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ERRATA TO: THE ALGEBRA OF SLICE FUNCTIONS

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Abstract. We correct the statement and proof of [4, Proposition 4.10] and straighten out [4, Example 4.13] accordingly. We take this chance to correct a sentence within [4, Examples 1.13].

This note corrects a few errors in the article [4]. The first correction concerns a single sentence within [4, Examples 1.13, page 4736]: we must add to the sentence “The set $\mathcal{Z}$ is the intersection of $\mathcal{J}$ with the hyperplane $q_0 = 0$” the words “minus $\mathbb{R} + \epsilon \text{Im}(\mathbb{H})$.” For a proof of this fact, see [3, page 5513].

The second correction originates from [2, Remark 6.3]. We propose a new statement and proof of [4, Proposition 4.10]. In $\mathbb{R}_3$, let us adopt the notation $\omega_\pm := \frac{1}{2}(1 \pm i_{123})$. We recall that $\omega_+ \omega_- = 0 = \omega_- \omega_+$. We notice that $\omega_+^2 = \omega_-^2 = 1$, and that $\omega_+ - \omega_- = i_{123}$. As a consequence, $\mathbb{R}_3 = \omega_+ \mathbb{R}_2 + \omega_- \mathbb{R}_2$.

Proposition 4.10. If $A = \mathbb{R}_3$, if $f \in S(\Omega)$ and if $x \in \Omega \setminus \mathbb{R}$, then one of the following happens:

1. $V(f) \cap S_x = \emptyset$;
2. $V(f) \cap S_x = \{y\}, f_+(x) \in C_A^+$ and $y = \text{Re}(x) - f_+(x)f_+(x)^{-1}$;
3. $V(f) \cap S_x$ is not empty and $f_+(x) \in \omega_+ \mathbb{R}_2^*$; for all $y \in V(f) \cap S_x$, it holds $V(f) \cap S_x = \{\omega_+ y + \omega_- z : z \in S_x \cap \mathbb{R}_2\}$;
4. $V(f) \supseteq S_x$ and $f_+(x) = 0$.

In each of the aforementioned cases, respectively,

1. $S_x$ does not intersect $V(f^c)$ nor $V(N(f))$;
2. $S_x \subseteq V(N(f))$ and $V(f^c) \cap S_x = \{f_+(x)^{-1} y f_+(x)\}$;
3. $S_x \subseteq V(f^c)$ and $V(f^c) \cap S_x = \{h^{-1} y h : y \in V(f) \cap S_x\}$, where $h \in \mathbb{R}_2^*$ is such that $f_+(x) = \omega_+ h$;
4. $S_x$ is included both in $V(f^c)$ and in $V(N(f))$.

Proof. Let us assume $S_x = \alpha + \beta S_{R_3}$, whence $f(\alpha + \beta I) = a_1 + I a_2$ for each $I \in S_{R_3}$, where $a_1 = f_+(x), a_2 = \beta f_+(x)$. We can apply [4, Theorem 4.1], taking into account the following facts: $\mathbb{R}_3$ is nonsingular; $\mathbb{R}_3$ is compatible; $C_A^+$ is $\mathbb{R}_3$ minus the set $\omega_+ \mathbb{R}_2^* \cup \omega_- \mathbb{R}_2^*$ of its zero divisors. We derive the following properties.

- If $f_+(x) = 0$ then one of the following holds:

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• If \( f'_s(x) \) is a right zero divisor then one of the following holds:
  - \( S_x \) intersects \( V(f) \) at least one point \( y = \alpha + \beta u \) and the intersection is the set of all \( y' = \alpha + \beta v \in S_x \) such that \( (v - u)f'_s(x) = 0 \). Moreover, \( S_x \subseteq V(N(f)) \).
  - \( S_x \) does not intersect \( V(f) \).

We now observe that, when \( f'_s(x) \) is a right zero divisor then one of the following holds:
  - \( S_x \) intersects \( V(f) \) at least one point \( y = \alpha + \beta u \) and the intersection is the set of all \( y' = \alpha + \beta v \in S_x \) such that \( (v - u)f'_s(x) = 0 \). Moreover, \( S_x \subseteq V(N(f)) \).
  - \( S_x \) does not intersect \( V(f) \).

- \( S_x \) is included in \( V(f), V(f^c), V(N(f)) \) and \( V(N(f^c)) \).
- \( S_x \) does not intersect \( V(f), V(f^c) \) nor \( V(N(f)) \).

We now observe that, when \( S_x \) is not included in \( V(N(f)) \), then it does not intersect \( V(N(f)) \). Suppose, indeed, \( N(f)(\alpha + \beta I) = n(a_1) - n(a_2) + I t(a_1 a_2) \) to vanish for some \( I \in S_{R_2}. \) Since the functions \( n, t : R_3 \to R_3 \) take values in the center \( \mathbb{R} + e_{123} \mathbb{R} \) of the algebra, there exist \( a, b, c, d \in \mathbb{R} \) such that \( n(a_1) - n(a_2) = a + e_{123} b \) and \( t(a_1 a_2^2) = c + e_{123} d \), whence \( a + e_{123} b = -I(c + e_{123} d) \). By squaring, we obtain \( a^2 + b^2 + 2 e_{123} a b = -c^2 - d^2 - 2 e_{123} c d \), whence \( a^2 + b^2 = -c^2 - d^2 \). It follows that \( a = b = c = d = 0 \) and that \( n(a_1) - n(a_2) = t(a_1 a_2^2) = 0 \). As a consequence, \( S_x \) is included in \( V(N(f)) \).

We are left with studying in detail the case when \( a_2 = \beta f'_s(x) \) is a right zero divisor. We recall that, in \( R_3 \), both the set of left zero divisors and the set of right zero divisors coincide with \( \omega \mathbb{R}^2 \cup \omega - \mathbb{R}^2 \). We suppose henceforth that \( a_2 = \omega \pm h \) with \( h \in \mathbb{R}^2 \).

We first assume \( y = \alpha + \beta u \in V(f) \cap S_x \). We will prove that \( V(f) \cap S_x = \{ \omega \pm y + \omega \pm z : z \in S_x \cap R_2 \} \) by solving the equation \( (v - u) a_2 = 0 \) or, equivalently, \( (v - u) a_2 = 0 \) for \( v \in S_{R_2} \). This equation is equivalent to \( v - u = \omega \pm k \) for some \( k \in R_2 \). We remark that \( \omega \pm u = \omega \pm u \pm \) for appropriate \( u \in R_2 \). Thus,

\[
v = u + \omega v k = \omega u + \omega v + \omega k = \omega u + \omega v + \omega (u v + k)
\]

for some \( k \in R_2 \). Equivalently, \( v = \omega \pm u \pm + \omega \pm u' = \omega \pm u + \omega \pm u' \) for some \( u' \in R_2 \).

Such a \( v \) belongs to \( S_{R_2} \) if and only if,

\[
0 = t(v) = \omega \pm t(u) + \omega \pm t(u') = \omega \pm t(u'),
\]

\[
1 = n(v) = \omega \pm n(u) + \omega \pm n(u') = \omega \pm + \omega \pm n(u'),
\]

where we took into account the fact that \( t(u) = 0 \) and \( n(u) = 1 \). Thus, \( v \) belongs to \( S_{R_2} \) if and only if \( t(u') = 0, n(u') = 1 \), i.e., \( u' \in S_{R_2} \). It follows that the solutions of \( (v - u) a_2 = 0 \) in \( S_{R_2} \) are exactly the Clifford numbers \( v = \omega \pm u + \omega \pm u' \) with \( u' \in S_{R_2} \). This proves that the elements of \( V(f) \cap S_x \) are the points \( \alpha + \beta v = \omega \pm (\alpha + \beta u) + \omega \pm (\alpha + \beta u') = \omega \pm y + \omega \pm z \) with \( z \in S_x \cap R_2 \).

Under the same assumption \( y = \alpha + \beta u \in V(f) \cap S_x \), not only \( S_x \subseteq V(N(f)) \) as we already stated; it also holds \( y' := h^{-1} y' h \in V(f^c) \cap S_x \). To prove this fact, we first observe that \( y' \) belongs to \( S_x \) by [4, Remark 1.16]: indeed, \( h \in C_{S_x} \). We then observe that \( f(y) = a_1 + u a_2 = 0 \) implies \( a_1 = -u a_2 \), whence \( f'(y') = a_1^2 - h^{-1} u a_2^2 = a_1^2 - h^{-1} u h \omega k = a_1^2 - h^{-1} u h = a_1^2 u - a_1^2 u = 0 \). Taking into account the equalities \( f = (f^c)^c \) and \( \beta(f^c)'(x) = a_2^2 \), we conclude that \( V(f^c) \cap S_x = \{ h^{-1} y' h : y \in V(f) \cap S_x \} \).

Now let us assume, instead, \( V(f) \cap S_x = \emptyset \). We remark that \( V(f^c) \cap S_x = \emptyset \): if \( f^c \) had a zero in \( S_x \), then \( (f^c)^c = f \) would have a zero in \( S_x \) by what we already
proved. We conclude the proof by checking that $V(N(f)) \cap S_a = \emptyset$. Suppose by contradiction $V(N(f)) \cap S_a \neq \emptyset$, whence $S_a \subseteq V(N(f))$. Then $n(a_1) = n(a_2)$ and $t(a_1 a_2^*) = 0$. The fact that $n(a_1) = n(a_2) = \omega_a n(h)$ implies that $a_1 = \omega_a k$, with $k \in \mathbb{R}_2^2$ having $n(k) = n(h)$. Since $1 = n(k)n(h)^{-1} = n(kh^{-1})$ and $0 = t(a_1 a_2^*) = \omega_2 t(kh^+ c) = \omega_2 t(kh^{-1} n(h))$, if we set $w := -kh^{-1}$, then $n(w) = 1$ and $t(w) = 0$. Thus, $w \in S_{R^2}$ and $f(\alpha + \beta w) = \omega_2 k + w\omega_2 h = \omega_2 (k + wh) = 0$, which contradicts the hypothesis $V(f) \cap S_a = \emptyset$. \hfill \qed

Consequently, we apply a third correction. Namely, we modify [4, Example 4.13] as follows.

**Example 4.13.** Let $A = \mathbb{R}_3$ and let $f(x) = (e_1 - \frac{\text{Im}(x)}{\text{Im}(x^2)}) \omega_-$. Then $f$ is slice regular in $Q_A \setminus \mathbb{R}$ and $f$ is constant in $C^+_I$ for each $I \in S_{\mathbb{R}_2}$. By direct computation, $f(e_1) = 0$ and $f'_1(e_1) = -\omega_-$. By Proposition 4.10, it holds

$$V(f) = \bigcup_{u \in S_{\mathbb{R}_2}} C_{-e_1 + \omega_+ u}^+.$$

For instance, $C_{e_1}^+, C_{e_2}^+$ are both included in $V(f)$ because $\omega_- e_1 + \omega_+ e_1 = e_1$ and $\omega_- e_1 + \omega_+ (-e_1) = (\omega_- - \omega_+) e_1 = -e_1 e_1 = -e_1^2 e_2 = e_2 e_3 = e_{23}$. An example of $g \in SR(Q_A \setminus \mathbb{R})$ with the same zero set as $f$, but which is not constant along the half-slices $C^+_I$, can be constructed following [1] and letting $g(x) = x \cdot f(x) = x f(x)$.

We take this chance to point out that [5, Example 9.6] must be corrected along the same lines. This is done in [2, Example 6.6], with an approach that is slightly different from ours.

**References**


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