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LIPSCHITZ STABLE DETERMINATION OF POLYGONAL CONDUCTIVITY INCLUSIONS IN A 2-DIMENSIONAL LAYERED MEDIUM FROM THE DIRICHLET-TO-NEUMANN MAP

ELENA BERETTA, ELISA FRANCINI, AND SERGIO VESSELLA

ABSTRACT. Using a distributed representation formula of the Gateaux derivative of the Dirichlet-to-Neumann map with respect to movements of a polygonal conductivity inclusion, [14], we extend the results obtained in [9], proving global Lipschitz stability for the determination of a polygonal conductivity inclusion, embedded in a 2-dimensional layered medium, from knowledge of the Dirichlet-to-Neumann map.

1. INTRODUCTION

In this paper we consider the inverse problem of determining a polygonal conductivity inclusion in a 2-dimensional layered medium Ω . We address the issue of stable reconstruction from knowledge of the Dirichlet-to-Neumann (DN) map, proving a quantitative Lipschitz stability estimate. This extends the results obtained in [9], where Lipschitz stability was proved in the case of one or more well separated polygonal inclusions embedded in a homogeneous medium. There, a crucial step to prove Lipschitz stability was an accurate investigation of the differentiability properties of the DN map and a lower bound of the derivative of the DN map with respect to movements of a polygonal conductivity inclusion $\mathcal{P}^0$. In [9] the authors used a *boundary integral representation* of the directional derivative, obtained rigorously in [13], via an accurate study of the blow-up rate of the gradient of solutions in a neighborhood of the vertices of the polygon.

In the case of an arbitrary polygonal regular partition of the domain Ω , the behaviour of the solutions in a neighbourhood of the points of intersection of the sides of the polygons depend on how the sides of elements of the partition intersect not allowing in general the derivation of the boundary representation of the derivative, [10]. On the other hand, as shown in [14], [21], [19], one can prove the existence of a *distributed representation* of the Gateaux derivative of the DN map.

The novelty of our approach is to use the derivative of the DN map in its distributed form and via this representation to establish a lower bound for the derivative, which represents the key to derive the desired Lipschitz stability estimates. The main idea to derive this bound is the following: we compute the distributed derivative of the DN map along a pair of singular solutions $G_0(\cdot, y)$ and $G_0(\cdot, z)$ for the conductivity operator with singularities $y, z \in \mathbb{R}^2 \setminus \bar{\Omega}$ close to $\partial\Omega$ and we, then, study the properties of this function that we denote by $S_0(y, z)$. First, an integration by parts allows us to express $S_0(y, z)$ in terms of the *boundary representation* of the derivative away from the vertices. Then, using the

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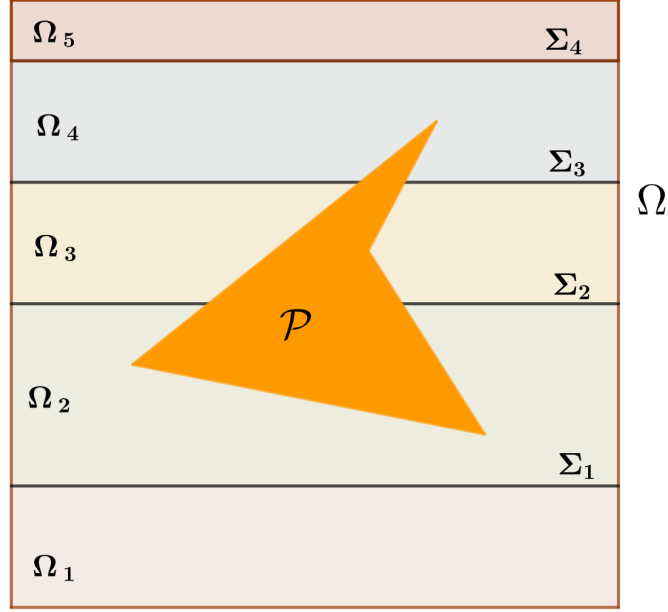


Figure 1. An admissible geometry.

fact that the Green function is solution of the conductivity equation, and exploiting its behaviour near the discontinuity interfaces, we establish for $S_0(y, z)$ quantitative estimates of smallness, [2], along a walkway connecting the part of the boundary where the data are localized to the sides of the polygon $\mathcal{P}^0$, far from its vertices and its intersections with the interfaces, allowing $S_0(y, z)$ to blow up, at a certain controlled rate, when getting close to its sides. This fact, coupled with the asymptotic behaviour of the Green's function in the walkway when approaching the side of $\mathcal{P}^0$, leads to the desired lower bound of the derivative.

To our knowledge, stability for polygonal inclusions from finitely many boundary measurements has been derived only in the case of polygonal inclusions embedded in an homogeneous medium: in [7] where the authors derive a local stability result and in [23] where the logarithmic stability estimate for convex polygons is global but the measurement depends on the unknown polygon. Also, the results obtained in [3] where Lipschitz stability has been proved for conductivities belonging to a finite dimensional subspace do not apply in our case. On the other hand, in several applications, like the geophysical one, many measurements are at disposal justifying the use of the full DN map, [6].

The result can be extended also to a more general piecewise linear partition of the domain Ω provided the distance of the vertices of the polygon is at controlled positive distance from the layers of the partition. This kind of geometrical setting originates from applications, for example in geophysics exploration, where a layered medium (the Earth) under inspection contains heterogeneities in the form of rough bounded subregions (for example subsurface salt bodies) with different conductivity properties [18], [17].

We think that the ideas introduced here will pave the way to new stability results for relevant higher dimensional problems. In fact, in a paper in preparation, we prove Lipschitz stability in the case of an arbitrary polyhedral Lipschitz conductivity inclusion imbedded in a homogeneous medium and we expect to extend the result also in the case of polyhedra embedded in a layered medium. Moreover, we expect to be able to generalize these results to the framework of linear elasticity.

The paper is organized as follows: in Section 2 we state our main assumptions and the main results. In Section 3 we derive a first rough stability estimate in order to establish that, if the difference of the DN map corresponding to two polygons are close, then the polygons have the same number of sides and are close in the Hausdorff distance. In Section 4 we construct a suitable Lipschitz vector field and prove some crucial regularity results. In Section 5 we derive a representation of the Gateaux derivative of the DN map and prove its continuity. In Section 6 we derive a lower bound of the derivative and give the proof of the main results stated in section 2. Finally, Section 7 is devoted to a discussion on possible generalizations and open problems.

2. ASSUMPTIONS AND MAIN RESULT

In this section we state the main assumptions and the main results. For sake of simplicity we fix a domain $\Omega = (-L, L) \times (-L, L) \subset \mathbb{R}^2$ and let $\omega_0, \dots, \omega_m$ be real numbers such that $\omega_0 = -L < \omega_1 < \dots < \omega_m = L$ and let

$$\bar{\Omega} = \cup_{i=1}^m \bar{\Omega}_i, \quad \text{where } \Omega_i = \{(x, y) \in \Omega : \omega_{i-1} < y < \omega_i\}.$$

Let us consider a background conductivity of the form

$$(2.1) \quad \gamma_b(x) = \sum_{i=1}^m \gamma_i \chi_{\Omega_i},$$

for positive given constants $\gamma_i, i = 1, \dots, m$. Let us denote by Σ_i the interface $(-L, L) \times \{\omega_i\}$. We assume

$$(2.2) \quad \text{dist}(\Sigma_i, \Sigma_{i+1}) \geq d_0, \forall i = 0, \dots, m-1.$$

for some $d_0 > 0$. We denote by $\text{dist}(\cdot, \cdot)$ the Euclidean distance between non empty subsets A, B in $\mathbb{R}^2$ i.e $\text{dist}(A, B) = \inf\{\|x - y\|, x \in A, y \in B\}$.

Let k be a positive given constant $k \neq \gamma_i, i = 1, \dots, m$, let $\mathcal{P} \subset \Omega$ be a closed polygon and set

$$\gamma_{\mathcal{P}} = \gamma_b(x) + (k - \gamma_b(x))\chi_{\mathcal{P}}(x).$$

Let us denote by $\Lambda_{\mathcal{P}}$ the DN map related to $\gamma_{\mathcal{P}}$, that is the map

$$\begin{aligned} \Lambda_{\mathcal{P}} : H^{1/2}(\partial\Omega) &\rightarrow H^{-1/2}(\partial\Omega) \\ f &\rightarrow \gamma_{\mathcal{P}} \frac{\partial u}{\partial n}|_{\partial\Omega}, \end{aligned}$$

where $u \in H^1(\Omega)$ is the unique solution to

$$(2.3) \quad \begin{cases} \text{div}(\gamma_{\mathcal{P}} \nabla u) &= 0 \text{ in } \Omega, \\ u &= f \text{ on } \partial\Omega, \end{cases}$$

and n is the unit outer normal direction to $\partial\Omega$. The norm of the DN map in the space of linear operators $\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$ is defined by

$$\|\Lambda_{\mathcal{P}}\|_* = \sup \left\{ \|\Lambda_{\mathcal{P}}\phi\|_{H^{-1/2}(\partial\Omega)} / \|\phi\|_{H^{1/2}(\partial\Omega)} : \phi \neq 0 \right\}.$$

Let us now define the class of polygons we are dealing with.

Let $\mathcal{A}$ the set of closed, simply connected, polygons $\mathcal{P} \subset \Omega$ whose boundary is a simple curve such that:

$$(2.4) \quad \mathcal{P} \text{ has at most } N_0 \text{ sides each one with length greater than } d_0;$$

$$(2.5) \quad \partial\mathcal{P} \text{ is of Lipschitz class with constants } r_0 \text{ and } K_0;$$

there exists a constant $\beta_0 \in (0, \pi/2]$ such that the angle β at each vertex of $\mathcal{P}$ satisfies the conditions

$$(2.6) \quad \beta_0 \leq \beta \leq 2\pi - \beta_0 \text{ and } |\beta - \pi| \geq \beta_0;$$

$$(2.7) \quad \text{dist}(\mathcal{P}, \partial\Omega) \geq d_0.$$

We also assume that for every vertex P_k of $\mathcal{P}$ we have

$$(2.8) \quad \text{dist}(P_k, \Sigma_i) \geq d_0/2, \forall i = 1, \dots, m-1.$$

Notice that we do not assume convexity of the polygon.

In the sequel $\mathcal{P}^0 \Delta \mathcal{P}^1 := (\mathcal{P}^0 \setminus \mathcal{P}^1) \cup (\mathcal{P}^1 \setminus \mathcal{P}^0)$ will indicate the symmetric difference of the two polygons $\mathcal{P}^0$ and $\mathcal{P}^1$.

We also assume that we can distinguish the inclusion from the background, so we assume there is a positive constant c_0 such that

$$(2.9) \quad |k - \gamma_i| \geq c_0, \text{ for } i = 1, \dots, m.$$

The constants $N_0, d_0, r_0, K_0, L, \beta_0, c_0$, and m will be referred to as the *a priori data*.

In the sequel we will introduce a number of constants that we will usually denote by C . The values of these constants might differ from one line to the other but they will only be determined by the *a priori data* and will always be greater than 1.

Finally, let us recall the definition of the Hausdorff distance between two sets A and B :

$$d_H(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} \text{dist}(x, y), \sup_{y \in B} \inf_{x \in A} \text{dist}(y, x) \right\}.$$

The following stability results hold:

Theorem 2.1. *Let $\mathcal{P}^0, \mathcal{P}^1 \in \mathcal{A}$ and let $k, \gamma_i, i = 1, \dots, m$ satisfy assumption (2.9). There exists C depending only on the *a priori data* such that*

$$d_H(\partial\mathcal{P}^0, \partial\mathcal{P}^1) \leq C \|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_*.$$

Theorem 2.2. *Let $\mathcal{P}^0, \mathcal{P}^1 \in \mathcal{A}$. There exists a positive constant C depending only on the *a priori data* such that,*

$$(2.10) \quad \|\gamma_{\mathcal{P}^0} - \gamma_{\mathcal{P}^1}\|_{L^1(\Omega)} \leq C \|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_*.$$

Remark 2.3. *Our main results can be extended thoroughly to the case where Ω is a bounded domain with Lipschitz boundary.*

3. LOGARITHMIC STABILITY ESTIMATES

In this section we first state a stability result obtained in [16], where the authors establish a logarithmic stability result for conductivities in $H^s(\Omega)$, for $s \in (0, 1/2)$, that we can apply in our framework since, for $\mathcal{P} \in \mathcal{A}$, we have that $\gamma_{\mathcal{P}} \in H^s(\Omega)$ for $s \in (0, 1/2)$ (see [11], [12] and [25]). This allows us to prove that, if the DN maps corresponding to two polygons are close, then the polygons have the same number of sides and are close in the Hausdorff distance (Proposition 3.4).

Proposition 3.1. (3.1 in [16]) *There exists $\alpha < 1/4$ and $C > 1$ depending only on the a priori data, such that, for $\mathcal{P}^0, \mathcal{P}^1 \in \mathcal{A}$,*

$$(3.1) \quad \|\gamma_{\mathcal{P}^0} - \gamma_{\mathcal{P}^1}\|_{L^2(\Omega)} \leq C |\ln \|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_*|^{-\alpha},$$

if $\|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_* < 1/2$.

Corollary 3.2. *Under the assumptions of Theorem 2.1, there exists $\alpha < 1/4$ and $C > 1$ depending only on the a priori data, such that, if $\|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_* < 1/2$, then*

$$(3.2) \quad |\mathcal{P}^1 \Delta \mathcal{P}^0| \leq \frac{C}{c_0} |\ln \|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_*|^{-2\alpha}.$$

Proof. It follows immediately from (3.1) and from assumption (2.9), that gives

$$\begin{aligned} \|\gamma_{\mathcal{P}^0} - \gamma_{\mathcal{P}^1}\|_{L^2(\Omega)}^2 &= \int_{\Omega} (\gamma_{\mathcal{P}^0} - \gamma_{\mathcal{P}^1})^2 \\ &= \int_{\mathcal{P}^1 \setminus \mathcal{P}^0} (k - \gamma_b(x))^2 + \int_{\mathcal{P}^0 \setminus \mathcal{P}^1} (k - \gamma_b(x))^2 \\ &\geq c_0^2 |\mathcal{P}^1 \Delta \mathcal{P}^0|. \end{aligned}$$

□

We can now follow the same strategy as in [9] and notice that the a priori assumptions on the set of polygons $\mathcal{A}$ give a relation between $|\mathcal{P}^1 \Delta \mathcal{P}^0|$, $d_H(\partial \mathcal{P}^0, \partial \mathcal{P}^1)$ and the distance between the endpoints of the polygons.

Let us recall here some results from [9].

Lemma 3.3. (Lemma 3.2 in [9]) *Given two polygons $\mathcal{P}^0$ and $\mathcal{P}^1$ in $\mathcal{A}$, we have*

$$d_H(\partial \mathcal{P}^0, \partial \mathcal{P}^1) \leq C \sqrt{|\mathcal{P}^0 \Delta \mathcal{P}^1|},$$

where C depends only on the a priori data.

Proposition 3.4. (Proposition 3.3 in [9]) *Given the set of polygons $\mathcal{A}$ there exist δ_0 and C_0 depending only on the a priori data such that, if for some $\mathcal{P}^0, \mathcal{P}^1 \in \mathcal{A}$ we have*

$$d_H(\partial \mathcal{P}^0, \partial \mathcal{P}^1) \leq \delta_0,$$

then $\mathcal{P}^0$ and $\mathcal{P}^1$ have the same number N of vertices $\{P_i^0\}_{i=1}^N$ and $\{P_i^1\}_{i=1}^N$, respectively, that can be ordered in such a way that

$$(3.3) \quad \text{dist}(P_i^0, P_i^1) \leq C_0 d_H(\partial \mathcal{P}^0, \partial \mathcal{P}^1) \text{ for every } i = 1, \dots, N.$$

More precisely $\delta_0 = \min\{K_0 r_0, d_0 \frac{\sin \beta_0}{16}\}$ and $C_0 = \sqrt{1 + \frac{16}{\sin^2 \beta_0}}$.

By Corollary 3.2, Lemma 3.3 and Proposition 3.4 we have the following

Proposition 3.5. *(Proposition 3.4 in [9]) Under the same assumptions of Theorem 2.1 there exist positive constants ε_0 , α , $C > 1$ depending only on the a priori data, such that, if*

$$\varepsilon := \|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_* < \varepsilon_0,$$

then $\mathcal{P}^0$ and $\mathcal{P}^1$ have the same number N of vertices $\{P_j^0\}_{j=1}^N$ and $\{P_j^1\}_{j=1}^N$, respectively.

Moreover, the vertices can be ordered so that

$$(3.4) \quad \text{dist}(P_j^0, P_j^1) \leq \omega(\varepsilon), \quad \forall j = 1, \dots, N,$$

where $\omega(\varepsilon) = C|\ln \varepsilon|^{-\alpha}$.

4. MAPPING $\mathcal{P}^0$ TO $\mathcal{P}^1$

In this section we construct a suitable Lipschitz vector field mapping the polygon $\mathcal{P}^0$ to $\mathcal{P}^1$. In this section we assume that

$$d_H(\partial\mathcal{P}^0, \partial\mathcal{P}^1) \leq \frac{\delta_0}{5},$$

where δ_0 is as in Proposition 3.4 and let $\{P_j^0\}_{j=1}^N$ and $\{P_j^1\}_{j=1}^N$ be the vertices of $\mathcal{P}^0$ and $\mathcal{P}^1$. Let us also consider the intersections of $\partial\mathcal{P}^0$ with

$$\Sigma = \cup_{i=1}^{m-1} \Sigma_i,$$

and denote them by $\{P_j^0\}_{j=N+1}^{N+k_0}$.

Since, by (3.3),

$$\text{dist}(P_i^0, P_i^1) \leq C_0 \frac{\delta_0}{5} \leq \frac{d_0 \sin \beta_0}{80} \sqrt{1 + \frac{16}{\sin^2 \beta_0}} \leq \frac{\sqrt{17}d_0}{80} < \frac{d_0}{2}, \quad i = 1, \dots, N,$$

by assumption (2.8), for each side of $\mathcal{P}^0$ intersecting Σ , the corresponding side in $\mathcal{P}^1$ intersects Σ .

This means that the intersections of $\partial\mathcal{P}^1$ with Σ are given by $\{P_j^1\}_{j=N+1}^{N+k_0}$ and

$$\text{dist}(P_j^0, P_j^1) \leq C_0 d_H(\partial\mathcal{P}^0, \partial\mathcal{P}^1), \quad \text{for } j = N+1, \dots, N+k_0.$$

Let us reorder the indices in

$$\{P_j^0\}_{j=1}^M \quad \text{and} \quad \{P_j^1\}_{j=1}^M,$$

such that $M = N + k_0 \leq mN$, $P_j^0 P_{j+1}^0$ is a segment in $\partial\mathcal{P}^0$, and

$$\text{dist}(P_j^0, P_j^1) \leq C_0 d_H(\partial\mathcal{P}^0, \partial\mathcal{P}^1) \quad \text{for } j = 1, \dots, M.$$

Let $W \subset\subset \Omega$ be a tubular neighborhood of $\partial\mathcal{P}^0$ of width $\frac{d_0}{4}$ so that

$$\text{dist}(W, \partial\Omega) \geq \frac{d_0}{2},$$

and $W \supset \partial\mathcal{P}^0$.

Let us now define a map $\mathcal{U} \in W^{1,\infty}(\mathbb{R}^2)$ in such a way that the following properties are satisfied (cf. [13] for a similar construction):

$$(4.1) \quad \mathcal{U}(P_j^0) = P_j^1 - P_j^0 \quad \text{for every } j = 1, \dots, M,$$

$$(4.2) \quad \text{supp}(\mathcal{U}) \subset \overline{W},$$

$$(4.3) \quad \mathcal{U} \text{ is piecewise affine continuous on the line segments } P_j^0 P_{j+1}^0, j = 1, \dots, M, \\ \text{and also on } (W \cap \Sigma) \setminus \mathcal{P}^0.$$

Notice that on Σ the map is parallel to Σ .

$$(4.4) \quad |\mathcal{U}| + \frac{d_0}{8} |D\mathcal{U}| \leq C_0 d_H(\partial\mathcal{P}^0, \partial\mathcal{P}^1),$$

where we denote by $D\mathcal{U}$ the Jacobian matrix of $\mathcal{U}$. From now on I will denote the 2×2 identity matrix.

Proposition 4.1. *The map*

$$\Phi_t = I + t\mathcal{U}, \quad \text{for } t \in [0, 1],$$

has the following properties:

$$(4.5) \quad \Phi_t \text{ is piecewise affine continuous on } \partial\mathcal{P}^0 \cup \Sigma \setminus \mathcal{P}^0;$$

$$(4.6) \quad \Phi_t \in W^{1,\infty}(\Omega) \text{ is invertible and} \\ |D\Phi_t - I|, |D\Phi_t^{-1} - I| \leq C t d_H(\partial\mathcal{P}^0, \partial\mathcal{P}^1);$$

$$(4.7) \quad \Phi_t(\Omega_i \setminus \mathcal{P}^0) \subset \Omega_i \text{ for all } i = 1, \dots, m;$$

$$(4.8) \quad \left| \frac{d}{dt} \Phi_t \right|, \left| \frac{d}{dt} \Phi_t^{-1} \right| \leq C d_H(\partial\mathcal{P}^0, \partial\mathcal{P}^1);$$

$$(4.9) \quad \left| \frac{d}{dt} D\Phi_t \right|, \left| \frac{d}{dt} D\Phi_t^{-1} \right| \leq C d_H(\partial\mathcal{P}^0, \partial\mathcal{P}^1);$$

$$(4.10) \quad \left| \frac{d}{dt} D\Phi_t^{-1} + D\mathcal{U} \right| \leq C t d_H^2(\partial\mathcal{P}^0, \partial\mathcal{P}^1), \quad \left| \frac{d}{dt} (D\Phi_t^{-1})^T + D\mathcal{U}^T \right| \leq C t d_H^2(\partial\mathcal{P}^0, \partial\mathcal{P}^1).$$

Proof. (4.5) follows directly from the definition of $\mathcal{U}$.

In order to prove (4.6) we notice that

$$|D\Phi_t - I| = t |D\mathcal{U}| \leq 8t \frac{C_0}{d_0} d_H(\partial\mathcal{P}^0, \partial\mathcal{P}^1),$$

hence

$$|D\Phi_t - I| \leq \frac{8C_0\delta_0}{5d_0} \leq \frac{\sqrt{17}}{10} < \frac{1}{2},$$

hence Φ_t is invertible for every $t \in [0, 1]$. Moreover, by the Implicit Map Theorem it follows that $D\Phi_t^{-1}(y) = (D\Phi_t)^{-1}(D\Phi_t^{-1}(y))$ and the analyticity in the parameter t of $(D\Phi_t)^{-1}$ gives

$$|D\Phi_t^{-1} - I| \leq 8t \frac{C_0}{d_0} d_H(\partial\mathcal{P}^0, \partial\mathcal{P}^1).$$

Let us now prove (4.7) noticing that if $P_j^0 \in \Sigma$, then $\mathcal{U}(P_j^0)$ is parallel to Σ , hence $\Phi_t(\Sigma) \subset \Sigma$. The invertibility of the map Φ_t then gives (4.7).

Estimates (4.8) and (4.9) follows directly from (4.4) and finally we obtain (4.10) again by analyticity of $(D\Phi_t)^{-1}$ with respect to t :

$$\left| \frac{d}{dt} D\Phi_t^{-1} + DU \right| \leq Ct |DU|^2 \leq Ctd_H^2(\partial\mathcal{P}^0, \partial\mathcal{P}^1),$$

and analogously, since $(D\Phi_t^{-1}(y))^T = (D\Phi_t)^{-T}(\Phi_t^{-1}(y))$,

$$\left| \frac{d}{dt} (D\Phi_t^{-1})^T + DU^T \right| \leq Ct |DU|^2 \leq Ctd_H^2(\partial\mathcal{P}^0, \partial\mathcal{P}^1),$$

which concludes the proof. $\square$

5. DIFFERENTIABILITY PROPERTIES OF THE DN MAP

In this section we will first show the existence of the material derivative of u , $\dot{u} := \frac{d(u \circ \Phi_t)}{dt}|_{t=0}$ in a variational setting. This will allow us to prove the differentiability of the DN map and the explicit computation of the distributed form of the Gateaux derivative. Finally, via its representation we will prove that the derivative is continuous.

Let $\Phi_t(x) = x + t\mathcal{U}(x)$ and let us define

$$\mathcal{P}^t = \Phi_t(\mathcal{P}^0),$$

and notice that, by (4.6),

$$\gamma_{\mathcal{P}^t}(x) = \gamma_{\mathcal{P}^0}(\Phi_t^{-1}(x)).$$

Notice that the notation is consistent, that is $\gamma_{\mathcal{P}^0}(\Phi_1^{-1}(x)) = \gamma_{\mathcal{P}^1}(x)$. Let us denote by u_t the solution to

$$(5.1) \quad \begin{cases} \operatorname{div}(\gamma_{\mathcal{P}^t} \nabla u_t) &= 0 \text{ in } \Omega, \\ u_t &= f \text{ on } \partial\Omega. \end{cases}$$

Let us define

$$A(t) = (D\Phi_t^{-1}) (D\Phi_t^{-1})^T \det(D\Phi_t),$$

and

$$\mathcal{A} = \frac{d}{dt} A(t) \Big|_{t=0} = \operatorname{div}(\mathcal{U})I - (DU + DU^T).$$

Let us now show a result corresponding to Lemma 2.1 in [14].

Lemma 5.1. *The solution u_t to problem (5.1) has a material derivative at $t = 0$, $\dot{u} \in H_0^1(\Omega)$, that solves*

$$(5.2) \quad \int_{\Omega} \gamma_{\mathcal{P}^0} \nabla \dot{u} \cdot \nabla \psi = - \int_{\Omega} \gamma_{\mathcal{P}^0} \mathcal{A} \nabla u_0 \cdot \nabla \psi,$$

for every $\psi \in H_0^1(\Omega)$.

Proof. Let $\tilde{f} \in H^1(\Omega)$ be an extension of f such that $\operatorname{supp}(\tilde{f}) \cap \operatorname{supp}(\mathcal{U}) = \emptyset$. Let $w_t = u_t - \tilde{f}$ for $t \in [0, 1]$. Notice that $w_t \in H_0^1(\Omega)$ and $\operatorname{div}(\gamma_{\mathcal{P}^t} \nabla w_t) = -\operatorname{div}(\gamma_{\mathcal{P}^t} \nabla \tilde{f})$ in Ω , that is

$$(5.3) \quad \int_{\Omega} \gamma_{\mathcal{P}^t} \nabla w_t \cdot \nabla \psi = - \int_{\Omega} \gamma_{\mathcal{P}^t} \nabla \tilde{f} \cdot \nabla \psi, \quad \forall \psi \in H_0^1(\Omega).$$

Let $\tilde{w}_t = w_t \circ \Phi_t$; since $\Phi_t(x) = x$ close to $\partial\Omega$, we have that $\tilde{w}_t \in H_0^1(\Omega)$ satisfies

$$(5.4) \quad \int_{\Omega} \gamma_{\mathcal{P}^0} A(t) \nabla \tilde{w}_t \cdot \nabla \psi = - \int_{\Omega} \gamma_{\mathcal{P}^0} \nabla \tilde{f} \cdot \nabla \psi \quad \forall \psi \in H_0^1(\Omega).$$

where we have used the fact that $A = I$ on the support of $\tilde{f}$ and the fact that $\gamma_{\mathcal{P}^t}(\Phi_t(x)) = \gamma_{\mathcal{P}^0}(x)$. By subtracting (5.3) for $t = 0$ from (5.4) and dividing by t , we get

$$(5.5) \quad \int_{\Omega} \gamma_{\mathcal{P}^0} A(t) \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right) \cdot \nabla \psi = \int_{\Omega} \gamma_{\mathcal{P}^0} \frac{I - A(t)}{t} \nabla w_0 \cdot \nabla \psi, \quad \forall \psi \in H_0^1(\Omega).$$

Now we proceed as in the proof of Lemma 2.1 in [14]. Applying (4.1) we have that $\|A(t)\|_{L^\infty(\Omega)} \geq C$ for all $t \in [0, 1]$ and that $A(t)$ is differentiable at $t = 0$. Hence, choosing $\psi = \frac{\tilde{w}_t - w_0}{t} \in H_0^1(\Omega)$ as test function and applying Cauchy-Schwarz inequality, we get

$$\left\| \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right) \right\|_{L^2(\Omega)} \leq C \|\nabla w_0\|_{L^2(\Omega)},$$

which means that

$$\frac{\tilde{w}_t - w_0}{t} \text{ is bounded in } H_0^1(\Omega),$$

therefore there is a weakly convergent sequence in $H_0^1(\Omega)$ and its weak limit is the material derivative $\dot{w}$ of w . By passing to the limit in (5.5) we get that

$$(5.6) \quad \int_{\Omega} \gamma_{\mathcal{P}^0} \nabla \dot{w} \cdot \nabla \psi = - \int_{\Omega} \gamma_{\mathcal{P}^0} \mathcal{A} \nabla w_0 \cdot \nabla \psi, \quad \forall \psi \in H_0^1(\Omega).$$

We also have strong convergence. In fact, if we take $\psi = \frac{\tilde{w}_t - w_0}{t}$ in (5.5) we get

$$\int_{\Omega} \gamma_{\mathcal{P}^0} A(t) \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right) \cdot \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right) = \int_{\Omega} \gamma_{\mathcal{P}^0} \frac{I - A(t)}{t} \nabla w_0 \cdot \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right)$$

and, using weak convergence of $\frac{\tilde{w}_t - w_0}{t}$ in $H^1(\Omega)$, strong convergence of $\frac{I - A(t)}{t} \nabla w_0$ in $L^2(\Omega)$, and by (5.6), where we choose $\dot{w}$ as test function, we obtain

$$\lim_{t \rightarrow 0} \int_{\Omega} \gamma_{\mathcal{P}^0} A(t) \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right) \cdot \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right) = - \int_{\Omega} \gamma_{\mathcal{P}^0} \mathcal{A} \nabla w_0 \cdot \nabla \dot{w} = \int_{\Omega} \gamma_{\mathcal{P}^0} \nabla \dot{w} \cdot \nabla \dot{w}.$$

Finally, by

$$\begin{aligned} \int_{\Omega} \gamma_{\mathcal{P}^0} \left| \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right) \right|^2 &= \int_{\Omega} \gamma_{\mathcal{P}^0} A(t) \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right) \cdot \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right) \\ &\quad + \int_{\Omega} \gamma_{\mathcal{P}^0} (I - A(t)) \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right) \cdot \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right), \end{aligned}$$

and since

$$\int_{\Omega} \gamma_{\mathcal{P}^0} (I - A(t)) \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right) \cdot \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right) \leq Ct,$$

we obtain

$$\left\| \sqrt{\gamma_{\mathcal{P}^0}} \nabla \left(\frac{\tilde{w}_t - w_0}{t} \right) \right\|_{L^2(\Omega)}^2 \rightarrow \|\sqrt{\gamma_{\mathcal{P}^0}} \nabla \dot{w}\|_{L^2(\Omega)}^2.$$

This last convergence with the weak convergence in $H^1(\Omega)$ and coercivity finally gives

$$\left\| \nabla \left(\frac{\tilde{w}_t - w_0}{t} - \dot{w} \right) \right\|_{L^2(\Omega)} \rightarrow 0,$$

as $t \rightarrow 0$. Finally, by the Poincaré inequality, $\frac{\tilde{w}_t - w_0}{t} \rightarrow \dot{w}$ in $H_0^1(\Omega)$.

Let us now go back to u_t . Since $u_t = w_t + \tilde{f}$ and $\tilde{u}_t = \tilde{w}_t + \tilde{f}(\Phi_t)$, we have

$$\frac{\tilde{u}_t - u_0}{t} = \frac{\tilde{w}_t - w_0}{t} + \frac{\tilde{f}(\Phi_t) - \tilde{f}}{t}.$$

But, we notice that $\tilde{f}(\Phi_t) - \tilde{f} \equiv 0$ in Ω , since $\text{supp}(\tilde{f}) \subset \{x : \Phi_t(x) = x\}$, hence $\dot{u} = \dot{w}$. $\square$

5.1. Derivative of the forward map. Let us now evaluate the Gateaux derivative of the DN map along the direction of the vector field $\mathcal{U}$.

Let $f, g \in H^{1/2}(\partial\Omega)$ and let u_t solve (5.1) and v_t be the unique solution to

$$(5.7) \quad \begin{cases} \text{div}(\gamma_{\mathcal{P}^t} \nabla v_t) &= 0 \text{ in } \Omega, \\ v_t &= g \text{ on } \partial\Omega. \end{cases}$$

Let

$$F(t, f, g) = \langle \Lambda_{\mathcal{P}^t} f, g \rangle = \int_{\Omega} \gamma_{\mathcal{P}^t} \nabla u_t \cdot \nabla v_t = \langle \gamma_b \frac{\partial u_t}{\partial n}, g \rangle,$$

and analogously let

$$F(0, f, g) = \langle \Lambda_{\mathcal{P}_0} f, g \rangle = \int_{\Omega} \gamma_{\mathcal{P}_0} \nabla u_0 \cdot \nabla v_0 = \langle \gamma_b \frac{\partial u_0}{\partial n}, g \rangle.$$

Hence,

$$\frac{F(t, f, g) - F(0, f, g)}{t} = \langle \gamma_b \frac{\partial}{\partial n} \left(\frac{u_t - u_0}{t} \right), g \rangle.$$

Since $\gamma_b \frac{\partial u_t}{\partial n} = \gamma_b \frac{\partial \tilde{u}_t}{\partial n}$, because $\text{supp}(\mathcal{U})$ is far from $\partial\Omega$, by

$$\frac{\tilde{u}_t - u_0}{t} \rightarrow \dot{u},$$

in $H^1(\Omega)$, and recalling (5.2), we get

$$\frac{F(t, f, g) - F(0, f, g)}{t} \rightarrow \langle \gamma_{\mathcal{P}_0} \frac{\partial \dot{u}}{\partial n}, g \rangle = \int_{\Omega} \gamma_{\mathcal{P}_0} \nabla \dot{u} \cdot \nabla v_0 = - \int_{\Omega} \gamma_{\mathcal{P}_0} \mathcal{A} \nabla u_0 \cdot \nabla v_0,$$

as $t \rightarrow 0$. Hence

$$F'(0, f, g) = - \int_{\Omega} \gamma_{\mathcal{P}_0} \mathcal{A} \nabla u_0 \cdot \nabla v_0,$$

where we denote by F' the derivative with respect to t .

Remark 5.2. *Using a similar argument, one can show that $F(t, f, g)$ is differentiable at any $t_0 \in [0, 1]$ and that*

$$F'(t_0, f, g) = - \int_{\Omega} \gamma_{\mathcal{P}^{t_0}} \mathcal{A}_{t_0} \nabla u_{t_0} \cdot \nabla v_{t_0},$$

for

$$\mathcal{A}_{t_0} = \frac{d}{dt} \left((D\Phi_{t_0, t}^{-1}) (D\Phi_{t_0, t}^{-1})^T \det(D\Phi_{t_0, t}) \right) |_{t=t_0},$$

where $\Phi_{t_0,t} = I + t\mathcal{U}_{t_0}$ and $\mathcal{U}_{t_0}$ is a $W^{1,\infty}(\Omega)$ map satisfying (4.2), (4.4) and such that $\mathcal{U}_{t_0}(P_j^0 + t_0(P_j^1 - P_j^0)) = P_j^1 - P_j^0 \forall j = 1, \dots, M$ and finally, u_{t_0}, v_{t_0} are solutions of (5.1), (5.7) respectively for conductivity $\gamma_{\mathcal{P}^{t_0}}$.

5.2. Continuity of the Gateaux derivative. We now want to prove the following result

Proposition 5.3. *There exist constants $C, \beta > 0$ depending only on the a priori data, such that, for all $t \in [0, 1]$,*

$$(5.8) \quad |F'(t, f, g) - F'(0, f, g)| \leq C \|f\|_{H^{1/2}(\partial\Omega)} \|g\|_{H^{1/2}(\partial\Omega)} t^\beta d_H^{1+\beta}(\partial\mathcal{P}^0, \partial\mathcal{P}^1).$$

Proof.

$$\begin{aligned} F'(t, f, g) - F'(0, f, g) &= - \int_{\Omega} \gamma_{\mathcal{P}^t} \mathcal{A}_t \nabla u_t \cdot \nabla v_t + \int_{\Omega} \gamma_{\mathcal{P}^0} \mathcal{A} \nabla u_0 \cdot \nabla v_0 \\ &= - \int_{\Omega} (\gamma_{\mathcal{P}^t} - \gamma_{\mathcal{P}^0}) \mathcal{A} \nabla u_0 \cdot \nabla v_0 - \int_{\Omega} \gamma_{\mathcal{P}^t} (\mathcal{A}_t - \mathcal{A}) \nabla u_t \cdot \nabla v_t - \\ &\quad - \int_{\Omega} \gamma_{\mathcal{P}^t} \mathcal{A} \nabla (u_t - u_0) \cdot \nabla v_0 - \int_{\Omega} \gamma_{\mathcal{P}^t} \mathcal{A} \nabla u_t \cdot \nabla (v_t - v_0) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Let us start estimating I_1 :

$$(5.9) \quad |I_1| = \left| \int_{\mathcal{P}^t \triangle \mathcal{P}^0} (k - \gamma_b) \mathcal{A} \nabla u_0 \cdot \nabla v_0 \right|$$

$$(5.10) \quad \leq C \|\mathcal{A}\|_{L^\infty(\Omega)} \|\nabla u_0\|_{L^2(\mathcal{P}^t \triangle \mathcal{P}^0)} \|\nabla v_0\|_{L^2(\mathcal{P}^t \triangle \mathcal{P}^0)}.$$

By Meyers' Theorem, [26], $u_0, v_0 \in W_{loc}^{1,p}(\Omega)$, with $p > 2$, so that, applying Hölder inequality we get

$$\|\nabla u_0\|_{L^2(\mathcal{P}^t \triangle \mathcal{P}^0)}^2 \leq |\mathcal{P}^t \triangle \mathcal{P}^0|^{1-\frac{2}{p}} \|\nabla u_0\|_{L^p(\mathcal{P}^t \triangle \mathcal{P}^0)}^2 \leq |\mathcal{P}^t \triangle \mathcal{P}^0|^{1-\frac{2}{p}} \|\nabla u_0\|_{L^2(\Omega)}^2,$$

which implies

$$(5.11) \quad \|\nabla u_0\|_{L^2(\mathcal{P}^t \triangle \mathcal{P}^0)} \leq C t^{\frac{1}{2}-\frac{1}{p}} d_H^{\frac{1}{2}-\frac{1}{p}} \|f\|_{H^{1/2}(\partial\Omega)},$$

where we have set, to simplify the notation, $d_H := d_H(\partial\mathcal{P}^0, \partial\mathcal{P}^1)$ and, analogously,

$$(5.12) \quad \|\nabla v_0\|_{L^2(\mathcal{P}^t \triangle \mathcal{P}^0)} \leq C t^{\frac{1}{2}-\frac{1}{p}} d_H^{\frac{1}{2}-\frac{1}{p}} \|g\|_{H^{1/2}(\partial\Omega)}.$$

Moreover, by (4.4) $\|\mathcal{A}\|_{L^\infty(\Omega)} \leq C d_H$, hence,

$$(5.13) \quad |I_1| \leq C t^{1-\frac{2}{p}} d_H^{2-\frac{2}{p}}.$$

For estimating I_2 we need to estimate $\mathcal{A}_t - \mathcal{A}$. Let us rewrite

$$\begin{aligned} \mathcal{A}_t - \mathcal{A} &= \left(\frac{d}{dt} \det D\Phi_t \right) D\Phi_t^{-1} D\Phi_t^{-T} - (\operatorname{div} \mathcal{U}) I \\ &\quad + \det D\Phi_t \left(\frac{d}{dt} D\Phi_t^{-1} \right) D\Phi_t^{-T} + D\mathcal{U} + (\det D\Phi_t) D\Phi_t^{-1} \left(\frac{d}{dt} D\Phi_t^{-T} \right) + D\mathcal{U}^T \\ &= J_1 + J_2 + J_3. \end{aligned}$$

where

$$J_1 = \frac{d}{dt} \det D\Phi_t D\Phi_t^{-1} D\Phi_t^{-T} - (\operatorname{div} \mathcal{U})I,$$

$$J_2 = \det D\Phi_t \left(\frac{d}{dt} D\Phi_t^{-1} \right) D\Phi_t^{-T} + D\mathcal{U},$$

and

$$J_3 = \det D\Phi_t D\Phi_t^{-1} \left(\frac{d}{dt} D\Phi_t^{-T} \right) + D\mathcal{U}^T.$$

From Proposition 4.1 and applying (4.4) we have the following bound on J_1 and we get (5.14)

$$|J_1| \leq C \left(\left| \frac{d}{dt} \det D\Phi_t - (\operatorname{div} \mathcal{U})I \right| + (\operatorname{div} \mathcal{U})(|D\Phi_t^{-1} - I| + |D\Phi_t^{-T} - I|) \right) \leq Ctd_H^2,$$

and again by Proposition 4.1 we derive

$$(5.15) \quad |J_2| \leq C(\|D\mathcal{U}\|_{L^\infty(\Omega)} |D\Phi_t^{-T} - I| + \left| \frac{d}{dt} D\Phi_t^{-1} + D\mathcal{U} \right|) \leq Ctd_H^2,$$

and, arguing as for J_2 ,

$$(5.16) \quad |J_3| \leq Ctd_H^2.$$

Hence, collecting (5.14), (5.15) and (5.16), we have

$$(5.17) \quad \|\mathcal{A}_t - \mathcal{A}\|_{L^\infty(\Omega)} \leq Ctd_H^2,$$

which gives

$$(5.18) \quad |I_2| \leq Ctd_H^2 \|f\|_{H^{1/2}(\partial\Omega)} \|g\|_{H^{1/2}(\partial\Omega)}.$$

For the last two terms I_3, I_4 we note that, by (4.4),

$$(5.19) \quad |I_3| \leq C\|\mathcal{A}\|_{L^\infty(\Omega)} \|\nabla(u_t - u_0)\|_{L^2(\Omega)} \|\nabla v_0\|_{L^2(\Omega)} \leq Ct^{\frac{1}{2} - \frac{1}{p}} d_H^{\frac{3}{2} - \frac{1}{p}} \|f\|_{H^{1/2}(\partial\Omega)} \|g\|_{H^{1/2}(\partial\Omega)},$$

where we have used the energy estimate

$$\|u_t - u_0\|_{H^1(\Omega)} \leq C\|\nabla u_0\|_{L^2(\mathcal{P}^t \triangle \mathcal{P}^0)},$$

and the bounds (5.11) and (5.12). Analogously,

$$(5.20) \quad |I_4| \leq C\|\mathcal{A}\|_{L^\infty(\Omega)} \|\nabla(v_t - v_0)\|_{L^2(\Omega)} \|\nabla u_0\|_{L^2(\Omega)} \leq Ct^{\frac{1}{2} - \frac{1}{p}} d_H^{\frac{3}{2} - \frac{1}{p}} \|f\|_{H^{1/2}(\partial\Omega)} \|g\|_{H^{1/2}(\partial\Omega)}.$$

Finally, collecting (5.13), (5.18), (5.19), and (5.20), and setting $\beta = \frac{1}{2} - \frac{1}{p} > 0$ the claim follows. $\square$

6. BOUND FROM BELOW FOR THE DERIVATIVE

In this section we will derive the key lower bound for the operator norm of the Gateaux derivative,

$$\|F'(0)\|_* := \sup \left\{ \frac{|F(0, f, g)|}{\|f\|_{H^{1/2}(\partial\Omega)} \|g\|_{H^{1/2}(\partial\Omega)}} : f, g \neq 0 \right\}.$$

In order to simplify the exposition of the proof we will assume that Ω contains only one interior layer $\Sigma = (-L, L) \times \{0\}$ and we denote by $\Omega_- = (-L, L) \times (-L, 0)$ and $\Omega_+ = (-L, L) \times (0, L)$ the two layers with corresponding conductivity γ_- and γ_+ .

For seek of clarity let us describe the main steps of our strategy:

We introduce the Green's function for the conductivity operator $\operatorname{div}(\gamma_{\mathcal{P}^0} \nabla \cdot)$ in a slightly enlarged domain Ω_0 and investigate its asymptotic behaviour near the discontinuity interfaces comparing it with the one of a biphase explicit fundamental solution, see Proposition 6.1.

As a second step we consider the derivative $F'(0, G_0(\cdot, y), G_0(\cdot, z))$ computed along a pair of Green's functions $G_0(\cdot, y)$ and $G_0(\cdot, z)$ for the conductivity operator $\operatorname{div}(\gamma_{\mathcal{P}^0} \nabla \cdot)$ with singularities at $y, z \in \Omega_0 \setminus \bar{\Omega}$.

We study the properties of $F'(0, G_0(\cdot, y), G_0(\cdot, z))$ by using its integral representation and we show that $F'(0, G_0(\cdot, y), G_0(\cdot, z))$ can be extended to a suitable subset of Ω . In particular, via regularity of solutions to elliptic equations with discontinuous coefficients (Proposition 6.2) and the estimates of the Green's function of Proposition 6.1 we prove quantitative unique continuation estimates for $F'(0, G_0(\cdot, y), G_0(\cdot, z))$. The peculiarity of this estimate is that we allow the solution to blow up, at a certain controlled rate when getting close to the sides of the polygon $\mathcal{P}^0$. This fact, coupled with the asymptotic behaviour of the Green's function close to the side of the polygon leads to the desired lower bound of $\|F'(0)\|_*$.

Let

$$\Omega_0 = \Omega \cup \{(-d_0, d_0) \times [L, L + 2d_0]\},$$

and let us extend $\gamma_{\mathcal{P}^0}$ to Ω_0 by setting $\gamma_{\mathcal{P}^0} = \gamma_+$ in $(-d_0, d_0) \times [L, L + 2d_0]$.

We now state some useful estimates of the Green function $G_0(x, y)$ corresponding to the operator $\operatorname{div}(\gamma_{\mathcal{P}^0} \nabla \cdot)$ and to the domain Ω_0 . Our purpose is now to study the exact behaviour of the Green's function close to the discontinuities of $\gamma_{\mathcal{P}^0}$. This is done by proving that the behaviour close to the interface discontinuities is the same as that one of a biphase fundamental solution, indicated by $\bar{\Gamma}$, which is explicitly computable.

Let us fix a point $y \in \Omega_0 \setminus \mathcal{P}^0$ and let $0 < r_1 = \operatorname{dist}(y, \{P_i^0\}_{i=1}^M \cup \partial\Omega_0)$. There is a constant $\tilde{C}$ (depending only on the a priori data) such that $B_{r_1/\tilde{C}}(y)$ contains at most a segment line dividing the disk into two parts with different conductivity. This means that, for a suitable choice of the coordinate system centered at y , there exists $a \in [-r_1/\tilde{C}, r_1/\tilde{C}]$ such that

$$(6.1) \quad \gamma_{\mathcal{P}^0}(x) = \gamma^1 + (\gamma^2 - \gamma^1) \chi_{\{x_2 > a\}}(x) \quad \text{for every } x = (x_1, x_2) \in B_{r_1/\tilde{C}}(y),$$

where γ^1 and γ^2 can assume one of the values γ_+ , γ_- and k .

Let us now define a new coefficient $\gamma_y(x)$ by extending trivially to the whole $\mathbb{R}^2$ the values that the coefficient $\gamma_{\mathcal{P}_0}$ assumes in $B_{r_1/\tilde{C}}(y)$. We take

$$\gamma_y(x) = \gamma^1 + (\gamma^2 - \gamma^1)\chi_{\{x_2 > a\}}(x) \quad \text{for every } x = (x_1, x_2) \in \mathbb{R}^2$$

with the same values of γ^1 and γ^2 and the same coordinate system as in (6.1). For some points y , $\gamma_y(x)$ is a constant function.

Let us denote by $\bar{\Gamma}$ the biphasic fundamental solution to

$$\operatorname{div}(\gamma_y(\cdot)\nabla\bar{\Gamma}(\cdot, y)) = \delta_y \text{ in } \mathbb{R}^2.$$

Proposition 6.1. *For each $c_1 > 1$, there exists a constant $C > 0$ depending only on the a priori data and on c_1 such that for any $y \in \Omega_0 \setminus \mathcal{P}^0$ with $\operatorname{dist}(y, \{P_i^0\}_{i=1}^M \cup \partial\Omega_0) \geq d_0/c_1$, then*

$$(6.2) \quad \|G_0(\cdot, y) - \bar{\Gamma}(\cdot, y)\|_{H^1(\Omega_0)} \leq C,$$

and, for any $\rho > 0$,

$$(6.3) \quad \|G_0(\cdot, y)\|_{H^1(\Omega_0 \setminus B_\rho(y))} \leq C \left| \ln \frac{D}{\rho} \right|^{1/2},$$

where D depends only on L and d_0 .

Let P be a point on $\partial\mathcal{P}^0$ such that $\operatorname{dist}(P, \{P_i^0\}_{i=1}^M) \geq d_0/c_1$ for some $c_1 > 1$ and let $y_r = P + rn(P)$, where $n(P)$ is the unit outer normal to $\partial\mathcal{P}^0$ and $r > 0$. Then for $r < \frac{d_0}{c_1\tilde{C}}$ and for $x \in \mathcal{P}^0 \cap B(P, d_0/c_1\tilde{C})$, we have

$$(6.4) \quad |\nabla G_0(x, y_r) - \nabla \bar{\Gamma}(x, y_r)| \leq C,$$

and

$$\nabla \bar{\Gamma}(x, y_r) = \frac{2}{\gamma^1 + k} \nabla \Gamma(x, y_r),$$

where $\gamma^1 = \gamma_+$ if $B(P, d_0/d_0/c_1\tilde{C})$ intersects Ω^+ or $\gamma^1 = \gamma_-$ if $B(P, d_0/d_0/c_1\tilde{C})$ intersects Ω^- and where $\Gamma(x, y)$ is the fundamental solution for the Laplacian operator.

Proof. The proof of (6.2) can be derived with similar arguments as in Proposition 3.4 in [8].

Observe that from (6.2) we have

$$(6.5) \quad \begin{aligned} \|G_0(\cdot, y)\|_{H^1(\Omega_0 \setminus B_\rho(y))} &\leq \|G_0(\cdot, y) - \bar{\Gamma}(\cdot, y)\|_{H^1(\Omega_0)} + \|\bar{\Gamma}(\cdot, y)\|_{H^1(\Omega_0 \setminus B_\rho(y))} \\ &\leq C + \|\bar{\Gamma}(\cdot, y)\|_{H^1(\Omega_0 \setminus B_\rho(y))}. \end{aligned}$$

Now,

$$(6.6) \quad \|\bar{\Gamma}(\cdot, y)\|_{H^1(\Omega_0 \setminus B_\rho(y))} \leq \|\bar{\Gamma}(\cdot, y)\|_{H^1(B_D(y) \setminus B_\rho(y))},$$

where $D = \sqrt{4L^2 + (2L + 2d_0)^2}$. Now, using the explicit representation for the bi-phase fundamental solution (see for example (4.26) in [5]) we have that

$$|\bar{\Gamma}(x, y)| \leq C |\ln |x - y||, \quad |\nabla \bar{\Gamma}(x, y)| \leq C |x - y|^{-1},$$

where C depends on γ^+ and γ^- and an easy computation shows that

$$(6.7) \quad \|\bar{\Gamma}(\cdot, y)\|_{H^1(B_D(y) \setminus B_\rho(y))} \leq C \left| \ln \frac{D}{\rho} \right|^{1/2}.$$

Hence, from last inequality, (6.6) and (6.5), inequality (6.3) follows. Finally, (6.4) follows from similar arguments as in Proposition 3.4 in [8]. $\square$

We will also need the following regularity result for solutions of elliptic equations with piecewise constant coefficients. The proposition below is a special case of Proposition 1.6 in [22].

Proposition 6.2. *Let B_r be the disk of radius $r > 0$ centered at the origin, and let $B_r^\pm$ be the upper and the lower half disk and let γ^1 and γ^2 be two positive constants.*

Let $v \in H^1(B_r)$ be a solution to

$$(6.8) \quad \operatorname{div} \left(\left(\gamma^1 + (\gamma^2 - \gamma^1) \chi_{B_r^+} \right) \nabla v \right) = 0 \text{ in } B_r.$$

Then $v \in C^\infty(\overline{B_r^+}) \cap C^\infty(\overline{B_r^-})$ and for every $\delta > 0$ there is a constant C depending only on γ^1 , γ^2 and δ such that

$$(6.9) \quad \|\nabla v\|_{L^\infty(B_{(1-\delta)r})} \leq C \|v\|_{L^2(B_r)}.$$

We are now ready to prove the main result of this section the proof of which will be divided in four steps.

Proposition 6.3. *There exists a constant $m_0 > 0$, depending only on the a priori data, such that*

$$(6.10) \quad \|F'(0)\|_* \geq m_0 d_H.$$

Proof. Let us set

$$V = (P_1^1 - P_1^0, P_2^1 - P_2^0, \dots, P_M^1 - P_M^0).$$

We observe that

$$(6.11) \quad C^{-1} d_H \leq |V| \leq C d_H,$$

for C depending only on the a priori data: the upper bound follows from Proposition 3.4, while the lower bound is just an immediate consequence of the fact that the two polygons have the same number of ordered vertices, hence the distance between sides of the polygons is bounded by the distance of the corresponding vertices.

Let us first normalize by the length of vector $|V|$ by setting

$$\tilde{\mathcal{U}} = \frac{\mathcal{U}}{|V|}, \quad \tilde{\mathcal{A}} = \frac{\mathcal{A}}{|V|},$$

and

$$H(f, g) = - \int_{\Omega} \gamma_{\mathcal{P}^0} \tilde{\mathcal{A}} \nabla u_0 \cdot \nabla v_0,$$

so that

$$F'(0, f, g) = |V| H(f, g).$$

Let

$$(6.12) \quad m_1 = \|H\|_* = \sup \left\{ \frac{|H(f, g)|}{\|f\|_{H^{1/2}(\partial\Omega)} \|g\|_{H^{1/2}(\partial\Omega)}} : f, g \neq 0 \right\},$$

be the operator norm of H , so that

$$(6.13) \quad |H(f, g)| \leq m_1 \|f\|_{H^{1/2}(\partial\Omega)} \|g\|_{H^{1/2}(\partial\Omega)} \text{ for every } f, g \in H^{1/2}(\partial\Omega).$$

First step. To start with we choose special boundary values f and g , by setting, for $y, z \in \Omega_0 \setminus \overline{\Omega}$, $\tilde{f} = G_0(\cdot, y)|_{\partial\Omega}$ and $\tilde{g} = G_0(\cdot, z)|_{\partial\Omega}$. With these choices, the corresponding solutions are

$$(6.14) \quad G_0(x, y) \quad \text{and} \quad G_0(x, z),$$

that we will continue to denote by u_0 and v_0 for sake of brevity. Notice that, since $y, z \in \Omega_0 \setminus \overline{\Omega}$, then $u_0, v_0 \in H^1(\Omega)$. Let us define

$$S_0(y, z) = - \int_{\Omega} \gamma_{\mathcal{P}^0} \tilde{\mathcal{A}} \nabla u_0 \cdot \nabla v_0 = - \int_{\Omega} \gamma_{\mathcal{P}^0} \tilde{\mathcal{A}} \nabla G_0(\cdot, y) \cdot \nabla G_0(\cdot, z).$$

Notice that, for $y, z \in D_0 = [-\frac{d_0}{2}, \frac{d_0}{2}] \times [L + \frac{d_0}{2}, L + \frac{3d_0}{2}]$,

$$S_0(y, z) = H(G_0(\cdot, y)|_{\partial\Omega}, G_0(\cdot, z)|_{\partial\Omega}),$$

hence, by (6.13), by (6.3) of Proposition 6.1 with $\rho = d_0/4$, and, finally, by the Trace Theorem we have

$$(6.15) \quad |S_0(y, z)| \leq C m_1 \left| \ln \frac{4D}{d_0} \right| \text{ for } y, z \in D_0,$$

where C depends on the a priori data.

Second step. We now use the properties of the function S_0 to go from the distributed integral representation in Ω to a boundary representation formula on $\partial\mathcal{P}^0$ far from the vertices.

By assumptions (2.4) and (2.8), there exists a constant C_0 depending only on the a priori data, such that

$$\text{dist}(P_j^0, P_l^0) > \frac{2d_0}{C_0}, \text{ if } j \neq l,$$

and

$$B\left(P_j^0, \frac{d_0}{C_0}\right) \text{ does not intersect sides of } \mathcal{P}^0 \text{ that do not contain } P_j^0.$$

Let $\mathcal{B} = \cup_{j=1}^M B\left(P_j^0, \frac{d_0}{4C_0}\right)$ and let us write

$$(6.16) \quad S_0(y, z) = - \int_{\Omega \setminus \mathcal{B}} \gamma_{\mathcal{P}^0} \tilde{\mathcal{A}} \nabla u_0 \cdot \nabla v_0 - \int_{\mathcal{B}} \gamma_{\mathcal{P}^0} \tilde{\mathcal{A}} \nabla u_0 \cdot \nabla v_0.$$

Since $\gamma_{\mathcal{P}^0}$ is piece-wise constant, the solutions u_0 and v_0 are harmonic in each domain where $\gamma_{\mathcal{P}^0}$ is constant, that is in $\Omega_+ \setminus \mathcal{P}^0$, $\Omega_- \setminus \mathcal{P}^0$ and $\mathcal{P}^0$. In each of these sets

$$\tilde{\mathcal{A}} \nabla u_0 \cdot \nabla v_0 = -\text{div}(b),$$

where

$$b = \left(\tilde{\mathcal{U}} \cdot \nabla u_0 \right) \nabla v_0 + \left(\tilde{\mathcal{U}} \cdot \nabla v_0 \right) \nabla u_0 - (\nabla v_0 \cdot \nabla u_0) \tilde{\mathcal{U}},$$

(see [14] for details). We can write

$$(6.17) \quad \int_{\Omega \setminus \mathcal{B}} \gamma_{\mathcal{P}^0} \tilde{\mathcal{A}} \nabla u_0 \cdot \nabla v_0 = -\gamma_+ \int_{\Omega_+ \setminus (\mathcal{P}^0 \cup \mathcal{B})} \operatorname{div}(b) - \gamma_- \int_{\Omega_- \setminus (\mathcal{P}^0 \cup \mathcal{B})} \operatorname{div}(b) - k \int_{\mathcal{P}^0 \setminus \mathcal{B}} \operatorname{div}(b).$$

For any function u defined in Ω , let us use the following notation:

$$u^+ = u|_{\overline{\Omega}_+ \setminus \mathcal{P}^0}, \quad u^- = u|_{\overline{\Omega}_- \setminus \mathcal{P}^0}, \quad u^i = u|_{\overline{\mathcal{P}^0}}.$$

Let us now denote by n the unit outer normal to $\partial\Omega$, to $\partial\mathcal{P}^0$, and to the balls in $\mathcal{B}$ and by e_2 the unit vertical normal vector.

Integrating by parts and recalling that $\operatorname{supp}(\tilde{\mathcal{U}}) \subset W$, hence $b = 0$ on $\partial\Omega$, we have

$$(6.18) \quad \int_{\Omega_+ \setminus (\mathcal{P}^0 \cup \mathcal{B})} \operatorname{div}(b^+) = - \int_{(\partial\mathcal{B} \cap \Omega_+) \setminus \mathcal{P}^0} b^+ \cdot n - \int_{(\partial\mathcal{P}^0 \cap \Omega_+) \setminus \mathcal{B}} b^+ \cdot n - \int_{\Sigma \setminus \mathcal{P}^0} b^+ \cdot e_2,$$

$$(6.19) \quad \int_{\Omega_- \setminus (\mathcal{P}^0 \cup \mathcal{B})} \operatorname{div}(b^-) = - \int_{(\partial\mathcal{B} \cap \Omega_-) \setminus \mathcal{P}^0} b^- \cdot n - \int_{(\partial\mathcal{P}^0 \cap \Omega_-) \setminus \mathcal{B}} b^- \cdot n + \int_{\Sigma \setminus \mathcal{P}^0} b^- \cdot e_2,$$

and

$$(6.20) \quad \int_{\mathcal{P}^0 \setminus \mathcal{B}} \operatorname{div}(b^i) = \int_{\partial\mathcal{P}^0 \setminus \mathcal{B}} b^i \cdot n - \int_{\partial\mathcal{B} \cap \mathcal{P}^0} b^i \cdot n.$$

By (6.17), (6.18), (6.19) and (6.20) we get

$$(6.21) \quad \int_{\Omega \setminus \mathcal{B}} \gamma_{\mathcal{P}^0} \tilde{\mathcal{A}} \nabla u_0 \cdot \nabla v_0 = \int_{\partial\mathcal{B}} \gamma_{\mathcal{P}^0} b \cdot n + \int_{\Sigma \setminus \mathcal{P}^0} [\gamma_{\mathcal{P}^0} b \cdot e_2] - \int_{\partial\mathcal{P}^0 \setminus \mathcal{B}} [\gamma_{\mathcal{P}^0} b \cdot n],$$

where $[\]$ denotes the jump.

By definition, since $\mathcal{U}$ is zero on $\Sigma \setminus W$, affine on $(W \cap \Sigma) \setminus \mathcal{P}^0$ and (4.1) holds for $P_j^0 \in \Sigma$, we have $\tilde{\mathcal{U}} \cdot e_2 = 0$ on Σ . By transmission conditions for u_0 and v_0 on Σ , we have $\gamma_+ \nabla u_0^+ \cdot e_2 = \gamma_- \nabla u_0^- \cdot e_2$ and $\gamma_+ \nabla v_0^+ \cdot e_2 = \gamma_- \nabla v_0^- \cdot e_2$ while $\tilde{\mathcal{U}} \cdot \nabla u_0^+ = \tilde{\mathcal{U}} \cdot \nabla u_0^-$ and $\tilde{\mathcal{U}} \cdot \nabla v_0^+ = \tilde{\mathcal{U}} \cdot \nabla v_0^-$. For these reasons,

$$(6.22) \quad [\gamma_{\mathcal{P}^0} b \cdot e_2] = 0 \text{ on } \Sigma.$$

In a similar way, by transmission conditions on $\partial\mathcal{P}^0$, we can write

$$(6.23) \quad [\gamma_{\mathcal{P}^0} b \cdot n] = (\tilde{\mathcal{U}} \cdot n) (k - \gamma_+) \mathcal{M}^+ \nabla u_0^i \cdot \nabla v_0^i \text{ on } \partial\mathcal{P}^0 \setminus \mathcal{B} \cap \Omega_+,$$

where $\mathcal{M}^+$ is a tensor with eigenvectors n and $n^\perp$ with eigenvalues $\frac{k}{\gamma_+}$ and 1, respectively, and

$$(6.24) \quad [\gamma_{\mathcal{P}^0} b \cdot n] = (\tilde{\mathcal{U}} \cdot n) (k - \gamma_-) \mathcal{M}^- \nabla u_0^i \cdot \nabla v_0^i \text{ on } \partial\mathcal{P}^0 \setminus \mathcal{B} \cap \Omega_-,$$

where $\mathcal{M}^-$ is a tensor with eigenvectors n and $n^\perp$ with eigenvalues $\frac{k}{\gamma_-}$ and 1.

Hence, finally, by (6.16), (6.21), (6.22), (6.23) and (6.24),

$$(6.25) \quad S_0(y, z) = - \int_{\mathcal{B}} \gamma_{\mathcal{P}^0} \tilde{\mathcal{A}} \nabla u_0 \cdot \nabla v_0 - \int_{\partial\mathcal{B}} \gamma_{\mathcal{P}^0} b \cdot n + \int_{\partial\mathcal{P}^0 \setminus \mathcal{B}} (\tilde{\mathcal{U}} \cdot n) (k - \gamma_{\mathcal{P}^0}) \mathcal{M} \nabla u_0^i \cdot \nabla v_0^i$$

where $\mathcal{M} = \mathcal{M}^+ \chi_{\Omega_+ \cap \partial\mathcal{P}^0} + \mathcal{M}^- \chi_{\Omega_- \cap \partial\mathcal{P}^0}$.

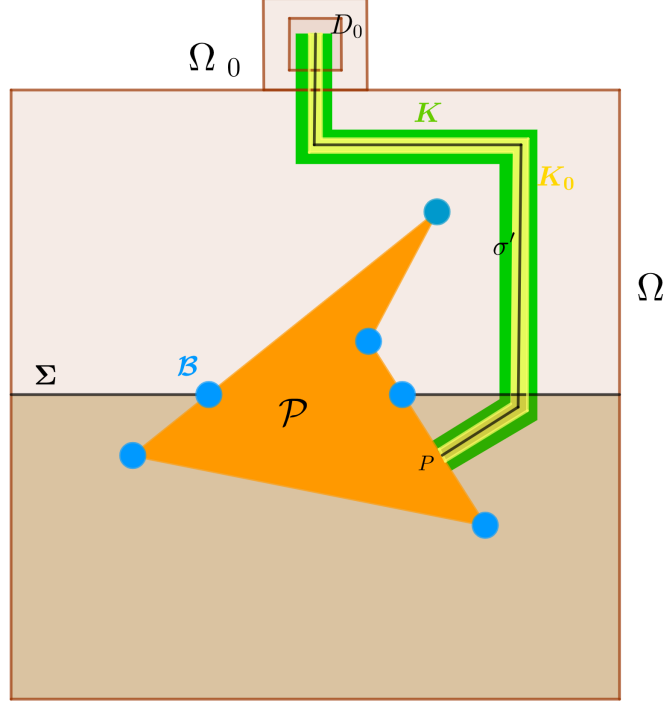


Figure 2. The path from D_0 to P .

Third step. We now use the properties of S_0 to propagate estimate (6.15) up to points that are close to the sides of $\mathcal{P}^0$ but far from the vertices.

From formula (6.25) we deduce that S_0 is well defined for $y, z \in \Omega_0 \setminus \overline{\mathcal{P}^0 \cup \mathcal{B}}$ and, recalling (6.14), we have that S_0 solves the equation

$$(6.26) \quad \operatorname{div}(\gamma_{\mathcal{P}^0} \nabla S_0) = 0 \text{ in } \Omega_0 \setminus \overline{\mathcal{P}^0 \cup \mathcal{B}},$$

both with respect to y and to z .

Let us now consider any segment $P_j^0 P_{j+1}^0$ of $\partial \mathcal{P}^0$ and let P be the mid-point of such segment.

By Assumptions on (2.6)-(2.8) the disk $B(P, \frac{d_0}{C_0})$ intersects $\partial \mathcal{P}^0$ only on the side containing $P_j^0 P_{j+1}^0$ and it does not intersect Σ .

Let σ be a simple curve joining the point $P + \frac{d_0}{C_0} n(P)$ to the point $Q_0 = (0, L + d_0)$ such that $\sigma \subset \Omega_0$ and $\operatorname{dist}(\sigma, \mathcal{P}^0) \leq \frac{d_0}{C_0}$. Let $\sigma' = \sigma \cup \{P + tn(P) : t \in [0, \frac{d_0}{C_0}]\}$.

Let $K = \left\{x \in \Omega_0 \setminus \mathcal{P}^0 : \operatorname{dist}(x, \sigma') < \frac{d_0}{4C_0}\right\}$ and $K_0 = \left\{x \in \Omega_0 \setminus \mathcal{P}^0 : \operatorname{dist}(x, \sigma') < \frac{d_0}{8C_0}\right\}$. See Figure 2 for an example.

The function S_0 solves in K the equation $\operatorname{div}(\gamma_{\mathcal{P}^0} \nabla S_0) = 0$ with respect to both y and z . Let us now estimate function $S_0(y, z)$ for points $y, z \in K$.

First of all by (6.3) we notice that, since $\operatorname{dist}(K, \mathcal{B}) \geq \tilde{r}_0 := \frac{d_0}{4C_0}$, then

$$\|\nabla u_0\|_{L^2(\mathcal{B})} \leq \|G_0(\cdot, y)\|_{H^1(\Omega_0 \setminus B_{\tilde{r}_0}(y))} \leq C,$$

where C depends only on the a priori constants. Since the same holds for v_0 , we have

$$(6.27) \quad \left| \int_{\mathcal{B}} \gamma_{\mathcal{P}^0} \tilde{\mathcal{A}} \nabla u_0 \cdot \nabla v_0 \right| \leq C \|\nabla u_0\|_{L^2(\mathcal{B})} \|\nabla v_0\|_{L^2(\mathcal{B})} \leq C.$$

Let us now notice that we can cover $\partial\mathcal{B}$ by a finite number of balls of radius $\frac{d_0}{16C_0}$ such that, in each ball of radius $\frac{d_0}{8C_0}$ concentric to them, the function u_0 solves an equation of the form (6.8) with $h = g = 0$.

By combining (6.9) with (6.3) and the fact that the distance between K and the largest balls covering $\partial\mathcal{B}$ is bigger than $\tilde{r}_0$, then $\|\nabla u_0\|_{L^\infty(\partial\mathcal{B})}$ and $\|\nabla v_0\|_{L^\infty(\partial\mathcal{B})}$ are bounded by a constant depending only on the a priori constants and, hence,

$$(6.28) \quad \left| \int_{\partial\mathcal{B}} \gamma_{\mathcal{P}^0} b \cdot n \right| \leq C.$$

For a similar reason, for points on $\partial\mathcal{P}^0 \setminus (\mathcal{B} \cup B(P, \frac{d_0}{C_0}))$ we can bound the L^∞ norms of u_0 and v_0 for point y and z in K and

$$(6.29) \quad \left| \int_{\partial\mathcal{P}^0 \setminus (\mathcal{B} \cup B(P, \frac{d_0}{C_0}))} (\tilde{\mathcal{U}} \cdot n) (k - \gamma_{\mathcal{P}^0}) \mathcal{M} \nabla u_0^i \cdot \nabla v_0^i \right| \leq C.$$

Now, finally, we consider

$$\int_{\partial\mathcal{P}^0 \cap B(P, \frac{d_0}{C_0})} (\tilde{\mathcal{U}} \cdot n) (k - \gamma_{\mathcal{P}^0}) \mathcal{M} \nabla u_0^i \cdot \nabla v_0^i,$$

and notice that, if y, z are at positive fixed distance from $\partial\mathcal{P}^0 \cap B(P, \frac{d_0}{C_0})$, then we can again use (6.3) and (6.9) and bound the L^∞ norm of ∇u_0^i and ∇v_0^i . On the other hand, for points y and z close to $\partial\mathcal{P}^0 \cap B(P, \frac{d_0}{C_0})$ we can use (6.4) and the explicit formula for Γ to finally get that

$$(6.30) \quad \left| \int_{\partial\mathcal{P}^0 \cap B(P, \frac{d_0}{C_0})} (\tilde{\mathcal{U}} \cdot n) (k - \gamma_{\mathcal{P}^0}) \mathcal{M} \nabla u_0^i \cdot \nabla v_0^i \right| \leq C (d_y d_z)^{-1/2},$$

where $d_y = \text{dist}(y, \mathcal{P}^0)$ and $d_z = \text{dist}(z, \mathcal{P}^0)$.

By putting together (6.25) and (6.27)-(6.30) we finally get, for every $y, z \in K$

$$(6.31) \quad |S_0(y, z)| \leq C (d_y d_z)^{-1/2}.$$

Let us now recall that S_0 solves (6.26) with respect to both variables, that is bounded by (6.15) if $y, z \in D_0 = [-\frac{d_0}{2}, \frac{d_0}{2}] \times [L + \frac{d_0}{2}, L + \frac{3d_0}{2}]$ and is bounded by (6.31) for $y, z \in K$.

Let us fix z in D_0 and consider S_0 as a function of y only.

We take $Q_0 = (0, L + d_0)$ and $\tilde{r} = \frac{d_0}{2}$, so that, by (6.15) and recalling (6.12)

$$(6.32) \quad \|S_0(\cdot, z)\|_{L^2(B_{\tilde{r}}(Q_0))} \leq C m_1.$$

Let us set $\tilde{K} = \left\{ x \in K : d(x, \mathcal{P}^0) \geq \frac{d_0}{4C_0} \right\}$ and $\tilde{K}_0 = \left\{ x \in K_0 : d(x, \mathcal{P}^0) \geq \frac{d_0}{2C_0} \right\}$.

By (6.31) we have that there is a constant C depending only on the a priori data, such that

$$(6.33) \quad \|S_0(\cdot, z)\|_{L^2(\tilde{K})} \leq C.$$

We now apply Theorem 5.1 in [2] that gives an estimate on smallness propagation for solutions of elliptic differential equations and that, in the 2 dimensional setting, holds for L^∞ coefficients (for a similar result, for coefficients with jumps, in any dimension see [15]). Notice that, in the present case, the right hand side of the equation is null and we take $\Omega = \tilde{K}$. There exist positive constants δ and C depending only on the a priori data, such that, by (6.32) and (6.33) we have

$$(6.34) \quad \|S_0(\cdot, z)\|_{L^2(\tilde{K}_0)} \leq Cm_1^\delta.$$

In particular, since the disk centered at $\bar{Q} = P + \frac{5d_0}{8C_0}n(P)$ of radius $R_1 = \frac{d_0}{8C_0}$ is contained in $\tilde{K}_0$, we have

$$(6.35) \quad \|S_0(\cdot, z)\|_{L^2(B_{R_1}(\bar{Q}))} \leq Cm_1^\delta \quad \forall z \in D_0.$$

Now let us notice that, for fixed z , S_0 is a harmonic function in $\{x \in K : \text{dist}(x, \mathcal{P}^0) \leq \frac{d_0}{C_0}\}$ hence, since (6.35) holds, by the mean value property and Hölder inequality

$$(6.36) \quad |S_0(y, z)| \leq Cm_1^\delta,$$

for $y \in B_{R_1/2}(\bar{Q})$ and $z \in D_0$.

Let us now fix $y \in B_{R_1/2}(\bar{Q})$ and consider the function $S_0(y, \cdot)$. By the same procedure as before we have that

$$(6.37) \quad \|S_0(y, \cdot)\|_{L^2(B_{R_1}(\bar{Q}))} \leq Cm_1^{\delta^2} \quad \forall y \in B_{R_1/2}(\bar{Q}).$$

We now apply a classical three sphere inequality for harmonic functions (see, for example, [4, Appendix E.2]) to the function $S_0(\cdot, z)$ in the spheres

$$B_{\bar{R}_1}(\bar{Q}) \subset B_{\bar{R}_2}(\bar{Q}) \subset B_{\bar{R}_3}(\bar{Q}),$$

for

$$\bar{R}_1 = R_1/2, \quad \bar{R}_2 = \frac{d_0}{2C_0} - \frac{r}{2} \quad \text{and} \quad \bar{R}_3 = \frac{d_0}{2C_0} - \frac{r}{4},$$

where r will be chosen afterwards. We have that, for every $z \in B_{\bar{R}_1}(\bar{Q})$ and for

$$(6.38) \quad \theta_r = \frac{\ln(\bar{R}_3/\bar{R}_2)}{\ln(\bar{R}_3/\bar{R}_1)},$$

then

$$(6.39) \quad \|S_0(\cdot, z)\|_{L^2(B_{\bar{R}_2}(\bar{Q}))} \leq \|S_0(\cdot, z)\|_{L^2(B_{\bar{R}_1}(\bar{Q}))}^{\theta_r} \|S_0(\cdot, z)\|_{L^2(B_{\bar{R}_3}(\bar{Q}))}^{1-\theta_r},$$

hence, by (6.31) and (6.37) we have

$$(6.40) \quad \|S_0(\cdot, z)\|_{L^2(B_{\bar{R}_2}(\bar{Q}))} \leq C \left(\frac{1}{r}\right)^{\frac{1-\theta_r}{2}} m_2^{\theta_r} \quad \forall z \in B_{\bar{R}_1}(\bar{Q}),$$

where (see (6.37))

$$m_2 = Cm_1^{\delta^2}.$$

Recalling that $S_0(\cdot, z)$ is a harmonic function, by the mean value properties and Hölder inequality, we have, for $y_r = P + rn(P)$

$$(6.41) \quad |S_0(y_r, z)| \leq C \left(\frac{1}{r} \right)^{1+\frac{1-\theta_r}{2}} m_2^{\theta_r} \quad \forall z \in B_{\bar{R}_1}(\bar{Q}).$$

We now consider $S_0(y_r, \cdot)$ in the same disks as before and we finally get

$$(6.42) \quad |S_0(y_r, y_r)| \leq C \left(\frac{1}{r} \right)^{1+\theta_r+(1-\theta_r)(\frac{\theta_r}{2}+1)} m_2^{\theta_r^2}.$$

From (6.38) it is straightforward to see that

$$\left(\frac{1}{r} \right)^{\theta_r} = 1 + o(1), \quad \text{as } r \rightarrow 0,$$

which implies

$$(6.43) \quad |S_0(y_r, y_r)| \leq C \frac{1}{r^2} m_2^{\theta_r^2}.$$

Fourth step. We now use the fact that S_0 blows up when approaching any side of the polygon far from the vertices to conclude the proof.

By using estimate (6.4) and proceeding similarly as in [9, (3.21)] we have for $r \leq \frac{d_0}{8C_0}$ and $y_r = P + rn(P)$

$$(6.44) \quad |S_0(y_r, y_r)| \geq \frac{C\tilde{\mathcal{U}}(P) \cdot n(P)}{r} - C |\ln r|,$$

and, by comparing (6.43) with (6.44) we get

$$(6.45) \quad \left| \tilde{\mathcal{U}}(P) \cdot n(P) \right| \leq C \left(r |\ln r| + \frac{1}{r} m_2^{\theta_r^2} \right).$$

If $m_2 \leq \exp(-(48)^4)$ i.e.

$$(6.46) \quad m_1 \leq \left(\frac{\exp(-(48)^4)}{C} \right)^{1/\delta^2} := C_1^{-1},$$

(where C_1 depends only on a priori data), then we can pick up

$$r = 96\bar{R}_1 |\ln(m_2)|^{-1/4} = \frac{6d_0}{C_0} |\ln(m_2)|^{-1/4}$$

in (6.45) and, after some straightforward estimations, we end up with the following bound

$$\left| \tilde{\mathcal{U}}(P) \cdot n(P) \right| \leq C |\ln(m_2)|^{-1/5},$$

which, recalling the definition of m_2 , can be written as

$$(6.47) \quad \left| \tilde{\mathcal{U}}(P) \cdot n(P) \right| \leq \omega_0(m_1),$$

where $\omega_0(t)$ is an increasing concave function such that $\lim_{t \rightarrow 0^+} \omega_0(t) = 0$.

Notice that, with a similar procedure, this estimate can be obtained for each point in a neighborhood of P on the side $P_j^0 P_{j+1}^0$. Since $\tilde{\mathcal{U}}$ is affine on $P_j^0 P_{j+1}^0$, we get that the estimate holds at the endpoint of the segment as well, hence

$$|\tilde{\mathcal{U}}(P_j^0) \cdot n_j| \leq \omega_0(m_1),$$

and

$$|\tilde{\mathcal{U}}(P_{j+1}^0) \cdot n_j| \leq \omega_0(m_1),$$

where n_j denotes the outer normal direction to $P_j^0 P_{j+1}^0$ and ω_0 is the function in (6.47) multiplied by a constant depending on the a priori parameters.

In particular, by (4.1), for each P_j^0 , we have

$$(6.48) \quad \left| \frac{P_j^0 - P_j^1}{|V|} \cdot n \right| \leq \omega_0(m_1),$$

for each n normal to sides through P_j^0 .

Let j_0 be such that $|P_{j_0}^0 - P_{j_0}^1| = \max_{j=1,\dots,M} |P_j^0 - P_j^1|$. By construction $P_{j_0}^0$ is an endpoint of $\mathcal{P}^0$ (and not an intersection with Σ) and $\frac{|P_{j_0}^0 - P_{j_0}^1|}{|V|} \geq \frac{1}{M}$. Moreover, since there are two linearly independent unit directions n for which (6.48) holds for $j = j_0$, then it holds for every unit direction, and, by choosing $\tilde{n}$ parallel to $P_{j_0}^0 - P_{j_0}^1$ we get

$$\frac{1}{M} \leq \frac{|P_{j_0}^0 - P_{j_0}^1|}{|V|} = \left| \frac{P_{j_0}^0 - P_{j_0}^1}{|V|} \cdot \tilde{n} \right| \leq \omega_0(m_1),$$

from which

$$(6.49) \quad m_1 \geq \omega_0^{-1}\left(\frac{1}{M}\right).$$

Hence, recalling (6.46),

$$m_1 \geq \min\left(\omega_0^{-1}\left(\frac{1}{M}\right), C_1^{-1}\right).$$

Finally, from (6.11) and the fact that

$$\|F'(0)\|_* = |V| \|H\|_*$$

we get

$$\|F'(0)\|_* \geq m_0 d_H,$$

with $m_0 = C^{-1} \min\left(\omega_0^{-1}\left(\frac{1}{M}\right), C_1^{-1}\right)$. □

Remark 6.4. *We performed the proof in the simpler case of a single interface Σ , but the same proof works in the case of multiple interfaces Σ_i for $i = 1, \dots, m$, as long as we control the distance between interfaces (see Assumption (2.2)) and only consider polygons whose vertices are far from the interfaces (see Assumption (2.8)). In that case the proof is only technically more involved to write down in the integration by parts that lead to formula (6.25). See Figure 3. The case where the vertices are allowed to get close to an interface or touch it is more complicated because the definition of the map $\mathcal{U}$ is not at all straightforward, but not impossible to treat and will be object of future work.*

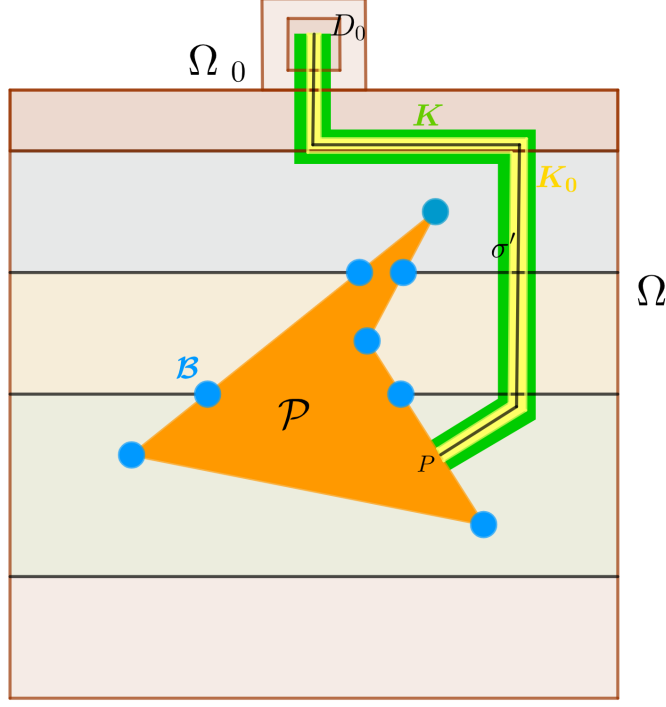


Figure 3. The path from D_0 to P in the general case

6.1. Lipschitz stability estimate. In this section we conclude the proof of Theorem 2.1. From Proposition 6.3 we have that there exist f_0 and g_0 in $H^{1/2}(\partial\Omega)$ such that

$$|F'(0, f_0, g_0)| \geq \frac{m_0}{2} d_H \|f_0\|_{H^{1/2}(\partial\Omega)} \|g_0\|_{H^{1/2}(\partial\Omega)}.$$

Let us first assume that $\|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_* \leq \varepsilon_0$, and let f_0 and g_0 be the functions that satisfy (6.10) in Proposition 6.3. By (6.10) and by (5.8), we have

$$\begin{aligned} |\langle (\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1})(f_0), g_0 \rangle| &= |F(1, f_0, g_0) - F(0, f_0, g_0)| = \left| \int_0^1 F'(t, f_0, g_0) dt \right| \\ &\geq |F'(0, f_0, g_0)| - \int_0^1 |F'(t, f_0, g_0) - F'(0, f_0, g_0)| dt \\ &\geq (m_0 - C d_H^\beta) d_H \|f_0\|_{H^{1/2}(\partial\Omega)} \|g_0\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

By (3.4), there exists $\varepsilon_1 \in (0, \varepsilon_0)$ depending only on a priori constant such that, if

$$\|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_* \leq \varepsilon_1,$$

then

$$(m_0 - C d_H^\beta) \geq m_0/2,$$

and

$$(6.50) \quad d_H = d_H(\partial\mathcal{P}^0, \partial\mathcal{P}^1) \leq \frac{2}{m_0} \|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_*,$$

and the claim follows if $\|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_* \leq \varepsilon_1$ (where ε_1 depend only on the a priori data).

If, now, $\|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_* \geq \varepsilon_1$ (and, hence, also if $\|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_* \geq \varepsilon_0$), since the following trivial inequality holds

$$d_H(\partial\mathcal{P}^0, \partial\mathcal{P}^1) \leq 2L,$$

we easily derive

$$(6.51) \quad d_H(\partial\mathcal{P}^0, \partial\mathcal{P}^1) \leq 2L \leq 2L \frac{\|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_*}{\varepsilon_1} \leq C \|\Lambda_{\mathcal{P}^0} - \Lambda_{\mathcal{P}^1}\|_*.$$

□

We now proceed with the proof of Theorem (2.2)

Proof. As in the proof of Theorem 2.1 we distiguish two cases.

In the first case $\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_* > \varepsilon_1$, hence

$$\begin{aligned} \|\gamma_0 - \gamma_1\|_{L^1(\Omega)} &= C |\mathcal{P}^0 \Delta \mathcal{P}^1| \leq C |\Omega| \\ &\leq C' \frac{\|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_*}{\varepsilon_1}. \end{aligned}$$

In the second case, from (6.50) and Proposition 3.4, it follows that $\mathcal{P}^0$ and $\mathcal{P}^1$ have the same number of vertices that can be ordered so that

$$(6.52) \quad \text{dist}(P_i^0, P_i^1) \leq C \|\Lambda_{\gamma_0} - \Lambda_{\gamma_1}\|_*.$$

Let $d_1 = \max_{i=1, \dots, N} \text{dist}(P_i^0, P_i^1)$. We have that

$$\mathcal{P}^1 \subset (\mathcal{P}^0)_{d_1} = \{x \in \mathbb{R}^2 : \text{dist}(x, \mathcal{P}^0) \leq d_1\},$$

and, hence,

$$\mathcal{P}^1 \setminus \mathcal{P}^0 \subset (\mathcal{P}^0)_{d_1} \setminus \mathcal{P}^0,$$

that gives

$$|\mathcal{P}^1 \setminus \mathcal{P}^0| \leq |(\mathcal{P}^0)_{d_1} \setminus \mathcal{P}^0| \leq 2N_0 d_0 d_1,$$

so that

$$|\mathcal{P}^0 \Delta \mathcal{P}^1| \leq C d_1,$$

and, by (6.52) we have (2.10). □

7. CONCLUSIONS

We have derived Lipschitz stability estimates for polygonal conductivity inclusions in a layered medium in terms of the DN map, using its differentiability properties.

In order to reduce as much as possible the technicalities of the proofs we have considered straight layers though we expect that the result can be generalized to the case of piecewise linear layers.

Also, let us observe that our result extends thoroughly to the case of finitely many conductivity polygonal inclusions $\mathcal{P}_j, j = 1, \dots, M$ contained in Ω if they are at controlled

distance from $\partial\Omega$ and from each other, and all vertices are at a given positive distance from the layers. In fact, under these assumptions we can still propagate the information on the data from $\partial\Omega$ to $\Omega \setminus \bigcup_{j=1,\dots,M} \overline{\mathcal{P}_j}$, establish a lower bound for the derivative of F and, using similar arguments as in the proof of the main result, derive a Lipschitz estimate.

To reduce as much as possible the technicalities of the paper, we have used a logarithmic stability result known in the literature, [16]. On the other hand, we can extend our main Lipschitz stability result in terms of the local DN map either adapting the results obtained in [1], or observing that we can use the Green's functions with localized trace on the boundary, used in the proof of Proposition 6.3, to obtain a logarithmic stability in terms of the local DN map, see [20] for a similar result.

Finally, it is relevant for the geophysical application we have in mind to extend the results of stability and reconstruction to the 3-D setting possibly considering an inhomogeneous and/or anisotropic medium. This case is not at all straightforward since differentiability properties of the DN map in this case are not known.

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