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Properties of characters of π-separable groups

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A Hair perhaps divides the False and True;
Yes; and a single Alif were the clue —
Could you but find it — to the Treasure-house,
And peradventure to The Master too;

Khayyam/FitzGerald

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Preface

If G is a π-separable group, where π is a set of primes, it is possible to define a peculiar set of characters $\mathrm{B}_\pi(G)$, subset of the set of the irreducible characters.

This set was first defined by Martin Isaacs in 1984, with the aim of finding a generalization of the Brauer characters. In fact, these B_π-characters have the property that they remain irreducible, and distinct, when restricted to the π-elements of the group, and the restriction of the characters in $\mathrm{B}_{p'}(G)$ to the p-regular elements coincides exactly with the irreducible Brauer characters of the group G.

Thus, B_π-characters were originally a generalization to a set of primes of a canonical set of lifts for the irreducible Brauer characters. However, the theory behind them appeared, from the very beginning, to be rich and elegant, and it was further investigated by several authors, mainly Martin Isaacs, Gabriel Navarro and Thomas Wolf. Moreover, the new Isaacs' book on *Character theory of solvable groups*, which dedicates an entire first part to π-separable groups, may raise a renewed interest in the theory in the next years.

Despite the work of so many authors, however, some aspects of the theory remain open to further investigations. The aim of this thesis is to cover at least some of them.

In Chapter 1 we summarize the basic theory of B_π-characters, as developed by Isaacs and, later, by Navarro, and we see an example of how B_π-characters can be identified without using the algorithm that defines them. In this chapter there are almost no original results; however, it is a useful introduction to the theory for an unfamiliar reader.

In Chapter 2, we study problems related with the zeros of irreducible characters. We see that, in supersolvable groups, B_π-characters behave exactly like ordinary irreducible characters, but this is no longer true if we consider solvable groups. However, we will see that a sort of parallelism between ordinary and B_π-character theory is restored if we replace the set $\mathrm{B}_\pi(G)$ with the larger set $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$.

This unexpected similarity still holds when, in Chapter 4, we study the relation between normal structure of the group G and prime numbers dividing character degrees. One of the main results of Chapter 4, in particular, is that a prime number divides the degree of a character in $\mathrm{Irr}(G)$ if and only if it divides the degree of a character in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$. This may be surprising, since in general the degrees of characters in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ are considerably less, in number, then the degrees of the irreducible characters.

Character degrees are a central topic also in Chapter 5. In fact, in this chapter we use the degrees of the characters in $\mathrm{B}_p(G)$ to find a bound for the p-length of a p-solvable group. Moreover, we will see that the bound still holds if we only consider B_p-characters having values in some restriction of the field of algebraic integers. This is interesting because, if we forget about B_p-characters, we still have a non-trivial bound concerning degrees of ordinary irreducible characters with a restricted field of values. Furthermore, the technique we use to control the field of values of a B_p-character leads to some original results concerning *cut groups*, objects which arise in the study of the ring of unities of a group algebra. Thus, we have an example of how the study of B_π-characters can be used to prove results which initially seemed to be unrelated with it. This is particularly satisfying, since, as Martin Isaacs once wrote[1], a theory "should have the power to answer questions that it did not ask".

In Chapter 6 we talk about character correspondence and the McKay conjecture. Most of the theory related with B_π-character correspondence was already developed at the time we approached the problem; we offer, however, a different point of view on some aspects of it. Then, we temporarily forget about B_π-characters and we study the existence of a natural Mckay correspondence, realized by the character restriction, such that the corresponding characters in the normalizer of the Sylow subgroup are linear.

[1] 18, Preface.

Finally, we mention that Chapter 3 is a stand-alone in the thesis, since it does not involve B_π-characters. Indeed, it is the result of a joint work with M. J. Felipe and V. Ortiz-Sotomayor, of the Universitat Politècnica de València, and it is a prosecution of the study of zeros of irreducible characters which we began in Chapter 2. In particular, in Chapter 3 we see what can be said about the zeros of irreducible characters lying in normal subgroups of G, and what happens to the group structure when we impose conditions on the conjugacy class size of those elements. Even if not directly related with the main topic of the thesis, we believe that it may be interesting for some of the few people who will read it.

There are many people who helped me during my three years as a PhD student.

As first, I have to thank my supervisor, professor Silvio Dolfi, for introducing me to the research in character theory and for correcting the many mistakes I make, when I search for results with excessive optimism.

I thank the reviewers of this thesis, Emanuele Pacifici and Joan Tent, for their useful comments. Joan's observations, in particular, have been precious for correcting some errors which were still present in the thesis.

I thank all the people of the Algebra group in the University of Florence, the stimulating environment they created was crucial in my growth as a mathematician. I also thank Victor and Maria José, of the Universitat Politècnica de València, for welcoming me during the three months I spent in Spain.

Finally, I thank all the people who have previously worked on the character theory of π-separable groups, both for developing such an interesting theory and for having left some problems unsolved for me to study.

Chapter 1

Review of the π-Theory

In this chapter, the reader is introduced to the theory of characters of π-separable groups. At first, the definition of π-separability is given, as well as other basic definitions. Then, we give the definition of π-special characters, as given by Gajendragadkar in [8], and of π-factorable characters. We proceed by defining the B_π-characters and explaining their basic properties, as shown by Isaacs in [16]. Still following [16], we explain how B_π-characters have been used, in π-separable groups, to find a generalization of Brauer characters. Finally, we briefly see that the theory of B_π-characters becomes simpler in groups of odd order.

1.1 Finite π-separable groups

Let π be a set of primes and denote as π' its complementary set. We say that a natural number $m \in \mathbb{N}$ is a π-number if all its prime divisors are in π. For a natural number $n \in \mathbb{N}$, we denote as n_π its π-part, i.e., the largest π-number to divide n. A finite group is said to be a π-group if its order is a π-number.

A finite group G is said to be *solvable* if, given a *composition series*, i.e., a subnormal series of maximal length, each factor group in the series is a group of prime order. Equivalently, a finite group G is solvable if and only if, for any normal series of maximal length, each factor group in the series is an abelian group (of prime power order).

The concept can be generalized by requiring the condition to hold only for some primes.

Definition 1.1.1. A finite group G is said to be π-*solvable* if, given a composition series, each factor group in the series is either a π'-group or a group of prime order. Equivalently, G is π-solvable if and only if, for any normal series of maximal length, each factor group in the series is either a π'-group or an abelian group (of prime power order).

Now, a further generalization of the concept follows quite naturally.

Definition 1.1.2. A finite group G is π-separable if and only if, for any normal series of maximal length, each factor group in the series is either a π-group or a π'-group.

Clearly, a group is π-separable if and only if it is π'-separable. Moreover, a π-solvable group is always π-separable and a solvable group is π-solvable for any set of primes π. Moreover, a $\{p\}$-separable group is clearly also $\{p\}$-solvable and, in this case, we simplify the notation and we write that it is p-solvable.

Definition 1.1.3. A subgroup H of a finite group G is a Hall π-subgroup if $|H| = |G|_\pi$, or, equivalently, if H is a π-group and $|G : H|$ is a π'-number.

A finite group G is not always guaranteed to have a Hall π-subgroup. If G is π-separable, however, then a Hall π-subgroup always exists and any two of them are conjugate.

1.2 π-special and π-factorable characters

It is well known that the existence of the extension, to a group G, of a G-invariant character ψ of a normal subgroup is strongly linked with the primes dividing the order and the degree of ψ. A family of characters can be defined, which *behaves well* in this sense.

Definition 1.2.1 (*[8, Definition 2.1]*)**.** If G is a finite group, a character $\chi \in \mathrm{Irr}(G)$ is said to be π-*special* if its degree and its order are π-numbers and, for any subnormal subgroup M of G and any irreducible constituent φ of χ_M, $o(\varphi)$ is a π-number.

Sometimes, we will write $\mathrm{X}_\pi(G)$ to describe the subset of $\mathrm{Irr}(G)$ of all the π-special characters. Notice that, if G is a π'-group and $\varphi \in \mathrm{X}_\pi(G)$, since both $\chi(1)$ and $o(\chi)$ have to be π-numbers, it follows that $\chi(1) = o(\chi) = 1$ and, therefore, $\mathrm{X}_\pi(G) = \{1_G\}$.

The behaviour of π-special characters when induced from, or restricted to, normal subgroups is well described by the following propositions.

Proposition 1.2.2 (*[8, Proposition 4.1]*)**.** *Let G be a finite group and $\chi \in \mathrm{X}_\pi(G)$. If M is a subnormal subgroup of G, then every irreducible constituent of χ_M is a π-special character.*

Proposition 1.2.3 (*[8, Proposition 4.5]*)**.** *Let G be a finite group and let $N \lhd G$ such that G/N is a π-group. If $\psi \in \mathrm{X}_\pi(N)$, then every irreducible constituent of ψ^G is a π-special character.*

Proposition 1.2.4 (*[8, Proposition 4.3]*)**.** *Let G be a finite group and let $N \lhd G$ such that G/N is a π'-group. If $\psi \in \mathrm{X}_\pi(N)$ is G-invariant, then it extends to G and there exists a unique extension which is a π-special character. If ψ is not G-invariant, none of the irreducible constituents of ψ^G is a π-special character.*

There is no need for the group to be π-separable in order to define π-special characters. However, if G is a π-separable group, we have further interesting properties.

Theorem 1.2.5 (*[8, Theorem 7.2]*)**.** *Let G be a π-separable group and let $\alpha, \beta \in \mathrm{Irr}(G)$, with α π-special character and β π'-special character. Then, $\alpha\beta$ is an irreducible character of G and this factorization is unique.*

An irreducible character which can be written as a product of a π-special and a π'-special character is said to be a *π-factorable character.*

We now consider *character pairs* (H, θ), where H is a subgroup of some fixed group G and θ is an irreducible character of H. We say that $(H, \theta) \leq (K, \varphi)$ if $H \leq K$ and θ is an irreducible constituent of φ_H. This defines a partial order on the set of character pairs.

Definition 1.2.6 (*[16, Definition 3.1]*)**.** Let G be π-separable. A *π-factorable subnormal pair* of G is a character pair (S, θ), where $S \lhd \lhd G$ and θ is a π-factorable character. We write $\mathfrak{F}_\pi(G)$ to denote the set of π-factorable subnormal pairs in G.

Theorem 1.2.7 (*[16, Theorem 3.2]*). *Let G be π-separable and let $\chi \in \mathrm{Irr}(G)$ then there exists a π-factorable subnormal pair (S,θ) of G such that it is maximal in $\mathfrak{F}_\pi(G)$ and $(S,\theta) \leq (G,\chi)$. Moreover, if (R,η) is another such pair, then $R = S^g$ and $\eta = \theta^g$ for some $g \in G$.*

Theorem 1.2.8 (*[16, Theorem 4.4 and Lemma 4.5]*). *Let G be π-separable and let (S,μ) be a maximal π-factorable subnormal pair. Let $T = I_G(S,\theta)$, where $I_G(S,\theta) = I_{\mathrm{N}_G(S)}(\theta)$. Then the induction defines a bijection between $\mathrm{Irr}(T \mid \mu)$ and $\mathrm{Irr}(G \mid \mu)$. Moreover, if $S < G$, then also $T < G$.*

If G is π-separable and $\chi \in \mathrm{Irr}(G)$, by Theorem 1.2.7 we have that there exists $(S,\mu) \in \mathfrak{F}_\pi(G)$ maximal such that $(S,\mu) \leq (G,\chi)$. If $T = I_G(S,\mu)$, by Theorem 1.2.8 there exists $\xi \in \mathrm{Irr}(T \mid \mu)$ such that $\xi^G = \chi$. This process associates, to the pair (G,χ) a specific pair (T,μ), determined uniquely up to conjugacy in G, which is called a *standard inducing pair* for (G,χ).

If χ is already π-factorable, then $(S,\mu) = (G,\chi)$ and, therefore, also $T = G$. Otherwise, $S < G$ and, by Theorem 1.2.8, also $T < G$. In this case, we can repeat the process and find a standard inducing pair for (T,ξ). If we continue this way until we reach a π-factorable pair, which will happen eventually, since the group is finite, we have

$$(G,\chi) = (T_0,\xi_0) > (T_1,\xi_1) > \dots > (T_k,\xi_k),$$

where (T_i,ξ_i) is a standard inducing pair for (T_{i-1},ξ_{i-1}) and ξ_k is π-factorable. At each stage, the pair (T_i,ξ_i) is determined up to conjugacy in T_{i-1}, in particular, the terminal pair (T_k,μ_k) is determined up to conjugacy in G.

Definition 1.2.9 (*[16, Definition 4.6]*). Let G be π-separable and let $\chi \in \mathrm{Irr}(G)$. Any pair (W,μ) with μ π-factorable, which results from repeatedly constructing standard inducing pairs, beginning with (G,χ), is said to be a *nucleus* for χ. The set of nuclei for χ is denoted as $\mathrm{nuc}(\chi)$.

If $(W,\mu) \in \mathrm{nuc}(\chi)$, then μ is said to be a *nucleus character* for χ and $\mu^G = \chi$.

Corollary 1.2.10. *Let G be π-separable and suppose $\chi \in \mathrm{Irr}(G)$ is primitive, then χ is π-factorable.*

1.3 The B_π-characters

We can now give the definition of B_π-characters.

Definition 1.3.1 (*[16, Definition 5.1]*)**.** Let $\chi \in \mathrm{Irr}(G)$, where G is a π-separable group, and let $(W, \mu) \in \mathrm{nuc}(\chi)$, which is unique up to conjugation for elements of G. If μ is a π-special character, we say that χ is a B_π-*character*. We denote as $\mathrm{B}_\pi(G)$ the set of B_π-characters of the group G.

Clearly, the principal character 1_G is always a B_π-character, for every set of primes π. Moreover, for any π, $\mathrm{B}_\pi(G) \cap \mathrm{B}_{\pi'}(G) = \{1_G\}$.

As the definition may suggest, there exists a strong relation between π-special and B_π-characters.

Lemma 1.3.2 (*[16, Lemma 5.4]*)**.** *Let G be a π-separable group and let $\chi \in \mathrm{Irr}(G)$. The following are equivalent.*

i) χ is π-special.

ii) $\chi \in \mathrm{B}_\pi(G)$ and $\chi(1)$ is a π-number.

iii) $\chi \in \mathrm{B}_\pi(G)$ and χ is π-factorable.

As a consequence, we have that, if G is a π-separable group, $\mathrm{X}_\pi(G) \subseteq \mathrm{B}_\pi(G) \subseteq \mathrm{Irr}(G)$.

It is interesting, and useful, to study the behaviour of the B_π-characters in relation with normal subgroups. As expected, this behaviour will be similar to the one of π-special characters.

Theorem 1.3.3. *Let G be π-separable and let $M \trianglelefteq G$. If $\chi \in \mathrm{B}_\pi(G)$, then every irreducible constituent of χ_M belongs to $\mathrm{B}_\pi(M)$.*

On the other hand, if $\psi \in \mathrm{B}_\pi(M)$, then there exist some characters in $\mathrm{B}_\pi(G)$ lying over ψ. In particular, if G/M is a π-group, then every character in the set $\mathrm{Irr}(G \mid \psi)$ belongs to $\mathrm{B}_\pi(G)$ while, if G/M is a π'-group, then there exists a unique character in the set $\mathrm{Irr}(G \mid \psi)$ which belongs to $\mathrm{B}_\pi(G)$.

Proof. It is a direct consequence of [16, Theorem 6.2] and [16, Theorem 7.1] □

Corollary 1.3.4 (*[16, Corollary 5.3]*)**.** *If G is π-separable, $\mathrm{O}_{\pi'}(G)$ is in the kernel of every character $\chi \in \mathrm{B}_\pi(G)$.*

The main property of B_π-characters, however, concerns their restriction to Hall π-subgroups.

Theorem 1.3.5 (*[16, Theorem 8.1]*)**.** *Let $\chi \in \mathrm{B}_\pi(G)$, with G π-separable, and let $H \in \mathrm{Hall}_\pi(G)$. Then the following hold.*

a) For each $\alpha \in \mathrm{Irr}(H)$, $\alpha(1) \geq [\alpha, \chi_H]\chi(1)_\pi$.

b) There exists at least one irreducible constituent α of χ_H such that $\alpha(1) = \chi(1)_\pi$.

c) If α is as in b), then $[\chi_H, \alpha] = 1$, and $[\psi_H, \alpha] = 0$ for any $\psi \in \mathrm{B}_\pi(G)$, $\psi \neq \chi$.

Corollary 1.3.6 (*[16, Corollary 8.2]*)**.** *Let G be π-separable and let H be a Hall π-subgroup of G. Then, restriction defines an injection from the set of π-special characters of G into* $\mathrm{Irr}(H)$.

Characters like the ones in Theorem 1.3.5, point *b)*, play an important role in the theory of characters of π-separable groups. We refer to them as *Fong characters.*

Definition 1.3.7 (*[16, Definition 8.6]*)**.** Let G be a π-separable group and let H be a Hall π-subgroup of G. We say that $\alpha \in \mathrm{Irr}(H)$ is a *Fong character* of H in G if there exists $\chi \in \mathrm{B}_\pi(G)$ such that α is a constituent of χ_H and $\alpha(1) = \chi(1)_\pi$. We say that α is associated with χ.

As a first consequence of Theorem 1.3.5, we have informations about the field of values of the characters in $\mathrm{B}_\pi(G)$.

If n is a natural number, we write $\mathbb{Q}_n$ to refer to the *n-cyclotomic extension* of $\mathbb{Q}$, i.e., the extension of the field of rational numbers obtained by adjoining a primitive n-root of unity ζ_n to $\mathbb{Q}$. If π is a set of primes, $\mathbb{Q}_\pi$ denotes the extension of the field of rational numbers obtained by adjoining all complex n-th roots of unity of $\mathbb{Q}$, for all π-numbers n.

Corollary 1.3.8 (*[16, Corollary 12.1]*)**.** *If $\chi \in \mathrm{B}_\pi(G)$, then it has values in $\mathbb{Q}_\pi$, i.e., for every $x \in G$, $\chi(x) \in \mathbb{Q}_\pi$.*

If H is an Hall π-subgroup of a π-separable group G, it is in general hard to determine whether a character $\varphi \in \mathrm{Irr}(H)$ is a Fong character associated with some $\chi \in \mathrm{B}_\pi(G)$. The task, however, becomes easier under some extra assumptions.

Theorem 1.3.9 (*([21, Corollary 6.1] or [18, Theorem 5.13])*)**.** *Let H be a Hall π-subgroup of a π-separable group G and let $\varphi \in \mathrm{Irr}(H)$. If φ is primitive, then it is a Fong character associated with some character $\chi \in \mathrm{B}_\pi(G)$. Moreover, $\eta \in \mathrm{Irr}(H)$ is a Fong character associated with χ if and only if φ and η are $\mathrm{N}_G(H)$-conjugated.*

Furthermore, if a character in $\mathrm{Irr}(H)$ is not only primitive but linear, we can rely to an even stronger result, which allows us to determine the associated B_π-character without the use of the Isaacs' algorithm we described before Definition 1.2.9.

Theorem 1.3.10. *Let G be a π-solvable group and let H be a Hall π-subgroup of G. Let $\varphi \in \mathrm{Irr}(H)$. Then, there exists a unique maximal subgroup W of G such that φ extends to W, and φ has a unique extension $\hat{\varphi}$ to W which is π-special.*

Moreover, if φ is linear, then $\chi = (\hat{\varphi})^G$ is an irreducible character in $\mathrm{B}_\pi(G)$ and φ is a Fong character associated with χ.

Finally, if $\psi \in \mathrm{Irr}(G)$ and $\psi(1)$ is a π'-number, then there exists $\varphi \in \mathrm{Lin}(H)$ and $W, \hat{\varphi}$ as in the first paragraph such that $\psi = (\hat{\varphi}\beta)^G$, for some π'-special character $\beta \in \mathrm{Irr}(W)$.

Proof. The first part is a consequence of [19, Theorem A] and of [22, Theorem F], while the second part follows from [21, Theorem B] and [25, Theorem 2.2] and the third part follows from [25, Theorem 3.6]. □

Finally, it is easy to work with Fong characters when the Hall π-subgroup is normal.

Proposition 1.3.11. *Let G be a finite group with a normal Hall π-subgroup H, then each irreducible character of H is a Fong character in G. Moreover, if $\varphi \in \mathrm{Irr}(H)$, $I = \mathrm{I}_G(\varphi)$ and $\eta \in \mathrm{Irr}(I \mid \varphi)$, then $\chi = \eta^G \in \mathrm{Irr}(G)$ is a B_π-character if and only if η is a π-special character.*

Proof. Let $\varphi \in \mathrm{Irr}(H)$, then $(H, \varphi) \in \mathfrak{F}_\pi(G)$ since H is normal in G. Let $\chi \in \mathrm{Irr}(G)$ lying over φ and let $(M, \gamma) \in \mathfrak{F}_\pi(G)$ maximal such that $(H, \varphi) \leq (M, \gamma) \leq (G, \chi)$. Then, $\gamma = \alpha\beta$, with α π-special, and thus $\alpha_H = \varphi$.

Let $I = \mathrm{I}_G(\varphi)$ and let $J = \mathrm{I}_G(M, \gamma)$; then, $J \leq I$, since φ extends to $\alpha \in \mathrm{Irr}(M)$ and, thus, if $g \in \mathrm{N}_G(M)$ fixes α, it also fixes φ. By iterating the Isaacs' algorithm, with (M, γ) in place of (G, χ), we obtain $(W, \mu) \in \mathrm{nuc}(\chi)$ such that $(H, \varphi) \leq (W, \mu) \leq (I, \eta)$, for some $\eta \in \mathrm{Irr}(I \mid \varphi)$ such that $\mu^I = \eta$ and, thus, $\eta^G = \chi$.

Now, if η is π-special, then η_H is irreducible by Corollary 1.3.6 and, thus, also $\eta_W \in \mathrm{Irr}(W)$. It follows that $(W, \mu) = (I, \eta)$, μ is π-special and, therefore, $\chi \in \mathrm{B}_\pi(G)$.

On the other hand, if $\chi \in \mathrm{B}_\pi(G)$, then φ is a Fong character associated with χ, since $\chi(1)_\pi = \varphi(1)$. As a consequence, $[\eta_H, \varphi] \leq [\chi_H, \varphi] = 1$ and, thus, $\eta_H = \varphi$ and $\eta_W = \mu$. It follows that $(W, \mu) = (I, \eta)$ and η is π-special, because so is μ.

Finally, by Proposition 1.2.4, $\mathrm{X}_\pi(I \mid \varphi)$ is nonempty, thus, φ is a Fong character in G. □

1.4 Restriction to π-elements

Let χ be a character of a finite group G, not necessarily irreducible; it is possible to restrict χ, as a class function, to the conjugacy classes of π-elements of G. Let χ^* be this restriction. We say that χ^* is a *π-partial character.*

A π-partial character χ^* is said to be *reducible* if it can be written as a sum of two other π-partial characters, i.e., if there exist $\psi, \theta \in \mathrm{Char}(G)$ such that $\chi^* = \psi^* + \theta^*$. If χ^* is not reducible, we say that it is *irreducible* and we denote as $\mathrm{I}_\pi(G)$ the set of irreducible π-partial characters.

If $\pi = p'$, so that G is a p-solvable group, Fong-Swan theorem asserts that $\mathrm{I}_{p'}(G) = \mathrm{IBr}_p(G)$, i.e., the p'-partial characters are exactly the irreducible p-Brauer characters. As a consequence, $\mathrm{I}_{p'}(G)$ is a base set for the class functions of G defined on p-regular elements.

For a generic set of primes π, we cannot rely on Brauer theory any more. However, it is still possible that irreducible π-partial characters may form a basis for the class functions defined on π-elements.

This was the motivation for Isaacs to first study, in [16], the character theory of π-separable groups, and to introduce the B_π-characters.

A first related result is a direct consequence of Theorem 1.3.5.

Corollary 1.4.1 (*[16, Corollary 9.1]*)**.** *Let G be π-separable, then the functions χ^* are distinct and linearly independent for $\chi \in \mathrm{B}_\pi(G)$.*

The key result, then, is to find the relation between the number of B_π-characters and the number of conjugacy classes of π-elements.

Theorem 1.4.2 (*[16, Theorem 9.3]*)**.** *Let G be π-separable. Then $|\mathrm{B}_\pi(G)|$ is equal to the number of conjugacy classes of π-elements. The restriction χ^* of $\chi \in \mathrm{B}_\pi(G)$ to π-elements are distinct and they form a basis for the class functions on π-elements of G.*

An interesting consequence of the proof of Theorem 1.4.2 is a version for B_π-characters of Brauer theorem on group actions.

Corollary 1.4.3 (*[16, Lemma 9.6]*)**.** *Let G be a π-separable group and let $\sigma \in \mathrm{Aut}(G)$. Then σ fixes the same number of $\chi \in \mathrm{B}_\pi(G)$ as it fixes classes of π-elements.*

An other consequence of Theorem 1.4.2 concerns the kernels of B_π-characters. We have already seen in Corollary 1.3.4 that $\mathrm{O}_{\pi'}(G)$ is in the kernel of every B_π-character. Now, however, we can have a more precise result.

Corollary 1.4.4. *If G is a π-separable group, $\mathrm{O}_{\pi'}(G) = \bigcap\{\ker(\chi) \mid \chi \in \mathrm{B}_\pi(G)\}$.*

Proof. Let $K = \bigcap\{\ker(\chi) \mid \chi \in \mathrm{B}_\pi(G)\}$, then we already know from Corollary 1.3.4 that $\mathrm{O}_{\pi'}(G) \leq K$. On the other hand, notice that $K \lhd G$. Thus, if $\psi \in \mathrm{B}_\pi(K)$, then by Theorem 1.3.3 there exists $\chi \in \mathrm{B}_\pi(G)$ lying over ψ. Since $K \leq \ker(\chi)$, we have that $\psi = 1_K$. Thus, $|\mathrm{B}_\pi(K)| = 1$ and it follows by Theorem 1.4.2 that $\{1\}$ is the only conjugacy class of π-elements of K. Thus, $K \leq \mathrm{O}_{\pi'}(G)$ and the thesis follows. □

Finally, using Theorem 1.4.2, it is proved that there exists a bijection between $\mathrm{B}_\pi(G)$ and $\mathrm{I}_\pi(G)$.

Corollary 1.4.5 (*[16, Corollary 10.2]*). *Let G be π-separable. Then, restriction to π-elements realizes a bijection between* $\mathrm{B}_\pi(G)$ *and* $\mathrm{I}_\pi(G)$. *In particular,* $\mathrm{I}_\pi(G)$ *is a basis for the class functions on π-elements of G.*

If we take $\pi = p'$, the Corollary 1.4.5 provides a family of lifts for the irreducible Brauer characters. In fact, the classical Fong-Swan theorem [32, Theorem 10.1] already proved that, in a p-solvable group, every irreducible Brauer character coincides with the restriction of an irreducible ordinary character to p-regular elements. Only in [23], however, it was provided a set of *lifts* for the Brauer characters which *behaves well* in relation with normal subgroups.

Corollary 1.4.6 (*[16, Corollary 10.3]*). *If G is a p-solvable group, restriction to p-regular elements realizes a bijection between* $\mathrm{B}_{p'}(G)$ *and* $\mathrm{IBr}_p(G)$.

However, Corollary 1.4.6 can also be seen as a consequence of a more general result.

Theorem 1.4.7 (*[16, Theorem 11.1]*). *Let G be π-separable and let $p \notin \pi$ be a prime. Then, restriction to p-regular elements defines an injection* $\mathrm{B}_\pi(G) \mapsto \mathrm{IBr}_p(G)$.

1.5 Groups of odd order

It is worth to talk briefly about the behaviour of B_π-characters in groups of odd order. In fact, in those groups the theory behind B_π-characters happens to be a lot simpler, and it suggests intriguing similarities between ordinary and B_π-characters, as we will see in the next chapters.

Properties of B_π-characters in groups of odd order have been studied extensively in [15]. The main results of that paper are the following.

The reader shall remember that, due to Feit-Thompson Theorem, a group of odd order is solvable. Therefore, there is no need to assume the group to be π-separable.

Theorem 1.5.1 (*[15, Theorem C]*). *Let G be a group of odd order and let K be a π-complement, i.e., a Hall π'-subgroup. Let $\chi \in \mathrm{Irr}(G)$. Then,* $\chi \in \mathrm{B}_\pi(G)$ *if and only if* $[(1_K)^G, \chi]$ *is odd.*

Theorem 1.5.2 (*[15, Theorem D]*)**.** *Let G be a group of odd order and let K be a π-complement, i.e., a Hall π'-subgroup. Let $\chi \in \mathrm{X}_\pi(G)$. Then, 1_K is the only irreducible constituent of χ_K of odd multiplicity.*

To prove those results, some interesting technique is used. One of the key steps of the proof, in particular, is the following lemma.

Lemma 1.5.3 (*[15, Lemma 3.1]*)**.** *Let G be of odd order and suppose π is a set of primes. Let χ be an irreducible character of G, then $\chi \in \mathrm{B}_\pi(G)$ if and only if χ have values in $\mathbb{Q}_\pi$.*

There exists a version of Lemma 1.5.3 also for groups of even order, assuming that $2 \in \pi$.

Lemma 1.5.4 (*[15, Lemma 3.3]*)**.** *Let G be π-separable with $2 \in \pi$. Let χ be an irreducible character of G, then $\chi \in \mathrm{B}_\pi(G)$ if and only if χ have values in $\mathbb{Q}_\pi$ and the restriction of χ to π-elements lie in $\mathrm{I}_\pi(G)$.*

A useful consequence of Lemmas 1.5.3 and 1.5.4 is the following.

Corollary 1.5.5 (*[15, Corollary 3.5]*)**.** *Let G be π-separable and assume either that $2 \in \pi$ or that $|G|$ is odd. Let $N \lhd G$, $\theta \in \mathrm{Irr}(N)$ and let $T = \mathrm{I}_G(\theta)$. Suppose $\psi \in \mathrm{Irr}(T \mid \theta)$ and let $\chi = \psi^G$, so that $\chi \in \mathrm{Irr}(G)$. Then, $\chi \in \mathrm{B}_\pi(G)$ if and only if $\psi \in \mathrm{B}_\pi(T)$.*

1.6 An example: characters of $\mathrm{SL}(2,3) \ltimes (\mathbb{Z}_3)^2$

We conclude the chapter with an example of B_π-characters of a solvable group.

Let $G = \mathrm{SL}(2,3) \ltimes (\mathbb{Z}_3)^2$, with $\mathrm{SL}(2,3)$ acting naturally on $(\mathbb{Z}_3)^2$. The group structure is well known: $|G| = 216$, G' has index 3 in G and $(\mathbb{Z}_3)^2$ is a subgroup of G' of index 8.

We compute the character table of G using the software GAP.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
$\|C_i\|$	1	36	12	24	36	12	24	54	8	9
$o(x)$	1	6	3	3	6	3	3	4	3	2
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	α	$\bar{\alpha}$	$\bar{\alpha}$	$\bar{\alpha}$	α	α	1	1	1
χ_3	1	$\bar{\alpha}$	α	α	α	$\bar{\alpha}$	$\bar{\alpha}$	1	1	1
χ_4	2	1	-1	-1	1	-1	-1	0	2	-2
χ_5	2	α	-$\bar{\alpha}$	-$\bar{\alpha}$	$\bar{\alpha}$	-α	-α	0	2	-2
χ_6	2	$\bar{\alpha}$	-α	-α	α	-$\bar{\alpha}$	-$\bar{\alpha}$	0	2	-2
χ_7	3	0	0	0	0	0	0	-1	3	3
χ_8	8	0	2	-1	0	2	-1	0	-1	0
χ_9	8	0	2α	$-\alpha$	0	$2\bar{\alpha}$	$-\bar{\alpha}$	0	-1	0
χ_{10}	8	0	$2\bar{\alpha}$	$-\bar{\alpha}$	0	2α	$-\alpha$	0	-1	0

Here, $\alpha = \frac{-1+\sqrt{-3}}{2} = e^{\frac{2\pi}{3}i}$.

We can determine which characters of G are in $\mathrm{B}_{\{2\}}(G)$, and which are in $\mathrm{B}_{\{3\}}(G)$, without recurring to Isaacs' algorithm.

Clearly, $\chi_1 = 1_G$ is both a $\mathrm{B}_{\{2\}}$-character and a $\mathrm{B}_{\{3\}}$-character. Characters χ_2 and χ_3 are linear of order 3, thus, they are $\{3\}$-special characters.

Characters χ_4, χ_5 and χ_6 restrict irreducibly to a character $\psi \in \mathrm{Irr}(G')$ and have $(\mathbb{Z}_3)^2$ in their kernel. It follows that ψ is a character of the 2-group $G'/(\mathbb{Z}_3)^2$ and, therefore, is a $\{2\}$-special character. Since ψ is invariant in G, we know from Proposition 1.2.4 that there exists only one irreducible $\{2\}$-special character lying over ψ. This character can only be χ_4, since it is the only one having values in $\mathbb{Q}_{\{2\}}$ (see Corollary 1.3.8). On the other hand, χ_5 and χ_6 are nor in $\mathrm{B}_{\{2\}}(G)$ neither in $\mathrm{B}_{\{3\}}(G)$.

The character χ_7 has degree 3 and has $(\mathbb{Z}_3)^2$ in its kernel. It follows that is is induced by a linear $\{2\}$-special character of G' and, thus, it is in $\mathrm{B}_{\{2\}}(G)$ by Theorem 1.3.3.

Finally, χ_8, χ_9 and χ_{10} are all extensions to G of a character $\theta \in \mathrm{Irr}(G')$ of degree 8, which is induced by a non-principal linear character of the 3-subgroup $(\mathbb{Z}_3)^2$. Therefore, θ is in $\mathrm{B}_{\{3\}}(G')$ and it follows from Theorem 1.3.3 that χ_8, χ_9 and χ_{10} are in $\mathrm{B}_{\{3\}}(G)$.

Thus, $\mathrm{B}_{\{2\}}(G) = \{1_G, \chi_4, \chi_7\}$, $\mathrm{B}_{\{3\}}(G) = \{1_G, \chi_2, \chi_3, \chi_8, \chi_9, \chi_{10}\}$.

Chapter 2

Zeros of B_π-characters

In this chapter, we study problems related with the zeros of irreducible characters. After a brief review of the existing results concerning the zeros of ordinary characters, we search for similar ones for π-special and B_π-characters. In particular, at first we study the zeros of π-special characters under the strong hypothesis that the group is supersolvable. Then, we characterize the elements where no B_π-character *vanishes*, i.e., where the character value is not zero, in solvable and supersolvable groups. Finally, we see what happens when no B_π-character and no $\mathrm{B}_{\pi'}$-character vanishes on every p-element of a π-separable group, for some prime p.

2.1 Review of the theory of vanishing elements

Let $\chi \in \mathrm{Char}(G)$ and $x \in G$. If $\chi(x) = 0$, we say that χ *vanishes* on x. It is known from a famous theorem of Burnside (see [14, Theorem 3.15]) that every nonlinear irreducible character of a group G vanishes on some element of the group. It is interesting, however, to understand *where* an irreducible character vanishes, and where it does not.

We also mention that Burnside's Theorem has been later improved in [29], using the Classification of finite simple groups.

Theorem 2.1.1 (*[29, Theorem B]*)**.** *If G is a finite group and $\chi \in \mathrm{Irr}(G)$ is nonlinear, there exists $g \in G$ of prime-power order such that $\chi(g) = 0$.*

We first see some results published in [33]. In that paper, it is studied where primitive characters vanish. A first interesting result of [33], however, does not require a character to be primitive, or even irreducible.

Theorem 2.1.2 (*[33, Theorem 2.1]*)**.** *Let χ be a character of a finite group G and suppose that a prime p does not divide $\chi(1)$. If $x \in G$ has p-power order, then $\chi(x) \neq 0$.*

The main result of [33] is the following.

Theorem 2.1.3 (*[33, Corollary 2.4]*)**.** *Let G be a solvable group and let $\chi \in \mathrm{Irr}(G)$ be primitive. Suppose $\chi(1)$ is a π-number, where π is a set of primes. Let $x \in G$, then $\chi(x) = 0$ if and only if $\chi(x_\pi) = 0$, where x_π is the π-part of x.*

We recall that the *π-part* of an element $x \in G$, with G finite, is the unique element $x_\pi \in G$ such that $o(x_\pi)$ is a π-number and $x = x_\pi y$, for some $y \in G$ such that $o(y)$ is a π'-number.

Corollary 2.1.4. *Let G be a solvable group and let $\chi \in \mathrm{Irr}(G)$ be primitive. Let $x \in G$. If $(o(x), \chi(1)) = 1$, then $\chi(x) \neq 0$.*

In [33], moreover, it is presented also an interesting consequence of Theorem 1.5.2 for groups of odd order.

Theorem 2.1.5. *Let G be a group of odd order and let $\chi \in \mathrm{Irr}(G)$ be π-special. If x is a π'-element of G, then $\chi(x) \neq 0$.*

The most influential paper on zeros of irreducible characters is probably [26], where *nonvanishing elements* are studied.

Definition 2.1.6 (*[26]*)**.** An element $x \in G$ is said to be *nonvanishing* if no irreducible character vanishes on x.

A group G is said to be *supersolvable* if each *chief factor* of G is of prime order. We recall that, for $M, N \lhd G$, $M < N$, N/M is said to be a *chief factor* of G if there is no $H \lhd G$ such that $M < H < N$.

It is clear from the definition that a supersolvable group is also solvable. On the other hand, it is not difficult to prove that a nilpotent group is supersolvable.

If G is a finite group and M, N are two normal nilpotent subgroup of G, then also the product MN is normal and nilpotent in G. For a group G, the *Fitting subgroup* of G, denoted by $\mathrm{F}(G)$, is its maximal normal nilpotent subgroup.

It exists a characterization of nonvanishing elements in supersolvable groups, involving the Fitting subgroup.

Theorem 2.1.7 (*[26, Theorem B]*)**.** *If G is a supersolvable group, all the nonvanishing elements of G lie in $\mathrm{Z}(\mathrm{F}(G))$. In particular, if G is nilpotent, then the nonvanishing elements of G are central.*

If G is solvable, however, the situation is less transparent.

Theorem 2.1.8 (*[26, Theorem D]*)**.** *Let x be a nonvanishing element of the solvable group G, then the image of x in $G/\mathrm{F}(G)$ has 2-power order. In particular, if x has odd order, then $x \in \mathrm{F}(G)$. In any case, if G is not nilpotent, then x lies in the penultimate term of the ascending Fitting series.*

It is conjectured that a nonvanishing element of a solvable group always lies in Fitting subgroup. However, the conjecture is still unproven.

A different approach to the study of nonvanishing elements was presented in [7], where it was studied what happens where every p-element of a group is nonvanishing. The results in [7] rely on the Classification of finite simple groups.

Theorem 2.1.9 (*[7, Theorem A]*)**.** *Let G be a finite group and p a prime number. If every p-element of G is nonvanishing, then G has a normal Sylow p-subgroup.*

Using similar techniques, the result has been later improved in [28].

Theorem 2.1.10 (*[28, Theorem B]*)**.** *Let G be a finite group, let p be a prime and let $P \in \mathrm{Syl}_p(G)$. The following conditions are equivalent:*

a) p does not divide the degree of any irreducible constituent of $(1_P)^G$;

b) $\chi(x) \neq 0$ for all irreducible constituents χ of $(1_P)^G$ and all $x \in P$;

c) $P \lhd G$.

Notice that Theorem 2.1.10 links the normality of a Sylow p-subgroup with both nonvanishing properties and primes dividing character degrees. The relation between character degrees and normal structure of the group will be the main topic of Chapter 4.

2.2 Zeros of π-special characters in supersolvable groups

Theorem 2.1.5 says that a π-special character in a group of odd order never vanishes on a π'-element. It would be tempting to try to generalize the result for solvable group of every order; however, [33, Example 3.3] proves that such generalization is impossible, even when $2 \in \pi$.

Therefore, when considering groups of even order, it is necessary to strengthen our hypothesis in other ways.

A character χ of a group G is said to be *monomial* if $\chi = \lambda^G$, where λ is a linear character of some (not necessarily proper) subgroup of G. A group G is said to be an *M-group* if every irreducible character of G is monomial.

It is known that M-groups are solvable (see [14, Corollary 5.13]); otherwise, the class of M-groups remains hard to study, also because it is not closed for subgroups. There is however a sufficient condition for a group to be a M-group: in fact, G is a M-group if each chief factor of every subgroup of G has nonsquare order (see [14, Theorem 6.22]). In particular, the condition is verified when G is a supersolvable group, since in this case every chief factor has prime order.

We prove first that, under slightly weaker conditions then the supersolvability, every π-special character is monomial.

Lemma 2.2.1. *Let π be a set of primes and let G be a solvable group such that, if F is any subgroup of G, L/K is a chief factor of F and $|L/K|$ is a π-number, then $|L/K|$ is not a square. Suppose, furthermore, that $2 \in \pi$ or $|G|$ is odd. Let $\chi \in \mathrm{Irr}(G)$ be a π-special character. Then, there exists $W \leq G$ and $\lambda \in \mathrm{Irr}(W)$ such that λ is a linear π-special character and $\lambda^G = \chi$.*

Proof. Let $\chi \in \mathrm{Irr}(G)$ be a nonlinear π-special character and let $M \lhd G$ be minimal for the property of χ_M being irreducible. We can assume

$M > 1$, since χ is nonlinear. Let $N \lhd G$ such that M/N is a chief factor of G. Then, M/N is a chief factor of π-order, since otherwise $(\chi_M)_N = \chi_N$ would be irreducible, too, because χ is π-special. It follows by hypothesis that $|M/N|$ is nonsquare and, thus, $\chi_N = \sum_{i=1}^{t} \varphi_i$ for some distinct $\varphi_i \in \mathrm{Irr}(N)$ and $t > 1$, by [14, Theorem 6.18]. Let $\varphi = \varphi_1$, one has that $T = \mathrm{I}_G(\varphi) < G$.

Moreover, if $\mu \in \mathrm{Irr}(T \mid \varphi)$ is such that $\mu^G = \chi$, since either $2 \in \pi$ or $|G|$ is odd it follows from Corollary 1.5.5 that μ is a B_π-character and, since $\mu(1)$ is a π-number, because so is $\chi(1)$, by Lemma 1.3.2 we also have that μ is π-special.

It follows that, by induction on the group order, there exists $W \leq T$, $\lambda \in \mathrm{Lin}(W)$ π-special such that $\lambda^T = \mu$. Then, $\lambda^G = \chi$ and the lemma is proved. □

Corollary 2.2.2. *Let G be solvable, let $\chi \in \mathrm{B}_\pi(G)$ and let (V, μ) be a nucleus for χ. Suppose that, if F is a subgroup of V, L/K is a chief factor of F and $|L/K|$ is a $\pi \cup \{2\}$-number, then $|L/K|$ is not a square. Let $x \in G$ be a π'-element and suppose that $o(x)$ and $\chi(1)$ are coprime.*

Furthermore, suppose that at least one of these conditions hold:

(i) $o(x)$ is odd;

(ii) G has a normal 2-complement.

Then, $\chi(x) \neq 0$.

Moreover, if condition (i) holds, there exists $W \leq G$ such that $|G : W| = \chi(1)$ and, given $T = \{g_1, ..., g_k\}$ a system of right coset representatives of W in G, then $\chi(x) = \big|\{g \in T \mid gxg^{-1} \in W\}\big|$.

Proof. Let $\delta = \pi \cup \{2\}$, then μ is δ-special, too, and by Lemma 2.2.1 there exists $W \leq V$ and $\lambda \in \mathrm{Lin}(W)$ δ-special such that $\lambda^V = \mu$ and, thus, $\lambda^G = \chi$. Since $\chi(1) = |G : W|$ is coprime with $o(x)$ and the group G is solvable, W can be chosen such that $x \in W$.

If conditions (i) holds, then $o(\lambda)$ and $o(x)$ are coprime and, thus, $\lambda(x^g) = 1$ for every $g \in G$ such that $x^g \in W$, since $o(x^g) = o(x)$ is an odd π'-number and λ has values in $\mathbb{Q}_\pi$. Thus, $\chi(x) = \sum_{g \in T} \lambda^\circ(gxg^{-1})$ and $\lambda^\circ(gxg^{-1}) = 1$ if $gxg^{-1} \in W$, zero otherwise. It follows the thesis.

Suppose now that condition (ii) holds and suppose x has even order. Let x_2 be the 2-part of x, such that x uniquely factors as $x = x_2 y$ with

$o(y)$ an odd π'-number, so a δ'-number. Let $g \in G$ such that $x^g \in W$, then $\lambda(x^g) = \lambda(x_2{}^g)$ since λ is linear and $o(y^g) = o(y)$ is coprime with $o(\lambda)$.

Now, let one consider $a \in T$. If axa^{-1} is not an element of W, then $\lambda^\circ(axa^{-1}) = 0$. Suppose that axa^{-1} is in W and let P be a Sylow 2-subgroup of W which contains x_2, Q a Sylow 2-subgroup of W which contains ax_2a^{-1}. Since all the Sylow 2-subgroups of W are W-conjugates, there exists $w \in W$ such that $Q^w = P$. By replacing a with $w^{-1}a$, one has that ax_2a^{-1} is in P, too.

Now, P is a 2-Sylow of G, too, because by hypothesis $\chi(1) = |G : W|$ is odd. Moreover, condition (ii) is equivalent of P being conjugacy-closed in G, i.e., two elements of P which are conjugate in G are also conjugate in P (see [20, Theorem 5.25]). It follows that, if $axa^{-1} \in W$, then $\lambda(axa^{-1}) = \lambda(x)$ and, thus, $\chi(x)$ is a multiple of $\lambda(x) = \lambda(x_2)$ and in particular it is nonzero. □

Note that the hypothesis on G of Corollary 2.2.2 holds if the group is supersolvable, while the hypothesis that $o(x)$ is coprime with $\chi(1)$ holds, in particular, if χ is π-special.

Corollary 2.2.3. *If G is a supersolvable group and $\chi \in \mathrm{Irr}(G)$ a π-special character, then χ does not vanish on any π'-element of G.*

2.3 π-nonvanishing elements in π-supersolvable groups

In this section, we find a version for B_π-characters of the results in the first part of [26]. In particular, we prove a variant of Theorem 2.1.7 for elements of supersolvable and of π-supersolvable groups where no B_π-character vanishes.

We recall that a group is said to be *π-supersolvable* if each chief factor is either a π'-group or a group of prime order.

Definition 2.3.1. An element $x \in G$, for G π-separable, is called *π-nonvanishing* if $\chi(x) \neq 0$ for each χ in $\mathrm{B}_\pi(G)$.

The concept of Fitting subgroup can be generalized for a set of primes. The *π-Fitting subgroup* of a finite group G is defined as the normal subgroup $\mathrm{F}_\pi(G)$ such that $\mathrm{F}_\pi(G)/\,\mathrm{O}_{\pi'}(G) = \mathrm{F}(G/\,\mathrm{O}_{\pi'}(G))$.

It is known that $x \in G$ belongs to $\mathrm{F}(G)$ if and only if it centralizes every chief factor of G (see [13, Theorem 4.2]). An other known property of $\mathrm{F}(G)$ is that, if G is solvable, $\mathrm{C}_G(\mathrm{F}(G)) = \mathrm{Z}(\mathrm{F}(G))$ (see [20, Exercise 3B.14]). Before we begin to study π-nonvanishing elements, we need to prove that a similar properties exist also for $\mathrm{F}_\pi(G)$. We believe the following results to be already known but we still provide a proof, since we did not find any reference of them in the literature.

Lemma 2.3.2. *Let $x \in G$ be a π'-element, with G π-separable; x centralizes every π-chief factor of G if and only if $x \in \mathrm{O}_{\pi'}(G)$.*

Proof. One direction clear, so let us suppose x centralizes every π-chief factor of G and let N be a minimal normal subgroup of G. By induction on the order of G, we have that $x \in O \trianglelefteq G$ such that $O/N = \mathrm{O}_{\pi'}(G/N)$. If N is a π'-group, then $O = \mathrm{O}_{\pi'}(G)$ and we are done. Thus, assume that N is a π-group; Let $C = \mathrm{C}_O(N) \trianglelefteq G$ and observe that, by Shur-Zassenhaus theorem, $C = N \times \mathrm{O}_{\pi'}(C)$. As $x \in C$, we conclude $x \in \mathrm{O}_{\pi'}(C) \leq \mathrm{O}_{\pi'}(G)$. □

Proposition 2.3.3. *Let G be π-separable, then $x \in \mathrm{F}_\pi(G)$ if and only if it centralizes every π-chief factor of G.*

Proof. For sake of simplicity, we may refer to the group $G/\mathrm{O}_{\pi'}(G)$ as $\bar{G}$ and, in a similar fashion, we may call $\bar{x}$ and $\bar{N}$ the images in $\bar{G}$, under the canonical epimorphism, of an element $x \in G$ and of a subgroup $\mathrm{O}_{\pi'}(G) \leq N < G$.

One direction of the proof is trivial, since if $x \in \mathrm{F}_\pi(G)$, then it follows that $\bar{x} \in \mathrm{F}_\pi(G)/\mathrm{O}_{\pi'}(G) = \mathrm{F}(G/\mathrm{O}_{\pi'}(G))$ centralizes every chief factor of $G/\mathrm{O}_{\pi'}(G)$.

Let now C be the intersection of all the centralizers of π-chief factors of G, then $C \trianglelefteq G$. By Lemma 2.3.2, we know that $\mathrm{O}_{\pi'}(G) \leq C$ and $\bar{C} = C/\mathrm{O}_{\pi'}(G)$ is a π-group. Thus, $\bar{C}$ centralizes every chief factor of $\bar{G} = G/\mathrm{O}_{\pi'}(G)$, since a normal π-group centralizes every π'-chief factor and C centralizes every π-chief factor by definition. It follows that $\bar{C} \leq \mathrm{F}(\bar{G})$ and, thus, $C \leq \mathrm{F}_\pi(G)$. □

Proposition 2.3.4. *Let G be π-solvable, then $\mathrm{C}_G(\mathrm{F}_\pi(G)) = \mathrm{Z}(\mathrm{F}_\pi(G))$.*

Proof. Let $C = \mathrm{C}_G(\mathrm{F}_\pi(G))$ and $F = \mathrm{F}_\pi(G)$, let $H = CF \trianglelefteq G$ and suppose $|H : F| > 1$. Let $F < N \leq H$ such that N/F is a chief factor of G, then either N/F is a π'-group or it is a p-group for some $p \in \pi$. In the former case, let $\delta = \pi'$, while in the latter let $\delta = \pi' \cup \{p\}$. Notice that, in either cases, $\mathrm{O}_\delta(N) \leq F$. Now, let $\bar{N} = N/\mathrm{O}_\delta(N)$ and let $\bar{F}$ be the image of F in $\bar{N}$ under the canonical epimorphism. It follows that $\bar{N}/\bar{F}$ and $\bar{F}$ are of coprime order and, thus, there exists $\bar{K}$ complement of $\bar{F}$ in $\bar{N}$. However, since $\bar{K}$ is centralized by $\bar{F}$, we have that $\bar{K} = \mathrm{O}_\delta(\bar{N}) = 1$ and, thus, $N = F$. Therefore, $|H : F| = 1$ and $F = H$. It follows that $C \leq F$ and $C = \mathrm{Z}(F)$. □

We can now proceed to studying π-nonvanishing elements in then context of π-supersolvable groups. We do so by adjusting some techniques of [26] to our situation.

Lemma 2.3.5 *([26, Lemma 2.1])*. *Let $H \subseteq G$ and suppose all the members of the coset Hg are conjugate in G. If $\chi \in \mathrm{Char}(G)$ and χ_H has no principal constituents, then $\chi(g) = 0$.*

Lemma 2.3.6. *Let G be π-separable, let $x \in G$ be π-nonvanishing, $M \leq G$ and suppose that $[M, x] \trianglelefteq G$ and it is centralized by M. Then, $[M, x]$ is a π'-group.*

Proof. One proceeds as in the proof of [26, Lemma 2.2].

Let $H = [M, x]$, since H is centralized by M, then the map $m \mapsto [m, x]$ is a homomorphism from M onto H. Thus, for $h \in H$, one has $h^{-1} = [m, x]$ for some $m \in M$. Then, $h = [x, m] = x^{-1}x^m$ and one has $xh = x^m$. Thus, all elements of the coset xH are G-conjugate and, since $H \trianglelefteq G$ by hypothesis, it follows by Lemma 2.3.5 that $\chi(x) = 0$ for any character $\chi \in \mathrm{Irr}(G)$ such that $H \nsubseteq \ker(\chi)$. Since, however, x is π-nonvanishing, this never happens for any B_π-character and, thus, $[M, x] \leq \ker(\chi)$ for each $\chi \in \mathrm{B}_\pi(G)$. Then one concludes that it is a π'-group for Corollary 1.4.4. □

Theorem 2.3.7. *Let G be π-supersolvable. If $x \in G$ is π-nonvanishing, then $x\,\mathrm{O}_{\pi'}(G) \in \mathrm{Z}(\mathrm{F}_\pi(G)/\mathrm{O}_{\pi'}(G))$. In particular, $x \in \mathrm{F}_\pi(G)$.*

Proof. Let $F = \mathrm{F}_\pi(G)$ and let N be minimal normal in G. We can clearly assume that $\mathrm{O}_{\pi'}(G) = 1$, thus, $|N| = p$ for some prime $p \in \pi$. By induction on $|G|$, one has that $\bar{x} \in \bar{G} = G/N$ centralizes $\bar{F}$, since $\bar{F}$ is a normal π-subgroup of $\mathrm{F}_\pi(\bar{G})$. Thus, $[x, F] \leq N$.

Since $|N| = p$, it follows that $[x, F] \lhd G$. Moreover, F centralizes N and, thus, it centralizes $[x, F]$. It follows from Lemma 2.3.6 that $[x, F]$ is a π'-group and, therefore, $[x, F] = 1$.

This proves that $x \in \mathrm{C}_G(F) = \mathrm{Z}(F)$, when the last equivalence holds for Proposition 2.3.4. □

Corollary 2.3.8. *Let G be solvable and π-supersolvable and suppose that either $2 \in \pi$ or G has a normal 2-complement. Let $x \in G$ be a π'-element and suppose $o(x)$ is coprime with the degree of every character in $\mathrm{B}_\pi(G)$. Then, $x \in \mathrm{O}_{\pi'}(G)$.*

Proof. By Corollary 2.2.2, x is π-nonvanishing and the thesis then follows from Theorem 2.3.7. □

2.4 Nonvanishing elements in solvable groups

In this section, we study analogues of the results of [26] for solvable groups.

We begin with a variant of [26, Lemma 2.3].

Lemma 2.4.1. *Let G be π-separable, let $x \in G$ be a π-nonvanishing element and let $N \trianglelefteq G$, then x fixes some member of each orbit of the action of G on $\mathrm{B}_\pi(N)$.*

Proof. Let $\psi \in \mathrm{B}_\pi(N)$, one knows by Theorem 1.3.3 that there exists $\chi \in \mathrm{B}_\pi(G \mid \psi)$. Let T be the stabilizer of ψ in G, then, by Clifford's correspondence, there exists $\theta \in \mathrm{Irr}(T|\psi)$ such that $\theta^G = \chi$. Since x is π-nonvanishing, $\theta^G(x) \neq 0$ and it follows that there exists at least one conjugate x^g of x which lies in T. It follows that x stabilizes ψ^{-g}. □

We use Lemma 2.4.1 to prove an analogue of [26, Theorem 2.4] involving B_π-characters, and we will also prove an analogue result of [26, Theorem D]. However, in order to do so we need to adopt stronger hypothesis on the nonvanishing elements.

Theorem 2.4.2. *Suppose G is solvable but not nilpotent and let $x \in G$ be both a π- and a π'-nonvanishing element. Then, x is in the penultimate term of the ascending Fitting series.*

We recall that the *ascending Fitting series* of a π-solvable group G is a series

$$1 = F_0 < F_1 < ... < F_k = G$$

such that, for each $i > 0$, $F_i = \mathrm{F}(G/F_{i-1})$.

Proof. We prove it in the same way [26, Theorem 2.4] is proved.

Let $N \lhd G$ be the second-from-the-last term of the Fitting series of G. Then $\bar{x}$ is nonvanishing in $\bar{G} = G/N$ and one needs to show that $\bar{x} \in \mathrm{F}(\bar{G})$. Thus, let one assume that $N = 1$ and $G/\mathrm{F}(G)$ is nilpotent and prove that $x \in \mathrm{F}(G)$. In particular, by [13, Theorem 4.2], it is enough to prove that $x \in C$ for $C = \mathrm{C}_G(K/L)$, where K/L is any chief factor of G.

Let $\bar{G} = G/C$, since C contains $\mathrm{F}(G)$ one has that $\bar{G}$ is nilpotent and, thus, if we apply Theorem 2.3.7 two times, we have that $[x, \bar{G}] \leq \mathrm{O}_\pi(\bar{G}) \cap \mathrm{O}_{\pi'}(\bar{G}) = 1$ and it follows that $\bar{x} \in \mathrm{Z}(\bar{G})$. Moreover, $\bar{G}$ acts on $\mathrm{Irr}(K/L)$ and its orbit are exactly the orbits of the action of G. Since K/L is either a π-group or a π'-group, by Lemma 2.4.1 x fixes some member of each of those orbits; it follows that so does $\bar{x}$ and, being $\bar{x}$ central in $\bar{G}$, it fixes each member of $\mathrm{Irr}(K/L)$. It follows that x fixes all members of $\mathrm{Irr}(K/L)$, too, and thus also each conjugacy class of K/L. However, K/L is abelian, so each conjugacy class contains just one element and x centralizes K/L, as required. □

The key passage to prove [26, Theorem D] is the following theorem.

Theorem 2.4.3 (*[26, Theorem 4.2]*)**.** *Let $x \in \mathrm{F}(G)$, where G acts faithfully and irreducibly on a finite vector space V, and assume that x fixes a point in each G-orbit in V. Then $x^2 = 1$. Also, if x lies in an abelian normal subgroup of G, then $x = 1$.*

We use Theorem 2.4.3 to find a version of [26, Theorem D] for π-nonvanishing elements.

Theorem 2.4.4. *Let G be a solvable group and let $x \in G$ be both π- and π'-nonvanishing. Then, the image of x in $G/\mathrm{F}(G)$ has 2-power order.*

Proof. Let $x = x_2 u$, where x_2 is the 2-part of x and u is of odd order; we need to prove that $u \in \mathrm{F}(G)$. For this purpose, it is enough to show that $u \in C$, where $C = \mathrm{C}_G(K/L)$ and K/L is an (abelian) arbitrary chief factor of G.

Now, x fixes some member of each G-orbit in $\mathrm{Irr}(K/L)$, by Lemma 2.4.1. The same is true for u and, if we consider $\bar{G} = G/C$, also for $\bar{u} \in \bar{G}$. Moreover, the solvability of G implies that $C > 1$, thus, $|\bar{G}| < |G|$ and, by induction on group order, one has that $\bar{u} \in \mathrm{F}(\bar{G})$. Then, Theorem 2.4.3 applies and we deduce that $\bar{u}^2 = 1$. Since u has odd order, however, then $\bar{u} = 1$ and $u \in C$, as desired. □

One may wonder if it is true that the image of a π-nonvanishing element in $G/\mathrm{F}_\pi(G)$ has 2-power order, as the analogies between nonvanishing and π-nonvanishing elements may suggest. However, for π-nonvanishing elements this version of [26, Theorem D] is false, as the following example proves. The example also proves that there is not a version for π-nonvanishing elements for [26, Theorem 2.4].

Example 2.4.5. Let $G = \mathrm{SL}(2,3) \ltimes (\mathbb{Z}_3)^2$, with $\mathrm{SL}(2,3)$ acting naturally on $(\mathbb{Z}_3)^2$. From the discussion in Section 1.6, we see that G has two conjugacy classes of nonvanishing elements, $C_1 = \{1\}$ and C_9, whit this last one counting 8 elements. Moreover, no $\mathrm{B}_{\{3\}}$-character vanishes on classes C_3, C_4, C_6 and C_7, which makes a total of 81 $\{3\}$-nonvanishing elements in the group.

On the other hand, we see that $\mathrm{F}(G) = \mathrm{F}_{\{3\}}(G) = (\mathbb{Z}_3)^2$, which therefore has only 9 elements. It follows that there are $\{3\}$-nonvanishing elements of order 3 which do not lie in $\mathrm{F}_{\{3\}}(G)$.

The author has found no examples of group with elements which are both π-nonvanishing and π'-nonvanishing, but vanishing.

2.5 Vanishing elements in minimal normal subgroups

In this section, we will see some preliminary results, which will be used later in the thesis. Let us recall at first one of the tools used in [7].

We recall that a group M is said to be characteristically simple if it has no proper nontrivial $\mathrm{Aut}(M)$-invariant subgroups. A group M is characteristically simple exactly when it is the direct product of isomorphic simple groups.

Lemma 2.5.1 *([7, Lemma 2.8]).* *Let A be an abelian group which acts faithfully by automorphisms on a group M. Assume that $|A|$ and $|M|$ are coprime. If M is characteristically simple, then there exists $\theta \in \mathrm{Irr}(M)$ such that $\mathrm{I}_A(\theta) = 1$.*

A first consequence of Lemma 2.5.1 is the following lemma, borrowed from [7], which we adapt for B_π-characters.

Lemma 2.5.2. *Let G be a group, let N be a minimal normal π'-subgroup and let $M \trianglelefteq G$ such that M/N is an abelian π-group. Let $A < G$ be a complement of N in M. Furthermore, suppose that $\mathrm{O}_\pi(M) = 1$. Then, there exists a character $\chi \in \mathrm{B}_{\pi'}(G)$ which vanishes on every nontrivial element of A and such that $\chi(1)$ is divided by $|M : N|$.*

Note that the group A always exists, by the Schur–Zassenhaus theorem.

Proof. Since $\mathrm{C}_A(N) \lhd M$ and A is a π-group, $\mathrm{C}_A(N) \leq \mathrm{O}_\pi(M) = 1$; thus, A acts faithfully on N. By Lemma 2.5.1, there exists some character $\tau \in \mathrm{Irr}(N)$ such that $\eta = \tau^M \in \mathrm{Irr}(M)$. In particular, $|M : N|$ divides $\eta(1)$. Since $\tau \in \mathrm{B}_{\pi'}(N) = \mathrm{Irr}(N)$ and N is normal in M, by Theorem 1.3.3 the character η is in $\mathrm{B}_{\pi'}(M)$, too. Moreover, note that η vanishes on A and that the same is true for η^g for each $g \in G$. Then, every character in $\mathrm{B}_{\pi'}(G \mid \eta)$, which is a nonempty set because of Theorem 1.3.3, vanishes on every nontrivial element of A and its degree is divided by $|M : N|$. □

We may see now a first application of Lemma 2.5.2. The following result is in the spirit of the ones of the previous section; however, the techniques we use are more similar to the ones we will use in the next section.

Proposition 2.5.3. *Let G be π-separable and let H be a Hall π-subgroup of G. Let $x \in \mathrm{Z}(H)$ be a π'-nonvanishing element. Then, $x \in \mathrm{Z}(\mathrm{O}_\pi(G)) \leq \mathrm{Z}(\mathrm{F}(G))$.*

Proof. Suppose $x \neq 1$ and proceed by induction on $|G|$. Assume at first that $\mathrm{O}_\pi(G) = 1$ and let N be a minimal normal subgroup of G. Then, N is a π'-group. Let $O \lhd G$ such that $O/N = \mathrm{O}_\pi(G/N)$ and let $Z \lhd G$ such that $Z/N = \mathrm{Z}(O/N)$; by induction, $x \in Z$, so $Z > 1$. Since Z/N is a π-group, by the Schur–Zassenhaus theorem we have $Z = NA$, for some complement A of N in Z such that $x \in A$.

Now, by Lemma 2.5.2, with Z in place of M, there exists a character $\chi \in \mathrm{B}_{\pi'}(G)$ which vanishes on every nontrivial element of A and, in particular, on x. This, however, contradicts the fact that x is π'-nonvanishing. It follows that $K = \mathrm{O}_\pi(G) > 1$.

Now, by induction, one has that $\bar{x} \in \mathrm{O}_\pi(G/K) = 1$ and, thus, $x \in K$. Finally, $x \in \mathbf{Z}(H) \cap K \leq \mathbf{Z}(K)$. □

Let us now see some other results, adapted from [7], which allow us to handle nonabelian simple groups. Some of these results rely on the Classification of finite simple groups.

Lemma 2.5.4 (*[7, Proposition 2.2]*)**.** *Let* $N = S_1 \times \cdots \times S_k$ *be a minimal normal subgroup of* G*, where* $S_i \cong S$*, a nonabelian simple group. If* $\theta \in \mathrm{Irr}(S)$ *extends to* $\mathrm{Aut}(S)$*, then* $\theta \times \cdots \times \theta \in \mathrm{Irr}(N)$ *extends to* G*.*

Lemma 2.5.5. *Let* S *be a nonabelian simple group and let* $p \mid |S|$*. Then either there exists* $\chi \in \mathrm{Irr}(S)$ *of* p*-defect zero or there exists a* p*-element* $x \in S$ *and a character* $\chi \in \mathrm{Irr}(S)$ *such that* χ *extends to* $\mathrm{Aut}(S)$ *and* χ *vanishes on* x*.*

Proof. It is the consequence of Proposition 2.1, Lemma 2.3 and Proposition 2.4 of [7]. □

Proposition 2.5.6. *Let* $N \lhd G$ *be a nonabelian minimal normal* π*-subgroup of the* π*-separable group* G *and let* $p \mid |N|$*. Then, there exists a* p*-element* $x \in N$ *such that* $\chi(x) = 0$ *for some* $\chi \in \mathrm{B}_\pi(G)$*.*

Proof. The Proposition is proved using arguments from the proof of [7, Theorem A].

Let $N = S_1 \times \cdots \times S_k$ with $S_i \cong S$ for each i, where S is a non-abelian simple group. Assume at first that there exists $\theta \in \mathrm{Irr}(S)$ of p-defect zero, then $\psi = \theta \times \cdots \times \theta \in \mathrm{Irr}(N)$ is of p-defect zero, too. By [14, Theorem 8.17], ψ vanishes on every p-element of N. Let $\chi \in \mathrm{B}_\pi(G)$ lying over ψ, which exists by Theorem 1.3.3, then all the irreducible constituents of χ_N are conjugated to ψ and it follows that χ vanishes on every p-element of N.

By Lemma 2.5.5, one can assume now that there exists a p-element $x \in S$ and a character $\theta \in \mathrm{Irr}(S)$ such that θ extends to $\mathrm{Aut}(S)$ and

χ vanishes on x. Let $\psi = \theta \times \cdots \times \theta \in \mathrm{Irr}(N)$, then ψ vanishes on $y = (x, \cdots, x)$ and it extends to G by Lemma 2.5.4. In particular, ψ is G invariant. It follows that, if χ is a B_π-character of G which lies over ψ, which exists again by Theorem 1.3.3, then χ_N is a multiple of ψ and it vanishes on the p-element y. □

2.6 Groups where every p-element is nonvanishing

In this section, we study variants for the B_π-characters of the results in [7].

We have already seen in Proposition 2.5.3 that a π'-nonvanishing element lying in the centre of a Hall π-subgroup is contained in $\mathrm{O}_\pi(G)$. From this, it easily follows that, if the Hall π-subgroup is nilpotent and every π-element is π'-nonvanishing, then the Hall π-subgroup is normal in G. However, a similar result can be also obtained imposing weaker conditions on the group.

Proposition 2.6.1. *Let G be a π-solvable group. If every p-element of G is π'-nonvanishing for each $p \in \pi$, then G has a normal Hall π-subgroup.*

Note that, if the Hall π-subgroup H is normal, then it is in the kernel of every character in $\mathrm{B}_{\pi'}(G)$, so this is also a sufficient condition.

Proof. One argues by induction on $|G|$.

Let G not be a π'-group and, at first, suppose $\mathrm{O}_\pi(G) = 1$. Let N be a minimal normal subgroup of G, then N is a π'-group. By induction, G/N has a normal Hall π-subgroup M/N. Thus, there exists $K < M$, $K \lhd G$ such that K/N is a chief factor for G. Since G is π-solvable, then K/N is an abelian π-group. By Schur-Zassenhaus theorem, $K = NA$ for some complement A of N in K.

Now, from Lemma 2.5.2 it follows that there exists a character $\chi \in \mathrm{B}_{\pi'}(G)$ which vanishes on every nontrivial element of the π-subgroup A of G, in contradiction with the hypothesis.

It follows that, if G is not a π'-group, then $O = \mathrm{O}_\pi(G) > 1$. By induction, the Hall π-subgroup H/O of G/O is normal. However, $H/O \leq \mathrm{O}_\pi(G/O) = 1$, thus, $H = O$ and it is normal in G. □

A reader may have noticed that, from the results of Proposition 2.6.1 and Theorem 2.1.2, it follows that, if G is a π-solvable group and each $\mathrm{B}_{\pi'}$-character is π'-special, then G has a normal Hall π-subgroup. However, in Chapter 4 we will see that the same is true also without assuming the group to be π-solvable.

We shall now prove a version for π-nonvanishing elements of [7, Theorem A]. However, as we have done in Theorem 2.4.4, we need to assume the elements to be both π- and π'-nonvanishing.

Since it requires no extra effort, we prove a slightly more general version of the theorem, in the spirit of the results we will see in the next chapter.

Theorem 2.6.2. *Let G be a π-separable group, let $N \trianglelefteq G$ and let p be any prime. Suppose that no character in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ vanishes on any p-element of N. Then, N has a normal Sylow p-subgroup.*

Proof. One argues by induction on $|G| + |N|$.

Assume p divides $|N|$, otherwise the theorem is trivial. Moreover, we can assume, without loss of generality, that $p \in \pi$.

If $O = \mathrm{O}_p(N) > 1$ and P is a Sylow p-subgroup of N, then by induction the Sylow p-subgroup P/O of N/O is normal. However, $P/O \leq \mathrm{O}_p(N/O) = 1$, thus, $P = O$ is normal in N.

Suppose then that $\mathrm{O}_p(N) = 1$ and let $M \leq N$ be a minimal normal subgroup of G. If $p \mid |M|$, then M is a nonabelian π-group and, by Proposition 2.5.6, there exists a character in $\mathrm{B}_\pi(G)$ which vanishes on a p-element of M, in contradiction with the hypothesis. Thus, we can assume p does not divide $|M|$.

By induction, let P_0/M be a normal Sylow p-subgroup of N/M, which is nontrivial because p divides $|N : M|$, and let $C/M = \mathrm{Z}(P_0/M)$. Note that $C \trianglelefteq G$, since $P_0 \triangleleft G$. Let A be a complement of M in C. Then, by Lemma 2.5.2, there exists a character in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ which vanishes on every nontrivial element of A. However, since A is a nontrivial p-subgroup of N, this contradicts the hypothesis. □

Corollary 2.6.3. *Let G be π-separable and let $N \triangleleft G$ be a π-subgroup. If no B_π-character of G vanishes on any p-element of N, then N has a normal Sylow p-subgroup. In particular, if every element of prime power order of N is π-nonvanishing, then N is nilpotent.*

Proof. Since N is in the kernel of every $\mathrm{B}_{\pi'}$-character, every element of N is π'-nonvanishing and the thesis follows directly from Theorem 2.6.2. □

Clearly, if in Theorem 2.6.2 we take $N = G$, we have a version for π-nonvanishing elements of [7, Theorem A].

Theorem 2.6.4. *Let G be a π-separable group and let p be any prime. Suppose that no character in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ vanishes on any p-element of G. Then, G has a normal Sylow p-subgroup.*

It could be noticed that, from Theorem 2.6.4 and Theorem 2.1.2 it follows that the group G has a normal (abelian) Sylow p-subgroup if and only if p does not divide the degree of any character in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$. However, we will give a different proof of this fact in Chapter 4.

Chapter 3

Nonvanishing elements in normal subgroups

As a digression from the main theme of the thesis, in this chapter we talk about nonvanishing elements of a group which are contained in a normal subgroup. We first assume that all the p elements of a normal subgroup N of a group G are nonvaishing in G and we see that N has a normal Sylow p-subgroup. Then we weaken our hypothesis on N by allowing some elements to be vanishing in G and imposing conditions on class sizes of these elements.

3.1 Introduction and preliminaries

The results in this chapter are from a joint work with M. J. Felipe and V. Ortiz-Sotomayor, of the Universitat Politècnica de València.

In Chapter 2 we studied the properties of elements of a π-separable group G where no B_π-character vanishes. Let N be a normal subgroup of G, in this chapter we investigate the properties of elements of N where no irreducible character of G vanishes.

The result presented in this chapter do not involve B_π-characters, since it would need to add the hypothesis of the group to be π-separable, which otherwise we can avoid. Therefore, this chapter needs to be seen more as a digression from the main theme of the thesis.

However, we mention that, with the further hypothesis of the group being π-separable, most of the results presented here still hold if we consider elements of a normal subgroup where no character in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ vanishes.

A first result can be obtained as a consequence of Theorem 2.6.2, from Chapter 2.

Theorem 3.1.1. *Let N be a normal subgroup of a group G, and let p be a prime. If $\chi(x) \neq 0$ for every p-element $x \in N$ and for all $\chi \in \mathrm{Irr}(G)$, then N has a normal Sylow p-subgroup.*

In particular, if $\chi(x) \neq 0$ for every prime power order element $x \in N$ and for all $\chi \in \mathrm{Irr}(G)$, then N is nilpotent.

Proof. It follows directly from Theorem 2.6.2, when we take $\pi = \pi(G)$, i.e., we take as π the set of all the primes dividing $|G|$. □

Let us denote as $\mathrm{Van}(G)$ the set of all vanishing elements of the group G.

A straightforward consequence of Burnside's Theorem is that, if we have $\mathrm{Van}(G) = \emptyset$, then G is abelian. Concerning fact, it is worth noting that, in general, it is not true that a normal subgroup N is abelian if and only if every element of N is non-vanishing in G, i.e. if $N \cap \mathrm{Van}(G) = \emptyset$.

In fact, if $G = Q_8$ is a quaternion group of 8 elements and N is a normal subgroup of G isomorphic to a cyclic group of order 4, then N is abelian and $N \cap \mathrm{Van}(G) \neq \emptyset$. On the other hand, by [26, Theorem 5.1], for any prime p there exists a group G having a normal non-abelian Sylow p-subgroup, and every p-element of G is non-vanishing.

Moreover, one may notice that, from Theorem 2.1.7, which appears as Theorem B in [26], it follows that $G \setminus \mathrm{Z}(G) = \mathrm{Van}(G)$ for any nilpotent group G. Observe, however, that if a normal subgroup N is nilpotent, then $N \setminus \mathrm{Z}(G)$ may not coincide with $\mathrm{Van}(G) \cap N$. For instance, one can consider as G the normaliser in a Suzuki group of degree 8 of a Sylow 2-subgroup of it, and N the Sylow 2-subgroup. It holds that $\mathrm{Van}(G) \cap N = \emptyset$ although clearly $N \setminus \mathrm{Z}(G) \neq \emptyset$.

In fact, in general, $\mathrm{Van}(N) \neq \mathrm{Van}(G) \cap N$. It can happen, however, under some conditions.

Lemma 3.1.2. [12, Corollary 1.3] *Let H be a subgroup of a group G. Assume that $G = H\,\mathrm{C}_G(x)$ for some $x \in H$. Then $x \in \mathrm{Van}(G)$ if and only if $x \in \mathrm{Van}(H)$.*

In the chapter, we also generalize some results on nonvanishing elements by imposing conditions on the class size of vanishing elements of the group. For this reason, we collect here some preliminary results regarding conjugacy class sizes. We start with the next elementary properties which are frequently used, sometimes with no comment.

Lemma 3.1.3. *Let N be a normal subgroup of a group G, and let p be a prime. We have:*

(a) $\left|x^N\right|$ divides $\left|x^G\right|$, for any $x \in N$.

(b) $\left|(xN)^{G/N}\right|$ divides $\left|x^G\right|$, for any $x \in G$.

(c) If $xN \in G/N$ is a p-element, then $xN = yN$ for some p-element $y \in G$.

Lemma 3.1.4. *Let N be a normal subgroup of a group G, and let $H \in \mathrm{Hall}_\pi(N)$ for a set of primes π. If $x \in H$ is such that $\left|x^G\right|$ is a π-number, then x lies in $\mathrm{O}_\pi(N)$.*

Proof. Since $\left|x^N\right|$ divides $\left|x^G\right|$, then $(\left|x^N\right|, |N : H|) = 1$. It follows $N = H\,\mathrm{C}_N(x)$ and so $\langle x^N \rangle \leq \mathrm{O}_\pi(N)$. □

We also recall a generalisation of the above lemma when $N = G$ and $\pi = \{p\}$.

Lemma 3.1.5. [3, Lemma 3] *Let $x \in G$. If $\left|x^G\right|$ is a power of a prime p, then $[x^G, x^G]$ is a p-group.*

We end this section with the main result of [5], which will be necessary for proving Theorem 3.3.7. We present here an adapted version for our context of vanishing G-conjugacy classes.

Proposition 3.1.6. [5, Main Theorem] *Let G be a group which contains a non-trivial normal p-subgroup N, for a given prime p. Then $\left|x^G\right|$ is a multiple of p for each $x \in N \cap \mathrm{Van}(G)$.*

3.2 Nonvanishing elements and normal Sylow p-subgroups

In the Introduction of Chapter 2 we have mentioned Theorem 2.1.10, which appears as Theorem B in [28], which links the normality of a Sylow subgroup P of a group G with the fact that no irreducible constituent of $(1_P)^G$ vanishes on the elements of P.

The aforementioned Theorem 3.1.1, on the other hand, fails to provide such link. In this section, we refine Theorem 3.1.1 in order to have an analogue of Theorem 2.1.10 for nonvanishing elements in normal subgroups.

Theorem 3.2.1. *Let N be a normal subgroup of a group G, and let P be a Sylow p-subgroup of G for some prime p. Let $P_0 = P \cap N$ and $\beta \in \mathrm{Irr}(P/P_0)$. Then the following conditions are pairwise equivalent:*

(i) P_0 is a normal Sylow p-subgroup of N;

(ii) $\chi(x) \neq 0$ for all irreducible constituents χ of $(1_{P_0})^G$ and all $x \in P_0$;

(iii) $\chi(x) \neq 0$ for all irreducible constituents χ of β^G and all $x \in P_0$.

To prove the theorem, we will use some of the results we already mentioned in Section 2.5. Some of them, however, need to be adapted to the hypothesis of Theorem 3.2.1.

For this purpose, we mention here some preliminary results from [28], which rely on the Classification of finite simple groups.

Lemma 3.2.2. [28, Lemma 2.2] *Let G be a finite group, p a prime, and $P \in \mathrm{Syl}_p(G)$. If $\chi \in \mathrm{Irr}(G)$ has p-defect zero, then χ is a constituent of $(1_P)^G$ and vanishes on the non-trivial p-elements of G.*

Lemma 3.2.3. [28, Theorem 2.1] *Let S be a finite non-abelian simple group, p a prime, and $P \in \mathrm{Syl}_p(S)$. Then either S has a p-defect zero character, or there exists a constituent $\theta \in \mathrm{Irr}(S)$ of the permutation character $(1_P)^S$ such that θ extends to $\mathrm{Aut}(S)$ and $\theta(x) = 0$ for some p-element x of S.*

Now, we can prove the next proposition, inspired by the proof of [28, Theorem B].

Proposition 3.2.4. *Let M be a non-abelian minimal normal subgroup of a group G, and let p be a prime divisor of $|M|$. Let H be a subgroup of G such that $H \cap M \in \mathrm{Syl}_p(M)$. Let $\beta \in \mathrm{Irr}(H/H \cap M)$ Then, there exists $\chi \in \mathrm{Irr}(G)$ such that χ is a constituent of β^G and it vanishes on some p-element of M.*

In particular, if $H = P \in \mathrm{Syl}_p(G)$, then there exists $\chi \in \mathrm{Irr}(G)$ such that χ is a constituent of $(1_P)^G$ and it vanishes on some p-element of M.

Proof. We have $M = S_1 \times \cdots \times S_k$, where all S_i are isomorphic to a non-abelian simple group S with $p \in \pi(S)$. If $\theta \in \mathrm{Irr}(S)$ is of p-defect zero, then $\eta = \theta \times \cdots \times \theta \in \mathrm{Irr}(M)$ and η is also of p-defect zero. By Lemma 3.2.2 applied to M we have $[\eta, (1_{H\cap M})^M] \neq 0$ and η vanishes on the non-trivial p-elements of M.

Since $\beta \in \mathrm{Irr}(H/H\cap M)$, we have $[\beta_{H\cap M}, 1_{H\cap M}] \neq 0$. Then $(\beta^{HM})_M = (\beta_{H\cap M})^M = \beta(1)(1_{H\cap M})^M$ and $[\eta, (\beta^{HM})_M] = [\eta^{HM}, \beta^{HM}] \neq 0$. Hence there exists $\tau \in \mathrm{Irr}(HM)$ such that $[\tau, \eta^{HM}] \neq 0 \neq [\tau, \beta^{HM}]$. Let $\chi \in \mathrm{Irr}(G)$ over τ. Then χ_M is sum of G-conjugate characters of η. Therefore χ vanishes on the non-trivial p-elements of M and $[\chi, \beta^G] = [\chi_H, \beta] \neq 0$.

Suppose now that S does not have a character of p-defect zero. We have that, by Lemma 3.2.3, there exists $\theta \in \mathrm{Irr}(S)$ such that $[\theta, (1_{H\cap S})^S] \neq 0$ (note $H \cap S \in \mathrm{Syl}_p(S)$) which extends to $\mathrm{Aut}(S)$, and there exists a p-element $x \in S$ such that $\theta(x) = 0$. Thus $1 \neq y = (x, \ldots, x) \in M$ is a p-element and $\eta = \theta \times \cdots \times \theta$ vanishes on y, and certainly $[\eta_{H\cap M}, 1_{H\cap M}] \neq 0$. Since $[\beta_{H\cap M}, 1_{H\cap M}] \neq 0$, arguing as in the previous paragraph, we may affirm that there exists $\tau \in \mathrm{Irr}(HM)$ over η and over β. Let $\chi \in \mathrm{Irr}(G)$ be over η, so $[\chi, \beta^G] \neq 0$. By Lemma 2.5.4, η extends to G. Let $\hat{\eta}$ be an extension of η. By Gallagher, $\chi = \hat{\eta}\rho$ for some $\rho \in \mathrm{Irr}(G/M)$. Therefore, χ lies over β and $\chi(y) = \eta(y)\rho(1) = 0$. □

Theorem 3.2.5. *Let N be a normal subgroup of a group G, and let P_0 be a Sylow p-subgroup of N for some prime p. Let H be a subgroup of G such that $H \cap N = P_0$, and let $\beta \in \mathrm{Irr}(H/P_0)$. Then, P_0 is normal in N (and therefore in G) if and only if all irreducible constituents of β^G do not vanish on any p-element of N.*

Proof. Suppose $P_0 \trianglelefteq N$. Let χ be a constituent of β^G for $\beta \in \mathrm{Irr}(H/H\cap N)$. We have $[\beta_{P_0}, 1_{P_0}] \neq 0$, so $[\chi_{P_0}, 1_{P_0}] \neq 0$. Since $P_0 \trianglelefteq G$, then $\chi(x) \neq 0$ for all p-elements $x \in N$. Conversely, we consider that all irreducible constituents of β^G, where $\beta \in \mathrm{Irr}(H/H \cap N)$, do not vanish on any p-element of N, and we claim that P_0 is normal in N.

Suppose that the claim is false, and let us consider a counterexample which minimises $|G|$. Let M be a minimal normal subgroup of G such that $M \leqslant N$. We check that the hypotheses are inherited by $\overline{G} = G/M$. Certainly $\overline{H} \cap \overline{N} = N/M \cap HM/M = (H \cap N)M/M \in \mathrm{Syl}_p(N/M)$. Since $\beta \in \mathrm{Irr}(H/H \cap N)$, then $\beta \in \mathrm{Irr}(H/H \cap M)$ so $\overline{\beta} \in \mathrm{Irr}(HM/M)$. Besides, $H \cap N \leqslant \ker \beta$, so $\overline{H \cap N} \leqslant \ker \overline{\beta}$. Let $\overline{\chi} \in \mathrm{Irr}(\overline{G})$ be an irreducible constituent of $\overline{\beta}^{\overline{G}}$ and $\overline{x} \in \overline{N}$ a p-element. Then we may assume that $x \in N \smallsetminus M$ is a p-element and, since $[\overline{\chi}, \overline{\beta}^{\overline{G}}] \neq 0$, then it is easy to see that $[\chi_H, \beta] \neq 0$ and $\overline{\chi}(\overline{x}) = \chi(x) \neq 0$. By minimality, we get $\overline{P_0} \trianglelefteq \overline{G}$, so $P_0 M \trianglelefteq G$.

Let us assume that p divides the order of M. If M is a p-group, then $M \leqslant P_0$ and $P_0 = P_0 M \trianglelefteq G$, a contradiction. Hence M is non-abelian. Since $\beta \in \mathrm{Irr}(H/H \cap M)$, in virtue of Lemma 3.2.4 there exists $\chi \in \mathrm{Irr}(G)$ such that $[\chi, \beta^G] \neq 0$ and $\chi(x) = 0$ for some p-element $x \in M \leqslant N$, a contradiction again.

Thus p does not divide the order of M and $\mathrm{O}_p(N) = 1$. Let K/M be a chief factor of G such that $K \leq P_0 M \trianglelefteq G$, so K/M is an abelian p-group. Note $K = M(K \cap P_0)$ and $K \cap P_0 \in \mathrm{Syl}_p(K)$ is abelian. By Frattini's argument, $G = K\,\mathrm{N}_G(K \cap P_0) = M\,\mathrm{N}_G(K \cap P_0)$, so $\mathrm{C}_{K \cap P_0}(M) \trianglelefteq G$ and $\mathrm{C}_{K \cap P_0}(M) \leqslant \mathrm{O}_p(N) = 1$. Therefore $K \cap P_0$ is an abelian p-group which acts coprimely and faithfully on M, and M is characteristically simple. By Lemma 2.5.1 and Clifford theory, there exists $\theta \in \mathrm{Irr}(M)$ such that $\eta = \theta^K$ is irreducible. In particular, η and all its conjugates vanish on $K \smallsetminus M$. Therefore, if we prove that there exists $\chi \in \mathrm{Irr}(G)$ which lies over both η and β we will reach the final contradiction.

Let T be the inertia subgroup for θ in $P_0 M \trianglelefteq G$. Since $(|T/M|, |M|) = 1$ we have that θ extends to $\hat{\theta} \in \mathrm{Irr}(T)$ by [14, Corollary 6.28]. Further, p does not divide $\hat{\theta}(1)$ so $\hat{\theta}_{P_0 \cap T}$ has at least one linear constituent λ. As $T = M(P_0 \cap T)$, then $P_0 \cap T \cong T/M$ and we can see λ also as a character of T/M. By Gallagher, $\nu = \bar{\lambda}\hat{\theta}$ is an irreducible character of T, where $\bar{\lambda}$ is the complex conjugate of λ. Moreover, $\nu_M = \theta$ and by Clifford correspondence

$\nu^{P_0M} \in \mathrm{Irr}(P_0M)$. Hence

$$0 \neq [1_{P_0\cap T}, \overline{\lambda}_{P_0\cap T}\hat{\theta}_{P_0\cap T}] = [1_{P_0\cap T}, \nu_{P_0\cap T}] = [(\nu_{P_0\cap T})^{P_0}, 1_{P_0}] = \\ = [(\nu^{P_0T})_{P_0}, 1_{P_0}] = [(\nu^{P_0M})_{P_0}, 1_{P_0}] = [\nu^{P_0M}, (1_{P_0})^{P_0M}],$$

while $(\beta^{HN})_N = \beta(1)(1_{P_0})^N = \beta(1)((1_{P_0})^{P_0M})^N$, so

$$[(\beta^{HN})_N, (\nu^{P_0M})^N] = [(\beta^{HN})_N, \nu^N] = [\beta^{HN}, \nu^{HN}] \neq 0.$$

Therefore there exists $\tau \in \mathrm{Irr}(HN)$ over β and over ν. Let $\chi \in \mathrm{Irr}(G)$ over τ, so $[\chi, \beta^G] \neq 0$. Moreover, χ lies over θ, and then χ lies over $\eta = \hat{\theta}$. Thus χ_K is a sum of G-conjugate characters of η. Hence $\chi(x) = 0$ for all $x \in K \cap P_0$ and this is a final contradiction. □

Theorem 3.2.1 is now a corollary of the above result when we take H a Sylow p-subgroup of G (for Theorem 3.2.1 (iii)) and $H = P_0$ (for Theorem 3.2.1 (ii)). Moreover, when $N = G$ in Theorem 3.2.1, then we obtain Theorem 2.1.10.

Example 3.2.6. (1) Note that in Theorem 3.2.1 it is possible to choose $\beta \in \mathrm{Irr}(P/P_0)$ distinct from 1_P, in contrast to Theorem 2.1.10: Let G be a symmetric group of degree 4 and let N be an alternating group of degree 4. Take $P \in \mathrm{Syl}_2(G)$. Then there exists a non-trivial irreducible character $\beta \in \mathrm{Irr}(P)$ with $P_0 = P \cap N \leqslant \ker\beta$. Additionally, the irreducible constituents of β^G do not vanish on the p-elements of N, so the hypotheses in Theorem 3.2.1 (iii) are fulfilled.

(2) The following equivalence, similar to Theorem 2.1.10 (i)-(iii), is not true: P_0 is a normal Sylow p-subgroup of N if and only if p does not divide $\chi(1)$ for all irreducible constituents of $(1_{P_0})^G$: Consider G and N as above. Then $(1_{P_0})^G$ has three distinct irreducible constituents, being one of them of degree 2.

Both examples have been checked using the software GAP.

Let consider now a set of primes π instead of a single prime p. As a consequence of Theorem 3.1.1, we give in the following proposition extra information on the structure of a π-complement of G when N contains a Hall π-subgroup of it.

Proposition 3.2.7. *Let N be a normal subgroup of a group G such that every prime power order π-element of N is non-vanishing in G, for a set of primes π. Then N has a nilpotent normal Hall π-subgroup.*

Further, if $|G : N|$ is a π'-number, then any π-complement F of G verifies that $F\,\mathrm{Z}(G)$ is self-normalising.

Proof. Certainly, in virtue of Theorem 3.1.1 we have that N has a nilpotent normal Hall π-subgroup, say H. In fact, if $|G : N|$ is not divisible by any prime in π, then H is a normal Hall π-subgroup of G. Let F be a π-complement of H in G, so $G = HF$. We aim to show that $F\,\mathrm{Z}(G) = \mathrm{N}_G(F\,\mathrm{Z}(G))$. Take a prime power order element $x \in \mathrm{N}_H(F\,\mathrm{Z}(G))$. Then $F^x\,\mathrm{Z}(G) = (F\,\mathrm{Z}(G))^x = F\,\mathrm{Z}(G)$, so there exists some $y \in F\,\mathrm{Z}(G)$ such that $F^x = F^y = F$. Thus, $x \in \mathrm{N}_H(F) \leqslant \mathrm{C}_H(F)$ because $[\mathrm{N}_H(F), F] \leqslant H \cap F = 1$. Therefore $G = HF = H\,\mathrm{C}_G(x)$. Since $x \notin \mathrm{Van}(G)$ by assumption, then Lemma 3.1.2 yields that $x \notin \mathrm{Van}(H)$. Since H is nilpotent, from Theorem 2.1.7 it follows that $x \in \mathrm{Z}(H) \cap \mathrm{C}_G(F) \leqslant \mathrm{Z}(G)$. As this argument is valid for every prime power order element in $\mathrm{N}_H(F\,\mathrm{Z}(G))$, then $\mathrm{N}_H(F\,\mathrm{Z}(G)) \leqslant \mathrm{Z}(G)$. Finally, note that $\mathrm{N}_G(F\,\mathrm{Z}(G)) = \mathrm{N}_G(F\,\mathrm{Z}(G)) \cap HF = F(\mathrm{N}_H(F\,\mathrm{Z}(G))) = F\,\mathrm{Z}(G)$, as wanted. □

Corollary 3.2.8. *Let G be a group such that all the p-elements are non-vanishing. Then G has a normal Sylow p-subgroup, and $F\,\mathrm{Z}(G)$ is self-normalising for any p-complement F of G.*

3.3 Lengths of G-conjugacy classes of vanishing elements

We now study the problem under a weaker assumption: instead of asking that no element of a given order is vanishing, we impose a condition on class size of the vanishing element.

In general, the group G is not asked to be δ-separable, for some set of primes δ. We mention, however, that if G is δ-separable, then the same results are still valid if we replace the condition of being vanishing with the condition of being δ- or δ'-vanishing.

We start by showing an extension of Lemma 3.1.5 for a set of primes π and a G-conjugacy class. The proof is inspired by [2, Theorem C] under the weaker hypothesis of the π-separability of the normal subgroup N.

Proposition 3.3.1. *Let N be a normal π-separable subgroup of a group G. If $x \in N$ is such that $\left|x^G\right|$ is a π-number, then $[x^G, x^G] \leqslant \mathrm{O}_\pi(N)$. In particular, $x\,\mathrm{O}_\pi(N)/\,\mathrm{O}_\pi(N) \in \mathrm{Z}(\mathrm{F}(N/\,\mathrm{O}_\pi(N)))$.*

Indeed, if π consists of a single prime p, then the same statement is valid even if N is not p-soluble.

Proof. In order to prove the first claim, let us consider a counterexample which minimises $|G| + |N|$. One can clearly assume $\mathrm{O}_\pi(N) = 1$, so we aim to get the contradiction $[x^G, x^G] = 1$. Let us suppose firstly that $\langle x \rangle$ is subnormal in G. Then $x \in \mathrm{F}(G)$. As $\mathrm{F}(G)$ is a π'-group and $\left|x^G\right|$ is a π-number, then clearly $x \in \mathrm{Z}(\mathrm{F}(G))$ and $\langle x^G \rangle \leqslant \mathrm{Z}(\mathrm{F}(G))$, so $[x^G, x^G] = 1$.

Next we assume that the normal subgroup $M := \langle x^G \rangle$ is proper in N. Then by minimality we obtain $[x^M, x^M] = 1$, and it follows that $x \in \mathrm{Z}(\langle x^M \rangle)$. In particular, $\langle x \rangle$ is subnormal in M, and therefore in G, which contradicts the previous paragraph. Hence $M = N$.

Let $K := \mathrm{O}_{\pi'}(N)$. Since N is π-separable, then K is non-trivial. It follows from the class size hypothesis that K centralises x^G, so K is central in $N = \langle x^G \rangle$. As $[x^G, x^G]K/K \leqslant \mathrm{O}_\pi(N/K)$ by minimality, and $\mathrm{O}_\pi(N/K) = \mathrm{O}_\pi(N)K/K$ because K is central N, we deduce $[x^G, x^G] = N' \leqslant K \leqslant \mathrm{Z}(N)$. Therefore N is a nilpotent π'-group. Since $\left|x^G\right|$ is a π-number, we obtain $x \in \mathrm{Z}(N)$ and $[x^G, x^G] = 1$.

Next we concentrate on the second assertion. Let $\overline{G} := G/\,\mathrm{O}_\pi(N)$. Then, $[\overline{x}^{\overline{G}}, \overline{x}^{\overline{G}}] = 1$ by the first claim. It follows that $\langle \overline{x} \rangle \trianglelefteq \mathrm{Z}(\langle \overline{x}^{\overline{G}} \rangle) \trianglelefteq \overline{G}$, so $\langle \overline{x} \rangle \leqslant \mathrm{F}(\overline{G}) \cap \overline{N} \leqslant \mathrm{F}(\overline{N})$. As $\mathrm{F}(\overline{N})$ is a normal π'-subgroup of $\overline{G}$ and $|\overline{x}^{\overline{G}}|$ is a π-number, then necessarily $\overline{x} \in \mathrm{Z}(\mathrm{F}(\overline{N}))$.

Finally, observe that the last statement follows from Lemma 3.1.5, since $[x^G, x^G] \leqslant \mathrm{O}_p(G) \cap N \leqslant \mathrm{O}_p(N)$. □

Example 3.3.2. Note that the π-separability assumption in the previous result cannot be removed, even when $N = G$: Consider any non-trivial element in the centre of a Sylow p-subgroup of a non-abelian simple group and $\pi = p'$, for a prime divisor p of its order.

For a normal subgroup N of a group G, note that if xN is a vanishing (prime power order) element of G/N, then we can assume that x is also a vanishing (prime power order) element of G. This is because there exists a bijection between $\mathrm{Irr}(G/N)$ and the set of all characters in $\mathrm{Irr}(G)$

containing N in their kernel. This fact will be used in the sequel with no reference.

As an application of the above proposition and mainly Theorem 3.1.1, we prove the following.

Theorem 3.3.3. *Let N be a normal subgroup of a group G, and let π be any set of prime numbers.*

(1) *Suppose that $\left|x^G\right|$ is a π'-number for every prime power order π-element $x \in N$ which is vanishing in G. If N is π-separable, then $N/\operatorname{O}_{\pi'}(N)$ has a nilpotent normal Hall π-subgroup. In particular, the Hall π-subgroups of N are nilpotent.*

(2) *Suppose that $\left|x^G\right|$ is a π-number for every prime power order π-element $x \in N$ which is vanishing in G. If $\operatorname{Hall}_\pi(N) \neq \emptyset$, then N has a normal Hall π-subgroup. Additionally, if all $\left|x^G\right|$ are also π-numbers for the prime power order π'-elements $x \in N$ that are vanishing in G, then the Hall π'-subgroups of N are nilpotent.*

Proof. (1) Assume that N is π-separable, and that $\left|x^G\right|$ is a π'-number for every prime power order π-element $x \in \operatorname{Van}(G) \cap N$. Let us prove that $N/\operatorname{O}_{\pi'}(N)$ has a normal Sylow p-subgroup for each prime $p \in \pi$. Certainly, whenever $\operatorname{O}_{\pi'}(N) \neq 1$, the assertion follows by induction, considering the groups $G/\operatorname{O}_{\pi'}(N)$ and $N/\operatorname{O}_{\pi'}(N)$. Therefore we may assume that $\operatorname{O}_{\pi'}(N) = 1$. Let $Z_p := \operatorname{Z}(\operatorname{O}_p(N))$. In virtue of Proposition 3.3.1, we have that all the p-elements of $\operatorname{Van}(G) \cap N$ lie in $\operatorname{Z}(\operatorname{F}(N))$, and thus in Z_p. Therefore, if we denote $\overline{G} := G/Z_p$, then it follows that no prime power order p-element of $\overline{N}$ is vanishing in $\overline{G}$. Now Theorem 3.1.1 yields that $\overline{N}$ has a normal Sylow p-subgroup $\overline{P}$, where $P \in \operatorname{Syl}_p(N)$. Since Z_p is a p-group, then P is normal in N clearly and we get the claim. As this is valid for each prime $p \in \pi$, then $N/\operatorname{O}_{\pi'}(N)$ has a nilpotent normal Hall π-subgroup, as wanted.

(2) Assume that N has Hall π-subgroups, and that $\left|x^G\right|$ is a π-number for every prime power order π-element $x \in \operatorname{Van}(G) \cap N$. We claim that N has a normal Hall π-subgroup. Clearly we may assume $\operatorname{O}_\pi(N) = 1$. Let $H \in \operatorname{Hall}_\pi(N)$, and let $p \in \pi$. If $x \in N \cap \operatorname{Van}(G)$ is a p-element, then $x \in P \in \operatorname{Syl}_p(N)$. Hence there exists $g \in N$ such that $x^g \in P^g \in \operatorname{Syl}_p(H)$. Now Lemma 3.1.4 yields $x^g \in \operatorname{O}_\pi(N) = 1$. Thus there are no p-elements

in $N \cap \mathrm{Van}(G)$, and by Theorem 3.1.1 we get that N has a normal Sylow p-subgroup. Since this is valid for every prime $p \in \pi$, then N has a (nilpotent) normal Hall π-subgroup, as desired.

Next we show that N has nilpotent Hall π'-subgroups under the additional assumption that the prime power order π'-elements in $N \cap \mathrm{Van}(G)$ have also G-class sizes not divisible by any prime in π'. Note that N is π-separable because it has a normal Hall π-subgroup, say H. If we take any prime power order element $xH \in (N/H) \cap \mathrm{Van}(G/H)$, then we may suppose that $x \in N \cap \mathrm{Van}(G)$ is a prime power order element, so by assumptions $|x^G|$ is a π-number. Thus $|(xH)^{G/H}|$ is also a π-number. Therefore every $|(xH)^{G/H}|$ is a π-number for each prime power order π'-element $xH \in (N/H) \cap \mathrm{Van}(G/H)$, so by assertion (1) the π'-group N/H is nilpotent. Since N/H is isomorphic to a Hall π'-subgroup of N, the proof is completed. □

Example 3.3.4. We remark that the π-separability assumption in Theorem 3.3.3 (1) is necessary for the first claim. Let G be a symmetric group of degree 5, and let N be an alternating group of degree 5. Consider $\pi = \{3\}$. Then all the 3-elements in $N \cap \mathrm{Van}(G)$ have conjugacy class size equal to 20. Nevertheless, $N/\mathrm{O}_{\pi'}(N) = N$ does not have a normal Sylow 3-subgroup.

Example 3.3.5. It is not difficult to find groups satisfying the assumptions of Theorem 3.3.3. For instance, let $G = A\Gamma(2^3)$ be an affine semilinear group of order 168, and let N be the Hall $3'$-subgroup of G. If we consider $\pi = \{7\}$, then the pair (N, G) satisfies the hypotheses of Theorem 3.3.3 (1). Concerning Theorem 3.3.3 (2), if π is any set of prime numbers, $G = \mathrm{O}_\pi(G) \times \mathrm{O}_{\pi'}(G)$ and $N = \mathrm{O}_\pi(G)$, then the pair (N, G) certainly holds the hypotheses.

The next theorem combines the arithmetical conditions on the vanishing G-class sizes of Theorem 3.3.3.

Theorem 3.3.6. *Let N be a normal π-separable subgroup of a group G. Assume that $|x^G|$ is either a π-number or a π'-number for every prime power order π-element $x \in \mathrm{Van}(G) \cap N$. Then $N/\mathrm{O}_{\pi'}(N)$ has a normal Hall π-subgroup. Thus N has π-length at most 1.*

Proof. First, we claim that $O := \mathrm{O}_{\pi,\pi'}(N)$ contains a Sylow p-subgroup of N, for a prime $p \in \pi$. Let $x \in \mathrm{Van}(G) \cap N$ be a p-element. If $\left|x^G\right|$ is a π-number, then x lies in $\mathrm{O}_\pi(N)$ because of Lemma 3.1.4, so clearly $x \in O$. If $\left|x^G\right|$ is a π'-number, then by Proposition 3.3.1 we get $x\,\mathrm{O}_{\pi'}(N) \in \mathrm{F}(N/\,\mathrm{O}_{\pi'}(N))$, and again x lies in O. It follows that $\overline{N} := N/O$ contains no vanishing p-element of G/O, so $\overline{N}$ has a normal Sylow p-subgroup $\overline{P}$ in virtue of Theorem 3.1.1. Since $p \in \pi$ and clearly $\mathrm{O}_\pi(\overline{N}) = 1$, thus $\overline{P} = 1$.

Therefore O contains a Sylow p-subgroup of N for every $p \in \pi$, and thus $O/\,\mathrm{O}_{\pi'}(N)$ is a Hall π-subgroup of $N/\,\mathrm{O}_{\pi'}(N)$. □

The main theorem of [2] examines groups such that all their π-elements have prime power class sizes. The next result is a "vanishing version" of that theorem for prime power order elements and in the context of G-conjugacy classes.

Theorem 3.3.7. *Let N be a normal subgroup of a group G. Assume that $\left|x^G\right|$ is a prime power for each prime power order π-element $x \in N$ that is vanishing in G. Then $N/\,\mathrm{O}_{\pi'}(\mathrm{F}(N))$ has a normal Hall π-subgroup.*

In particular, if π is the set of prime divisors of $|N|$, then $N/\,\mathrm{F}(N)$ is nilpotent.

Proof. We claim that $\overline{N} := N/\,\mathrm{F}(N)$ has a normal Hall π-subgroup, and therefore $N/\,\mathrm{O}_{\pi'}(\mathrm{F}(N))$ so does because $\mathrm{F}(N)/\,\mathrm{O}_{\pi'}(\mathrm{F}(N))$ is a π-group. Arguing by contradiction, and in virtue of Proposition 3.2.7, we may assume that $\overline{N} \cap \mathrm{Van}(\overline{G})$ contains a non-trivial q-element for some prime $q \in \pi$, say $\overline{x}$. Hence we may suppose that $x \in (N \cap \mathrm{Van}(G)) \setminus \mathrm{F}(N)$ is a q-element. By assumptions, we have that $\left|x^G\right|$ is a power of some prime p. Observe that, since $x \notin \mathrm{F}(N)$, then $q \neq p$ due to Lemma 3.1.4. Now the last statement of Proposition 3.3.1 yields $(\langle x^G\rangle)' \leqslant \mathrm{O}_p(N) \leqslant \mathrm{F}(N)$, so $\overline{\langle x\rangle}$ is a subnormal nilpotent subgroup of $\overline{N}$. It follows that $\overline{x} \in \mathrm{F}(\overline{N})$, and as $\overline{x}$ is a q-element, then $\overline{x} \in \mathrm{O}_q(\overline{N})$. Now $\left|\overline{x}^{\overline{G}}\right|$ is a multiple of q by Proposition 3.1.6, and then $\left|x^G\right|$ so is, a contradiction.

Finally, if $\pi = \pi(N)$, then with a similar argument we deduce that there is no prime power order element in $N/\,\mathrm{F}(N)$ vanishing in $G/\,\mathrm{F}(N)$. Hence Theorem 3.1.1 applies and $N/\,\mathrm{F}(N)$ is nilpotent. □

3.4 Some consequences on vanishing conjugacy classes

New interesting contributions on the lengths of vanishing classes of a group G emerge from Theorem 3.3.3, Theorem 3.3.6 and Theorem 3.3.7 when $N = G$.

Theorem 3.4.1. *Let G be a π-separable group. If $\left|x^G\right|$ is a π'-number for every prime power order π-element $x \in \mathrm{Van}(G)$, then $G/\operatorname{O}_{\pi'}(G)$ has a nilpotent normal Hall π-subgroup. Therefore, G has nilpotent Hall π-subgroups, and its π-length is at most 1.*

Theorem 3.4.2. *Let G be a finite group such that $\mathrm{Hall}_\pi(G) \neq \emptyset$. Assume $\left|x^G\right|$ is a π-number for every prime power order π-element $x \in \mathrm{Van}(G)$. Then G has a normal Hall π-subgroup.*

Further, if the prime power order π'-elements in $\mathrm{Van}(G)$ have also class size a π-number, then the Hall π'-subgroups of G are nilpotent.

Theorem 3.4.3. *Let G be a group. Suppose that $\left|x^G\right|$ is either a π-number or a π'-number for every prime power order π-element $x \in \mathrm{Van}(G)$. Then $G/\operatorname{O}_{\pi'}(\operatorname{F}(G))$ has a normal Hall π-subgroup. In particular, G has π-length at most 1.*

Theorem 3.4.4. *Let G be a group. Assume that $\left|x^G\right|$ is a prime power for every vanishing element x of G of prime power order. Then G' is nilpotent.*

Proof. Arguing as in the proof of Theorem 3.3.7 we can see that $G/\operatorname{F}(G)$ has no prime power order vanishing elements. Thus, Theorem 2.1.1 applies and $G/\operatorname{F}(G)$ is abelian, so G' is nilpotent. □

Chapter 4

B_π-character degrees and normal subgroups

The classical Ito-Michler theorem and Thompson's theorem on character degrees prove that there exists a deep connection between the normal structure of a group G and the primes dividing the degrees of characters in $\mathrm{Irr}(G)$. In this chapter, we will see some variants, involving B_π-characters, of these two theorems. In particular, we will see that the theorems are still true if we consider the set $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ instead of $\mathrm{Irr}(G)$.

4.1 Character degrees and normal subgroups

One of the main topics of the theory of characters of finite groups is to find connections between the normal structure of a group and the primes dividing the degrees of its irreducible characters.

The most famous result of this type is the classical theorem of Ito-Michler.

Theorem 4.1.1 (Ito-Michler). *Let G be a finite group and let p a prime number, then G has a normal abelian Sylow p-subgroup if and only if p does not divide the degree of any character in* $\mathrm{Irr}(G)$.

There exists many variants of this theorem, one of them involves irreducible Brauer character and it is due to Michler himself.

Theorem 4.1.2 (Michler)**.** *Let G be a finite group and let p a prime number, then G has a normal Sylow p-subgroup if and only if p does not divide the degree of any character in* $\mathrm{IBr}_p(G)$*.*

In more recent years, some more variants of Ito-Michler Theorem have been found.

A first result, that we already cited in Chapter 2, links the normality of the Sylow subgroup with the degree of only some of the irreducible characters.

Theorem 4.1.3 (*[28, Theorem B]*)**.** *Let G be a finite group, let p be a prime and let $P \in \mathrm{Syl}_p(G)$. Then, $P \lhd G$ if and only if p does not divide the degree of any irreducible constituent of $(1_P)^G$.*

Another result of this type takes into account p-rational characters, i.e., those characters whose field of values is contained in a cyclotomic field $\mathbb{Q}_n$ for some n not divisible by p.

Theorem 4.1.4 (*[35, Theorem A]*)**.** *Let G be a finite group and let p be a prime. If p does not divide the degree of any p-rational character in* $\mathrm{Irr}(G)$*, then G has a normal Sylow p-subgroup.*

In Section 1.4 we have seen that, in a p-solvable group G, the characters in $\mathrm{IBr}_p(G)$ are lifted by the characters in $\mathrm{B}_{p'}(G)$, therefore they have the same character degrees. Moreover, we have seen in Section 1.5 that, if the group G is of odd degree and $P \in \mathrm{Syl}_P(G)$, $\mathrm{B}_{p'}$-characters are exactly the constituents of $(1_P)^G$ of odd multiplicity, and exactly the p-rational characters of G.

This may suggest that there are analogues of Theorems 4.1.3 and 4.1.4 also for B_π-characters.

The next theorem, due to Thompson, is a sort of dual of the theorem of Ito-Michler, since it describes what happens when a prime divides the degree of each nonlinear irreducible character of a group.

Theorem 4.1.5 (*[14, Corollary 12.2]*)**.** *Let G be a finite group and p a prime. If p divides the degree of every nonlinear irreducible character of G, then G has a normal p-complement.*

In [38], it is studied a variant of the above theorem which involves more then one prime. Let $\mathrm{Irr}_{\pi'}(G)$ be the set of irreducible characters which degree is not divided by any prime in π, for some set of primes π. Then, the condition that p divides the degree of every irreducible nonlinear character of G is equivalent to asking that $\mathrm{Irr}_{p'}(G) = \mathrm{Lin}(G)$.

Theorem 4.1.6. *[38, Corollary 3] Let G be a π-separable group and let H be a Hall π-subgroup, then $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$ if and only if $G' \cap \mathrm{N}_G(H) = H'$.*

Notice that, if Thompson's theorem holds for a group G and a prime p, then G is p-solvable and, thus, Theorem 4.1.6 applies, too.

4.2 Variants of Ito-Michler theorem

In this section, we see some variants of Ito-Michler theorem for B_π-characters. The techniques we use are similar to the ones of Chapter 2. In particular, we use the results we cited in Section 2.5.

As a first result of the section, we present a different proof of [18, Theorem 3.17] using Lemma 2.5.2.

We recall that a B_π-character χ belongs to $\mathrm{X}_\pi(G)$, i.e., it is π-special, if and only if its degree is a π-number.

Theorem 4.2.1 (*[18, Theorem 3.17]*)**.** *Let G be π-separable, then $\mathrm{B}_\pi(G) = \mathrm{X}_\pi(G)$ if and only if G has a normal π-complement.*

Proof. Note at first that, if G has a normal π-complement, then it follows that $\mathrm{B}_\pi(G) = \mathrm{X}_\pi(G)$. Thus, there is only one implication to be proved.

Let us assume that $\mathrm{B}_\pi(G) = \mathrm{X}_\pi(G)$ and prove the thesis by induction on $|G|$. At first, let us assume that $\mathrm{O}_{\pi'}(G) = 1$.

Let N be a minimal normal subgroup of G and suppose it to be a π-group. Since the hypothesis are preserved by factor groups, if H is a Hall π'-subgroup of G, then by induction HN is normal in G. In particular, it follows that there exists $K \lhd G$ such that K/N is a π'-chief factor of G. Since $|N|$ and $|K/N|$ are coprime, at least one of them is odd and, thus, since an odd group is solvable and both N and K/N are minimal normal subgroups, of G and G/N respectively, at least one of them is abelian.

Suppose K/N is abelian and let A be an abelian complement of N in K. Since we have assumed $\mathrm{O}_{\pi'}(G) = 1$, it follows by Lemma 2.5.2 that there exists a character in $\mathrm{B}_\pi(G)$ which degree is divided by the π'-number $|K : N|$, contradicting the hypothesis.

Suppose now that K/N is not abelian, so N has to be, and let $\lambda \in \mathrm{Irr}(N) = \mathrm{B}_\pi(N)$. If λ is not K-invariant, then the degree of some $\theta \in \mathrm{B}_\pi(K \mid \lambda)$ is divided by some primes in π' and, therefore, so is the degree of some character $\chi \in \mathrm{B}_\pi(G \mid \theta)$, in contradiction with the hypothesis. It follows that K fixes every character of N and, thus, it also centralizes N, since N is abelian. If B is a complement of N in K, it follows that it is normal in K. In particular, $1 < B = \mathrm{O}_{\pi'}(K) \leq \mathrm{O}_{\pi'}(G)$.

Therefore, we have that $\mathrm{O}_{\pi'}(G) \neq 1$ and the thesis follows by induction, because $\mathrm{B}_\pi(G/\mathrm{O}_{\pi'}(G)) = \mathrm{B}_\pi(G)$. □

In Chapter 2, we have anticipated that we can tell whether a group has a normal Sylow p-subgroup from the degrees of the characters in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$. We actually have an exact equivalent of Ito-Michler theorem for π-separable groups.

Theorem 4.2.2. *Let G be a π-separable group and p be any prime. Then G has a normal abelian Sylow p-subgroup if and only if p does not divide the degree of any character in* $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$.

Proof. It can be observed that there is little to prove in one direction, being it a consequence of the Ito-Michler theorem. Thus, we assume that p does not divide the degree of any character in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ and we first prove that there exists a normal Sylow p-subgroup. We argue by induction on $|G|$.

Let N be a minimal normal subgroup of G. Without loss of generality, we can assume N to be a π-group. If $p \mid |N|$, then $p \in \pi$. By induction, let K/N be a normal Sylow p-subgroup of G/N, then K is a normal π-subgroup of G which contains a Sylow p-subgroup $P \in \mathrm{Syl}_p(G)$. If P is normal abelian in K, then it is also in G. Otherwise, there exists $\theta \in \mathrm{Irr}(K) = \mathrm{B}_\pi(K)$ such that $p \mid \theta(1)$ and, by Theorem 1.3.3, there exists $\chi \in \mathrm{B}_\pi(G)$ lying over θ. As a consequence, $p \mid \chi(1)$, in contradiction with the hypothesis.

Therefore, we can assume p does not divide $|N|$. Since N is arbitrarily chosen, we can assume that $\mathrm{O}_p(G) = 1$. As in the previous paragraph, let

K/N be a normal Sylow p-subgroup of G/N, which is nontrivial because p divides $|G:N|$, and let $C/N = \mathrm{Z}(K/N)$. Note that $C \trianglelefteq G$. Let A be a complement of N in C. Then, by Lemma 2.5.2, there exists a character χ in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ such that $|C:N|$ divides $\chi(1)$. However, since $|C:N|$ is a power of p, this would contradict the hypothesis.

Finally, if P is a normal Sylow p-subgroup of G and $\gamma \in \mathrm{Irr}(P)$, then by Theorem 1.3.3 there exists $\chi \in \mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ lying over γ and, thus, $\gamma(1) \mid \chi(1)$. Since $p \nmid \chi(1)$, then γ is linear. It follows that $\mathrm{Irr}(P) = \mathrm{Lin}(P)$ and, thus, P is abelian. □

Since the conditions on the group of Theorem 4.2.2 are the same as in the Theorem of Ito-Michler, a corollary easily follows.

Corollary 4.2.3. *Let G be a π-separable group and let p be any prime. Then, p divides the degree of some characters in $\mathrm{Irr}(G)$ if and only if it divides the degree of some characters in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$.*

Finally, we recall that, in a p-solvable group G, the set $\mathrm{B}_{p'}(G)$ is a family of lifts for the irreducible Brauer characters.

Corollary 4.2.4. *Let G be a p-solvable group. If a prime q divides the degree of some characters in $\mathrm{Irr}(G)$ and it does not divide the degree of any irreducible Brauer character, then q divides the degree of some characters in $\mathrm{B}_p(G)$.*

4.3 Groups where only one B_π-character is not π-special

In the previous section, we have seen what happens when $\mathrm{B}_\pi(G) = \mathrm{X}_\pi(G)$. We now see what can we say on a group when $|\mathrm{B}_\pi(G) \setminus \mathrm{X}_\pi(G)| = 1$.

A first result is easy to prove.

Proposition 4.3.1. *Let G be π-separable and suppose $\mathrm{B}_\pi(G) \setminus \mathrm{X}_\pi(G) = \{\chi\}$. If $\chi(1)$ is a π'-number, then $\mathrm{O}^\pi(G) = G$.*

Proof. Let $K = \mathrm{O}^\pi(G)$, then χ_K is irreducible and it follows from Gallagher theorem that $\chi\psi \in \mathrm{B}_\pi(G) \setminus \mathrm{X}_\pi(G)$ for each $\psi \in \mathrm{Irr}(G/K)$. Then, $|\mathrm{Irr}(G/K)| = 1$ and $G = K$. □

In order to study the problem without the hypothesis that $\chi(1)$ is a π'-number, we need to further assume that $\pi = \{p\}$. In order to simplify the notation, we write $\mathrm{B}_p(G)$ and $\mathrm{X}_p(G)$ instead of $\mathrm{B}_{\{p\}}(G)$ and $\mathrm{X}_{\{p\}}(G)$.

Lemma 4.3.2. *Let G be a p-solvable group and let $P \in \mathrm{Syl}_p(G)$. Let $H, N \trianglelefteq G$ such that $N \leq H$, N is a minimal normal p-group (and, thus, it is also abelian) and H/N is a normal p-complement for G/N. Suppose that $\mathrm{B}_p(G) \setminus \mathrm{X}_p(G) = \{\chi\}$ and that $\chi(1) = qn$ with q power of p, $p \nmid n$ and $q, n > 1$. Then:*

A) $H = \bigcap\{\ker\eta \mid \eta \in \mathrm{X}_p(G)\}$;

B) $n = |N| - 1$;

C) $\sum_{\eta\in \mathrm{X}_p(G)} \eta^2(1) = q^2$.

Proof. Let $\psi \in \mathrm{B}_p(H)$ be an irreducible constituent of χ_H and $\mu \in \mathrm{Irr}(N)$ be and irreducible constituent of ψ_N. Note that μ is a Fong character for ψ and, since μ is linear, then $\psi(1) = n$.

Step 1: all the nonprincipal characters in $\mathrm{Irr}(N)$ are constituent of χ_N. Moreover, A) holds.

Let $C = \mathrm{C}_N(H)$, then $C \lhd G$ and $C \leq N$, thus $C = 1$ or $C = N$, since N is minimal normal in G. If $C = N$, however, then $H = N \times H_0$ and H_0 is a normal p-complement of G, thus, $\mathrm{B}_p(G) = \mathrm{X}_p(G)$, a contradiction. It follows that $C = 1$ and, thus, no nonprincipal character in $\mathrm{Irr}(N)$ is H-invariant. From this, in particular, one has that $\mathrm{X}_p(H) = \{1_H\}$, since any p-special character would restrict irreducibly to N (see Corollary 1.3.6). By Proposition 1.2.2, H is in the kernel of every p-special character in G. On the other hand, $H = \bigcap\{\ker\eta \mid \eta \in \mathrm{Irr}(G/H)\} \geq \bigcap\{\ker\eta \mid \eta \in \mathrm{X}_p(G)\}$ and A) holds.

Moreover, it follows that, for every nonprincipal $\xi \in \mathrm{Irr}(N)$, there exists a character $\theta \in \mathrm{B}_p(H \mid \xi)$ such that $\theta(1)_{p'} > 1$. However, since χ is the only character in $\mathrm{B}_p(G)$ such that $\chi(1)_{p'} > 1$, every such character θ has to be a constituent of χ_H and Step 1 follows.

Step 2: ψ is G-invariant. Moreover, B) holds.

Let $\chi_H = e\sum_{i=1}^m \psi_i$, where $m = |G : \mathrm{I}_G(\psi)|$ is a power of p, and let $\psi_N = \sum_{j=1}^n \mu_j$, since all the characters μ_j are Fong characters for ψ and so they have multiplicity 1. Moreover, since all the characters ψ_i are distinct B_p-characters and N is a Sylow p-subgroup for H, if $\psi_{i_1} \neq \psi_{i_2}$ then $[\psi_{i_1 N}, \psi_{i_2 N}] = 0$. It follows that $\chi_N = e \sum_{k=1}^{mn} \mu_k$. However, by Step 1, one has that each character in $\mathrm{Irr}(N) \setminus \{1_N\}$ is a constituent of χ_N, thus $mn = |\mathrm{Irr}(N)| - 1 = |N| - 1$. Since $p \nmid |N| - 1$, then $p \nmid m$ and thus $m = 1$ and $n = |N| - 1$. Moreover, $e = q$.

Step 3: C) holds.

All the irreducible components of ψ^G are in $\mathrm{B}_p(G)$, for Theorem 1.3.3, and the p'-part of their degree is not 1, thus $\psi^G = q\chi$. It follows that χ and ψ are fully ramified and, thus, $|G : H| = q^2$. However, by A), one has that H is the intersection of all the kernels of the p-special characters in G; it follows that $\mathrm{X}_p(G) = \mathrm{Irr}(G/H)$ and thus C) holds. □

Corollary 4.3.3. *Let G be p-solvable. If $\mathrm{B}_p(G) \setminus \mathrm{X}_p(G) = \{\chi\}$, then $\chi(1)_p{}^2 \leq |G : \mathrm{O}^p(G)|$.*

Proof. Suppose it is not true and let G be a minimal counterexample, such that $|G : \mathrm{O}^p(G)| < \chi(1)_p{}^2$. Then clearly $\chi(1)_p > 1$. Let N be a minimal normal subgroup of G and notice that $|G/N : \mathrm{O}^p(G/N)| \leq |G : \mathrm{O}^p(G)|$, thus, $|G/N : \mathrm{O}^p(G/N)| < \chi(1)_p{}^2$. Therefore, by the minimality of G, χ is not a character of G/N; it follows that N is a p-group and that G/N has a normal p-complement H/N, by Theorem 4.2.1, since $\mathrm{B}_p(G/N) = \mathrm{X}_p(G/N)$. Notice that $H \geq \mathrm{O}^p(G)$, because $|G : H|$ is a p-number.

Now, by Lemma 4.3.2, points A) and C), one has that

$$\chi(1)_p{}^2 = \sum_{\eta \in \mathrm{X}_p(G)} \eta^2(1) = \sum_{\theta \in \mathrm{Irr}(G/H)} \theta^2(1) = |G : H| \leq |G : \mathrm{O}^p(G)|$$

and the corollary is proved. □

4.4 Variants on Thompson theorem

In this section, we see a refinement of Theorem 4.1.6 involving B_π-characters. In particular, we study what happens in the group when $\mathrm{B}_\pi(G) \cap \mathrm{Irr}_{\pi'}(G) \subseteq \mathrm{Lin}(G)$ and when $\mathrm{X}_{\pi'}(G) \subseteq \mathrm{Lin}(G)$, keeping in mind that $\mathrm{X}_{\pi'}(G) = \mathrm{B}_{\pi'}(G) \cap \mathrm{Irr}_{\pi'}(G)$ by Lemma 1.3.2.

For the section, we need a corollary to a variant of the famous McKay conjecture, due to T. Wolf.

Corollary 4.4.1. *Let G be a π-separable group, let H be a Hall π-subgroup of G and let $N = \mathrm{N}_G(H)$. Then:*

$$|\{\chi \in \mathrm{B}_\pi(G) \,|\, \chi(1) \text{ is a } \pi'\text{-number}\}| = |\{\psi \in \mathrm{B}_\pi(N) \,|\, \psi(1) \text{ is a } \pi'\text{-number}\}|\,,$$

$$|\mathrm{X}_{\pi'}(G)| = |\mathrm{X}_{\pi'}(N)| = |\mathrm{Irr}(N/H)|\,.$$

Proof. It is a direct consequence of [42, Theorem 1.15]. □

At first, an easy lemma is needed, which uses the properties of the Fong characters associated with a B_π-character.

Lemma 4.4.2. *Let G be a π-separable group and let H be a Hall π-subgroup, then $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G) \subseteq \mathrm{Lin}(G)$ if and only if every linear character of H extends to G.*

Proof. Let λ be a linear character in H. By Theorem 1.3.9, λ is the Fong character associated with some character $\chi \in \mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G)$. It follows that, if χ is linear, then it extends λ, while on the other hand if λ extends to G, then by Theorem 1.3.10 it has a linear π-special extension, which coincides with χ by Theorem 1.3.5. □

We can now prove a first result, which underlines a relation between the families of characters $\mathrm{Irr}(G)$ and $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ for what concerns the hypothesis of Thompson's theorem.

Proposition 4.4.3. *Let G be a π-separable group. Then,* $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$ *if and only if* $\mathrm{Irr}_{\pi'}(G) \cap \big(\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)\big) \subseteq \mathrm{Lin}(G)$*.*

Proof. One direction is obviously true. Thus, let one assume $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G) \subseteq \mathrm{Lin}(G)$ and suppose there exists a nonlinear character $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)$ is a π'-number. By Theorem 1.3.10 there exists $W \leq G$, $\alpha \in \mathrm{X}_\pi(W)$ linear and $\beta \in \mathrm{X}_{\pi'}(W)$ such that $\chi = (\alpha\beta)^G$, W contains a Hall π-subgroup H of G and it is the maximal subgroup of G such that α_H extends to W. However, by Lemma 4.4.2, α_H extends to G, thus $W = G$. It follows that β is a nonlinear π'-special character of G and this contradict the fact that every character in $\mathrm{X}_{\pi'}(G) = \mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_{\pi'}(G)$ is linear, in contrast with the hypothesis. □

At this point, we can already see what happens when the set $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G)$, or the set $\mathrm{X}_{\pi'}(G)$, contains only the principal character.

Corollary 4.4.4. *Let G be a π-separable group and let H be a Hall π-subgroup, then*

i) $\mathrm{Irr}_{\pi'}(G) = \{1_G\}$ *if and only if* $\mathrm{Irr}_{\pi'}(G) \cap \left(\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)\right) = \{1_G\}$*;*

ii) $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G) = \{1_G\}$ *if and only if $H = H'$;*

iii) $\mathrm{X}_{\pi'}(G) = \{1_G\}$ *if and only if H is self-normalizing;*

iv) $\mathrm{Irr}_{\pi'}(G) = \{1_G\}$ *if and only if $H = H'$ and H is self-normalizing.*

Proof. For point (i), only one direction is needed. Suppose, thus, that $\mathrm{Irr}_{\pi'}(G) \cap \left(\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)\right) = \{1_G\} \subseteq \mathrm{Lin}(G)$, then by Proposition 4.4.3 $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$. It follows that every character in $\mathrm{Irr}_{\pi'}(G)$ can be factorized as a product $\alpha\beta$, with $\alpha \in \mathrm{Irr}_{\pi'}(G) \cap \mathrm{X}_\pi(G)$ and $\beta \in \mathrm{X}_{\pi'}(G)$; however, the two sets of characters both coincide with $\{1_G\}$ by hypothesis.

Point (ii) follows directly from Lemma 4.4.2. In fact, if $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G) = \{1_G\} \subseteq \mathrm{Lin}(G)$, then every character in $\mathrm{Lin}(H)$ extends to G and, by Theorem 1.3.10, it has an extension in $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G)$, thus $\mathrm{Lin}(H) = \{1_H\}$. On the other hand, if $\mathrm{Lin}(H) = \{1_H\}$, then there are no nonprincipal linear Fong characters of H in G and it follows that $|\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G)| = 1$ and the thesis follows.

Finally, point (iii) is a direct consequence of Corollary 4.4.1 and point (iv) follows from points (i), (ii) and (iii). □

Let us now proceed by studying variants of Theorem 4.1.6.

Proposition 4.4.5. *Let G be a π-separable group and let H be a Hall π-subgroup. Then,* $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G) \subseteq \mathrm{Lin}(G)$ *if and only if $G' \cap H = H'$.*

Proof. From Lemma 4.4.2 we know that the property that every character in $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G)$ is linear is equivalent to the fact that every character of H/H' extends to G. Thus, suppose that for every $\lambda \in \mathrm{Irr}(H/H')$ there exists $\chi \in \mathrm{Lin}(G)$ such that $\chi_H = \lambda$. It follows that

$$H' \leq G' \cap H = \bigcap_{\chi \in \mathrm{Lin}(G)} \ker(\chi_H) = \bigcap_{\lambda \in \mathrm{Lin}(H)} \ker(\lambda) = H'$$

and, therefore, $G' \cap H = H'$.

On the other hand, suppose $G' \cap H = H'$; then, if K/G' is a Hall π'-subgroup of G/G', one can write

$$\frac{G}{G'} = \frac{HG'}{G'} \times \frac{K}{G'} = \frac{H}{H'} \times \frac{K}{G'}$$

and, thus, every $\lambda \in \mathrm{Irr}(H/H')$ extends to $\lambda \times 1_{K/G'} \in \mathrm{Irr}(G)$. □

Proposition 4.4.6. *Let G be a π-separable group, let H be a Hall π-subgroup for G and let $N = \mathrm{N}_G(H)$. Then, $\mathrm{X}_{\pi'}(G) \subseteq \mathrm{Lin}(G)$ if and only if $G' \cap N \leq H$.*

Proof. Assume at first that $\mathrm{X}_{\pi'}(G) \subseteq \mathrm{Lin}(G)$; therefore, if $\chi \in \mathrm{X}_{\pi'}(G)$, then χ_N is linear. Suppose that, for some $\chi, \psi \in \mathrm{X}_{\pi'}(G)$, $\chi_N = \psi_N$; then $N \leq \ker(\chi\bar{\psi}) \lhd G$. It follows that $\ker(\chi\bar{\psi}) = G$, by Frattini argument, and thus $\chi = \psi$. Therefore, the restriction realizes an injection from $\mathrm{X}_{\pi'}(G)$ to $\mathrm{X}_{\pi'}(N)$ and, since $|\mathrm{X}_{\pi'}(G)| = |\mathrm{X}_{\pi'}(N)|$ by Corollary 4.4.1, it is actually a bijection. It follows that every character in $\mathrm{Irr}(N/H)$ is the restriction of a linear character of G, thus we have that

$$G' \cap N = \bigcap_{\chi \in \mathrm{Lin}(G)} \ker(\chi_N) \leq \bigcap_{\lambda \in \mathrm{Irr}(N/H)} \ker(\lambda) = H.$$

On the other hand, suppose that $G' \cap N \leq H$. Let X be a complement for H in N and note that X is abelian. Moreover, note that NG' is normal in G and it contains N, thus $G = NG'$ by the Frattini argument. It follows that

$$\frac{G}{G'} \cong \frac{N}{G' \cap N} = X \times \frac{H}{G' \cap H}$$

and, thus, there is a bijection between characters in $\mathrm{Irr}(X) = \mathrm{Irr}(N/H)$ and characters in $\mathrm{X}_{\pi'}(G/G')$. However, by Corollary 4.4.1 we have that $|\mathrm{X}_{\pi'}(G/G')| = |\mathrm{Irr}(N/H)| = |\mathrm{X}_{\pi'}(G)|$ and, thus, it follows that every π'-special character in G is linear. □

We shall now summarize these last results in a single theorem.

Theorem 4.4.7. *Let G be a π-separable group, let H be a Hall π-subgroup for G and let $N = \mathrm{N}_G(H)$. Then,*

a) $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_\pi(G) \subseteq \mathrm{Lin}(G)$ *if and only if $G' \cap H = H'$;*

b) $\mathrm{Irr}_{\pi'}(G) \cap \mathrm{B}_{\pi'}(G) \subseteq \mathrm{Lin}(G)$ *if and only if* $G' \cap N \leq H$.

It is now easy to see that Theorem 4.1.6 can also be obtained as a corollary of Theorem 4.4.7 and of Proposition 4.4.3.

Corollary 4.4.8 (*[38, Corollary 3]*)**.** *Let* G *be a* π*-separable group and let* H *be a Hall* π*-subgroup for* G *and* $N = \mathrm{N}_G(H)$*. Then,* $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$ *if and only if* $G' \cap N = H'$.

Proof. If N is the normalizer in G of a Hall π-subgroup H, Theorem 4.4.7 and Proposition 4.4.3 provide that $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$ if and only if both $G' \cap N \leq H$ and $G' \cap H = H'$ and the two conditions on G happen simultaneously if and only if $G' \cap N = H'$. □

4.5 Some examples

Considering the nature of the results presented in this chapter, a natural question a reader may ask is whether the set $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ is actually strictly smaller then $\mathrm{Irr}(G)$. This happens quite often. In fact, we have seen in Theorem 1.4.2 that $|\mathrm{B}_\pi(G)|$ is equal to the number of conjugacy classes of π-elements of G. Therefore, $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G) = \mathrm{Irr}(G)$ if and only if each element of the π-separable group G is either a π-element or a π'-element. This is proved in [11, Lemma 4.2] to happen if and only if G is a Frobenius or a 2-Frobenius group and each Frobenius complement and Frobenius kernel is either a π-group or a π'-group.

Let us call $\mathrm{cd}(G)$ the set of irreducible character degrees of G and let us refer as $\mathrm{cd}^{\mathrm{B}_\pi}(G)$ and $\mathrm{cd}^{\mathrm{B}_{\pi'}}(G)$ to the sets of character degrees of, respectively, B_π-characters and $\mathrm{B}_{\pi'}$-characters. Even when $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ is strictly smaller then $\mathrm{Irr}(G)$, it may happen that $\mathrm{cd}(G) = \mathrm{cd}^{\mathrm{B}_\pi}(G) \cup \mathrm{cd}^{\mathrm{B}_{\pi'}}(G)$. This happens, for example, if we consider the group $\mathrm{SL}(2,3) \ltimes (\mathbb{Z}_3)^2$ with $\pi = \{2\}$, as we have seen in Section 1.6, or the group $(\mathrm{C}_3 \ltimes \mathrm{C}_7) \wr \mathrm{C}_2$, with $\pi = \{7\}$.

However, for a π-separable group G, in general we have that $\mathrm{cd}(G) \neq \mathrm{cd}^{\mathrm{B}_\pi}(G) \cup \mathrm{cd}^{\mathrm{B}_{\pi'}}(G)$. A first, obvious example of this fact is when $G = H \times K$, with H a π-group and K a π'-group, both nonabelian. In this case, in fact, we have that $\mathrm{B}_\pi(G) = \mathrm{Irr}(H)$ and $\mathrm{B}_{\pi'}(G) = \mathrm{Irr}(K)$.

Let us see some less trivial examples.

Example 4.5.1. A first example is derived directly form the trivial one. Let $G = H \times K$, with H a π-group and K a π'-group, both nonabelian, and let $\Gamma = G \wr \mathrm{C}_2$. Suppose $2 \in \pi$. Let θ be a nonlinear character in $\mathrm{Irr}(H)$ and let η be a nonlinear character in $\mathrm{Irr}(K)$. The character $(\theta \times 1_K) \times (1_H \times \eta)$ is an irreducible character of the base group $G \times G$ and it is not Γ-invariant, therefore it induces irreducibly to a character $\chi \in \mathrm{Irr}(\Gamma)$.

We have that the π-part of the degree of χ is $\chi(1)_\pi = 2\theta(1) > 2$ and the π'-part of the degree is $\chi(1)_{\pi'} = \eta(1) > 1$. Suppose there exists $\psi \in \mathrm{B}_\pi(\Gamma) \cup \mathrm{B}_{\pi'}(\Gamma)$ such that $\chi(1) = \psi(1)$ and let $\lambda_1 \times \lambda_2$ be a constituent of $\psi_{G\times G}$. Since $2 \in \pi$ and $\psi(1)$ is not a π-number, we have $\lambda_1(1)_{\pi'} \cdot \lambda_2(1)_{\pi'} = \psi(1)_{\pi'} > 1$, thus, $\lambda_1, \lambda_2 \in \mathrm{B}_{\pi'}(G)$, being G the direct product of a π-group and a π'-group. As a consequence, $\lambda_1(1)_\pi \cdot \lambda_2(1)_\pi = 1$ while $\psi(1)_\pi > 2$, a contradiction.

It follows that $\mathrm{cd}^{\mathrm{B}_\pi}(\Gamma) \cup \mathrm{cd}^{\mathrm{B}_{\pi'}}(\Gamma)$ is strictly smaller then $\mathrm{cd}(\Gamma)$.

The group in Example 4.5.1 still involves a group of type $H \times K$, with H a π-group and K a π'-group, both nonabelian. However, there exist examples which are not derived from the trivial one.

Example 4.5.2. Let $G = \mathrm{SL}(2,3) \ltimes (\mathbb{Z}_3)^2$, as in the example of Section 1.6. Computing the character table of G, in Section 1.6 we have seen that $\mathrm{cd}^{\mathrm{B}_{\{3\}}}(G) = \{1, 8\}$ and $\mathrm{cd}^{\mathrm{B}_{\{2\}}}(G) = \{1, 2, 3\}$.

Now, let $\Gamma = G \wr \mathrm{C}_2$, let $\theta \in \mathrm{B}_{\{3\}}(G)$ of degree 8 and let $\eta \in \mathrm{B}_{\{2\}}(G)$ of degree 3. The character $\theta \times \eta \in \mathrm{Irr}(G \times G)$ induces irreducibly to a character $\chi \in \mathrm{Irr}(\Gamma)$ and $\chi(1) = 48$.

Suppose there exists $\psi \in \mathrm{B}_{\{2\}}(\Gamma) \cup \mathrm{B}_{\{3\}}(\Gamma)$ such that $\psi(1) = \chi(1)$ and let $\lambda_1 \times \lambda_2$ be an irreducible constituent of $\psi_{G\times G}$. Then, λ_1 and λ_2 are either both in $\mathrm{B}_{\{2\}}(G)$ or they are both in $\mathrm{B}_{\{3\}}(G)$. Moreover, since $\psi(1) = 48$, then $\lambda_1(1) \cdot \lambda_2(1) \in \{24, 48\}$. However, neither 24 nor 48 can be written as a product of two numbers in $\mathrm{cd}^{\mathrm{B}_{\{2\}}}(G)$ or as a product of two numbers in $\mathrm{cd}^{\mathrm{B}_{\{3\}}}(G)$. It follows that $48 \in \mathrm{cd}(\Gamma)$ but $48 \notin \mathrm{cd}^{\mathrm{B}_{\{2\}}}(\Gamma) \cup \mathrm{cd}^{\mathrm{B}_{\{3\}}}(\Gamma)$.

Chapter 5

A bound for the p-length

In this chapter, we will see how the set of character degrees of B_p-characters having values in $\mathbb{Q}_p$ can provide a bound for the p-length of a p-solvable group. In order to do so, we will see how we can control the field of values of a B_p-character.

5.1 Bounds for the p-length from the character table

In this chapter, we will see how the theory of the B_π-characters can be used to study a problem which is apparently unrelated with it.

Let G be a finite group and let $\mathrm{cd}_{p'}(G)$ be the set of irreducible character degrees not divisible by a prime p. In a recent paper [10], it is proved that, if $\mathrm{cd}_{p'}(G) = \{1, m\}$, then the group G is solvable and $\mathrm{O}^{pp'pp'}(G) = 1$.

If G is a finite p-solvable group; the *p-length* of G, denoted as $\ell_p(G)$, is the minimum possible number of factors that are p-groups in any normal series for G in which each factor is either a p-group or a p'-group. It is not hard to prove that it is equal to the number of factors which are p-groups in the upper p-series of G. Therefore, the aforementioned result of [10] provides that, if $\left|\mathrm{cd}_{p'}(G)\right| = 2$, then G is solvable and $\ell_p(G) \leq 2$.

Let $E \subseteq \mathbb{C}$ be a field and let G be a finite group. We call $\mathrm{Irr}_E(G)$ the set of irreducible characters which have values in E. Then, we define the set

$$\mathrm{cd}_{E,p'}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}_E(G) \text{ and } p \nmid \chi(1)\}.$$

We will see that, for some $E \subset \mathbb{C}$, $|\mathrm{cd}_{E,p'}(G)|$ can provide a bound for the p-length of a p-solvable group G.

We recall that, if n is a natural number, we write $\mathbb{Q}_n$ for the n-cyclotomic extension of $\mathbb{Q}$, i.e., the extension of the field of rational numbers obtained by adjoining a primitive n-root of unity ζ_n to $\mathbb{Q}$. Notice that, in this notation, $\mathbb{Q}_2 = \mathbb{Q}$.

It is proved in [36] that the 2-length of a solvable group G is bounded by the number of rational-valued irreducible characters of odd degree. This result was later improved in [39] and [40].

Theorem 5.1.1 *([40, Theorem A]).* *Let G be a p-solvable group and let $\ell = \ell_p(G)$. Then, G has at least 2^ℓ irreducible characters of degree coprime to p and field of values contained in $\mathbb{Q}_p$.*

We mention that [40, Theorem A] is for us of particular interest, because it is proved using techniques which involve B_p-characters.

In this chapter, however, we find a bound that does not depend from the number of irreducible characters but from the number of distinct irreducible character degrees.

Theorem 5.1.2. *Let G be a p-solvable group and let $\ell_p(G)$ its p-length, then $\ell_p(G) \leq |\mathrm{cd}_{\mathbb{Q}_p,p'}(G)|$. In particular, if G is solvable, then $\ell_2(G) \leq |\mathrm{cd}_{\mathbb{Q},2'}(G)|$.*

We will see that, as in [40], the bound will be a consequence of a (stronger) one which depends from the degrees of B_p-characters.

Anyway, a Corollary easily follows from Theorem 5.1.2.

Corollary 5.1.3. *Let G be a p-solvable group and let $\ell_p(G)$ its p-length, then $\ell_p(G) \leq |\mathrm{cd}_{p'}(G)|$.*

5.2 The field of values of the B_p-characters

We have seen in Corollary 1.3.8 that the field of values of a character $\chi \in \mathrm{B}_\pi(G)$ is contained in $\mathbb{Q}_\pi$. Here, we will see that, under some further assumptions, the field of values of χ is contained in some smaller extension of $\mathbb{Q}$.

Proposition 5.2.1. *Let G be a π-separable group and H a Hall π-subgroup of G. Let $\chi \in \mathrm{B}_\pi(G)$ and let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}|\mathbb{Q})$. If $\varphi \in \mathrm{Irr}(H)$ is a Fong character associated with χ and $\varphi^\sigma = \varphi$, then also $\chi^\sigma = \chi$.*

In particular, if φ is rational-valued, then so is χ.

Moreover, if $\pi = \{p\}$ and $o(\sigma)$ is a power of p, then σ fixes χ if and only if it fixes some of the Fong characters associated with χ.

Proof. Suppose that there exists a Fong character $\varphi \in \mathrm{Irr}(H)$ associated with χ such that $\varphi^\sigma = \varphi$. Since χ^σ is again a B_π-character (see Chapter 1) and it lies over φ, it follows from the uniqueness part of Theorem 1.3.5 that $\chi^\sigma = \chi$.

In particular, since a character is rational-valued if and only if it is fixed by every $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|})$, we have that χ is rational valued if φ is.

Assume now that $\pi = \{p\}$, $H = P$ is a Sylow p-subgroup of G and $o(\sigma)$ is a power of p. If $\chi^\sigma = \chi$, then σ permutes the Fong characters associated with χ. Suppose none of these Fong characters is fixed by σ. If $C_1, ..., C_t$ are the orbits of this action, then $p \mid |C_i|$ for each $i = 1, ..., t$.

Now, let $\chi(1)_p = p^a$, so that p^a is the maximal power of p dividing $\chi(1)$, and notice that, by Theorem 1.3.5, we can write

$$\chi_P = \sum_{i=1}^{t} \sum_{\varphi \in C_i} \varphi + \Delta,$$

where either Δ is zero or the degree of each irreducible constituent of Δ is divided by p^{a+1}. Moreover, by definition, $\varphi(1) = p^a$ for each Fong character φ associated with χ.

Let φ_i be a representative for C_i for each i, then $\chi(1) = \sum_{i=1}^{t} |C_i| \varphi_i(1) + \Delta(1)$. Since p^{a+1} divides $\Delta(1)$ and p divides $|C_i|$ for every i, it follows that p^{a+1} divides $\chi(1)$, in contradiction with the maximality of p^a.

It follows that $|C_i| = 1$ for at least one index i and, thus, σ fixes at least one Fong character of χ. □

The condition on the order of the automorphism, in the second part of Proposition 5.2.1, is actually less strong then how it may seem. In fact, we know from Corollary 1.3.8 that a B_p-character has values in $\mathbb{Q}_\pi$, with $\pi = \{p\}$. Therefore, if $\chi \in \mathrm{B}_p(G)$, its field of values is contained in $\mathbb{Q}_{p^a}$, where $p^a = |G|_p$.

Now, it is known from Galois Theory that $\mathrm{Gal}(\mathbb{Q}_{p^a}|\mathbb{Q}) \cong C_{p-1} \times C_{p^{a-1}}$, for p odd, and $\mathrm{Gal}(\mathbb{Q}_{2^a}|\mathbb{Q}) \cong C_2 \times C_{2^{a-2}}$. Moreover, we also know that $\mathrm{Gal}(\mathbb{Q}_{p^a}|\mathbb{Q}) = \{\sigma_i \mid (i,p) = 1\}$, where σ_i is determined by $\sigma_i(\omega) = \omega^i$ for any root of unity ω (see, for instance, [31, Corollary 7.8]).

Let $\mathbb{Q}_{p^b}$ be a subfield of $\mathbb{Q}_{p^a}$, for some $0 < b \leq a$, and notice that every automorphism of $\mathbb{Q}_{p^b}$ can be extended to an automorphism of $\mathbb{Q}_{p^a}$. It follows that the restriction of automorphisms from $\mathbb{Q}_{p^a}$ to $\mathbb{Q}_{p^b}$ is a projection from $\mathrm{Gal}(\mathbb{Q}_{p^a}|\mathbb{Q})$ to $\mathrm{Gal}(\mathbb{Q}_{p^b}|\mathbb{Q})$ with kernel $\mathrm{Gal}(\mathbb{Q}_{p^a}|\mathbb{Q}_{p^b})$. It follows that

$$\frac{\mathrm{Gal}(\mathbb{Q}_{p^a}|\mathbb{Q})}{\mathrm{Gal}(\mathbb{Q}_{p^a}|\mathbb{Q}_{p^b})} \cong \mathrm{Gal}(\mathbb{Q}_{p^b}|\mathbb{Q})$$

and, from the description of $\mathrm{Gal}(\mathbb{Q}_{p^a}/\mathbb{Q})$ of the previous paragraph, we have that $\left|\mathrm{Gal}(\mathbb{Q}_{p^a}/\mathbb{Q}_{p^b})\right| = p^{a-b}$ (actually, assuming $b \geq 2$ if $p = 2$, it is possible to prove that $\mathrm{Gal}(\mathbb{Q}_{p^a}|\mathbb{Q}_{p^b}) \cong C_{p^{a-b}}$).

After this discussion, we can prove the following corollary.

Corollary 5.2.2. *Let G be a p-solvable group and P a Sylow p-subgroup of G. Let $\chi \in \mathrm{B}_p(G) \cap \mathrm{Irr}_{p'}(G)$ and let $\lambda \in \mathrm{Lin}(P)$ be a Fong character associated with χ. Let $p^b > 1$ be a power of p. Then, χ has values in $\mathbb{Q}_{p^b}$ if and only if $o(\lambda) \mid p^b$.*

Notice that this result can also be seen as a consequence of [25, Corollary D].

Proof. We know from the previous discussion that the field of values of every B_p-character of G is contained in $\mathbb{Q}_{p^a}$, for some $p^a = |G|_p$. As a consequence, a B_p-character has values in $\mathbb{Q}_{p^b}$ if and only if it is fixed by every morphism in $\mathrm{Gal}(\mathbb{Q}_{p^a}|\mathbb{Q}_{p^b})$, and every such morphism has p-power order.

It follows from Proposition 5.2.1 that χ has values in $\mathbb{Q}_{p^b}$ if and only if so does some Fong character $\rho \in \mathrm{Lin}(P)$ associated with χ. However, by Corollary 1.3.9, $\lambda = \rho^n$ for some $n \in \mathrm{N}_G(P)$ and they have the same field of values; thus, χ has values in $\mathbb{Q}_{p^b}$ if and only if so does λ.

Finally, since λ is linear, it has values in $\mathbb{Q}_{p^b}$ if and only if $o(\lambda) \mid p^b$. □

We conclude the section by talking briefly about an interesting consequence of Corollary 5.2.2 when applied to the study of *cut groups*.

The concept of cut groups arises form the study of group rings. In fact, a finite group G is said to be a *cut group* if $\mathbb{Z}G$ only contains trivial central units (see [1, Definition 1.1] for details). The reason why we are interested in these groups, however, is that, by [1, Proposition 2.2], G is a cut group if and only if, for each $\chi \in \mathrm{Irr}(G)$, $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{-d})$ for some non-negative integer d, where $\mathbb{Q}(\chi)$ is the field of values of χ. In particular, for every $\chi \in \mathrm{Irr}(G)$, $|\mathbb{Q}(\chi) : \mathbb{Q}| \leq 2$.

An open problem related to cut groups is to determine whether the Sylow 3-subgroup of a solvable cut group is again a cut group, another one is to say whether $\mathrm{O}_5(G)$ and $\mathrm{O}_7(G)$ of a cut group G are elementary abelian. The following corollary does not answer to any of the two questions; however, it is somehow related to both of them.

Corollary 5.2.3. *Let G be a p-solvable cut group and let P be a Sylow p-subgroup of G. If p is odd, P/P' is elementary abelian. If $p = 2$, then the exponent of every element in P/P' is at most 4.*

Proof. It is enough to prove that every nonprincipal character in $\mathrm{Lin}(P)$ has order p, for p odd, and order 2 or 4 for p even. Let $\lambda \in \mathrm{Lin}(P) \setminus \{1_P\}$, then by Theorem 1.3.9 it is a Fong character associated with some $\chi \in \mathrm{B}_p(G)$. Since G is cut, $|\mathbb{Q}(\chi) : \mathbb{Q}| = 2$. If p is odd, then χ has values in $\mathbb{Q}_p$, because it is fixed by every $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|_p}|\mathbb{Q})$ of odd order and, thus, by every $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|_p}|\mathbb{Q}_p)$; then, the thesis follows from Corollary 5.2.2.

For $p = 2$, the argument is a little more complex. Since $|\mathbb{Q}(\chi) : \mathbb{Q}| = 2$, we have that $|\mathrm{Gal}(\mathbb{Q}(\chi)|\mathbb{Q})| = 2$, as intermediate extensions of cyclotomic fields are Galois extensions. Let $\tau \in \mathrm{Gal}(\mathbb{Q}_{|G|}|\mathbb{Q})$ be the complex conjugation. If τ fixes χ then, for Proposition 5.2.1 and since all the Fong characters associated with χ are conjugated in $\mathrm{N}_G(P)$, it fixes also λ. However, λ is linear and, thus, $o(\lambda) = 2$.

Suppose, now, that τ does not fix χ. Then, $\mathrm{Gal}(\mathbb{Q}(\chi)|\mathbb{Q}) = \{\mathrm{id}, \tau|_{\mathbb{Q}(\chi)}\}$. It follows that, for any $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|_2}|\mathbb{Q})$, either $\sigma|_{\mathbb{Q}(\chi)} = \mathrm{id}$ or $\sigma|_{\mathbb{Q}(\chi)} = \tau|_{\mathbb{Q}(\chi)}$. Thus, either σ or $\sigma\tau$ fixes χ and, by the same arguments as before, either σ or $\sigma\tau$ fixes also λ. As a consequence, $\mathrm{Gal}(\mathbb{Q}(\lambda)|\mathbb{Q}) = \{\mathrm{id}, \tau|_{\mathbb{Q}(\lambda)}\}$ and, since λ is linear, it follows that $o(\lambda) = 4$. □

5.3 A bound from the B_p-character degrees

We will prove Theorem 5.1.2 as a consequence of the following theorem, concerning the B_p-characters.

Theorem 5.3.1. *Let G be a p-solvable group, let $\ell_p(G)$ its p-length and let*

$$\mathrm{cd}^{\mathrm{B}_p}_{\mathbb{Q}_p,p'}(G) = \{\chi(1) \mid \chi \in \mathrm{B}_p(G) \cap \mathrm{Irr}_{\mathbb{Q}_p}(G) \text{ and } p \nmid \chi(1)\},$$

with $\mathbb{Q}_p$ being the p-cyclotomic extension of $\mathbb{Q}$. Then, $\ell_p(G) \leq \left|\mathrm{cd}^{\mathrm{B}_p}_{\mathbb{Q}_p,p'}(G)\right|$. In particular, if G is solvable, then $\ell_2(G) \leq \left|\mathrm{cd}^{\mathrm{B}_2}_{\mathbb{Q},2'}(G)\right|$.

We need at first to prove two preliminary results.

If a linear character of a subgroup extends to the whole group G, it is not always true that there exists an extension which preserves the order. However, it happens to be possible under special conditions.

Lemma 5.3.2. *Let $M \lhd G$ and suppose M is complemented in G, i.e., there exists $H \leq G$ such that $G = HM$ and $H \cap M = 1$. If $\lambda \in \mathrm{Lin}(M)$ is invariant in G, then there exists $\varphi \in \mathrm{Lin}(G)$ such that θ extends λ and $o(\theta) = o(\lambda)$.*

Proof. Since $\ker\lambda \lhd G$, there is no loss to assume $\ker\lambda = 1$. Since λ is linear, it follows that $M' = 1$ and, thus, M is abelian. However, since λ is G-invariant and faithful, we have that H acts trivially on M. It follows that $G = H \times M$ and the thesis follows. □

We already know from Theorem 1.3.3 how B_π-characters behave in relation with normal subgroups. However, here a refinement is needed, in order to have more precise informations on character degrees.

Proposition 5.3.3. *Let G be a π-separable group and let $M \trianglelefteq G$. Let $\lambda \in \mathrm{Lin}(M)$ such that $o(\lambda)$ is a π-number and assume also that λ extends to $\nu \in \mathrm{Lin}(HM)$, where H is a Hall π-subgroup of G. Then, there exists a character $\chi \in \mathrm{B}_\pi(G)$ such that χ lies over λ and $\chi(1)_\pi = 1$. Moreover, ν_H is a Fong character associated with χ.*

Proof. Let $\nu \in \mathrm{Lin}(HM)$ be an extension of λ to HM and let $\varphi = \nu_H$, then φ is a linear character of the Hall π-subgroup H and, by Theorem 1.3.9, there exists $\chi \in \mathrm{B}_\pi(G)$ which lies over φ and $\chi(1)_\pi = \varphi(1) = 1$. Thus, φ is a Fong character associated with χ.

Moreover, if $K = H \cap M$, then $\xi = \varphi_K$ is a Fong character associated with λ and it also lies under χ. Let θ be an irreducible constituent of χ_M which lies over ξ, then θ is a B_π-character, because of Theorem 1.3.3, and it follows that $\theta = \lambda$ for the uniqueness part of Theorem 1.3.5. Thus, χ lies over λ and the proof is concluded. □

Now, we are ready to prove Theorem 5.3.1.

Proof of Theorem 5.3.1. In order to simplify the notation, since there is no ambiguity, we will write $\mathrm{Bcd}_{p'}(G)$ to refer to $\mathrm{cd}^{\mathrm{B}_p}_{\mathbb{Q}_p,p'}(G)$.

Let G be a counterexample of minimal order to the theorem and let K be a minimal normal subgroup of G, then $\ell_p(G) > \left|\mathrm{Bcd}_{p'}(G)\right|$ and $\ell_p(G/K) \leq \left|\mathrm{Bcd}_{p'}(G/K)\right|$ because of the minimality in the choice of G. Moreover, since K is minimal normal, $\ell_p(G/K) \leq \ell_p(G) \leq \ell_p(G/K)+1$, while $\left|\mathrm{Bcd}_{p'}(G/K)\right| \leq \left|\mathrm{Bcd}_{p'}(G)\right|$ because $\mathrm{Bcd}_{p'}(G/K) \subseteq \mathrm{Bcd}_{p'}(G)$. It follows that

$$\ell_p(G/K) \leq \left|\mathrm{Bcd}_{p'}(G/K)\right| \leq \left|\mathrm{Bcd}_{p'}(G)\right| < \ell_p(G) \leq \ell_p(G/K)+1.$$

As a consequence, $\ell_p(G) = \ell_p(G/K)+1$ and $\mathrm{Bcd}_{p'}(G/K) = \mathrm{Bcd}_{p'}(G)$.

Therefore, we have that K is a p-group and $\mathrm{O}_{p'}(G/K) \neq K$, since otherwise $\ell_p(G/K) = \ell_p(G)$. Since K is arbitrarily chosen, it follows that $\mathrm{O}_{p'}(G) = 1$. Moreover, if $N/K = \mathrm{O}_{p'}(G/K)$, we have that $N > K$ and $\mathrm{Bcd}_{p'}(G/N) = \mathrm{Bcd}_{p'}(G/K) = \mathrm{Bcd}_{p'}(G)$.

Let Y be a complement for K in N. By the Frattini argument, we have that $G = \mathrm{N}_G(Y)N = \mathrm{N}_G(Y)K$. Moreover,

$$\mathrm{N}_G(Y) \cap K = \mathrm{C}_K(Y) = \mathrm{O}_p(\mathrm{Z}(N)) \lhd G.$$

Since, however, K is a minimal normal subgroup of G and $K \not\leq \mathrm{Z}(N)$ (otherwise, $Y \leq \mathrm{O}_{p'}(G) = 1$), we have $\mathrm{N}_G(Y) \cap K = 1$. Hence, K is complemented in G and, by Lemma 5.3.2, every $\lambda \in \mathrm{Irr}(K)$ has an extension to its inertia subgroup G_λ of the same order.

Since we have proved that $\mathrm{C}_K(Y) = 1$, the only element in K to be centralized by Y is 1. Since K is abelian, it follows that no nonprincipal character in $\mathrm{Irr}(K)$ is N-invariant. Thus, for any $\lambda \in \mathrm{Irr}(K)$, $N \not\leq G_\lambda$.

On the other hand, let P be a Sylow p-subgroup of the group G, then $\mathrm{Z}(P) \cap K \neq 1$. Therefore, the action of P on K fixes more then one element and, thus, there exists $\lambda \in \mathrm{Irr}(K) \setminus \{1_K\}$ which is P-invariant.

Let $\lambda \in \mathrm{Irr}(K)$ be a nonprincipal P-invariant character and let $T = G_\lambda$, so that $P \leq T$ and λ has an extension to T of order p (notice that $o(\lambda) = p$ because K is elementary abelian). It follows from Proposition 5.3.3 that there exist some characters in $\mathrm{B}_p(G) \cap \mathrm{Irr}_{p'}(G)$ lying over λ with an associated Fong character of order p and, by Corollary 5.2.2, they have values in $\mathbb{Q}_p$. Among these characters, let χ be the one of maximal degree. By the third paragraph of the proof, $\mathrm{Bcd}_{p'}(G/N) = \mathrm{Bcd}_{p'}(G)$, thus, there exists $\psi \in \mathrm{B}_p(G/N)$ having values in $\mathbb{Q}_p$ such that $\psi(1) = \chi(1)$. Let $\gamma \in \mathrm{Lin}(PN/N)$ be a Fong character associated with ψ. If we consider ψ and γ as characters of, respectively, G and PN, we have that $\varepsilon = \gamma_P$ is a Fong character associated with ψ (as a character in $\mathrm{B}_p(G)$), since it is a constituent of ψ_P and $\varepsilon(1) = \psi(1)_p = 1$. Notice that $o(\varepsilon) = p$ because of Corollary 5.2.2.

Let W be the unique maximal subgroup of G such that ε extends to W and notice that $N \leq W$. Moreover, let $\hat{\lambda}$ be an extension of λ to T of order p, let $\upsilon = \hat{\lambda}_P$ and let V be the unique maximal subgroup of G such that the linear character $\rho = \varepsilon\upsilon$ extends to V. Notice that both W and V exist by Theorem 1.3.10 and, for the last part of that theorem, $\psi(1) = |G : W|$.

Now, let $\theta \in \mathrm{Irr}(V)$ be the unique p-special extension of ρ to V, then $\theta_K = \rho_K = \lambda$, as $K \leq \ker(\varepsilon)$. Therefore, λ extends to V and it follows that $V \leq T$. Moreover, if $\xi = \hat{\lambda}_V$, then $(\theta\bar{\xi})_P = \rho\bar{\upsilon} = \varepsilon$; since W is the unique maximal extension subgroup for ε, it follows that $W \geq V$. However, for the second part of Theorem 1.3.10, we have that $\theta^G \in \mathrm{B}_p(G)$ and, by Corollary 5.2.2, θ^G have values in $\mathbb{Q}_p$, because $o(\rho) = p$. Since θ lies over λ, the choice of χ to be of maximal degree leads to

$$|G : V| = \theta^G(1) \leq \chi(1) = \psi(1) = |G : W|\,.$$

Therefore, $V = W$ and W is contained in T.

However, we showed earlier in the proof that $N \not\leq T$. On the other hand, we also have that $N \leq W \leq T$ and this leads to a contradiction. □

We can now prove Theorem 5.1.2.

Proof of Theorem 5.1.2. Since $\mathrm{cd}_{\mathbb{Q}_p,p'}(G) \supseteq \mathrm{cd}^{\mathrm{B}_p}_{\mathbb{Q}_p,p'}(G)$, the conclusion is true for Theorem 5.3.1. $\square$

To conclude the chapter, let us see some example to show that, in general, for a p-solvable group G, $\left|\mathrm{cd}_{p'}(G)\right| \neq \left|\mathrm{cd}_{\mathbb{Q}_p,p'}(G)\right| \neq \left|\mathrm{cd}^{\mathrm{B}_p}_{\mathbb{Q}_p,p'}(G)\right|$.

Example 5.3.4. Let $G = \mathrm{A}_5 \times \mathrm{C}_7$ and $p = 7$. We can see that $\mathrm{cd}_{7'}(G) = \{1, 3, 4, 5\}$, $\mathrm{cd}_{\mathbb{Q}_7,7'}(G) = \{1, 4, 5\}$ and $\mathrm{cd}^{\mathrm{B}_7}_{\mathbb{Q}_7,7'}(G) = \{1\}$. Clearly, $\ell_7(G) = 1$.

Example 5.3.5. Let G be the semidirect product of $\mathrm{SL}(2, 3)$ acting naturally on $(\mathbb{Z}_3)^2$ and let $p = 3$. In Section 1.6 we have seen that $\mathrm{cd}_{3'}(G) = \mathrm{cd}_{\mathbb{Q}_3,3'}(G) = \{1, 2, 8\}$ and $\mathrm{cd}^{\mathrm{B}_3}_{\mathbb{Q}_3,3'}(G) = \{1, 2\}$. Notice that, in this case, $\ell_3(G) = 2$.

Chapter 6

Character correspondence in π-separable groups

In this chapter we talk about correspondence of characters between the group and the normalizer of one of its Sylow or Hall subgroups. After a brief introduction on the McKay conjecture and its variants, we first talk about the restriction of B_π-characters to the normalizer of a Hall π-subgroup, and we define the *upper-Fong* characters. Then, we focus on ordinary irreducible characters and we investigate when there exists a McKay natural correspondence such that the corresponding characters in the normalizer are linear.

6.1 The McKay conjecture

One of the most famous problems in character theory of finite groups is for sure the Mckay conjecture. Initially proposed in 1972 for characters of odd degree, it correlates irreducible characters of a group with the ones of the normalizer of a Sylow subgroup.

Conjecture 6.1.1 (McKay, 1972)**.** *Let G be a group and p a prime, and let P be a Sylow p-subgroup of G. Then,* $\left|\mathrm{Irr}_{p'}(G)\right| = \left|\mathrm{Irr}_{p'}(\mathrm{N}_G(P))\right|$.

The conjecture, however, is already proved to be true for solvable groups. In fact, it was reduced to simple groups in [24] and later it was proved in [30] for $p = 2$. The conjecture is also verified to be true, for every p, for many classes of groups; however, it is in general still open.

As we may expect for a problem of this fame, many variants of the McKay conjecture have been proposed in the last decades. Here, we only mention one involving B_π-characters, which was proved in [42].

Theorem 6.1.2 (*[42, Theorem 1.15]*)**.** *Let π and ω be two sets of primes and let G be both π-separable and ω-separable. Let H be a Hall ω-subgroup of G and let $N = \mathrm{N}_G(H)$. Then:*

$$|\{\chi \in \mathrm{B}_\pi(G) \,|\, \chi(1) \textit{ is a } \omega'\textit{-number}\}| = |\{\psi \in \mathrm{B}_\pi(N) \,|\, \psi(1) \textit{ is a } \omega'\textit{-number}\}|\,.$$

In particular, from this theorem it follows a result we already mentioned in Section 4.4.

Corollary 6.1.3. *Let G be a π-separable group, let H be a Hall π-subgroup of G and let $N = \mathrm{N}_G(H)$. Then:*

$$|\mathrm{B}_\pi(G) \cap \mathrm{Irr}_{\pi'}(G)| = |\mathrm{B}_\pi(N) \cap \mathrm{Irr}_{\pi'}(N)| \textit{ and } |\mathrm{X}_{\pi'}(G)| = |\mathrm{X}_{\pi'}(N)|\,.$$

Theorem 6.1.2 leads to further refinements of the McKay conjecture. In fact, the character theory of π-separable groups has been used in [25] to prove some variants of the conjecture for p-solvable groups. In particular, in [25] it was first introduced the so called *Galois-McKay conjecture*, proved for p-solvable groups in the same paper and recently reduced to simple groups in [34].

We are not going to talk about this specific conjecture, since it is not strongly related with the arguments of the thesis. However, we believe it was worth to be mentioned, since it is an interesting application of the theory of the B_π-characters.

If $\left|\mathrm{Irr}_{p'}(G)\right| = \left|\mathrm{Irr}_{p'}(\mathrm{N}_G(P))\right|$, then it is possible to define a bijection between the two sets of characters. It is natural to ask if there exists a *canonical*, or *natural*, bijection, i.e., a correspondence which can be established choice-freely.

Despite the fact that it is proved to exists for several classes of groups, in general no choice free correspondence exists between the two sets $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathrm{N}_G(P))$, even for solvable groups.

For what concerns solvable groups, however, it is interesting to notice that in [17] a choice free correspondence is established for groups of odd order. Moreover, in [37, Theorem D] a choice free correspondence is established for p-solvable groups under the stronger assumptions that $\mathrm{N}_G(P) = P\,\mathrm{C}_G(P)$.

If we further strengthen our hypothesis on $\mathrm{N}_G(P)$, the hypothesis of p-solvability can be dropped.

Theorem 6.1.4 (*[37, Corollary B]*)**.** *Let G be a finite group, let p be odd, and let $P \in \mathrm{Syl}_p(G)$. Suppose that $\mathrm{N}_G(P) = P$. Then there is a natural bijection $\chi \mapsto \chi^*$ between $\mathrm{Irr}_{p'}(G)$ and the linear characters of P. In fact, if $\chi \in \mathrm{Irr}_{p'}(G)$ and $\lambda \in \mathrm{Lin}(P)$, then χ and λ correspond under bijection if and only if*

$$\chi_P = \lambda + \Delta$$

where Δ is either zero or a character whose irreducible constituents all have degree divisible by p.

Notice that Theorem 6.1.4 gives an example of choice free correspondence realized by character restriction.

6.2 The *upper-Fong characters*

We first see what happens when we restrict B_π-characters to the normalizer of a Hall π-subgroup.

From now on, if not otherwise specified, we assume G to be a π-separable group, $\chi \in \mathrm{B}_\pi(G)$, H to be a Hall π-subgroup for G and $N = \mathrm{N}_G(H)$.

Let $\varphi \in \mathrm{Irr}(H)$ be a Fong character associated with χ, then there exists an irreducible constituent ψ of χ_N which lies over φ. In particular, $\psi(1)_\pi = \chi(1)_\pi$.

Definition 6.2.1. Let G be a π-separable group, let H be a Hall π-subgroup for G and let N be the normalizer of H in G. Let $\chi \in \mathrm{B}_\pi(G)$, if $\psi \in \mathrm{Irr}(N)$ is an irreducible constituent of χ_N such that $\psi(1)_\pi = \chi(1)_\pi$, then ψ is an *upper-Fong character* associated with χ.

Lemma 6.2.2. *Let G be a π-separable group, $H \in \mathrm{Hall}_\pi(G)$ and $N = \mathrm{N}_G(H)$. Let $\chi \in \mathrm{B}_\pi(G)$ and let $\psi \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N. Then the following are equivalent:*

a) ψ is an upper-Fong character associated with χ;

b) ψ lies over a Fong character of H in G associated with χ;

c) $\psi(1)_\pi$ is minimal among the irreducible constituents of χ_N.

Moreover, if ψ is an upper-Fong character lying over $\varphi \in \mathrm{Irr}(H)$, then ψ is the unique constituent of χ_N lying over φ and $[\chi_N, \psi] = 1$. Finally, the sum $\sum_i \psi_i(1)_{\pi'}$ on the upper-Fong characters associated with χ is equal the number of Fong characters associated with χ.

Proof. It all follows directly from Theorem 1.3.5.

Let φ be an irreducible constituent of ψ_H; since $H \lhd N$ and N/H is a π'-group, it follows that $\psi(1)_\pi = \varphi(1)$ and, thus, $\psi(1)_\pi = \chi(1)_\pi$ if and only if $\varphi(1) = \chi(1)_\pi$. Thus, a) is equivalent to b). Moreover, since φ is a Fong character if and only if is of minimal degree among the irreducible constituents of χ_H, it follows that b) is equivalent to c).

Finally, one has that $[\chi_N, \psi] \leq [\chi_H, \varphi] = 1$ and also $[\psi_H, \varphi] = 1$, thus, ψ is the unique constituent of χ_N lying over φ, $\psi_H = \sum_i^m \varphi_i$ and $\psi(1)_{\pi'} = m$ is the number of Fong characters associated with χ which lie under ψ. Then the thesis follows. □

One may wonder if an upper-Fong character is itself a B_π-character. We do not have a general answer to this question; however, we can answer positively under some extra assumptions.

At first, we see an interesting consequence of Proposition 1.3.11.

Proposition 6.2.3. *Let G be a π-separable group, $H \in \mathrm{Hall}_\pi(G)$ and $N = \mathrm{N}_G(H)$, and let $\psi \in \mathrm{Irr}(N)$ be an upper-Fong character for the group G lying over $\varphi \in \mathrm{Irr}(H)$. Let $I = \mathrm{I}_N(\varphi)$, then there exists $\eta = \alpha\beta \in \mathrm{Irr}(I)$ such that $\eta^N = \psi$, α is the (unique) π-special extension of φ to I and β is either 1_I or a linear π'-special character of order 2. In particular, if either $2 \in \pi$ or $2 \nmid |I : I'|$, then ψ is in $\mathrm{B}_\pi(N)$.*

Proof. Since H is normal in N, we know from Clifford theory that there exists $\eta \in \mathrm{Irr}(I \mid \varphi)$ such that $\eta^N = \psi$ and, by Gallagher theorem and

Proposition 1.2.4, $\eta = \alpha\beta$ with α being the unique π-special extension of φ to I and $\beta \in \mathrm{Irr}(N/H)$. In particular, β is a π'-special character, because N/H is a π'-group, and it is linear, since $\beta(1) \leq [\psi_H, \varphi] = 1$.

Now, suppose ψ is associated with a character $\chi \in \mathrm{B}_\pi(G)$ and let σ be any element of $\mathrm{Gal}(\mathbb{Q}_{|G|_{\pi'}}|\mathbb{Q})$. Then, $0 \neq [\chi_N, \psi] = [\chi_N, \psi^\sigma]$, since by Corollary 1.3.8 χ has values in $\mathbb{Q}_\pi$ and, thus, it is invariant for σ. Moreover, $0 \neq [\psi_H, \varphi] = [{\psi^\sigma}_H, \varphi]$ because φ is a character of a π-group. It follows that ψ^σ is an irreducible constituent of χ_N and $\psi^\sigma(1)_\pi = \psi(1)_\pi = \chi(1)_\pi$, thus ψ^σ is an upper-Fong character associated with χ which lies over φ, by Lemma 6.2.2, and it follows that $\psi^\sigma = \psi$ by uniqueness.

Now, since σ fixes both ψ and φ, then η is fixed, too, by uniqueness of the Clifford correspondence. Since, however, $\eta = \alpha\beta$ and α, being π-special, is σ-invariant, we have that $\beta^\sigma = \beta$ and, since this holds for any $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|_{\pi'}}|\mathbb{Q})$ and β is a linear π'-special character, it follows that either $\beta = 1_I$ or $o(\beta) = 2$. Finally, by Proposition 1.2.4, if $\beta = 1_I$, then ψ is a B_π-character. □

Corollary 6.2.4. *If either $|G|$ is odd or $2 \in \pi$, then all the upper-Fong characters are B_π-characters.*

We now see that an upper-Fong characters is a B_π-character also when it lies over a primitive Fong character φ. We see this as a direct consequence of the theory developed in [25] and, later, in [6].

Before we proceed, however, notice that, if H is a Hall π-subgroup of a π-separable group G, $N = \mathrm{N}_G(H)$ and $\varphi \in \mathrm{Irr}(H)$ is a primitive Fong character associated with some $\chi \in \mathrm{B}_\pi(G)$, then by Theorem 1.3.9 the Fong characters associated with χ are an orbit for the action of N on $\mathrm{Irr}(H)$. In particular, under these hypothesis, χ has a unique upper-Fong character. We now see that it is also a B_π-character.

Theorem 6.2.5. *Let G be a π-separable group, let H be a Hall π-subgroup of a π-separable group G, let N be the normalizer of H in G and let $\varphi \in \mathrm{Irr}(H)$ be a primitive Fong character associated with some $\chi \in \mathrm{B}_\pi(G)$. If ψ is the (unique) upper-Fong character associated with χ, then $\psi \in \mathrm{B}_\pi(N)$.*

Moreover, under this hypothesis, $\psi(1)$ divides $\chi(1)$, and $\frac{\chi(1)}{\psi(1)}$ divides $|G : N|$.

Proof. By [25, Theorem 2.2], there exists $(W, \mu) \in \mathrm{nuc}(\chi)$ such that $H \leq W$ and $\mu_H = \varphi$. Moreover, if $I = \mathrm{I}_N(\varphi)$, by the proof of [25, Theorem

3.5] it follows that $W \cap N = I$. Finally, by Clifford theory, ${\mu_I}^N$ is an irreducible constituent of χ_N and ${\mu_I}^N(1)_\pi = \varphi(1) = \chi(1)_\pi$, thus $\psi = {\mu_I}^N$ and, since μ_I is π-special (because it is π-factorized by Proposition 6.2.3 and its order divides the order of μ), it follows from Proposition 6.2.3 that $\psi \in \mathrm{B}_\pi(N)$.

In order to prove the last assertion, observe that $\psi(1)_\pi = \chi(1)_\pi$ and $\psi(1)_{\pi'} = |N : W \cap N|$, $\chi(1)_{\pi'} = |G : W|$. Moreover, $|G : W \cap N| = |G : W|\,|W : W \cap N| = |G : N|\,|N : W \cap N|$ and $|G : N|$ is the number of Hall π-subgroups in G, while, on the other hand, $|W : W \cap N| = |W : \mathrm{N}_W(H)|$ is the number of Hall π-subgroups in W. Then, it follows from [41, Corollary 1.2] that $|W : W \cap N|$ divides $|G : N|$. Thus, $|N : W \cap N|$ divides $|G : W|$ and $\frac{|G:W|}{|N:W\cap N|}$ divides $|G : N|$. □

Corollary 6.2.6. *Let G be a π-separable group and let N be the normalizer of a Hall π-subgroup of G. Let $\mathcal{B}(G)$ and $\mathcal{B}(N)$ be the sets of B_π-characters of, respectively, G and N which lie over primitive Fong characters. Then, the character restriction realizes a bijection between $\mathcal{B}(G)$ and $\mathcal{B}(N)$. In fact, $\chi \in \mathcal{B}(G)$ and $\psi \in \mathcal{B}(N)$ correspond under bijection if and only if ψ is the unique upper-Fong character associated with χ.*

Finally, notice that the correspondence in Corollary 6.2.6 preserves the π-part of character degrees, i.e., $\chi(1)_\pi = \psi(1)_\pi$ when χ and ψ correspond under bijection. This leads to the following corollary.

Corollary 6.2.7 (*[6, Theorem 2.2]*)**.** *Let G be a π-separable group and let N be the normalizer of a Hall π-subgroup of G. Then, the character restriction realizes a bijection between the sets $\mathrm{B}_\pi(G) \cap \mathrm{Irr}_{\pi'}(G)$ and $\mathrm{B}_\pi(N) \cap \mathrm{Irr}_{\pi'}(N)$.*

6.3 A *linear* McKay correspondence

We now temporarily forget about B_π-characters to talk about a different problem related with character correspondence.

As we have mentioned at the beginning of this chapter, it is not always possible to find a choice free McKay correspondence, even for solvable groups. On the other hand, if we further assume that $\mathrm{N}_G(P) = P\,\mathrm{C}_G(P)$, a choice free correspondence has been described in [37] for p-solvable groups. The correspondence, however, is not given by the character restriction, i.e., the character corresponding to $\chi \in \mathrm{Irr}_{p'}(G)$ is not necessarily the only irreducible constituent of $\chi_{\mathrm{N}_G(P)}$ with a degree not divisible by p.

One may wonder if it is possible to have a correspondence given by the character restriction, eventually under some stronger assumptions on the group structure and characters.

Problem 6.3.1. Let G be a finite group, let p be a prime number and let N be the normalizer of a Sylow p-subgroup of G. Under which assumptions on G, for each $\chi \in \mathrm{Irr}_{p'}(G)$ there exists exactly one irreducible constituent χ^* of χ_N which belongs to $\mathrm{Irr}_{p'}(N)$ *and* χ^* *is linear*?

We first herd about this problem from Carolina Vallejo, from Universidad Autónoma de Madrid, who also explained us some very basic characterization, suggested by Gabriel Navarro, which we report here.

In this section, we see some necessary condition for the existence of the correspondence in Problem 6.3.1, while we will see a sufficient condition in the next section.

For the sake of synthesis, let us establish an hypothesis which will be referred repeatedly.

Hypothesis 6.3.2. Let G be a finite group, χ an irreducible character in $\mathrm{Irr}_{p'}(G)$, p a prime number, P a Sylow p-subgroup of G and $N = \mathrm{N}_G(P)$ the normalizer of P in G, and suppose that the restriction of χ to N is $\chi_N = \lambda + \Delta$, where λ is a linear character and Δ is either zero or such that every irreducible constituent of it has a degree divisible by p.

If Hypothesis 6.3.2 holds, we will refer sometimes to the character λ of the hypothesis also as χ^*.

Here it is presented a partial characterization of the groups where Hypothesis 6.3.2 holds. We first state an easy lemma suggested, originally, by Gabriel Navarro.

Lemma 6.3.3. *Let G be a group where Hypothesis 6.3.2 holds for every $\chi \in \mathrm{Irr}_{p'}(G)$ and let P be a p-Sylow and N its normalizer in G. Then:*

1. $\mathrm{Irr}_{p'}(N) = \mathrm{Lin}(N)$ *and, for each $\lambda \in \mathrm{Lin}(N)$, there exists $\chi \in \mathrm{Irr}_{p'}(G)$ such that $\chi^* = \lambda$;*

2. *$N = X \times P$ for some abelian p'-group X.*

Proof. For point 1), one only has to notice that, if $\lambda \in \mathrm{Irr}_{p'}(N)$, then λ^G is a p'-number. It follows that λ^G has at least one irreducible constituent χ in $\mathrm{Irr}_{p'}(G)$ and the thesis follows by Hypothesis 6.3.2: since λ is an irreducible constituent of χ_N, then $\lambda = \chi^*$ and it is linear.

For point 2), we only have to notice that, since $\mathrm{Irr}_{p'}(N) = \mathrm{Lin}(N)$, it follows from Thompson theorem on character degrees (Theorem 4.1.5 in this thesis) that N has a normal p-complement X. Since both X and P are normal in N, it follows that $N = P \times X$. Finally, since $p \nmid \xi(1)$ for every $\xi \in \mathrm{Irr}(X)$, it follows from 1) that every irreducible character of X is linear; thus, X is abelian. □

A further characterization of the subgroup X is possible.

Proposition 6.3.4. *Let G, N and X be as in Lemma 6.3.3, then, for each $x \in X$, $x^G \cap X = \{x\}$.*

Proof. Let $\lambda \in \mathrm{Irr}(X)$ and let $\chi \in \mathrm{Irr}_{p'}(G)$ such that $\chi^* = \lambda \times 1_P$. Since Hypothesis 6.3.2 holds and $N = X \times P$, then $\chi_N = \lambda \times 1_P + \Delta$, with $\Delta = \sum_i \mu_i \times \varepsilon_i$ for some $\mu_i \in \mathrm{Irr}(X)$ and some nonlinear $\varepsilon_i \in \mathrm{Irr}(P)$.

Now, let $\mathbf{R}$ be the ring of algebraic integers and let M be a maximal ideal containing $p\mathbf{R}$, so that $F = \mathbf{R}/M$ is a field of characteristic p. For any $x \in X$, $\mu_i \times \varepsilon_i(x) = \varepsilon_i(1)\mu_i(x) \equiv 0 \pmod{M}$, since $p \mid \varepsilon_i(1)$; thus, it follows from the previous paragraph that $\chi(x) \equiv \lambda(x) \pmod{M}$. Since, for any $y \in X$ G-conjugate to x, $\chi(x) = \chi(y)$, we have that $\lambda(x) \equiv \lambda(y) \pmod{M}$ and, since $\lambda(x), \lambda(y) \in \mathbb{Q}_m$ for some $p \nmid m$, it follows from [14, Lemma 15.1] that $\lambda(x) = \lambda(y)$.

Thus, if $x, y \in X$ are G-conjugate, then $\lambda(x) = \lambda(y)$ for every $\lambda \in \mathrm{Irr}(X)$ and, since X is abelian, it follows that $x = y$. □

Another result relates Hypothesis 6.3.2 with the character restriction to the derived subgroup of a Sylow subgroup.

Proposition 6.3.5. *Let $\chi \in \mathrm{Irr}_{p'}(G)$, then Hypothesis 6.3.2 holds for χ if and only if $[\chi_{P'}, 1_{P'}] = 1$.*

Proof. Let us consider the restriction of χ to N. Since p does not divide the degree of χ, then it does not divide the degree of at least one irreducible

constituent of χ_N. Thus, let $\chi_N = \lambda + \psi_1 + \ldots + \psi_k$, where λ, ψ_i are characters in $\mathrm{Irr}(N)$ and $p \nmid \lambda(1)$.
Now, consider that:

1. since $p \nmid \lambda(1)$ and $P \lhd N$, all the irreducible constituents of λ_P are linear, therefore they all lie over $1_{P'}$ and it follows that λ is linear if and only if $[\lambda_{P'}, 1_{P'}] = 1$;

2. $p \mid \psi_i(1)$ if and only if it divides the degree of all the irreducible constituents of $(\psi_i)_P$, thus, if and only if $[(\psi_i)_{P'}, 1_{P'}] = 0$.

Since $[\chi_{P'}, 1_{P'}] = [\lambda_{P'}, 1_{P'}] + \sum_{i=1}^{k} [(\psi_i)_{P'}, 1_{P'}]$, the thesis follows. □

Thus, it may be useful to better know how P' behaves in the group G. What we are able to do is characterize its normalizer when Hypothesis 6.3.2 holds for every character in $\mathrm{Irr}_{p'}(G)$.

Proposition 6.3.6. *Let G be a group and let P be a Sylow p-subgroup of G. Let N be the normalizer of P in G and let M be the normalizer of P'. Suppose that for every $\chi \in \mathrm{Irr}_{p'}(G)$ Hypothesis 6.3.2 holds. Then, $\mathrm{Irr}_{p'}(M) = \mathrm{Lin}(M)$ and $M = NK$ for some normal p'-subgroup K of M. Moreover, $N \cap K = \{1\}$ and $M' = P'K$.*

Observe that $N \leq M$, since P' is normal in N.

Proof. At first, let us prove that $\mathrm{Irr}_{p'}(M) = \mathrm{Lin}(M)$. Let $\psi \in \mathrm{Irr}_{p'}(M)$, then there exists a character $\chi \in \mathrm{Irr}_{p'}(G)$ which lies over ψ. Since Hypothesis 6.3.2 holds for χ, then so it does for ψ and it follows from Proposition 6.3.5 that $[\psi_{P'}, 1_{P'}] = 1$. Since however P' is normal in M, we have that ψ is a linear character in $\mathrm{Irr}(M/P')$. Therefore, $\mathrm{Irr}_{p'}(M) = \mathrm{Lin}(M)$.

Now, by Thompson theorem on character degrees, it follows that M has a normal p-complement and, in particular, it is p-solvable. Then it follows from Theorem 4.1.6 that $M' \cap N = P'$. Now, since M has a normal p-complement, so does M'; let K be the normal p-complement of M', then $M' = P'K$. Finally, K is normal in M, too, and, since $M = NM'$ for the Frattini argument and $M' \cap N = P'$, it follows that $M = NK$ and $N \cap K = \{1\}$. □

6.4 A sufficient condition for the linear McKay correspondence

In this section, we find a sufficient condition for Problem 6.3.1. It is, however, a strong condition to impose. In fact, we assume X to be a Hall subgroup of G.

Theorem 6.4.1. *Let G be a π-separable group and $P \in \mathrm{Syl}_p(G)$. Let $N = \mathrm{N}_G(P)$ and suppose $N = P \times X$ with $X \in \mathrm{Hall}_\pi(G)$ and abelian. Moreover, assume that either G is solvable or p is odd. If $\chi \in \mathrm{Irr}_{p'}(G)$, then the restriction of χ to N is $\chi_N = \lambda + \Delta$, where λ is a linear character and Δ is either zero or such that every irreducible constituent of it has a degree divisible by p.*

To prove Theorem 6.4.1, we use some techniques from [37], like the following lemma.

Lemma 6.4.2. *Let G be a finite group, let $P \in \mathrm{Syl}_p(G)$ and let $N = \mathrm{N}_G(P)$. Suppose $M \lhd G$ and that, for some $g \in G$ and $\mu \in \mathrm{Irr}(M)$, μ and μ^g are both P-invariant. Then $\mu^g = \mu^n$ for some $n \in N$.*

Proof. It is part of the proof of [37, Lemma 2.1]. □

Lemma 6.4.3. *Let G be a finite group and let $P \in \mathrm{Syl}_p(G)$. Let $N = \mathrm{N}_G(P)$ and suppose $N = P \times X$ with X abelian. Let $M \lhd G$ be a p'-group and suppose that, for some $g \in G$ and $\mu \in \mathrm{Irr}(M)$, μ and μ^g are both P-invariant. Then, $\mu = \mu^g$.*

Proof. From Lemma 6.4.2 we know that $\mu^g = \mu^x$ for some $x \in X$. Let $C = \mathrm{C}_M(P)$ and notice that $C = X \cap M$. Let μ^* and $(\mu^x)^*$ be the Glaubermann correspondents of μ and μ^x in $\mathrm{Irr}(C)$, then consider $(\mu^*)^x$ and notice that it is the only constituent of $(\mu^x)_C$ with multiplicity non divisible by p, therefore $(\mu^x)^* = (\mu^*)^x$ by [14, Theorem 13.1 c)]. However, $(\mu^*)^x = \mu^*$ because X is abelian. It follows that $\mu^g = \mu$, since the Glaubermann correspondence is injective. □

We are now ready to prove Theorem 6.4.1.

Proof of Theorem 6.4.1. At first, notice that if $X = 1$ then the thesis is true for Theorem 6.1.4, for p odd, and [37, Theorem D], for G solvable.

Suppose $X > 1$ and let $M \lhd G$ minimal, then we can assume M to be a p'-group, since $\mathrm{O}_{p'}(G) \neq 1$ by [37, Theorem 3.2]. Let $\chi \in \mathrm{Irr}_{p'}(G)$, suppose M is not in $\ker\chi$ (otherwise, for the character χ the thesis follows by induction, since $\mathrm{N}_{G/M}(PM/M) = NM/M$) and let ξ be an irreducible constituent of χ_M which is P-invariant. By Lemma 6.4.3, ξ is unique among the irreducible constituents of χ_M.

On the other hand, however, since $\xi \neq 1_M$, by the Glaubermann correspondence we have that $X \cap M \neq 1$; therefore, since G is π-separable, M is a π-group and it follows that $M \leq X$. In particular, we have that ξ extends to X. Moreover, since $P \leq X$, it follows that every character of M is P-invariant; thus, ξ is the unique irreducible constituent of χ and it is also G-invariant.

Let r be any prime and let R/M be a Sylow r-subgroup of G. If $r \in \pi$, then $R \leq X$ up to conjugation and ξ extends to R. On the other hand, if $r \notin \pi$, then $r \nmid o(\xi)$ and thus ξ extends to R because it is G-invariant. It follows, by [14, Theorem 6.29], that ξ extends to G. Therefore, $\chi = \nu\psi$ for some ν extension of ξ and $\psi \in \mathrm{Irr}(G/M)$. By induction, there exists a unique irreducible constituent λ of ψ_N such that $p \nmid \lambda(1)$, and λ is linear. It follows that the character $\nu_N\lambda$ is the unique irreducible constituent of χ_N whose degree is not divided by p, and of course it is linear. $\square$

Conclusion and future developments

In this thesis, we studied some aspects of the theory of characters of π-separable groups which were previously never studied, and we found some original results. Some problems, however, remain open and some new questions arise.

In Chapter 2 we proved that there are strong analogies between zeros of irreducible and B_π-characters. In particular, we saw that, in a supersolvable group, there are necessary conditions for an element to be π-nonvanishing which are similar to the known necessary conditions for an element to be nonvanishing. Moreover, in a solvable group, there exist necessary conditions for an element to be both π-nonvanishing and π'-nonvanishing and these conditions are the same we know exist for nonvanishing elements. Since, however, these are necessary conditions, and not sufficient, we are still not able to proceed further with the analogies between the two sets of characters.

Thus, a question remains unanswered: let G be a π-separable group, let $x \in G$ and suppose there exists a character $\chi \in \mathrm{Irr}(G)$ such that $\chi(x) = 0$, then is it true that there exists also a character $\psi \in \mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ such that $\psi(x) = 0$?

In Chapter 4 we saw that strong analogies between the sets of characters $\mathrm{Irr}(G)$ and $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ continue to appear. In particular, we proved that the degrees of characters in $\mathrm{Irr}(G)$ and in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$ are divided by the same primes. When we study the prime degree graph, however, the situation is less transparent. In fact, most of the techniques which are used to study the prime degree graph, in solvable groups, cannot be applied if we consider only the degrees of characters in $\mathrm{B}_\pi(G) \cup \mathrm{B}_{\pi'}(G)$. There exist some results about the prime degree graph of B_π-characters, see for example [27]; however, in general we still do not have an optimal bound for the diameter of a connected component of the prime graph of the character degrees in $\mathrm{cd}^{\mathrm{B}_\pi \cup \mathrm{B}_{\pi'}}(G)$.

A related open problem is to understand *how much different* the two sets $\mathrm{cd}(G)$ and $\mathrm{cd}^{\mathrm{B}_\pi \cup \mathrm{B}_{\pi'}}(G)$ are. In fact, we have seen in the examples of Section4.5 that, in general, $\mathrm{cd}(G) \neq \mathrm{cd}^{\mathrm{B}_\pi \cup \mathrm{B}_{\pi'}}(G)$; thus, is it possible to find a bound for $\left|\mathrm{cd}(G) \setminus \mathrm{cd}^{\mathrm{B}_\pi \cup \mathrm{B}_{\pi'}}(G)\right|$? To answer to this question, results of Chapter 5 may be of use, since they provide a lower bound for $\left|\mathrm{cd}_{p'}^{\mathrm{B}_p}(G)\right|$. The generalization of this bound to π containing more then one prime, however, appears to be hard. In fact, we believe that, in general, the π-length of a π-separable group do not provide a lower bound for $\left|\mathrm{cd}_{\pi'}^{\mathrm{B}_\pi}(G)\right|$. It may still provide a bound for $\left|\mathrm{cd}^{\mathrm{B}_\pi}(G)\right|$; however, in this situation we cannot rely on the fact that the Fong characters are linear and, thus, it is much harder to work with them.

In Chapter 6, we described the behaviour of B_π-characters when restricted to the normalizer of a Hall π-subgroup. In particular, we defined the upper-Fong characters and we proved that they are B_π-characters under some further assumptions. However, may be possible that they are always B_π-characters but we are still not able to prove it, nor to find counterexamples. If it was true, it would provide an elegant, alternative way to prove the McKay natural correspondence in case of primitive Fong characters.

The theory behind B_π-characters may be difficult to approach, and it is hard to master. Nevertheless, the study of B_π-characters can be rewarding, both when they are applied to other problems, independent from the theory, and when we focus on the most profound aspects of the theory itself. I am grateful I have had the possibility to do research in this fascinating area of the character theory of finite groups.

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