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CHARACTERS OF π'-DEGREE AND SMALL CYCLOTOMIC FIELDS

EUGENIO GIANNELLI, NGUYEN NGOC HUNG, A. A. SCHAEFFER FRY, AND CAROLINA VALLEJO

Abstract. We show that every finite group of order divisible by 2 or q, where q is a prime number, admits a $\{2, q\}'$-degree nontrivial irreducible character with values in $\mathbb{Q}(e^{2\pi i/q})$. We further characterize when such character can be chosen with only rational values in solvable groups. These results follow from more general considerations on groups admitting a $\{p, q\}'$-degree nontrivial irreducible character with values in $\mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(e^{2\pi i/q})$, for any pair of primes p and q. Along the way, we completely describe simple alternating groups admitting a $\{p, q\}'$-degree nontrivial irreducible character with rational values.

1. Introduction

One of the main problems in Finite Group Representation Theory is to understand fields of values of characters, by which we mean the smallest field containing all values of a given character. A classical result of Burnside states that groups of odd order do not possess nontrivial irreducible characters with real fields of values. Actually, this property characterizes odd-order groups in an elementary way, see Theorem 2.3 below. It is also true that a group G has even order if, and only if, G possesses a nontrivial irreducible character with rational field of values [NT08, Theorem 8.2]. However, unlike the real case, the proof of this simply-stated result already requires the Classification of the Finite Simple Groups [GLS94] (CFSG for short), evidencing the deep nature of rationality phenomena in character theory.

R. Gow conjectured that every finite group of even order has a nontrivial irreducible character with odd degree and rational field of values. In 2008, G. Navarro and P.H. Tiep [NT08, Theorem B] finally confirmed this prediction. Later (but appearing first in the literature [NT06]), they generalized their result by proving that every finite group of order divisible by a prime q admits a nontrivial irreducible character of degree coprime to q with values in the rather small cyclotomic extension $\mathbb{Q}(e^{2i\pi/q})$. The study of fields of values of irreducible characters of degree not divisible by a given prime is a subject interesting in its own right [ILNT19], which is recently blooming thanks to a growing interest in the Galois refinement of the McKay conjecture proposed by G. Navarro in [Nav04]. This refined conjecture has been reduced to a question on simple groups in [NSV20], which makes it important to understand the fields of values for such groups.

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Usually, extensions of results from one prime to a set of primes π fail (at least without assuming separability properties in the group), and the behavior of finite groups with respect to properties related to π is no longer smooth. However, in [GSV19] the first, third, and fourth-named authors show that every nontrivial group possesses a nontrivial irreducible character of degree not divisible by any prime in π, where π is any set consisting of at most two primes. In the above-mentioned context of character fields of values, it is natural to consider further restrictions on the values of such π'-degree characters.

In Theorem A, we show that every finite group of order divisible by 2 or q possesses a nontrivial irreducible character of $\{2,q\}'$-degree with field of values contained in $\mathbb{Q}(e^{2i\pi/q})$, a surprising result that generalizes both [NT06] and [NT08, Theorem B] in the fashion of [GSV19].

Theorem A. *Let G be a finite group, let q be a prime and write $\pi = \{2,q\}$. Then G possesses a nontrivial π'-degree irreducible character with field of values contained in $\mathbb{Q}(e^{2\pi i/q})$ if, and only if,* $\gcd(|G|, 2q) > 1$.

The obvious problem suggested by Theorem A is to try to understand when the irreducible character it identifies can be chosen to be rational, that is, when such character can be chosen to have only rational values. In other words, for a group G of even order and an odd prime q, we would like to characterize when G has a π'-degree rational character, where $\pi = \{2,q\}$. This is not always the case, in contrast to what happens if we allow small cyclotomic field extensions of $\mathbb{Q}$ as fields of values, as described by Theorem A. For example, the only rational linear character of A_4 is the trivial one. A complete answer to this problem appears difficult to achieve and at the time of this writing, we do not know what form such a classification would take. However, in the case where G is a solvable group (or an alternating group, see Theorem D below), we can completely solve this problem.

Theorem B. *Let G be a solvable group, q be a prime and set $\pi = \{2,q\}$. Then G admits a nontrivial rational irreducible character of π'-degree if, and only if, H/H' has even order, where $H \in \mathrm{Hall}_\pi(G)$.*

We care to remark that the statement of Theorem B does not hold outside solvable groups, as shown by A_5 and $\pi = \{2,3\}$ (see Remark 5.11).

Our proof of Theorem A relies on the Classification of the Finite Simple Groups. In fact, for alternating groups and generic groups of Lie type, the arguments naturally extend from a pair $\{2,q\}$ of primes to any pair $\{p,q\}$. Hence we obtain Theorem A as a corollary of the following statement, which classifies finite groups admitting a π'-degree character with values in certain cyclotomic extensions of $\mathbb{Q}$, for any set π consisting of two primes. Note that the seemingly random exceptions in Theorem C below suggest that the use of the CFSG is perhaps unavoidable in the present context. From now on, we will use $\mathbb{Q}(\chi)$ to denote the field of values of a character χ.

Theorem C. *Let G be a finite group and $\pi = \{p,q\}$ be a set of primes such that either p or q divides $|G|$. Assume that:*

- *(i) $\pi \neq \{3,5\}$ or G does not have a composition factor isomorphic to the Tits group ${}^2F_4(2)'$.*
- *(ii) $\pi \neq \{23,43\}, \{29,43\}$ or G does not have a composition factor isomorphic to the Janko group J_4.*

Then G possesses a nontrivial irreducible character χ of π'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$.

Theorem C has been used in [HMM20] to obtain a lower bound for the number of *almost p-rational* irreducible characters of p'-degree in a finite group G. A character χ is said to be almost p-rational if $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/n})$ for some nonnegative integer n with p-part at most p. The aforementioned refinement of the McKay conjecture [Nav04, Conjecture A] would imply that the number of almost p-rational irreducible characters of p'-degree of G is at least the number of conjugacy classes in the group $\mathbf{N}_G(P)/\Phi(P)$, where $P \in \mathrm{Syl}_p(G)$ and $\Phi(P)$ is its Frattini subgroup. Therefore, we expect that any finite group of order divisible by p has *many* almost p-rational irreducible characters of p'-degree. Notice that Theorems A and C are consistent with this new consequence of the Galois refinement of the McKay conjecture, as the π'-degree characters identified by them are almost p-rational regardless of the prime q.

Finally, and as briefly mentioned before stating Theorem B, we are able to completely determine which simple alternating groups admit a nontrivial rational irreducible character of π'-degree, for any set π consisting of exactly two primes. We will write $\mathrm{Irr}_{\pi'}(G)$ to denote the set of irreducible characters of G of π'-degree.

Theorem D. *Let $n \geqslant 5$ be a natural number and let p, q be distinct primes. Let $\pi = \{p, q\}$. The alternating group A_n admits a nontrivial rational irreducible character of π'-degree for all those $n \in \mathbb{N}$ that do not satisfy any of the following conditions (up to possibly interchanging the primes p and q).*

(i) $n = p^m = 2q^k + 1$, for some $m, k \in \mathbb{N}_{\geqslant 1}$ such that m is odd.
(ii) $n = 2p^m = q^k + 1$, for some $m, k \in \mathbb{N}_{\geqslant 1}$ such that k is odd.

Moreover, in case (i), $\mathbb{Q}(\phi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ for all $\phi \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$. On the other hand, in case (ii), $\mathbb{Q}(\psi) \subseteq \mathbb{Q}(e^{2\pi i/q})$ for all $\psi \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$.

This paper is structured as follows: In Section 2, we prove Theorems A and C assuming Theorem 2.1 on finite simple groups. In Section 3, we prove Theorem D, which in particular yields the alternating group case of Theorem 2.1. In Section 4, we prove Theorem 2.1 for sporadic groups and simple groups of Lie type, thus completing the proof of Theorem 2.1. Finally, we prove Theorem B in Section 5.

2. Proofs of Theorems A and C

Given a character χ of a finite group and a field extension F of $\mathbb{Q}$, we say that χ is *F-valued* if $\mathbb{Q}(\chi) \subseteq F$. Recall that χ is always $\mathbb{Q}(e^{2\pi i/|G|})$-valued. In the special cases where F is the field of rational or real numbers, we will sometimes just say that χ is rational or real, respectively. In particular, rational characters are real. Moreover, given a prime number p, we say that χ is *p-rational* if $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/n})$ for some nonnegative integer n not divisible by p [Isa06, Definition 6.29].

The aim of this section is to prove Theorems A and C of the introduction. In order to do so, we assume the following result on finite simple groups. This will be shown to hold in Sections 3 and 4.

Theorem 2.1. *Let S be a nonabelian simple group and $\pi = \{p, q\}$ be a set of primes. Assume that (S, π) is not one of $({}^2F_4(2)', \{3, 5\})$, $(J_4, \{23, 43\})$, or $(J_4, \{29, 43\})$. Then there exists $\mathbf{1}_S \neq \chi \in \mathrm{Irr}(S)$ of π'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$.*

We start with a lemma.

Lemma 2.2. *Let $M \lhd G$ such that $|G : M| = r$ an odd prime. Let $\theta \in \mathrm{Irr}(M)$ with $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(e^{2\pi i/p})$ for some prime $p \neq r$. Then there exists $\chi \in \mathrm{Irr}(G)$ lying over θ with $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$.*

Proof. If the stabilizer in G of θ is $G_\theta = M$ then by the Clifford correspondence [Isa06, Theorem 6.11], we have $\theta^G \in \mathrm{Irr}(G)$ with $\mathbb{Q}(\theta^G) \subseteq \mathbb{Q}(\theta) \subseteq \mathbb{Q}(e^{2\pi i/p})$, as required. Therefore, we may assume that θ is G-invariant.

Note that θ is r-rational. It follows from [Isa06, Theorem 6.30] that θ^G has a unique r-rational irreducible constituent χ. Indeed, θ is extendible to χ, and hence $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(\chi)$.

For each $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}(\theta))$, obviously χ^σ is also an r-rational character of G lying over θ. Therefore, by the uniqueness of χ, we have $\chi^\sigma = \chi$. Then χ is $\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}(\theta))$-fixed, which implies that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\theta)$. We have shown that $\mathbb{Q}(\chi) = \mathbb{Q}(\theta) \subseteq \mathbb{Q}(e^{2\pi i/p})$, as desired. □

We will often use the following classic result of Burnside. We include its elementary proof to emphasize the difference between reality and rationality of characters mentioned in the introduction.

Theorem 2.3 (Burnside)**.** *A finite group G has even order if, and only if, some nontrivial $\chi \in \mathrm{Irr}(G)$ is real.*

Proof. Let us assume that $|G|$ is even. By [Isa06, Corollary 2.7] and [Isa06, Corollary 2.23.(b)], we can write

$$|G| = |G : G'| + \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi(1) > 1}} \chi(1)^2 .$$

If $|G : G'|$ is even, then G has a normal subgroup H of index 2 and the only nontrivial irreducible character of G/H is rational. It follows that G has a nontrivial real character. Otherwise, the sum of the squares of the degrees of nonlinear characters of G is odd. Then the action of the complex conjugation on characters must leave some nonlinear irreducible character χ of G invariant. Therefore χ is a nontrivial real irreducible character of G.

The proof of the converse is also elementary, see [Isa06, Problem 3.16]. □

Theorem 2.4. *Let G be a finite group and $\pi = \{p, q\}$ be a set of primes. Then G possesses a nontrivial irreducible character χ of π'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$ if, and only if, $\gcd(|G|, 2pq) > 1$, provided that we are not in one of the following situations:*

(i) G has a composition factor isomorphic to the Tits group ${}^2F_4(2)'$ and $\pi = \{3, 5\}$.
(ii) G has a composition factor isomorphic to the Janko group J_4 and π is one of $\{23, 43\}$ or $\{29, 43\}$.

Proof. First assume that G is a finite group with $\gcd(|G|, 2pq) = 1$. Let $\chi \in \mathrm{Irr}(G)$ such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$. Since $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/|G|})$, we have χ is rational-valued. As G is of odd order, it follows from Theorem 2.3 that χ is trivial.

Next we assume that $\gcd(|G|, 2pq) > 1$. We aim to show that G has a nontrivial irreducible character χ of π'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$.

Let $G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = 1$ be a composition series of G and let $0 \leqslant k \leqslant n-1$ be the smallest such that G_k/G_{k+1} is either nonabelian simple or cyclic of order $2, p$, or q. In particular, G_i/G_{i+1} is cyclic of order coprime to $2pq$ for every $i < k$.

If G_k/G_{k+1} is cylic of order $2, p$, or q, then obviously G_k/G_{k+1} has a nontrivial irreducible character θ of π'-degree such that $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(e^{2\pi i/q})$. On the other hand, when $G_k/G_{k+1} =: S$ is nonabelian simple, Theorem 2.1 implies that there exists $\mathbf{1}_S \neq \theta \in \mathrm{Irr}(S)$ of π'-degree such that $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(e^{2\pi i/q})$.

Viewing the above θ as a character of G_k, we now know that G_k possesses a nontrivial irreducible character θ_k of π'-degree such that $\mathbb{Q}(\theta_k) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\theta_k) \subseteq \mathbb{Q}(e^{2\pi i/q})$. Using Lemma 2.2, we obtain $\theta_{k-1} \in \mathrm{Irr}(G_{k-1})$ lying over θ_k with $\mathbb{Q}(\theta_{k-1}) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\theta_{k-1}) \subseteq \mathbb{Q}(e^{2\pi i/q})$. Moreover, following the proof of Lemma 2.2, we see that $\theta_{k-1}(1) = \theta_k(1)$ or $\theta_{k-1}(1) = |G_{k-1} : G_k|\theta_k(1)$, which guarantees that θ_{k-1} is of π'-degree. Repeating this process k times, we can produce a nontrivial irreducible character $\chi := \theta_0$ of π'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$. □

Theorems A and C follow immediately from Theorem 2.4.

3. Alternating groups

The aim of this section is to prove Theorem 2.1 for alternating groups. In order to do so, we completely describe alternating groups possessing a rational-valued π'-degree character. This is done by proving Theorem D of the introduction, which might be of independent interest.

We begin by recalling that irreducible characters of the symmetric group S_n are labelled by partitions of n [JK81, Chapter 2]. We denote by χ^λ the irreducible character of S_n corresponding to the partition λ of n. We will sometimes use the notation $\lambda \vdash n$ to mean that λ is a partition of n. Similarly we will write $\lambda \vdash_{p'} n$ to say that $\chi^\lambda(1)$ is coprime to p. Given a partition λ of n, we denote by λ' its conjugate. If $\lambda \neq \lambda'$ then $(\chi^\lambda)_{\mathsf{A}_n} \in \mathrm{Irr}(\mathsf{A}_n)$. On the other hand, if $\lambda = \lambda'$ then $(\chi^\lambda)_{\mathsf{A}_n} = \phi + \phi^g$ for some $\phi \in \mathrm{Irr}(\mathsf{A}_n)$ and $g \in \mathsf{S}_n \smallsetminus \mathsf{A}_n$.

Assuming that the reader is familiar with the basic combinatorial concepts involved in the representation theory of symmetric groups (as explained for instance in [Ol94, Chapter 1]), we recall some important facts that will play a crucial role in our proofs. Given $\lambda \vdash n$ and $i, j \in \mathbb{N}$ we denote by $h_{ij}(\lambda)$ the length of the *hook of* λ corresponding to node (i, j). For $e \in \mathbb{N}$, we let $\mathcal{H}^e(\lambda)$ be the set consisting of all those nodes (i, j) of λ such that e divides $h_{ij}(\lambda)$. Moreover, we let $C_e(\lambda)$ denote the e-core of λ.

For any natural number m, we denote by $\nu_p(m)$ the exponent of the maximal power of p dividing m. The following lemma follows from [Ol94, Proposition 6.4].

Lemma 3.1. *Let p be a prime and let n be a natural number with p-adic expansion $n = \sum_{j=0}^{k} a_j p^j$. Let λ be a partition of n. Then $\nu_p(\chi^\lambda(1)) = 0$ if, and only if, $|\mathcal{H}^{p^k}(\lambda)| = a_k$ and $C_{p^k}(\lambda) \vdash_{p'} n - a_k p^k$.*

A consequence of Lemma 3.1 is highlighted by the following statement.

Lemma 3.2. *Let p be a prime and let $n = p^k + \varepsilon$ for some $\varepsilon \in \{0, 1\}$. Let $\lambda \vdash n$ be such that $\chi^\lambda(1) > 1$. Then χ^λ is an irreducible character of p'-degree of S_n if and only if $h_{11}(\lambda) = p^k$.*

A second useful consequence of [Ol94, Proposition 6.4] is stated in the following lemma.

Lemma 3.3. *Let $n = 2^k + \varepsilon$ for some $\varepsilon \in \{0, 1\}$, and let $\lambda \vdash n$. Then $\nu_2(\chi^\lambda(1)) = 1$ if and only if $\mathcal{H}^{2^k}(\lambda) = \varnothing$ and $|\mathcal{H}^{2^{k-1}}(\lambda)| = 2$.*

We conclude this brief background summary by recalling a well-known fact on cyclotomic extensions of the rational numbers [BEW98, Lemma 1.2.1].

Lemma 3.4. *If p is an odd prime number, then $\mathbb{Q}(\sqrt{p}) \subseteq \mathbb{Q}(e^{2\pi i/p})$ if and only if $p \equiv 1 \bmod 4$. On the other hand, $\mathbb{Q}(\sqrt{-p}) \subseteq \mathbb{Q}(e^{2\pi i/p})$ if and only if $p \equiv 3 \bmod 4$.*

We are now ready to prove the main result of this section, which is Theorem D in the introduction.

Proof of Theorem D. Assume first that both primes p and q divide the order of A_n. Equivalently, we have $p, q \leqslant n$. Let $n = \sum_{i=1}^{t} a_i p^{m_i} = \sum_{j=1}^{r} b_j q^{k_j}$ be the p-adic and respectively q-adic expansions of n. Here $m_1 > m_2 > \cdots > m_t \geqslant 0$ and $k_1 > k_2 > \cdots k_r \geqslant 0$. Without loss of generality, we can assume that $b_1 q^{k_1} < a_1 p^{m_1}$. We consider $\lambda \in \mathcal{P}(n)$ to be defined by:

$$\lambda = (n - b_1 q^{k_1}, n - a_1 p^{m_1} + 1, 1^{b_1 q^{k_1} - (n - a_1 p^{n_1} + 1)}).$$

As done in the proof of [GSV19, Theorem 2.8], we observe that $\chi^\lambda \in \mathrm{Irr}_{\pi'}(\mathsf{S}_n)$ and that $\chi^\lambda(1) \neq 1$ unless $n = a_1 p^{m_1} = b_1 q^{k_1} + 1$. We also claim that $\lambda \neq \lambda'$. This follows by observing that $\lambda = \lambda'$ would imply that

$$b_1 q^{k_1} - (n - a_1 p^{m_1}) = n - b_1 q^{k_1} - 1 \text{ and that } n - a_1 p^{m_1} \in \{0, 1\}.$$

Then we would have that $b_1 q^{k_1} = n - b_1 q^{k_1} - 1$ if $n - a_1 p^{m_1} = 0$ or that $b_1 q^{k_1} = n - b_1 q^{k_1}$ if $n - a_1 p^{m_1} = 1$. Both these situations can not occur. We conclude that $\chi := (\chi^\lambda)_{\mathsf{A}_n} \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$ and that $\mathbb{Q}(\chi) = \mathbb{Q}$.

Let us now consider the case where $n = ap^m = bq^k + 1$, for some $m, k \in \mathbb{N}$, some $1 \leqslant a \leqslant p - 1$ and some $1 \leqslant b \leqslant q - 1$.

If $b \geqslant 3$, then we consider $\mu = ((b-1)q^k + 1, 1^{q^k})$. Since $h_{11}(\mu) = ap^m$, $h_{12}(\mu) = (b-1)q^k$ and $h_{21}(\mu) = q^k$, we deduce that $\chi^\mu \in \mathrm{Irr}_{\pi'}(\mathsf{S}_n)$ by Lemma 3.1. Since $b \geqslant 3$ we also have that $\mu \neq \mu'$ and hence that $\chi := (\chi^\mu)_{\mathsf{A}_n} \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$ is nontrivial and such that $\mathbb{Q}(\chi) = \mathbb{Q}$.

If $b \in \{1, 2\}$ and $a \geqslant 3$ then we consider $\nu = ((a-1)p^m, 2, 1^{p^m - 2})$. Since $h_{11}(\nu) = bq^k$, $h_{12}(\nu) = (a-1)p^m$ and $h_{21}(\nu) = p^m$, we deduce that $\chi^\nu \in \mathrm{Irr}_{\pi'}(\mathsf{S}_n)$ by Lemma 3.1. As above, $a \geqslant 3$ implies that $\nu \neq \nu'$ and hence that $\chi := (\chi^\nu)_{\mathsf{A}_n} \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$ is nontrivial and such that $\mathbb{Q}(\chi) = \mathbb{Q}$.

Let us now study the situation where $a, b \in \{1, 2\}$. Since $ap^m = bq^k + 1$ we observe that the only cases to consider are $(a, b) \in \{(1, 1), (1, 2), (2, 1)\}$.

• If $(a, b) = (1, 2)$ then $n = p^m = 2q^k + 1$ and hence $p \neq 2$. Since $2 = b \leqslant q - 1$ we also have that $q \neq 2$. By Lemma 3.2 we deduce that $\chi^\lambda \in \mathrm{Irr}_{p'}(\mathsf{S}_n)$ if and only if $\lambda = (d, 1^{n-d})$ is a hook partition. Moreover, since q is odd, again from Lemma 3.1 we observe that the only hook partitions of n that label characters of S_n of degree coprime to q are $(n), (1^n)$ and $\zeta = (1 + q^k, 1^{q^k}) = \zeta'$. We also observe that m must be odd in this situation, as $p^m = 2q^k + 1 \equiv 3 \bmod 4$. It follows that A_n admits exactly two distinct nontrivial irreducible characters of π'-degree: the two irreducible constituents ϕ_1, ϕ_2 of $(\chi^\zeta)_{\mathsf{A}_n}$. By [JK81, 2.5.13] we observe that their fields of values are equal to $\mathbb{Q}(\sqrt{-p^m})$ and strictly contain $\mathbb{Q}$. Moreover, since m is odd then $p \equiv\ p^m = 2q^k + 1 \equiv\ 3 \bmod 4$. Hence using Lemma 3.4 we observe that for all $i \in \{1, 2\}$ we have

$$\mathbb{Q}(\phi_i) = \mathbb{Q}(\sqrt{-p^m}) = \mathbb{Q}(\sqrt{-p}) \subseteq \mathbb{Q}(e^{2\pi i/p}).$$

• If $(a, b) = (2, 1)$ then $n = 2p^m = q^k + 1$ and hence $q \neq 2$. The situation is similar to the one described above. Using Lemma 3.2 we notice that the non-linear irreducible characters of S_n of degree coprime to q are labelled by all partitions λ such that $h_{11}(\lambda) = q^k$ and $h_{22}(\lambda) = 1$. Since $2 = a \leqslant p - 1$, we have that $p \neq 2$. Therefore, Lemma 3.1 implies that

the only partition that labels a non-linear irreducible character of S_n of degree coprime to p and to q is $\eta = (p^m, 2, 1^{p^m-2}) = \eta'$. As before we deduce that A_n admits exactly two distinct nontrivial irreducible characters of π'-degree: the two irreducible constituents ψ_1 and ψ_2 of $(\chi^\eta)_{\mathsf{A}_n}$. By [JK81, 2.5.13] we observe that for all $i \in \{1,2\}$ we have that $\mathbb{Q}(\psi_i) = \mathbb{Q}(\sqrt{q^k})$. It follows that for any $i \in \{1,2\}$, $\mathbb{Q}(\psi_i)$ strictly contains $\mathbb{Q}$ if and only if k is odd. In this case, for all $i \in \{1,2\}$ we have that $\mathbb{Q}(\psi_i) = \mathbb{Q}(\sqrt{q})$. Moreover, since $p \neq 2$ then $q \equiv 1 \bmod 4$. Therefore $\mathbb{Q}(\psi_i) \subseteq \mathbb{Q}(e^{2\pi i/q})$, by Lemma 3.4.

• If $(a,b) = (1,1)$ then exactly one of p or q is equal to 2.

If $q = 2$ then $n = p^m = 2^k + 1$. By Lemma 3.2 we deduce that $\chi^\lambda \in \mathrm{Irr}_{p'}(\mathsf{S}_n)$ if and only if $\lambda = (d, 1^{n-d})$ is a hook partition. Lemma 3.2 shows that $(n), (1^n)$ are the only hook partitions labelling an odd-degree character of S_n. Moreover, using Lemma 3.3 we observe that the partition $\zeta = (1 + 2^{k-1}, 1^{2^{k-1}}) = \zeta'$, is the only hook partition of n such that $\nu_2(\chi^\zeta(1)) = 1$. We deduce that the two irreducible constituents ϕ_1 and ϕ_2 of $(\chi^\zeta)_{\mathsf{A}_n}$ are the only nontrivial irreducible characters of π'-degree of A_n. By [JK81, 2.5.13] we observe that $\mathbb{Q}(\phi_1) = \mathbb{Q}(\phi_2) = \mathbb{Q}(\sqrt{p^m})$. Hence $\mathbb{Q}(\phi_1)$ (and $\mathbb{Q}(\phi_2)$) strictly contain $\mathbb{Q}$ if and only if m is odd. In this case, $p \equiv p^m \equiv 2^k + 1 \equiv 1 \bmod 4$. Therefore Lemma 3.4 implies that for all $i \in \{1,2\}$ we have that

$$\mathbb{Q}(\phi_i) = \mathbb{Q}(\sqrt{p^m}) = \mathbb{Q}(\sqrt{p}) \subseteq \mathbb{Q}(e^{2\pi i/p}).$$

If $p = 2$ then $n = 2^m = q^k + 1$ and we notice that k is necessarily odd. Moreover, Lemma 3.2 implies that the only $\{2,q\}'$-degree irreducible characters of S_n are the linear ones. On the other hand, Lemma 3.3 shows that $\eta = (2^{m-1}, 2, 1^{2^{m-1}-2}) = \eta'$ is the only partition labelling a q'-degree irreducible character of S_n such that $\nu_2(\chi^\eta(1)) = 1$. Arguing as above, we deduce that A_n admits exactly two distinct nontrivial irreducible characters of π'-degree: the two irreducible constituents ψ_1 and ψ_2 of $(\chi^\eta)_{\mathsf{A}_n}$. By [JK81, 2.5.13] we observe that for all $i \in \{1,2\}$, we have $\mathbb{Q}(\psi_i) = \mathbb{Q}(\sqrt{-q^k})$. It follows that for any $i \in \{1,2\}$, $\mathbb{Q}(\psi_i)$ strictly contains $\mathbb{Q}$. Since k is odd, for all $i \in \{1,2\}$ we have $\mathbb{Q}(\psi_i) = \mathbb{Q}(\sqrt{-q})$. Moreover, since $p = 2$ we have $q \equiv 3 \bmod 4$. Therefore $\mathbb{Q}(\psi_i) \subseteq \mathbb{Q}(e^{2\pi i/q})$, by Lemma 3.4.

To complete the proof we need to treat the easier case where pq does not divide $|\mathsf{A}_n| = n!/2$. In this setting, we just need to show that there exists a rational-valued π'-degree irreducible character of A_n (as conditions (i) and (ii) of the statement of Theorem D can not be satisfied).

If $p, q > n$, then every irreducible character of A_n has π'-degree. Since $n \geqslant 5$, we have $(n-1,1) \neq (n-1,1)'$ and therefore $\chi = (\chi^{(n-1,1)})_{\mathsf{A}_n}$ is a nontrivial irreducible character of π'-degree of A_n such that $\mathbb{Q}(\chi) = \mathbb{Q}$. Otherwise, up to possibly interchanging p and q we can assume that $p \leqslant n < q$. In this case, every irreducible character of A_n has degree coprime to q. Let $n = a + pw$ for some $0 \leqslant a \leqslant p-1$ and $w \geqslant 1$. Consider $\lambda = (n-(a+1), a+1) \in \mathcal{P}(n)$. Since $n \geqslant 5$ we notice that necessarily $\lambda \neq \lambda'$. Moreover, Lemma 3.1 implies that p does not divide $\chi^\lambda(1)$. It follows that $\chi = (\chi^\lambda)_{\mathsf{A}_n}$ is a non-trivial irreducible character of π'-degree of A_n such that $\mathbb{Q}(\chi) = \mathbb{Q}$. □

A straightforward consequence of Theorem D is that Theorem 2.1 holds for alternating groups.

Corollary 3.5. *Let $\pi = \{p,q\}$ be a set of two primes and let $n \geqslant 5$. Then A_n possesses a nontrivial irreducible character χ of π'-degree such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/q})$.*

Proof. If n does not satisfy conditions (i) and (ii) of Theorem D, then A_n has a nontrivial rational character. If n satisfies condition (i), then the proof of Theorem D shows that there exists $\phi \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$ such that $\phi(1) > 1$ and such that $\mathbb{Q}(\phi) \subseteq \mathbb{Q}(e^{2\pi i/p})$. On the other hand, if n satisfies condition (ii), we have shown in the proof of Theorem D that there exists $\psi \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$ such that $\psi(1) > 1$ and such that $\mathbb{Q}(\psi) \subseteq \mathbb{Q}(e^{2\pi i/q})$. □

4. Simple groups of Lie type

In this section, we prove Theorem 2.1 for simple groups of Lie type and sporadic simple groups. The following reduces us to the case of simple groups of Lie type with non-exceptional Schur multipliers. The list of finite simple groups with exceptional Schur multipliers is available in [GLS98, Table 6.1.3].

Proposition 4.1. *Let S be a simple group of Lie type with an exceptional Schur multiplier, or let S be a sporadic group. Assume that S is not the Janko group J_4 or the Tits group ${}^2F_4(2)'$. Then S satisfies Theorem 2.1. Further, the Tits group ${}^2F_4(2)'$ satisfies Theorem 2.1 for $\pi \neq \{3,5\}$, and the Janko group J_4 satisfies Theorem 2.1 for $\pi \notin \{\{23,43\},\{29,43\}\}$.*

Proof. This can be confirmed using GAP and the Atlas [GAP, Atl]. In particular, the character tables for the groups under consideration are available in the GAP Character Table Library, and we make use of the `Conductor` command in GAP, which returns the smallest natural number m for which a character in a stored character table takes its values in $\mathbb{Q}(e^{2\pi i/m})$. □

When S is a simple group of Lie type, the required character $\chi \in \mathrm{Irr}(S)$ of π'-degree we produce will be a *semisimple character*. Let us recall some brief background on these characters from [Ca85, DM91, GM20]. (We refer the reader in general to these references for more on the character theory of groups of Lie type.)

Let $\mathcal{G}$ be a connected reductive algebraic group in characteristic p and F a Frobenius endomorphism of $\mathcal{G}$. For each rational maximal torus $\mathcal{T}$ of $\mathcal{G}$ and character $\theta \in \mathrm{Irr}(\mathcal{T}^F)$, Deligne–Lusztig's twisted induction $R_{\mathcal{T}}^{\mathcal{G}}$ is used to define the Deligne-Lusztig character $R_{\mathcal{T}}^{\mathcal{G}}(\theta)$. Let $\mathcal{G}^*$ be an algebraic group with a Frobenius endomorphism F^* such that $(\mathcal{G}, F)$ is dual to $(\mathcal{G}^*, F^*)$. Set $G := \mathcal{G}^F$ and $G^* := (\mathcal{G}^*)^{F^*}$.

Recall that if $(\mathcal{T}, \theta)$ is G-conjugate to $(\mathcal{T}', \theta')$, then $R_{\mathcal{T}}^{\mathcal{G}}(\theta) = R_{\mathcal{T}'}^{\mathcal{G}}(\theta')$. Moreover, by [DM91, Proposition 13.13], the G-conjugacy classes of pairs $(\mathcal{T}, \theta)$ are in one-to-one correspondence with the G^*-conjugacy classes of pairs $(\mathcal{T}^*, s)$, where s is a semisimple element of G^* and $\mathcal{T}^*$ is a rational maximal torus containing s. Due to this correspondence, we can use the notation $R_{\mathcal{T}^*}^{\mathcal{G}}(s)$ for $R_{\mathcal{T}}^{\mathcal{G}}(\theta)$. For each conjugacy class (s) of semisimple elements in G^* such that $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected, one can define a so-called semisimple character of G as follows:

$$\chi_{(s)} := \frac{1}{|W(s)|} \sum_{w \in W(s)} \varepsilon_{\mathcal{G}} \varepsilon_{\mathcal{T}_w^*} R_{\mathcal{T}_w^*}^{\mathcal{G}}(s),$$

where $W(s)$ is the Weyl group of $\mathbf{C}_{\mathcal{G}^*}(s)$, $\mathcal{T}_w^*$ is a torus of $\mathcal{G}^*$ of type w, and $\varepsilon_{\mathcal{G}} = \pm 1$, depending on whether the relative rank of $\mathcal{G}$ is even or odd, see [DM91, Definition 14.40]. Moreover, $\chi_{(s)}$ is irreducible and

$$\chi_{(s)}(1) = |G^* : \mathbf{C}_{G^*}(s)|_{p'},$$

where we recall that p is the defining characteristic of G and $n_{p'}$ denotes the p'-part of a positive integer n.

Lemma 4.2. *With the notation as above, let $s \in G^*$ be a semisimple element such that $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected. Suppose s has order k and $\sigma \in \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/|G|})/\mathbb{Q})$ satisfies $\sigma(\xi) = \xi^m$ for every kth root of unity ξ, where m is an integer relatively prime to k. Then $\chi_{(s)}^\sigma = \chi_{(s^m)}$. In particular, $\mathbb{Q}(\chi_{(s)}) \subseteq \mathbb{Q}(e^{2\pi i/k})$.*

Proof. Let $\mathcal{T}^*$ be a rational maximal torus of $\mathcal{G}^*$ containing s. Let $\mathcal{T}$ be a rational maximal torus of $\mathcal{G}$ and $\theta \in \mathrm{Irr}(\mathcal{T}^F)$ such that the G-conjugacy class of $(\mathcal{T}, \theta)$ corresponds to the G^*-conjugacy class of $(\mathcal{T}^*, s)$ under the correspondence described above. Then $(\mathcal{T}, \theta^m)$ corresponds to $(\mathcal{T}^*, s^m)$, and the multiplicative order of θ in the group $\mathrm{Irr}(\mathcal{T}^F)$ is the same as the order of s. Therefore, the values of θ are in $\mathbb{Q}(e^{2\pi i/k})$.

We recall the character formula for $R_{\mathcal{T}'}^{\mathcal{G}}(\theta)$, which we simplify as $R_{\mathcal{T}',\theta}$:

$$R_{\mathcal{T}',\theta}(g) = \frac{1}{|\mathbf{C}_{\mathcal{G}}^0(t)^F|} \sum_{x \in \mathcal{G}^F} \theta(x^{-1}tx) Q_{x\mathcal{T}'x^{-1}}^{\mathbf{C}_{\mathcal{G}}^0(t)}(u),$$

where t is semisimple, u is unipotent, and $g = tu = ut$ is the Jordan decomposition of $g \in G$. Also, $\mathbf{C}_{\mathcal{G}}^0(t)$ is the connected component of $\mathbf{C}_{\mathcal{G}}(t)$ and $Q_{x\mathcal{T}x^{-1}}^{\mathbf{C}_{\mathcal{G}}^0(t)}$ are Green functions of $\mathbf{C}_{\mathcal{G}}^0(t)$, see [Ca85, Theorem 7.2.8]. As the linear character θ is $\mathbb{Q}(e^{2\pi i/k})$-valued and the Green functions are rational-valued, we have $R_{\mathcal{T}',\theta}^\sigma = R_{\mathcal{T}',\theta^m}$ and $\mathbb{Q}(R_{\mathcal{T}',\theta}) \subseteq \mathbb{Q}(e^{2\pi i/k})$ for every rational maximal torus $\mathcal{T}'$ of $\mathcal{G}$. The conclusion now follows from the definition of $\chi_{(s)}$. □

In particular, if $\mathbf{Z}(\mathcal{G})$ is connected, then $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected for every semisimple $s \in \mathcal{G}^*$. (See, for example, [DM91, Remark 13.15(ii)].) If $\mathbf{Z}(\mathcal{G})$ is not connected, we may embed $\mathcal{G}$ into another connected reductive group $\widetilde{\mathcal{G}}$ satisfying $\mathbf{Z}(\widetilde{\mathcal{G}})$ is connected, via a so-called regular embedding $\iota\colon \mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$. We record here some of the properties of regular embeddings. For proofs and a nice detailed discussion, we refer the reader to [GM20, Section 1.7]. In this situation, ι may be chosen so that, identifying $\mathcal{G}$ with its image $\iota(\mathcal{G})$ in $\widetilde{\mathcal{G}}$, the Frobenius endomorphism F is naturally the restriction of a Frobenius endomorphism on $\widetilde{\mathcal{G}}$, which we will also denote by F. Then writing $\widetilde{G} := \widetilde{\mathcal{G}}^F$, we have $G \lhd \widetilde{G}$, the quotient group $\widetilde{G}/G$ is abelian of order prime to p, and the restriction $\mathrm{Res}_G^{\widetilde{G}}(\chi)$ to G of any $\chi \in \mathrm{Irr}(\widetilde{G})$ is multiplicity-free.

Further, the regular embedding ι induces a dual map $\iota^*\colon \widetilde{\mathcal{G}}^* \to \mathcal{G}^*$ that maps $\widetilde{G}^* := (\widetilde{\mathcal{G}}^*)^{F^*}$ onto G^* and whose kernel is central and F^*-fixed. With this in place, following the treatment in [GM20, Corollary 2.6.18], we may define the semisimple characters of G corresponding to a given semisimple $s \in G^*$ to be the irreducible constituents of the restriction $\mathrm{Res}_G^{\widetilde{G}}(\chi_{(\widetilde{s})})$ of a semisimple character $\chi_{(\widetilde{s})}$ of $\widetilde{G}$, where $\widetilde{s}$ is a semisimple element of $\widetilde{G}^*$ such that $\iota^*(\widetilde{s}) = s$. Fixing χ_s to be one such semisimple character of G corresponding to s, we again have

$$\chi_s(1) = |G^* : \mathbf{C}_{G^*}(s)|_{p'}. \tag{4.1}$$

Further, the set $\mathrm{Irr}(G)$ may be partitioned into subsets known as rational Lusztig series, denoted $\mathcal{E}(G, s)$, which are indexed by G^*-conjugacy classes of semisimple elements $s \in G^*$ (see, e.g. [DM91, Proposition 14.41]). In particular, the semisimple characters of G corresponding to s lie in the set $\mathcal{E}(G, s)$. When $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected, $\chi_{(s)}$ is the unique semisimple character in $\mathcal{E}(G, s)$. To avoid confusion, we will use the notation $\chi_{(s)}$ only in the case $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected.

In the case $\mathbf{C}_{\mathcal{G}^*}(s)$ is disconnected, and hence Lemma 4.2 does not apply, the following, extracted from [SFT18], will be useful. Here, we write $\mathcal{E}(G,s)^\sigma$ for the set $\{\chi^\sigma \mid \chi \in \mathcal{E}(G,s)\}$.

Lemma 4.3. *Let $G = \mathcal{G}^F$ be a group of Lie type defined in characteristic p as above. Let $s \in G^*$ be semisimple and let $\sigma \in \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/|G|})/\mathbb{Q})$. Then*

(i) If s has prime order q and m is an integer coprime to q such that $\sigma(\xi) = \xi^m$ for each qth root of unity ξ, then $\mathcal{E}(G,s)^\sigma = \mathcal{E}(G,s^m)$.

(ii) If $\mathcal{E}(G,s)^\sigma = \mathcal{E}(G,s)$ and $\chi^\sigma(u) = \chi(u)$ for all unipotent elements $u \in G$ (that is, elements of order a power of p) and all $\chi \in \mathrm{Irr}(G)$, then every semisimple character in $\mathcal{E}(G,s)$ is fixed by σ.

Proof. Part (i) is a direct application of [SFT18, Lemma 3.4]. Part (ii) is an application of [SFT18, Lemma 3.8], since the Gelfand-Graev characters of G (see [DM91, Definition 14.21]) are induced from characters on the unipotent radical, and hence take nonzero values only on unipotent elements. □

We remark that the same proof as [SFT18, Lemma 3.4], which is analogous to that of Lemma 4.2, yields the corresponding statement of Lemma 4.3(i) for arbitrary $|s|$, but we will not need this here.

Now, if S is a simple group of Lie type not isomorphic to an alternating group or one listed in Proposition 4.1, then $S = G/Z(G)$ for $G = \mathcal{G}^F$, where $\mathcal{G}$ is simple of simply connected type. The following will be used in conjunction with Lemma 4.3 in many cases in which $\mathbf{C}_{\mathcal{G}^*}(s)$ is not connected. Although it can be extracted from the proofs of [GSV19, Lemma 3.3 and Theorem 3.5], we rewrite the proof for the convenience of the reader.

Lemma 4.4. *Let $S = G/\mathbf{Z}(G)$ be as in the previous paragraph, where G is defined in characteristic p. Let $q \neq p$ be another prime, let Q be a Sylow q-subgroup of G^*, and moreover assume that $|\mathbf{Z}(G)|$ is either a power of q or coprime to q. Then there exists a nontrivial semisimple character χ_s in a series $\mathcal{E}(G,s)$ of G satisfying the following properties:*

(i) $s \in Z(Q)$ and has order q;

(ii) $Z(G)$ is in the kernel of χ_s;

In particular, χ_s may be viewed as a nontrivial member of $\mathrm{Irr}_{\{p,q\}'}(S)$.

Proof. Let $s \in Z(Q)$ have order q. Then $\mathbf{C}_{G^*}(s)$ contains a Sylow q-subgroup of G^*, and hence any semisimple character in $\mathcal{E}(G,s)$ is a member of $\mathrm{Irr}_{\{p,q\}'}(G)$, using (4.1). Now, our exclusions imply that $|Z(G)| = [G^* : (G^*)']$. If $|Z(G)|$ is a power of q, then elementary character theory yields every member of $\mathrm{Irr}_{q'}(G)$ is trivial on $Z(G)$ (see, e.g., [GSV19, Lemma 3.4]). If $q \nmid |Z(G)|$, the equality $|Z(G)| = [G^* : (G^*)']$ implies $Q \leqslant (G^*)'$, so $s \in (G^*)'$. By [NT13, Lemma 4.4(ii)], any character of $\mathcal{E}(G,s)$ is then trivial on $Z(G)$. In either case, this yields that any semisimple character in $\mathcal{E}(G,s)$ satisfies the statement. □

Proposition 4.5. *Let S be a simple group of Lie type. Then Theorem 2.1 holds for S.*

Proof. We may assume S is not one of the groups listed in Proposition 4.1 nor isomorphic to an alternating group. Further, thanks to [NT06, NT08], we may assume that $p \neq q$.

Let S be of the form $G/\mathbf{Z}(G)$, where $G = \mathcal{G}^F$ is the set of fixed points of a simple connected reductive algebraic group of simply connected type defined in characteristic r, under a Frobenius endomorphism F. Note that the Steinberg character St_G of G has degree a power of r, is rational-valued, and is trivial on $\mathbf{Z}(G)$. Hence, we may assume that $r = p$ is one of the primes in π.

Throughout, let $Q \in \mathrm{Syl}_q(G^*)$. If p is an odd prime, let $\eta \in \{\pm 1\}$ be such that $p \equiv \eta \pmod 4$, and note that $\mathbb{Q}(\sqrt{\eta p}) \subseteq \mathbb{Q}(e^{2\pi i/p})$, using Lemma 3.4.

By [MT11, Exercise 20.16], we see that $\mathbf{C}_{\mathcal{G}^*}(s)$ is connected whenever $|s|$ is relatively prime to $|\mathbf{Z}(\mathcal{G})|_{p'}$. Then if $q \nmid |\mathbf{Z}(\mathcal{G})|$, we have $q \nmid |\mathbf{Z}(G)|$ (see [GLS98, Theorem 1.12.5 and Table 6.1.2]) and the character χ_s constructed in Lemma 4.4 is actually $\chi_{(s)}$ and takes values in $\mathbb{Q}(e^{2\pi i/q})$, using Lemma 4.2. Hence we assume $q \mid |\mathbf{Z}(\mathcal{G})|$.

Let $\mathcal{G}$ be of type A_{n-1}. Then $\mathcal{G} = \mathrm{SL}_n$ and $\widetilde{\mathcal{G}} = \mathrm{GL}_n$, and we may write $n = a_1 q + \cdots a_t q^t$ with $0 \leqslant a_i < q$ for $1 \leqslant i \leqslant t$. We write $\widetilde{G} = \mathrm{GL}_n^\epsilon(p^a)$ and $G = \mathrm{SL}_n^\epsilon(p^a)$, where $\epsilon \in \{\pm 1\}$ and $\epsilon = 1$ gives the untwisted version $\mathrm{SL}_n(p^a)$, and $\epsilon = -1$ gives the twisted version $\mathrm{SU}_n(p^a)$. Recall that $\mathbf{C}_{\widetilde{\mathcal{G}}^*}(\widetilde{s})$ is connected for any semisimple $\widetilde{s} \in \widetilde{G}^*$. Further, note that $\mathbf{Z}(G) = G \cap \mathbf{Z}(\widetilde{G})$, $\widetilde{G}^* \cong \widetilde{G}$, $S = \mathrm{PSL}_n^\epsilon(p^a) \cong (G^*)'$, $G = (\widetilde{G})' \cong (\widetilde{G}^*)'$, and $G^* \cong \widetilde{G}/\mathbf{Z}(\widetilde{G}) = \mathrm{PGL}_n^\epsilon(p^a)$. Throughout, we will make these identifications. Let $\widetilde{Q} \in \mathrm{Syl}_q(\widetilde{G})$. Then by [CF64, We55], we have $\widetilde{Q} = \prod_{i=1}^t Q_i^{a_i}$, where the $Q_i \in \mathrm{Syl}_q(\mathrm{GL}_{q^i}^\epsilon(p^a))$ are embedded diagonally in $\widetilde{G}$. Let $k = \min\{i | a_i > 0\}$, so that $n_q = q^k$.

First, assume that n is not a power of q. Let $s' \in \mathbf{Z}(Q_k)$ have order q. If $n \neq 2q^k$, define $\widetilde{s} \in \mathbf{Z}(\widetilde{Q})$ to be of the form $\mathrm{diag}(s', I_{n-q^k})$. If $q \mid (p^a - \epsilon)$, then s' may further be chosen to be of the form $\mu I_{q^k} \in \mathbf{Z}(\mathrm{GL}_{q^k}^\epsilon(p^a))$, where $\mu \in C_{p^a-\epsilon} \leqslant \mathbb{F}_{p^{2a}}^\times$ has order q. Then $\det(\widetilde{s}) = \det(s') = \mu^{q^k} = 1$. Otherwise, $q \nmid |\widetilde{G}/G|$, so $\widetilde{Q} \leqslant G$. In either case, $\widetilde{s} \in G = (\widetilde{G}^*)'$, so the corresponding semisimple character $\chi_{(\widetilde{s})}$ of $\widetilde{G}$ is trivial on $\mathbf{Z}(\widetilde{G})$, by [NT13, Lemma 4.4]. If q is odd and $n = 2q^k$, we may instead let $\widetilde{s} \in \mathbf{Z}(\widetilde{Q})$ be of the form $\mathrm{diag}(\mu I_{q^k}, \mu^{-1} I_{q^k})$ if $q \mid (p^a - \epsilon)$ and $\mathrm{diag}(s', I_{q^k})$ if $q \nmid (p^a - \epsilon)$, and we again see that $\widetilde{s} \in G$. Further, since the conjugacy classes of semisimple elements of $\widetilde{G}$ are determined by their eigenvalues, we see $\widetilde{s}$ is not $\widetilde{G}$-conjugate to $\widetilde{s}z$ for any nontrivial $z \in \mathbf{Z}(\widetilde{G}^*)$. But the irreducible characters of $\widetilde{G}/G$ are in bijection with elements of $\mathbf{Z}(\widetilde{G}^*)$, and $\chi_{(\widetilde{s})} \otimes \widehat{z} = \chi_{(\widetilde{s}z)}$ for $\widehat{z} \in \mathrm{Irr}(\widetilde{G}/G)$ corresponding to $z \in \mathbf{Z}(\widetilde{G}^*)$ (see [DM91, 13.30]). Hence $\chi_{(\widetilde{s})}$ is also irreducible when restricted to G. By Lemma 4.2, $\chi_{(\widetilde{s})}$ has values in $\mathbb{Q}(e^{2\pi i/q})$, so this yields a member of $\mathrm{Irr}(S)$ with degree prime to both p and q and with values in $\mathbb{Q}(e^{2\pi i/q})$, as desired.

Now assume that $n = q^k$. Then [GSV19, Lemma 3.4] yields that any irreducible character of G with degree prime to q is trivial on the center, which has size $\gcd(n, p^a - \epsilon)$. Hence it suffices to show there exists a member of $\mathrm{Irr}(G)$ with degree prime to p and to q whose values lie in $\mathbb{Q}(e^{2\pi i/p})$ or $\mathbb{Q}(e^{2\pi i/q})$.

Let χ_s be as in Lemma 4.4, where we let $Q = \widetilde{Q}\mathbf{Z}(\widetilde{G})/\mathbf{Z}(\widetilde{G}) \in \mathrm{Syl}_q(G^*)$, and let $\widetilde{s} \in \widetilde{Q}\mathbf{Z}(\widetilde{G})$ be such that $\widetilde{s}\mathbf{Z}(\widetilde{G}) = s$. Then notice $\widetilde{s}^q \in \mathbf{Z}(\widetilde{G})$. Let $\zeta = e^{2\pi i/|\widetilde{s}|}$, so that the semisimple character $\widetilde{\chi}_{(\widetilde{s})}$ of $\widetilde{G}$ corresponding to $\widetilde{s}$ takes its values in $\mathbb{Q}(\zeta)$, by Lemma 4.2, and lies over the $\{p,q\}'$-degree character χ_s of G. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}(e^{2\pi i/q}))$. Then σ maps ζ to ζ^m for some m with $\gcd(m, |\widetilde{s}|) = 1$. Further, $m \equiv 1 \pmod q$, since σ fixes qth roots of unity. In particular, $\widetilde{s}^m = \widetilde{s}z$ for some $z \in \mathbf{Z}(\widetilde{G})$. Then using Lemma 4.2, we have $\widetilde{\chi}_{(\widetilde{s})}^\sigma = \widetilde{\chi}_{(\widetilde{s}^m)} = \widetilde{\chi}_{(\widetilde{s}z)}$, and hence $\widetilde{\chi}_{(\widetilde{s})}^\sigma$ also lies over χ_s. In particular, $\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi}_{(\widetilde{s})})^\sigma = \mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi}_{(\widetilde{s})}^\sigma) = \mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi}_{(\widetilde{s}z)}) = \mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi}_{(\widetilde{s})} \otimes \widehat{z}) = \mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi}_{(\widetilde{s})})$. So, $\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi}_{(\widetilde{s})})$ is fixed by each such σ, and therefore has values in $\mathbb{Q}(e^{2\pi i/q})$.

If q is odd and $\chi \in \mathrm{Irr}(G)$, then [SFV19, Theorem 6.1] yields that $\mathbb{Q}(\chi) = \mathbb{Q}(\mathrm{Res}_G^{\widetilde{G}}\widetilde{\chi})$ for any $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{G})$ lying over χ. Hence in this case, $\mathbb{Q}(\chi_s) \subseteq \mathbb{Q}(e^{2\pi i/q})$ as well. If $q = 2$, then

the above yields $\mathbb{Q}(\mathrm{Res}_G^{\widetilde{G}}(\widetilde{\chi}_{(\tilde{s})})) = \mathbb{Q}$, so [SFV19, Theorem 6.1] yields that $\mathbb{Q}(\chi_s) \subseteq \mathbb{Q}(\sqrt{\eta p})$, completing the proof in this case.

Hence we may assume $\mathcal{G}$ is not of type A_{n-1}, and therefore $|\mathbf{Z}(\mathcal{G})|$ is a power of q. Let χ_s be the character constructed in Lemma 4.4. In this case, note that either $q = 2$ or $(\mathcal{G}, q) = (E_6, 3)$. In the latter case, G is the simply connected type group $E_6^\epsilon(p^a)_{sc}$, where $\epsilon \in \{\pm 1\}$, $\epsilon = 1$ corresponds to the untwisted version, and $\epsilon = -1$ corresponds to the twisted version. Using [TZ04, Theorem 1.8 and Lemma 2.6], we see that any irreducible character of G takes integer values on unipotent elements, and hence Lemma 4.3 yields that χ_s is fixed by any Galois automorphism σ that fixes the field $\mathbb{Q}(e^{2\pi i/3})$, and hence has values in $\mathbb{Q}(e^{2\pi i/3})$.

We may therefore take $q = 2$, p odd, and $\mathcal{G}$ to be of type B_n, C_n, D_n, or E_7. Here $s^2 = 1$, and by Lemma 3.4, it suffices to show that χ_s takes values in $\mathbb{Q}(\sqrt{\eta p})$. The data available in CHEVIE and [Lüb07] yield that the odd-degree characters of ${}^3D_4(p^a)$ and $E_7(p^a)$ are all rational-valued, and hence we assume S is $B_n(p^a)$ with $n \geqslant 2$, $C_n(p^a)$ with $n \geqslant 3$, $D_n(p^a)$ with $n \geqslant 4$, or ${}^2D_n(p^a)$ with $n \geqslant 4$.

If S is $C_n(p^a)$, then $G = \mathrm{Sp}_{2n}(p^a)$, and [SFT20, Theorem B] yields that χ_s has values in $\mathbb{Q}(\sqrt{\eta p})$. If S is $B_n(p^a)$, $D_n(p^a)$, or ${}^2D_n(p^a)$, [TZ04, Corollary 8.3 and Lemma 2.6] yields $\chi(u) \in \mathbb{Q}(\sqrt{\eta p})$ for every unipotent element u of G and every $\chi \in \mathrm{Irr}(G)$. Then Lemma 4.3 implies that χ_s takes its values in $\mathbb{Q}(\sqrt{\eta p})$ as well, completing the proof. □

5. Rational characters of π'-degree in solvable groups

In this section, we prove Theorem B. Namely, we characterize when a solvable group G has a π'-degree rational character, where $\pi = \{2, q\}$ is a pair of primes. We first note that if G has a normal Hall $\{2, q\}$-subgroup, then the solution to the characterization problem is pretty simple in Lemma 5.1 below. We thank G. Navarro for pointing out a simplified version of a previous argument.

In what follows, we assume that the reader is familiar with the theory of characters and normal subgroups as in [Isa06, Chapter 6], for instance.

Lemma 5.1. *Let G be a finite group and $p < q$ be two primes. Set $\pi = \{p, q\}$. Suppose that $H \lhd G$ where $H \in \mathrm{Hall}_\pi(G)$ and G/H has odd order. Then G has a nontrivial irreducible character χ of π'-degree with values in $\mathbb{Q}(e^{2\pi i/p})$ if, and only if, H/H' has order divisible by p. Moreover, if $\lambda \in \mathrm{Irr}(H)$ lies under χ, then $o(\lambda) = p$.*

Proof. Notice that to prove both implications, we may assume that H is abelian. If H has order divisible by p, then let $\mathbf{1}_H \neq \lambda \in \mathrm{Irr}(H)$ be linear with $o(\lambda)$ equal to p. By [Isa06, Corollary 6.27], let $\hat{\lambda} \in \mathrm{Irr}(G_\lambda)$ be the only extension of λ such that $o(\hat{\lambda}) = o(\lambda)$. In particular, $\mathbb{Q}(\hat{\lambda}) = \mathbb{Q}(e^{2\pi i/p})$ and hence $\mathbf{1}_G \neq \chi = (\hat{\lambda})^G \in \mathrm{Irr}(G)$ (by [Isa06, Theorem 6.11]) has values in $\mathbb{Q}(e^{2\pi i/p})$. Since G/H is a π'-group, it follows that χ has π'-degree, as wanted.

Suppose now that $\mathbf{1}_G \neq \chi \in \mathrm{Irr}(G)$ has π'-degree, $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p})$, and p does not divide $|H|$. If $p = 2$, then χ is a nontrivial rational irreducible character of an odd-order group. This would contradict Theorem 2.3. If $p > 2$, then $|H| = q^b$ and since G/H is a π'-group, then G is a p'-group. In particular $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(e^{2\pi i/p}) \cap \mathbb{Q}(e^{2\pi i/|G|}) = \mathbb{Q}$. Since G/H is an odd-order group by hypothesis, and $q > p > 2$ also by hypothesis, then we get again a contradiction with Theorem 2.3.

Finally, let $\lambda \in \mathrm{Irr}(H)$ be a constituent of χ_H. Then λ is linear and nontrivial. Indeed, if $\lambda = \mathbf{1}_H$, then χ would be a rational irreducible character of G/H of odd order, hence $\chi = \mathbf{1}_G$ by Burnide's theorem, contradicting one of the assumptions on χ. We consider

$G_\lambda^* = \{g \in G \mid \lambda^g = \lambda^\sigma \text{ for some } \sigma \in \mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})\} \leqslant G$, sometimes known as the *semi-inertia* group of λ in G. By [NT10, Lemma 2.3] the map given by $g \mapsto \sigma$, whenever $g \in G_\lambda^*$ and $\lambda^g = \lambda^\sigma$, defines a monomorphism $G_\lambda^*/G_\lambda \to \mathrm{Gal}(\mathbb{Q}(\lambda)/\mathbb{Q})$. Write $E = \mathbb{Q}(\lambda)$. Since λ is a linear character of a π-group, then E is obtained by adjoining a root of unity of order $o(\lambda) = p^a q^b$ to $\mathbb{Q}$, where $a, b \geqslant 0$ are not both zero. Write $\xi = e^{2\pi i/p}$. We claim that $\mathbb{Q}(\xi) \subseteq E$. Otherwise, $o(\lambda) = q^b$ with $b \geqslant 1$, and $E \cap \mathbb{Q}(\xi) = \mathbb{Q}$. Take any $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ and extend it to $\overline{\sigma} \in \mathrm{Gal}(E(\xi)/\mathbb{Q}(\xi))$ by elementary Galois theory. Since $\chi^{\overline{\sigma}} = \chi$, then $\lambda^\sigma = \lambda^{\overline{\sigma}} = \lambda^g$ for some $g \in G$. Note that $g \in G_\lambda^*$. In particular,

$$|G_\lambda^*/G_\lambda| = |\mathrm{Gal}(E(\xi)/\mathbb{Q}(\xi))| = |\mathrm{Gal}(E/\mathbb{Q})| = (q-1)q^{b-1}$$

contradicts the oddness hypothesis on $|G/H|$ as $2 \leqslant p < q$. Now we know that $\mathbb{Q}(\xi) \subseteq E$. The above argument actually proves that $\mathrm{Gal}(E/\mathbb{Q}(\xi))$ injects into G_λ^*/G_λ, since

$$|\mathrm{Gal}(E/\mathbb{Q}(\xi))| = \frac{|\mathrm{Gal}(E/\mathbb{Q}|)}{|\mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})|} = \begin{cases} p^{a-1} & \text{if } b = 0, \\ p^{a-1}(q-1)q^{b-1} & \text{otherwise.} \end{cases}$$

The fact that $|G/H|$, hence $|G_\lambda^*/G_\lambda|$, is odd and coprime to p forces $a = 1$ and $b = 0$, that is, $o(\lambda) = p$. □

We are working in slightly more generality than needed in this section. We note that if $p = 2$, then the condition on the order of G/H is trivially satisfied in Lemma 5.1. However, it is a necessary condition in general: if p is an odd prime, then the group $G = \mathsf{C}_p \rtimes \mathsf{C}_{p-1}$ where the action is faithful has a π'-degree rational irreducible character for every $\pi = \{p, q\}$ with q a divisor of $p - 1$.

For an arbitrary group G and a set of primes π, we will denote by $\mathcal{X}_{\pi',p}(G)$ the set of π'-degree irreducible characters of G with values in $\mathbb{Q}(e^{2\pi i/p})$. If G is π-separable, then the set $\mathcal{X}_{\pi',p}(G)$ consists entirely of monomial characters, given that $|\mathbf{N}_G(H)/H|$ is odd for $H \in \mathrm{Hall}_\pi(G)$. In [NV12] this fact is proven in the case where π consists of a single prime, but it generalizes to arbitrary π, see Theorem 5.4 below. Recall that by work of P. Hall and S. A. Čunhihin, Hall π-subgroups in π-separable groups *behave like* Sylow subgroups.

Lemma 5.2. *Let G be π-separable, $H \in \mathrm{Hall}_\pi(G)$, and $\chi \in \mathrm{Irr}_{\pi'}(G)$. If $M \lhd G$, then χ_M contains some H-invariant irreducible constituent and any two of them are $\mathbf{N}_G(H)$-conjugate.*

Proof. Let $\theta \in \mathrm{Irr}(M)$ be a constituent of χ_M. By the Clifford correspondence [Isa06, Theorem 6.11], $|G : G_\theta|$ is a π'-number. Since G is π-separable, then $H \subseteq G_\theta^x = G_{\theta^x}$ for some $x \in G$, and θ^x is an H-invariant constituent of χ_M. Write $\varphi = \theta^x$. Suppose that φ^y is also H-invariant. Then H and $H^{y^{-1}}$ are Hall π-subgroups of G_φ. In particular, $H^{y^{-1}} = H^g$ for some $g \in G_\varphi$. Note that $gy \in \mathbf{N}_G(H)$ and $\varphi^y = \varphi^{gy}$. □

The following is a nice application of Theorem 2.3 in the context of coprime group actions.

Lemma 5.3. *Let a solvable group H act coprimely on a group M with $\mathbf{C}_M(H)$ of odd order. Then $\mathrm{Irr}(M)$ contains no nontrivial real H-invariant character.*

Proof. Assume first that H is solvable. By the Glauberman correspondence [Isa06, Theorem 13.1], there is natural bijection between H-invariant characters in $\mathrm{Irr}(M)$ and the set $\mathrm{Irr}(\mathbf{C}_M(H))$. In particular, the number of real H-invariant characters in $\mathrm{Irr}(M)$ is the same as the number of real characters in $\mathrm{Irr}(\mathbf{C}_M(H))$ by [Isa06, Problem 13.1]. In this case, the result follows from Theorem 2.3 and the hypothesis on the order of $\mathbf{C}_M(H)$.

If H is nonsolvable then, by Feit-Thompson's odd order theorem [FT63], the group M has odd order, and the result follows from Theorem 2.3. □

Theorem 5.4. *Let π be a set of primes. Let G be π-separable and $H \in \mathrm{Hall}_\pi(G)$. Suppose that $\mathbf{N}_G(H)/H$ has odd order. For every $\chi \in \mathcal{X}_{\pi',p}(G)$, there is a pair (U, λ) with $H \leqslant U \leqslant G$ and $\lambda \in \mathrm{Irr}(U)$ linear such that $\mathbb{Q}(\lambda) \in \mathbb{Q}(e^{2\pi i/p})$ and $\chi = \lambda^G$. Moreover, any other such pair is G-conjugate to (U, λ).*

Proof. We first prove the existence of a pair by induction on $|G|$. We may assume that $H < G$ and $G' > 1$ since otherwise χ would be linear and there is nothing to prove. We may assume that $\ker(\chi) = 1$ by working in $G/\ker(\chi)$.

Let $M \lhd G$. We claim that if for some H-invariant irreducible constituent θ of χ_M, $\overline{\theta}$ also lies under χ, then $\theta = \overline{\theta}$. Indeed, if θ and $\overline{\theta}$ are H-invariant irreducible constituents of χ_M, then $\overline{\theta} = \theta^x$ with $x \in \mathbf{N}_G(H)$ by Lemma 5.2. Write $N = \mathbf{N}_G(H)$. Note that $x^2 \in N_\theta$. Since $|N : H|$ is odd, then $\langle xH \rangle = \langle x^2 H \rangle$, so $x \in N_\theta$ and the claim follows.

We may assume that $\mathbf{O}_{\pi'}(G) = 1$. Otherwise, write $K = \mathbf{O}_{\pi'}(G) > 1$. Let $\theta \in \mathrm{Irr}(K)$ be H-invariant lying under χ by Lemma 5.2. Note that $\mathbb{Q}(\chi_K) = \mathbb{Q}$ because $\mathbb{Q}(\chi_K) \subseteq \mathbb{Q}(e^{2\pi i/p})$ and K is a π'-group. Hence $\overline{\theta}$ lies under χ and by the paragraph above $\overline{\theta} = \theta$. Note that $\mathbf{C}_K(H) \cong \mathbf{N}_{KH}(H)/H$ can be seen as a subgroup of $\mathbf{N}_G(H)/H$, and hence $\mathbf{C}_K(H)$ has odd order. By Lemma 5.3, we obtain that $\theta = \mathbf{1}_K$, contradicting the fact that χ is faithful.

Let $M = \mathbf{O}_\pi(G)$. We show that $M > 1$ is abelian. By [Isa08, Theorem 3.21] we have that $M > 1$. Let $\mu \in \mathrm{Irr}(M)$ be under χ. Since χ has π'-degree, we have that μ is linear. Then M' is contained in the kernel of every G-conjugate of μ. Since χ is faithful, we conclude that $M' = 1$.

Let L be a minimal normal subgroup of G contained in G'. Since G is π-separable and $\mathbf{O}_{\pi'}(G) = 1$, then $L \subseteq M$ is q-elementary abelian for some $q \in \pi$.

Let $\nu \in \mathrm{Irr}(L)$ be H-invariant lying under χ. Note that ν is linear. We show that $\mathbb{Q}(\nu) \subseteq \mathbb{Q}(e^{2\pi i/p})$. If $q \neq p$, then $\mathbb{Q}(\chi_L) = \mathbb{Q}(e^{2\pi i/p}) \cap \mathbb{Q}(e^{2\pi i/|L|}) = \mathbb{Q}$. Hence $\overline{\nu}$ and ν lie under χ, and by the second paragraph of this proof $\overline{\nu} = \nu$. In particular $\mathbb{Q}(\nu) = \mathbb{Q} \subseteq \mathbb{Q}(e^{2\pi i/p})$ (in this case $q = 2$ necessarily). If L is p-elementary abelian, then $\mathbb{Q}(\nu) = \mathbb{Q}(e^{2\pi i/p})$.

Next we show that ν extends to its stabilizer G_ν and consequently $G_\nu < G$. In order to show that ν extends to G_ν, it is enough to show it extends to every Sylow subgroup of G_ν/L by [Isa06, Theorem 6.26]. Let $R/L \in \mathrm{Syl}_r(G_\lambda/L)$. If $r \neq q$, then ν extends to R because L is a q-group and [Isa06, Corollary 6.27]. If $r = q$, then R is a q-group. Since χ has q'-degree, then χ_R has some linear constituent and hence ν extends to R. Note that if $G_\nu = G$, and $\tau \in \mathrm{Irr}(G)$ is an extension of ν then $L \subseteq G' \subseteq \ker(\tau)$. Then $\nu = \mathbf{1}_L$, a contradiction with $\ker(\chi) = 1$.

Let $\psi \in \mathrm{Irr}(G_\nu)$ lying over ν be the Clifford correspondent of χ as in [Isa06, Theorem 6.11]. Since both χ and ν have values in $\mathbb{Q}(e^{2\pi i/p})$, so does ψ. We have $H \subseteq G_\nu < G$, and by induction there exists a pair (U, λ) of the desired kind inducing ψ, hence also inducing χ.

For the uniqueness up to G-conjugacy, see the second part of the proof of [Val16, Theorem 2.6]. □

The fact that under the hypothesis of Theorem 5.4, the set $\mathcal{X}_{\pi',p}(G)$ consists entirely of monomial characters allows us to construct a map between $\mathcal{X}_{\pi',p}(G)$ and $\mathcal{X}_{\pi',p}(\mathbf{N}_G(H))$ as in [Isa90, Theorem C]. In order to prove that such map is bijective, we will need a π-version of [IN08, Theorem 3.3], which relies on π-versions of [IN08, Theorem 2.1 and Corollary 2.2].

All these versions hold without restrictions on the set π. Their proofs can be obtained by just mimicking the proofs in [IN08] and using the following easy observation.

Lemma 5.5. *Let $H \leqslant G$ with $(|G:H|, |H|) = 1$. Suppose that a linear character $\mu \in \mathrm{Irr}(H)$ extends to G. Then there is an extension $\lambda \in \mathrm{Irr}(G)$ of μ with $o(\lambda) = o(\mu)$.*

Proof. Since μ extends to G, note that $H' \subseteq H \cap G' \subseteq \ker(\mu)$. In particular, μ seen as a character of $H/H \cap G' \cong HG'/G' \lhd G/G'$ extends to G/G'. By [Isa06, Corollary 6.27], μ extends to some $\lambda \in \mathrm{Irr}(G/G')$ with $o(\lambda) = o(\mu)$ and the statement follows. □

Theorem 5.6. *Let G be a π-separable group and $H \in \mathrm{Hall}_\pi(G)$. If $K \lhd H$, then there is a unique subgroup $V \leqslant G$ maximal with the property that $H \subseteq \mathbf{N}_G(V)$ and $V \cap H = K$.*

Proof. Mimic the proof of [IN08, Theorem 2.1] using the corresponding properties of Hall π-subgroups in π-separable groups. □

Corollary 5.7. *Let G be a π-separable group and $H \in \mathrm{Hall}_\pi(G)$. If $\mu \in \mathrm{Irr}(H)$ is linear, then there exists a unique subgroup $U \leqslant G$ containing H and maximal such that μ extends to U.*

Proof. Mimic the proof of [IN08, Corollary 2.2] using Lemma 5.5 to guarantee the existence of an extension of λ with suitable order. □

Theorem 5.8. *Let G be π-separable and $U \leqslant G$ be a subgroup of π'-index. Let $\lambda \in \mathrm{Irr}(U)$ be linear and such that $o(\lambda)$ is a π-number, Assume that λ does not extend to any subgroup of G containing U. Then $\lambda^G \in \mathrm{Irr}(G)$.*

Proof. Mimic the proof of [IN08, Theorem 3.3] using Corollary 5.7. □

Theorem 5.9. *Let $p < q$ be two primes and set $\pi = \{p, q\}$. Let G be a π-separable group and $H \in \mathrm{Hall}_\pi(G)$. Write $N = \mathbf{N}_G(H)$ and suppose that N/H has odd order. Define a map*

$$\Omega\colon \mathcal{X}_{\pi',p}(G) \to \mathcal{X}_{\pi',p}(N)$$

in the following way: If $\chi \in \mathcal{X}_{\pi',p}(G)$, choose a pair (U, λ) where $H \leqslant U \leqslant G$ and $\lambda \in \mathrm{Irr}(U)$ is linear such that $\mathbb{Q}(\lambda) \subseteq \mathbb{Q}(e^{2\pi i/p})$ and $\lambda^G = \chi$, then set $\Omega(\chi) = (\lambda_{U\cap N})^N$. Then Ω is a well-defined bijection.

Proof. We are basically going to follow the proof of [Val16, Theorem 2.13]. Given $\chi \in \mathcal{X}_{\pi',p}(G)$, the existence of the a pair (U, λ) as in the statement is guaranteed by Theorem 5.4. Set $\Omega(\chi) = (\lambda_{U\cap N})^N$. By [Isa90, Lemma 2.3.(a)] we have that $\Omega(\chi) \in \mathrm{Irr}(N)$, so in particular $\Omega(\chi) \in \mathcal{X}_{\pi',p}(N)$. Since any other inducing pair for χ is G-conjugate to (U, λ) and

$$(\lambda_{U\cap N})^N = (\lambda^g_{U^g\cap N})^N$$

for every $g \in G$, we see that Ω is a well-defined map $\mathcal{X}_{\pi',p}(G) \to \mathcal{X}_{\pi',p}(N)$. Now [Isa90, Lemma 2.3.(b)] guarantees Ω is injective. It remains to prove that Ω is surjective. Let $\theta \in \mathcal{X}_{\pi',p}(N)$ and $\mu \in \mathrm{Irr}(H)$ lie under θ. Note that μ is linear. By Lemma 5.1, we have that $o(\mu) = p$. Let $\varphi \in \mathrm{Irr}(N_\mu)$ be the Clifford correspondent of θ lying over μ as in [Isa06, Theorem 6.11]. Since both θ and μ have values in $\mathbb{Q}(e^{2\pi i/p})$ so does φ. In particular, $\mathbb{Q}(\varphi) = \mathbb{Q}(e^{2\pi i/p})$. Let $\hat{\mu} \in \mathrm{Irr}(N_\mu)$ be the canonical extension given by [Isa06, Corollary 6.27], so that $o(\hat{\mu}) = o(\mu) = p$. By Gallagher's theorem [Isa06, Corollary 6.17], we have $\varphi = \beta\hat{\mu}$, with $\beta \in \mathrm{Irr}(N_\mu/H)$. Note that $\mathbb{Q}(\beta) \subseteq \mathbb{Q}(e^{2\pi i/p})$ must be rational as N_μ/H is a π'-group. Since $|N_\mu : H|$ is odd by hypothesis, we get $\beta = \mathbf{1}_{N_\mu}$ by Theorem 2.3. Hence $(\hat{\mu})^N = \theta$. By Corollary 5.7 and Lemma

5.5, there exists a subgroup $U \leqslant G$ maximal with the property that μ extends to $\lambda \in \mathrm{Irr}(U)$ and $o(\lambda) = p$. Note that $U/\ker(\lambda) = \ker(\lambda)N_\mu/\ker(\lambda) \cong N_\mu/\ker(\hat{\mu})$ and $\lambda_{N_\mu} = \hat{\mu}$. By Theorem 5.8, we have $\chi = \lambda^G \in \mathcal{X}_{\pi',p}(G)$. Hence $\Omega(\chi) = (\lambda_{U\cap N})^N = (\hat{\mu})^N = \theta$. □

As an immediate consequence of Lemma 5.1 and Theorem 5.9, we can derive the following result.

Corollary 5.10. *Let $p < q$ be two primes and set $\pi = \{p, q\}$. Let G be a π-separable group and $H \in \mathrm{Hall}_\pi(G)$. Assume that $\mathbf{N}_G(H)/H$ has odd order. Then G has a nontrivial irreducible character of π'-degree with values in $\mathbb{Q}(e^{2\pi i/p})$ if, and only if, H/H' has order divisible by p.*

We can finally proof Theorem B of the Introduction.

Proof of Theorem B. If $2 < q$, then the statement of Theorem B is equivalent to the $p = 2$ case in Corollary 5.10, since the condition on the order of $\mathbf{N}_G(H)/H$ becomes superfluos and π-separability of G is equivalent to solvability of G by Burnside's p^aq^b and Feit-Thompson's odd order theorems ([Isa06, Theorem 3.10] and [FT63], respectively).

We are left to deal with the much easier case where $2 = q$. In this case, $H \in \mathrm{Syl}_2(G)$ and the condition H/H' has even order is equivalent to G having even order. Then one of the implication follows from Theorem 2.3, while the other is [NT08, Lemma 3.1]. □

Remark 5.11. We observe that the statement of Theorem B does not hold outside the realm of solvable groups. For example, let $\pi = \{2, 3\}$ and consider the alternating group A_5. By Theorem D, we know that A_5 has a rational π'-degree irreducible character (take for instance $(\chi^{(3,2)})_{\mathsf{A}_5}$). On the other hand H/H' has order 3 for every $H \in \mathrm{Hall}_\pi(\mathsf{A}_5)$. This follows by observing that all Hall π-subgroups of A_5 are isomorphic to A_4.

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