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Regularity results for a class of doubly nonlinear very singular parabolic equations

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ABSTRACT

The aim of this paper is to present several properties of the nonnegative weak solutions to a class of very singular equations whose prototype is

$$u_t = \operatorname{div}\left(u^{m-1}|Du|^{p-2}Du\right), \qquad p > 1 \text{ and } 3-p < m+p < 2.$$

Namely, we prove L^r_{loc} and $L^r_{\text{loc}} - L^\infty_{\text{loc}}$ estimates and Harnack estimates. Note that $3-p = m+p$ is a critical value: under this threshold the energy estimates hold with a reverse sign.

1. Introduction

In this paper we deal with equations that have as prototype

$$u_t = \operatorname{div}\left(u^{m-1}|Du|^{p-2}Du\right), \tag{1.1}$$

with $p > 1$ and $3-p < m+p < 2$ which correspond to take $-1 < \gamma < 0$, where $\gamma = \frac{m+p-2}{p-1}$, being this structure typical of doubly nonlinear equations.

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Many of our results and of the techniques we apply here hold also in the case $p > 1$, $m < 1$ and $3 - p < m + p \neq 2 < 3$, which correspond to take $0 < |\gamma| < 1$; and all the results presented in this paper are also valid for $\gamma = 0$, for more details see [31].

Note that when $p = 2$ we have the Fast Diffusion Equation

$$u_t = m^{-1}\Delta(u^m), \qquad m < 1 \tag{1.2}$$

while, when $m = 1$, we have to deal with the Singular p-Laplacian Equation

$$u_t = \operatorname{div}\big(|Du|^{p-2}Du\big), \qquad 1 < p < 2. \tag{1.3}$$

We can formally write equation (1.1) as

$$u_t - \operatorname{div}\big(|Dw|^{p-2}Dw\big) = 0, \tag{1.4}$$

where $w = \frac{u^\gamma - 1}{\gamma}$, and this shows that when $m + p \to 2$, which corresponds to $\gamma \to 0$, the equations converge formally to the logarithmic limit

$$u_t - \Delta_p(\log u) = u_t - \operatorname{div}\big(|D(\log u)|^{p-2}D\log u\big) = 0, \qquad p > 1. \tag{1.5}$$

The singular Eqs. (1.2) and (1.3) have been widely studied, as they have a great mathematical interest and are also connected to several applications. But the more they are becoming singular the less is known. Consider, for instance, the singular equation (1.3). It is known that any bounded solution is regular (see, for instance, the monograph [18]), but Harnack inequality, the existence of potential and the boundedness of any weak solution happen only in the supercritical range $p > \frac{2N}{N+2}$, with N standing for the dimension of the space domain (see, for instance, the monograph [24]) and these phenomena look like to be highly correlated (see, for instance, the review papers [27] and [28]). When p is in the subcritical range ($1 < p \leq \frac{2N}{N+2}$) only some suitable $L^r - L^\infty$ estimates can be proved (see [22]) and the Harnack estimates are degenerating in a very weak form (see [2,22,23] and [35]). The same results hold for the fast diffusion equation (1.2) (for the subcritical case see also [3]) and for doubly nonlinear parabolic Eqs. (1.1) with $2 < m+p < 3$ [34] being the threshold $m = \frac{N-2}{N}$, for Eq. (1.2), and $m + p = 3 - \frac{p}{N}$, for Eq. (1.1). The doubly nonlinear case is not yet totally understood, although many results are available (see, for instance [32–34,37–39] and [40]). The case $m+p = 3$, the so called Trudinger's equation, is still object of intensive studies (see, for instance [42]). The regularity theory is fully developed only for the supercritical case (for an overview, see [52]).

Regarding the case $m + p \leq 2$ not very much is known. The most important case concerns the Fast Diffusion Equation which becomes $u_t = \Delta(\log u)$ when $m \to 0$. This equation comes from differential geometry when describing Ricci flow on complete $\mathbb{R}^2$ (see, for instance, [5,36], [11,14,54] and [15]). In [19] the Cauchy problem, with a nonnegative initial datum $u_o \in L^1_{loc}(\mathbb{R}^2)$, was deeply analyzed obtaining necessary and sufficient conditions for the existence of the solutions. The case $N \geq 3$ was widely studied by Daskalopoulos and del Pino, and they proved the existence of radial solution with prescribed growth to infinity and the nonexistence of integrable solutions even with an integrable initial datum (see, for instance, [13]). Not very much is known about bounded domains (see, for instance, [44]). Under a suitable definition, it is possible to show the existence of noncontinuous solutions [48]. Such kind of equations have applications in kinetic theory of gases (see, for instance, [6,17] and [45]), in thin film dynamics (see, for instance, [4,16] and [53]), and in general diffusion processes with singular diffusivity (see, for instance, [46]). For more details about this equation see, for instance, [21], [29,30,49] and [50]. The general case $m + p = 2$ and $p > 1$, was studied in [31]. We stress out that not much is done for $m < 0$. This problem was first faced in [47] (where the nonexistence of solution is studied) and [12] (where also the existence is considered). Several papers stemmed from these two foundational papers. Among them we quote the recent paper [41] where wellposedness and large-time behavior for a class of weighted ultrafast diffusion equations is studied. In that paper one can find not only updated references but also recent applications of these equations for problems related to quantization for probability measures.

However, to our knowledge, there is only a paper where equation (1.2) is studied for $m < 0$ with DeGiorgi regularity approach: in [20] where the authors considered the case $-1 < m < 0$. We recall that the value $m = -1$ represents a critical threshold because under that value the energy estimates (on which the DeGiorgi method is based) reverse the sign. More precisely, in that paper uniform local upper and lower bounds were derived for the solutions u_m to the equations $u_t = \Delta(u^m)$, for $-1 < m < 1$ and $m \neq 0$. It was also proved the stability of the case $m = 0$ i.e. the logarithmic case: the authors proved that, when $m \to 0$, the solutions $u_m \to u_o$ where u_o is the solution of (1.5) with $p = 2$.

In this paper, we start the analysis of very singular doubly nonlinear equation, i.e. when $3-p < m+p < 2$ and $m\langle 1,\ p\rangle 1$. Besides their intrinsic mathematical interest, such kind of equations appear in recent models of morphogenesis (see [1] and [10]); in [7,8] and [9] one can find results regarding the wellposedness of the model as well as qualitative properties of the equations. Here we prove suitable $L^r - L^\infty$ estimates, a result of expansion of positivity and Harnack estimates. The proofs of many of the results hold also in the case $2 < m + p < 3$ and $m\langle 1, p\rangle 1$. In particular, we prove a *weak* form of Harnack, for which the derived constant depends on the ratio of some integral norms of the weak solution u (see Section 7). One should not be surprised by this dependence as it is now well-known that, within the critical and subcritical ranges $2 < m+p \leq 3 - \frac{p}{N}$, for $p < N$, no Harnack inequalities hold with an absolute constant. Another interesting feature, which we are willing to address in a nearby future, concerns the regularity of u since the properties we present are somehow related to regularity results.

For the sake of simplicity we prove our results only for the prototype equations but, working with the prototype equations is immaterial and the same results hold for equations with variable coefficients (we will be more precise in the next sections). All these results are proved having, as starting point, the techniques introduced in [20] and in [31]. In a forthcoming paper we will prove a stability result: i.e. when $m + p \to 2$ the solutions to (1.4) converge to the solution of (1.5).

2. Setting the framework

Let Ω be an open set of $\mathbf{R}^N$, for $N \geq 1$, and T a positive real number, then define $\Omega_T = \Omega \times (0, T]$. Let p and m verify

$$p > 1, \quad m < 1 \quad \text{and} \quad 3 - p < m + p \neq 2 < 3,$$

which corresponds to take $\gamma = \dfrac{m+p-2}{p-1}$ in the range $0 < |\gamma| < 1$.

We say that a nonnegative function u is a local weak sub(super)solution to

$$u_t - \operatorname{div}\big(u^{m-1}|Du|^{p-2}Du\big) = 0, \tag{E}$$

if

$$u \in C(0,T; L^2_{\text{loc}}(\Omega)), \quad u^\gamma \in L^p_{\text{loc}}(0,T; W^{1,p}_{\text{loc}}(\Omega)), \quad u^{\frac{m-1}{p-1}}|Du| \in L^p_{\text{loc}}(\Omega_T)$$

and

$$\int_K u\psi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \Big(-u\psi_t + u^{m-1}|Du|^{p-2}Du \cdot D\psi\Big)dxdt \leq (\geq) 0 \tag{2.1}$$

for every compact set $K \subset \Omega$, for every sub-interval $[t_1, t_2] \subset (0, T]$ and for all nonnegative smooth test functions ψ.

We say that u is a local weak solution if it is both a local weak sub and supersolution.

In the sections to come, we consider the extra regularity assumption

$$w \in L^{\infty}_{\text{loc}}(0,T;L^{\alpha}_{\text{loc}}(\Omega)), \qquad \text{for some } \alpha > N+p, \tag{2.2}$$

where

$$w = \frac{u^{\gamma}-1}{\gamma}.$$

This assumption on w allows us to get good energy estimates and from them derive a DeGiorgi type Lemma, crucial to the expansion of positivity. We refer to [44] for some comments regarding this assumption when working with the Dirichlet problem related to (1.5) for $p=2$.

It is well known that the time derivative u_t, in general, only makes sense in a distributional context so, and in order to use u as test function, one can overcome this difficulty by considering, for instance, the regularization

$$u^{\star}(x,t) = \frac{1}{\sigma}\int_0^t e^{\frac{s-t}{\sigma}} u(x,s)\, ds, \quad \sigma > 0, \tag{2.3}$$

used by Kinnunen and Lindqvist [43] (see Lemma 2.11 for regularity results on this average) when studying several properties for the porous medium equation, and then pass to the limit as $\sigma \to 0$ to recover u. This average only needs to consider values of $u(x,t)$ taken in Ω_T, it is defined at each point, for continuous or bounded and semicontinuous functions u, and verifies

$$(u^{\star})_t = \frac{u-u^{\star}}{\sigma}.$$

The average inequality for a nonnegative weak supersolution u in Ω_T to Eq. (E) is the following: for every compact set $K \subset \Omega$, for every sub-interval $[t_1,t_2] \subset (0,T]$

$$\int_{t_1}^{t_2}\int_K \Big((u^{\star})_t\psi + \Big(u^{m-1}|Du|^{p-2}Du\Big)^{\star}\cdot D\psi\Big)dxdt \geq 0 \tag{2.4}$$

for all nonnegative test functions $\psi \in L^q_{\text{loc}}(0,T;L^q(K)) \cap L^p_{\text{loc}}(0,T;W^{1,p}_0(K))$, where $q>1$ will be fixed according to the regularity of u: when deriving the energy estimates (3.1) we take $q=2$ whereas for the L^r estimates we consider $q=r$.

Let us now introduce some notation regarding the sets (cubes and cylinders) we will be working with. We denote by $K_{\rho}(y)$ the cube of $\mathbf{R}^N$ centered at y with edge 2ρ. If $y=0$, we simply write K_{ρ} instead of $K_{\rho}(0)$. Let $\theta > 0$. We define the cylinders

$$Q^-_{\rho}(\theta) = K_{\rho} \times (-\theta\rho^p, 0], \qquad Q^+_{\rho}(\theta) = K_{\rho}\times(0,\theta\rho^p]$$

and, for $(y,s) \in \mathbf{R}^N \times \mathbf{R}$,

$$(y,s)+Q^-_{\rho}(\theta) = K_{\rho}(y)\times(s-\theta\rho^p, s], \qquad (y,s)+Q^+_{\rho}(\theta) = K_{\rho}(y)\times(s, s+\theta\rho^p].$$

3. Energy estimates

The energy estimates satisfied by the weak (super)solutions are one of the main tools in regularity theory, and are the scope of the following result.

Proposition 3.1. *Let u be a nonnegative, local weak supersolution to (E) in Ω_T. Then for every cylinder $(y,s)+Q^-_{\rho}(\theta) \subset \Omega_T$, $k>0$ and every nonnegative smooth cutoff function ζ vanishing on the boundary of*

$K_\rho(y)$, it holds

$$
\begin{aligned}
&\sup_{s-\theta\rho^p<t\le s}\int_{K_\rho(y)}(u-k)_-^2\zeta^p(x,t)dx+k^{m-1}\iint_{(y,s)+Q_\rho^-(\theta)}|D[(u-k)_-\zeta]|^p dxdt\\
&\quad\le\frac{2^p}{\gamma+1}k\left\{\int_{K_\rho(y)}(u-k)_-\zeta^p(x,s-\theta\rho^p)dx+p\iint_{(y,s)+Q_\rho^-(\theta)}(u-k)_-\zeta|\zeta_t|dxdt\right\}\\
&\qquad+2^{p-1}k^{m-1}\iint_{(y,s)+Q_\rho^-(\theta)}(u-k)_-^p|D\zeta|^p dxdt\\
&\qquad+2^{2p-1}(p-1)^{p-1}k^{-\frac{m-1}{p-1}}\iint_{(y,s)+Q_\rho^-(\theta)}\Big(\frac{u^\gamma-k^\gamma}{\gamma}\Big)_-^p|D\zeta|^p\ dxdt.
\end{aligned}\tag{3.1}
$$

Analogous estimates hold in the cylinder $(y,s)+Q_\rho^+(\theta)\subset\Omega_T$.

Proof. Without loss of generality, we assume $(y,s)=(0,0)$. Due to the regularity constraints on u, the time derivative u_t may not exist in the Sobolev's sense and we have to consider the regularized function $u^\star$ defined in (2.3).

Now, in (2.4) we consider the integration over $Q_\tau=K_\rho\times(-\theta\rho^p,\tau]$, where $-\theta\rho^p<\tau\le 0$, and take

$$\psi=-\left(\frac{u_\varepsilon^\gamma-k^\gamma}{\gamma}\right)_-\zeta^p,$$

where ζ is a nonnegative smooth cutoff function vanishing on the boundary of K_ρ and u_ε is defined as follows

$$u_\varepsilon=\begin{cases}u & \text{if } u\ge\varepsilon\\ \varepsilon & \text{if } u<\varepsilon,\end{cases}\tag{3.2}$$

being $0<\varepsilon<k$. Notice that, since the function $f(s)=\frac{s^\gamma}{\gamma}$ is increasing in $[0,+\infty)$ for every $0<|\gamma|<1$, we have

$$\frac{u_\varepsilon^\gamma}{\gamma}=\left(\frac{u^\gamma-\varepsilon^\gamma}{\gamma}\right)_++\frac{\varepsilon^\gamma}{\gamma}$$

which implies that $|D(\frac{u_\varepsilon^\gamma}{\gamma})|$ belongs to $L^p_{\rm loc}(\Omega_T)$. We also have $\psi\in L^2_{loc}(\Omega_T)$ due to (2.2), therefore the function ψ is an admissible test function and we get

$$-\iint_{Q_\tau}(u^\star)_t\left(\frac{u_\varepsilon^\gamma-k^\gamma}{\gamma}\right)_-\zeta^p-\iint_{Q_\tau}(u^{m-1}|Du|^{p-2}Du)^\star\cdot D\left[\left(\frac{u_\varepsilon^\gamma-k^\gamma}{\gamma}\right)_-\zeta^p\right]dxdt\le 0.$$

We observe that u_ε converge to u a.e. as $\varepsilon\to 0$. Then, by dominated convergence,

$$-\iint_{Q_\tau}(u^\star)_t\left(\frac{u_\varepsilon^\gamma-k^\gamma}{\gamma}\right)_-\zeta^p dxdt\longrightarrow-\iint_{Q_\tau}(u^\star)_t\left(\frac{u^\gamma-k^\gamma}{\gamma}\right)_-\zeta^p dxdt$$

The last term verifies

$$
\begin{aligned}
-\iint_{Q_\tau}(u^\star)_t\left(\frac{u^\gamma-k^\gamma}{\gamma}\right)_-\zeta^p dxdt&=-\iint_{Q_\tau}(u^\star)_t\left(\frac{k^\gamma-(u^\star)^\gamma-u^\gamma+(u^\star)^\gamma}{\gamma}\right)\zeta^p\chi_{[\frac{u^\gamma}{\gamma}<\frac{k^\gamma}{\gamma}]}dxdt\\
&=\iint_{Q_\tau}\partial_t\left(\int_{u^\star}^k\left(\frac{k^\gamma-s^\gamma}{\gamma}\right)\ ds\right)\zeta^p\chi_{[\frac{u^\gamma}{\gamma}<\frac{k^\gamma}{\gamma}]}dxdt\\
&\quad-\iint_{Q_\tau}(u^\star)_t\Big(\frac{(u^\star)^\gamma-u^\gamma}{\gamma}\Big)\zeta^p\chi_{[\frac{u^\gamma}{\gamma}<\frac{k^\gamma}{\gamma}]}dxdt\\
&\ge\iint_{Q_\tau}\partial_t\left(\int_{u^\star}^k\left(\frac{k^\gamma-s^\gamma}{\gamma}\right)\ ds\right)\zeta^p\chi_{[\frac{u^\gamma}{\gamma}<\frac{k^\gamma}{\gamma}]}dxdt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{K_\rho} \left(\int_{u^\star(x,\tau)}^{k} \left(\frac{k^\gamma - s^\gamma}{\gamma} \right) \, ds \right) \zeta^p(x,\tau) \chi_{[u<k]} dx \\
&- \int_{K_\rho} \left(\int_{u^\star(x,-\theta\rho^p)}^{k} \left(\frac{k^\gamma - s^\gamma}{\gamma} \right) \, ds \right) \zeta^p(x,-\theta\rho^p) \chi_{[u<k]} dx \\
&- p \iint_{Q_\tau} \left(\int_{u^\star}^{k} \left(\frac{k^\gamma - s^\gamma}{\gamma} \right) \, ds \right) \zeta^{p-1} \zeta_t \chi_{[u<k]} dx dt.
\end{aligned}
$$

The estimate was obtained due to the fact that

$$
(u^\star)_t \frac{(u^\star)^\gamma - u^\gamma}{\gamma} = \frac{u - u^\star}{\sigma} \frac{(u^\star)^\gamma - u^\gamma}{\gamma} < 0
$$

since $f(s) = \dfrac{s^\gamma}{\gamma}$ is an increasing function. We then pass to the limit as $\sigma \to 0$ to obtain the inferior bound

$$
\int_{K_\rho \times \{\tau\}} \left(\int_u^k \left(\frac{k^\gamma - s^\gamma}{\gamma} \right) \, ds \right) \zeta^p \chi_{[u<k]} dx - \int_{K_\rho \times \{-\theta\rho^p\}} \left(\int_u^k \left(\frac{k^\gamma - s^\gamma}{\gamma} \right) \, ds \right) \zeta^p \chi_{[u<k]} dx
$$

$$
-p \iint_{Q_\tau} \int_u^k \left(\frac{k^\gamma - s^\gamma}{\gamma} \right) \, ds \zeta^{p-1} \zeta_t \chi_{[u<k]} dx dt.
$$

In order to obtain upper and lower bounds to $\int_u^k \left(\frac{k^\gamma - s^\gamma}{\gamma} \right) ds$, on the set $[u < k] \cap Q_\tau$, we proceed as follows. On the one hand and since $\gamma > -1$ and $f(s) = s^\gamma/\gamma$ is an increasing function,

$$
\begin{aligned}
\int_u^k \left(\frac{k^\gamma - s^\gamma}{\gamma} \right) ds &= \frac{k^{\gamma+1}}{\gamma+1} - \frac{k^\gamma}{\gamma} u + \frac{u^\gamma}{\gamma} \frac{u}{\gamma+1} \\
&< \frac{k^{\gamma+1}}{\gamma+1} - \frac{k^\gamma}{\gamma} u \left(1 - \frac{1}{\gamma+1} \right) = \frac{k^\gamma}{\gamma+1} (u-k)_- \ .
\end{aligned}
$$

On the other hand and since $\gamma < 1$,

$$
\frac{k^\gamma - s^\gamma}{\gamma} = \int_s^k \xi^{\gamma-1} \, d\xi \geq k^{\gamma-1}(k-s)_+
$$

and then

$$
\int_u^k \left(\frac{k^\gamma - s^\gamma}{\gamma} \right) ds \geq k^{\gamma-1} \int_u^k (k-s)_+ \, ds = \frac{k^{\gamma-1}}{2} (u-k)_-^2 .
$$

Therefore the parabolic term can be estimated from below by

$$
\begin{aligned}
&\frac{k^{\gamma-1}}{2} \int_{K_\rho} (u-k)_-^2 \zeta^p(x,\tau) dx - \frac{k^\gamma}{\gamma+1} \int_{K_\rho} (u-k)_- \zeta^p(x,-\theta\rho^p) dx \\
&\quad - p \frac{k^\gamma}{\gamma+1} \iint_{Q_\tau} (u-k)_- \zeta^{p-1} |\zeta_t| dx dt.
\end{aligned}
$$

Concerning the elliptic term we get

$$
\begin{aligned}
&- \iint_{Q_\tau} (u^{m-1}|Du|^{p-2}Du)^\star \cdot D \left[\left(\frac{u_\varepsilon^\gamma - k^\gamma}{\gamma} \right)_- \zeta^p \right] dx dt \\
&\quad = - \iint_{Q_\tau} (u^{m-1}|Du|^{p-2}Du)^\star \cdot D \left[\left(\frac{u_\varepsilon^\gamma - k^\gamma}{\gamma} \right)_- \right] \zeta^p dx dt \\
&\quad\quad - p \iint_{Q_\tau} (u^{m-1}|Du|^{p-2}Du)^\star \cdot D\zeta \left(\frac{u_\varepsilon^\gamma - k^\gamma}{\gamma} \right)_- \zeta^{p-1} dx dt = I_1 + I_2.
\end{aligned}
$$

We observe that

$$D\left(\frac{u_\varepsilon^\gamma - k^\gamma}{\gamma}\right)_- = D\left(\frac{u^\gamma - k^\gamma}{\gamma}\right)_- \chi_{[\varepsilon<u<k]} = -u^{\frac{m-1}{p-1}} Du\ \chi_{[\varepsilon<u<k]},$$

thereby as $\varepsilon \to 0$

$$I_1 = \iint_{Q_\tau} (u^{m-1}|Du|^{p-2}Du)^\star \cdot u^{\frac{m-1}{p-1}} Du\ \chi_{[\varepsilon<u<k]}\zeta^p dxdt$$
$$\longrightarrow \iint_{Q_\tau} (u^{m-1}|Du|^{p-2}Du)^\star \cdot u^{\frac{m-1}{p-1}} Du\ \chi_{[u<k]}\zeta^p dxdt$$

since $u^{\frac{m-1}{p-1}}|Du| \in L^p_{\mathrm{loc}}(\Omega_T)$. Taking the limit as $\sigma \to 0$, using the properties stated in Lemma 2.11 in [43] we obtain

$$\iint_{Q_\tau} u^{\frac{p(m-1)}{p-1}}|D(u-k)_-|^p \zeta^p dxdt.$$

The estimate of I_2 is easier, as we do not have to differentiate u^γ hence, letting first ε and then σ to 0, we are led to

$$\begin{aligned}
&-p\iint_{Q_\tau} u^{m-1}|Du|^{p-2}Du\cdot D\zeta\left(\frac{u^\gamma-k^\gamma}{\gamma}\right)_- \zeta^{p-1}dxdt\\
&\geq -p\iint_{Q_\tau} u^{m-1}|D(u-k)_-|^{p-1}\zeta^{p-1}|D\zeta|\left(\frac{u^\gamma-k^\gamma}{\gamma}\right)_- dxdt\\
&\geq -(p-1)\delta^{-\frac{p}{p-1}}\iint_{Q_\tau} u^{\frac{p(m-1)}{p-1}}|D(u-k)_-|^p\zeta^p dxdt\\
&\quad -\delta^p\iint_{Q_\tau}|D\zeta|^p\left(\frac{u^\gamma-k^\gamma}{\gamma}\right)_-^p dxdt,
\end{aligned}$$

where we applied Young's inequality with δ. Therefore the elliptic term is estimated from below by

$$\left(1-(p-1)\delta^{-\frac{p}{p-1}}\right)\iint_{Q_\tau} u^{\frac{p(m-1)}{p-1}}|D(u-k)_-|^p\zeta^p dxdt - \delta^p\iint_{Q_\tau}|D\zeta|^p\left(\frac{u^\gamma-k^\gamma}{\gamma}\right)_-^p dxdt.$$

Since $m<1$, we deduce $u^{\frac{p(m-1)}{p-1}} \geq k^{\frac{p(m-1)}{p-1}}$, whenever $u \leq k$. Finally, choosing $\delta = (2(p-1))^{\frac{p-1}{p}}$ and taking the supremum over τ, we obtain (3.1). $\square$

4. De Giorgi-type lemma

Let us now consider u to be a nonnegative, locally bounded, local, weak supersolution to (E) in Ω_T and let $\theta > 0$ be such that the cylinder

$$(y,s) + Q^-_{8\rho}(\theta) = K_{8\rho}(y) \times (s-\theta(8\rho)^p, s]$$

is contained in Ω_T. Take $M > 0$ such that

$$\operatorname*{essinf}_{(y,s)+Q^-_{8\rho}(\theta)} u < M \leq \operatorname*{ess\,sup}_{(y,s)+Q^-_{8\rho}(\theta)} u. \tag{4.1}$$

We remark that in [33], for $2 < m+p < 3$, a DeGiorgi type Lemma was derived not considering the extra regularity assumption (2.2). However it is also useful within this range when performing the limiting procedure $m+p \to 2$.

Set

$$L_\gamma = \left[\sup_{t_0-\theta_0(8\rho)^p<t<t_0} \fint_{K_{8\rho}(x_0)} \left(\frac{M^\gamma - u^\gamma}{\gamma M^\gamma}\right)_+^\alpha dx\right]^{\frac{1}{\alpha}} \tag{4.2}$$

and finally take $\Lambda_\gamma = \max\{1, L_\gamma\}$.

Lemma 4.1. *For every ξ and a in the interval $(0,1)$, there exists a positive number ν, depending on $M,\theta,\xi,a,N,p,\Lambda_\gamma$ and independent of γ such that if*

$$|[u \le \xi M] \cap (y,s) + Q^-_{2\rho}(\theta)| \le \nu |Q^-_{2\rho}(\theta)|$$

then

$$u \ge a\xi M \qquad a.e.\ in\ (y,s) + Q^-_\rho(\theta).$$

Proof. We limit ourselves to the case $(y,s) = (0,0)$, which is admissible via a translation argument. Introduce the decreasing sequences of numbers

$$\rho_n = \rho + \frac{\rho}{2^n} \in (\rho, 2\rho], \qquad k_n = \xi_n M, \quad \text{where} \quad \xi_n = a\xi + \frac{1-a}{2^n}\xi \in (a\xi, \xi]$$

and construct the sequences of nested cubes and cylinders

$$K_n = K_{\rho_n}, \qquad Q_n = K_n \times (-\theta\rho_n^p, 0],$$

for $n = 0,1,2,\ldots$, over which we define the cutoff function $\zeta(x,t) = \zeta_1(x)\zeta_2(t)$ verifying

$$\zeta_1 = \begin{cases} 1 & \text{in} \quad K_{n+1} \\ 0 & \text{in} \quad \mathbf{R}^N \setminus K_n \end{cases} \qquad |D\zeta_1| \le \frac{1}{\rho_n - \rho_{n+1}} = \frac{2^{n+1}}{\rho},$$

and

$$\zeta_2 = \begin{cases} 0 & \text{if} \quad t \le -\theta\rho_n^p \\ 1 & \text{if} \quad t \ge -\theta\rho_{n+1}^p \end{cases} \qquad 0 \le (\zeta_2)_t \le \frac{2^{p(n+1)}}{\theta\rho^p}.$$

For the above choices, the energy estimates (3.1) now read

$$\sup_{-\theta\rho_n^p < t \le 0} \int_{K_n} (u-k_n)_-^2 \zeta^p(x,t)dx + k_n^{m-1} \iint_{Q_n} |D[(u-k_n)_-\zeta]|^p dx d\tau$$
$$\le c_p \frac{2^{pn}}{\rho^p} k_n^{m+p-1} \left\{ \left(1 + \frac{k_n^{3-m-p}}{(\gamma+1)\theta}\right)|A_n| + L_\gamma^p |A_n|^{1-\frac{p}{\alpha}} |Q_n|^{\frac{p}{\alpha}} \right\},$$

where $c_p = \max\{2^{3p-1}(p-1)^{p-1}, 2^{2p}p, 2^{2p-1}\}$ and $A_n = [u < k_n] \cap Q_n$. To estimate the last integral in the energy estimates, we start by applying Holder's inequality (recalling that $\alpha > p$), then we use the definition (4.2) of L_γ and finally we exploit the fact that $f(s) = (s^\gamma - u^\gamma)/(\gamma s^\gamma)$ is an increasing function; and we arrive at

$$k_n^{-\frac{m-1}{p-1}} \iint_{Q_n} \left(\frac{u^\gamma - k_n^\gamma}{\gamma}\right)_-^p |D\zeta|^p \, dxdt$$
$$\le k_n^{-\frac{m-1}{p-1}} \frac{2^{p(n+1)}}{\rho^p} \iint_{Q_n} \left(\frac{k_n^\gamma - u^\gamma}{\gamma}\right)^p \chi_{[u<k_n]} dxdt$$
$$\le \frac{2^{p(n+1)}}{\rho^p} k_n^{-\frac{m-1}{p-1}+\gamma p} \left(\sup_{(-\theta\rho_n^p, 0]} \fint_{K_n} \left(\frac{k_n^\gamma - u^\gamma}{\gamma k_n^\gamma}\right)_+^\alpha dx \right)^{p/\alpha} |A_n|^{1-\frac{p}{\alpha}} |Q_n|^{\frac{p}{\alpha}}$$
$$= \frac{2^{p(n+1)}}{\rho^p} k_n^{m+p-1} \Lambda_\gamma^p |A_n|^{1-\frac{p}{\alpha}} |Q_n|^{\frac{p}{\alpha}}.$$

The cases $p \ge 2$ and $1 < p < 2$ are studied separately. At first we consider $p \ge 2$.

Observe that $(u-k_n)_-^2 \ge k_n^{2-p}(u-k_n)_-^p$, hence from the previous estimate we arrive at

$$\sup_{-\theta\rho_n^p < t \le 0} \int_{K_n} (u-k_n)_-^p \zeta^p(x,t)dx + k_n^{m+p-3} \iint_{Q_n} |D[(u-k_n)_-\zeta]|^p dx d\tau$$
$$\le c_p \frac{2^{pn}}{\rho^p} k_n^{m+2p-3} \left\{ \left(1 + \frac{k_n^{3-m-p}}{(\gamma+1)\theta}\right)|A_n| + L_\gamma^p |A_n|^{1-\frac{p}{\alpha}} |Q_n|^{\frac{p}{\alpha}} \right\},$$

Now we need to recall the definition of A_n, apply consecutively Hölder's inequality, then apply the Sobolev embedding (see Proposition 3.1, chapter I, in [18]) and finally use the previous estimate to get

$$\begin{aligned}\left(\frac{1-a}{2^{n+1}}\right)^p (\xi M)^p|A_{n+1}| &= (k_n-k_{n+1})^p|A_{n+1}|\\ &\le \iint_{Q_{n+1}} (u-k_n)_-^p \chi_{[u<k_{n+1}]}dxd\tau\\ &\le \iint_{Q_{n+1}} (u-k_n)_-^p dxd\tau\\ &\le \left(\iint_{Q_n} [(u-k_n)_-\zeta]^{p\frac{N+p}{N}}dxd\tau\right)^{\frac{N}{N+p}} |A_n|^{\frac{p}{N+p}}\\ &\le C_{N,p}\left(\iint_{Q_n} |D[(u-k_n)_-\zeta]|^p dxd\tau\right)^{\frac{N}{N+p}}\\ &\quad\times\left(\sup_{-\theta\rho_n^p<t\le 0}\int_{K_n} |(u-k_n)_-\zeta|^p(x,t)dx\right)^{\frac{p}{N+p}}|A_n|^{\frac{p}{N+p}}\\ &\le C_0\frac{2^{pn}}{\rho^p}\, k_n^{(N+m+2p-3)\frac{p}{N+p}}\left\{\left(1+\frac{k_n^{3-m-p}}{(\gamma+1)\theta}\right)|A_n| + L_\gamma^p|A_n|^{1-\frac{p}{\alpha}}|Q_n|^{\frac{p}{\alpha}}\right\}|A_n|^{\frac{p}{N+p}}\end{aligned}$$

where $C_0 = C_{N,p}\times c_p$ depends only upon N and p.

Having in mind that $|\gamma|<1$ and therefore $m<1$, $m+p<3$ and $k_n\le \xi M$, we can go even further and get the upper bound

$$\begin{aligned}\left(\frac{1-a}{2^{n+1}}\right)^p (\xi M)^p|A_{n+1}| &\le C_0\frac{2^{pn}}{(\xi M)^{\frac{p(3-m-p)}{N+p}}\rho^p}(\xi M)^p\left(1+\frac{(\xi M)^{3-m-p}}{(\gamma+1)\theta}\right)\Lambda_\gamma^p\\ &\quad\times\left\{|A_n|^{1+\frac{p}{N+p}} + |A_n|^{1+\frac{p}{N+p}-\frac{p}{\alpha}}|Q_n|^{\frac{p}{\alpha}}\right\}\end{aligned}$$

and then, being $Y_n := \frac{|A_n|}{|Q_n|}$, we get

$$Y_{n+1}\le \tilde{C}_o\frac{b_o^n}{(1-a)^p}\left(\frac{\theta}{(\xi M)^{3-m-p}}\right)^{\frac{p}{p+N}}\left(1+\frac{(\xi M)^{3-m-p}}{(\gamma+1)\theta}\right)\Lambda_\gamma^p Y_n^{1+\beta}\ ,$$

for $\tilde{C_o} = 2^{N+3p+1}C_o$ (just depending on N and p), $b_o = 2^{2p}>1$ and $\beta = \frac{p}{N+p}-\frac{p}{\alpha}>0$.

If $1<p<2$, we start by considering the average level $\tilde{k}_n = \frac{k_n+k_{n+1}}{2}$ and observing that

$$\begin{aligned}\int_{K_n}(u-k_n)_-^2\zeta^p(x,t)dx &\ge \int_{K_n\cap[u<\tilde{k}_n]}(u-k_n)_-^{2-p}(u-k_n)_-^p\zeta^p(x,t)dx\\ &\ge\left(\frac{(1-a)\xi M}{2^{n+2}}\right)^{2-p}\int_{K_n}(u-\tilde{k}_n)_-^p\zeta^p(x,t)dx\end{aligned}$$

and also, since $\iint_{Q_n}|D[(u-k_n)_-\zeta]|^p dxd\tau \ge \iint_{Q_n}|D[(u-\tilde{k}_n)_-\zeta]|^p dxd\tau$, $k_n\le\xi M$ and $3-m-p>0$, we have

$$\begin{aligned}\iint_{Q_n}|D[(u-\tilde{k}_n)_-\zeta]|^p dxd\tau &\le c_p\frac{2^{pn}}{\rho^p}k_n^p\left\{\left(1+\frac{k_n^{3-m-p}}{(\gamma+1)\theta}\right)|A_n| + L_\gamma^p|A_n|^{1-\frac{p}{\alpha}}|Q_n|^{\frac{p}{\alpha}}\right\}\\ &\le c_p\frac{2^{pn}}{\rho^p}(\xi M)^p\left\{\left(1+\frac{(\xi M)^{3-m-p}}{(\gamma+1)\theta}\right)|A_n| + L_\gamma^p|A_n|^{1-\frac{p}{\alpha}}|Q_n|^{\frac{p}{\alpha}}\right\}.\end{aligned}$$

Then, arguing as before in the case $p > 2$,

$$
\begin{aligned}
\left(\frac{1-a}{2^{n+2}}\right)^p (\xi M)^p |A_{n+1}| &= (\tilde{k}_n - k_{n+1})^p |A_{n+1}| \\
&\leq \iint_{Q_{n+1}} (u-\tilde{k}_n)_-^p \chi_{[u<k_{n+1}]} dx d\tau \\
&\leq \iint_{Q_{n+1}} (u-\tilde{k}_n)_-^p \, dx d\tau \\
&\leq \left(\iint_{Q_n} [(u-\tilde{k}_n)_-\zeta]^{p\frac{N+p}{N}} dx d\tau\right)^{\frac{N}{N+p}} |A_n|^{\frac{p}{N+p}} \\
&\leq C \left(\iint_{Q_n} |D[(u-\tilde{k}_n)_-\zeta]|^p dx d\tau\right)^{\frac{N}{N+p}} \\
&\quad \times \left(\sup_{-\theta\rho_n^p < t \leq 0} \int_{K_n} |(u-\tilde{k}_n)_-\zeta|^p (x,t) dx\right)^{\frac{p}{N+p}} |[u<\tilde{k}_n] \cap Q_n|^{\frac{p}{N+p}} \\
&\leq C_{N,p} \left(c_p \frac{2^{pn}}{\rho^p} (\xi M)^p \left\{ \left(1 + \frac{(\xi M)^{3-m-p}}{(\gamma+1)\theta}\right) |A_n| + \Lambda_\gamma^p |A_n|^{1-\frac{p}{\alpha}} |Q_n|^{\frac{p}{\alpha}} \right\}\right)^{\frac{N}{N+p}} \\
&\quad \times \left(\sup_{-\theta\rho_n^p < t \leq 0} \int_{K_n} |(u-\tilde{k}_n)_-\zeta|^p (x,t) dx\right)^{\frac{p}{N+p}} |A_n|^{\frac{p}{N+p}}.
\end{aligned}
$$

Combining the two previous estimates we arrive at

$$
Y_{n+1} \leq \tilde{C}_1 b_1^n \left(\frac{\theta}{(\xi M)^{3-m-p}}\right)^{\frac{p}{N+p}} \left(\frac{1}{1-a}\right)^{\frac{p(N+2)}{N+p}} \Lambda_\gamma^p \left(1 + \frac{(\xi M)^{3-m-p}}{(\gamma+1)\theta}\right) Y_n^{1+\beta}
$$

where $\tilde{C}_1 = C_{N,p} \times c_p \times 2^{N-1+\frac{2(2-p)p}{N+p}}$, $b_1 = 2^{\frac{(2-p)p}{N+p}} > 1$ and β the same as before.

So, in both cases, by choosing conveniently $C = C(N,p)$, we have a recursive algebraic estimate of the type

$$
Y_{n+1} \leq \frac{C}{(1-a)^{\frac{p(N+2)}{N+p}}} \Lambda_\gamma^p \left(1 + \frac{(\xi M)^{3-m-p}}{(\gamma+1)\theta}\right) \left(\frac{\theta}{(\xi M)^{3-m-p}}\right)^{\frac{p}{N+p}} 2^{2pn} Y_n^{1+\beta},
$$

therefore, from a fast geometric convergence result (see Lemma 4.1, chap.1, in [18]), one has $Y_n \to 0$, as $n \to \infty$, if

$$
Y_0 \leq \left\{ \frac{C}{(1-a)^{\frac{p(N+2)}{N+p}}} \Lambda_\gamma^p \left(1 + \frac{(\xi M)^{3-m-p}}{(\gamma+1)\theta}\right) \left(\frac{\theta}{(\xi M)^{3-m-p}}\right)^{\frac{p}{N+p}} \right\}^{-\frac{1}{\beta}} 2^{-\frac{2p}{\beta^2}}.
$$

By taking

$$
\nu = \left(\frac{(1-a)^{\frac{p(N+2)}{N+p}}}{C 2^{\frac{2p}{\beta}}} \frac{1}{\Lambda_\gamma^p} \left(\frac{(\gamma+1)\theta}{(\gamma+1)\theta + (\xi M)^{3-m-p}}\right) \left(\frac{(\xi M)^{3-m-p}}{\theta}\right)^{\frac{p}{N+p}} \right)^{\frac{1}{\beta}} \tag{4.3}
$$

the condition on Y_0 is verified and the proof is complete. □

5. L^r_{loc} And $L^r_{\text{loc}} - L^\infty_{\text{loc}}$ estimates

In this section we present and prove local estimates involving L^r and L^∞ norms within the singular range $3 - p < m + p \neq 2 < 3$, $m < 1$. Note that they were already derived for $2 < m + p < 3$ in [51] and in [32]; and for $m + p = 2$: with $p = 2$ in [16] and within the wider range $p > 1$ in [31]. So the truly missing case is

$m+p<2$, corresponding to $\gamma<0$. The proofs presented here work for all $0<|\gamma|<1$ and contain a precise trace of the various constants, which in particular are shown to be independent of γ.

Consider the extra regularity assumption

$$u \in L^{\infty}_{\text{loc}}(0,T; L^r_{\text{loc}}(\Omega)), \qquad \text{for } r > \max\left\{1, \frac{N}{p}\right\} \tag{5.1}$$

together with

$$\lambda_r = rp + N(m+p-3) > 0 \qquad\qquad r+m+p-3>0. \tag{5.2}$$

This assumption allows us to turn the qualitative information on the local boundedness of u into a quantitative estimate (see Theorems 5.3 and 5.5).

Proposition 5.1 (L^r_{loc} *Estimates Backwards in Time*)**.** *Let u be a nonnegative, locally bounded, local weak solution to (E) in Ω_T satisfying (5.1) for some r as in (5.2). Assume that the cylinder $K_{2\rho}(y)\times[s,t]$ is included in Ω_T. Then there exists a positive constant C, depending only upon m, p, N and r, such that*

$$\sup_{s\le\tau\le t}\int_{K_\rho(y)} u^r(x,\tau)dx \le C\left(\int_{K_{2\rho}(y)} u^r(x,s)dx + \left[\frac{(t-s)^r}{\rho^{\lambda_r}}\right]^{\frac{1}{3-m-p}}\right).$$

The expression of the constant C is explicitly given in the proof.

Proof. We follow the strategy of the proof of Proposition A.3.1 in [24]. Let $0\le\zeta(x)\le1$ be a smooth function defined in $K_{(1+\bar\sigma)\rho}(y)$, where $s<\tau\le t$ and $0<\bar\sigma<1$, such that

$$\zeta = 1 \text{ in } K_\rho(y), \quad \zeta=0 \text{ in } \Omega\setminus K_{(1+\bar\sigma)\rho}(y), \quad |D\zeta|\le\frac{1}{\bar\sigma\rho}.$$

Let $k>0$ be a constant to be chosen and q such that

$$\max\{r-1,1\}<q<r.$$

Consider the average inequality (2.4) over the cylinder $Q_\tau = K_{(1+\bar\sigma)\rho}(y)\times(s,\tau]$ and take $\varphi = f(u)\zeta^p$ as test function, where

$$f(u) = u^{r-1}\left(\frac{(u-k)_+}{u}\right)^q.$$

Then one computes $f'(u) = u^{r-2}\left(\frac{(u-k)_+}{u}\right)^{q-1}\left[(r-1)\left(\frac{u-k}{u}\right)+q\frac{k}{u}\right]$ which easily implies

$$(r-1)u^{r-2}\left(\frac{(u-k)_+}{u}\right)^q \le f'(u) \le qu^{r-2}\left(\frac{(u-k)_+}{u}\right)^{q-1}.$$

Finally, set

$$F(u) = \int_k^u f(v)dv.$$

We then get

$$\iint_{Q_\tau}(u^\star)_t f(u)\zeta^p dxdt + \iint_{Q_\tau}\left(u^{m-1}|Du|^{p-2}Du\right)^\star\cdot D(f(u)\zeta^p)dxdt \ge 0.$$

Both the integrals in the previous inequality are finite. This can be seen arguing as follows. By (5.1), u belongs to $L^r_{\text{loc}}(\Omega_T)$ and therefore $(u^\star)_t$ is in $L^r_{\text{loc}}(\Omega_T)$ and u^{r-1} is in $L^{r'}_{\text{loc}}(\Omega_T)$, being r' the conjugate exponent of r. Since $f(u)\le u^{r-1}$, this shows that the first integral converges. As for the second one, since $u^{m-1}|Du|^{p-1}$ belongs to $L^{p'}_{\text{loc}}(\Omega_T)$, where $\frac1p+\frac1{p'}=1$, it remains to prove that $|D(f(u)\zeta^p)|\in L^p_{\text{loc}}(\Omega_T)$. To this aim, we compute the derivative $D(f(u)\zeta^p) = f'(u)Du\,\zeta^p + f(u)D(\zeta^p)$, recall the bound for $f'(u)$ and notice that

$$u^{r-2} \le \begin{cases} M^{r-2} & \text{if } r \ge 2 \\ k^{r-2} & \text{if } r < 2 \,, \end{cases}$$

where $M = \|u\|_{L^\infty(K_{2\rho}(y)\times[s,t])}$, and that $|Du| \in L^p_{\rm loc}(\Omega_T)$ since $u^{\frac{m-1}{p-1}}|Du|$ belongs to $L^p_{\rm loc}(\Omega_T)$ and u is locally bounded.

Next, we estimate the two integrals separately.

As for the parabolic term we have

$$\begin{aligned}\iint_{Q_\tau} (u^\star)_t f(u)\zeta^p dxdt &= \iint_{Q_\tau} (u^\star)_t \left(f(u^\star) + f(u) - f(u^\star)\right)\zeta^p dxdt \\ &= \iint_{Q_\tau} \partial_t(F(u^\star))\zeta^p + \iint_{Q_\tau} \frac{u-u^\star}{\sigma}\left(f(u) - f(u^\star)\right)\zeta^p dxdt \\ &\ge \int_{K_{(1+\bar\sigma)\rho}(y)\times\{\tau\}} F(u^\star)\zeta^p\, dx - \int_{K_{(1+\bar\sigma)\rho}(y)\times\{s\}} F(u^\star)\zeta^p\, dx.\end{aligned}$$

The inequality relies on the fact that $f(u)$ is an increasing function and that the cutoff function ζ is independent of t.

Letting $\sigma \to 0$ we get the lower bound

$$\int_{K_{(1+\bar\sigma)\rho}(y)} F(u(x,\tau))\zeta^p(x)\, dx - \int_{K_{(1+\bar\sigma)\rho}(y)} F(u(x,s))\zeta^p(x)\, dx.$$

As for the elliptic term, passing to the limit as $\sigma \to 0$ we obtain

$$\begin{aligned}&\iint_{Q_\tau} u^{m-1}|Du|^p f'(u)\zeta^p dxdt + p\iint_{Q_\tau} u^{m-1}|Du|^{p-2}Du\cdot D\zeta\, f(u)\zeta^{p-1}dxdt \\ &\ge (r-1)\iint_{Q_\tau} u^{m-1}\frac{f(u)}{u}|Du|^p\zeta^p dxdt - p\iint_{Q_\tau} u^{m-1}|Du|^{p-1}\,|D\zeta|\,f(u)\zeta^{p-1}dxdt\end{aligned}$$

and then, by means of Young's inequality with δ, the lower bound is obtained

$$\begin{aligned}&(r-1)\left(1 - \frac{p-1}{(r-1)\delta^{p/(p-1)}}\right)\iint_{Q_\tau} u^{m-2}f(u)|Du|^p\zeta^p dxdt - \delta^p\iint_{Q_\tau} u^{m+p+r-3}|D\zeta|^p dxdt \\ &\ge -\left(2\,\frac{p-1}{r-1}\right)^{p-1}\iint_{Q_\tau} u^{m+p-2}f(u)|D\zeta|^p dxdt\end{aligned}$$

by taking $\delta = \left(2\,\frac{p-1}{r-1}\right)^{\frac{p-1}{p}}$ and discarding the nonnegative term containing the gradient of u.

Combining the previous estimates, recalling the definition of ζ and applying Hölder's inequality, we arrive at

$$\begin{aligned}\sup_{s<\tau\le t}\int_{K_\rho(y)} F(u(x,\tau))dx &\le \sup_{s<\tau\le t}\int_{K_{(1+\bar\sigma)\rho}(y)} F(u(x,\tau))\zeta^p(x)dx \\ &\le \int_{K_{(1+\bar\sigma)\rho}(y)} F(u(x,s))\zeta^p(x)dx + \left(2\,\frac{p-1}{r-1}\right)^{p-1}\iint_{Q_\tau} u^{m+p-2}f(u)|D\zeta|^p dxdt \\ &\le \int_{K_{(1+\bar\sigma)\rho}(y)} F(u(x,s))\zeta^p(x)dx + \frac{1}{\bar\sigma^p\rho^p}\left(2\,\frac{p-1}{r-1}\right)^{p-1}\iint_{Q_\tau} u^{m+p+r-3}dxdt,\end{aligned}$$

where the last inequality comes from the estimate $f(v) \le v^{r-1}$ which implies $F(u) \le u^r$. Then we obtain

$$\sup_{s<\tau\le t}\int_{K_\rho(y)} F(u(x,\tau))dx \le \int_{K_{(1+\bar\sigma)\rho}(y)} u^r(x,s)dx + \frac{1}{\bar\sigma^p\rho^p}\left(2\,\frac{p-1}{r-1}\right)^{p-1}\iint_{Q_\tau} u^{m+p+r-3}dxdt.$$

A lower bound for the left hand side can be obtained by the following argument. Write the classical inequality $z^a \le \delta + C_\delta z^b$, which holds all $z \ge 0$, $0 < a < b$ and $\delta > 0$, for $z = \left(\frac{v-k}{v}\right)$, $a = r-1$ and $b = q$ to get

$$\left(\frac{v-k}{v}\right)^{r-1} \le \delta + C_\delta \left(\frac{v-k}{v}\right)^{q},$$

where $C_\delta > 0$ depends on δ, r, q. Multiplying both sides by v^{r-1} and integrating from k to u we have

$$\frac{(u-k)^r}{r} \le \delta \, \frac{u^r - k^r}{r} + C_\delta F(u).$$

It follows that

$$u^r \le 2^{r-1}\Big((u-k)^r + k^r\Big) \le 2^{r-1}\delta \, (u^r - k^r) + 2^{r-1} r C_\delta F(u) + 2^{r-1}k^r$$

and therefore, by choosing $\delta = 2^{-r}$,

$$u^r \le (2^r - 1)k^r + \bar{C}F(u)$$

with $\bar{C} = \bar{C}(r,q)$. Integrating over $K_\rho(y) \cap [u > k]$

$$\int_{K_\rho(y)\cap[u>k]} u^r(x,s)dx \le \bar{C}\int_{K_\rho(y)} F(u(x,s))dx + (2^r-1)k^r|K_\rho|,$$

and then

$$\sup_{s<\tau\le t} \fint_{K_\rho(y)} u^r(x,\tau)dx \le \bar{C} \sup_{s<\tau\le t}\fint_{K_\rho(y)} F(u(x,\tau))dx + 2^r k^r.$$

Now we choose k according to

$$k^r = \frac{1}{2^{r+1}} \sup_{s<\tau\le t}\fint_{K_\rho(y)} u^r(x,\tau)dx,$$

which yields the desired lower bound

$$\sup_{s<\tau\le t}\fint_{K_\rho(y)} u^r(x,\tau)dx \le 2\bar{C} \sup_{s<\tau\le t}\fint_{K_\rho(y)} F(u(x,\tau))dx.$$

Then

$$\begin{aligned}
\sup_{s<\tau\le t}\int_{K_\rho(y)} u^r(x,\tau)dx &\le 2\bar{C}\int_{K_{(1+\bar\sigma)\rho}(y)} u^r(x,s)dx + \frac{2\bar{C}}{\bar\sigma^p\rho^p}\left(2\,\frac{p-1}{r-1}\right)^{p-1}\iint_{Q_\tau} u^{m+p+r-3}dxdt \\
&\le 2\bar{C}\int_{K_{(1+\bar\sigma)\rho}(y)} u^r(x,s)dx \\
&\quad + \frac{2\bar{C}}{\bar\sigma^p\rho^p}\left(2\,\frac{p-1}{r-1}\right)^{p-1}\left(\sup_{s<\tau\le t}\int_{K_{(1+\bar\sigma)\rho}(y)\times\{\tau\}} u^r dx\right)^{\frac{r+m+p-3}{r}} (t-s)^{\frac{r+m+p-3}{r}}|Q_t|^{1-\frac{r+m+p-3}{r}} \\
&\le 2\bar{C}\int_{K_{(1+\bar\sigma)\rho}(y)} u^r(x,s)dx \\
&\quad + 2^{p-1+2N(3-m-p)/r}\left(\frac{p-1}{r-1}\right)^{p-1}\frac{2\bar{C}}{\bar\sigma^p}\left(\frac{(t-s)^r}{\rho^{\lambda_r}}\right)^{\frac{1}{r}}\left(\sup_{s<\tau\le t}\int_{K_{(1+\bar\sigma)\rho}(y)\times\{\tau\}} u^r dx\right)^{\frac{r+m+p-3}{r}} \\
&\le 2\bar{C}\int_{K_{(1+\bar\sigma)\rho}(y)\times\{s\}} u^r dx \\
&\quad + 2^{p+2N}\left(\frac{p-1}{r-1}\right)^{p-1}\frac{\bar{C}}{\bar\sigma^p}\left(\frac{(t-s)^r}{\rho^{\lambda_r}}\right)^{\frac{1}{r}}\left(\sup_{s<\tau\le t}\int_{K_{(1+\bar\sigma)\rho}(y)\times\{\tau\}} u^r dx\right)^{\frac{r+m+p-3}{r}}
\end{aligned}$$

recalling the conditions presented in (5.2).

To obtain an iterative relation regarding the values $\sup \int u^r$, we start by considering the sequence of radii,

$$\rho_n = \rho \sum_{i=1}^{n} \frac{1}{2^i} = \rho\Big(1 - \frac{1}{2^n}\Big), \qquad \text{being} \quad \rho_{n+1} = (1+\sigma_n)\rho_n,$$

and defining

$$Y_n = \sup_{s<\tau\le t} \int_{K_{\rho_n}(y)\times\{\tau\}} u^r\, dx.$$

Hence, from the previous integral estimate applied to the sequences above, we arrive at the recursive inequality

$$Y_n \le 2\bar{C} \int_{K_{2\rho}(y)\times\{s\}} u^r\, dx + C 2^{np} \left(\frac{(t-s)^r}{\rho^{\lambda_r}}\right)^{\frac{1}{r}} Y_{n+1}^{1-\frac{3-m-p}{r}},$$

where $C = \bar{C}\, 2^{2N+p} \left(\frac{p-1}{r-1}\right)^{p-1}$. Applying Young's inequality with ϵ and exponents $\frac{r}{3-m-p}$ and its conjugate $\frac{r}{m+p+r-3}$, we obtain

$$Y_n \le 2\bar{C} \int_{K_{2\rho}(y)\times\{s\}} u^r\, dx + C_\varepsilon ilon \left(\frac{(t-s)^r}{\rho^{\lambda_r}}\right)^{\frac{1}{3-m-p}} 2^{\frac{npr}{3-m-p}} + \epsilon\, Y_{n+1},$$

where $C_\varepsilon ilon = \frac{3-m-p}{r} \left(\frac{r\epsilon}{m+p+r-3}\right)^{\frac{m+p+r-3}{r}} C$. By iteration we get for $n \ge 2$

$$Y_1 \le 2\bar{C} \sum_{i=0}^{n-2} \epsilon^i \int_{K_{2\rho}(y)\times\{s\}} u^r\, dx + C_\varepsilon ilon \left(\frac{(t-s)^r}{\rho^{\lambda_r}}\right)^{\frac{1}{3-m-p}} \sum_{i=0}^{n-2} \big(2^{\frac{rp}{3-m-p}} \epsilon\big)^i + \epsilon^{n-1} Y_n,$$

Taking $\bar{\epsilon}$ so that $2^{\frac{rp}{3-m-p}}\bar{\epsilon} = 2^{-1}$, letting $n \to +\infty$ and since $u \in L^\infty_{\mathrm{loc}}(0,T; L^r_{\mathrm{loc}}(\Omega))$

$$Y_1 \le \frac{2\bar{C}}{1-\bar{\epsilon}} \int_{K_{2\rho}(y)\times\{s\}} u^r\, dx + C_{\bar{\epsilon}} \left(\frac{(t-s)^r}{\rho^{\lambda_r}}\right)^{\frac{1}{3-m-p}}. \quad \square$$

Remark 5.2. We stress that the constant C of Proposition 5.1 goes to infinity as $r \searrow 1$.

In what follows we present and prove a sup estimate for the weak solutions to (E), known in the literature as $L^r_{loc} - L^\infty_{loc}$ estimate.

Theorem 5.3 ($L^r_{loc} - L^\infty_{loc}$ *Estimate*)**.** *Let u be a nonnegative, locally bounded, local weak solution to (E) in Ω_T satisfying (5.1), (5.2). Then there exists a positive constant C, depending only upon N, p, m and r, such that*

$$\sup_{K_{\frac{\rho}{2}}(y)\times[s,t]} u \le C \left\{ \left(\frac{\rho^p}{t-s}\right)^{\frac{N}{\lambda_r}} \left(\frac{1}{\rho^N(t-s)} \int_{-t+2s}^{t} \int_{K_\rho(y)} u^r\right)^{\frac{p}{\lambda_r}} + \left(\frac{t-s}{\rho^p}\right)^{\frac{1}{3-m-p}} \right\}$$

for all cylinders

$$K_{2\rho}(y) \times [s-(t-s), s+(t-s)] \subset \Omega_T.$$

The constant C is quantitatively determined in the proof.

Proof. Assume $(y,s) = (0,0)$ and for fixed $\sigma \in (0,1)$ set

$$\rho_n = \sigma\rho + \frac{1-\sigma}{2^n}\rho, \quad t_n = -\sigma t - \frac{1-\sigma}{2^n} t, \quad n = 0,1,2,\ldots$$

Consider the sequence of nested and shrinking cylinders $Q_n = K_{\rho_n} \times (t_n, t)$ with common vertex $(0, t)$ and observe that, by construction

$$Q_0 = K_\rho \times (-t, t) \quad \text{and} \quad Q_\infty = K_{\sigma\rho} \times (-\sigma t, t).$$

Set

$$M = \sup_{Q_0} u \quad \text{and} \quad M_\sigma = \sup_{Q_\infty} u.$$

We first prove an estimate of M_σ in terms of M.

Consider cutoff functions $\xi \in C_0^\infty(Q_n)$, verifying $\xi(x,t) = \xi_1(x)\xi_2(t) \in [0,1]$

$$\xi_1 = 1 \text{ in } K_{\rho_{n+1}}, \qquad \xi_1 = 0 \text{ in } \mathbf{R}^N \setminus K_{\rho_n}, \qquad |D\xi_1| \le \frac{2^{n+1}}{(1-\sigma)\rho}$$
$$\xi_2 = 1\ ,\ \tau \ge t_{n+1}, \qquad \xi_2 = 0\ ,\ \tau \le t_n, \qquad 0 \le (\xi_2)_t \le \frac{2^{n+1}}{(1-\sigma)t}.$$

Finally define the sequence of levels

$$k_n = k\left(1 - \frac{1}{2^{n+1}}\right), \quad n = 0, 1, 2, \ldots$$

where $k > 0$ is to be chosen.

In what follows we derive integral estimates to the truncated functions $(u - k_{n+1})_+$. So we will be working on sets where $u > k_{n+1} > k/2 > 0$, therefore u is bounded away from zero and one can apply the chain rule for all r satisfying (5.1) and (5.2). To be accurate one should start by considering the average inequality (2.4) and then pass to the limit as $\sigma \to 0$. We decided to proceed formally at this stage since the arguments to be used do not bring any kind of novelty regarding what has been done previously.

Consider first $1 < p < 2$. Formally multiply equation (E) by $(u - k_{n+1})_+^{r-1}\xi^p$, where $r > 1$ satisfies (5.2), and integrate over the cylinders $K_{\rho_n} \times (t_n, \tau)$, for $\tau \in (t_n, t]$.

The parabolic term is easily estimated from below by

$$\frac{1}{r}\int_{K_{\rho_n}} (u - k_{n+1})_+^r \xi^p(x,\tau)dx - \frac{p}{r}\frac{2^{n+1}}{(1-\sigma)t}\iint_{Q_n} (u - k_{n+1})_+^r\ dxdt.$$

As for the elliptic term, we integrate by parts and then use Young's inequality (with ϵ) to arrive at the inferior bound

$$\begin{aligned}
&(r-1)\iint_{Q_n} u^{m-1}|D(u-k_{n+1})_+|^p (u-k_{n+1})_+^{r-2}\xi^p\ dxdt \\
&- p\iint_{Q_n} u^{m-1}|D(u-k_{n+1})_+|^{p-1}\xi^{p-1}(u-k_{n+1})_+^{r-1}|D\xi|\ dxdt \\
\ge& \frac{r-1}{2}\iint_{Q_n} u^{m-1}|D(u-k_{n+1})_+|^p (u-k_{n+1})_+^{r-2}\xi^p\ dxdt \\
&- \left(2\frac{p-1}{r-1}\right)^{p-1}\iint_{Q_n} u^{m-1}(u-k_{n+1})_+^{p+r-2}|D\xi|^p\ dxdt\ .
\end{aligned}$$

By recalling that $u \le M$ and $3 - m - p > 0$

$$\begin{aligned}
\left|D(u-k_{n+1})_+^{\frac{r}{p}}\right|^p &= \left(\frac{r}{p}\right)^p (u-k_{n+1})_+^{r-p}|D(u-k_{n+1})_+|^p \\
&\le \left(\frac{r}{p}\right)^p M^{3-m-p}u^{m-1}|D(u-k_{n+1})_+|^p(u-k_{n+1})_+^{r-2}
\end{aligned}$$

and then, by noticing that $\frac{k}{2} < k_{n+1} < u \leq M$, we get

$$\begin{aligned}
&\iint_{Q_n} u^{m.-1}|D(u-k_{n+1})_+|^p (u-k_{n+1})_+^{r-2}\xi^p \ dxdt \\
\geq & \frac{1}{M^{3-m-p}}\left(\frac{p}{r}\right)^p \iint_{Q_n} \left|D(u-k_{n+1})_+^{\frac{r}{p}}\right|^p \xi^p \ dxdt \\
\geq & \frac{1}{2^{(p-1)}M^{3-m-p}}\left(\frac{p}{r}\right)^p \iint_{Q_n} \left|D[(u-k_{n+1})_+^{\frac{r}{p}}\xi]\right|^p \ dxdt - \frac{1}{M^{3-m-p}}\left(\frac{p}{r}\right)^p \iint_{Q_n} (u-k_{n+1})_+^r |D\xi|^p \ dxdt \\
\geq & \frac{1}{2^{(p-1)}M^{3-m-p}}\left(\frac{p}{r}\right)^p \iint_{Q_n} \left|D[(u-k_{n+1})_+^{\frac{r}{p}}\xi]\right|^p \ dxdt - \left(\frac{2}{k}\right)^{3-m-p}\left(\frac{p}{r}\right)^p \iint_{Q_n} (u-k_{n+1})_+^r |D\xi|^p \ dxdt
\end{aligned}$$

and for

$$\begin{aligned}
&\iint_{Q_n} u^{m-1}(u-k_{n+1})_+^{p+r-2}|D\xi|^p \ dxdt \\
=&\iint_{Q_n} \left(\frac{u}{(u-k_{n+1})_+}\right)^{m-1}(u-k_{n+1})_+^{r+m+p-3}|D\xi|^p \ dxdt \\
\leq&\iint_{Q_n} (u-k_{n+1})_+^{r+m+p-3}|D\xi|^p \ dxdt, \quad \text{since } \ m<1 \ \text{ due to } \ |\gamma|<1 \\
\leq&\iint_{Q_n} (u-\tilde{k}_n)_+^{r+m+p-3}|D\xi|^p \chi_{[u>k_{n+1}]} \ dxdt, \quad \text{for } \ \tilde{k}_n = \frac{k_n+k_{n+1}}{2} \\
\leq&\left(\frac{2^{n+2}}{k}\right)^{3-m-p}\iint_{Q_n} (u-k_n)_+^r|D\xi|^p \ dxdt \ .
\end{aligned}$$

Combining all the previous estimates and taking

$$k \geq \left(\frac{t}{\rho^p}\right)^{\frac{1}{3-m-p}},$$

we obtain, for all $\tau \in (t_n, t]$,

$$\begin{aligned}
&\int_{K_{\rho_n}} (u-k_{n+1})_+^r \xi^p(x,\tau)dx + \frac{r(r-1)}{2^p M^{3-m-p}}\left(\frac{p}{r}\right)^p \iint_{Q_n} \left|D[(u-k_{n+1})_+^{\frac{r}{p}}\xi]\right|^p \ dxdt \\
\leq&\ \frac{2^{(n+1)(3-m)}}{(1-\sigma)^p\rho^p} 2^{3-m-p}\left\{p + \frac{r(r-1)}{2}\left(\frac{p}{r}\right)^p + 2r\left(\frac{p-1}{r-1}\right)^{p-1}\right\}\frac{1}{k^{3-m-p}}\iint_{Q_n} (u-k_n)_+^r \ dxdt \\
\leq&\ C_1 \frac{2^{2(n+1)}}{(1-\sigma)^p t}\iint_{Q_n} (u-k_n)_+^r \ dxdt,
\end{aligned}$$

for $C_1 = 2^{3-m-p}\left\{p + \frac{r(r-1)}{2}\left(\frac{p}{r}\right)^p + 2r\left(\frac{p-1}{r-1}\right)^{p-1}\right\}$.

By first applying Hölder's inequality (with exponent $(N+p)/N$), then Sobolev's embedding (with exponent $p(N+p)/N$ and a constant $\kappa = \kappa(N,p)$) and finally using the previous estimate we get

$$\begin{aligned}
X_{n+1} =& \iint_{Q_{n+1}} (u-k_{n+1})_+^r \ dxdt \leq \iint_{Q_n} (u-k_{n+1})_+^r \xi^p \ dxdt = \iint_{Q_n} \left((u-k_{n+1})_+^{\frac{r}{p}}\xi\right)^p \ dxdt \\
\leq& \left(\iint_{Q_n} \left((u-k_{n+1})_+^{\frac{r}{p}}\xi\right)^{p(N+p)/N} \ dxdt\right)^{N/(N+p)} |Q_n \cap [u>k_{n+1}]|^{p/(N+p)} \\
\leq& \ \kappa \left\{\left(\iint_{Q_n} \left|D[(u-k_{n+1})_+^{\frac{r}{p}}\xi]\right|^p \ dxdt\right)\left(\sup_{t_n \leq \tau \leq t}\int_{K_{\rho_n}} (u-k_{n+1})_+^r \xi^p(x,\tau)dx\right)^{p/N}\right\}^{N/(N+p)}
\end{aligned}$$

$$
\begin{aligned}
&\times |Q_n \cap [u > k_{n+1}]|^{p/(N+p)} \\
&\leq C_2 M^{(3-m-p)\frac{N}{N+p}} \frac{2^{2(n+1)}}{(1-\sigma)^p t} X_n \, |Q_n \cap [u > k_{n+1}]|^{p/(N+p)} \\
&\leq C_2 M^{(3-m-p)\frac{N}{N+p}} \frac{2^{2(n+1)}}{(1-\sigma)^p t} \left(\frac{2^{(n+2)r}}{k^r}\right)^{\frac{p}{N+p}} X_n^{1+\frac{p}{N+p}},
\end{aligned}
$$

where $C_2 = \kappa \times C_1 \times \left(\frac{2^p}{r(r-1)}\left(\frac{r}{p}\right)^p\right)^{\frac{N}{N+p}}$. The last inequality was obtained by noticing that

$$
X_n \geq \iint_{Q_n \cap [u > k_{n+1}]} (u - k_n)_+^r \, dxdt \geq \left(\frac{k}{2^{n+2}}\right)^r |Q_n \cap [u > k_{n+1}]| \, .
$$

From the previous estimate on X_{n+1} and by defining $Y_n = \frac{X_n}{|Q_n|}$, we have, for $C_3 = 2^{N+4+p+2r} C_2$,

$$
Y_{n+1} \leq C_3 \frac{M^{(3-m-p)\frac{N}{N+p}}}{k^{rp/(N+p)}(1-\sigma)^p} \left(\frac{\rho^p}{t}\right)^{\frac{N}{N+p}} b^n Y_n^{1+\frac{p}{N+p}}, \qquad b = 2^{2+rp/(N+p)} > 1.
$$

From a geometric convergence lemma, one has $Y_n \to 0$, as $n \to \infty$, if

$$
Y_0 \leq \left(C_3 \frac{M^{(3-m-p)\frac{N}{N+p}}}{k^{rp/(N+p)}(1-\sigma)^p} \left(\frac{\rho^p}{t}\right)^{\frac{N}{N+p}}\right)^{-\frac{N+p}{p}} b^{-(\frac{N+p}{p})^2}.
$$

This estimate together with the previous one, $k \geq \left(\frac{t}{\rho^p}\right)^{\frac{1}{3-m-p}}$, is verified once we take

$$
k = C_4 \frac{M^{(3-m-p)\frac{N}{pr}}}{(1-\sigma)^{\frac{N+p}{r}}} \left(⨌_Q u^r\right)^{1/r} \left(\frac{\rho^p}{t}\right)^{\frac{N}{pr}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{3-m-p}},
$$

for $C_4 = C_3^{\frac{N+p}{rp}} b^{\frac{(N+p)^2}{rp^2}}$.

For this choice of k we have

$$
M_\sigma = \sup_{Q_\infty} u \leq C_4 \frac{M^{(3-m-p)\frac{N}{pr}}}{(1-\sigma)^{\frac{N+p}{r}}} \left(⨌_Q u^r\right)^{1/r} \left(\frac{\rho^p}{t}\right)^{\frac{N}{pr}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{3-m-p}}. \tag{5.3}
$$

Now consider the sequences, $n = 0, 1, \ldots$

$$
\tilde{\rho}_n = \sigma\rho + (1-\sigma)\rho \sum_{i=1}^{n} \frac{1}{2^i} \qquad \text{and} \qquad \tilde{t}_n = -\sigma t - (1-\sigma) t \sum_{i=1}^{n} \frac{1}{2^i}
$$

for which $K_{\tilde{\rho}_n} \times (\tilde{t}_n, t) = \tilde{Q}_n \subset \tilde{Q}_{n+1}$, and define $M_n = \sup_{\tilde{Q}_n} u$. Applying (5.3) to the cylinders $\tilde{Q}_n$ and $\tilde{Q}_{n+1}$ and then Young's inequality (with $\epsilon = 1/2$), we arrive at

$$
M_n \leq \frac{1}{2} M_{n+1} + C_5 I
$$

for

$$
I = \frac{1}{(1-\sigma)^{\frac{(N+p)p}{\lambda r}}} \left(\frac{\rho^p}{t}\right)^{\frac{N}{\lambda r}} \left(⨌_Q u^r \, dxdt\right)^{\frac{p}{\lambda r}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{3-m-p}}
$$

and $C_5 = C_4^{\frac{pr}{\lambda r}} \left(\frac{s^{\frac{s}{s-1}}}{s-1}\left(\frac{1}{2}\right)^{\frac{N(3-m-p)}{\lambda r}}\right)^{-1}$, where $s = \frac{pr}{N(3-m-p)}$. By iteration

$$
M_0 \leq \left(\frac{1}{2}\right)^n M_n + C_5 I \sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^i
$$

and then, since $(M_n)_n$ is equibounded, when taking $n \to \infty$,

$$\sup_{Q_\sigma} u \le C_5 I$$

and the proof is complete once we take $\sigma = \frac{1}{2}$.

Now consider $p > 2$. Proceed in a formal way and multiply equation (E) by $(u-k_{n+1})_+\xi^p$ and then integrate over the cylinders $K_{\rho_n} \times (t_n, \tau)$, for $\tau \in (t_n, t]$.

While there are no substantial changes in the estimation of the parabolic term, the elliptic term is estimated from below as follows

$$\begin{aligned}
&\frac{1}{2^p}\iint_{Q_n} u^{m-1}|D(u-k_{n+1})_+\xi|^p\,dxdt - \left(\frac{1}{2}+(2(p-1))^{p-1}\right)\iint_{Q_n} u^{m-1}(u-k_{n+1})_+^p|D\xi|^p\,dxdt\\
&\ge \frac{M^{m-1}}{2^p}\iint_{Q_n}|D(u-k_{n+1})_+\xi|^p\,dxdt - \frac{1+2^p(p-1)^{p-1}}{2}\iint_{Q_n} u^{m+p-3}(u-k_{n+1})_+^2|D\xi|^p\,dxdt\\
&\ge \frac{M^{m-1}}{2^p}\iint_{Q_n}|D(u-k_{n+1})_+\xi|^p\,dxdt\\
&\quad - \frac{1+2^p(p-1)^{p-1}}{2}\left(\frac{2}{k}\right)^{3-m-p}\frac{2^{p(n+1)}}{(1-\sigma)^p\rho^p}\iint_{Q_n}(u-k_{n+1})_+^2\,dxdt\,.
\end{aligned}$$

By considering $k \ge \left(\frac{t}{\rho^p}\right)^{\frac{1}{3-m-p}}$, one gets, for all $\tau \in (t_n, t]$

$$\begin{aligned}
\int_{K_{\rho_n}}(u-k_{n+1})_+^2\xi^p(x,\tau)dx + \frac{M^{m-1}}{2^{p-1}}\iint_{Q_n}|D(u-k_{n+1})_+\xi|^p\,dxdt&\\
\le C_o\frac{2^{(n+1)p}}{(1-\sigma)^p t}\iint_{Q_n}(u-k_n)_+^2\,dxdt\,,&
\end{aligned} \tag{5.4}$$

where $C_o = p + (1+2^p(p-1)^{p-1})2^p$. Set $X_n = \iint_{Q_n}(u-k_n)_+^2\,dxdt$. Arguing as before we have

$$\begin{aligned}
X_{n+1} &\le \iint_{Q_n}(u-k_{n+1})_+^2\xi^2\,dxdt \le \left(\iint_{Q_n}((u-k_{n+1})_+\xi)^{\frac{p(N+p)}{N}}\,dxdt\right)^{\frac{2N}{p(N+p)}}|Q_n\cap[u>k_{n+1}]|^{1-\frac{2N}{p(N+p)}}\\
&\le \kappa\left\{\left(\iint_{Q_n}|D[(u-k_{n+1})_+\xi]|^p\,dxdt\right)\left(\sup_{t_n\le\tau\le t}\int_{K_{\rho_n}}(u-k_{n+1})_+^p\xi^p(x,\tau)dx\right)^{\frac{p}{N}}\right\}^{\frac{2N}{p(N+p)}}\\
&\quad\times|Q_n\cap[u>k_{n+1}]|^{1-\frac{2N}{p(N+p)}}\\
&\le \kappa\left\{\left(\iint_{Q_n}|D[(u-k_{n+1})_+\xi]|^p\,dxdt\right)\left(M^{p-2}\sup_{t_n\le\tau\le t}\int_{K_{\rho_n}}(u-k_{n+1})_+^2\xi^p(x,\tau)dx\right)^{\frac{p}{N}}\right\}^{\frac{2N}{p(N+p)}}\\
&\quad\times|Q_n\cap[u>k_{n+1}]|^{1-\frac{2N}{p(N+p)}}\\
&\le \kappa C_o^{2/p}2^{(p-1)\frac{2N}{p(N+p)}}M^{[1-m+(p-2)\frac{p}{N}]\frac{2N}{p(N+p)}}\left(\frac{2^{(n+1)p}}{(1-\sigma)^p t}\right)^{\frac{2}{p}}\left(\frac{2^{2(n+2)}}{k^2}\right)^{1-\frac{2N}{p(N+p)}}X_n^{\frac{2}{p}+1-\frac{2N}{p(N+p)}}.
\end{aligned}$$

Set $Y_n = \frac{X_n}{|Q_n|}$. Then

$$Y_{n+1} \le C_1 M^{[1-m+(p-2)\frac{p}{N}]\frac{2N}{p(N+p)}}\frac{2^{4n}}{(1-\sigma)^2}\left(\frac{\rho^p}{t}\right)^{\frac{2N}{p(N+p)}}\left(\frac{1}{k^2}\right)^{1-\frac{2N}{p(N+p)}}Y_n^{1+\frac{2}{N+p}}$$

for $C_1 = C_1(N,p)$. The fast geometric convergence lemma says that $Y_n \to 0$, when $n \to \infty$, if

$$Y_0 \le \left(C_1 M^s \frac{1}{(1-\sigma)^2} \left(\frac{\rho^p}{t}\right)^{\frac{2N}{p(N+p)}} k^{-2\left(1-\frac{2N}{p(N+p)}\right)} \right)^{-\frac{N+p}{2}} 2^{-(N+p)^2},$$

where $s = [1-m+(p-2)\frac{p}{N}]\frac{2N}{p(N+p)} > 0$.

Our next step will be to obtain a bound to Y_0 related to the L^r-norm of u, which will be accomplished by considering separately the two possible cases: $r \ge 2$ and $r < 2$. As for $r \ge 2$, we estimate

$$Y_0 = \iint_Q \left(u - \frac{k}{2}\right)_+^2 \le \left(\frac{k}{2}\right)^{2-r} \iint_Q u^r.$$

Thus we just need to choose k in order to satisfy the inequality

$$\left(\frac{k}{2}\right)^{2-r} \iint_Q u^r \le \left(C_1 M^s \frac{1}{(1-\sigma)^2} \left(\frac{\rho^p}{t}\right)^{\frac{2N}{p(N+p)}} k^{-2\left(1-\frac{2N}{p(N+p)}\right)} \right)^{-\frac{N+p}{2}} 2^{-(N+p)^2},$$

so we consider

$$k = C_2 \left\{ \iint_Q u^r \right\}^{\frac{p}{\varpi}} M^{\frac{(1-m)N+p(p-2)}{\varpi}} \left(\frac{\rho^p}{t}\right)^{\frac{N}{\varpi}} \left(\frac{1}{1-\sigma}\right)^{\frac{p(N+p)}{\varpi}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{3-m-p}},$$

where $\varpi = p(N+p+r-2) - 2N$, $C_2 = \left(C_1^{(N+p)/2} 2^{r-2+(N+p)^2}\right)^{\frac{p}{\varpi}}$. Thus $M_\sigma \le k$. By applying Young's inequality with exponent $\mu = \frac{[p(N+p+r-2)-2N]}{(1-m)N+p(p-2)} > 1$, we estimate k obtaining

$$M_\sigma \le \frac{1}{2} M + C_2^{\mu'} \bar{C} \left\{ \frac{1}{(1-\sigma)^{\frac{p(N+p)}{\lambda_r}}} \left(\frac{\rho^p}{t}\right)^{\frac{N}{\lambda_r}} \left(\iint_Q u^r\right)^{\frac{p}{\lambda_r}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{3-m-p}} \right\}$$

where $\bar{C} = \left(\mu'(\mu 2^{-1})^{\frac{\mu'}{\mu}}\right)^{-1}$ and μ' denotes the conjugate exponent of μ. By computing μ' we notice that $C_2^{\mu'} = \left(C_1^{(N+p)/2} 2^{r-2+(N+p)^2}\right)^{\frac{p}{\lambda_r}}$. Arguing as in the case $1 < p < 2$, we first apply the previous estimate to the sequence $(M_n)_n$ and then, by iteration and taking the limit as $n \to \infty$, we arrive at

$$\sup_{K_{\sigma\rho} \times (-\sigma t, t)} u \le C \frac{1}{(1-\sigma)^{\frac{p(N+p)}{\lambda_r}}} \left(\frac{\rho^p}{t}\right)^{\frac{N}{\lambda_r}} \left(\iint_Q u^r\right)^{\frac{p}{\lambda_r}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{3-m-p}},$$

where $C = C_2^{\mu'} \bar{C}$.

In the case $r < 2$, we start by noticing that

$$Y_0 \le \iint_Q u^2 \le M^{2-r} \iint_Q u^r$$

and consequently, being $C_3 = \left(C_1^{(N+p)/2} 2^{(N+p)^2}\right)^{\frac{p}{p(N+p+r-2)-2N}}$

$$\begin{aligned} M_\sigma &\le C_3 \left\{ \iint_Q u^r \right\}^{\frac{p}{[p(N+p)-2N]}} M^{\frac{(1-m)N+p(p-r)}{p(N+p)-2N}} \left(\frac{\rho^p}{t}\right)^{\frac{N}{p(N+p)-2N}} \left(\frac{1}{1-\sigma}\right)^{\frac{p(N+p)}{p(N+p)-2N}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{3-m-p}} \\ &\le \frac{1}{2} M + C_4 \tilde{C} \left\{ \frac{1}{(1-\sigma)^{\frac{p(N+p)}{\lambda_r}}} \left(\frac{\rho^p}{t}\right)^{\frac{N}{\lambda_r}} \left(\iint_Q u^r\right)^{\frac{p}{\lambda_r}} + \left(\frac{t}{\rho^p}\right)^{\frac{1}{3-m-p}} \right\}, \end{aligned}$$

with $C_4 = C_3^{\frac{p(N+p)-2N}{\lambda_r}}$ and $\tilde{C}$ is similar to $\bar{C}$ with μ given by $\frac{p(N+p)-2N}{(1-m)N+p(p-r)}$. The conclusion is as above. □

Remark 5.4. When $1 < p < 2$, the constant $C \to \infty$ when $r \to 1$ due to the presence of the factor $r-1$ in the denominator; for $p > 1$, C becomes unbounded when $\lambda_r \to 0$.

Due to Theorem 5.3 and Proposition 5.1 we can also prove (it is a straightforward proof) the following result.

Theorem 5.5. *Let u be a nonnegative, locally bounded, local weak solution to (E) in Ω_T satisfying (5.1), for $r > 1$ satisfying (5.2). Then there exists a positive constant C depending on N, p, m, r such that, for all cylinders $K_\rho(y) \times [2s-t, t] \subset \Omega_T$,*

$$\sup_{K_{\frac{\rho}{2}}(y)\times[s,t]} u \le C \left\{ \frac{1}{(t-s)^{\frac{N}{\lambda_r}}} \left(\int_{K_{2\rho}(y)} u^r(x, 2s-t)dx \right)^{\frac{p}{\lambda_r}} + \left(\frac{t-s}{\rho^p} \right)^{\frac{1}{3-m-p}} \right\}.$$

The constant $C \to \infty$ as either $r \to 1$ or $\lambda_r \to 0$.

Remark 5.6. The results of this section were obtained under the extra regularity assumption (5.1) verifying also (5.2). These conditions are not new and for that coherent with the literature. In fact, in [25,26] and [51] counterexamples were produced regarding the sharpness of such assumptions in the case of: the porous medium equation, for $0 < m < 1$; the p-Laplacian equation, for $1 < p < 2$; and the doubly nonlinear equation (E), for $m+p > 2$, $m+p > 3 - \frac{p}{N}$, $p > 1$. In the setting considered in this work and to the best of the authors knowledge, counterexamples were not yet discussed — most possibly because, if one follows the previous approaches, one has to rely on comparison and uniqueness results (not yet obtained within our range).

6. Expansion of positivity

The expansion of positivity is an essential tool in establishing regularity results, in particular Harnack type inequalities. It has already been established for $m+p = 2$ in [31] and for $2 < m+p < 3$ in [33]. Therefore, here we are focussed on $m+p < 2$, namely $\gamma < 0$.

Proposition 6.1. *Let u be a nonnegative, locally bounded, local weak supersolution to (E) satisfying (2.2), (5.1) with r as in (5.2). Assume that for some $M > 0$ as in (4.1) and parameters $a, \delta \in (0,1)$ there holds*

$$|[u(\cdot,\tau) \ge M] \cap K_\rho(y)| \ge a|K_\rho(y)| \tag{6.1}$$

for all τ such that

$$s - \delta M^{3-m-p}\rho^p \le \tau \le s\ .$$

Then there exists a constant $\sigma \in (0,1)$ that can be determined in terms of $\{a, \delta, \Lambda_\gamma, \gamma\}$ such that

$$u(\cdot, t) \ge \sigma M \qquad \text{in } K_{2\rho}(y)$$

for all times

$$s - \frac{\delta M^{3-m-p}}{8}\rho^p \le t \le s.$$

Following, for instance, the approach presented in [31], the proof of this result relies on the following lemma.

Lemma 6.2. *Under the assumptions of Proposition 6.1, for every $\nu > 0$ there exists $\sigma \in (0,1)$ depending on $N, p, a, \Lambda_\gamma, \nu, \delta$ and γ, such that*

$$|[u(\cdot,t) \le 2\sigma M] \cap K_{4\rho}(y)| \le \nu |K_{4\rho}|$$

for all

$$s - \frac{\delta M^{3-m-p}}{4}\rho^p < t \le s.$$

To simplify the notation, we perform the change of variables

$$x \to \frac{x-y}{\rho}, \qquad t \to 8^p \frac{t-(s-\delta M^{3-m-p}\rho^p)}{\delta M^{3-m-p}\rho^p}, \qquad u \to \frac{u}{M}$$

which transforms the cylinder $K_{8\rho}(y) \times (s-\delta M^{3-m-p}\rho^p, s]$into $Q_8^+ = K_8 \times (0, 8^p]$ and the new function (still denoted by u) into a weak solution to the equation

$$u_t - \delta 8^{-p}\mathrm{div}\left(u^{m-1}|Du|^{p-2}Du\right) = 0, \quad \text{in} \quad Q_8^+. \tag{6.2}$$

Moreover, assumption (6.1) yields

$$|[u(\cdot,t) \ge 1] \cap K_4| \ge \frac{a}{4^N}|K_4|$$

and consequently

$$|[u(\cdot,t) \ge 1] \cap K_8| \ge \frac{a}{8^N}|K_8| \tag{6.3}$$

for all $t \in (0, 8^p]$.

Note that assumption (5.1) is preserved under the change of variables and the quantities L_γ and Λ_γ, presented in (4.2), remain unchanged.

In this new setting Lemma 6.2 can be (re)stated as follows.

Lemma 6.3. *Let u be a nonnegative, locally bounded, local weak supersolution to Eq. (6.2) satisfying (6.3), (4.2), (5.1) and (5.2). Then for every $\nu > 0$ there exists $\sigma \in (0,1)$ depending upon $N, p, a, \Lambda_\gamma, \nu, \delta$ and γ, such that*

$$|[u(\cdot,t) \le \sigma] \cap K_4| \le \nu |K_4| \tag{6.4}$$

for all $\frac{3}{4}8^p < t \le 8^p$.

Proof. Assume that $u_t \in C(0, 8^p; L^1(K_8))$. Since u is a supersolution, for every nonnegative test function $\phi \in C(Q_8^+) \cap C(0, 8^p; W_0^{1,p}(K_8))$ we have for any $k > 0$

$$\int_{K_8} \frac{\partial}{\partial t}(k-u)_+ \phi\, dx + \frac{\delta}{8^p}\int_{K_8} u^{m-1}|D(k-u)_+|^{p-2}D(k-u)_+ \cdot D\phi\, dx \le 0.$$

Take $\phi = \left(\frac{k^\gamma - u^\gamma}{\gamma}\right)_+ \zeta^p$, where $k \in (0,1]$ and $\zeta \in C_0^\infty(Q_8^+)$ verifies $\zeta(x,t) = \zeta_1(x)\zeta_2(t) \in [0,1]$

$$\zeta_1 = 1 \text{ in } K_4, \qquad \zeta_1 = 0 \text{ in } \mathbf{R}^N \setminus K_8, \qquad |D\zeta_1| \le \frac{1}{4}$$

$$\zeta_2 = 1\ ,\ t \ge \frac{3}{4}8^p, \qquad \zeta_2 = 0\ ,\ t \le 0, \qquad 0 \le (\zeta_2)_t \le \frac{4}{3 \cdot 8^p}.$$

By Young's inequality and recalling that $f(s) = \frac{s^\gamma}{\gamma}$ is an increasing function, we get

$$\begin{aligned}&\frac{d}{dt}\int_{K_8} \Phi_k(u)\zeta^p(x,t)dx + \frac{\delta}{2 \cdot 8^p}\int_{K_8} \frac{|D(k-u)_+|^p}{u^{\frac{(1-m)p}{p-1}}}\zeta^p dx \\ &\quad \le \delta 8^{-p}(2(p-1))^{p-1}\int_{K_8} \Psi_k^p(u)|D\zeta|^p dx + p\int_{K_8} \Phi_k(u)\zeta^{p-1}\zeta_t dx\end{aligned}$$

where

$$\Phi_k(u) = \left(\int_u^k \frac{k^\gamma - s^\gamma}{\gamma} ds\right)_+ \qquad \text{and} \qquad \Psi_k(u) = \left(\frac{k^\gamma - u^\gamma}{\gamma}\right)_+.$$

Thus we have

$$\Phi_k(u) \le \int_0^k \frac{k^\gamma - s^\gamma}{\gamma} ds = \frac{k^{\gamma+1}}{\gamma+1}.$$

As for $\Psi_k(u)$ we have

$$\Psi_k(u) = k^\gamma \Big(\frac{1-(u/k)^\gamma}{\gamma}\Big)_+ \le k^\gamma \Big(\frac{1-u^\gamma}{\gamma}\Big)_+ \qquad \text{since } 0 < k \le 1\,.$$

Taking these estimates into account, applying Holder's inequality and recalling the definition of L_γ, given in (4.2), we arrive at

$$\begin{aligned}\frac{d}{dt}\int_{K_8} \Phi_k(u)\zeta^p(x,t)dx &+ \frac{\delta}{2\cdot 8^p}\int_{K_8} \frac{|D(k-u)_+|^p}{u^{\frac{(1-m)p}{p-1}}}\zeta^p dx \\ &\le |K_8| \left\{ \frac{p}{32^{3p-1}} k^{3-m-p} + \delta 2^{-2p-3}(p-1)^{p-1} L_\gamma \right\} k^{\gamma p} \le C_1\ \Lambda_\gamma^p\ k^{\gamma p}\end{aligned}$$

where C_1 is a constant depending only upon N and p.

The left hand side is estimated as follows: we start by noticing that

$$\int_{K_8} \frac{|D(k-u)_+|^p}{u^{\frac{(1-m)p}{p-1}}}\zeta^p dx = \int_{K_8} |D\,\Psi_k(u)|^p \zeta^p dx$$

and then, from Proposition 2.1 of Chapter I of [18] and (6.3), we get a constant $C_2 = C_2(N,p) > 0$ such that

$$\begin{aligned}\int_{K_8} |D\,\Psi_k(u)|^p\zeta^p dx &\ge C_2\, |[\Psi_k(u)=0]\cap K_4|^{\frac{N-1}{N}p} \int_{K_8} \Psi_k(u)^p\zeta^p dx, \\ &\ge C_2 a^p \left(\frac{|K_8|}{8^N}\right)^{\frac{N-1}{N}p} \int_{K_8} \Psi_k(u)^p\zeta^p dx \\ &= C_3 a^p \int_{K_8} \Psi_k(u)^p\zeta^p dx,\end{aligned}$$

for some $C_3 = C_3(N,p) > 0$. Hence we obtain

$$\frac{d}{dt}\int_{K_8} \Phi_k(u)\zeta^p(x,t)dx + C_4 a^p\delta \int_{K_8} \Psi_k^p(u)\zeta^p dx \le C_1 k^{\gamma p}\Lambda_\gamma^p, \tag{6.5}$$

being $C_4 = 2^{-1-3p}C_3 = C_4(N,p)$. Let us introduce the quantities

$$Y_n = \sup_{0<t<8^p} \int_{K_8} \chi_{[u(\cdot,t)<h^n]}\zeta^p(x,t)dx \tag{6.6}$$

where $h \in (0,1)$ is to be chosen. We

Claim. *given $\nu > 0$, there exist $h, \xi \in (0,1)$ depending on $N, p, a, \delta, \nu, \gamma$ and Λ_γ such that for every $n = 0, 1, \ldots$*

$$\textit{either} \quad Y_n \le \nu \quad \textit{or} \quad Y_{n+1} \le \max\{\nu, \xi Y_n\}. \tag{6.7}$$

Now (6.4) is a straightforward consequence of this claim. In fact, by iterating (6.7) we find $Y_n \le \max\{\nu, \xi^n Y_0\}$ for every $n \ge 1$. Choosing $\bar n$ such that $\xi^{\bar n} < \nu 2^{-N}$, we have $Y_{\bar n} \le \nu|K_4|$, since $Y_0 \le |K_8|$. By the definition of $Y_{\bar n}$ we get

$$\sup_{\frac34 8^p<t<8^p} \left|K_4 \cap [u(\cdot,t) < h^{\bar n}]\right| \le \sup_{0<t<8^p}\int_{K_8} \chi_{[u(\cdot,t)<h^{\bar n}]}\zeta^p(x,t)dx \le \nu|K_4|\ ,$$

which yields (6.4) with $\sigma = h^{\bar n}$.

The proof is complete once the claim is proved.

Proving the Claim. Fix $\nu > 0$, take $n \in \mathbf{N}$ and assume that $Y_n > \nu$, otherwise there is nothing to prove. By the definition of Y_{n+1}, for every $\varepsilon \in (0, \frac{\nu}{2})$ there exists $t_\varepsilon \in (0, 8^p]$ such that

$$\int_{K_8} \chi_{[u(\cdot,t_\varepsilon)<h^{n+1}]}\zeta^p(x,t_\varepsilon)dx \geq Y_{n+1} - \varepsilon.$$

At this point we have two alternatives, either

$$\frac{d}{dt}\int_{K_8} \Phi_{h^n}[u(\cdot,t_\varepsilon)]\zeta^p(x,t_\varepsilon)dx \geq 0$$

or

$$\frac{d}{dt}\int_{K_8} \Phi_{h^n}[u(\cdot,t_\varepsilon)]\zeta^p(x,t_\varepsilon)dx < 0.$$

Assume that the first alternative holds true. Then, by (6.5) we deduce that

$$\int_{K_8} \Psi^p_{h^n}[u(\cdot,t_\varepsilon)]\zeta^p(x,t_\varepsilon)dx \leq C\frac{h^{n\gamma p}}{a^p\delta}\Lambda^p_\gamma, \quad C = \frac{C_1}{C_4} = C(N,p).$$

On the set $[u(\cdot,t_\varepsilon) < h^{n+1}]$ we have

$$\Psi_{h^n}[u(\cdot,t_\varepsilon)] = h^{n\gamma}\Big(\frac{1-(u/h^n)^\gamma}{\gamma}\Big)_+ \geq h^{n\gamma}\Big(\frac{1-h^\gamma}{\gamma}\Big).$$

Therefore

$$\Big(\frac{1-h^\gamma}{\gamma}\Big)^p\int_{K_8} \chi_{[u(x,t_\varepsilon)<h^{n+1}]}\zeta^p(x,t_\varepsilon)dx \leq \int_{K_8} \Psi^p_{h^n}[u(x,t_\varepsilon)]\zeta^p(x,t_\varepsilon)dx \leq C\frac{1}{a^p\delta}\Lambda^p_\gamma$$

and also

$$Y_{n+1} \leq C\frac{1}{a^p\delta}\Big(\frac{\gamma}{1-h^\gamma}\Big)^p\Lambda^p_\gamma + \varepsilon.$$

So we just need to recall that $\varepsilon \in (0, \frac{\nu}{2})$ and then choose h sufficiently small (recall that $\gamma < 0$) so that

$$C\frac{1}{a^p\delta}\Big(\frac{\gamma}{1-h^\gamma}\Big)^p\Lambda^p_\gamma \leq \frac{\nu}{2} \tag{6.8}$$

to complete the proof, in the case the first alternative holds.

Now assume that the second alternative holds and define

$$t_* = \sup\Big\{t \in (0,t_\varepsilon) \mid \frac{d}{dt}\int_{K_8} \Phi_{h^n}[u(x,t)]\zeta^p(x,t)dx \geq 0\Big\}.$$

It follows that the function $t \to \int_{K_8} \Phi_{h^n}[u(x,t)]\zeta^p(x,t)dx$ has negative derivative in the interval $(t_*, t_\varepsilon]$ and this yields

$$\int_{K_8} \Phi_{h^n}[u(x,t_\varepsilon)]\zeta^p(x,t_\varepsilon)dx \leq \int_{K_8} \Phi_{h^n}[u(x,t_*)]\zeta^p(x,t_*)dx.$$

Due to the definition of t_* and arguing as in the first alternative we have

$$\int_{K_8} \Psi^p_{h^n}[u(x,t_*)]\zeta^p(x,t_*)dx \leq Ch^{n\gamma p}\frac{\Lambda^p_\gamma}{a^p\delta}, \quad C = C(N,p).$$

For every $s \in (0,1)$, on the set $[u(\cdot,t_*) < h^n(1-s)]$ we have

$$\Psi_{h^n}[u(\cdot,t_*)] \geq h^{n\gamma}\left(\frac{1-(1-s)^{|\gamma|}}{|\gamma|}\right)$$

and therefore

$$\int_{K_8} \chi_{[u(x,t_*)<h^n(1-s)]}\zeta^p(x,t_*)dx \le C\left(\frac{|\gamma|}{1-(1-s)^{|\gamma|}}\right)^p \frac{\Lambda_\gamma^p}{a^p\delta}.$$

By the definition of Y_n, since $[u < h^n(1-s)] \subseteq [u < h^n]$, we have

$$\int_{K_8} \chi_{[u(x,t_*)<h^n(1-s)]}\zeta^p(x,t_*)dx \le \min\left\{Y_n, C\left(\frac{|\gamma|}{1-(1-s)^{|\gamma|}}\right)^p \frac{\Lambda_\gamma^p}{a^p\delta}\right\}.$$

Being $s \to \left(\frac{|\gamma|}{1-(1-s)^{|\gamma|}}\right)^p$ a monotone decreasing function in $(0,1)$, there exists s_* such that

$$Y_n = C\left(\frac{|\gamma|}{1-(1-s_*)^{|\gamma|}}\right)^p \frac{\Lambda_\gamma^p}{a^p\delta},$$

hence

$$\int_{K_8} \chi_{[u(x,t_\varepsilon)<h^n(1-s)]}\zeta^p(x,t_*)dx \le \begin{cases} Y_n & \text{if } 0<s<s_* \\ C\left(\frac{|\gamma|}{1-(1-s)^{|\gamma|}}\right)^p \frac{\Lambda_\gamma^p}{a^p\delta} & \text{if } s_* \le s < 1 \end{cases}$$

In particular

$$\begin{aligned} s_* &= 1-\left\{1-|\gamma|\left(C\frac{\Lambda_\gamma^p}{a^p\delta Y_n}\right)^{\frac{1}{p}}\right\}^{\frac{1}{|\gamma|}} \\ &< 1-\left\{1-|\gamma|\left(C\frac{\Lambda_\gamma^p}{a^p\delta \nu}\right)^{\frac{1}{p}}\right\}^{\frac{1}{|\gamma|}}, \quad \text{since } Y_n > \nu \\ &= 1-(1-|\gamma|A)^{\frac{1}{|\gamma|}}, \qquad A = \frac{\Lambda_\gamma}{a}\left(\frac{C}{\delta\nu}\right)^{\frac{1}{p}} \\ &\le 1-e^{-A} = s_{**} \end{aligned} \tag{6.9}$$

since $g(s) = 1-(1-sA)^{\frac{1}{s}}$, $s>0$, is a monotone decreasing function verifying

$$\lim_{s\to 0^+} g(s) = 1-e^{-A}\ .$$

Thereby, on the one hand

$$\begin{aligned} \int_{K_8} \Phi_{h^n}[u(x,t_\varepsilon)]\zeta^p(x,t_\varepsilon)dx &\le \int_{K_8} \Phi_{h^n}[u(x,t_*)]\zeta^p(x,t_*)dx \\ &= \int_{K_8}\left(\int_0^{(h^n-u(x,t_*))_+} \chi_{[s<(h^n-u)_+]}\frac{h^{\gamma n}-(s+u)^\gamma}{\gamma}\,ds\right)\zeta^p(x,t_*)dx \\ &\le \int_{K_8}\left(\int_0^{h^n} \chi_{[s<(h^n-u)_+]}\frac{h^{\gamma n}-(s+u)^\gamma}{\gamma}\,ds\right)\zeta^p(x,t_*)dx \\ &\le \int_0^{hn} \frac{h^{\gamma n}-s^\gamma}{\gamma}\left(\int_{K_8}\chi_{[s<(h^n-u)_+]}\zeta^p(x,t_*)dx\right)ds \\ &= \int_0^1 h^{n(\gamma+1)}\frac{1-s^\gamma}{\gamma}\left(\int_{K_8}\chi_{[sh^n<(h^n-u)_+]}\zeta^p(x,t_*)dx\right)ds \\ &= \int_0^1 h^{n(\gamma+1)}\frac{1-s^\gamma}{\gamma}\left(\int_{K_8\cap[u<h^n(1-s)]}\zeta^p(x,t_*)dx\right)ds \\ &= \int_0^{s_*} h^{n(\gamma+1)}\frac{1-s^\gamma}{\gamma}\ Y_n\,ds + \int_{s_*}^1 h^{n(\gamma+1)}\frac{1-s^\gamma}{\gamma} C\frac{\Lambda_\gamma^p}{a^p\delta}\left(\frac{|\gamma|}{1-(1-s)^{|\gamma|}}\right)^p ds \end{aligned}$$

$$
\begin{aligned}
&< h^{n(\gamma+1)} Y_n \left(\int_0^1 \frac{1-s^\gamma}{\gamma}\, ds - \int_{s_*}^1 \frac{1-s^{|\gamma|}}{|\gamma|} \left[1 - A^p \left(\frac{|\gamma|}{1-(1-s)^{|\gamma|}} \right)^p \right] ds \right) \\
&= h^{n(\gamma+1)} Y_n \left(\frac{1}{\gamma+1} - \int_{s_*}^1 \frac{1-s^{|\gamma|}}{|\gamma|} \left[1 - A^p \left(\frac{|\gamma|}{1-(1-s)^{|\gamma|}} \right)^p \right] ds \right) \\
&\le h^{n(\gamma+1)} Y_n \left(\frac{1}{\gamma+1} - \int_{s_{**}}^1 \frac{1-s^{|\gamma|}}{|\gamma|} \left[1 - A^p \left(\frac{|\gamma|}{1-(1-s)^{|\gamma|}} \right)^p \right] ds \right) \\
&\le h^{n(\gamma+1)} Y_n \left(\frac{1}{\gamma+1} - \int_{s_{**}}^1 \frac{1-s^{|\gamma|}}{|\gamma|} \left[1 - \left(\frac{1-(1-s_{**})^{|\gamma|}}{1-(1-s)^{|\gamma|}} \right)^p \right] ds \right) \quad \text{due to (6.9)} \\
&= \frac{h^{n(\gamma+1)}}{\gamma+1} Y_n \left(1 - \int_{s_{**}}^1 f_\gamma(s) ds \right),
\end{aligned}
$$

for

$$
f_\gamma(s) = (\gamma+1) \frac{1-s^\gamma}{\gamma} \left[1 - \left(\frac{1-(1-s^{**})^{|\gamma|}}{1-(1-s)^{|\gamma|}} \right)^p \right].
$$

On the other hand

$$
\begin{aligned}
\int_{K_8} \Phi_{h^n}[u(x,t_\varepsilon)] \zeta^p(x,t_\varepsilon) dx &\ge \int_{K_8 \cap [u(x,t_\varepsilon) < h^{n+1}]} \Phi_{h^n}[u(x,t_\varepsilon)] \zeta^p(x,t_\varepsilon) dx \\
&= \int_{K_8 \cap [u(x,t_\varepsilon) < h^{n+1}]} \left(\int_{u(x,t_\varepsilon)}^{h^n} \frac{h^{\gamma n} - s^\gamma}{\gamma}\, ds \right)_+ \zeta^p(x,t_\varepsilon) dx \\
&\ge \int_{K_8 \cap [u(x,t_\varepsilon) < h^{n+1}]} \left(\int_{h^{n+1}}^{h^n} \frac{h^{\gamma n} - s^\gamma}{\gamma}\, ds \right)_+ \zeta^p(x,t_\varepsilon) dx \\
&= \frac{h^{n(\gamma+1)}}{\gamma+1} \left(1 - h - h\frac{1-h^\gamma}{\gamma} \right) \int_{K_8 \cap [u(x,t_\varepsilon) < h^{n+1}]} \zeta^p(x,t_\varepsilon) dx \\
&\ge \frac{h^{n(\gamma+1)}}{\gamma+1} \left(1 - h - h\frac{1-h^\gamma}{\gamma} \right) (Y_{n+1} - \varepsilon)\ .
\end{aligned}
$$

Combining the last two estimates we have

$$
\left(1 - h - h\frac{1-h^\gamma}{\gamma} \right) (Y_{n+1} - \varepsilon) \le Y_n \left(1 - \int_{s_{**}}^1 f_\gamma(s) ds \right).
$$

Set

$$
\epsilon_o = \int_{s_{**}}^1 f_\gamma(s) ds
$$

and note that ϵ_o depends on s_{**} and γ.

Then

$$
Y_{n+1} \le \frac{1-\epsilon_o}{1 - h - \frac{h - h^{\gamma+1}}{\gamma}} Y_n + \varepsilon.
$$

Now we just need to take h sufficiently small such that

$$
\frac{1-\epsilon_o}{1 - h - \frac{h - h^{\gamma+1}}{\gamma}} \le 1 - \frac{\epsilon_o}{2} \tag{6.10}
$$

and by letting $\varepsilon \to 0$ we finally get $Y_{n+1} \le \xi Y_n$, for a constant $\xi = 1 - \frac{\epsilon_o}{2} \in (0,1)$ depending only on $N, p, a, \delta, \Lambda_\gamma, \nu$ and γ and our claim is proved.

A final remark: the assumption $u_t \in C(0, 8^p; L^1(K_8))$ can be removed and one has to argue in a similar way as in [18], chapter IV, section 9. □

Proof of Proposition 6.1. We are now in position to prove Proposition 6.1. Consider any cylinder of the form $(y,t)+Q^-_{4\rho}(\theta)$, with

$$\theta=(\sigma M)^{3-m-p} \quad \text{and} \quad s-\frac{\delta M^{3-m-p}}{8}\rho^p\le t\le s$$

and σ to be fixed. Then the inclusion $(t-(4\rho)^p\theta,t]\subset(s-\frac{\delta M^{3-m-p}}{4}\rho^p,s]$ holds true for any t as above if and only if

$$s-\frac{\delta M^{3-m-p}}{8}\rho^p-(4\rho)^p(\sigma M)^{3-m-p}\ge s-\frac{\delta M^{3-m-p}}{4}\rho^p,$$

which we may assume, without loss of generality, by choosing σ smaller if necessary. From Lemma 6.2, we know that for every $\nu>0$ there exists $\sigma\in(0,1)$ such that

$$|(y,t)+Q^-_{4\rho}(\theta)\cap[u\le 2\sigma M]|\le\nu|Q^-_{4\rho}(\theta)|.$$

Let us fix ν according to Lemma 4.1 with $a=\dfrac12$ and ξM replaced by $2\sigma M$ and θ as above. Then by (4.3) we get

$$\nu=\left(C^{\frac{p(2+\beta)}{\beta}}\ \frac{1}{\Lambda_\gamma^{p}}\right)^{\frac1\beta},\qquad \beta=\frac{p}{N+p}-\frac{p}{\alpha}>0.$$

Thus we arrive at

$$u\ge\sigma M\quad \text{in } K_{2\rho}(y)\times(t-\theta(2\rho)^p,t].$$

The Proposition is completely proved once we recall the previous choices on t. □

7. Harnack inequality

Let u be a nonnegative, locally bounded, local weak solution to the singular equation (E) satisfying (2.2), (5.1) and (5.2). In the next few lines we fix the necessary notation for the Harnack inequality. Let $(x_0,t_0)\in\Omega_T$ and $\rho>0$ be such that $K_{8\rho}(x_0)\subset\Omega$, and introduce the quantity

$$\theta_0=\left(\varepsilon\left(\fint_{K_\rho(x_0)}u^r(\cdot,t_0)dx\right)^{\frac1r}\right)^{3-m-p}, \tag{7.1}$$

where $\varepsilon\in(0,1)$ is to be chosen. If $\theta_0>0$ assume that

$$(x_0,t_0)+Q^-_{8\rho}(\theta_0)=K_{8\rho}(x_0)\times(t_0-\theta_0(8\rho)^p,t_0]\subset\Omega_T,$$

and set

$$\eta=\left[\frac{\displaystyle\fint_{K_\rho(x_0)}u^r(x,t_0)dx}{\displaystyle\fint_{K_{4\rho}(x_0)}u^r(x,t_0-\theta_0\rho^p)dx}\right]^{\frac{p}{\lambda_r}} \tag{7.2}$$

where λ_r is defined in (5.2). We first establish the following intermediate result.

Proposition 7.1. *Suppose $0<\eta<1$. Then there exist constants $\varepsilon,\mu,\alpha_0\in(0,1)$, depending only upon the data $\{p,N\}$ and r and γ, such that*

$$|[u(\cdot,t)>\mu\eta M^*]\cap K_{2\rho}(x_0)|\ge\alpha_0\eta^r|K_{2\rho}|$$

for all $t\in(t_0-\frac12\theta_0\rho^p,t_0]$, where

$$\eta M^*=\frac{\theta_0^{\frac{1}{3-m-p}}}{\delta^*},$$

for a suitable $\delta^*\in(0,1)$ depending only on N,p,r,Λ_γ.

Proof. Assume that (x_0, t_0) coincides with the origin and write $K_\rho(0) = K_\rho$. By Proposition 5.1, considered for the cylinder $K_{2\rho} \times (s, 0)$, $s \in (-\theta_0\rho^p, 0]$, and recalling the definition (7.1) of θ_0, there exists $C > 0$ such that

$$\begin{aligned}\int_{K_\rho} u^r(x,0)dx &\le C\int_{K_{2\rho}} u^r(x,s)dx + C\left[\frac{(\theta_0\rho^p)^r}{\rho^{\lambda_r}}\right]^{\frac{1}{3-m-p}}\\ &= C\int_{K_{2\rho}} u^r(x,s)dx + C\varepsilon^r\int_{K_\rho} u^r(x,0)dx\end{aligned}$$

and then, by choosing ε in such a way that $C\varepsilon^r \le \frac{1}{2}$, one arrives at

$$\int_{K_{2\rho}} u^r(x,s)dx \ge \frac{1}{2C}\int_{K_\rho} u^r(x,0)dx \tag{7.3}$$

for all $s \in (-\theta_0\rho^p, 0]$. Observe that being ε fixed, the length θ_0 of the cylinder is completely determined.

Now consider the cylinder $K_{2\rho} \times (-\frac{1}{2}\theta_0\rho^p, 0]$ for which we apply Theorem 5.5. Recalling the definitions (7.1) and (7.2), of θ_0 and of η respectively, and recalling that $0 < \eta < 1$, one obtains

$$\begin{aligned}\sup_{K_{2\rho}\times(-\frac{1}{2}\theta_0\rho^p,0]} u &\le \frac{C'}{\theta_0^{\frac{N}{\lambda_r}}}\left(\fint_{K_{4\rho}} u^r(x,-\theta_0\rho^p)dx\right)^{\frac{p}{\lambda_r}} + C'\theta_0^{\frac{1}{3-m-p}}\\ &= \frac{C'}{\varepsilon^{\frac{N}{\lambda_r}(3-m-p)}}\frac{1}{\eta}\left(\fint_{K_\rho} u^r(x,0)dx\right)^{\frac{1}{r}} + C'\varepsilon\left(\fint_{K_\rho} u^r(x,0)dx\right)^{\frac{1}{r}}\\ &= C'\left(\frac{1}{\eta\varepsilon^{\frac{N}{\lambda_r}(3-m-p)}} + \varepsilon\right)\left(\fint_{K_\rho} u^r(x,0)dx\right)^{\frac{1}{r}}\\ &\le \frac{1}{\varepsilon'\eta}\left(\fint_{K_\rho} u^r(x,0)dx\right)^{\frac{1}{r}}, \qquad \varepsilon' = \frac{\varepsilon^{\frac{N(3-m-p)}{\lambda_r}}}{2C'}\end{aligned}$$

where $C' = C'(N, p, r)$. Let us define M^* according to

$$\varepsilon'\eta M^* = \left(\fint_{K_\rho} u^r(x,0)dx\right)^{\frac{1}{r}} = \frac{\theta_0^{\frac{1}{3-m-p}}}{\varepsilon}. \tag{7.4}$$

Then we have

$$\sup_{K_{2\rho}\times(-\frac{1}{2}\theta_0\rho^p,0]} u \le M^*.$$

Let $\mu \in (0, 1)$ to be chosen. Using (7.3) and (7.4) and the estimate above, for all $s \in (-\frac{1}{2}\theta_0\rho^p, 0]$ one gets

$$\begin{aligned}(\varepsilon'\eta M^*)^r &\le 2^{N+1}C\fint_{K_{2\rho}} u^r(x,s)dx\\ &\le 2^{N+1}C\left(\fint_{K_{2\rho}\cap[u\le\mu\eta M^*]} u^r(x,s)dx + \fint_{K_{2\rho}\cap[u>\mu\eta M^*]} u^r(x,s)dx\right)\\ &\le 2^{N+1}C\mu^r(\eta M^*)^r + 2^{N+1}C(M^*)^r\frac{|[u(\cdot,s) > \mu\eta M^*]\cap K_{2\rho}|}{|K_{2\rho}|}.\end{aligned}$$

Thereby

$$|[u(\cdot,s) > \mu\eta M^*]\cap K_{2\rho}| \ge \alpha_0\eta^r|K_{2\rho}|.$$

for all $s \in (-\frac{1}{2}\theta_0\rho^p, 0]$, where

$$\alpha_0 = \frac{\varepsilon'^r - \mu^r 2^{N+1}C}{2^{N+1}C}.$$

By choosing $\mu \in (0, 1)$ sufficiently small we can ensure that $\alpha_0 \in (0, 1)$. □

We are now ready to prove the following Harnack inequality.

Theorem 7.2. *Let u be a nonnegative, locally bounded local weak solution to the singular equation (E), satisfying (2.2), (5.1) and (5.2) in Ω_T. Introduce θ_0 as in (7.1) and assume that $\theta_0 > 0$. There exist a constant $\varepsilon \in (0,1)$ depending only on N, p, and a continuous, increasing function $f(\eta)$, defined in $\mathbf{R}^+$ and such that $f(\eta) \to 0$, as $\eta \to 0$, that can be determined a priori only in terms of $\{\alpha, N, p,\}$ and Λ_γ, such that*

$$\inf_{K_{4\rho}(x_0)\times(t_0-\frac{\theta_0}{16}\rho^p,t_0)} u \geq f(\eta) \sup_{K_{2\rho}(x_0)\times(t_0-\frac{\theta_0}{2}\rho^p,t_0)} u. \tag{7.5}$$

Proof. In fact, being the conditions of Proposition 7.1 verified we can then spread the positivity of u. For that, in Proposition 6.1 set $M = \mu\eta M^*$, $a = \alpha_0\eta^q$ and $\delta = \frac{\theta_0}{2M} = \frac{\varepsilon\varepsilon'}{2\mu}$. Such M verifies (4.1):

$$M = \mu\eta M^* = \frac{\mu}{\varepsilon'}\left(\fint_{K_\rho} u^r(x,0)dx\right)^{\frac{1}{r}} \leq \frac{\mu}{\varepsilon'}\sup_{K_\rho} u(\cdot,0) \leq \sup_{Q^-_{8\rho}(\theta_0)} u$$

since, from the previous choice of μ, we have $\mu < \varepsilon'$. Therefore, there exists a constant σ in $(0,1)$, depending upon the data $\{p, N\}$ and α_0, η and δ such that

$$u(\cdot,t) > \sigma\,\mu\eta M^* \qquad \text{in } K_{4\rho},$$

for all $t \in (-\frac{1}{16}\theta_0\rho^p, 0)$; thereby, recalling the estimate for M^*,

$$\inf_{K_{4\rho}\times(-\frac{\theta_0}{16}\rho^p,0)} u \geq f(\eta) \sup_{K_{2\rho}\times(-\frac{\theta_0}{2}\rho^p,0)} u\ , \qquad f(\eta) = \sigma\,\mu\eta. \quad \square$$

Remark 7.3. Inequality (7.5) is not a Harnack inequality per se, since η depends upon the solution itself. Therefore it can be regarded as a weak form of a Harnack estimate. Also the size of the cylinder depends on the solution, giving thereby the name intrinsic to the inequality.

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