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**VARIATIONAL p -HARMONIOUS FUNCTIONS:
EXISTENCE AND CONVERGENCE
TO p -HARMONIC FUNCTIONS**

E. W. CHANDRA, M. ISHIWATA, R. MAGNANINI, AND H. WADADE

ABSTRACT. In a recent paper, the last three authors showed that a game-theoretic p -harmonic function v is characterized by an asymptotic mean value property with respect to a kind of mean value $\nu_p^\varepsilon[v](x)$ defined variationally on balls $B_r(x)$. In this paper, in a domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, we consider the operator μ_p^ε , acting on continuous functions on $\overline{\Omega}$, defined by the formula $\mu_p^\varepsilon[v](x) = \nu_p^{r_\varepsilon(x)}[v](x)$, where $r_\varepsilon(x) = \min[\varepsilon, \text{dist}(x, \Gamma)]$ and Γ denotes the boundary of Ω . We first derive various properties of μ_p^ε such as continuity and monotonicity. Then, we prove the existence and uniqueness of a function $u^\varepsilon \in C(\overline{\Omega})$ satisfying the Dirichlet-type problem:

$$u(x) = \mu_p^\varepsilon[u](x) \text{ for every } x \in \Omega, \quad u = g \text{ on } \Gamma,$$

for any given function $g \in C(\Gamma)$. This result holds, if we assume the existence of a suitable notion of barrier for all points in Γ . That u^ε is what we call the *variational p -harmonious function* with Dirichlet boundary data g , and is obtained by means of a Perron-type method based on a comparison principle.

We then show that the family $\{u^\varepsilon\}_{\varepsilon>0}$ gives an approximation scheme for the viscosity solution $u \in C(\overline{\Omega})$ of

$$\Delta_p^G u = 0 \text{ in } \Omega, \quad u = g \text{ on } \Gamma,$$

where Δ_p^G is the so-called game-theoretic (or homogeneous) p -Laplace operator. In fact, we prove that u^ε converges to u , uniformly on $\overline{\Omega}$ as $\varepsilon \rightarrow 0$.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we introduce and study what we call *variationally p -harmonious functions*. Their definition is based on a variational notion of average for functions in a Lebesgue space. In fact, we define a p -mean of a function $v \in L^p(E)$ as a real number $\nu_p[v]$ such that

$$(1.1) \quad \|v - \nu_p[v]\|_{p,E} = \min_{\nu \in \mathbb{R}} \|v - \nu\|_{p,E}.$$

Here, $L^p(E)$ denotes the standard Lebesgue space for $1 \leq p \leq \infty$ (equipped with a suitably normalized norm), where E is a Lebesgue measurable set in \mathbb{R}^N with finite measure. The existence and uniqueness of $\nu_p[v]$ is guaranteed for $1 < p < \infty$, since it is the projection of v on the subspace of constant functions. Also, $\nu_\infty[v]$ can be explicitly determined. When $p = 1$, in order to uniquely determine $\nu_1[v]$, we need to assume that v is continuous in an open set E . In general, $\nu_p[v]$ is nonlinear in v , unless p equals 2, in which case we recover the standard mean value of a function.

If Ω is a bounded domain in \mathbb{R}^N , we can associate with ν_p the operator μ_p^ε that acts on a function $v \in C(\overline{\Omega})$ — the space of continuous functions on $\overline{\Omega}$ — by the rule

$$\mu_p^\varepsilon[v](x) = \nu_p^{r_\varepsilon(x)}[v](x) \text{ for } x \in \overline{\Omega},$$

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where $\nu_p^r[v]$ denotes the p -mean of v , when we choose E as the ball $B_r(x)$, centered at $x \in \mathbb{R}^N$ and with radius $r > 0$, and $r_\varepsilon(x) = \min[\varepsilon, \text{dist}(x, \Gamma)]$ for $x \in \overline{\Omega}$. Here, $\text{dist}(x, \Gamma)$ denotes the distance of a point $x \in \overline{\Omega}$ to the boundary Γ of Ω (see Section 2.1 for details). The use of the function r_ε is inspired by the work in [10, 11], in which, for the planar case, it is associated to another family of nonlinear means.

By Theorem 2.7 below, μ_p^ε maps the space $C(\overline{\Omega})$ into itself. Thus, it is natural to study the set of functions $v \in C(\Omega)$ satisfying the fixed-point equation:

$$v = \mu_p^\varepsilon[v] \quad \text{in } \Omega.$$

We call these functions *variationally p -harmonious* in Ω . Affine functions are p -harmonious for any $p \in [1, \infty]$. Classical harmonic functions are 2-harmonious. For $p \neq 2$, p -harmonic functions are not a subset of p -harmonious functions. Needless to say, functions that satisfy in Ω the inequalities $v \leq \mu_p^\varepsilon[v]$ or $v \geq \mu_p^\varepsilon[v]$ will be named in this paper *variationally p -subharmonious* or *p -superharmonious*.

Harmonious functions have been introduced in [16], for the case $p = \infty$, in relationship with the so-called absolutely minimizing Lipschitz extensions. Classes of p -harmonious functions for a general $p \in (1, \infty]$ have been introduced in [21, 22], and in [10, 11] also for the case $p = 1$ in the plane, in connection with dynamic programming principles associated with *tug-of-war* games (see also [24, 25]). We also refer the reader to [1, 6, 9, 10, 11, 17, 18, 19] for other related works that motivate these issues.

The classes of p -harmonious functions just mentioned are based on a definition of p -mean different from that given in (1.1). That mean, although defined explicitly, occasionally fails to fulfil certain desired properties, such as monotonicity and continuity. Instead, the p -mean defined variationally in (1.1), whose study in the context of p -harmonious functions was introduced in [13], naturally enjoys those properties. In particular, as we shall see in Section 5, it satisfies the structural requirements, codified in [8], which yield the convergence of the underlying dynamic programming principles to p -harmonic functions.

In Section 3, we will show that variationally p -harmonious functions satisfy the weak and strong comparison principle, despite they are not solutions of a partial differential equation. These results are a by-product of the variational definition in (1.1) and are corollaries of analogous ones derived in Section 2 for the operator μ_p^ε .

Once these local properties of variationally p -harmonious functions have been established, we consider the Dirichlet-type problem of finding a function $u = u^\varepsilon$ in $C(\overline{\Omega})$ such that

$$(1.2) \quad u = \mu_p^\varepsilon[u] \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma.$$

Here, g is any given function in $C(\Gamma)$. In order to obtain existence and uniqueness for this problem, we proceed in the wake of classical Perron's method for harmonic functions. In fact, we define two classes of continuous functions on $\overline{\Omega}$, \mathcal{S}_g and \mathcal{S}^g , which contain what we call the *variational p -subsolutions* and *p -supersolutions* of problem (1.2), respectively. Precisely, $v \in \mathcal{S}_g$ (resp. $v \in \mathcal{S}^g$) if and only if

$$v \leq \mu_p^\varepsilon[v] \quad (\text{resp. } v \leq \mu_p^\varepsilon[v]) \quad \text{in } \Omega, \quad v \leq g \quad (\text{resp. } v \geq g) \quad \text{on } \Gamma.$$

The desired solution u^ε of (1.2) is then obtained by checking that the functions defined by

$$\underline{u}^\varepsilon(x) = \sup_{v \in \mathcal{S}_g} v(x), \quad \overline{u}^\varepsilon(x) = \inf_{w \in \mathcal{S}^g} w(x) \quad \text{for } x \in \overline{\Omega}$$

coincide on $\overline{\Omega}$, provided Γ satisfies some sufficient regularity assumption. Then, u^ε can be defined by $u^\varepsilon \equiv \underline{u}^\varepsilon = \overline{u}^\varepsilon$ on $\overline{\Omega}$, and the assumed regularity assumptions on Γ guarantee that u^ε satisfies the boundary condition in (1.2), for every given $g \in C(\Gamma)$.

In fact, similarly to what done in the classical case, the existence of a solution of (1.2) is related to a suitable notion of barrier: given a point $x_0 \in \Gamma$, a *barrier* at x_0 for the problem (1.2) is a continuous function $w = w^{x_0}$ on $\overline{\Omega}$, which is variationally p -superharmonic in Ω , positive in $\overline{\Omega} \setminus \{x_0\}$, and such that $w(x_0) = 0$. If $x_0 \in \Gamma$ admits a barrier, we say that x_0 is a *regular point* for the Dirichlet problem (1.2). Our existence and uniqueness result then reads as follows.

Theorem 1.1 (Existence and uniqueness of p -harmonious functions). *Let Ω be a bounded domain in \mathbb{R}^N and take $\varepsilon_0 > 0$ such that Ω contains at least a ball of radius ε_0 . Then, for every $\varepsilon \in (0, \varepsilon_0]$, the Dirichlet-type problem (1.2) admits a unique solution u^ε of class $C(\overline{\Omega})$ for every $g \in C(\Gamma)$ if and only if all points in Γ are regular for (1.2).*

In Corollary 4.4, as it happens in the classical case, we will show that all points of Γ are regular if Γ satisfies the exterior sphere condition.

If we compare Theorem 1.1 to similar results obtained in the literature (see [11, 20, 22]), we can see some differences. Apart for the use of a different notion of p -mean, which is there given by the explicit definition

$$\eta_p^\varepsilon[v](x) = \frac{N+2}{N+p} \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} v \, dy + \frac{p-2}{2(N+p)} \left[\frac{\max_{B_\varepsilon(x)} v}{B_\varepsilon(x)} + \frac{\min_{B_\varepsilon(x)} v}{B_\varepsilon(x)} \right]$$

for $v \in C(\Omega)$, the role of the operator μ_p^ε used in (1.2) is replaced in [20, 22] by the mapping

$$C(\Omega_\varepsilon) \ni v \mapsto \eta_p^\varepsilon[v] \chi_\Omega + g^* \chi_{\Gamma_\varepsilon},$$

where $\Omega_\varepsilon = \Omega \cup \Gamma_\varepsilon$, $\Gamma_\varepsilon = \{y \in \mathbb{R}^N \setminus \Omega : \text{dist}(y, \Gamma) \leq \varepsilon\}$ and g^* is a continuous extension of the boundary data g to Γ_ε . Thus, the solution of the relevant Dirichlet-type problem obtained with these ingredients is in general discontinuous on Γ , differently from what stated in Theorem 1.1. On the other hand, the construction of a p -harmonious function in those cases is obtained by purely monotonicity arguments, whereas we need the existence of a barrier.

The final purpose of this paper is to show that the operator μ_p^ε gives an approximation scheme for the unique viscosity solution of the Dirichlet problem:

$$(1.3) \quad \Delta_p^G u = 0 \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma.$$

Here, Δ_p^G denotes the *game theoretic (or homogeneous) p -Laplacian*, (formally) defined by

$$(1.4) \quad \Delta_p^G u = \frac{\Delta u}{p} + \frac{p-2}{p} \frac{\langle \nabla^2 u \nabla u, \nabla u \rangle}{|\nabla u|^2}.$$

For $1 \leq p < \infty$, we have that

$$\Delta_p^G u = \frac{1}{p} \frac{\text{div}(|\nabla u|^{p-2} \nabla u)}{|\nabla u|^{p-2}}.$$

The normalizing factor p allows a formal definition in the limiting case $p = \infty$. With this choice, $\Delta_2^G = \Delta/2$, where Δ is the classical Laplace operator, and

$$\Delta_\infty u = \frac{\langle \nabla^2 u \nabla u, \nabla u \rangle}{|\nabla u|^2}.$$

We observe that Δ_p^G is uniformly elliptic. However, it has discontinuous coefficients and is not variational. Thus, the fulfilment of (1.3) must be intended in a suitable viscosity sense, which we will detail in Section 5. Notice that *game-theoretic p -harmonic functions*, i.e. the viscosity solutions of the first equation in (1.3), have

been proved to coincide with p -harmonic functions, i.e. the weak solutions of the p -Laplace equation, $\Delta_p u = 0$, where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

(See [14].) With these premises, we can state our convergence result.

Theorem 1.2 (Limits of p -harmonious functions). *Let Ω be a bounded domain containing a ball of radius $\varepsilon_0 > 0$. Suppose that there exists $\varepsilon_\Gamma \in (0, \varepsilon_0]$ such that, for every $0 < \varepsilon < \varepsilon_\Gamma$, Γ is made of regular points for the problem (1.2).*

For $0 < \varepsilon < \varepsilon_\Gamma$, let u^ε be the unique solution in $C(\overline{\Omega})$ of (1.2). Then, for every $g \in C(\Gamma)$, there exists $u \in C(\overline{\Omega})$ such that u^ε converges to u uniformly on $\overline{\Omega}$ as $\varepsilon \rightarrow 0$ and u is the unique viscosity solution of (1.3).

Results similar to this theorem have been obtained in [8, 20, 22], based on the p -mean $\eta_p^\varepsilon[v]$. In particular, this theorem was expected to hold since the average ν_p enjoys all the structural assumptions, proposed in [8], for the convergence of the underlying dynamic programming principle. Those assumptions are additivity with constants, 1-homogeneity and monotonicity for essentially bounded functions (see Section 2.1). However, Theorem 1.2 cannot be proved as a direct application of the general result contained in [8], since the relevant definitions of p -harmonious functions are different. Therefore, we adapt the argument used in [8] to the case of the operator μ_p^ε considered in this paper. We succeed in this goal with the help of an argument established in [5] and the proof of an asymptotic mean value property for μ_p^ε , which characterizes game-theoretic p -harmonic functions. This is based on the formula

$$(1.5) \quad \mu_p^\varepsilon[\phi](x) = \phi(x) + \frac{p}{2(N+p)} \Delta_p^G \phi(x) \varepsilon^2 + o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0,$$

which holds for any $x \in \Omega$ and any $\phi \in C^2(\Omega)$ with $\nabla \phi(x) \neq 0$. The formula (1.5) is obtained by adapting [13, Theorem 3.2].

As a final remark, we comment on the relationship between the barriers for (1.2) and those for (1.3). By combining Theorem 1.1 with Theorem 1.2, we can observe that the existence of a barrier for (1.2) for any ε implicitly shows the existence of a barrier also for (1.3) since the solvability of (1.2) or (1.3) is equivalent to the existence of a barrier in the corresponding sense together with the fact that the solvability of (1.2) for any ε implies the solvability of (1.3).

We summarize the contents of this paper. In Section 2, we present some old and new properties on the mean ν_p such as continuity and monotonicity of the operator μ_p^ε . In Section 3, we introduce what we call variationally p -harmonious functions and show that they satisfy weak and strong comparison principles. We shall prove existence and uniqueness for the Dirichlet problem for variationally p -harmonious functions in Section 4. Finally, in Section 5, we will show that limits of variationally p -harmonious functions are p -harmonic in a viscosity sense.

2. THE p -MEAN AND THE OPERATOR μ_p^ε

In this section, we collect some properties of the p -means that are instrumental in the proofs of Theorems 1.1 and 1.2.

2.1. Definition of p -mean and general properties. Let $E \subset \mathbb{R}^N$ be a measurable set of finite measure. For a measurable function $v : E \rightarrow \overline{\mathbb{R}}$, we shall use the notation:

$$\|v\|_{p,E} = \left(\frac{1}{|E|} \int_E |v|^p dx \right)^{1/p} \quad \text{for } 1 \leq p < \infty \quad \text{and}$$

$$\|v\|_{\infty,E} = \inf\{t \geq 0 : |\{x \in E : |v(x)| > t\}|\},$$

whenever these quantities are finite. Since E has finite measure, we can (and will) always assume that $|E| = 1$, without loss of generality.

Let $v \in L^p(E)$. We define a p -mean of v as a real number $\nu_p[v]$ such that

$$\|v - \nu_p[v]\|_{p,E} = \min_{\nu \in \mathbb{R}} \|v - \nu\|_{p,E}.$$

As shown in [13], $\nu_p[v]$ is uniquely defined for $1 < p \leq \infty$ and is a natural generalization of the classical mean value of v . In fact, $\nu_2[v]$ is the average of v on E . We also know that $\nu_1[v]$ may not be unique. For instance, if $E = [-2, 2]$ and $v = \mathcal{X}_{[-1,1]}$, then any number ν in $[0, 1]$ fulfils the definition when $p = 1$. However, $\nu_1[v]$ is uniquely defined if $v \in L^1(\Omega) \cap C(\Omega)$, when Ω is an open domain of finite measure in \mathbb{R}^N (see [23]). For the sake of brevity, from now on, in the case $p = 1$ we shall assume that $v \in L^1(\Omega) \cap C(\Omega)$.

It is useful to know that $\nu_p[v]$ is the unique root of the equation

$$(2.1) \quad \int_E |v - \nu|^{p-2} [v - \nu] dy = 0,$$

for $1 \leq p < \infty$, and

$$\nu_\infty[v] = \frac{1}{2} \left\{ \operatorname{esssup}_E v + \operatorname{essinf}_E v \right\}$$

(see [13]). For $p = 1$, we mean that $|t|^{p-2}t = \operatorname{sgn}(t)$, which equals 1 if $t > 0$, -1 if $t < 0$, and 0 if $t = 0$.

Remark 2.1. (i) It can be proved that $\nu_p[v] \rightarrow \nu_\infty[v]$ as $p \rightarrow \infty$. This fact can be obtained by taking the limit as $p \rightarrow \infty$ in the equality

$$\left\{ \int_{E_p^+} (v - \nu_p[v])^{p-1} dy \right\}^{\frac{1}{p-1}} = \left\{ \int_{E_p^-} (\nu_p[v] - v)^{p-1} dy \right\}^{\frac{1}{p-1}},$$

that follows from (2.1). Here, $E_p^\pm = \{y \in E : v \gtrless \nu_p[v]\}$.

(ii) Notice that while $\nu = \nu_p[v]$ is uniquely defined for any function v in $L^p(E)$ with $1 < p < \infty$, still, its characterization as solution of (2.1) should be analyzed in dependence on the values of p . In fact, notice that the integral

$$\int_E |v - \nu|^{p-2} dy$$

is always finite for $2 \leq p < \infty$, but may not be so for $1 < p < 2$. Thus, while for $2 \leq p < \infty$ the mean $\nu_p[v]$ can also be characterized as

$$\nu_p[v] = \frac{\int_E |v - \nu_p[v]|^{p-2} v dy}{\int_E |v - \nu_p[v]|^{p-2} dy}$$

(unless v is a.e. constant on E), such a characterization may fail to be true for $1 < p < 2$, unless we extend it by setting $\nu_p[v] = 0$, if the denominator is infinite.

For instance, if E is the unit disk B in \mathbb{R}^2 and $v(x, y) = x^3 - y^3$, by uniqueness we have that $\nu_p[v] = 0$, since (2.1) is satisfied by this value for any $1 < p < \infty$. Nevertheless, we have that

$$\int_B |v|^{p-2} dy = \infty \quad \text{and} \quad \int_B |v|^{p-2} v dy = 0 \quad \text{for} \quad 1 < p < \frac{4}{3}.$$

Elementary properties of $\nu_p[v]$ are the following (see [13, Proposition 2.7 and Proposition 2.5]): for every function $v \in L^p(E)$, it holds that

- (a) (additivity with constants) $\nu_p[v + c] = \nu_p[v] + c$ for every $c \in \mathbb{R}$;
- (b) (homogeneity) $\nu_p[\alpha v] = \alpha \nu_p[v]$ for every $\alpha \in \mathbb{R}$;
- (c) (monotonicity) $\nu_p[v_1] \leq \nu_p[v_2]$ if $v_1 \leq v_2$ a.e. in E .

For our purposes in this paper, we need to show that ν_p is *strictly monotonic*.

Lemma 2.2 (Strict monotonicity). *Let $E \subset \mathbb{R}^N$ be a measurable set with finite measure. Let $v_1, v_2 \in L^p(E)$, $1 \leq p \leq \infty$, and assume that $v_1 \leq v_2$ a.e. in E and that the measure of the set $E^\sharp = \{x \in E : v_1(x) < v_2(x)\}$ is positive. Then it holds that $\nu_p[v_1] < \nu_p[v_2]$. In particular, if $v_1 \leq v_2$ a.e. in E and $\nu_p[v_1] = \nu_p[v_2]$, we have that $v_1 = v_2$ a.e. in E .*

Proof. For $p = \infty$, our claim follows from an inspection. If $1 \leq p < \infty$, the assumptions on v_1 and v_2 and the fact that the function $\mathbb{R} \ni t \mapsto |t - \nu|^{p-2}(t - \nu)$ is strictly increasing give that

$$\int_{E^\sharp} |v_1 - \nu|^{p-2}(v_1 - \nu) dy < \int_{E^\sharp} |v_2 - \nu|^{p-2}(v_2 - \nu) dy$$

for every ν . Thus, the same holds with E^\sharp replaced by E , and our claims follow from the characterization (2.1). \square

In [13, Theorem 2.4], we proved the continuity of the operator ν_p . That result can be relaxed.

Lemma 2.3 (Continuity). *Let $E \subset \mathbb{R}^N$ be a measurable set with finite measure and fix $1 < p < \infty$. Let $v \in L^p(E)$ and let $\{v_n\}_{n \in \mathbb{N}}$ be a sequence of measurable functions that converges to v a.e. in E .*

Suppose that either

- (i) $\{v_n\}_{n \in \mathbb{N}}$ is increasing and $v_n \geq 0$ on E for every $n \in \mathbb{N}$, or
- (ii) there exists a function $w \in L^p(E)$ such that $|v_n| \leq w$ a.e. in E for every $n \in \mathbb{N}$.

Then $\nu_p[v_n] \rightarrow \nu_p[v]$ as $n \rightarrow \infty$.

Proof. (i) Set $\nu_n = \nu_p[v_n]$ and $\nu = \nu_p[v]$. The sequence of numbers ν_n converges or diverges to $+\infty$, since it increases. Let $\bar{\nu}$ be its limit. Since $v_n \leq v$, we have that $\nu_n \leq \nu$ and hence $\bar{\nu} \leq \nu$. Now, observe that

$$0 = \int_E |v_n - \nu_n|^{p-2}(v_n - \nu_n) dy \geq \int_E |v_n - \bar{\nu}|^{p-2}(v_n - \bar{\nu}) dy,$$

since the function $\mathbb{R} \ni s \mapsto |t - s|^{p-2}(t - s)$ is decreasing and $v_n \leq \bar{\nu}$. The integrand at right-hand side of the last inequality is bounded from below by the number $-|\bar{\nu}|^{p-2}\bar{\nu}$. We thus can apply the Monotone Convergence Theorem and pass to the limit to obtain that

$$\int_E |v - \nu|^{p-2}(v - \nu) dy = 0 \geq \int_E |v - \bar{\nu}|^{p-2}(v - \bar{\nu}) dy.$$

This inequality gives that $\bar{\nu} \geq \nu$, and the proof is complete.

(ii) The assumptions in (ii) give that $v_n \rightarrow v$ in $L^p(E)$ by the Dominated Convergence Theorem. Our claim then follows from [13, Theorem 2.4]. \square

Remark 2.4. When $p = \infty$, we obtain the convergence $\nu_\infty(v_n) \rightarrow \nu_\infty(v)$ as $n \rightarrow \infty$, if either $v_n \rightarrow v$ uniformly or, in the case (i), if E is a bounded open set and $v_n, v \in C(E)$ for every $n \in \mathbb{N}$. In fact, in this case the monotonicity of the sequence gives its uniform convergence, by Dini's monotone convergence theorem.

When $p = 1$, in order to obtain the desired convergence, we must require that $v_n, v \in L^1(\Omega) \cap C(\Omega)$, where Ω is an open domain of finite measure.

2.2. The operator μ_p^ε and its properties. Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $B_r(x)$ be a ball of radius r , centered at x and contained in Ω . We let $\nu_p^r[v](x)$ be the p -mean of v relative to the set $E = B_r(x)$. In order to well-define $\nu_p^r[v](x)$, we shall always assume that $v \in L_{\text{loc}}^p(\Omega)$ for $1 < p \leq \infty$ and $v \in C(\Omega) \cap L_{\text{loc}}^1(\Omega)$ when $p = 1$.

Next, for each fixed $\varepsilon > 0$, we set $r_\varepsilon(x) = \min[\varepsilon, \text{dist}(x, \Gamma)]$ and define:

$$\mu_p^\varepsilon[v](x) = \nu_p^{r_\varepsilon(x)}[v](x) \text{ for every } x \in \Omega.$$

We stress the fact that, differently from $\nu_p^r[v](x)$ that can be defined if x is far enough from Γ , $\mu_p^\varepsilon[v](x)$ is well-defined for any $x \in \Omega$. Notice that, from (2.1), we easily infer that $\nu_p^r[v](x)$ is the unique root ν of the equation

$$(2.2) \quad \int_B |v(x + rz) - \nu|^{p-2} [v(x + rz) - \nu] dz = 0$$

for $1 \leq p < \infty$, and

$$(2.3) \quad \nu_\infty^r[v](x) = \frac{1}{2} (\text{esssup}_{z \in B} v(x + rz) + \text{essinf}_{z \in B} v(x + rz)),$$

where B denotes the unit ball in \mathbb{R}^N centered at the origin. These facts tell us that we can assume that $\mu_p^\varepsilon[v] = v$ on Γ , at least formally. In fact, for $x \in \Gamma$, $r_\varepsilon(x) = 0$ and (2.2)-(2.3) give that $\nu = v(x)$ when $r = 0$. This issue will be discussed in more detail in Theorem 2.7 below.

The following result extends a formula obtained in [13, Theorem 3.2].

Proposition 2.5. *Let Ω be a domain in \mathbb{R}^N and let $\phi \in C^2(\Omega)$. Then, for every bounded open set A with $\bar{A} \subset \Omega$ and such that $\nabla\phi \neq 0$ on \bar{A} , it holds that*

$$\lim_{\varepsilon \rightarrow 0} \frac{\mu_p^\varepsilon[\phi] - \phi}{r_\varepsilon^2} = \frac{p}{2(N+p)} \Delta_p^G \phi \text{ uniformly on } \bar{A}.$$

Proof. Let x_ε a point in \bar{A} maximizing the difference

$$\left| \frac{\mu_p^\varepsilon[\phi] - \phi}{r_\varepsilon^2} - \frac{p}{2(N+p)} \Delta_p^G \phi \right|$$

on \bar{A} . Up to a subsequence, we have that $x_\varepsilon \rightarrow x$ for some $x \in \bar{A}$, and hence $r_\varepsilon(x_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The conclusion then follows from [13, Theorem 3.2]. \square

The following claim is a straightforward consequence of Lemma 2.2.

Corollary 2.6 (Monotonicity). *Let $\Omega \subset \mathbb{R}^N$ be a domain of finite measure. Let $v \in L^p(\Omega)$, $1 \leq p < \infty$, $v \in L^1(\Omega) \cap C(\Omega)$ for $p = 1$. Let $v_1, v_2 \in L^p(\Omega)$ be such that $v_1 \leq v_2$ a.e. in Ω . Then $\mu_p^\varepsilon[v_1] \leq \mu_p^\varepsilon[v_2]$ on $\bar{\Omega}$. Moreover, we have that $\mu_p^\varepsilon[v_1](x) < \mu_p^\varepsilon[v_2](x)$ whenever the set*

$$\{y \in B_{r_\varepsilon(x)}(x) : v_1(y) < v_2(y)\}$$

has positive measure.

Next, we show that μ_p^ε acts quite naturally on $\text{USC}(\bar{\Omega})$ and $\text{LSC}(\bar{\Omega})$, the classes of functions that are upper and lower semicontinuous on $\bar{\Omega}$, respectively. We state our result for the case of lower semicontinuous functions. The remaining case will easily follow.

Theorem 2.7 (Invariance semicontinuity property). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. If $v \in \text{LSC}(\bar{\Omega})$, then we have that $\mu_p^\varepsilon[v] \in \text{LSC}(\bar{\Omega}) \cap C(\Omega)$.*

Proof. We first prove that $\mu_p^\varepsilon[v] \in C(\Omega)$. We take a point $x \in \Omega$ and any sequence of points $x_n \in \Omega$ converging to x . Then, we observe that

$$\mu_p^\varepsilon[v](x) = \mu_p^1[w^x](0),$$

where $w^x : B \rightarrow \mathbb{R}$ is defined by

$$(2.4) \quad w^x(z) = v(x + r_\varepsilon(x)z) \text{ for } z \in B.$$

In other words, $\mu_p^\varepsilon[v](x) = \nu_p[w^x]$, when we choose E to be the unit ball B .

Being as $v \in L^p_{\text{loc}}(\Omega)$ and r_ε continuous, we infer that $w^{x_n} \rightarrow w^x$ in $L^p(B)$ as $n \rightarrow \infty$. Thus, [13, Theorem 2.4] gives that $\nu_p[w^{x_n}] \rightarrow \nu_p[w^x]$, and hence we conclude that $\mu_p^\varepsilon[v](x_n) \rightarrow \mu_p^\varepsilon[v](x)$.

We are left to prove the desired semicontinuity at points in Γ . Thus, we take any sequence of points $x_n \in \overline{\Omega}$ converging to some point $x \in \Gamma$. Observe that, since $\overline{\Omega}$ is compact, v admits its minimum $m > -\infty$ on it. Next, we proceed as before and consider the function w^{x_n} in (2.4). This is well-defined at points $x_n \in \Omega$. At points $x_n \in \Gamma$, instead, we consistently mean that $w^{x_n} = v(x_n)$. Since $\nu_p[w^{x_n}] = \nu_p[w^{x_n} - m] + m$ and $w^{x_n} \geq m$, we can always suppose that $w^{x_n} \geq 0$. By the semicontinuity of v and the continuity of r_ε , we know that

$$\liminf_{n \rightarrow \infty} w^{x_n}(z) \geq w^x(z) \text{ for any } z \in B.$$

Now, we have that

$$\inf_{k \geq n} (w^{x_k}) \rightarrow \liminf_{n \rightarrow \infty} (w^{x_n}) \text{ on } B \text{ as } n \rightarrow \infty,$$

and the convergence is monotone increasing. Thus, we can apply Lemma 2.3 (i) to obtain that

$$\lim_{n \rightarrow \infty} \nu_p \left[\inf_{k \geq n} (w^{x_k}) \right] = \nu_p \left[\liminf_{n \rightarrow \infty} (w^{x_n}) \right].$$

Notice that this holds even if there are infinitely many terms x_n that belong to Γ .

Therefore, we conclude that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mu_p^\varepsilon[v](x_n) &= \liminf_{n \rightarrow \infty} \nu_p[w^{x_n}] \geq \liminf_{n \rightarrow \infty} \nu_p \left[\inf_{k \geq n} (w^{x_k}) \right] = \\ &= \nu_p \left[\liminf_{n \rightarrow \infty} (w^{x_n}) \right] \geq \nu_p[w^x] = \mu_p^\varepsilon[v](x). \end{aligned}$$

This gives that $\mu_p^\varepsilon[v]$ is lower semicontinuous on $\overline{\Omega}$. \square

3. VARIATIONALLY p -HARMONIOUS FUNCTIONS

Slightly differently from what done in [8, 22], we give the following definitions. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Set $1 \leq p \leq \infty$ and fix $\varepsilon > 0$ such that Ω contains at least one ball of radius ε . We say that $v \in L^p(\Omega)$ is a *variationally p -subharmonic* (or *variationally p -superharmonic*) function in Ω if

$$v(x) \leq \mu_p^\varepsilon[v](x) \quad (\text{or } v(x) \geq \mu_p^\varepsilon[v](x)) \text{ for almost every } x \in \Omega.$$

A *variationally p -harmonious* function is both variationally p -subharmonic and p -superharmonic. We stress the fact that the term ‘‘variationally’’ refers to the variational definition of the operator μ_p^ε .

Example 3.1. (i) It is easily seen that constant functions are variationally p -harmonious for any $p \in [1, \infty]$.

(ii) For $\xi \in \mathbb{R}^N$, let $a(y) = \langle \xi, y \rangle + c$ be an affine function. We have that $a(y) = a_0(y) + \langle \xi, x \rangle + c$, where $a_0(y) = \langle \xi, y - x \rangle$ and $\langle \xi, x \rangle + c$ is constant. It is easy to infer that $\nu_p^r[a_0](x) = 0$, by the central symmetry of a_0 with respect to x . Thus, we conclude that

$$\nu_p^r[a](x) = \nu_p^r[a_0](x) + \nu_p^r[\langle \xi, x \rangle + c](x) = \langle \xi, x \rangle + c = a(x).$$

It is then clear that $\mu_p^\varepsilon[a] = a$.

Therefore, we infer that affine functions are always variationally p -harmonious. It is then evident that convex and concave functions are variationally p -subharmonic and p -superharmonic, respectively.

(iii) Fix an $\alpha > 0$ and set $\gamma_\alpha(y) = |y|^{-\alpha}$ for $y \neq 0$. Clearly, we have that $\gamma_\alpha \in L^p_{\text{loc}}(\mathbb{R}^N)$ if $\alpha p < N$ and $\gamma_\alpha \in L^p_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$ for any $p \in [1, \infty]$.

Next, let

$$\Delta_p^G \phi = \frac{1}{p} \left\{ \Delta \phi + (p-2) \frac{\langle \nabla^2 \phi \nabla \phi, \nabla \phi \rangle}{|\nabla \phi|^2} \right\}$$

denote the *game-theoretic p -laplacian* of a smooth function away from its critical points. We easily compute that

$$\Delta_p^G \gamma_\alpha(y) = \frac{\alpha[\alpha(p-1) + p - N]}{p} |y|^{-(\alpha+2)} \quad \text{for } y \neq 0.$$

By applying [13, formula (3.6) in Theorem 3.2], for any $x \neq 0$, we obtain that

$$\frac{\gamma_\alpha(x) - \nu_p^r[\gamma_\alpha](x)}{r^2} = -\frac{\alpha[\alpha(p-1) + p - N]}{2(N+p)} |x|^{-(\alpha+2)} + o(1)$$

as $r \rightarrow 0$.

Now, let Ω be a bounded open set such that $\overline{\Omega}$ does not contain the origin. Depending on the sign of $\alpha(p-1) + p - N$, there exists $r_\Omega > 0$ such that the inequalities $\gamma_\alpha \geq \nu_p^r[\gamma_\alpha]$ or $\gamma_\alpha \leq \nu_p^r[\gamma_\alpha]$ are uniformly satisfied in $\overline{\Omega}$ for $0 < r < r_\Omega$. Therefore, for $0 < r < r_\Omega$, γ_α is variationally p -subharmonic in Ω , if

$$p > \frac{\alpha + N}{\alpha + 1},$$

and variationally p -superharmonic in Ω , if

$$p < \frac{\alpha + N}{\alpha + 1}.$$

3.1. Comparison principles. Despite they are not solutions of any partial differential equation, variationally p -harmonic functions satisfy comparison principles.

Theorem 3.1 (Weak comparison principle). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let $v \in \text{USC}(\overline{\Omega})$ and $w \in \text{LSC}(\overline{\Omega})$ be p -subharmonic and p -superharmonic in Ω , respectively.*

If $v \leq w$ on Γ , then $v \leq w$ on $\overline{\Omega}$.

Proof. We proceed by contradiction. We have that $v - w \in \text{USC}(\overline{\Omega})$, and hence it attains its maximum M on $\overline{\Omega}$. Suppose that M is positive. Since $v \leq w$ on Γ , the set $A = \{x \in \Omega : v(x) - w(x) = M\}$ contains at least one point x_0 . Let Ω' be the connected component of Ω that contains x_0 . The set $A' = \{x \in \Omega' : v(x) - w(x) = M\}$ is thus non-empty and also closed in Ω' , being the pre-image of the closed set $\{M\} \subset \mathbb{R}$ under the continuous function $v - w : \Omega' \rightarrow \mathbb{R}$.

To show that A' is open, we take any $x \in A'$ and observe that

$$\mu_p^\varepsilon[v - M](x) \leq \mu_p^\varepsilon[w](x) \leq w(x)$$

since $v - M \leq w$ in Ω by construction and w is p -superharmonic. Thus, we infer that

$$w(x) \geq \mu_p^\varepsilon[v - M](x) = \mu_p^\varepsilon[v](x) - M \geq v(x) - M = w(x)$$

since v is p -subharmonic. In particular, $\mu_p^\varepsilon[w](x) = \mu_p^\varepsilon[v - M](x)$, which gives that $w \equiv v - M$ on $B_{r_\varepsilon(x)}(x)$, by Lemma 2.2. In other words, we have that $B_{r_\varepsilon(x)}(x)$ is contained in A' . Thus, we have proved that A' is also open and, since Ω' is connected, we have that $A' = \Omega'$. Thus, $v - w \equiv M$ on Ω' and, by continuity, we infer that $M \equiv v - w$ on the boundary of Ω' , which is contained in Γ . Therefore, we reach the contradiction $0 < M \leq 0$. \square

By more or less the same arguments, we can prove the following strong comparison principle.

Corollary 3.2 (Strong comparison principle). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $v, w \in C(\Omega)$ be p -subharmonic and p -superharmonic in Ω , respectively.*

Suppose that $v \leq w$ in Ω . Then, either $v < w$ in Ω or, else, $v \equiv w$ in Ω .

Proof. Suppose that $v - w = 0$ at some point in Ω . Then, we repeat the same argument in the proof of Theorem 3.1 with $A' = \{x \in \Omega : v(x) - w(x) = 0\}$ and $M = 0$. We thus infer that A' is non-empty, closed, and open. Since Ω is connected, we then obtain that $A' = \Omega$, and hence $v \equiv w$ in Ω . \square

4. EXISTENCE OF p -HARMONIOUS FUNCTIONS

In this section, we shall prove Theorem 1.1. We start with some definitions.

Let Ω be a bounded domain in \mathbb{R}^N with boundary Γ and let $g : \Gamma \rightarrow \mathbb{R}$ be a continuous function. For a fixed number $\varepsilon > 0$, we introduce two classes of functions:

$$\begin{aligned} \mathcal{S}_g &= \{v \in C(\overline{\Omega}) : v \leq \mu_p^\varepsilon[v] \text{ in } \Omega, v \leq g \text{ on } \Gamma\}, \\ \mathcal{S}^g &= \{v \in C(\overline{\Omega}) : v \geq \mu_p^\varepsilon[v] \text{ in } \Omega, v \geq g \text{ on } \Gamma\}. \end{aligned}$$

It is clear that $\mathcal{S}^g = -\mathcal{S}_{-g}$. The elements of \mathcal{S}_g and \mathcal{S}^g will be called *variational p -subsolutions* and, respectively, *p -supersolutions* of (1.2). These classes are non-empty since they contain the constant functions on $\overline{\Omega}$ defined by

$$v \equiv \min_{\Gamma} g \quad \text{and} \quad w \equiv \max_{\Gamma} g,$$

respectively.

Our aim is to obtain a solution of (1.2) as an envelope of functions in \mathcal{S}_g or in \mathcal{S}^g . In fact, we define the *lower and upper Perron solutions* $\underline{u}^\varepsilon$ and \overline{u}^ε of (1.2) by

$$\underline{u}^\varepsilon(x) = \sup_{v \in \mathcal{S}_g} v(x) \quad \text{and} \quad \overline{u}^\varepsilon(x) = \inf_{w \in \mathcal{S}^g} w(x) \quad \text{for } x \in \overline{\Omega}.$$

The following lemma collects some elementary properties of these functions.

Lemma 4.1. *Let Ω be a bounded domain in \mathbb{R}^N and let $g \in C(\Gamma)$. Then,*

(i) *it holds that*

$$\min_{\Gamma} g \leq \underline{u}^\varepsilon \leq \overline{u}^\varepsilon \leq \max_{\Gamma} g \quad \text{on } \overline{\Omega};$$

(ii) *$u \in C(\overline{\Omega})$ is a solution of (1.2) if and only if $u \in \mathcal{S}_g \cap \mathcal{S}^g$;*

(iii) *it holds that $u = \underline{u}^\varepsilon = \overline{u}^\varepsilon$ on $\overline{\Omega}$, if $u \in C(\overline{\Omega})$ is a solution of (1.2).*

Proof. (i) By the weak comparison principle (Theorem 3.1), we have that $v \leq w$ on $\overline{\Omega}$ for every $v \in \mathcal{S}_g$ and $w \in \mathcal{S}^g$, being as $v \leq g \leq w$ on Γ . Thus, $\underline{u}^\varepsilon \leq \overline{u}^\varepsilon$, clearly, by the properties of supremum and infimum. The other two inequalities easily follow from the fact that the two relevant constant functions belong to the classes \mathcal{S}_g or \mathcal{S}^g .

(ii) The statement is trivial by the definitions of \mathcal{S}_g and \mathcal{S}^g .

(iii) If $u \in C(\overline{\Omega})$ is a solution of (1.2), then (ii) gives that $u \in \mathcal{S}_g \cap \mathcal{S}^g$, and hence $\overline{u}^\varepsilon \leq u \leq \underline{u}^\varepsilon \leq \overline{u}^\varepsilon$ on $\overline{\Omega}$. \square

In analogy with the classical case and as already mentioned in the introduction, we shall say that a function w is a *barrier* at a point $x_0 \in \Gamma$ for the Dirichlet-type problem (1.2) if $w \in C(\overline{\Omega})$, w is superharmonic in Ω , and $w > 0$ in $\overline{\Omega} \setminus \{x_0\}$ with $w(x_0) = 0$. If $x_0 \in \Gamma$ admits a barrier, we say that x_0 is a *regular point* for (1.2).

Proposition 4.2. *If $x_0 \in \Gamma$ is a regular point for (1.2), then*

$$\underline{u}^\varepsilon(x_0) = \overline{u}^\varepsilon(x_0) = g(x_0).$$

Proof. The proof proceeds as in the classical case. Let $\eta > 0$. Since $g \in C(\Gamma)$, we can find $\delta > 0$ such that $|g - g(x_0)| < \eta$ on $\Gamma \cap B_\delta(x_0)$. Also, if w is a barrier at x_0 , the following number is well-defined:

$$M_\delta = \max_{\Gamma \setminus B_\delta(x_0)} \frac{|g - g(x_0)|}{w}.$$

Thus, we infer that

$$|g - g(x_0)| \leq \eta + M_\delta w \quad \text{on } \Gamma.$$

This fact gives that the functions $g(x_0) + \eta + M_\delta w$ and $g(x_0) - \eta - M_\delta w$ belong to the classes \mathcal{S}^g and \mathcal{S}_g , respectively. Hence, by definition

$$g(x_0) - \eta - M_\delta w \leq \underline{u}^\varepsilon \leq \bar{u}^\varepsilon \leq g(x_0) + \eta + M_\delta w \quad \text{on } \bar{\Omega}$$

and, in particular

$$g(x_0) - \eta \leq \underline{u}^\varepsilon(x_0) \leq \bar{u}^\varepsilon(x_0) \leq g(x_0) + \eta,$$

being as $w(x_0) = 0$. The desired conclusion then ensues, since $\eta > 0$ is arbitrary. \square

Proof of Theorem 1.1. (i) Let us assume that every point in Γ admits a barrier for (1.2). Lemma 4.1 and the definition of $\underline{u}^\varepsilon$ give that $\underline{u}^\varepsilon$ is bounded and lower semicontinuous on $\bar{\Omega}$, that is $\underline{u}^\varepsilon \in L^\infty(\bar{\Omega}) \cap \text{LSC}(\bar{\Omega})$. The assumption also gives that $\underline{u}^\varepsilon = g$ on Γ , thanks to Proposition 4.2.

Next, we define a sequence of functions by the following iterative scheme:

$$u_1 = \underline{u}^\varepsilon, \quad u_{j+1} = \mu_p^\varepsilon[u_j] \quad \text{on } \bar{\Omega}, \quad \text{for } j \geq 1.$$

Proposition 2.7 ensures that all the functions u_j belong to $L^\infty(\bar{\Omega}) \cap \text{LSC}(\bar{\Omega})$. Clearly, it holds that $u_j = g$ on Γ for every $j \geq 1$.

Now, since $v \leq \underline{u}^\varepsilon$ on $\bar{\Omega}$ for every $v \in \mathcal{S}_g$, we have that $v \leq \mu_p^\varepsilon[v] \leq \mu_p^\varepsilon[\underline{u}^\varepsilon]$ in Ω for every $v \in \mathcal{S}_g$, which gives $\underline{u}^\varepsilon \leq \mu_p^\varepsilon[\underline{u}^\varepsilon]$ in Ω . Hence, we can infer that $u_1 \leq \mu_p^\varepsilon[u_1] = u_2$ on $\bar{\Omega}$. Then by monotonicity, we have that $u_2 = \mu_p^\varepsilon[u_1] \leq \mu_p^\varepsilon[u_2] = u_3$ and, by iterating, we obtain that $u_j \leq u_{j+1}$ on $\bar{\Omega}$ for every $j \geq 1$. Thus, the sequence $\{u_j\}_{j \in \mathbb{N}}$ is increasing on $\bar{\Omega}$. By iteration, we also have that

$$\min_{\Gamma} g \leq u_j \leq \max_{\Gamma} g \quad \text{on } \bar{\Omega}, \quad \text{for every } j \geq 1,$$

because, by Lemma 4.1 (i), we see that

$$\min_{\Gamma} g \leq u_1 = \underline{u}^\varepsilon \leq \bar{u}^\varepsilon \leq \max_{\Gamma} g \quad \text{on } \bar{\Omega}.$$

Thus, the sequence $\{u_j\}_{j \in \mathbb{N}}$ converges pointwise on $\bar{\Omega}$ to a function u , which is lower semicontinuous on $\bar{\Omega}$ and such that

$$\min_{\Gamma} g \leq u \leq \max_{\Gamma} g \quad \text{on } \bar{\Omega} \quad \text{and} \quad u = g \quad \text{on } \Gamma.$$

Moreover, applying Lemma 2.3 gives that $\mu_p^\varepsilon[u_j] \rightarrow \mu_p^\varepsilon[u]$ as $j \rightarrow \infty$ in Ω , and hence $u = \mu_p^\varepsilon[u]$ in Ω , since $u_j = \mu_p^\varepsilon[u_j]$ in Ω , for every $j \geq 1$. As a result, the function u satisfies the Dirichlet problem (1.2).

By playing on the fact that $\mathcal{S}^g = -\mathcal{S}_{-g}$, we can determine a decreasing sequence of functions U_j , bounded and upper semicontinuous on $\bar{\Omega}$, which converge to another solution U of (1.2). Clearly, such a sequence is initialized by choosing $U_1 = \bar{u}^\varepsilon$, so that, by comparison, the relevant iterating schemes tell us that $U \geq u$ on $\bar{\Omega}$, being as $U_1 = \bar{u}^\varepsilon \geq \underline{u}^\varepsilon = u_1$ on $\bar{\Omega}$.

Thus, we are left to prove that $u = U$ in $\bar{\Omega}$. Notice first that $U - u$ is non-negative and belongs to $\text{USC}(\bar{\Omega})$, being as $U \in \text{USC}(\bar{\Omega})$ and $u \in \text{LSC}(\bar{\Omega})$. Hence, there exists $x_0 \in \bar{\Omega}$ such that

$$\max_{\bar{\Omega}}(U - u) = (U - u)(x_0).$$

Our goal is to prove that $(U - u)(x_0) = 0$.

By contradiction, we assume that $(U - u)(x_0) > 0$ and consider the set

$$\Omega^\sharp = \{x \in \overline{\Omega} : (U - u)(x) = (U - u)(x_0) > 0\}.$$

Since $U - u = g - g = 0$ on Γ , we have that Ω^\sharp is a subset of Ω . Theorem 2.7 then tells us that $U - u$ is continuous in Ω , and hence we can infer that Ω^\sharp is closed in Ω , because it is the pre-image of the singleton $\{(U - u)(x_0)\} \subset \mathbb{R}$.

Next, notice that $U - U(x_0) \leq u - u(x_0)$ in Ω , since $U - u \leq (U - u)(x_0)$ in Ω , so that, if we take a point $x \in \Omega$, we get that

$$\mu_p^\varepsilon[U](x) - U(x_0) = \mu_p^\varepsilon[U - U(x_0)](x) \leq \mu_p^\varepsilon[u - u(x_0)](x) = \mu_p^\varepsilon[u](x) - u(x_0),$$

by the monotonicity of μ_p^ε . Thus, we infer that

$$U(x) - U(x_0) = \mu_p^\varepsilon[U - U(x_0)](x) \leq \mu_p^\varepsilon[u - u(x_0)](x) = u(x) - u(x_0),$$

since both U and u are variationally p -harmonic in Ω .

Now, if $x \in \Omega^\sharp$, we know that $u(x) - u(x_0) = U(x) - U(x_0)$, and hence we obtain that

$$U(x) - U(x_0) = \mu_p^\varepsilon[U - U(x_0)](x) = \mu_p^\varepsilon[u - u(x_0)](x) = U(x) - U(x_0).$$

The definition of μ_p^ε and Lemma 2.2 then tell us that $U - U(x_0) \equiv u - u(x_0)$, and hence $U - u \equiv (U - u)(x_0)$ on the ball $B_r(x)$ with $r = r_\varepsilon(x) > 0$. As a consequence, we have shown that $B_r(x) \subset \Omega^\sharp$, that is Ω^\sharp is open, since x is arbitrarily chosen in Ω^\sharp . All in all, Ω^\sharp is a closed and open subset of Ω and, moreover, is non-empty, since it contains at least x_0 . Therefore, $\Omega^\sharp = \Omega$, because Ω is connected, that is $U - u \equiv (U - u)(x_0)$ in Ω .

In conclusion, we find a contradiction by taking a sequence of points $x_n \in \Omega$ that converge to any point $\bar{x} \in \Gamma$. In fact, since $U - u \in \text{LSC}(\overline{\Omega})$, we have that

$$0 < (U - u)(x_0) = \limsup_{n \rightarrow \infty} (U - u)(x_n) \leq (U - u)(\bar{x}) = g(\bar{x}) - g(\bar{x}) = 0.$$

□

Corollary 4.3. *Let $w : \overline{\Omega} \rightarrow \mathbb{R}$ be any function such that $\underline{u}^\varepsilon \leq w \leq \overline{u}^\varepsilon$ on $\overline{\Omega}$. Consider the sequence of functions defined by*

$$w_1 = w, \quad w_{j+1} = \mu_p^\varepsilon[w_j] \quad \text{on } \overline{\Omega} \quad \text{for } j = 1, 2, \dots.$$

Then, we have that

$$\lim_{j \rightarrow \infty} w_j \rightarrow u^\varepsilon = \underline{u}^\varepsilon = \overline{u}^\varepsilon \quad \text{on } \overline{\Omega}.$$

Proof. By comparison, we can infer that

$$u_j \leq w_j \leq U_j \quad \text{on } \overline{\Omega}$$

for any $j \in \mathbb{N}$. Thus, we have that

$$u \leq \liminf_{j \rightarrow \infty} w_j \leq \limsup_{j \rightarrow \infty} w_j \leq U = u,$$

which gives the desired claim. □

Corollary 4.4. *Let Ω be a bounded domain in \mathbb{R}^N , satisfying the uniform exterior sphere condition at any point in Γ . Then there exists $0 < \varepsilon_\Gamma \leq \varepsilon_0$ such that for every $0 < \varepsilon < \varepsilon_\Gamma$, all points in Γ are regular for (1.2).*

In particular, for every $g \in C(\Gamma)$, the Dirichlet-type problem (1.2) admits the unique solution $u^\varepsilon = \underline{u}^\varepsilon = \overline{u}^\varepsilon \in C(\overline{\Omega})$ defined by Perron's method.

Proof. Our assumption is that there exists a radius $R > 0$ such that, for every $x_0 \in \Gamma$, we can find a ball $B_R(y_0)$ such that $\overline{B_R(y_0)} \cap \overline{\Omega} = \{x_0\}$. Hence, we define a function w by

$$w(x) = \frac{1}{R^\alpha} - \frac{1}{|x - y_0|^\alpha} \quad \text{for } x \in \overline{\Omega}.$$

It is clear that $w \in C(\overline{\Omega})$, w is positive in $\overline{\Omega} \setminus \{x_0\}$, $w(x_0) = 0$. Finally, if we choose, say, $\alpha = (N + 1)/(p - 1)$, by Example 3.1 (iii), for $0 < \varepsilon < \min(\varepsilon_0, r_\Gamma)$, w is p -superharmonic in Ω .

Therefore, with these specifications, w is the desired barrier. \square

5. LIMITS OF VARIATIONALLY p -HARMONIOUS FUNCTIONS

In this section, we shall prove Theorem 1.2. We will first collect in Section 5.1 some known facts on the Dirichlet problem (1.3), that we recall here:

$$\Delta_p^G u = 0 \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma.$$

5.1. Generalized viscosity solutions for (1.3). We start with the classical definition of a viscosity solution of an elliptic degenerate equation. Consider a continuous mapping $F : \overline{\Omega} \times \mathbb{R} \times (\mathbb{R}^N \setminus \{0\}) \times \mathcal{S}^N \rightarrow \mathbb{R}$. Here, \mathcal{S}^N is the set of symmetric $N \times N$ matrices. The upper and lower semi-continuous envelopes F^* and F_* of F are the functions defined by

$$F^*(x, s, \xi, X) = \limsup_{(y, t, \eta, Y) \rightarrow (x, s, \xi, X)} F(y, t, \eta, Y)$$

for $(x, s, \xi, X) \in \overline{\Omega} \times \mathbb{R} \times (\mathbb{R}^N \setminus \{0\}) \times \mathcal{S}^N$, and by $F_* = -(-F)^*$.

We recall from [7, 15] that a bounded and upper (resp. lower) semi-continuous function u is a *viscosity subsolution* (resp. *supersolution*) of $F = 0$ in $\overline{\Omega}$ if, for any $(x, \phi) \in \overline{\Omega} \times C^2(\overline{\Omega})$ with $\nabla \phi(x) \neq 0$ and such that $u - \phi$ has a local maximum (resp. minimum) at x with $\phi(x) = u(x)$, it holds that

$$F_*(x, \phi(x), \nabla \phi(x), \nabla^2 \phi(x)) \leq 0 \quad (\text{resp. } F^*(x, \phi(x), \nabla \phi(x), \nabla^2 \phi(x)) \geq 0).$$

A *viscosity solution* of $F = 0$ in $\overline{\Omega}$ is a function $u \in C(\overline{\Omega})$ which is both a viscosity subsolution and supersolution of $F = 0$ in $\overline{\Omega}$.

Also, in [3, 4, 12], the following definitions are considered. A function u in $\text{USC}(\overline{\Omega}) \cap L^\infty(\Omega)$ (resp. in $\text{LSC}(\overline{\Omega}) \cap L^\infty(\Omega)$) is a *generalized viscosity subsolution* (resp. *supersolution*) of (1.3) if, for any $(x, \phi) \in \overline{\Omega} \times C^2(\overline{\Omega})$ with $\nabla \phi(x) \neq 0$ and such that $u - \phi$ has a local maximum (resp. minimum) at x with $u(x) = \phi(x)$, it holds that

$$\begin{aligned} -\Delta_p^G \phi(x) &\leq 0 \quad (\text{resp. } -\Delta_p^G \phi(x) \geq 0) && \text{for } x \in \Omega, \\ \min \{ -\Delta_p^G \phi(x), \phi(x) - g(x) \} &\leq 0 \\ (\text{resp. } \max \{ -\Delta_p^G \phi(x), \phi(x) - g(x) \} &\geq 0) && \text{for } x \in \Gamma. \end{aligned}$$

A *generalized viscosity solution* of (1.3) is a continuous function on $\overline{\Omega}$, which is both a generalized viscosity subsolution and supersolution. This weaker notion of viscosity solution appears naturally as the limiting situation of the dynamic programming principle with respect to (1.5) as is observed below. This formulation has been proved to be equivalent (we refer the reader to [2, 8]) to the standard definition of viscosity solution of $F = 0$ for the operator F defined by

$$F(x, s, \xi, X) = -\frac{\text{tr}(X)}{p} - \frac{p-2}{p} \frac{\langle X\xi, \xi \rangle}{|\xi|^2}$$

for $(x, s, \xi, X) \in \overline{\Omega} \times \mathbb{R} \times (\mathbb{R}^N \setminus \{0\}) \times \mathcal{S}^N$. Here, $\text{tr}(X)$ stands for the trace of the matrix X .

In order to prove Theorem 1.2, we follow an argument used in [5]. To this aim, we need to set up some further notation.

Proposition 5.1. *Let $G : \overline{\Omega} \times \mathbb{R} \times (\mathbb{R}^N \setminus \{0\}) \times \mathcal{S}^N \rightarrow \mathbb{R}$ be the mapping defined by*

$$G(x, s, \xi, X) = \begin{cases} -\frac{\operatorname{tr}(X)}{p} - \frac{p-2}{p} \frac{\langle X\xi, \xi \rangle}{|\xi|^2} & \text{if } x \in \Omega, \\ s - g(x) & \text{if } x \in \Gamma, \end{cases}$$

for $(x, s, \xi, X) \in \overline{\Omega} \times \mathbb{R} \times (\mathbb{R}^N \setminus \{0\}) \times \mathcal{S}^N$. Then, we compute:

$$G^*(x, s, \xi, X) = G_*(x, s, \xi, X) = -\frac{\operatorname{tr}(X)}{p} - \frac{p-2}{p} \frac{\langle X\xi, \xi \rangle}{|\xi|^2} \quad \text{if } x \in \Omega$$

and, if $x \in \Gamma$,

$$G^*(x, s, \xi, X) = \max \left\{ -\frac{\operatorname{tr}(X)}{p} - \frac{p-2}{p} \frac{\langle X\xi, \xi \rangle}{|\xi|^2}, s - g(x) \right\},$$

$$G_*(x, s, \xi, X) = \min \left\{ -\frac{\operatorname{tr}(X)}{p} - \frac{p-2}{p} \frac{\langle X\xi, \xi \rangle}{|\xi|^2}, s - g(x) \right\}.$$

Proof. The first formula for $G^* = G_* = G$ follows from the continuity of G at interior points of Ω . When $x \in \Gamma$ instead, we observe that, for every sufficiently small $\delta > 0$, the supremum of G on $(B_\delta(x) \cap \overline{\Omega}) \times B_\delta(s) \times B_\delta(\xi) \times B_\delta(X)$ (where the balls must be intended in the relevant Euclidean spaces) equals

$$\max \left\{ -\frac{\operatorname{tr}(X)}{p} - \frac{p-2}{p} \frac{\langle X\xi, \xi \rangle}{|\xi|^2}, s - g(x) \right\}.$$

The formula for G^* then ensues. The one for G_* easily follows from the formula $G_* = -(-G)^*$. \square

Remark 5.2. Let $x \in \overline{\Omega}$ and $\phi \in C^2(\overline{\Omega})$ with $\nabla\phi(x) \neq 0$. Then, we have that

$$G(x, \phi(x), \nabla\phi(x), \nabla^2\phi(x)) = \begin{cases} -\Delta_p^G \phi(x) & \text{if } x \in \Omega, \\ \phi(x) - g(x) & \text{if } x \in \Gamma, \end{cases}$$

and hence, we obtain for $x \in \Gamma$,

$$G^*(x, \phi(x), \nabla\phi(x), \nabla^2\phi(x)) = \max \{ -\Delta_p^G \phi(x), \phi(x) - g(x) \},$$

$$G_*(x, \phi(x), \nabla\phi(x), \nabla^2\phi(x)) = \min \{ -\Delta_p^G \phi(x), \phi(x) - g(x) \}.$$

By Remark 5.2 together with [8, Theorem 3.3] and the classical weak comparison principle for the viscosity solution of (1.3) (see [14, 17]), we immediately obtain the following weak comparison principle for $G = 0$ in $\overline{\Omega}$.

Theorem 5.3. *Let $u \in \operatorname{USC}(\overline{\Omega}) \cap L^\infty(\Omega)$ and $v \in \operatorname{LSC}(\overline{\Omega}) \cap L^\infty(\Omega)$. If u, v are respectively a generalized viscosity subsolution and a generalized viscosity supersolution of $G = 0$ in $\overline{\Omega}$, then we have that $u \leq v$ on $\overline{\Omega}$.*

5.2. An approximation scheme by p -harmonious functions. We introduce a family of mappings $\mathcal{A}_\varepsilon : \mathbb{R} \times \overline{\Omega} \times C(\overline{\Omega}) \rightarrow \mathbb{R}$ defined for $0 < \varepsilon < \varepsilon_0$ by

$$\mathcal{A}_\varepsilon(s, x, u) = \begin{cases} \frac{2(N+p)\varepsilon}{p} \frac{s - \mu_p^\varepsilon[u](x)}{r_\varepsilon(x)^2} & \text{if } x \in \Omega, \\ \varepsilon[s - g(x)] & \text{if } x \in \Gamma \end{cases}$$

for $(s, x, u) \in \mathbb{R} \times \overline{\Omega} \times C(\overline{\Omega})$. Here, $\varepsilon_0 > 0$ is taken so small so as to be sure that the set Ω_ε is still a domain for every $0 < \varepsilon < \varepsilon_0$.

Lemma 5.4. *If $x \in \overline{\Omega}$ and $\phi \in C^2(\overline{\Omega})$ is such that $\nabla\phi(x) \neq 0$, then we have that*

$$\begin{aligned} \limsup_{(\varepsilon, y, \delta) \rightarrow (0^+, x, 0)} \frac{\mathcal{A}_\varepsilon(\phi(y) + \delta, y, \phi + \delta)}{\varepsilon} &= G^*(x, \phi(x), \nabla\phi(x), \nabla^2\phi(x)), \\ \liminf_{(\varepsilon, y, \delta) \rightarrow (0^+, x, 0)} \frac{\mathcal{A}_\varepsilon(\phi(y) + \delta, y, \phi + \delta)}{\varepsilon} &= G_*(x, \phi(x), \nabla\phi(x), \nabla^2\phi(x)). \end{aligned}$$

Proof. By using elementary properties of p -means (see [13, Proposition 2.7]), for every $\delta \in \mathbb{R}$, we have that

$$\begin{aligned} \frac{\mathcal{A}_\varepsilon(\phi(y) + \delta, y, \phi + \delta)}{\varepsilon} &= \\ \frac{2(N+p)}{p} \frac{\phi(y) + \delta - \mu_p^\varepsilon[\phi + \delta](y)}{r_\varepsilon(y)^2} &= \frac{2(N+p)}{p} \frac{\phi(y) - \mu_p^\varepsilon[\phi](y)}{r_\varepsilon(y)^2} \quad \text{if } y \in \Omega, \end{aligned}$$

and

$$\frac{\mathcal{A}_\varepsilon(\phi(y) + \delta, y, \phi + \delta)}{\varepsilon} = \phi(y) + \delta - g(y) \quad \text{if } y \in \Gamma.$$

Then if $x \in \Omega$, by the uniform convergence claimed in Lemma 2.5 and the continuity of ϕ and g , we then simply infer that

$$\begin{aligned} \limsup_{(\varepsilon, y, \delta) \rightarrow (0^+, x, 0)} \frac{\mathcal{A}_\varepsilon(\phi(y) + \delta, y, \phi + \delta)}{\varepsilon} &= \\ \begin{cases} -\Delta_p^G \phi(x) & \text{if } x \in \Omega, \\ \max \{-\Delta_p^G \phi(x), \phi(x) - g(x)\} & \text{if } x \in \Gamma. \end{cases} \end{aligned}$$

The first claim is thus proved. The second claim follows similarly. \square

We are now ready to prove our final result.

Proof of Theorem 1.2. Up to re-defining ε_0 , Theorem 1.1 tells us that, for every $\varepsilon \in (0, \varepsilon_0)$, there exists a unique function $u^\varepsilon \in C(\overline{\Omega})$ that solves the equation

$$(5.1) \quad \mathcal{A}_\varepsilon(u^\varepsilon(x), x, u^\varepsilon) = 0 \quad \text{for } x \in \overline{\Omega}.$$

Also, by Lemma 4.1 (i), we have that u^ε is bounded in $\overline{\Omega}$ from below and above by the minimum and the maximum of g on Γ , and hence independently of ε . Thus, by this property, the functions defined for $x \in \overline{\Omega}$ by

$$u_*(x) = \liminf_{(\varepsilon, y) \rightarrow (0^+, x)} u^\varepsilon(y) \quad \text{and} \quad u^*(x) = \limsup_{(\varepsilon, y) \rightarrow (0^+, x)} u^\varepsilon(y)$$

are bounded in $\overline{\Omega}$. Also, u_* and u^* are lower semi-continuous and, respectively, upper semi-continuous in $\overline{\Omega}$, by a standard result.

The strategy of the proof is to show that u_* and u^* are a viscosity supersolution and a viscosity subsolution of $G = 0$ in $\overline{\Omega}$. In fact, if we prove that, the comparison principle in Theorem 5.3 would give that $u^* \leq u_*$ on $\overline{\Omega}$. Thus, since $u_* \leq u^*$ on $\overline{\Omega}$ by construction, we would infer that $u_* \equiv u^*$ on $\overline{\Omega}$. As a consequence, the function defined on $\overline{\Omega}$ by $u = u_* \equiv u^*$ would be continuous and a viscosity solution of $G = 0$ on $\overline{\Omega}$. Then, u would be the unique viscosity solution of (1.3) by [8, Theorem 3.3] and Remark 5.2.

We also point out that, while u^ε converges pointwise to u by construction, we can further prove that this convergence is uniform on $\overline{\Omega}$. In fact, u turns out to be the limit of the monotonic sequences defined for $k \in \mathbb{N}$ and $x \in \overline{\Omega}$ by

$$\begin{aligned} v_k(x) &= \sup \left\{ u^\varepsilon(y) \mid 0 < \varepsilon < 1/k, |y - x| < 1/k, y \in \overline{\Omega} \right\}, \\ w_k(x) &= \inf \left\{ u^\varepsilon(y) \mid 0 < \varepsilon < 1/k, |y - x| < 1/k, y \in \overline{\Omega} \right\}. \end{aligned}$$

The uniform convergence on $\overline{\Omega}$ of these sequences follows from Dini's monotone convergence theorem (see [26, Theorem 7.13]), since they are monotonic, $\overline{\Omega}$ is compact, v_k is lower semicontinuous, w_k is upper semicontinuous, and u is continuous on $\overline{\Omega}$. The uniform convergence of u^ε then follows by observing that

$$w_k(x) \leq u^\varepsilon(x) \leq v_k(x) \quad \text{for } x \in \overline{\Omega} \quad \text{and} \quad 0 < \varepsilon < 1/k.$$

Therefore, to complete the proof, we are left to show that u_* and u^* are a viscosity supersolution and a viscosity subsolution of $G = 0$ in $\overline{\Omega}$. We shall prove that fact only for u^* , since for u_* we can proceed similarly.

We preliminarily notice that, by [13, Theorem 2.5], the mapping \mathcal{A}_ε is decreasing in the third variable, in the sense that for $(s, x) \in \mathbb{R} \times \overline{\Omega}$ and $u, v \in C(\overline{\Omega})$, it holds

$$\mathcal{A}_\varepsilon(s, x, u) \geq \mathcal{A}_\varepsilon(s, x, v) \quad \text{if } u \leq v \quad \text{in } \overline{\Omega}.$$

Now, let $(x, \phi) \in \overline{\Omega} \times C^2(\overline{\Omega})$ with $\nabla\phi(x) \neq 0$ be such that $u^* - \phi$ has a local maximum at x with $u^*(x) = \phi(x)$. Without loss of generality, we can always suppose that the maximum is global and strict, that is $u^* - \phi < u^*(x) - \phi(x) = 0$ in $\overline{\Omega} \setminus \{x\}$. By a standard argument in the theory of viscosity solutions (see [15]), we know that there exists a sequence of elements $(\varepsilon_j, x_j) \in (0, \varepsilon_0) \times \overline{\Omega}$ such that, for each fixed $j \in \mathbb{N}$, x_j is a global maximum point for $u^{\varepsilon_j} - \phi$ and

$$(\varepsilon_j, x_j, u^{\varepsilon_j}(x_j)) \rightarrow (0, x, u^*(x)) \quad \text{as } j \rightarrow \infty.$$

If we set $\delta_j = u^{\varepsilon_j}(x_j) - \phi(x_j)$, then $\delta_j \rightarrow 0$ as $j \rightarrow \infty$ and $u^{\varepsilon_j} - \phi \leq \delta_j$ on $\overline{\Omega}$.

Now, we use (5.1) at x_j for $\varepsilon = \varepsilon_j$ and $u = u^{\varepsilon_j}$, and infer that

$$\begin{aligned} 0 &= \mathcal{A}_{\varepsilon_j}(u^{\varepsilon_j}(x_j), x_j, u^{\varepsilon_j}) = \\ &\quad \mathcal{A}_{\varepsilon_j}(\phi(x_j) + \delta_j, x_j, u^{\varepsilon_j}) \geq \mathcal{A}_{\varepsilon_j}(\phi(x_j) + \delta_j, x_j, \phi + \delta_j). \end{aligned}$$

Here, the last inequality follows from the aforementioned monotonicity of $\mathcal{A}_{\varepsilon_j}$. Then since $\nabla\phi(x) \neq 0$, we can apply Lemma 5.4 to obtain that

$$\begin{aligned} 0 &\geq \liminf_{j \rightarrow \infty} \frac{\mathcal{A}_{\varepsilon_j}(\phi(x_j) + \delta_j, x_j, \phi + \delta_j)}{\varepsilon_j} \geq \\ &\quad \liminf_{(\varepsilon, y, \delta) \rightarrow (0^+, x, 0)} \frac{\mathcal{A}_\varepsilon(\phi(y) + \delta, y, \phi + \delta)}{\varepsilon} = F_*(x, \phi(x), \nabla\phi(x), \nabla^2\phi(x)). \end{aligned}$$

Hence, u^* is a viscosity subsolution of $G = 0$ on $\overline{\Omega}$. The proof of Theorem 1.2 is complete. \square

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