k-1} B^{k,n}_{t+1} \Lambda^{2|t,n-2}_0 - \sum_{i=0}^{k-1}\left[B^{k-1,n-1}_i + B^{k,n-1}_i\right]\Lambda^{2|i,n-1}_{2|0} =^{Lemma\ 3.4.20}$$

$$\sum_{t=0}^{k-1} B^{k-1,n-2}_t \Lambda^{2|t,n-2}_0 - \sum_{i=0}^{k-1} B^{k-1,n-1}_i \Lambda^{2|i,n-1}_{2|0} =$$

$$\Gamma^{k-1,n-2}_0 - \Gamma^{k-1,n-2}_{2|0} =$$

$$\Gamma^{k-1,n-2}_0$$

We have finally to compute the coefficient $\Gamma^{k,n}_0$.

$$\Lambda^{k,\,n}_0 = \Lambda^{k,\,n-1}_0 - \Lambda^{k-1,\,n-2}_0 + (-1)^k\binom{n-k-1}{k-1}F^{2,n}_0 + \sum_{i=1}^{k}(-1)^{k-i+1}F^{1,n}_i\binom{n-i-k}{k-i}$$

and use it to expand the expression $\sum_{i=1}^{k} B^{k,n}_i \Lambda^{i,n}_0$.

$$
\begin{aligned}
\Gamma_0^{k,n} &= \sum_{i=0}^{k} B_i^{k,n}\Lambda_0^{2|i,n} = \\
&\sum_{i=0}^{k} B_i^{k,n}\Lambda_0^{2|i,n-1} - \sum_{i=0}^{k-1} B_{i+1}^{k,n}\Lambda_0^{2|i,n-2} + \sum_{i=1}^{k}(-1)^{i-1}B_i^{k,n}\binom{n-i-2}{i-1}F_0^{3,n} + \\
&+\sum_{i=0}^{k}\left[B_i^{k,n}\sum_{j=1}^{i+1}(-1)^{i-j+1}\Psi_j^{1,n}\binom{n-i-j}{i-j+1}\right] = \\
&\sum_{i=0}^{k} B_i^{k,n}\Lambda_0^{2|i,n-1} - \sum_{i=0}^{k-1} B_{i+1}^{k,n}\Lambda_0^{2|i,n-2} + F_0^{3,n}\left(\sum_{i=1}^{k}(-1)^{i-1}B_i^{k,n}\binom{n-i-2}{i-1}\right) + \\
&+\sum_{j=1}^{k+1}\Psi_j^{1,n}\left(\sum_{i=j-1}^{k}(-1)^{i-j+1}\binom{n-i-j}{i-j+1}B_i^{k,n}\right) =^{t\to i-j+1} \\
&\sum_{i=0}^{k} B_i^{k,n}\Lambda_0^{2|i,n-1} - \sum_{i=0}^{k-1} B_{i+1}^{k,n}\Lambda_0^{2|i,n-2} + F_0^{3,n}\left(\sum_{i=1}^{k}(-1)^{i-1}B_i^{k,n}\binom{n-i-2}{i-1}\right) + \\
&+\sum_{j=1}^{k+1}\Psi_j^{1,n}\left(\sum_{t=0}^{k-j+1}(-1)^{t}\binom{n-2j-t+1}{t}B_{t+j-1}^{k,n}\right) = \\
&\sum_{i=0}^{k} B_i^{k,n}\Lambda_0^{2|i,n-1} - \sum_{i=0}^{k-1} B_{i+1}^{k,n}\Lambda_0^{2|i,n-2} + F_0^{3,n}\left(\sum_{i=1}^{k}(-1)^{i-1}B_i^{k,n}\binom{n-i-2}{i-1}\right) + \\
&+\sum_{j=1}^{k+1}\Psi_j^{1,n}\left(\sum_{t=0}^{k-j+1}(-1)^{t}\binom{n-2j-t+1}{t}B_t^{k-j+1,n-2j+2}\right) =^{s=i-1} \\
&\sum_{i=0}^{k} B_i^{k,n}\Lambda_0^{2|i,n-1} - \sum_{i=0}^{k-1} B_{i+1}^{k,n}\Lambda_0^{2|i,n-2} + F_0^{3,n}\left(\sum_{s=0}^{k-1}(-1)^{s}B_{s+1}^{k,n}\binom{n-s-3}{s}\right) + \\
&+\sum_{j=1}^{k+1}\Psi_j^{1,n}\left(\sum_{t=0}^{k-j+1}(-1)^{t}\binom{n-2j-t+1}{t}B_t^{k-j+1,n-2j+2}\right) = \\
&\sum_{i=0}^{k} B_i^{k,n}\Lambda_0^{2|i,n-1} - \sum_{i=0}^{k-1} B_{i+1}^{k,n}\Lambda_0^{2|i,n-2} + F_0^{3,n}\left(\sum_{s=0}^{k-1}(-1)^{s}B_s^{k-1,n-2}\binom{(n-2)-s-1}{s}\right) + \\
&+\sum_{j=1}^{k}\Psi_j^{1,n}\left(\sum_{t=0}^{k-j+1}(-1)^{t}\binom{n-2j-t+1}{t}B_t^{k-j+1,n-2j+2}\right) + \Psi_{k+1}^{1,n} =^{Def} \\
&\sum_{i=0}^{k} B_i^{kn}\Lambda_0^{2|i,n-1} - \sum_{i=0}^{k-1} B_{i+1}^{k,n}\Lambda_0^{2|i,n-2} + \Psi_{k+1}^{1,n}.
\end{aligned}
$$

Now we can expand $B_i^{k,n}$ using the Lemma 3.4.18 and use the properties of the $B_i^{k,n}$ to complete the computation

$$
\begin{aligned}
\sum_{i=0}^{k} B_i^{k,n}\Lambda_0^{2|i,n-1} - \sum_{i=0}^{k-1} B_{i+1}^{k,n}\Lambda_0^{2|i,n-2} + \Psi_{k+1}^{1,n} &= \\
\sum_{i=0}^{k}\left[B_i^{k,n-1} + B_i^{k-1,n-1}\right]\Lambda_0^{2|i,n-1} - \sum_{i=0}^{k-1} B_i^{k-1,n-2}\Lambda_0^{2|i,n-2} + \Psi_{k+1}^{1,n} &= \\
\sum_{i=0}^{k} B_i^{k,n-1}\Lambda_0^{2|i,n-1} + \sum_{i=0}^{k-1} B_i^{k-1,n-1}\Lambda_0^{2|i,n-1} - \sum_{i=0}^{k-1} B_i^{k-1,n-2}\Lambda_0^{2|i,n-2} + \Psi_{k+1}^{1,n} &= \\
\Gamma_0^{k,n-1} + \left(\Gamma_0^{k-1,n-1} - \Gamma_0^{k-1,n-2}\right) + \Psi_{k+1}^{1,n}&.
\end{aligned}
$$

The case $n = 2k+1$ is very similar but in the last part we use extensively the lemmata proved before

$$
\begin{aligned}
\Gamma_0^{k,n} = \sum_{i=0}^{k} B_i^{k,n}\Lambda_0^{2|i,n} &= \\
\sum_{i=0}^{k-1} B_i^{k,n}\Lambda_0^{2|i,n-1} - \sum_{i=1}^{k-1} B_i^{k,n}\Lambda_0^{2|i-1,n-2} + F_0^{3,n}\left(\sum_{i=1}^{k}(-1)^{i-1}B_i^{k,n}\binom{n-i-2}{i-1}\right) &+ \\
+\sum_{i=0}^{k}\left[B_i^{k,n}\sum_{j=1}^{i+1}(-1)^{i-j+1}\Psi_j^{1,n}\binom{n-i-j}{i-j+1}\right] &=^{Prev.\ Case} \\
\sum_{i=0}^{k-1} B_i^{k,n}\Lambda_0^{2|i,n-1} - \sum_{i=1}^{k-1} B_i^{k,n}\Lambda_0^{2|i-1,n-2} + \Psi_{k+1}^{1,n} &=^{Lemma\ 3.4.18} \\
\sum_{i=0}^{k-1} B_i^{k,n-1}\Lambda_0^{2|i,n-1} + \sum_{i=0}^{k-1} B_i^{k-1,n-1}\Lambda_0^{2|i,n-1} - \sum_{i=0}^{k-1} B_i^{k-1,n-2}\Lambda_0^{2|i,n-2} + \Psi_{k+1}^{1,n} &=^{Lemma\ 3.4.20} \\
\Gamma_0^{k-1,n-1} - \Gamma_0^{k-1,n-2} + \Psi_{k+1}^{1,n}&.
\end{aligned}
$$

We can now complete the proof of Proposition 3.4.15 using as crucial tool the following Lemma, which proves equality between $\Gamma_1^{k,n}$ and $\Gamma_0^{k,n}$ and give us important closed formulae for these coefficients.

Lemma 3.4.21. *Let us denote with $\Psi(n,k)$ the rational function defined as follows*

$$\Psi(n,k) = \sum_{i=1}^{k+1} \Psi_i^{k-i+2,n} + F_0^{k+2,n}.$$

The following identities hold:

1.

$$\Gamma_0^{k,n} = \sum_{j=2k+1}^{n} \Psi(j,k).$$

2.

$$\Psi(n,k) = \frac{(t^2+q)(t-q)(t^{2(n-2k)-1}+1)}{t^{n-2k+1}}.$$

3.

$$\Psi(n,k) = \Psi(n-2,k-1).$$

4.

$$\Gamma_0^{k,n} = \Gamma_0^{k-1n-2} = \Gamma_1^{kn}.$$

5.

$$\Gamma_0^{k,n} = \frac{(t^2+q)(t-q)(t^{2(n-2k)}-1)}{t^{n-2k+1}(t-1)}.$$

Proof. Observe that, by definition

$$\Psi(n,k) = \Psi(n-1,k-1) + \Psi_{k+1}^{1,n}.$$

We will use extensively induction to prove the above statements.

1. Suppose there exists a minimal pair (n,k) such that 1) does not hold. If $n = 2k+1$ the following must hold

$$\Gamma_0^{k,n} = \Gamma_0^{k-1,n-1} - \Gamma_0^{k-1n-2} + \Psi_{k+1}^{1,n} =^{Ind.} \Psi(n-1,k-1) + \Psi_{k+1}^{1,n} = \Psi(n,k).$$

Otherwise for generic n we have a similar argument

$$\begin{aligned} \Gamma_0^{k,n} &= \\ \Gamma_0^{k-1,n-1} + (\Gamma_0^{k-1,n-1} - \Gamma_0^{k-1,n-2}) + \Psi_{k+1}^{1,n} &=^{Ind.} \\ \Gamma_0^{k,n-1} + \Psi(n-1,k-1) + \Psi_{k+1}^{1,n} &= \\ \Gamma_0^{k,n-1} + \Psi(n,k) = \sum_{j=2k+1}^{n} \Psi(j,k) \end{aligned}$$

Then such a pair (n,k) cannot exists and 1) is proved.

2. We will use again an inductive process. For $n = 3$ a direct computation shows that

$$\Gamma_0^{1,3} = F_0^{3,3} + \Psi_1^{2,3} + \Psi_2^{1,3} = \frac{(t^2+q)(t-q)(t+1)}{t^2}$$

Suppose there exists a minimal pair (n,k) such that 2) does not hold, then

$$\Psi(n,k) = \Psi(n-1,k-1) + \Psi_{k+1}^{1,n} =$$

$$\frac{(t^2+q)(t-q)(t^{2(n-2k+1)-1}+1)}{t^{n-2k+2}} + \frac{(t^2+q)(q-t)(t^{2n}-t^{4k})(t-1)}{t^{n+2k+2}} =$$

$$\frac{(t^2+q)(t-q)(t^{2(n-2k)-1}+1)}{t^{n-2k+1}}$$

3. It follows immediately from 2)

4. First of all let's observe that if $n = 2k+1$ the result is an immediate consequence of part 1) and 3) of the Lemma: The coefficients $\Gamma_0^{k,2k+1}$ and $\Gamma_0^{k-1,n-1}$ are exactly $\Psi(2k+1,k)$ and $\Psi(2(k-1)+1,k-1)$.

   Otherwise

$$\Gamma_0^{k,n} =$$

$$\Gamma_0^{k-1,n-1} + (\Gamma_0^{k-1,n-1} - \Gamma_0^{k-1,n-2}) + \Psi_{k+1}^{1,n} =$$

$$\Gamma_0^{k,n-1} + \Psi(n,k) =^{Ind.}$$

$$\Gamma_0^{k-1,n-3} + \Psi(n,k).$$

   Now we have the following equivalences

$$\Gamma_0^{k-1,n-3} + \Psi(n,k) = \Gamma_0^{k-1,n-2} \Longleftrightarrow \Psi(n,k) = \Gamma_0^{k-1,n-2} - \Gamma_0^{k-1,n-3} = \Psi(n-2,k-1)$$

   and the statement comes directly from 3).

5. By 4) it is enough prove the statement for $k = 1$. For $n = 3$ we showed that

$$\Gamma_0^{1,3} = F_0^{3,3} + \Psi_1^{2,3} + \Psi_2^{1,3} = \frac{(t^2+q)(t-q)(t+1)}{t^2}$$

   And now by induction

$$\Gamma_0^{1,n} = \Gamma_0^{1,n-1} + \Psi(n,1) =$$

$$\frac{(t^2+q)(t-q)(t^{2(n-3)}-1)}{t^{n-2}(t-1)} + \frac{(t^2+q)(t-q)(t^{2(n-2)-1}+1)}{t^{n-1}} =$$

$$\frac{(t^2+q)(t-q)}{t^{n-1}} \left[\frac{t(t^{2(n-3)}-1)}{(t-1)} + (t^{2(n-2)-1}+1) \right] =$$

$$\frac{(t^2+q)(t-q)(t^{2(n-2)}-1)}{t^{n-1}(t-1)}.$$

□

As an immediate Corollary of the Lemma, recalling that $\Gamma_h^{k,n} = \Gamma_{h-1}^{k-1,n-2}$ if $h > 0$ and reasoning by induction we obtain that

$$\Gamma_0^{kn} =^{Lem.} \Gamma_1^{k,n} = \Gamma_0^{k-1,n-2} =^{Ind.} \Gamma_j^{k-1,n-2} = \Gamma_{j+1}^{kn}$$

for $k-1 \geq j \geq 0$ and this proves the statement of Proposition 3.4.15. □

At this moment we know very well the structure of the coefficients of the C_j's in $\Gamma^{k,n}$ if j is smaller than k. For the other two unknown coefficients (i.e. the k-th and the $k+1$-th one) we want give now explicit formulae.

Remark 3.4.22. By Proposition 3.4.15, the following recurrences for $\Lambda_k^{2|k,n}$ and $\Lambda_{k+1}^{2|k,n}$ hold

$$\Lambda_k^{2|k,n} = \Lambda_{k-1}^{2|k-1,n-2} - \Lambda_{2|k-1}^{2|k,n-1} + F_0^{3,n}\binom{n-2k-1}{1}.$$

$$\Lambda_{k+1}^{2|k,n} = \Lambda_k^{2|k-1,n-2} - \Lambda_{2k}^{2|k,n-1} - F_0^{3,n}.$$

Now, the only contribution to the coefficient $\Gamma_{k+1}^{k,n}$ comes from R_k and then

$$\begin{aligned} \Gamma_{k+1}^{k,n} = \Lambda_{k+1}^{2|k,n} &= \\ \Lambda_k^{2|k-1,n-2} - \Lambda_{2|k}^{2|k,n-1} - F_0^{3,n} &= \\ \Lambda_k^{2|k-1,n-1} - \left[F_0^{2,n} + F_0^{3,n}\right] &= \\ \Gamma_k^{k-1,n-2} - \left[F_0^{2,n} + F_0^{3,n}\right]&. \end{aligned}$$

Similarly, to evaluate $\Gamma_k^{k,n}$ we have to describe explicitly the contribution to the coefficient

$$\begin{aligned} \Gamma_k^{k,n} = B_k^{k,n}\Lambda_k^{2|k,n} + B_{k-1}^{k,n}\Lambda_k^{k-1,n} &= \\ \Lambda_{k-1}^{2|k-1,n-2} - \Lambda_{2|k-1}^{2|k,n-1} + F_0^{3,n}\binom{n-2k-1}{1} + B_{k-1}^{k,n}\left(\Lambda_{k-1}^{2|k-2,n-2} - \Lambda_{2|k-1}^{2|k-1,n-1} - F_0^{3,n}\right) &= \\ \Lambda_{k-1}^{2|k-1,n-2} + B_{k-1}^{kn}\Lambda_{k-1}^{2|k-2,n-2} - \left[\Lambda_{2|k-1}^{2|k,n-1} + B_{k-1}^{k,n}\Lambda_{2|k-1}^{2|k-1,n-1}\right] + F_0^{3,n}\left[\binom{n-2k-1}{1} - B_{k-1}^{k,n}\right] &= \\ \Lambda_{k-1}^{2|k-1,n-2} + B_{k-1}^{k,n}\Lambda_{k-1}^{2|k-2,n-2} - F_0^{1,n-1}\left[B_{k-1}^{k,n} - \binom{n-2k-1}{1}\right] + F_0^{3,n}\left[\binom{n-2k-1}{1} - B_{k-1}^{k,n}\right] &= \\ \Gamma_{k-1}^{k-1,n-2} - \left(F_0^{2,n} + F_0^{3,n}\right)\left[B_{k-1}^{k,n} - \binom{n-2k-1}{1}\right] &= \\ \Gamma_{k-1}^{k-1,n-2} - \left(F_0^{2,n} + F_0^{3,n}\right)\left[B_{k-1}^{k,n} - \binom{n-2k}{1} + 1\right] &= \\ \Gamma_{k-1}^{k-1,n-2} - \left[F_0^{2,n} + F_0^{3,n}\right]&. \end{aligned}$$

The two formulae just proved allow us, using an iterated process, to obtain the following closed forms

Lemma 3.4.23.

$$\Gamma_k^{k,n} = -\frac{(t-q)(1+qt^{2(n-2k)-1})}{t^{n-2k}} - \frac{(1-q^2t^{2n-2k-1})}{t^{n-1}}\frac{(t^{2k}-1)}{(t-1)},$$

$$\Gamma_{k+1}^{k,n} = -\frac{(1-q^2t^{2(n-k-1)})}{t^{n-1}}\frac{(t^{2k+1}-1)}{(t-1)}.$$

Proof. We proved that

$$\Gamma_k^{k,n} = \Gamma_{k-1}^{k-1,n-2} - \left[F_0^{2,n} + F_0^{3,n}\right]$$
$$\Gamma_{k+1}^{k,n} = \Gamma_k^{k-1,n-2} - \left[F_0^{2,n} + F_0^{3,n}\right]$$

Substituting $\Gamma_{k-1}^{k-1,n-2}$ and $\Gamma_k^{k-1,n-2}$, the iterative expansion leads us to

$$\Gamma_k^{k,n} = \Gamma_0^{0,n-2k} - \sum_{i=0}^{k-1}\left[F_0^{2,n-2i} + F_0^{3,n-2i}\right],$$

$$\Gamma_{k+1}^{k,n} = \Gamma_1^{0,n-2k} - \sum_{i=0}^{k-1}\left[F_0^{2,n-2i} + F_0^{3,n-2i}\right] = -F_0^{2,n-2k} - \sum_{i=0}^{k-1}\left[F_0^{2,n-2i} + F_0^{3,n-2i}\right].$$

For $\Gamma_0^{0,n-2k}$ we have a close formula. A simple computations shows that

$$\sum_{i=H+1}^{n+1} F_0^{2,i} = \sum_{i=H}^{n} F_0^{1,i} = \sum_{i=H}^{n} \frac{(1-q^2t^{2i})}{t^i} =$$
$$\sum_{i=H}^{n}\frac{1}{t^i} - q^2\sum_{i=H}^{n} t^i =$$
$$\frac{1}{t^n}\sum_{i=0}^{n-H} t^i - q^2t^H\sum_{i=0}^{n-H} t^i =$$
$$\frac{(1-q^2t^{n+H})}{t^n}\left(\sum_{i=0}^{n-H} t^i\right) = \frac{(1-q^2t^{n+H})}{t^n}\frac{(t^{n-H+1}-1)}{(t-1)}$$

obtaining easily our thesis. ☐

We conclude this section with a very important relation between coefficients $\Gamma_k^{k,n}$ and $\Gamma_0^{k,n}$ that can now easily proved. It will simplify in a relevant way the calculations needed to prove the Reeder's Conjecture.

Corollary 3.4.24.

$$\Gamma_k^{k,n} = \Gamma_0^{k,n} + \frac{(q^2t^{2(k-1)}-1)(t^{2n-2k+1}-1)}{t^{n-1}(t-1)}$$

Proof. This is just an explicit calculation. We verified the identity using the program Sage. The computation is presented in the Appendix. ☐

Notation 3.4.25. We will call B_k^n the polynomial $\Gamma_k^{k,n} - \Gamma_0^{k,n}$ as in the Corollary above.

3.4.4 The base case $k = 1$

In this section we analyze the base case of our inductive reasoning. It introduces very well the strategy we are going to use for the general case.

Remark 3.4.26. To prove the Conjecture it is convenient to deal with the specialized version of the polynomials $C_{2|k}$ and then work with evaluated version of the recursion. We will use the same notation used above for the evaluated version of coefficients and polynomials.

First of all the case $n = 2k+1$, i.e. $n = 3$. We have shown that in this case the recurrence has the form

$$\Gamma^{1,3} \ : \ \Gamma^{1,3}_{2|1} C_{2|1} + \Gamma^{1,3}_{1} C_1 + \Gamma^{1,3}_{0} C_0 = 0.$$

Recall that there is a transition formula between C_1 and C_0, in particular

$$C_1 = \frac{q^3(q+1)(q^4+1)}{(q^{11}+1)} C_0.$$

We will call this coefficient $T^{1,3}_0$. Therefore we obtain

$$\Gamma^{1,3}_{2|1} C_{2|1} = -C_1 \left[\Gamma^{1,3}_{1} + \frac{\Gamma^{1,3}_{0}}{T^{1,3}_{0}} \right]$$

This leads us to a polynomial identity proved in the appendix:

$$\frac{\Gamma^{1,3}_{2|1} P_{1,3}}{(1+q^{11})(1-q^4)} \cdot \frac{H(2,1)}{H(2,2)} = \frac{\Gamma^{1,3}_{2|1} P_{1,3}}{(1+q^{11})(q^8-1)(q^8-1)} = -\left[\Gamma^{1,3}_{1} + \frac{\Gamma^{1,3}_{0}}{T^{1,3}_{0}} \right].$$

Now we are ready to prove the general case. The recursion is different from the previous one by the presence of a contribution given by C_2.

$$\Gamma^{1,n} \ : \ \Gamma^{1,n}_{2|1} C_{2|1} + \Gamma^{1,n}_{2} C_2 + \Gamma^{1,n}_{1} C_1 + \Gamma^{1,n}_{0} C_0$$

Remark 3.4.27. There is a transition $T^{n}_{2,1}$ coefficient between C_2 and C_1 as we proved in the previous case. This coefficient has the following explicit form

$$T^{n}_{2,1} = \frac{(t^{2(n-3)}-1)(t^{2n}-1)(1-qt)t^2}{(t^{2(n-1)}-1)(t^4-1)(1-qt^{2(n-1)-1})}.$$

This coefficient, suitably specialized, becomes

$$T^{n}_{2,1} = \frac{q^4(q^{4(n-3)}-1)(q^{4n}-1)(q^3+1)}{(q^{4(n-1)}-1)(q^8-1)(1+q^{4n-5})}.$$

Remark 3.4.28. To prove the Conjecture in the case C_2 we showed that the following equality holds

$$\Lambda^{2,n}_{2} C_2 = -\Lambda^{1,n-2}_{0} \left(C_1 + C_0 \right).$$

Now we have just to substitute these identities in the recurrence:

$$\begin{aligned}\Gamma^{1,n}_{2|1}C_{2|1} = -\left[\Gamma^{1,n}_{2}C_2 + \Gamma^{1,n}_{1}C_1 + \Gamma^{1,n}_{0}C_0\right] =\\ -\left[\Gamma^{1,n}_{2}C_2 + (B_1 + \Gamma^{1,n}_{0})C_1 + \Gamma^{1,n}_{0}C_0\right] =\\ -\left[\Gamma^{1,n}_{2}C_2 + B_1C_1 + \Gamma^{1,n}_{0}(C_1 + C_0)\right] =\\ -\left[\left(\Gamma^{1,n}_{2} - \frac{\Gamma^{1,n}_{0}\Lambda^{2,n}_{2}}{\Lambda^{1,n-2}_{0}}\right)C_2 + B_1C_1\right] =\\ -C_2\left[\Gamma^{1,n}_{2} - \frac{\Gamma^{1,n}_{0}\Lambda^{2,n}_{2}}{\Lambda^{1,n-2}_{0}} + \frac{B_1}{T^n_{2,1}}\right].\end{aligned}$$

Recalling formula (3.1.2) and substituting it in the above equality we obtain that in this case the Conjecture it is equivalent to prove a polynomial identity we proven in Appendix:

$$\frac{P_{1,n}(1-q^{4n+2})}{q^{2n+4}(q^3+1)(1-q^4)}\cdot\frac{H(n-2,2)}{H(n-1,2)} = -\left[\Gamma^{1,n}_{2} - \frac{\Gamma^{1,n}_{0}\Lambda^{2,n}_{2}}{\Lambda^{1,n-2}_{0}} + \frac{B_1}{T^n_{2,1}}\right].$$

3.4.5 Reeder's Conjecture for $\lambda = \omega_1 + \omega_{2k+1}$

The proof of Reeeder's Conjecture for the polynomials $C_{2|k}$ is the generalization of the reasoning done for the case $k = 1$. Let us start with some fundamental remarks.

Remark 3.4.29. The following formula holds between the polynomials $C_k(q,t)$:

$$C_{k+1}(q,t) = \frac{(t^{2(n-2k-1)}-1)(t^{2(n-k+1)}-1)(1-qt^{2k-1})t^2}{(t^{2(n-2k+1)}-1)(t^{2(k+1)}-1)(1-qt^{2(n-k)-1})}C_k(q,t). \tag{3.4.16}$$

The specialization $s = q$ and $t = q^2$ lead us to the version we will use in what follows:

$$C_{k+1}(-q,q^2) = C_k(-q,q^2)\frac{q^4(q^{4(n-k+1)}-1)(q^{4(n-2k-1)}-1)(q^{4k-1}+1)}{(q^{4(k+1)}-1)(q^{4(n-2k+1)}-1)(q^{4(n-k)-1}+1)}. \tag{3.4.17}$$

We will denote this coefficient with the notation T^n_k

Remark 3.4.30. To prove the case C_k we showed the following identity

$$\Lambda^{k+1,n}_{k+1}C_{k+1} = -\Lambda^{1,n-2k}_{0}\left(C_k + \cdots + C_0\right) \tag{3.4.18}$$

Remark 3.4.31. In Proposition 3.4.15 we proved that Stembridge's recurrence is equivalent to

$$\Gamma^{k,n}_{2|k}C_{2|k} + \Gamma^{k,n}_{k}C_k + \Gamma^{k,n}_{0}\left(C_{k-1} + \cdots + C_0\right) = 0$$

if $n = 2k+1$, and to

$$\Gamma^{k,n}_{2|k} C_{2|k} + \Gamma^{k,n}_{k+1} C_{k+1} + \Gamma^{k,n}_{k} C_k + \Gamma^{k,n}_{0} \left(C_{k-1} + \cdots + C_0\right) = 0$$

in all the other cases.

Now we are ready to prove the general case of Reeder's Conjecture, starting from the recurrences $\Gamma^{k,n}$ in the previous remark.

Let us start with the case $n = 2k+1$

$$\Gamma^{k,n}_{2|k} C_{2|k} = -\left[\Gamma^{k,n}_{k} C_k + \Gamma^{k,n}_{0} \left(C_{k-1} + \cdots + C_0\right)\right] = \\ -C_k, \left[\Gamma^{k,n}_{k} - \frac{\Gamma^{k,n}_{0} \Lambda^{k,n}_{k}}{\Lambda^{1,3}_{0}}\right]$$

Using the transition formulae, the conjecture is now equivalent to prove the identity

$$\frac{\Gamma^{k,n}_{2|k} \cdot P_{k,n} \cdot (t^{2(n-2k)} - 1)}{(1 + st^{2(n-k)-1})(1-t^2)(t^{2(n-2k+1)} - 1)(t^{2(k+1)} - 1)} = -\left[\Gamma^{k,n}_{k} - \frac{\Gamma^{k,n}_{0} \Lambda^{k,n}_{k}}{\Lambda^{1,3}_{0}}\right]$$

We are now ready for the general case:

$$\Gamma^{k,n}_{2|k} C_{2|k} = \\ -\left[\Gamma^{k,n}_{k+1} C_{k+1} + \Gamma^{k,n}_{k} C_k + \Gamma^{kn}_{0} \left(C_{k-1} + \cdots + C_0\right)\right] =$$

$$-\left[\Gamma^{k,n}_{k+1} C_{k+1} + \left(\Gamma^{k,n}_{0} + B_k\right) C_{k,n} + \Gamma^{k,n}_{0} \left(C_{k-1} + \cdots + C_0\right)\right] = \\ -\left[\Gamma^{k,n}_{k+1} C_{k+1} + B_k C_k + \Gamma^{k,n}_{0} \left(C_k + C_{k-1} + \cdots + C_0\right)\right] = \\ -\left[\left(\Gamma^{k,n}_{k+1} - \frac{\Gamma^{k,n}_{0} \Lambda^{k+1,n}_{k+1}}{\Lambda^{1,n-2k}_{0}}\right) C_{k+1} + B_k C_k\right] = \\ -C_{k+1}\left(\Gamma^{k,n}_{k+1} - \frac{\Gamma^{k,n}_{0} \Lambda^{k+1,n}_{k+1}}{\Lambda^{1,n-2}_{0}} + \frac{B_k}{T^n_k}\right)$$

Again our conjecture is equivalent again to a polynomial identity by 3.1.2

$$\frac{\Gamma^{k,n}_{2|k}\ P_{k,n}}{q^4(1+q^{4k+3})(1-q^4)} \cdot \frac{H(n-k-1,k+1)}{H(n-k,k+1)} = \\ \frac{\Gamma^{k,n}_{2|k}\ P_{k,n}}{q^4(1+q^{4k+3})(1-q^4)} \cdot \frac{(1-q^{4(n-2k)})}{(1-q^{4(n-k+1)})(1-q^{4(n-2k+1)})} = \\ -\left(\Gamma^{k,n}_{k+1} - \frac{\Gamma^{k,n}_{0} \Lambda^{1,n-2k}_{0}}{\Lambda^{k+1,n}_{k+1}} + \frac{B_k}{T^n_k}\right)$$

We proved the identities using the program Sage (see appendix), obtaining then our conjecture.

3.5 Appendix to Chapter 3: SAGE Computations.

In this sections we present the codex we used to prove rational functions identities we needed to prove the Reeder's Conjecture.

```
sage: Gammak(k,n)=-(t-q)*(1+q*t^(2*(n-2*k)-1))/t^(n-2*k)-
(1q^2*t^(2*n-2*k-1))*(t^(2*k)-1)/(t^(n-1)*(t-1))
sage: Gamma0(k,n)=(t^2+q)*(t-q)*(t^(2*(n-2*k))-1)/(t^(n-2*k+1)*(t-1))
sage: Bk(k,n)=Gammak(k,n)-Gamma0(k,n)
sage: Bk(k,n).factor()
-(q*t^k + t)*(q*t^k - t)*(t^(2*k) - t^(2*n + 1))*t^(-2*k - n - 1)/(t - 1)
sage: Bk(1,n).factor()
(q + 1)*(q - 1)*(t^(2*n) - t)/((t - 1)*t^n)
```

Here we defined Γ_0^{kN} and Γ_k^{kN} and proved the Lemma 3.4.20.

```
sage: Gammakk(k,n)=-(1-q^2*t^(2*n-2*k-2))*(t^(2*k+1)-1)/(t^(n-1)*(t-1))
sage: Gamma2k(n)=(1-q^2*t^(2*n))/t^n
sage: T10(n)=t*(t-q)*(t^(2*(n-1))-1)/((1-q*t^(2*n-1))*(t^2-1))

sage: S=Gamma2k(3)(q=-q,t=q^2)*T2kk(1,3)
sage: S.factor()
(q^8 + q^6 - q^5 + 2*q^4 - q^3 + q^2 + 1)*
(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)*
(q + 1)*(q - 1)/((q^4 + 1)*q^5)
sage: Q=Gammak(1,3)+Gamma0(1,3)/T10(3)
sage: Q(q=-q,t=q^2).factor()
(q^8 + q^6 - q^5 + 2*q^4 - q^3 + q^2 + 1)*
(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)
*(q + 1)*(q - 1)/((q^4 + 1)*q^5)
```

This is the implementation of Γ_{k+1}^{kN} and $\Gamma_0^{k,N}$ and the proof of polynomial identity in case $k = 1,\ N = 3$.

```
 sage: Pkn(k,n)=(q^2+q)*(q^(4*(n-k+1))-1)*(q^(4*(k+1))-1)+
 (q^8+q^3)*(q^(4*(n-k))-1)*(q^(4*k)-1)
sage: Lambda01(n)=-(t-q)*(t^(2*(n-1))-1)/(t^(n-1)*(t-1))
sage: Lambdak(k,n)=(t^(2*k-1)-q*t^(2*n))*(t^(2*k)-1)/(t^(n+2*k-1)*(t-1))
sage: T2kk(k,n)=Pkn(k,n)*(q^(4*(n-2*k))-1)/
```

```
((1-q^4)*(1+q^(4*(n-k)-1))*(q^(4*(n-2*k+1))-1)*(q^(4*(k+1))-1))
sage: T2kkk(k,n)=Pkn(k,n)*(q^(4*(n-2*k))-1)/
((1-q^4)*(q^4+q^(4*k+3))*(q^(4*(n-2*k-1))-1)*(q^(4*(n-k+1))-1))

sage: var('q', domain='positive')
q
sage: T(n)=Gammakk(1,n)(q=-q,t=q^2)-Gamma0(1,n)(q=-q,t=q^2)*
Lambdak(2,n)(q=-q,t=q^2)/
Lambda01(n-2)(q=-q,t=q^2)+Bk(1,n)(q=-q,t=q^2)/Tk(1,n)
sage:  S(n)=Gamma2k(n)(q=-q,t=q^2)*T2kkk(1,n)
sage: (S(n)-T(n)).factor()
0
```

In the previous excerpt of codex is verified the identity that proves the conjecture for $k=1$ in the general case. For the case $N=2k+1$ the computation is the following:

```
sage: var('q', domain='positive')
q
sage: T(k)=Gammak(k,2*k+1)(q=-q,t=q^2)-Gamma0(k,2*k+1)(q=-q,t=q^2)*
Lambdak(k,2*k+1)(q=-q,t=q^2)/Lambda01(3)(q=-q,t=q^2)
sage: S(k)=Gamma2k(2*k+1)(q=-q,t=q^2)*T2kk(k,2*k+1)
sage: (S(k)-T(k)).factor()
0
```

And finally, in the next excerpt of codex is verified the identity that proves the general case of the Conjecture:

```
sage: var('q', domain='positive')
q
sage: S(k,n)=Gamma2k(n)(q=-q,t=q^2)*T2kkk(k,n)
sage: T(k,n)=Gammakk(k,n)(q=-q,t=q^2)-Gamma0(k,n)(q=-q,t=q^2)*
Lambdak(k+1,n)(q=-q,t=q^2)/
Lambda01(n-2*k)(q=-q,t=q^2)+Bk(k,n)(q=-q,t=q^2)/Tk(k,n)
sage: (S(k,n)-T(k,n)).factor()
0
```

Chapter 4

More conjectures about the Exterior Algebra

In this chapter we analyze from a "qualitative" point of view the irreducible components in the exterior algebra. We recall a conjecture due to Kostant that describes in terms of their dominant weights the set of irreducible representations appearing in $\Lambda\mathfrak{g}$. This conjecture is proved for the case A in [5] and in [24]. We propose a very partial result in the case C. Moreover we conjecture that something similar to what conjectured by Kostant happens for the exterior algebra of the little adjoint representation ΛV_{θ_s}.

4.1 Some results on a Conjecture due to Kostant

Let $\mathfrak{g}$ be a simple Lie Algebra over the field $\mathbb{C}$. We recall that $\mathfrak{g}$ acts on itself by the adjoint action and, from a representation- theoretic point of view, it is the irreducible representation of highest weight θ. In [26] Kostant proved the following isomorphism of $\mathfrak{g}$ representations:

Theorem 4.1.1.

$$\Lambda\mathfrak{g} \simeq (V_\rho \otimes V_\rho)^{\oplus 2^{\mathrm{rk}\mathfrak{g}}}$$

Finding the irreducible representations appearing in $\Lambda\mathfrak{g}$ is then equivalent to find the isotypical components appearing in $V_\rho \otimes V_\rho$. Kostant proposed the following description of such irreducible representations:

Conjecture 4.1.2 (Kostant)**.** *The representation V_λ appears in the decomposition of $V_\rho \otimes V_\rho$ if and only if $\lambda \leq 2\rho$ in the dominance order.*

In [4] the authors deal with the more complicated problem of finding multiplicities in the general tensor product $V_\nu \otimes V_\mu$. They prove that in the case A_n such multiplicities are given by the integral points in some special polytopes. For such polytopes they give a complete description in terms of inequalities depending by the coordinates of μ

and ν with respect to the base given by fundamental weights. In [5] they find a combinatorial description for the integral points in the Berestein and Zelevinsky polytopes and as a corollary of such a construction prove, using elementary techniques of linear programming, that Kostant Conjecture holds for Lie algebras of type A. Moreover, in [4] they conjecture that tensor product decomposition can be described in terms of integral points of polytopes, called Berenstein and Zelevinsky polytopes, for each simple Lie Algebra. In [6] such a conjecture is proved in an equivalent form.

We will now recall some of their results in the case of Lie algebras of type C.

Remark 4.1.3. Every weight m in the root lattice can be described by a vector of non negative integers $(m_{12}, m^+_{12}, \dots, m_{n-1n}, m^+_{n-1n}, m_1, \dots, m_n)$ such that

$$m = \sum_{i<j} m_{ij}(e_i - e_j) + \sum_{i<j} m^+_{ij}(e_i + e_j) + \sum_i 2m_i e_i$$

We will say that such a sequence of integers $(m_{12}, m^+_{12}, \dots, m_{n-1n}, m^+_{n-1n}, m_1, \dots, m_n)$ is an $\mathfrak{sp}_{2n}\mathbb{C}$*-partition for* λ.

Considering now $m_{12}, m^+_{12}, \dots, m_{n-1n}, m^+_{n-1n}$ and $m_1, \dots, m_n$ as variables, we want use them to describe the inequalities that defines the Berestein and Zelevinsky polytope for a generical tensor product $V_\lambda \otimes V_\mu$.

We fix an index i in the ordered set $\{\bar{0} < 1 < \bar{1} < \cdots < n < \bar{n}\}$ and an integer $t \in \{1, \dots, n\}$. We define the linear operators $\mathscr{L}^t_i(m)$, $\mathscr{L}^{\bar{t}}_i(m)$, $\mathscr{N}^{t\,0/1}_i(m)$ and $\mathscr{N}^{\bar{t}\,0/1}_i(m)$, in the variables $m_{12}, m^+_{12}, \dots, m_{n-1n}, m^+_{n-1n}$ and $m_1, \dots, m_n$, in the following way:

$$\mathscr{L}^t_i(m) = \sum_{j=1}^{t-1} (M_{j,i+1} - M_{j\,i}) - (m_{t\,i} - m^+_{t\,i}) + m_{t\,i+1}.$$

$$\mathscr{L}^{\bar{t}}_i(m) = \sum_{j=1}^{t} (M_{j,i+1} - M_{j\,i}) + m_{t+1\,i+1}.$$

$$\mathscr{L}^{\bar{t}}_n(m) = -\sum_{j=1}^{t} M_{j\,n} + m_{t+1}.$$

$$\mathscr{L}^{\bar{t}}_n(m) = -\sum_{j=1}^{t} M_{j\,n} + m_t.$$

$$\mathscr{N}^{t\,0}_i(m) = m^+_{i\,i+1} + \sum_{j=i+1}^{t-1} \left(m^+_{i\,j+1} - m^+_{i+1\,j+1}\right) + (m^+_{i\,t+1} - m_{i+1\,t+1}).$$

$$\mathcal{N}_i^{\bar{t}\,0}(m) = m^+_{i\,i+1} + \sum_{j=i+1}^{t} \left(m^+_{i\,j+1} - m^+_{i+1\,j+1}\right).$$

$$\mathcal{N}_i^{n\,0}(m) = m^+_{i\,i+1} + \sum_{j=i+1}^{n-1} \left(m^+_{i\,j+1} - m^+_{i+1\,j+1}\right) + 2(m_i - m_{i+1}).$$

$$\mathcal{N}_i^{t\,1}(m) = m^+_{i\,i+1} + 2(m_i - m_{i+1}) + m_{i\,t} - m^+_{i\,t} + \sum_{j=i+1}^{t-1} \left(m^+_{i\,j+1} - m^+_{i+1\,j+1}\right) + \\ + \sum_{j=t}^{n-1} \left[(m_{i\,j+1} + m^+_{i\,j+1}) - (m_{i+1\,j+1} + m^+_{i+1\,j+1})\right]$$

$$\mathcal{N}_i^{\bar{t}\,1}(m) = m^+_{i\,i+1} + 2(m_i - m_{i+1}) + \sum_{j=i+1}^{t-1} \left(m^+_{i\,j+1} - m^+_{i+1\,j+1}\right) + \\ + \sum_{j=t}^{n-1} \left[(m_{i\,j+1} + m^+_{i\,j+1}) - (m_{i+1\,j+1} + m^+_{i+1\,j+1})\right]$$

$$\mathcal{L}_n^1(m) = -M_{1\,2} + m_1.$$

$$\mathcal{L}_1^{\bar{0}}(m) = m_{1\,2}.$$

$$\mathcal{L}_1^1(m) = m_{1\,2}.$$

$$\mathcal{L}_1^{\bar{1}}(m) = M_{1\,2}.$$

Where we denoted by M_{ij} the difference $m_{ij} - m^+_{ij}$. Now, if $\lambda = a_1\omega_1 + \cdots + a_n\lambda_n$ and $\mu = b_1\omega_1 + \cdots + b_n\omega_n$ are a pair of dominant weight, then the inequalities defining the Berestein and Zelevinsky polyotope associated to the tensor product $V_\lambda \otimes V_\mu$ are described in the following theorem:

Theorem 4.1.4. *The irreducible components of $V_\lambda \otimes V_\mu$ are in bijection with integral points of the polytope defined by the inequalities*

- $\mathcal{L}_i^s \leq a_i$
- $\mathcal{N}_i^{t,0} \leq b_i$

- $\mathscr{N}_i^{t,1} \leq b_i$
- $\mathscr{N}_1^{n,1} \leq b_n$

where j ranges in $\{1, \dots, n\}$ and $\bar{0} \leq s < j$ for the $\mathscr{L}_j^s$ operators, $1 \leq i \leq n-1$ and t ranges in $\bar{i} \leq t \leq n$ and $\bar{i} < t \leq n$ respectively for $\mathscr{N}_i^{t,0}$ and $\mathscr{N}_i^{t,1}$.

We are interested in study integral points contained in the Berestein and Zelevinsky polytope in the case of Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$ when $a_i = b_j = 1$ for all $i, j \leq n$.

Theorem 4.1.5 (Berestein and Zelevinsky)**.** *Let γ be a dominant weight. The (generalized) Littlewood Richardson coefficient $c_{\lambda\mu}^{\gamma}$ is equal to the number of $\mathfrak{sp}_{2n}\mathbb{C}$-partitions of $\lambda + \mu - \gamma$ contained in the Berestein and Zelevinski polytope.*

Kostant's Conjecture is then equivalent to find a $\mathfrak{sp}_{2n}$-partition of $2\rho - \lambda$ in the Berestein and Zelevinsky polytope, for each weight $\lambda \leq 2\rho$. We will call such $\mathfrak{sp}_{2n}\mathbb{C}$-partitions "*admissible*".

We are not able to find admissible $\mathfrak{sp}_{2n}$-partitions for each weight smaller than 2ρ. However we can prove the following result for the C_n type:

Theorem 4.1.6. *Let $\mu = \mu_1 e_1 + \dots + \mu_n e_n$ a weight smaller than 2ρ. If $\mu \leq 2\rho$ and $\mu_i \leq n - i + 1$ then V_μ appears in the irreducible decomposition of $\Lambda\mathfrak{g}$.*

The strategy to prove the theorem is quite simple: for each weight λ satisfying the hypothesis, we construct an $\mathfrak{sp}_{2n}$-partition for $2\rho - \lambda$ and we prove such a $\mathfrak{sp}_{2n}$-partition satisfies the Berestein and Zelevinsky inequalities. We have first of all recall some combinatorics of weights for C_n.

Remark 4.1.7. In the realization of C_n described in 1.1.4, the fundamental weight ω_i for the Lie algebra C_n is equal to $e_1 + \dots + e_i$. In e_i-notations ω_i is then the vector $(1, \dots 1, 0, \dots, 0)$ with the first i coordinates equal to 1.

Via this identification we can express a general integral weight $\lambda = a_1\omega_1 + \dots + a_n\omega_n$ in the form $\lambda = \lambda_1 e_1 + \dots + \lambda_n e_n$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

Each weight λ can be then expressed as a partition $\pi_\lambda = (\lambda_1, \dots, \lambda_n)$ and the differences $\lambda_i - \lambda_{i+1}$ are exactly equal to a_i. We will refer to the partition described in this way as the *shape* of λ and we will denote it by π_λ. Moreover with the symbol $|\lambda|$ we will denote the integer $\sum_{i=1}^n \lambda_i$

Example 4.1.8. Consider the weight ρ for C_n. It is equal to $\omega_1 + \dots + \omega_n$. Then it correspond to the partition $(n, n-1, \dots, 2, 1)$. Analogously, the weights $k\rho$ have shape $(kn, k(n-1), \dots, 2k, k)$.

Using this description we can classify the weights smaller than a fixed weight λ in terms of their partitions:

Lemma 4.1.9. *Let λ and μ be two dominant weights for C_n and let be $\pi_\lambda = (\lambda_1, \dots, \lambda_n)$ and $\pi_\mu = (\mu_1, \dots, \mu_n)$ the associated partitions. Then $\lambda \leq \mu$ if and only if the following conditions hold:*

1. $\sum_{i=1}^{k}(\lambda_i - \mu_i) \geq 0$ *for all* $1 \leq k \leq n$
2. $|\pi_\lambda| \equiv |\pi_\mu| \bmod 2$.

Proof. The weight μ is smaller than λ if and only if there exists a sequence $\{\alpha_1, \dots, \alpha_h\}$ of positive roots such that $\lambda - \sum \alpha_j = \mu$. Subtracting a root of the form $e_i - e_j$ to λ corresponds to remove a box from the i-th row of π_μ and add it to the j-th one. Conversely subtracting a root of the form $e_i + e_j$ corresponds to remove a box from the i-th and from j-th row. In both cases the parity remains the same. The sum of differences increase for each i if we subtract $e_i + e_j$, otherwise if we subtract $e_i - e_j$ the value $\sum_{i=1}^{k}(\lambda_i - \mu_i)$ increase if $k < j$ and remains the same if $k \geq j$, obtaining one implication. Suppose now that the two conditions are satisfied. Let j be the minimum index such that $\lambda_j - \mu_j$ is positive. Let us define $\pi_{\mu'}$ as the shape in wich we have moved boxes from lower rows of π_μ to the j-th row until $\lambda_j - \mu'_j = 0$ or $\lambda_s - \mu_s \geq 0$ for all $s > j$. Iterating the process we find the shape of a weight μ^k greater than μ in the dominance order that differs from λ only by roots of the form $e_i + e_j$, because of condition 2) and by the fact that "exchange " a box between two rows does not change the parity of $|\mu|$. By transitivity of the order and inductive reasoning on the quantity $D = \sum_{\{i|\lambda_i<\mu_i\}} \mu_i - \lambda_i$, we obtain the claim. □

An order on partitions induces an order on the weights, requiring the additional condition the parity of $|\pi_\lambda|$ and $|\pi_\mu|$ must be the same if λ and μ are in a same chain. The dominance ordering of partitions then corresponds to the classical dominance ordering of weights. We are interested in a different ordering.

Definition 4.1.10 (Inclusion order of partitions)**.** *Let π and σ be two partitions of lenght n. We say that $\pi \geq \sigma$ if and only if $\pi_i \geq \sigma_i$ for all $i \leq n$.*

Definition 4.1.11 (Inclusion order on weights)**.** *Let λ and μ be weights for C_n. We say that $\lambda \geq \mu$ if and only if*

- $|\lambda| \equiv |\mu| \bmod 2$
- $\pi_\lambda \geq \pi_\mu$ *in the inclusion order of partition.*

Example 4.1.12. Let us consider the weight 2ρ, the weights smaller than 2ρ in the inclusion order are those with $|\pi_\mu| \equiv |\pi_\lambda| \bmod 2$ and such that $\mu_i \leq 2(n - i + 1)$.

Observe that the latter condition is equivalent to say that $c_i = (2\rho - \mu)_i$ is non negative for each $i \leq n$.

Proof of Theorem 4.1.6. We have two main cases, depending on the parity of the $\{c_i\}_{i\leq n}$.

First of all let's suppose that all the c_i are all even. This can happen because, by Lemma 4.1.9 we have $|\lambda| \equiv |2\rho| \bmod 2$. We will construct a $\mathfrak{sp}_{2n}$-partition $m = (m_{12}, \dots, m_n)$ in an iterative way. We start setting m to be the null vector.

Step 1 If $c_n = 0$ we set $m_n = 0$, otherwise $m_n = 1$.

Step $h+1 = n-(i-1)+1$ Let $(m_{i\,i+1}, m^+_{i\,i+1}, \ldots, m_{i\,n}, m^+_{i\,n}, m_i)$ be the integers constructed at the $h = n-i+1$-th step, we set:

- $m_{i-1,j} = m^+_{i-1,j} = m_{i-1} = 0$ for all j if $c_{i-1} = 0$;
- $m_{i-1} = m_i$, $m_{i-1\,n-j} = m_{i\,n-j}$ and $m^+_{i-i\,n-j} = m^+_{i\,n-j}$ if $c_i \geq c_{i-1} > 0$, with j that ranges between 0 and $c_{i-1}/2 - 1$ if $m_i = 0$ and between 0 and $c_{i-1}/2 - 2$ if $m_i = 1$;
- If $c_{i-1} = c_i + 2$ we set $m_{i-1} = m_i$ and $m_{i-1\,j} = m_{i\,j}$ e $m^+_{i-1\,j} = m^+_{i\,j}$ for all $j > i$. Finally we set $m_{i-1\,i} = m^+_{i-1\,i} = 1$.

Observe now that the string of integers constructed above is a $\mathfrak{sp}_{2n}$-partition for $2\rho - \lambda$ because, for each i, the following equality holds:

$$\sum_{j=1}^{n} \left(m_{ij} + m^+_{ij}\right) + 2m_i + \sum_{j=1}^{i-1} \left(-m_{ji} + m^+_{ji}\right) = c_i$$

We have just to prove that such $\mathfrak{sp}_{2n}$-partition is admissible.

Observe that, by construction, we have $M_{ij}(m) = 0$ for all the pairs $(i,\, j)$ with $i < j$ and that each element of m is smaller or equal to 1. It is then obvious that $\mathscr{L}^t_j \leq 1$ for all $j \leq n$ and all $t \in \{1, \bar{1}, \ldots, n, \bar{n}\}$.

For what concerns the operators $\mathscr{N}^{t,0/1}_j$, observe that by our construction, if $j > i+1$, we have that $m_{ij} = m^+_{ij} \neq 0$ implies $m_{i+1,j} \leq m^+_{i+1j} \neq 0$ and then the quantity $m^+_{ij} - m^+_{i+1j} = m^+_{ij} - m_{i+1j}$ is non positive. The same remark holds for the $\{m_i\}$: $m_i \neq 0$ implies $m_{i+1} \neq 0$ and then $m_i - m_{i+1}$ is non positive.

This implies immediately that m is an admissible partition.

Otherwise, it can happen that there exists some odd c_i. If we set $I = \{i \,|\, c_i \text{ odd}\}$, by reasons of parity, we have that the cardinality of I is even.

We can construct an $\mathfrak{sp}_{2n}\mathbb{C}$ partition using the following iterative process:

Step 1 Let $\{\gamma_1 < \cdots < \gamma_{2k}\}$ be the set of indices such that c_i is odd. We pair together the j-th and the $k+j$-th index obtaining the set $P = \{(\gamma_1,\, \gamma_{k+1}),\, \ldots,\, (\gamma_k,\, \gamma_{2k})\}$.

Step 2 We construct the weight λ' starting from λ uging the pairs in P: if $(\gamma_j,\, \gamma_{j+k}) \in P$, set $\lambda'_{\gamma_j} = \lambda_{\gamma_j} + 1$ and $\lambda'_{\gamma_{j+k}} = \lambda_{\gamma_{j+k}} - 1$, otherwise $\lambda'_{\gamma_j} = \lambda_{\gamma_j}$.

Step 3 Observe that λ' is again smaller than 2ρ and it sets of differences $\{c'_i\}$ is composed by non negative even integers. Using the previous case we can then construct an admissible $\mathfrak{sp}_{2n}$-partition $m' = (m'_{ij},\, m'^+_{ij},\, m'^i)$.

Step 4 If $(\gamma_j,\, \gamma_{j+k})$ is a pair in P, we set $m_{\gamma_j \gamma_{j+k}} = m'_{\gamma_j \gamma_{j+k}} + 1$, otherwise $m_{\gamma_j \gamma_{j+k}} = m'_{\gamma_j \gamma_{j+k}}$.

From the fact that for each pair of indexes (γ_i, γ_{k+i}) we have

$$\lambda'_i - 1 = \lambda_i$$

$$\lambda'_{k+i} + 1 = \lambda_{k+1}$$

we obtain immediately that m is a $\mathfrak{sp}_{2n}\mathbb{C}$-partition for λ. We have to show that m is admissible. We will use inductive reasoning and the following remark:

Remark 4.1.13. Given an $\mathfrak{sp}_{2n}\mathbb{C}$-partition m for $2\rho - \lambda$, if we "forgive" about the roots $e_1 \pm e_j$ we obtain an $\mathfrak{sp}_{2n-2}\mathbb{C}$-partition m' for $2\rho_{C_{n-1}} - \lambda'$ for some weight $\lambda' \leq 2\rho_{C_{n-1}}$. We call this operation the "restriction" of m to C_{n-1}. Consequently it make sense use the notation $\mathscr{L}_j^s[n](m)$, $\mathscr{N}_i^{t\,0}[n](m)$ and $\mathscr{N}_i^{t\,1}[n](m)$ for the operators of Theorem 4.1.4 to underline that $\mathscr{L}_j^s[h](m)$ is the value of $\mathscr{L}_j^s(m)$ when m is restricted to C_h. Then the following recursive relations holds

$$\mathscr{L}_j^s(m) = (M_{1\,j+1} - M_{i\,j}) + \mathscr{L}_{j-1}^{s-1}[n-1](\bar{m}) \tag{4.1.1}$$

$$\mathscr{L}_n^s(m) = -M_{1\,n} + \mathscr{L}_{n-1}^{s-1}[n-1](\bar{m}) \tag{4.1.2}$$

$$\mathscr{N}_h^{s,\epsilon}(m) = \mathscr{N}_{h-1}^{s-1,\epsilon}[n-1](\bar{m}) \tag{4.1.3}$$

where we adopted the convention that $\overline{s} - 1 = \overline{s-1}$. Moreover here $\bar{m}$ is the restriction of m to $\mathfrak{sp}_{2(n-1)}$-partition via the inclusion $C_{n-1} \to C_n$.

It can be verified by easy computations that the construction produces an admissible partition for $n = 2$. By induction we can suppose

$$\mathscr{N}_h^{s,\epsilon} = \mathscr{N}_{h-1}^{s-1,\epsilon}[n-1] \leq 1$$

Now remark that:

1. the construction produces $m_{ij} > 1$ only if (i,j) is in P.
2. In the partition m we have that m_{ij}^+ is different from 0 only if $m_{ij} \neq 0$ for all i,j. Then we have $m_{ij} \leq 2$ and $m_{ij}^+ \leq 1$. In particular $m_{ij} > m_{ij}^+$if and only if $i = \gamma_h$ e $j = \gamma_{h+k}$. Furthermore $M_{ij} = 0$ except in the case $i \neq \gamma_h$ and $j \neq \gamma_{h+k}$. In such evenience we have $M_{ij} = 1$.
3. In m it holds $m_{ij}^+ \neq 0$ only if $m_{i+1j}^+ \neq 0$, except if $j = i+1$. The quantities $m_{ij}^+ - m_{i+1j}^+$ and $m_{ij}^+ - m_{i+1j}$ are then smaller or equal to zero if $j > i+1$.
4. Finally $m_i \neq 0$ only if $m_{i+1} \neq 0$. This implies $m_i - m_{i+1} \leq 0$ for all i. By our construction m_i is always smaller or equal to 1.

These considerations lead us to the conclusion that $\mathscr{N}_1^{t\,0}(m)$, $\mathscr{N}_1^{\bar{t}\,0}(m)$ and $\mathscr{N}_1^{n\,0}(m)$ are all smaller or equal to 1. Differently to find an upper bound to $\mathscr{N}_1^{\bar{t}\,1}(m)$ and $\mathscr{N}_1^{t\,1}(m)$ we have to evaluate $m_{1j} + m_{1j}^+ - (m_{2j} + m_{2j}^+)$. Such expression is always smaller or equal to zero except if $(1,j) \in P$, i.e. c_1 is odd and $j = \gamma_{k+1}$. In such a case $m_{1j} + m_{1j}^+ - (m_{2j} + m_{2j}^+) = 1$.

Now, if c_1 *and* c_2 *are both odd*, for $j = \gamma_{k+2}$ we have $m_{1j} + m^+_{1j} - (m_{2j} + m^+_{2j}) < 0$ and if follows that

$$\sum_{j=s}^{n-1} \left[m_{1j+1} + m^+_{1j+1} - (m_{2j+1} + m^+_{2j+1}) \right] \leq 0$$

obtaining $\mathscr{N}_1^{\bar{t}\,1}(m) \leq 1$. We remark that if $s \geq \gamma_{k+1}$ the above inequality is strict and recalling that $M_{1s} \leq 1$ we obtain consequently that, if c_1 and c_2 are odd, then $\mathscr{N}_1^{t\,1}(m) \leq 1$.

If c_1 *is even and* c_2 *is odd* similar considerations hold: the quantities $m_{1j+1} + m^+_{1j+1} - (m_{2j+1} + m^+_{2j+1})$ are always non positive and $M_{1j} = 0$ for all j. Again this implies $\mathscr{N}_1^{\bar{t}\,1}(m) \leq 1$ e $\mathscr{N}_1^{t\,1}(m) \leq 1$.

Conversely, if c_1 *is odd and* c_2 *is even*, we observe that:

- $m_{12} = m^+_{12} = 0$ because $c_1 < c_2 + 2$
- $\sum_{j=s}^{n-1} \left[m_{1j+1} + m^+_{1j+1} - (m_{2j+1} + m^+_{2j+1}) \right]$ is smaller than 1 by previous observations on the values that $m_{1j} + m^+_{1j} - (m_{2j} + m^+_{2j})$ can assume.
- if $\sum_{j=s}^{n-1} \left[m_{1j+1} + m^+_{1j+1} - (m_{2j+1} + m^+_{2j+1}) \right] = 1$ then $M_{1s} = 0$
- if $M_{1s} = 1$ it follows that $\sum_{j=s}^{n-1} \left[m_{1j+1} + m^+_{1j+1} - (m_{2j+1} + m^+_{2j+1}) \right] \leq 0$

By these inequalities and using the inductive hypothesis the remaining inequalities for the $\mathscr{N}_j^{\bar{t}\,\epsilon}(m)$ are proved. It remains to prove that the conditions of 4.1.4 holds for the operators $\mathscr{L}_j^s(m)$.

Some of these inequalities are trivial by the construction of m:

$$\mathscr{L}_n^{\bar{t}}(m) = -\sum_{i=1}^{t} M_{i\,n} + m_{t+1} \leq 1$$

It follows from the fact that $M_{i\,n} \geq 0$ for all i and from the fact that the m_i is smaller or equal to 1 for all i by construction.

Moreover $\mathscr{L}_j^s(m) = \mathscr{L}_j^{\overline{s-1}}(m) - M_{sj}$ by the same considerations about the expressions M_{ij} (they are always non negative) we can reduce to prove that $\mathscr{L}_j^{\overline{s-1}}(m) \leq 1$.

We recall that

$$\mathscr{L}_j^{\bar{t}}(m) = \sum_{i=1}^{t} (M_{i,j+1} - M_{i\,j}) + m_{t+1\,j+1}.$$

Denote with P_{+k} the set of indices γ_h with $h > k$.

We have four cases:

1. If *j and $j+1$ are not in P_{+k}* by above remarks we have $M_{hj} = M_{hj+1} = 0$ for all h and for all j. Moreover m_{t+1j+1} is smaller than 1 ($j+1 \notin P_{+k}$). Then the inequalities are verified in this case.

2. If *$j \in P_{+k}$ and $j+1 \notin P_{+k}$* , suppose $j = \gamma_{z+k}$, we have $M_{hj+1} = 0$ for all h and $M_{hj} = 0$ if and only if $h \neq \gamma_z$. Otherwise we have $M_{\gamma_z \gamma_{z+k}} = 1$. This implies $\sum_{i=1}^{t} (M_{i,j+1} - M_{ij}) = -1$ if $t \geq \gamma_z$. Otherwise $\sum_{i=1}^{t} (M_{i,j+1} - M_{ij}) = 0$. Actually the fact that, because $j+1 \notin P_{+k}$ we have $m_{t+1j+1} \leq 1$, implies the inequalities.

3. If *$j \notin P_{+k}$ and $j+1 \in P_{+k}$* Set $j+1 = \gamma_{z+k}$. We remark that in this case, supposing $j \neq \gamma_z$, we obtain $m_{jj+1} = 0$ because $c'_j < c'_{j+1} + 2$. Moreover if $j = \gamma_z$ we have again $c'_j < c'_{j+1} + 2$ and then $m_{jj+1} = 1$. We then obtain $m_{t+1j+1} = 1$ if $t = \gamma_z - 1$, otherwise $m_{t+1j+1} = 0$ holds. Now consider the quantity $\sum_{i=1}^{t} (M_{i,j+1} - M_{ij})$. By our remarks about the M_{ij} it is equal to 0 if $t < \gamma_z$ and equal to 1 if $t \geq \gamma_z$. The inequalities in this case then follows immediately.

4. *If $j \in P_{+k}$ and $j+1 \in P_{+k}$* we can suppose $j = \gamma_{z+k}$ and the $j+1 = \gamma_{z+1+k}$. We consequently have $M_{ij} \neq 0$ if and only if $i = \gamma_z$ and that $M_{ij+1} \neq 0$ if and only if $i = \gamma_{z+1+k}$. We then obtain that $\sum_{i=1}^{t} (M_{i,j+1} - M_{ij})$ is equal to -1 if $\gamma_z \leq t < \gamma_{z+1}$ and 0 otherwise. We remark finally that $m_{t+1j+1} = 2$ only if $t+1 = \gamma_{z+1}$, i.e if $\gamma_z \leq t < \gamma_{z+1}$. This complete the proof of inequalities in this last case.

It remains to show that $m_{1j} \leq 1$ and $m^+_{ii+1} \leq 1$ for all i and j. The second inequality is obvious by construction and the first is a consequence of the fat that, if c_j is odd, we have $c'_j < c'_{j+1} + 2$ and then by construction $m'_{jj+1} = 0$.

4.2 Little Adjoint Representation

Theorem 4.1.1 is generalized by Panyushev in [28] for isotropy representations of symmetric spaces. More precisely, given a $\mathbb{Z}_2$-graduation on a simple Lie algebra $\mathfrak{g}$ (over $\mathbb{C}$), it induces a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

Clearly $\mathfrak{g}_0$ has a structure of Lie algebra and $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module called the *isotropy representation* of $\mathfrak{g}_0$.

Panyushev proves that, for exterior algebras of $\mathfrak{g}_1$ an analog of 4.1.1 holds:

Theorem 4.2.1. *In the previous setting, there exists a $\mathfrak{g}_0$-representation $V_{\mathfrak{g}_1}$ and a non negative integer k such that*

$$\Lambda \mathfrak{g}_1 \simeq 2^k \left(V_{\mathfrak{g}_1} \oplus V_{\mathfrak{g}_1} \right)$$

Panyushev give a classification of the representations $V_{\mathfrak{g}_1}$ and examine very closely the case when the $\mathbb{Z}/2\mathbb{Z}$-grading is induced by involutions of Dynkin diagrams in simply laced types A_{2n+1}, D_n and E_6. In this framework $\mathfrak{g}_0$ is of type C_{n+1}, B_{n-1} and F_4 and $\mathfrak{g}_1$ is their little adjoint representations V_{θ_s}, where θ_s denotes the unique short dominant root. In this setting he proves the following structure theorem:

Theorem 4.2.2.
$$\Lambda V_{\theta_s} \simeq 2^{|\Delta_s|}\,(V_{\rho_s} \otimes V_{\rho_s})$$
where Δ_s *is the set of short simple roots.*

The analogy with the case of the exterior algebra of $\mathfrak{g}$ lead us to formulate the following conjecture:

Conjecture 4.2.3. V_λ *is an irreducible component of* ΛV_{θ_s} *if and only if* $\lambda \leq 2\rho_s$.

Or in other terms

Conjecture 4.2.4. V_λ *is an irreducible component of* $V_{\rho_s} \otimes V_{\rho_s}$ *if and only if* $\lambda \leq 2\rho_s$.

The Conjecture 4.2.3 can be easily proved for case B_n in different ways. We propose a very natural proof based on elementary facts about representations of B_n.

Proof of Conjecture 4.2.4 in the case B_n. First of all we remark that in the realization of root system of type B_n (see Remark 1.1.3) the highest short root is e_1 and it follows immediately that V_{θ_s} is the defining representation.

It is a well known fact that $\Lambda^i V_{\theta_s}$, for $i \leq n$, have weights equal respectively to ω_i if $i < n$ and to $2\omega_n$ if $i = n$. An immediate computation can shown that $\rho_s = \omega_n$ and that the weights smaller to $2\omega_n$ are exactly 0 and the ω_i. This proves the conjecture for the odd orthogonal groups. $\square$

Moreover, we will remark that the same result can be obtained finding explicit integral points in the Berestein and Zelevinsky polytope associated to $V_{\rho_s} \otimes V_{\rho_s}$.

Finally, we underline that we proved the conjecture for exceptional cases F_4 and G_2 by direct computations. There seems not to be an easy way to prove our conjecture for the case C_n because of the great number of weights smaller than $2\rho_s$.

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