An interpolating inequality for solutions of uniformly elliptic equations

AN INTERPOLATING INEQUALITY FOR SOLUTIONS OF UNIFORMLY ELLIPTIC EQUATIONS

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ABSTRACT. We extend an inequality for harmonic functions, obtained in [15, 17], to the case of solutions of uniformly elliptic equations in divergence form, with merely measurable coefficients. The inequality for harmonic functions turned out to be a crucial ingredient in the study of the stability of the radial symmetry for Alexandrov's Soap Bubble Theorem and Serrin's problem. The proof of our inequality is based on a mean value property for elliptic operators stated and proved in [8] and [7].

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \ge 2$, and denote its boundary by Γ. The volume of Ω and the $(N-1)$-dimensional Hausdorff measure of Γ will be denoted, indifferently, by $|\Omega|$ and $|\Gamma|$. Let $A(x)$ be an $N \times N$ symmetric matrix whose entries $a_{ij}(x)$, $i, j = 1, \dots, N$, are measurable functions in Ω. We assume that $A(x)$ satisfies the (uniform) ellipticity condition:

$$\lambda\,|\xi|^2 \le \langle A(x)\,\xi, \xi\rangle \le \Lambda\,|\xi|^2 \quad \text{for any} \quad x \in \Omega,\ \xi \in \mathbb{R}^N. \tag{1.1}$$

Here, λ and Λ are positive constants. Associated to $A(x)$ we consider a uniformly elliptic linear operator L in divergence form, defined formally by

$$Lv = \operatorname{div}[A(x)\,\nabla v], \tag{1.2}$$

for every $x \in \Omega$.

In what follows, we shall use two scaling invariant quantities: for $1 \le p \le \infty$ the number $\|v\|_{p,\Omega}$ will denote the L^p-norm of a measurable function $v : \Omega \to \mathbb{R}$ with respect to the normalized Lebesgue measure $dx/|\Omega|$ and, for $0 < \alpha \le 1$, we define the scaling invariant Hölder seminorm

$$[v]_{\alpha,\Omega} = \sup\left\{ \left(\frac{d_\Omega}{2}\right)^{\alpha} \frac{|v(x_1) - v(x_2)|}{|x_1 - x_2|^{\alpha}} \,:\, x_1, x_2 \in \overline{\Omega},\, x_1 \ne x_2 \right\}, \tag{1.3}$$

where d_Ω is the diameter of Ω. Also, the mean value of v on Ω will be indicated by v_Ω.

For $0 < \alpha \le 1$, we let $\Sigma_\alpha(\Omega)$ be the set of weak solutions v of class $C^{0,\alpha}(\overline{\Omega})$ of $Lv = 0$ in Ω. We denote by B_r and S_r the ball and sphere of radius r centered at the origin. To avoid unessential technicalities, we state here our main result in the case in which Ω is a ball. The case of general domains will be treated later on.

Theorem 1.1. *Take $p \in [1, \infty)$. There exists a positive constant K such that, for any $v \in \Sigma_\alpha(B_r)$, it holds that*

$$\max_{S_r} v - \min_{S_r} v \le K\,[v]_{\alpha,B_r}^{\frac{N}{N+\alpha\,p}}\,\|v - v_{B_r}\|_{p,B_r}^{\frac{\alpha p}{N+\alpha p}}. \tag{1.4}$$

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Moreover, (1.4) *is optimal in the sense that the equality sign holds for some* $v \in \Sigma_\alpha(B_r)$. *Finally, we have that*

$$K \le 2\left(1+\frac{\alpha p}{N}\right)\left(\frac{N}{\alpha p}\right)^{\frac{\alpha p}{N+\alpha p}}\left(\frac{C}{c}\right)^{\frac{\alpha N}{N+\alpha p}}, \tag{1.5}$$

where c, C, *with* $c \le C$, *are two constants that only depend on* N, λ *and* Λ.

We recall that, by De Giorgi-Nash-Moser's theorem, we have that a solution of $Lu = 0$ is locally of class $C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1]$ that depends on N, λ and Λ. Moreover that regularity can be extended up to the boundary provided u is Hölder-continuous on Γ and Γ is sufficiently smooth – e.g., Γ satisfies a uniform exterior cone condition (see [9, Theorem 8.29]) or, more in general, condition (A) defined in [11, pag. 6] (see [11, Theorem 1.1 of Chapter 4]).

The reader's attention should be focused on the quantitative character of (1.4). This says that the oscillation of a solution of an elliptic equation can be controlled, up to the boundary, by its L^p-norm in the domain, provided some a priori information is given on its Hölder seminorm.

The effectiveness of an inequality like (1.4) can be understood from an important application of it, that was first given in [14], and then refined in [13, 15, 16, 17] (see also [12] for a survey on those issues). There, rougher versions of (1.4) for harmonic functions were used to obtain quantitative rigidity estimates for the spherical symmetry in two celebrated problems in differential geometry and potential theory: Alexandrov's Soap Bubble Theorem and Serrin's overdetermined problem (see [1, 2, 3, 18] for the original rigidity results). The sharper version obtained in [15, 17] gives nearly optimal estimates for those problems.

Theorem 1.1 improves the result obtained in [17, Lemma 3.14] (and hence the previous ones) from various points of view. As already mentioned, it extends the analogous estimates obtained for harmonic functions to the case of a uniformly elliptic linear operator in divergence, form with *merely* measurable coefficients. Moreover, it removes the restriction of smallness of the term $\|v - v_\Omega\|_{p,B}$ that was present in the previous inequalities. In doing so, it clears up which are the essential ingredients to consider to obtain a best possible bound. Finally, It also relaxes the former Lipschitz assumption on the solutions to a weaker Hölder continuous a priori information.

The proof of the existence of the constant K in (1.4) is obtained by a quite standard variational argument. The necessary compactness of the optimizing sequence is derived from a rougher version of (1.4), that it is proved in Lemma 2.2. The proof of this lemma extends the arguments, first used in [14] and refined in [13, 15, 17] for harmonic functions, to the case of an elliptic operator. The crucial ingredient to do so is a mean value theorem for elliptic equations in divergence form (see Theorem 2.1) the proof of which is sketched in [8, Remark at page 9] and given with full details in [7, Theorem 6.3].

The proof of Theorem 1.1 is given in Section 2. There, we also provide a proof for the case of smooth domains. In this case, the constant K also depends on the ratio between the diameter and the radius of a uniform interior touching ball for the relevant domain. In Section 3, we show that the proof's scheme can be extended to two instances of non-smooth domains: those satisfying either the uniform interior cone condition or the so-called local John's condition. The dependence of K on the relevant parameters follows accordingly.

2. The inequality in a ball and in smooth domains

We recall the already mentioned result introduced by L. Caffarelli [8, Remark on page 9], the proof of which is provided in full details in [7, Theorem 6.3]. In what follows, $B_r(x_0)$ denotes the ball of radius r centered at x_0.

Theorem 2.1 (Mean Value Property for Elliptic Operators)**.** *Let Ω be an open subset of $\mathbb{R}^N$. Let L be the elliptic operator defined by* (1.1)-(1.2) *and pick any $x_0 \in \Omega$. Then, there exist two constants c, C that only depend on N, λ and Λ, and, for $0 < r < \operatorname{dist}(x_0, \Gamma)/C$, an increasing family of domains $D_r(x_0)$ which satisfy the properties:*

(i) $B_{cr}(x_0) \subset D_r(x_0) \subset B_{Cr}(x_0)$;

(ii) for any v satisfying $Lv \geq 0$, we have that

$$v(x_0) \leq \frac{1}{|D_r(x_0)|} \int_{D_r(x_0)} v(y)\, dy \leq \frac{1}{|D_\rho(x_0)|} \int_{D_\rho(x_0)} v(y)\, dy, \tag{2.1}$$

for any $0 < r < \rho < \operatorname{dist}(x_0, \Gamma)/C$.

Issues related to this theorem and the study of the geometric properties of the sets $D_r(x_0)$ have been recently studied by I. Blank and his collaborators in [4, 5, 6].

2.1. **The inequality for a ball.** We begin our presentation by considering the case of a ball. This will avoid extra technicalities. We will later show how to extend our arguments to other types of domains.

The following lemma gives a rough estimate for sub-solutions of the elliptic equation $Lv = 0$.

Lemma 2.2. *Take $p \geq 1$. Let $v \in C^{0,\alpha}(\overline{B_r})$, $0 < \alpha \leq 1$, be a weak solution of $Lv \geq 0$ in B_r. Then we have that*

$$\max_{S_r} v - \min_{S_r} v \leq 2\left(1 + \frac{\alpha p}{N}\right)\left(\frac{N}{\alpha p}\right)^{\frac{\alpha p}{N+\alpha p}} \left(\frac{C}{c}\right)^{\frac{\alpha N}{N+\alpha p}} [v]_{\alpha, B_r}^{\frac{N}{N+\alpha p}} \, \|v - v_{B_r}\|_{p, B_r}^{\frac{\alpha p}{N+\alpha p}}. \tag{2.2}$$

Proof. Without loss of generality, we can assume that $v_{B_r} = 0$. Let x_1 and x_2 be points on S_r that respectively minimize and maximize v on S_r and, for $0 < \sigma < r$, define the two points $y_j = x_j - \sigma x_j/r$, $j = 1, 2$. Notice that x_j/r is the exterior unit normal vector to S_r at the point x_j.

By (1.3) and the fact that $2r$ is the diameter of B_r, we have that

$$|v(x_j)| \leq |v(y_j)| + [v]_{\alpha, B_r} \left(\frac{\sigma}{r}\right)^\alpha, \ j = 1, 2. \tag{2.3}$$

Being $0 < \sigma < r$, we have that $B_\sigma(y_j) \subset \Omega$. Thus, we apply Theorem 2.1 by choosing $x_0 = y_j$, $j = 1, 2$, and $r = \sigma/C$. By item (i), we have that

$$B_{\frac{c}{C}\sigma}(y_j) \subset D_{\frac{\sigma}{C}}(y_j) \subset B_\sigma(y_j) \subset B_r, \quad j = 1, 2. \tag{2.4}$$

Also, item (ii) gives that

$$|v(y_j)| \leq \frac{1}{|D_{\frac{\sigma}{C}}(y_j)|} \int_{D_{\frac{\sigma}{C}}(y_j)} |v|\, dy \leq \tag{2.5}$$

$$\frac{1}{|D_{\frac{\sigma}{C}}(y_j)|^{1/p}} \left[\int_{D_{\frac{\sigma}{C}}(y_j)} |v|^p\, dy\right]^{1/p} \leq |B|^{-\frac{1}{p}} \left(\frac{C}{c\,\sigma}\right)^{N/p} \left(\int_{B_r} |v|^p\, dy\right)^{1/p}.$$

The second inequality is a straightforward application of Hölder's inequality and, in the last inequality, we used (2.4), that also gives that

$$|D_{\frac{\sigma}{C}}(y_j)| \geq |B| \left(\frac{c}{C}\right)^N \sigma^N.$$

Putting together (2.3) and (2.5) yields that

$$\max_{S_r} v - \min_{S_r} v \le 2 \left[\left(\frac{C}{c}\right)^{N/p} \|v\|_{p,B_r} \left(\frac{\sigma}{r}\right)^{-N/p} + [v]_{\alpha,B_r} \left(\frac{\sigma}{r}\right)^{\alpha} \right], \tag{2.6}$$

for every $0 < \sigma < r$.

Therefore, by minimizing the right-hand side of the last inequality, we can conveniently choose

$$\frac{\sigma^*}{r} = \left[\frac{N}{\alpha p} \left(\frac{C}{c}\right)^{N/p} \frac{\|v\|_{p,B_r}}{[v]_{\alpha,B_r}} \right]^{p/(N+\alpha p)} \tag{2.7}$$

and obtain (2.2) if $\sigma^* < r$.

On the other hand, if $\sigma^* \ge r$, by (1.3) we can write:

$$\max_{S_r} v - \min_{S_r} v \le 2^{\alpha} [v]_{\alpha,B_r} \le 2^{\alpha} [v]_{\alpha,B_r} \left(\frac{\sigma^*}{r}\right)^{\alpha}.$$

Thus, (2.7) gives

$$\max_{S_r} v - \min_{S_r} v \le 2^{\alpha} \left(\frac{N}{\alpha p}\right)^{\frac{\alpha p}{N+\alpha p}} \left(\frac{C}{c}\right)^{\frac{\alpha N}{N+\alpha p}} [v]_{\alpha,B_r}^{\frac{N}{N+\alpha p}} \|v\|_{p,B_r}^{\frac{\alpha p}{N+\alpha p}}.$$

Therefore, (2.2) always holds true, being $2^{\alpha} \le 2(1 + \alpha p/N)$. □

Proof of Theorem 1.1. Lemma 2.2 tells us that (1.4) and (1.5) hold with

$$K = \sup\Big\{ \max_{S_r} v - \min_{S_r} v : v \in \Sigma_{\alpha}(B_r) \text{ with } [v]_{\alpha,B_r}^{\frac{N}{N+\alpha p}} \|v - v_{B_r}\|_{p,B_r}^{\frac{\alpha p}{N+\alpha p}} \le 1 \Big\}.$$

We are thus left to prove the existence of a $v \in \Sigma_{\alpha}(B_r)$ that attains the supremum. Again, we assume that $v_{B_r} = 0$ in the supremum and take a maximizing sequence of functions v_n, that is

$$[v_n]_{\alpha,B_r}^{\frac{N}{N+\alpha p}} \|v_n\|_{p,B_r}^{\frac{\alpha p}{N+\alpha p}} \le 1 \quad \text{and} \quad \max_{S_r} v_n - \min_{S_r} v_n \to K \quad \text{as} \quad n \to \infty.$$

Observe that

$$\|v_n\|_{p,B_r} \le 2^{\frac{\alpha N}{N+\alpha p}}, \ n \in \mathbb{N},$$

since

$$\|v\|_{p,B_r} = \|v - v_{B_r}\|_{p,B_r} \le 2^{\alpha} [v]_{\alpha,B_r}, \ v \in \Sigma_{\alpha}(B_r).$$

We can then extract a subsequence of functions, that we will still denote by v_n, that weakly converge in $L^p(B_r)$ to a function $v \in L^p(B_r)$. By the mean value property of Theorem 2.1, the sequence converges uniformly to v on the compact subsets of B_r, and hence v satisfies the mean value property of Theorem 2.1 in B_r. The same theorem then gives that $Lv = 0$ in B_r.

Next, we fix $x_1, x_2 \in B_r$ with $x_1 \ne x_2$. Since

$$r^{\alpha} \frac{|v_n(x_1) - v_n(x_2)|}{|x_1 - x_2|^{\alpha}} \le [v_n]_{\alpha,B_r} \le \|v_n\|_{p,B_r}^{-\frac{\alpha p}{N}},$$

the local uniform convergence and the semicontinuity of the L^p-norm with respect to weak convergence give that

$$r^{\alpha} \frac{|v(x_1) - v(x_2)|}{|x_1 - x_2|^{\alpha}} \le \|v\|_{p,B_r}^{-\frac{\alpha p}{N}}.$$

Since x_1 and x_2 are arbitrary, we infer that $[v]_{\alpha,B_r} \|v\|_{p,B_r}^{\frac{\alpha p}{N}} \le 1$. This means that v extends to a function of class $C^{0,\alpha}(\overline{B_r})$.

If we now prove that $v_n \to v$ uniformly on S_r, we will have that

$$K = \lim_{n\to\infty} \left(\max_{S_r} v_n - \min_{S_r} v_n \right) = \max_{S_r} v - \min_{S_r} v,$$

and the proof would be complete. For any $x \in S_r$ and $y \in B_r$, we can easily show that

$$\limsup_{n\to\infty} |v_n(x) - v(x)| \le r^{-\alpha} |x-y|^\alpha \limsup_{n\to\infty} [v_n]_{\alpha,B_r} + |v(y) - v(x)| \le$$
$$r^{-\alpha} |x-y|^\alpha \|v\|_{p,B_r}^{-\frac{\alpha p}{N}} + |v(y) - v(x)|.$$

Since $y \in B_r$ is arbitrary and v is continuous up to S_r, the right-hand side can be made arbitrarily small, and hence we infer that v_n converges to v pointwise on S_r. The convergence turns out to be uniform on S_r. In fact, if $x_n \in S_r$ maximizes $|v_n - v|$ on S_r then by compactness $x_n \to x$ as $n \to \infty$ for some $x \in S_r$, modulo a subsequence. Thus,

$$\max_{S_r} |v_n - v| = |v_n(x_n) - v(x_n)| \le$$
$$r^{-\alpha} |x_n - x|^\alpha [v_n]_{\alpha,B_r} + |v_n(x) - v(x)| + |v(x) - v(x_n)|,$$

and the right-hand side vanishes as $n \to \infty$, by the continuity of v and the pointwise convergence of v_n. The proof is complete. □

2.2. **The inequality for smooth domains.** The extension of Theorem 1.1 to the case of bounded domains with boundary Γ of class C^2 is not difficult. We recall that such domains satisfy a *uniform interior sphere condition.* In other words, there exists $r_i > 0$ such that for each $z \in \Gamma$ there is a ball of radius r_i contained in Ω the closure of which intersects Γ only at z.

Theorem 2.3. *Take $p \in [1, \infty)$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ of class C^2 and let L be the elliptic operator defined by* (1.1)-(1.2).

If $v \in \Sigma_\alpha(\Omega)$, then

$$\max_\Gamma v - \min_\Gamma v \le K\, [v]_{\alpha,\Omega}^{\frac{N}{N+\alpha p}} \, \|v - v_\Omega\|_{p,\Omega}^{\frac{\alpha p}{N+\alpha p}} \tag{2.8}$$

for some optimal constant K. Moreover, it holds that

$$K \le \max\left[2\left(1 + \frac{\alpha p}{N}\right), \left(\frac{d_\Omega}{r_i}\right)^\alpha \right] \left(\frac{N}{\alpha p}\right)^{\frac{\alpha p}{N+\alpha p}} \left(\frac{C}{c}\right)^{\frac{\alpha N}{N+\alpha p}}. \tag{2.9}$$

Proof. The proof runs similarly to that of Theorem 1.1. We just have to make some necessary changes to the proof of Lemma 2.2,

We take x_1 and x_2 in Γ that respectively minimize and maximize v on Γ and define the corresponding y_1, y_2 by $y_j = x_j - \sigma\nu(x_j)$, $j = 1, 2$, where $\nu(x_j)$ is the exterior unit normal vector to Ω at the point x_j. This time we use the restriction $0 < \sigma < r_i$, so that $B_\sigma(y_j) \subset \Omega$, $j = 1, 2$.

Next, we must replace (2.3) by

$$|v(x_j)| \le |v(y_j)| + [v]_{\alpha,\Omega} \left(\frac{2\sigma}{d_\Omega}\right)^\alpha, \ j = 1, 2, \tag{2.10}$$

and (2.5) by

$$|v(y_j)| \le |B|^{-\frac{1}{p}} \left(\frac{C}{c\,\sigma}\right)^{N/p} \left(\int_\Omega |v|^p \, dy\right)^{1/p}, \ j = 1, 2.$$

Thus, we arrive at

$$\max_\Gamma v - \min_\Gamma v \le 2 \left[\left(\frac{C}{c}\right)^{N/p} \|v\|_{p,\Omega} \left(\frac{2\sigma}{d_\Omega}\right)^{-N/p} + [v]_{\alpha,\Omega} \left(\frac{2\sigma}{d_\Omega}\right)^\alpha \right] \tag{2.11}$$

for $0 < \sigma < r_i$, in place of (2.6). Here, we used that $|\Omega| \le |B| \, (d_\Omega/2)^N$.

By minimizing the right-hand side of (2.11), this time we can choose

$$\frac{2\sigma^*}{d_\Omega} = \left[\frac{N}{\alpha p} \left(\frac{C}{c} \right)^{N/p} \frac{\|v\|_{p,\Omega}}{[v]_{\alpha,\Omega}} \right]^{p/(N+\alpha p)},$$

and obtain (2.8) and (2.9) if $\sigma^* < r_i$.

On the other hand, if $\sigma^* \ge r_i$, (1.3) gives:

$$\max_\Gamma v - \min_\Gamma v \le 2^\alpha \, [v]_{\alpha,\Omega} \le \left(\frac{2\sigma^*}{r_i} \right)^\alpha [v]_{\alpha,\Omega} =$$
$$\left(\frac{N}{\alpha p} \right)^{\frac{\alpha p}{N+\alpha p}} \left(\frac{C}{c} \right)^{\frac{\alpha N}{N+\alpha p}} \left(\frac{d_\Omega}{r_i} \right)^\alpha [v]_{\alpha,\Omega}^{\frac{N}{N+\alpha\, p}} \, \|v\|_{p,\Omega}^{\frac{\alpha p}{N+\alpha p}}.$$

Again, (2.8) and (2.9) hold true. □

Remark 2.4. Theorem 2.3 can be compared with [17, Lemma 3.14], that was proved for the Laplace operator. In that case, we have that $c = C = 1$ and the seminorm in (1.3) can be replaced by the maximum of $(d_\Omega/2)\,|\nabla v|$ on Γ, provided Γ is sufficiently smooth.

3. The inequality for two classes of non-smooth domains

In this section, for future reference, we consider and carry out some details for two cases of domains with non-smooth boundary.

3.1. **Domains with corners.** Given $\theta \in [0, \pi/2]$ and $h > 0$, we say that Ω satisfies the (θ, h)-*uniform interior cone condition*, if for every $x \in \Gamma$ there exists a finite right spherical cone $\mathcal{C}_x$ (with vertex at x and axis in some direction e_x), having opening width θ and height h, such that

$$\mathcal{C}_x \subset \overline{\Omega} \quad \text{and} \quad \overline{\mathcal{C}}_x \cap \Gamma = \{x\}.$$

Theorem 3.1. *Take $p \in [1, \infty)$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain satisfying the (θ, h)-uniform interior cone condition and let L be the elliptic operator defined by (1.1)-(1.2).*

If $v \in \Sigma_\alpha(\Omega)$, then (2.8) holds true for some optimal constant K. Moreover, we have that

$$K \le \max\left[2\left(1 + \frac{\alpha p}{N}\right), \left(\frac{d_\Omega}{h}\right)^\alpha (1+\sin\theta)^\alpha \right] \left(\frac{N}{\alpha p}\right)^{\frac{\alpha p}{N+\alpha p}} \left(\frac{C}{c\,\sin\theta}\right)^{\frac{\alpha N}{N+\alpha p}}. \tag{3.1}$$

Proof. The proof runs similarly to that of Theorem 2.3. We just have to take care of the bound for K.

Let x_1 and x_2 be the usual extremum points for v on Γ. This time, instead, we define the two points $y_j = x_j - \sigma e_{x_j}$, $j = 1, 2$, for

$$0 < \sigma < \frac{h}{1 + \sin\theta}.$$

Notice that, in view of the (θ, h)-uniform interior cone condition, the ball $B_{\sigma \sin\theta}(y_j)$ is contained in Ω. Thus, by proceeding as in the proof of Theorem 2.3 (this time applying Theorem 2.1 with $r = \frac{\sin\theta}{C}\sigma$ and $x_0 = y_j$, $j = 1, 2$), we arrive at the inequality

$$\max_\Gamma v - \min_\Gamma v \le 2 \left[\left(\frac{C}{c\,\sin\theta} \right)^{N/p} \|v\|_{p,\Omega} \left(\frac{2\sigma}{d_\Omega} \right)^{-N/p} + [v]_{\alpha,\Omega} \left(\frac{2\sigma}{d_\Omega} \right)^\alpha \right],$$

for every $0 < \sigma < h/(1+\sin\theta)$. Hence, this time we can choose

$$\frac{2\sigma^*}{d_\Omega} = \left[\frac{N}{\alpha p}\left(\frac{C}{c\,\sin\theta}\right)^{N/p}\frac{\|v\|_{p,\Omega}}{[v]_{\alpha,\Omega}}\right]^{p/(N+\alpha p)},$$

and obtain (2.8) and (3.1) if $\sigma^* < h/(1+\sin\theta)$.

On the other hand, if $\sigma^* \ge h/(1+\sin\theta)$, by (1.3) we have that

$$\max_\Gamma v - \min_\Gamma v \le 2^\alpha\,[v]_{\alpha,\Omega} \le \left(\frac{2\sigma^*}{h}\right)^\alpha (1+\sin\theta)^\alpha\,[v]_{\alpha,\Omega} =$$

$$\left(\frac{N}{\alpha p}\right)^{\frac{\alpha p}{N+\alpha p}}\left(\frac{C}{c\,\sin\theta}\right)^{\frac{\alpha N}{N+\alpha p}}\left(\frac{d_\Omega}{h}\right)^\alpha (1+\sin\theta)^\alpha\,[v]_{\alpha,\Omega}^{\frac{N}{N+\alpha p}}\,\|v\|_{p,\Omega}^{\frac{\alpha p}{N+\alpha p}}.$$

Again, (2.8) and (3.1) hold true. □

3.2. **Locally John's domains.** Following [10, Definition 3.1.12], we say that a bounded domain $\Omega\subset\mathbb{R}^N$ satisfies the (b_0,R)*-local John condition* if there exist two constants, $b_0>1$ and $R>0$, with the following properties. For every $x\in\Gamma$ and $r\in(0,R]$ we can find $x_r\in B_r(x)\cap\Omega$ such that $B_{r/b_0}(x_r)\subset\Omega$. Also, for each z in the set $\Delta_r(x)$ defined by $B_r(x)\cap\Gamma$, we can find a rectifiable path $\gamma_z:[0,1]\to\overline{\Omega}$, with length $\le b_0 r$, such that $\gamma_z(0)=z$, $\gamma_z(1)=x_r$, and

$$\operatorname{dist}(\gamma_z(t),\Gamma) > \frac{|\gamma_z(t)-z|}{b_0}\quad\text{for any}\quad t>0. \tag{3.2}$$

The constants b_0, R, the point x_r, and the curve γ_z are respectively called *John's constants, John's center (of $\Delta_r(x)$), and John's path.* The class of domains satisfying the local John condition is huge and contains, among others, the so-called *non-tangentially accessible* domains (see [10, Lemma 3.1.13]).

Theorem 3.2. *Take $p\in[1,\infty)$. Let $\Omega\subset\mathbb{R}^N$ be a bounded domain satisfying the (b_0,R)-local John condition and let L be the elliptic operator defined by* (1.1)-(1.2).

If $v\in\Sigma_\alpha(\Omega)$, then (2.8) *holds true for some optimal constant K. Moreover, we have that*

$$K \le \max\left[2\left(1+\frac{\alpha p}{N}\right),\left(\frac{d_\Omega b_0}{R}\right)^\alpha\right]\left(\frac{N}{\alpha p}\right)^{\frac{\alpha p}{N+\alpha p}}\left(\frac{Cb_0}{c}\right)^{\frac{\alpha N}{N+\alpha p}}. \tag{3.3}$$

Proof. Let x one of the usual extremum points for v on Γ. Let γ_x be a John's path from x to the John's center x_R of $\Delta_R(x)$. Since $B_{r/b_0}(x_R)\subset\Omega$ we have that

$$|x-x_R| \ge \operatorname{dist}(x_R,\Gamma) > \frac{R}{b_0}.$$

Thus, for $0<\sigma<R/b_0$, we can find a point y on the John's curve γ_x such that $|x-y|=\sigma$. Hence, by (1.3) we have that (2.10) still holds true.

In view of (3.2) we have that $B_{\sigma/b_0}(y)\subset\Omega$. Thus, as done to obtain (2.5) (this time applying Theorem 2.1 with $r=\sigma/(Cb_0)$ and $x_0=y$), we get that

$$|v(y)| \le \left(\frac{|\Omega|}{|B|}\right)^{\frac{1}{p}}\left[\frac{C\,b_0}{c\,\sigma}\right]^{N/p}\|v\|_{p,\Omega}.$$

This, (2.10), and the inequality $|\Omega|\le|B|\,(d_\Omega/2)^N$ then yield that

$$\max_\Gamma v - \min_\Gamma v \le 2\left[\left(\frac{C\,b_0}{c}\right)^{N/p}\left(\frac{2\,\sigma}{d_\Omega}\right)^{-N/p}\|v\|_{p,\Omega} + [v]_{\alpha,\Omega}\left(\frac{2\,\sigma}{d_\Omega}\right)^\alpha\right],$$

for every $0 < \sigma < R/b_0$. Hence, this time we can choose

$$\frac{2\,\sigma^*}{d_\Omega} = \left[\frac{N}{\alpha\, p} \left(\frac{C\, b_0}{c} \right)^{N/p} \frac{\|v\|_{p,\Omega}}{[v]_{\alpha,\Omega}} \right]^{p/(N+\alpha p)}, \tag{3.4}$$

and have that (2.8) and (3.3) hold true if $\sigma^* < R/b_0$.

On the other hand if $\sigma^* \geq R/b_0$, since by (1.3) it holds that

$$\max_\Gamma v - \min_\Gamma v = v(x_1) - v(x_2) \leq 2^\alpha\, [v]_{\alpha,\Omega} \left(\frac{\sigma^* b_0}{R} \right)^\alpha \leq [v]_{\alpha,\Omega}\, (2\,\sigma^*)^\alpha \left(\frac{b_0}{R} \right)^\alpha,$$

by (3.4) we immediately get

$$\max_\Gamma v - \min_\Gamma v \leq \left(\frac{d_\Omega b_0}{R} \right)^\alpha \left(\frac{N}{\alpha p} \right)^{\frac{\alpha p}{N+\alpha p}} \left(\frac{C b_0}{c} \right)^{\frac{\alpha N}{N+\alpha p}} [v]_{\alpha,\Omega}^{\frac{N}{N+\alpha\, p}}\, \|v\|_{p,\Omega}^{\alpha p/(N+\alpha p)}.$$

Hence, (2.8) and (3.3) still hold true. □

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