Introduction to: Topological degree and fixed point theories in differential and difference equations

INTRODUCTION TO: TOPOLOGICAL DEGREE AND FIXED POINT THEORIES IN DIFFERENTIAL AND DIFFERENCE EQUATIONS.

MARIA PATRIZIA PERA AND MARCO SPADINI

This issue is devoted to classical topological methods in nonlinear analysis in both finite and infinite dimensions. These methods include fixed point theory, topological degree of vector fields and maps, fixed point index, and coincidence degree theory.

Such tools have proved themselves to be indispensable in the study of qualitative properties of solutions to various types of analytical problems, especially those pertaining questions of existence and multiplicity of solutions. Topological methods are a well-established part of nonlinear analysis where they not only find uses as tools for various investigations, but also are object of study *per se*. A nice and very accessible introduction to some of these methods can be found, for instance, in the books [1, 2]. For a more general introduction to this class of methods see, e.g., [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] or one of the many general reference monographs on nonlinear analysis.

Indeed, topological tools are to be chosen when other, more direct, analytical devices cannot be applied. This is usually done by framing problems under scrutiny as existence questions of solutions to some equations in appropriate spaces. For instance, for the analysis of forced oscillations of constrained mechanical systems, fixed point index on a finite dimensional differentiable manifold should be called upon ([13, 14, 15] see also, e.g., [16, 17, 18, 19]). For more complicated boundary value problems involving functional equations, Leray-Schauder degree [20, 21, 22], some of its generalizations as for instance [23, 24, 25], or the coincidence degree in Banach spaces [26, 7, 27] can be more appropriate or, when seeking solutions to problems dealing with difference equations, fixed point theorems in Fréchet spaces of sequences are the way to go (see, e.g., [28]).

Speaking loosely, we may say that a common feature of the topological techniques in mathematical analysis and dynamical systems theory is that they derive information about the problem under consideration (be it some sort of equation, inequality, inclusion or estimate) from the topological structure of the space where the problem lives or of the operators involved. As an illustration of this fact one may consider the well-known Misiurewicz-Przytycki theorem that establishes a lower bound for the topological entropy of a C^1 self map of a smooth compact orientable manifold in terms of the topological degree of this map[1] (see, e.g., [29]). Usually it pays to place the problems in a wisely chosen space. A case in point is [30] where the classical Brouwer degree is applied to the study of the eigenvalue problem for

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[1]Namely, for a C^1 map $f\colon M \to M$, $h_{\mathrm{top}}(f) \geq \ln|\deg(f)|$.

square real matrices yielding a result about global continuation in nonlinear spectral theory that, in turn, can be applied to a Rabinowitz-type global continuation property of the solutions of a perturbed motion equation with friction.

A broad but somewhat fuzzy distinction can be made between two groups of topological tools. One class consisting of those devices that provide existence results directly on the grounds of how the involved functions interact with the topology of the space they operate upon; examples in this group are Brouwer or Schauder or Kakutani fixed point theorems [31, 22, 32], the Ważewski theorem [33, 34] or the Birkhoff twist-map theorem [35, 36, 37, 38]. A second group being constituted by more complex constructions that attach some invariant to map and then relate it to topological properties of the space. Examples of such constructions are the Brouwer or Schauder degrees and their close kin the fixed point index [31, 22], the coincidence degree [26, 7], and also the Maslov index [39, 40], the Conley index and Morse decomposition of attractor-repeller pairs[2] [41, 42, 43], the equivariant degree [44, 45] and the Nielsen number [46].

A pertinent example of the latter sort of constructions is provided by a popular proof of the so-called *hairy-ball theorem* that establishes the existence of a zero (a critical point in a different language) of any continuous tangent vector field to the 2-sphere $\mathbf{S}^2$. In this proof the degree (or rotation or characteristic) of a general tangent vector field on $\mathbf{S}^2$ is established to be nonzero by the use of the Poincaré-Hopf theorem that relates the Euler characteristic of the sphere ($\chi(\mathbf{S}^2) = 2$) to the degree of the tangent vector field[3]; the existence of a zero of the vector field follows from the properties of the degree. It is worth noticing that many of these invariants are built using powerful algebraic topology techniques but sometimes more direct analytic, or even axiomatic, constructions are available; this is the case, for instance, of the topological degree [48, 49, 50, 51, 52].

We called the above distinction "fuzzy" because of the strong theoretical links that exist between many of the mentioned tools. Also, it should be noticed that it is not unusual, in applications, to only partially use the information provided by the topological tools. As an example, consider that to prove the existence of fixed points one only has to check that the fixed point index is nonzero. Only in some special situations one makes use of the full force of the tools at hand. For instance in [53, 54] information about the genericity of forced oscillations induced by a periodic forcing on a compact boundaryless manifold is obtained by comparing the fixed point index of a suitable translation operator (in the sense of [55]) with the Betti numbers of the manifold via the Morse inequalities (see, e.g., [56]).

It is important to point out that topology alone may not be enough to provide the needed information. Consider, for instance, the well-known Yang-Yau estimate of the first eigenvalue of the Laplacian on a closed orientable surface [57], which depends on both the genus of the surface and the chosen metric[4]. Indeed, topological methods need sometimes to be supplemented by other techniques. A typical situation is that of continuation theorems: In that case, in order to establish existence results one needs to find some *a priori* estimates. This can sometimes be done analytically, in other situations one needs to use the geometry of the problem.

[2]These concepts are strongly related to the Ważeski's theory.

[3]The Poincaré-Hopf theorem [47, 31] says that the degree of a tangent vector field to a compact boundaryless manifold M coincides with the Euler characteristic $\chi(M)$ of M.

[4]A similar estimate for nonorientable surfaces has been recently proved by Karpukhin [58].

As an example of the latter situation, consider the question of existence of forced oscillations on an even dimensional sphere $\mathbf{S}^{2n}$ solved in [59] for $n = 1$ ad in [60] for $n > 1$ via a continuation argument[5]. In order to solve this problem an *a priori* estimate on the speed of solutions depending on the initial condition was found using both the geometry of the sphere and a generalized notion of winding number (this is another topological tool, see [62] for an informative and entertaining account). Another example of a quite different situation in which topological information must be joined by other of analytic nature is the problem of existence and multiplicity of harmonic solutions of a small periodic perturbation of an autonomous differential equation (possibly with delay), see [63, 64, 65, 66]. In this situation, assuming the unperturbed vector field is C^1 at its zeros, what turns out to play a key role is a notion of T-resonance[6] (see, e.g., [67, 68, 69]). Indeed, in the merely Lipschitz case, it would also be possible to obtain some results in the same spirit by using a nice theorem of J. A. Yorke [70] which is of a geometric nature[7].

Lately, the field of topological methods in mathematical analysis and dynamical systems theory has experienced a dramatic growth due to the increasing interest in nonlinear phenomena. It is impossible to provide here an even remotely complete list of the recent developments. The purpose of this theme issue is to bring together scientists working with seemingly diverse methods, but with common foundations. However, given the amplitude of the spectrum of topological methods, in order to maintain some degree of homogeneity, we focus on applications to equations and inclusions of different types of topological degree and fixed point theory. In a loose sense, the approach taken has its roots in Poincaré's work devoted to nonlinear differential equations (for an exposition see, e.g., [71]) and in Leray and Schauder classical work [72]. In doing so, we severely curb the scope of the issue disregarding many important developments based on degree theory. For instance, we omit recent progresses in the field of variational inequalities (see, e.g., [73, 74] and the many references therein) or contributions to the study of existence and stability of equations (see, e.g., [75, 76, 77]; see also the papers [78, 79] that have direct applications to optimal control theory [80, 81]). However, in spite of the restriction that we have to operate, given the strong relations mentioned above between all topological methods we feel that this theme issue offers a valuable service to all mathematicians working in the field of topological methods in nonlinear analysis.

The present theme issue consists of nine contributions authored by some of the most influential and active mathematicians in the field of topological methods in nonlinear analysis. Together, these papers cover a wide range of topics showing the great flexibility of these methods.

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[5] See [61] for the case of retarded periodic forcing.

[6] This is a notion involving the linearized equation at a zero. It can be checked by looking at the eigenvalues of the Fréchet differential of the unperturbed vector field at its zeros.

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(M. P. Pera and M. Spadini) Dipartimento di Matematica e Informatica "Ulisse Dini", Università degli Studi di Firenze, Via S. Marta 3, 50139, Firenze, Italy

Email address, M. Spadini: `marco.spadini@unifi.it`

Email address, M. P. Pera: `mariapatrizia.pera@unifi.it`