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Existence and uniqueness of solution for multidimensional parabolic PDAEs arising in semiconductor modeling *

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Abstract

This paper concerns with a compact network model combined with distributed models for semiconductor devices. For linear RLC networks containing distributed semiconductor devices, we construct a mathematical model that joins the differential-algebraic initial value problem for the electric circuit with multi-dimensional parabolic-elliptic boundary value problems for the devices. We prove an existence and uniqueness result, and the asymptotic behavior of this mixed initial boundary value problem of partial differential-algebraic equations.

1 Introduction

The main ingredient used in circuit simulation is the lumped network equations for the simulation of the network designs. The application of the Modified Nodal Analysis (MNA) formalism yields a system of differential-algebraic equations (DAEs), that can be classified using the index concept [1]. The index determines the number of inherent derivatives that are needed to derive the ordinary differential equation. Different index cases can only be distinguished by structural means for the classical MNA equations. Using for instance the tractability index [2], one decomposes the set of variables accordingly and projects parts of the equations.

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The transition from microelectronics to nanoelectronics requires a more systematic study of the coupling effects. These effects are particularly relevant in integrated circuit modeling, and their relevance increases with the decreasing of the scales. Relevant examples are, e.g., electrothermal coupling [3–5], electromagnetic coupling [6, 7], or electric network-device coupling [8–10].

In the modeling of the coupling effects there is a basic set of “lumped” differential-algebraic equations, generally the electric network equations, and a set of “distributed” partial differential equations, which model phenomena which arise at a different, finer scale. The coupling involves an interplay between integrated quantities coming from the distributed model, which enter the lumped equations, and lumped variables which are related to boundary data for the distributed equations. This leads directly to coupled systems of differential-algebraic equations (DAEs) for the electric network and partial differential equations (PDEs) for the semiconductor devices. The coupling has two parts. On the one hand, an additional source term occurs in the current balance of the electric network. On the other hand, the boundary conditions of the device equations depend on the time-dependent node potentials, which are genuine unknowns of the electric network.

In this paper, we focus on the electric network-device coupling. We consider an electric network which contains semiconductor devices, modeled by multi-dimensional, parabolic, drift-diffusion equations.

The coupling of the drift-diffusion model with the electric network equations have been extensively studied. The well-posedness of the resulting coupled system has been investigated for the steady-state (elliptic) one-dimensional case in [3, 8] and in the index-1 and index-2 multidimensional case respectively in [11] and [12]. The uniqueness for data close to equilibrium in the steady-state (elliptic) one-dimensional case is proved in [13]. In [14] is given a systematical approach for the decomposition in case of index greater than 2.

Other fundamental works in the direction of the study of coupled model within the framework of abstract differential-algebraic equations are [2, 15, 16]. The more recent publication [17] summarizes the state of the art of such coupled models in the simulation of electric circuits.

We also mention that, in practical applications, the electric network-device coupled system needs to be discretized in space for the distributed components of the system. In this respect, the analysis of the resulting semidiscretized DAE, as done in [18], is relevant and deeply connected to our analysis.

The present paper is strongly related to [9], where the authors prove existence and uniqueness result for the time-dependent (parabolic-elliptic) one-dimensional case. Here, we extend the above results to the multi-dimensional case. Namely, we establish existence and uniqueness of solution for an multi-dimensional index-1 parabolic partial differential-algebraic equation, and we discuss the asymptotic behaviour for long times.

The work is organized as follows. Section 2 covers the modeling of the coupled system; both subsystems are described in detail and the coupling terms are defined. In Section 3 we state the main result. First, we prove the local existence and uniqueness of solution and then using some a priori estimates we extend the solution globally. Section 4 deals with the asymptotic behaviour of the solution when time tends to infinity.

2 Coupled circuit-device model

In this section we consider an electric network which contains semiconductor devices. This system can be described by a coupled model, which consists of the electric network equations, the semiconductor device equations, and their coupling conditions. The resulting equations, although arising from a specific application, are susceptible of different generalizations and extensions.

2.1 Network models for electric circuits.

An electric network is described by the electrical potentials at the nodes and by the currents through the branches. Using the approach of Modified Nodal Analysis (MNA) [19, 20], the electric network equations can be obtained by the Kirchhoff current law, replacing the constitutive equations for the currents through branches with capacitors and resistances, and by the constitutive equations for the remaining components.

We consider a linear RLC network, that is, a network which connects n_C linear capacitors, n_L inductors and n_R resistors, and n_I independent voltage and n_V current sources. We assume that the network connects also semiconductor devices connected with circuit by Ohmic contacts. We assume that the network has m nodes plus the ground node, where the potential is zero. We denote by $\mathbf{u}(t) \in \mathbb{R}^m$ the node potentials, by $\mathbf{i}_L(t) \in \mathbb{R}^{n_L}$ the currents through inductors, by $\mathbf{i}_V(t) \in \mathbb{R}^{n_V}$ the currents through voltage sources, by $\mathbf{v}_V(t) \in \mathbb{R}^{n_V}$ the independent voltage sources, and by $\mathbf{i}_I(t) \in \mathbb{R}^{n_I}$ the independent current sources. The linear MNA system of equations can be written as

$$\mathbf{A}_C \mathbf{C} \mathbf{A}_C^\top \frac{d}{dt} \mathbf{u} + \mathbf{A}_R \mathbf{G} \mathbf{A}_R^\top \mathbf{u} + \mathbf{A}_L \mathbf{i}_L + \mathbf{A}_V \mathbf{i}_V + \boldsymbol{\ell}_D(\mathbf{x}) = \mathbf{A}_I \mathbf{i}_I(t), \quad (2.1a)$$

$$\mathbf{L} \frac{d}{dt} \mathbf{i}_L - \mathbf{A}_L^\top \mathbf{u} = \mathbf{O}, \quad (2.1b)$$

$$-\mathbf{A}_V^\top \mathbf{u} = \mathbf{v}_V(t), \quad (2.1c)$$

which is a differential-algebraic equation for the unknowns $\mathbf{u}$, $\mathbf{i}_L$, $\mathbf{i}_V$. Here, $\mathbf{A}_C \in \mathbb{R}^{m \times n_C}$, $\mathbf{A}_R \in \mathbb{R}^{m \times n_R}$, $\mathbf{A}_L \in \mathbb{R}^{m \times n_L}$, $\mathbf{A}_V \in \mathbb{R}^{m \times n_V}$, $\mathbf{A}_I \in \mathbb{R}^{m \times n_I}$, are incidence matrices, which describe the topology of the network. Moreover $\mathbf{C} \in \mathbb{R}^{n_C \times n_C}$, $\mathbf{G} \in \mathbb{R}^{n_G \times n_G}$ and $\mathbf{L} \in \mathbb{R}^{n_L \times n_L}$ denote the capacitance matrix, the conductance matrix and the inductance matrix, respectively, which are regular. The term $\boldsymbol{\ell}_D(\mathbf{x})$ in (2.1a) represents the coupling with semiconductor devices and it will be described in details later on.

We supplement equation (2.1) with consistent initial data

$$\mathbf{u}(t_0) = \mathbf{u}_0, \quad \mathbf{i}_L(0) = \mathbf{i}_{L0}, \quad \mathbf{i}_V(0) = \mathbf{i}_{V0}, \quad (2.2)$$

this consistency will be discussed in the next section.

2.2 The d -dimensional diode model.

For simplicity we consider a network containing only one semiconductor device. The case of circuits with many devices can be dealt using the same arguments, but the notation would

be much heavier. We consider a semiconductor device with $m_D + 1$ terminals. The device can be modeled by means of a domain $\Omega \subset \mathbb{R}^d$, characterized by a doping profile $N(\mathbf{x})$, with $\mathbf{x} \in \Omega$. We neglect all thermal effects, and assume that two carriers are responsible for the diode's output current, that is, electrons with negative charge $-q$, and holes with positive charge q . The behavior of the device is described in terms of number densities of electrons and holes, denoted by $n(\mathbf{x}, t), p(\mathbf{x}, t)$, current densities for electrons and holes, denoted by $\mathbf{J}_n(\mathbf{x}, t), \mathbf{J}_p(\mathbf{x}, t)$, and electrostatic potential, denoted by $\psi(\mathbf{x}, t)$, $t \in [0, T]$. These variables satisfy the following drift-diffusion system [21, 22]:

$$-\nabla \cdot (\epsilon_s \nabla \psi) = q(p - n + N), \quad (2.3a)$$

$$-q\partial_t n + \nabla \cdot \mathbf{J}_n = qR, \quad (2.3b)$$

$$q\partial_t p + \nabla \cdot \mathbf{J}_p = -qR, \quad (2.3c)$$

for $\mathbf{x} \in \Omega$ and $t \in [0, T]$. In (2.3a), ϵ_s is the semiconductor permittivity. The net recombination rate $R(n, p)$ on the right-hand side of equations (2.3b) and (2.3c) describes the generation or recombination of carriers due to various scattering effects. The current densities in (2.3b) and (2.3c) are given by:

$$\mathbf{J}_n = q(D_n \nabla n - \mu_n n \nabla \psi) \quad \text{and} \quad \mathbf{J}_p = -q(D_p \nabla p + \mu_p p \nabla \psi), \quad (2.3d)$$

where D_n, D_p , respectively, are the electrons and holes diffusion coefficients, and μ_n, μ_p , respectively, are the electrons and holes mobilities.

We supplement the system (2.3) with the initial conditions at time $t = 0$

$$n(\mathbf{x}, 0) = n_0(\mathbf{x}), \quad p(\mathbf{x}, 0) = p_0(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega. \quad (2.4)$$

Since we consider a semiconductor device containing $m_D + 1$ terminals, the boundary $\partial\Omega$ of the domain Ω contains $m_D + 1$ open, disconnected subsets $\Gamma_{D,i}$, $i = 0, 1, \dots, m_D$, representing the terminals of the device

$$\partial\Omega = \Gamma_D \cup \Gamma_N, \quad \Gamma_D = \bigcup_{i=0}^{m_D} \Gamma_{D,i}, \quad \Gamma_N = \partial\Omega \setminus \Gamma_D. \quad (2.5)$$

On Γ_D we assume Dirichlet boundary conditions, that is

$$n(\mathbf{x}, t) = n_D(\mathbf{x}), \quad p(\mathbf{x}, t) = p_D(\mathbf{x}), \quad \mathbf{x} \in \Gamma_D \quad (2.6a)$$

$$\psi(\mathbf{x}, t) = \psi_{bi}(\mathbf{x}) + u_{D,i}(t), \quad \mathbf{x} \in \Gamma_{D,i}, \quad i = 0, 1, \dots, m_D \quad (2.6b)$$

where

$$n_D = \frac{N}{2} + \sqrt{\left(\frac{N}{2}\right)^2 + n_i^2}, \quad p_D = -\frac{N}{2} + \sqrt{\left(\frac{N}{2}\right)^2 + n_i^2},$$

and

$$\psi_{bi} = U_T \ln \frac{n_D}{n_i},$$

where U_T is the thermal potential. The external potentials $u_{D,i}$, $i = 0, \dots, m_D$ coincide with the electric potentials at the nodes of the network which correspond to the terminals.

They represent the coupling with the circuit, and will be described later. On the remaining boundaries we assume homogeneous Neumann boundary conditions,

$$\nabla\psi(\mathbf{x}, t) \cdot \boldsymbol{\nu} = \mathbf{J}_n(\mathbf{x}, t) \cdot \boldsymbol{\nu} = \mathbf{J}_p(\mathbf{x}, t) \cdot \boldsymbol{\nu} = 0, \quad \mathbf{x} \in \Gamma_N, \quad t \in [0, T]. \quad (2.6c)$$

In (2.6c), $\boldsymbol{\nu}$ denotes the outer normal to the boundary.

2.3 Coupling conditions

In this section we explain the coupling conditions between the network and the device.

We identify the total current through $\Gamma_{D,i}$ with

$$J_{D,i} := - \int_{\Gamma_{D,i}} (-\epsilon_s \partial_t \nabla \psi + \mathbf{J}_n + \mathbf{J}_p) \cdot \boldsymbol{\nu}_i \, d\sigma, \quad i = 0, 1, \dots, m_D, \quad (2.7)$$

which is the sum of the so-called displacement current – that is, the time derivative of the electric field – with the currents due to carriers.

It is convenient to express the electric currents through the Ohmic contacts by using the auxiliary functions w_i , $i = 0, 1, \dots, m_D$, defined by the following elliptic boundary value problem:

$$\begin{cases} -\nabla \cdot (\epsilon_s \nabla w_i) = 0, & \text{in } \Omega, \\ w_i = \delta_{ij}, & \text{on } \Gamma_{D,j}, \quad j = 0, 1, \dots, m_D, \\ \frac{\partial w_i}{\partial \nu} = 0, & \text{on } \Gamma_N, \end{cases} \quad (2.8)$$

where δ_{ij} is Kronecker's delta ($\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$). Then, we can write:

$$J_{D,i}(t) = - \int_{\Omega} \nabla w_i \cdot (-\epsilon_s \partial_t \nabla \psi + \mathbf{J}_n + \mathbf{J}_p) \, d\mathbf{x}. \quad (2.9)$$

Furthermore, we can make explicit the dependence of the currents on the external applied potentials by decomposing the electric potential with the help of the auxiliary functions w_i . We introduce the stationary part ψ^* of the electric potential, defined as the solution of the problem

$$\begin{cases} -\nabla \cdot (\epsilon_s \nabla \psi^*) = qN, & \text{in } \Omega, \\ \psi^* = \psi_{\text{bi}}, & \text{on } \Gamma_D, \\ \boldsymbol{\nu} \cdot \nabla \psi^* = 0, & \text{on } \Gamma_N. \end{cases} \quad (2.10)$$

We define the linear functional $\mathcal{L}(\rho)$, which to the function ρ associates the function $\varphi \equiv \mathcal{L}(\rho)$, solution of the problem:

$$\begin{cases} -\nabla \cdot (\epsilon_s \nabla \varphi) = \rho, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \Gamma_D, \\ \boldsymbol{\nu} \cdot \nabla \varphi = 0, & \text{on } \Gamma_N. \end{cases} \quad (2.11)$$

Then we can write the electrostatic potential as

$$\psi = \psi^* + \sum_{i=0}^{m_D} w_i u_{D,i} + \mathcal{L}(qp - qn).$$

Using this decomposition, the electric current $J_{D,i}$ becomes

$$J_{D,i} = \sum_{j=0}^{m_D} C_{D,ij} \frac{du_{D,j}}{dt} + \mathcal{J}_{D,i}(u_{D,0}, u_{D,1}, \dots, u_{D,m_D}; \mathbf{J}), \quad i = 0, 1, \dots, m_D, \quad (2.12)$$

where

$$C_{D,ij} = \int_{\Omega} \epsilon_s \nabla w_i \cdot \nabla w_j d\mathbf{x}, \quad i, j = 0, 1, \dots, m_D, \quad (2.13)$$

and, for given regular functions n , p , ψ and given data n_D , p_D , we have introduced the functional $\mathcal{J}_{D,i}$ which to any values $u_{D,i}$, $i = 0, 1, \dots, m_D$, associates

$$\mathcal{J}_{D,i}(u_{D,0}, u_{D,1}, \dots, u_{D,m_D}; \mathbf{J}) = - \int_{\Omega} \nabla w_i \cdot (\epsilon_s \nabla \mathcal{L}(\nabla \cdot \mathbf{J}) + \mathbf{J}) d\mathbf{x} \quad i = 0, 1, \dots, m_D, \quad (2.14)$$

where $\mathbf{J} = \mathbf{J}_n + \mathbf{J}_p$, and n , p , ψ , n_D , p_D satisfy (2.3), (2.6). To write (2.12) in compact form, we introduce the vectors

$$\mathbf{J}_D = \begin{bmatrix} J_{D,0} \\ J_{D,1} \\ \vdots \\ J_{D,m_D} \end{bmatrix}, \quad \mathbf{u}_D = \begin{bmatrix} u_{D,0} \\ u_{D,1} \\ \vdots \\ u_{D,m_D} \end{bmatrix}, \quad \mathcal{J}_D = \begin{bmatrix} \mathcal{J}_{D,0} \\ \mathcal{J}_{D,1} \\ \vdots \\ \mathcal{J}_{D,m_D} \end{bmatrix},$$

and the matrix $\hat{\mathbf{C}}_D = (C_{D,ij})_{i,j=0,1,\dots,m_D} \in \mathbb{R}^{(m_D+1) \times (m_D+1)}$. Then, we can write

$$\mathbf{J}_D = \hat{\mathbf{C}}_D \frac{d\mathbf{u}_D}{dt} + \mathcal{J}_D(\mathbf{u}_D; \mathbf{J}). \quad (2.15)$$

The currents satisfy the identity

$$\sum_{i=0}^{m_D} J_{D,i} = 0, \quad (2.16)$$

which is a consequence of the equation

$$\nabla \cdot (-\epsilon_s \partial_t \nabla \psi + \mathbf{J}_n + \mathbf{J}_p) = 0, \quad (2.17)$$

obtained by taking the time derivative of the Poisson equation (2.3a) and using the continuity equations (2.3b) and (2.3c). Thus, the currents through the interfaces are not independent, due to (2.16), which implies

$$J_{D,0} = - \sum_{i=1}^{m_D} J_{D,i}.$$

Moreover, it is possible to prove that

$$\sum_{j=0}^{m_D} C_{D,ij} = 0, \quad i = 0, 1, \dots, m_D. \quad (2.18)$$

It is convenient to introduce the vectors

$$\mathbf{I}_D = \begin{bmatrix} J_{D,1} \\ \vdots \\ J_{D,m_D} \end{bmatrix}, \quad \mathbf{v}_D = \begin{bmatrix} u_{D,1} - u_{D,0} \\ \vdots \\ u_{D,m_D} - u_{D,0} \end{bmatrix}, \quad \mathcal{I}_D = \begin{bmatrix} \mathcal{J}_{D,1} \\ \vdots \\ \mathcal{J}_{D,m_D} \end{bmatrix}, \quad (2.19)$$

and the matrices $\mathbf{C}_D = (C_{D,ij})_{i,j=1,\dots,m_D} \in \mathbb{R}^{m_D \times m_D}$ and

$$\hat{\mathbf{A}}_D = \begin{bmatrix} -1 & -1 & \cdots & -1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Then we can write

$$\mathbf{J}_D = \hat{\mathbf{A}}_D \mathbf{I}_D, \quad \mathbf{v}_D = \hat{\mathbf{A}}_D^\top \mathbf{u}_D, \quad (2.20)$$

and (2.15) can be replaced by

$$\mathbf{I}_D = \mathbf{C}_D \frac{d\mathbf{v}_D}{dt} + \mathcal{I}_D(\mathbf{v}_D; \mathbf{J}). \quad (2.21)$$

Combining (2.20) and (2.21), and comparing the result with (2.15), we have the identity $\hat{\mathbf{C}}_D = \hat{\mathbf{A}}_D \mathbf{C}_D \hat{\mathbf{A}}_D^\top$. In (2.21), $\mathcal{I}_D(\mathbf{u}_D; \mathbf{J})$ has been replaced with $\mathcal{I}_D(\mathbf{v}_D; \mathbf{J})$, because the current is invariant for translations of the electric potential. In other words, if the triple (n, p, ψ) satisfy (2.3) with boundary conditions (2.6), then also the triple $(n, p, \bar{\psi} = \psi - u_{D,0})$ satisfy (2.3) with boundary conditions (2.6a) and

$$\bar{\psi}(\mathbf{x}, t) = \psi_{\text{bi}}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_{D,0}, \quad \bar{\psi}(\mathbf{x}, t) = \psi_{\text{bi}}(\mathbf{x}) + v_{D,i}(t), \quad \mathbf{x} \in \Gamma_{D,i}, \quad i = 1, \dots, m_D,$$

and the current densities $\mathbf{J}_n, \mathbf{J}_p$ in (2.3d) are the same for both triples.

Circuit-to-device coupling conditions

To relate the potentials $\mathbf{u}_D$, applied at the $m_D + 1$ contacts of the device, with the network potentials $\mathbf{u}$, we need to introduce a contact-to-node incidence matrix, which relates the device's contacts to the network nodes, selecting the node which corresponds to each contact. For this reason, we will call this matrix *selection matrix*, and denote it by $\mathbf{S}_D = (s_{D,ij}) \in \mathbb{R}^{m \times (m_D + 1)}$. The sifting matrix is defined by:

$$s_{D,ij} = \begin{cases} 1, & \text{if the contact } j \text{ is connected to the node } i, \\ 0, & \text{otherwise.} \end{cases} \quad (2.22)$$

By virtue of this definition, we can write

$$\mathbf{u}_D = \mathbf{S}_D^\top \mathbf{u}. \quad (2.23)$$

We refer to relation (2.23) as the *circuit-to-device coupling condition*, that can be also written using the voltage drop $\mathbf{v}_D$ defined in (2.19):

$$\mathbf{v}_D = \mathbf{A}_D^\top \mathbf{u}, \quad (2.24)$$

with $\mathbf{A}_D = \mathbf{S}_D \hat{\mathbf{A}}_D$.

Device-to-circuit coupling conditions

Using the selection matrix $\mathbf{S}_D$, we can express the term ℓ_D appearing in (2.1a) as:

$$\ell_D = \mathbf{S}_D \mathbf{J}_D = \mathbf{A}_D \mathbf{I}_D. \quad (2.25)$$

Using the representation (2.21), we get

$$\ell_D = \mathbf{A}_D \mathbf{C}_D \mathbf{A}_D^\top \frac{d\mathbf{u}}{dt} + \mathbf{A}_D \mathcal{I}_D(\mathbf{A}_D^\top \mathbf{u}; \mathbf{J}). \quad (2.26)$$

We refer to this relation as the *device-to-circuit coupling condition*.

2.4 Circuit unknowns decomposition and topological conditions

The device-to-circuit coupling condition (2.26) can be used in the circuit equations (2.1a), obtaining

$$(\mathbf{A}_C \mathbf{C} \mathbf{A}_C^\top + \mathbf{A}_D \mathbf{C}_D \mathbf{A}_D^\top) \frac{d\mathbf{u}}{dt} + \mathbf{A}_R \mathbf{G} \mathbf{A}_R^\top \mathbf{u} + \mathbf{A}_L \mathbf{i}_L + \mathbf{A}_V \mathbf{i}_V + \mathbf{A}_D \mathcal{I}_D(\mathbf{A}_D^\top \mathbf{u}; \mathbf{J}) = \mathbf{A}_I \mathbf{i}_I(t).$$

Then, the circuit equations (2.1)-(2.2) can be written in the compact form

$$\mathbf{E} \frac{d\mathbf{x}}{dt} = \mathbf{A} \mathbf{x} + \mathcal{F}_D(\mathbf{x}; \mathbf{J}) + \mathbf{s}(t), \quad (2.27)$$

$$\mathbf{x}(0) = \mathbf{x}_0, \quad (2.28)$$

which is a differential-algebraic equation for the unknown

$$\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \mathbf{i}_L \\ \mathbf{i}_V \end{bmatrix} \in \mathbb{R}^{n_o}, \quad n_o = m + n_L + n_V.$$

The matrices in (2.27) are given by

$$\mathbf{E} = \begin{bmatrix} \mathbf{A}_C \mathbf{C} \mathbf{A}_C^\top + \mathbf{A}_D \mathbf{C}_D & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{L} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -\mathbf{A}_R \mathbf{G} \mathbf{A}_R^\top & -\mathbf{A}_L & -\mathbf{A}_V \\ \mathbf{A}_L^\top & \mathbf{O} & \mathbf{O} \\ \mathbf{A}_V^\top & \mathbf{O} & \mathbf{O} \end{bmatrix},$$

$$\mathcal{F}_D(\mathbf{x}; \mathbf{J}) = -\boldsymbol{\pi}^\top \mathbf{A}_D \mathcal{I}_D(\mathbf{A}_D^\top \boldsymbol{\pi} \mathbf{x}; \mathbf{J}), \quad \mathbf{s}(t) = \begin{bmatrix} \mathbf{A}_I \mathbf{i}_I(t) \\ \mathbf{O} \\ \mathbf{v}_V(t) \end{bmatrix}, \quad (2.29)$$

where $\boldsymbol{\pi} = [\mathbf{I} \ \mathbf{O} \ \mathbf{O}] \in \mathbb{R}^{m \times n_o}$ is a projection matrix which selects the first block of the circuit unknown $\mathbf{x}$, so that $\mathbf{u} = \boldsymbol{\pi}\mathbf{x}$.

Using a well-established procedure [23], we decompose the circuit's unknowns into a differential component and an algebraic component. We describe the decomposition procedure and, at the same time, introduce index-1 topological conditions. First we write the system in the equivalent form:

$$\mathbf{E}_1 \left(\mathbf{P} \frac{d\mathbf{y}}{dt} + \mathbf{Q}\mathbf{z} \right) = \mathbf{A}_1\mathbf{y} + \mathcal{F}_D(\mathbf{P}\mathbf{y} + \mathbf{Q}\mathbf{z}; \mathbf{J}) + \mathbf{s}(t), \quad (2.30)$$

where $\mathbf{y} = \mathbf{P}\mathbf{x}$, $\mathbf{z} = \mathbf{Q}\mathbf{x}$, with $\mathbf{Q}$ projector onto $\ker(\mathbf{E})$, $\mathbf{P} = \mathbf{I} - \mathbf{Q}$ the complementary projector of $\mathbf{Q}$, and

$$\mathbf{E}_1 = \mathbf{E} - \mathbf{A}\mathbf{Q}, \quad \mathbf{A}_1 = \mathbf{A}\mathbf{P}. \quad (2.31)$$

We have explicitly

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{CD} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} \mathbf{P}_{CD} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \end{bmatrix}, \quad (2.32)$$

with $\mathbf{Q}_{CD}$ projector onto $\ker[\mathbf{A}_C \ \mathbf{A}_D]^\top$, and $\mathbf{P}_{CD} = \mathbf{I} - \mathbf{Q}_{CD}$. Then the index-1 conditions should ensure that $\mathbf{E}_1$ is invertible and $\mathcal{F}_D$ depends only on $\mathbf{y}$. If this is the case, we can decompose (2.30) as follows:

$$\frac{d\mathbf{y}}{dt} = \mathbf{P}\mathbf{E}_1^{-1} (\mathbf{A}_1\mathbf{y} + \mathcal{F}_D(\mathbf{P}\mathbf{y}; \mathbf{J}) + \mathbf{s}(t)), \quad (2.33)$$

$$\mathbf{z} = \mathbf{Q}\mathbf{E}_1^{-1} (\mathbf{A}_1\mathbf{y} + \mathcal{F}_D(\mathbf{P}\mathbf{y}; \mathbf{J}) + \mathbf{s}(t)). \quad (2.34)$$

We can see that the first equation is decoupled from the second and can be solved for $\mathbf{y}$.

It is possible to characterize the invertibility of $\mathbf{E}_1$ in terms of network topological conditions, involving only incidence matrices, as done for DAEs [23]. In this way we obtain the following PDAE index-1 conditions:

$$\ker(\mathbf{A}_D, \mathbf{A}_C, \mathbf{A}_R, \mathbf{A}_V)^\top = \{0\}, \quad (2.35)$$

$$\ker \mathbf{Q}_{CD}^\top \mathbf{A}_V = \{0\}. \quad (2.36)$$

These conditions also ensure that $\mathcal{F}_D$ depends only on $\mathbf{y}$, since we have

$$\mathcal{F}_D(\mathbf{P}\mathbf{y} + \mathbf{Q}\mathbf{z}; \mathbf{J}) = -\boldsymbol{\pi}^\top \mathbf{A}_D \mathcal{I}_D(\mathbf{A}_D^\top \boldsymbol{\pi}(\mathbf{P}\mathbf{y} + \mathbf{Q}\mathbf{z}); \mathbf{J}),$$

and $\mathbf{A}_D^\top \boldsymbol{\pi}\mathbf{Q} = [\mathbf{A}_D^\top \mathbf{Q}_{CD} \ \mathbf{O} \ \mathbf{O}] = \mathbf{O}$ by construction.

The index-1 conditions affects also the algebraic equation. In fact, we have

$$\mathbf{Q}\mathbf{E}_1^{-1} = \bar{\mathbf{E}}_1^{-1} \mathbf{Q}^\top, \quad \bar{\mathbf{E}}_1 = \mathbf{E} - \mathbf{Q}^\top \mathbf{A}\mathbf{Q}, \quad (2.37)$$

which implies $\mathbf{Q}\mathbf{E}_1^{-1} \boldsymbol{\pi}^\top \mathbf{A}_D = 0$, and thus

$$\mathbf{Q}\mathbf{E}_1^{-1} \mathcal{F}_D(\mathbf{P}\mathbf{y}; \mathbf{J}) = 0. \quad (2.38)$$

In turns, this yields a simplification of the decomposition (2.34), that is,

$$\mathbf{z} = \mathbf{Q}\mathbf{E}_1^{-1}(\mathbf{A}_1\mathbf{y} + \mathbf{s}(t)). \quad (2.39)$$

Finally, we discuss the consistency of the initial data (2.28). From the above discussion, we see that we can only supplement the (2.27) with the initial data

$$\mathbf{y}(t_0) = \mathbf{y}_0 = \mathbf{P}\mathbf{x}_0. \quad (2.40)$$

Then the algebraic part $\mathbf{z}_0 = \mathbf{Q}\mathbf{x}_0$ of the initial data (2.28) must satisfy the consistency condition

$$\mathbf{z}_0 = \mathbf{Q}\mathbf{E}_1^{-1}(\mathbf{A}_1\mathbf{P}\mathbf{y}_0 + \mathbf{s}(t_0)). \quad (2.41)$$

3 Global existence and uniqueness

The main results of this paper is the existence and uniqueness of the solution of the problem (2.1)-(2.3) with boundary and initial conditions (2.6) and (2.4) and coupling conditions (2.23) and (2.26). We summarize here the equations for the reader's convenience:

$$\mathbf{E}\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathcal{F}_D(\mathbf{x}; \mathbf{J}) + \mathbf{s}(t), \quad \text{in } (0, T], \quad (3.1a)$$

$$\mathbf{x}(0) = \mathbf{x}_0, \quad (3.1b)$$

$$-\nabla \cdot (\epsilon_s \nabla \psi) = q(p - n + N), \quad (3.1c)$$

$$-q\partial_t n + \nabla \cdot \mathbf{J}_n = qR, \quad \mathbf{J}_n = q(D_n \nabla n - \mu_n n \nabla \psi), \quad \text{in } (0, T] \times \Omega, \quad (3.1d)$$

$$q\partial_t p + \nabla \cdot \mathbf{J}_p = -qR, \quad \mathbf{J}_p = -q(D_p \nabla p + \mu_p p \nabla \psi), \quad (3.1e)$$

$$n = n_D, \quad p = p_D, \quad \text{in } (0, T] \times \Gamma_D, \quad (3.1f)$$

$$\psi = \psi_{\text{bi}} + u_{D,i}, \quad \text{in } (0, T] \times \Gamma_{D,i}, \quad i = 0, 1, \dots, m_D, \quad (3.1g)$$

$$\nabla \psi \cdot \boldsymbol{\nu} = \mathbf{J}_n \cdot \boldsymbol{\nu} = \mathbf{J}_p \cdot \boldsymbol{\nu} = 0 \quad \text{in } (0, T] \times \Gamma_N, \quad (3.1h)$$

$$n(\mathbf{x}, 0) = n_0(\mathbf{x}), \quad p(\mathbf{x}, 0) = p_0(\mathbf{x}) \quad \text{in } \Omega, \quad (3.1i)$$

$$\mathbf{u}_D = \mathbf{S}_D^\top \boldsymbol{\pi} \mathbf{x}, \quad (3.1j)$$

$$\mathcal{F}_D(\mathbf{x}; \mathbf{J}) = -\boldsymbol{\pi}^\top \mathbf{A}_D \mathcal{I}_D(\mathbf{A}_D^\top \boldsymbol{\pi} \mathbf{x}; \mathbf{J}), \quad \mathbf{J} = \mathbf{J}_n + \mathbf{J}_p. \quad (3.1k)$$

In the first subsection we state the main problem in a formal way, in the next subsection we prove local existence and uniqueness, and in the final subsection we prove the a-priori estimates that allow to extend to solution globally in time.

3.1 Notations and assumptions

Let us start introducing the notation on functional spaces. We denote by $L^r = L^r(\Omega)$ and $W^{k,r} = W^{k,r}(\Omega)$ the usual spaces of functions, with norms $\|\cdot\|_r$ and $\|\cdot\|_{k,r}$, respectively, and we denote $H^k = W^{k,2}$, [25]. We also use the space L_+^2 of all functions in L^2 which are

nonnegative almost everywhere. Let $[0, T]$ be a bounded time interval. For any Banach space V , we denote by $C([0, T]; V)$, $L^r([0, T]; V)$, and $H^k([0, T]; V)$ the usual spaces of functions defined on $[0, T]$ with values in V . For the purposes of this paper, we introduce the following Banach spaces:

$$\begin{aligned} W_0^{2,r} &= \{u \in W^{2,r} \mid u|_{\Gamma_D} = 0, \boldsymbol{\nu} \cdot \nabla u|_{\Gamma_N} = 0\}, \\ X &= \{u \in H^1 \mid u|_{\Gamma_D} = 0\}, \\ Y &= C([0, T]; L^2) \cap L^2([0, T]; X) \cap H^1([0, T]; X^*), \\ W &= Y \times Y \times \{C([0, T]; W_0^{2,2}) \cap L^2([0, T]; W_0^{2,r}) \cap H^1([0, T]; X)\}, \\ \mathcal{C}_L &= C([0, T]; \mathbb{R}^{m+n_L}), \\ \mathcal{C}_V &= C([0, T]; \mathbb{R}^{m+n_V}), \end{aligned}$$

where X^* is the dual space of X .

We make the following assumptions on (3.1):

- (A1) For the network equations, we assume index-1 topological conditions (2.35)-(2.36), and consistent initial conditions $\mathbf{x}_0$,

$$\mathbf{Q}\mathbf{x}_0 = \mathbf{Q}\mathbf{E}_1^{-1}(\mathbf{A}_1\mathbf{P}\mathbf{x}_0 + \mathbf{s}(t_0)); \quad (3.2)$$

- (A2) For the source terms in the network equations, we assume $\mathbf{s}(t) \in C((0, T])$;

- (A3) For the doping profile, we assume $N \in L^r$, $d < r < 6$;

- (A4) We assume constant mobilities and diffusivities, related by the Einstein relations:

$$D_n = U_T \mu_n, \quad D_p = U_T \mu_p, \quad (3.3)$$

where U_T is the thermal voltage;

- (A5) We assume that net recombination rate $R(n, p)$ has the Shockley-Read-Hall form

$$R(n, p) = r(n, p)(np - n_i^2), \quad r(n, p) = \frac{1}{\tau_p(n + n_i) + \tau_n(p + n_i)}, \quad (3.4)$$

where n_i is the constant intrinsic concentration, and τ_n, τ_p are positive constants. In particular, $R(n, p)$ is Lipschitz continuous, that is, there exists $L_R > 0$ such that for all positive n_1, n_2, p_1, p_2 , we have

$$|R(n_1, p_1) - R(n_2, p_2)| \leq L_R(|n_1 - n_2| + |p_1 - p_2|); \quad (3.5)$$

- (A6) For the initial data, we assume $n_0, p_0 \in L_+^2$;

- (A7) For the boundary $\partial\Omega = \bigcup_{j=0}^{m_D} \Gamma_{D,j} \cup \Gamma_N$, we assume that $\Gamma_{D,j}$ is closed in $\partial\Omega$, $\text{meas}(\Gamma_{D,j}) > 0$, $j = 0, 1, \dots, m_D$. Moreover, $\partial\Omega$ is such that the auxiliary functions w_j , $j = 0, 1, \dots, m_D$, are well-defined by (2.8), and the map $\mathcal{L}(\rho)$ defined by (2.11) is an isomorphism from L^r onto $W_0^{2,r}$.

Assumption (A7) ensures that all the terms in equation (3.1a) are well-defined. The index-1 topological conditions in assumption (A1) allows to replace (3.1a)-(3.1b) with:

$$\frac{d\mathbf{y}}{dt} = \mathbf{P}\mathbf{E}_1^{-1}(\mathbf{A}_1\mathbf{y} + \mathcal{F}_D(\mathbf{P}\mathbf{y}; \mathbf{J}) + \mathbf{s}(t)), \quad \text{in } (0, T], \quad (3.6a)$$

$$\mathbf{y}(0) = \mathbf{y}_0 = \mathbf{P}\mathbf{x}_0, \quad (3.6b)$$

$$\mathbf{z} = \mathbf{Q}\mathbf{E}_1^{-1}(\mathbf{A}_1\mathbf{y} + \mathbf{s}(t)), \quad (3.6c)$$

with $\mathbf{y} = \mathbf{P}\mathbf{x}$, $\mathbf{z} = \mathbf{Q}\mathbf{x}$.

We introduce a special steady-state solution (n^e, p^e, ψ^e) of (3.1c)-(3.1e) which corresponds to zero external electric voltage source $\mathbf{v}_V$ and electric current source $\mathbf{i}_I$, where the potentials applied to the diode vanish. Furthermore, this solution shall satisfy the conditions

$$n^e = n_i \exp(\psi^e/U_T), \quad p^e = n_i \exp(-\psi^e/U_T). \quad (3.7)$$

A steady-state solution corresponding to these constraints represents a physical state in total thermodynamic equilibrium. Assumption (A7) ensures that the equilibrium voltage $\psi^e \in W^{2,r}$ is uniquely determined by the following nonlinear elliptic problem:

$$\begin{aligned} -\nabla \cdot (\epsilon_s \nabla \psi) &= qN - qn_i (\exp(\psi/U_T) - \exp(-\psi/U_T)), \\ \psi(\mathbf{x}, t) &= \psi_{bi}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_{D,i}, \quad i = 0, 1, \dots, m_D. \end{aligned} \quad (3.8)$$

Moreover, we have $n^e, p^e \in W^{1,r}$.

Definition 1 (Solution to the coupled system) *A solution of the system of equation (3.1) is a tuple $(n, p, \psi, \mathbf{x})$ such that:*

- i) $\mathbf{x} = \mathbf{y} + \mathbf{z}$, with $\mathbf{y} \in \mathcal{C}_L$ satisfying (3.6a)-(3.6b), and $\mathbf{z} \in \mathcal{C}_V$ given by (3.6c);
- ii) $(n - n^e, p - p^e, \psi - \psi^e) \in Y$, $n, p \in C([0, T]; L_+^2)$ satisfy the initial conditions (3.1i);
- iii) the triple (n, p, ψ) satisfies the Poisson equation (3.1c) for all $t \geq 0$;
- iv) for all test functions $\xi_n, \xi_p \in Y$, n and p satisfy the weak formulation

$$\int_0^T [(\partial_t n, \xi_n) + (D_n \nabla n - \mu_n n \nabla \psi, \nabla \xi_n) + (R, \xi_n)] dt = 0, \quad (3.9a)$$

$$\int_0^T [(\partial_t p, \xi_p) + (D_p \nabla p + \mu_p p \nabla \psi, \nabla \xi_p) + (R, \xi_p)] dt = 0, \quad (3.9b)$$

where $(\cdot, \cdot)$ is the usual pairing between X^* and X .

3.2 Local existence and uniqueness

We start by proving the following result.

Theorem 1 (Local existence and uniqueness) *Under the assumptions (A1)-(A7) the problem (3.1) admits a unique solution, provided $T > 0$ is sufficiently small.*

Proof. We introduce the Banach space

$$Z = (n^e, p^e) + C([0, T]; L^2 \times L^2) \cap L^2([0, T]; X^2)$$

and fix the pair of functions $(\hat{n}, \hat{p}) \in Z$ and $\hat{\mathbf{y}} \in \mathcal{C}_L$, with $\hat{n}(\cdot, 0) = n_0(\cdot)$, $\hat{p}(\cdot, 0) = p_0(\cdot)$, and $\hat{\mathbf{y}}(0) = \mathbf{y}_0$. We consider the following linearized problem for (n, p) :

$$\int_0^T [(\partial_t n, \xi_n) + (D_n \nabla n + \mu_n \hat{n}^+ \hat{\mathbf{E}}, \nabla \xi_n) + (\hat{R}^+, \xi_n)] dt = 0, \quad (3.10a)$$

$$\int_0^T [(\partial_t p, \xi_p) + (D_p \nabla p - \mu_p \hat{p}^+ \hat{\mathbf{E}}, \nabla \xi_p) + (\hat{R}^+, \xi_p)] dt = 0, \quad (3.10b)$$

for all test functions $\xi_n, \xi_p \in Y$. We have indicated with $\hat{R}^+ = R(\hat{n}^+, \hat{p}^+)$, in which we have used the notation the notation $g^+ := \max(g, 0)$, for all function g and $\hat{\mathbf{E}} = -\nabla \hat{\psi}$ where $\hat{\psi}$ is the solution of the Poisson equation with source term $q(\hat{p}^+ - \hat{n}^+ + N)$ and boundary conditions

$$\begin{aligned} \psi(\mathbf{x}, t) &= \psi_{\text{bi}}(\mathbf{x}) + \hat{u}_{D,i}(t), \quad \mathbf{x} \in \Gamma_{D,i}, \quad i = 0, 1, \dots, m_D, \\ \hat{\mathbf{u}}_D &= \mathbf{S}_D^\top \boldsymbol{\pi} \mathbf{P} \hat{\mathbf{y}}. \end{aligned}$$

We consider also the problem for $\mathbf{y}$

$$\frac{d\mathbf{y}}{dt} = \mathbf{P} \mathbf{E}_1^{-1} \left(\mathbf{A}_1 \hat{\mathbf{y}} + \mathcal{F}_D(\mathbf{P} \hat{\mathbf{y}}; \hat{\mathbf{J}}) + \mathbf{s}(t) \right) \quad (3.10c)$$

with

$$\hat{\mathbf{J}} = q(D_n \nabla \hat{n} - D_p \nabla \hat{p} + (\mu_n \hat{n} + \mu_p \hat{p}) \hat{\mathbf{E}}).$$

The system of decoupled equations (3.10) admits a unique solution $(n, p, \mathbf{y})$ which satisfies $(n, p) \in (n^e, p^e) + Y^2$, $\mathbf{y} \in \mathcal{C}_L$, and have the initial data $n(\cdot, 0) = n_0(\cdot)$, $p(\cdot, 0) = p_0(\cdot)$, $y(0) = y_0$, the same initial data (3.6b) and (3.1i) of the original problem. In fact, the existence of a unique solution $\mathbf{y}$ to (3.10c) follows immediately by time integration of the equation over $[0, t]$. Because of the continuous embedding of H^1 into L^4 and the Poisson equation [26], we have the following estimates

$$\|\mu_n \hat{n}^+ \hat{\mathbf{E}}\|_2 \leq c \|\hat{n}\|_4 \|\hat{\mathbf{E}}\|_4 \leq c \|\hat{n}\|_{1,2} \|\hat{\mathbf{E}}\|_{1,2} \leq c \|\hat{n}\|_{1,2} (1 + |\hat{\mathbf{y}}| + \|\hat{n}\|_2 + \|\hat{p}\|_2).$$

A similar estimates follows for the hole distribution. This means the terms $\mu_n \hat{n}^+ \hat{\mathbf{E}}$ and $\mu_p \hat{p}^+ \hat{\mathbf{E}}$ are in L^2 so we can apply the standard results, see for example [25], for linear parabolic equations with discontinuous coefficients to get the existence of a unique solution $(n, p, \mathbf{y})$ to (3.10a)–(3.10b). We define an operator $\mathcal{Q}$ from $Z \times \mathcal{C}_L$ to $Z \times \mathcal{C}_L$ which, to any $(\hat{n}, \hat{p}, \hat{\mathbf{y}}) \in Z \times \mathcal{C}_L$ associates the solution $(n, p, \mathbf{y})$ of the problem (3.10)

$$(\hat{n}, \hat{p}, \hat{\mathbf{y}}) \rightarrow (n, p, \mathbf{y}) =: \mathcal{Q}(\hat{n}, \hat{p}, \hat{\mathbf{y}}).$$

For all $(n, p, \mathbf{y}) \in Z \times \mathcal{C}_L$, we introduce the norm

$$\|(n, p, \mathbf{y})\|^2 = \max_{0 \leq t \leq T} (\|n(\cdot, t)\|_2^2 + \|p(\cdot, t)\|_2^2 + |\mathbf{y}(t)|^2) + \int_0^T (\|\nabla n(\cdot, t)\|_2^2 + \|\nabla p(\cdot, t)\|_2^2) d\tau.$$

We prove that $\mathcal{Q}$ is strictly contractive with respect to this norm, for T small enough, in the set

$$S_a = \{(n, p, \mathbf{y}) \in Z \times \mathcal{C}_L : n(\cdot, t) = n_0(\cdot), p(\cdot, t) = p_0(\cdot), \mathbf{y}(0) = \mathbf{y}_0, |||(n, p, \mathbf{y})|||^2 \leq a\}$$

where the constant a is such that $a > \|n_0\|_2^2 + \|p_0\|_2^2 + |\mathbf{y}_0|^2$. We consider two triples $(\hat{n}_\ell, \hat{p}_\ell, \hat{\mathbf{y}}_\ell) \in S_a$ for $\ell = 1, 2$ and we set

$$(n_\ell, p_\ell, \mathbf{y}_\ell) = \mathcal{Q}(\hat{n}_\ell, \hat{p}_\ell, \hat{\mathbf{y}}_\ell), \quad \hat{\mathbf{E}}_\ell = \hat{\mathbf{E}}(\hat{n}_\ell, \hat{p}_\ell, \hat{\mathbf{y}}_\ell), \quad \text{for } \ell = 1, 2$$

where $\hat{\mathbf{E}}_\ell = -\nabla \hat{\psi}_\ell$ depends on the solution $\hat{\psi}_\ell$ of the Poisson equation with source term $q(\hat{p}_\ell^+ - \hat{n}_\ell^+ + N)$ and the boundary term is related to $\hat{\mathbf{y}}_\ell$. We write (3.10a), with $n = n_1$ and $n = n_2$, both with test function $\xi_n = n_1 - n_2$. Subtracting the two equations and integrating over the time interval $[0, t]$, we obtain

$$\begin{aligned} & \frac{1}{2} \|n_1 - n_2\|_2^2 + D_n \int_0^t \|\nabla(n_1 - n_2)\|_2^2 d\tau \\ &= -\mu_n \int_0^t \left(\hat{n}_1^+ \hat{\mathbf{E}}_1 - \hat{n}_2^+ \hat{\mathbf{E}}_2, \nabla(n_1 - n_2) \right) d\tau - \int_0^t \left(\hat{R}_1^+ - \hat{R}_2^+, n_1 - n_2 \right) d\tau \\ &\leq \mu_n \int_0^t \|\hat{n}_1^+ \hat{\mathbf{E}}_1 - \hat{n}_2^+ \hat{\mathbf{E}}_2\|_2 \|\nabla(n_1 - n_2)\|_2 d\tau + \int_0^t \|\hat{R}_1^+ - \hat{R}_2^+\|_2 \|n_1 - n_2\|_2 d\tau \end{aligned} \quad (3.11)$$

in which $\hat{R}_\ell^+ = R(\hat{n}_\ell^+, \hat{p}_\ell^+)$, $\ell = 1, 2$.

To estimate the first term on the right-hand side of (3.11), we first observe that, using the Gagliardo-Nirenberg inequality and the weighted Young inequality, for any function $u \in H^1$ we have

$$\|u\|_r \leq c \|u\|_2^{1-\eta} \|u\|_{1,2}^\eta + c \|u\|_2 \leq c(\delta) \|u\|_2 + \delta \|u\|_{1,2}, \quad (3.12)$$

where $\delta > 0$ can be chosen arbitrarily small, and $\eta = d\left(\frac{1}{2} - \frac{1}{r}\right) \in (0, 1)$, if we choose $2 < r < 6$. Moreover, for $r > d$, we can write

$$\|\hat{\mathbf{E}}_1 - \hat{\mathbf{E}}_2\|_\infty \leq c \|\hat{\mathbf{E}}_1 - \hat{\mathbf{E}}_2\|_{1,r} \leq c(|\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2| + \|\hat{n}_1 - \hat{n}_2\|_r + \|\hat{p}_1 - \hat{p}_2\|_r).$$

and

$$\|\hat{\mathbf{E}}_2\|_6 \leq c \|\hat{\mathbf{E}}_2\|_{1,2} \leq c(1 + |\hat{\mathbf{y}}_2| + \|\hat{n}_2\|_2 + \|\hat{p}_2\|_2) \leq c(a).$$

Then we have

$$\begin{aligned} \|\hat{n}_1^+ \hat{\mathbf{E}}_1 - \hat{n}_2^+ \hat{\mathbf{E}}_2\|_2 &\leq \|\hat{n}_1\|_2 \|\hat{\mathbf{E}}_1 - \hat{\mathbf{E}}_2\|_\infty + \|\hat{\mathbf{E}}_2\|_6 \|\hat{n}_1 - \hat{n}_2\|_3 \\ &\leq c(a) [c|\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2| + c(\delta)(\|\hat{n}_1 - \hat{n}_2\|_2 + \|\hat{p}_1 - \hat{p}_2\|_2) \\ &\quad + \delta(\|\hat{n}_1 - \hat{n}_2\|_{1,2} + \|\hat{p}_1 - \hat{p}_2\|_{1,2})]. \end{aligned} \quad (3.13)$$

So, we can estimate

$$\begin{aligned}
& \mu_n \int_0^t \|\hat{n}_1^+ \hat{\mathbf{E}}_1 - \hat{n}_2^+ \hat{\mathbf{E}}_2\|_2 \|\nabla(n_1 - n_2)\|_2 d\tau \\
& \leq c(a, \delta) \int_0^t (|\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2|^2 + \|\hat{n}_1 - \hat{n}_2\|_2^2 + \|\hat{p}_1 - \hat{p}_2\|_2^2) d\tau \\
& \quad + \delta \int_0^t (\|\hat{n}_1 - \hat{n}_2\|_{1,2}^2 + \|\hat{p}_1 - \hat{p}_2\|_{1,2}^2) d\tau + \delta \int_0^t \|\nabla(n_1 - n_2)\|_2^2 d\tau \\
& \leq \max\{c(a, \delta)T, \delta\} \|(\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2, \hat{n}_1 - \hat{n}_2, \hat{p}_1 - \hat{p}_2)\| + \delta \int_0^t \|\nabla(n_1 - n_2)\|_2^2 d\tau.
\end{aligned}$$

For the second term on the right-hand side of (3.11), using assumption (A5), we get

$$\begin{aligned}
\int_0^t \|\hat{R}_1^+ - \hat{R}_2^+\|_2 \|n_1 - n_2\|_2 d\tau & \leq \int_0^t L_R(\|\hat{n}_1 - \hat{n}_2\|_2 + \|\hat{p}_1 - \hat{p}_2\|_2) \|n_1 - n_2\|_2 d\tau \\
& \leq \int_0^t [c(\delta)(\|\hat{n}_1 - \hat{n}_2\|_2^2 + \|\hat{p}_1 - \hat{p}_2\|_2^2) + \delta \|n_1 - n_2\|_2^2] d\tau \\
& \leq c(\delta)T \|(\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2, \hat{n}_1 - \hat{n}_2, \hat{p}_1 - \hat{p}_2)\| + \delta \int_0^t \|n_1 - n_2\|_2^2 d\tau.
\end{aligned}$$

Combining the previous estimates, we get

$$\begin{aligned}
& \frac{1}{2} \|n_1 - n_2\|_2^2 + D_n \int_0^t \|\nabla(n_1 - n_2)\|_2^2 d\tau \\
& \leq \max\{c(a, \delta)T, \delta\} \|(\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2, \hat{n}_1 - \hat{n}_2, \hat{p}_1 - \hat{p}_2)\| + \delta \int_0^t \|n_1 - n_2\|_{1,2}^2 d\tau.
\end{aligned} \tag{3.14}$$

An analogous estimate holds for $p_1 - p_2$, that is,

$$\begin{aligned}
& \frac{1}{2} \|p_1 - p_2\|_2^2 + D_p \int_0^t \|\nabla(p_1 - p_2)\|_2^2 d\tau \\
& \leq \max\{c(a, \delta)T, \delta\} \|(\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2, \hat{n}_1 - \hat{n}_2, \hat{p}_1 - \hat{p}_2)\| + \delta \int_0^t \|p_1 - p_2\|_{1,2}^2 d\tau.
\end{aligned} \tag{3.15}$$

Next, we write (3.6a) for $\mathbf{y} = \mathbf{y}_1$, $\mathbf{y} = \mathbf{y}_2$, subtract the resulting equations, and multiply the result by $\mathbf{y}_1 - \mathbf{y}_2$. We note that

$$\mathbf{P}^\top \mathbf{E}_1 \mathbf{P} = \mathbf{P}^\top \mathbf{E}_1^\top \mathbf{P}, \tag{3.16}$$

which, together with $\mathbf{y}_1 - \mathbf{y}_2 = \mathbf{P}(\mathbf{y}_1 - \mathbf{y}_2)$, implies

$$\begin{aligned}
& \frac{d}{dt} \left(\frac{1}{2} (\mathbf{y}_1 - \mathbf{y}_2)^\top \mathbf{E}_1 (\mathbf{y}_1 - \mathbf{y}_2) \right) = (\mathbf{y}_1 - \mathbf{y}_2)^\top \mathbf{E}_1 \frac{d}{dt} (\mathbf{y}_1 - \mathbf{y}_2) \\
& = (\mathbf{y}_1 - \mathbf{y}_2)^\top \mathbf{E}_1 \left\{ \mathbf{P} \mathbf{E}_1^{-1} \left(\mathbf{A}_1 \hat{\mathbf{y}}_1 + \mathcal{F}_D(\mathbf{P} \hat{\mathbf{y}}_1; \hat{\mathbf{J}}_1) \right) - \mathbf{P} \mathbf{E}_1^{-1} \left(\mathbf{A}_1 \hat{\mathbf{y}}_2 + \mathcal{F}_D(\mathbf{P} \hat{\mathbf{y}}_2; \hat{\mathbf{J}}_2) \right) \right\} \\
& = (\mathbf{y}_1 - \mathbf{y}_2)^\top \left\{ \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} \mathbf{A}_1 (\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2) + \mathcal{F}_D(\mathbf{P} \hat{\mathbf{y}}_1; \hat{\mathbf{J}}_1) - \mathcal{F}_D(\mathbf{P} \hat{\mathbf{y}}_2; \hat{\mathbf{J}}_2) \right\},
\end{aligned}$$

with

$$\hat{\mathbf{J}}_\ell = q(D_n \nabla \hat{n}_\ell + \mu_n \hat{n}_\ell \hat{\mathbf{E}}_\ell - D_p \nabla \hat{p}_\ell + \mu_p \hat{p}_\ell \hat{\mathbf{E}}_\ell), \quad \ell = 1, 2.$$

After integrating with respect to time, we find

$$\begin{aligned} & \frac{1}{2}(\mathbf{y}_1 - \mathbf{y}_2)^\top \mathbf{E}_1(\mathbf{y}_1 - \mathbf{y}_2) \\ &= \int_0^t (\mathbf{y}_1 - \mathbf{y}_2)^\top \left\{ \left(\mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} \mathbf{A}_1(\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2) + \mathcal{F}_D(\mathbf{P}\hat{\mathbf{y}}_1; \hat{\mathbf{J}}_1) - \mathcal{F}_D(\mathbf{P}\hat{\mathbf{y}}_2; \hat{\mathbf{J}}_2) \right) \right\} d\tau \\ &\leq \int_0^t |\mathbf{y}_1 - \mathbf{y}_2|^2 d\tau + c \int_0^t \left(|\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2|^2 + |\mathcal{F}_D(\mathbf{P}\hat{\mathbf{y}}_1; \hat{\mathbf{J}}_1) - \mathcal{F}_D(\mathbf{P}\hat{\mathbf{y}}_2; \hat{\mathbf{J}}_2)|^2 \right) d\tau. \end{aligned}$$

From the definition of $\mathcal{F}_D$, we have

$$|\mathcal{F}_D(\mathbf{P}\hat{\mathbf{y}}_1; \hat{\mathbf{J}}_1) - \mathcal{F}_D(\mathbf{P}\hat{\mathbf{y}}_2; \hat{\mathbf{J}}_2)|^2 \leq c \sum_{i=1}^{m_D} |\mathcal{J}_{D,i}(\mathbf{P}\hat{\mathbf{y}}_1; \hat{\mathbf{J}}_1) - \mathcal{J}_{D,i}(\mathbf{P}\hat{\mathbf{y}}_2; \hat{\mathbf{J}}_2)|^2.$$

We notice that since $\hat{n}_1 - \hat{n}_2 = 0$ on Γ_D and $\boldsymbol{\nu} \cdot \nabla(\hat{n}_1 - \hat{n}_2) = 0$ on Γ_N we have that

$$\mathcal{L}(\Delta(\hat{n}_1 - \hat{n}_2)) = -\frac{\hat{n}_1 - \hat{n}_2}{\epsilon_s},$$

where $\mathcal{L}$ is defined in (2.11). Using the linearity of $\mathcal{J}_{D,i}(\mathbf{u}_D; \mathbf{J})$ with respect to $\mathbf{J}$, we find

$$\mathcal{J}_{D,i}(\mathbf{P}(\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2); D_n \nabla(\hat{n}_1 - \hat{n}_2)) = \mathcal{J}_{D,i}(\mathbf{P}(\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2); D_p \nabla(\hat{p}_1 - \hat{p}_2)) = 0,$$

and then

$$\begin{aligned} & |\mathcal{J}_{D,i}(\mathbf{P}\hat{\mathbf{y}}_1; \hat{\mathbf{J}}_1) - \mathcal{J}_{D,i}(\mathbf{P}\hat{\mathbf{y}}_2; \hat{\mathbf{J}}_2)| \\ &\leq q\mu_n |\mathcal{J}_{D,i}(\mathbf{P}(\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2); \hat{n}_1 \hat{\mathbf{E}}_1 - \hat{n}_2 \hat{\mathbf{E}}_2)| + q\mu_p |\mathcal{J}_{D,i}(\mathbf{P}(\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2); \hat{p}_1 \hat{\mathbf{E}}_1 - \hat{p}_2 \hat{\mathbf{E}}_2)|. \end{aligned} \quad (3.17)$$

Moreover, for any vector function $\mathbf{J}(\mathbf{x})$ such that $\boldsymbol{\nu} \cdot \mathbf{J} = 0$ on Γ_N , we have

$$\begin{aligned} |\mathcal{J}_{D,i}(\mathbf{u}_D; \mathbf{J})| &= \left| \int_{\Omega} \nabla w_i \cdot (\epsilon_s \nabla \mathcal{L}(\nabla \cdot \mathbf{J}) + \mathbf{J}) d\mathbf{x} \right| \\ &\leq \|\nabla w_i\|_2 (\|\epsilon_s \nabla \mathcal{L}(\nabla \cdot \mathbf{J})\|_2 + \|\mathbf{J}\|_2) \leq c \|\mathbf{J}\|_2. \end{aligned} \quad (3.18)$$

In fact, if we pose $\varphi = \mathcal{L}(\nabla \cdot \mathbf{J})$, we have by definition

$$-\int_{\Omega} \varphi \nabla \cdot (\epsilon_s \nabla \varphi) d\mathbf{x} = \int_{\Omega} \varphi \nabla \cdot \mathbf{J} d\mathbf{x},$$

which, integrating by parts and using the boundary conditions for φ , and the condition for $\mathbf{J}$, yields

$$\int_{\Omega} \epsilon_s |\nabla \varphi|^2 d\mathbf{x} = -\int_{\Omega} (\nabla \varphi) \cdot \mathbf{J} d\mathbf{x} \leq \delta \|\nabla \varphi\|_2^2 + c(\delta) \|\mathbf{J}\|_2^2.$$

Choosing the positive constant δ small enough, we get

$$\|\nabla \varphi\|_2 \leq c \|\mathbf{J}\|_2, \quad (3.19)$$

which implies the last inequality in (3.18). We can apply this result to (3.17), obtaining

$$|\mathcal{J}_{D,i}(\mathbf{P}\hat{\mathbf{y}}_1; \hat{\mathbf{J}}_1) - \mathcal{J}_{D,i}(\mathbf{P}\hat{\mathbf{y}}_2; \hat{\mathbf{J}}_2)|^2 \leq c(\|\hat{n}_1 \hat{\mathbf{E}}_1 - \hat{n}_2 \hat{\mathbf{E}}_2\|_2^2 + \|\hat{p}_1 \hat{\mathbf{E}}_1 - \hat{p}_2 \hat{\mathbf{E}}_2\|_2^2), \quad (3.20)$$

which, in turns, yields

$$\begin{aligned} |\mathcal{F}_D(\mathbf{P}\hat{\mathbf{y}}_1; \hat{\mathbf{J}}_1) - \mathcal{F}_D(\mathbf{P}\hat{\mathbf{y}}_2; \hat{\mathbf{J}}_2)|^2 &\leq c(a)(c|\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2|^2 + c(\delta)(\|\hat{n}_1 - \hat{n}_2\|_2^2 + \|\hat{p}_1 - \hat{p}_2\|_2^2) \\ &\quad + \delta(\|\hat{n}_1 - \hat{n}_2\|_{1,2}^2 + \|\hat{p}_1 - \hat{p}_2\|_{1,2}^2)). \end{aligned}$$

Since the matrix $\mathbf{E}_1$ is positive definite, there exist a constant $c_E > 0$ such that

$$\mathbf{y}^\top \mathbf{E}_1 \mathbf{y} \geq c_E |\mathbf{y}|^2.$$

Then, using the previous estimates, we find

$$\begin{aligned} c_E |\mathbf{y}_1 - \mathbf{y}_2|^2 &\leq \frac{1}{2} (\mathbf{y}_1 - \mathbf{y}_2)^\top \mathbf{E}_1 (\mathbf{y}_1 - \mathbf{y}_2) \\ &\leq \max\{c(a, \delta)T, \delta\} \|(\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2, \hat{n}_1 - \hat{n}_2, \hat{p}_1 - \hat{p}_2)\| + T \max_{0 \leq t \leq T} |\mathbf{y}_1 - \mathbf{y}_2|^2. \end{aligned} \quad (3.21)$$

Finally, we can combine (3.14), (3.15) and (3.21), and choose T and δ small enough so that

$$\|(\mathbf{y}_1 - \mathbf{y}_2, n_1 - n_2, p_1 - p_2)\| \leq c_Q \|(\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2, \hat{n}_1 - \hat{n}_2, \hat{p}_1 - \hat{p}_2)\|,$$

for some positive constant $c_Q < 1$. So we have proved that there exists a time $T > 0$, such that $\mathcal{Q}$ becomes a contraction on S_a , then the existence of a unique fixed point $(n_*, p_*, \mathbf{y}_*) \in S_a$ it is ensured by the Banach's fixed point theorem. To show that this fixed point is the unique solution to (2.1)-(2.3), it is sufficient to prove the nonnegativity of n_* and p_* . To this end, we consider $n_*^- = \min(n_*, 0) \in Y$. Writing (3.10a), for the test function $\xi_n = n_*^-$, we obtain

$$\begin{aligned} \frac{1}{2} \|n_*^-(t)\|_2^2 + D_n \int_0^t \|\nabla n_*^-\|_2^2 d\tau &= \int_0^t \int_\Omega (-\mu_n n_*^+ \mathbf{E}_* \cdot \nabla n_*^- - R(n_*^+, p_*^+) n_*^-) d\mathbf{x} d\tau \\ &= \int_0^t \int_\Omega r(n_*^+, p_*^+) n_i^2 n_*^- d\mathbf{x} d\tau \leq 0. \end{aligned}$$

This means that n_*^- vanishes almost everywhere; that is, n_* is nonnegative almost everywhere. In the same way we can prove the nonnegativity of p_* . $\blacksquare$

To prove the general existence and uniqueness of solution of the problem (3.1) we need to ensure that it is possible to prolong to arbitrary time intervals the solution, whose existence is proved in theorem 1.

3.3 A-priori estimates and global existence

In this section we prove some a priori estimates which allow for the prolongation of solution of problem (3.1). The a priori estimates are based on a modification of the energy functional associated to the coupled system, which is the sum of the energy associated to the device and the energy associated to the circuit.

The physical energy associated to the device is defined as

$$W_D = \int_{\Omega} \left\{ qU_T \left[n \left(\log \frac{n}{n_i} - 1 \right) + p \left(\log \frac{p}{n_i} - 1 \right) \right] + \frac{\epsilon_s}{2} |\nabla \psi|^2 \right\} d\mathbf{x}. \quad (3.22)$$

Using equation (3.1), we obtain

$$\begin{aligned} \frac{dW_D}{dt} = & - \int_{\Omega} \left(\frac{\mathbf{J}_n^2}{q\mu_n n} + \frac{\mathbf{J}_p^2}{q\mu_p p} \right) d\mathbf{x} - \int_{\Omega} qU_T R \log \frac{np}{n_i^2} d\mathbf{x} \\ & + \sum_{i=0}^{m_D} \int_{\Gamma_{D,i}} \psi_{bi}(\epsilon_s \partial_t \nabla \psi) \cdot \boldsymbol{\nu}_i d\sigma + \mathbf{u}_D^\top \mathbf{J}_D. \end{aligned} \quad (3.23)$$

The energy associated to the circuit is

$$W_C = \frac{1}{2} \mathbf{y}^\top \mathbf{E}_1 \mathbf{y} - \frac{1}{2} \mathbf{y}^\top \boldsymbol{\pi}^\top \mathbf{A}_D \mathbf{C}_D \mathbf{A}_D^\top \boldsymbol{\pi} \mathbf{y} = \frac{1}{2} \mathbf{u}^\top \mathbf{A}_C \mathbf{C} \mathbf{A}_C^\top \mathbf{u} + \frac{1}{2} \mathbf{i}_L^\top \mathbf{L} \mathbf{i}_L. \quad (3.24)$$

Using equation (3.1), we get

$$\frac{dW_C}{dt} = \mathbf{y}^\top \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}(t)) - \mathbf{u}_D^\top \mathbf{J}_D. \quad (3.25)$$

Here, recalling (2.38), we have used the identity

$$\mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} \mathcal{F}_D(\mathbf{P} \mathbf{y}) = \mathbf{E}_1 (\mathbf{P} + \mathbf{Q}) \mathbf{E}_1^{-1} \mathcal{F}_D(\mathbf{P} \mathbf{y}) = \mathcal{F}_D(\mathbf{P} \mathbf{y}).$$

Combining (3.23) and (3.25), for the total energy of the system $W = W_D + W_C$ we obtain

$$\begin{aligned} \frac{dW}{dt} = & - \int_{\Omega} \left(\frac{\mathbf{J}_n^2}{q\mu_n n} + \frac{\mathbf{J}_p^2}{q\mu_p p} \right) d\mathbf{x} - \int_{\Omega} qU_T R \log \frac{np}{n_i^2} d\mathbf{x} \\ & + \sum_{i=0}^{m_D} \int_{\Gamma_{D,i}} \psi_{bi}(\epsilon_s \partial_t \nabla \psi) \cdot \boldsymbol{\nu}_i d\sigma + \mathbf{y}^\top \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}(t)). \end{aligned} \quad (3.26)$$

Finally, to take care of the energy associated to the electric potential, we use the equilibrium solution (n^e, p^e, ψ^e) defined by (3.7), (3.8), to define a shifted device energy W_D^* :

$$W_D^* = \int_{\Omega} \left\{ qU_T \left[n \left(\log \frac{n}{n^e} - 1 \right) + n^e + p \left(\log \frac{p}{p^e} - 1 \right) + p^e \right] + \frac{\epsilon_s}{2} |\nabla(\psi - \psi^e)|^2 \right\} d\mathbf{x}. \quad (3.27)$$

We get

$$\frac{dW_D^*}{dt} = \frac{dW_D}{dt} - \int_{\Omega} \nabla \cdot (\psi^e \epsilon_s \partial_t \nabla \psi) d\mathbf{x} = \frac{dW_D}{dt} - \sum_{i=0}^{m_D} \int_{\Gamma_{D,i}} \psi_{bi}(\epsilon_s \partial_t \nabla \psi) \cdot \boldsymbol{\nu} d\mathbf{x},$$

thus, the shifted total energy $W^* = W_D^* + W_C$ satisfies

$$\frac{dW^*}{dt} = - \int_{\Omega} \left(\frac{\mathbf{J}_n^2}{q\mu_n n} + \frac{\mathbf{J}_p^2}{q\mu_p p} \right) d\mathbf{x} - \int_{\Omega} qU_T R \log \frac{np}{n_i^2} d\mathbf{x} + \mathbf{y}^\top \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}(t)). \quad (3.28)$$

We are now ready to modify the functional W^* , in order to control the possible degeneracy for $n = 0$ or $p = 0$. For any $\alpha > 0$, we define the Liapunov functional

$$\begin{aligned} \mathcal{H}_\alpha = & \int_{\Omega} q U_T \left\{ (n + \alpha) \left(\log \frac{n + \alpha}{n^e + \alpha} - 1 \right) + n^e \right\} d\mathbf{x} \\ & + \int_{\Omega} q U_T \left\{ (p + \alpha) \left(\log \frac{p + \alpha}{p^e + \alpha} - 1 \right) + p^e \right\} d\mathbf{x} + \int_{\Omega} \frac{\epsilon_s}{2} |\nabla(\psi - \psi^e)|^2 d\mathbf{x} + W_C. \end{aligned} \quad (3.29)$$

The parameter α is needed to ensure that $\mathcal{H}_\alpha$ is well defined, since n and p may vanish locally. Formally, as α tends to zero, the Liapunov functional $\mathcal{H}_\alpha$ tends to the total physical energy W^* of the coupled system. This assertion can be stated in a precise way by observing that the function

$$g(z) = \begin{cases} z \log z & \text{if } z > 0, \\ 0 & \text{if } z = 0 \end{cases} \quad (3.30)$$

is continuous for $z \in [0, \infty)$. Then the functional $\mathcal{H} = \lim_{\alpha \rightarrow 0} \mathcal{H}_\alpha$ is well defined, and we have $\mathcal{H} = W^*$. We define the following functions $\psi_n(\mathbf{x}, t)$, $\psi_p(\mathbf{x}, t)$, $\psi_n^e(\mathbf{x})$ and $\psi_p^e(\mathbf{x})$:

$$\psi_n = \psi - U_T \log \frac{n + \alpha}{n_i}, \quad \psi_p = \psi + U_T \log \frac{p + \alpha}{n_i}, \quad (3.31)$$

$$\psi_n^e = \psi^e - U_T \log \frac{n^e + \alpha}{n_i}, \quad \psi_p^e = \psi^e + U_T \log \frac{p^e + \alpha}{n_i}. \quad (3.32)$$

Lemma 1 *For the functional $\mathcal{H}_\alpha$ defined in (3.29), we have*

$$\begin{aligned} \mathcal{H}_\alpha(t) = & \mathcal{H}_\alpha(0) + \int_0^t \int_{\Omega} (\mathbf{J}_n \cdot \nabla(\psi_n - \psi_n^e) + \mathbf{J}_p \cdot \nabla(\psi_p - \psi_p^e)) d\mathbf{x} d\tau \\ & - \int_0^t \int_{\Omega} q U_T R \log \frac{(n + \alpha)(p + \alpha)}{(n^e + \alpha)(p^e + \alpha)} d\mathbf{x} d\tau + \int_0^t \mathbf{y}^\top \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}(\tau)) d\tau. \end{aligned} \quad (3.33)$$

Proof. Using (3.1) we compute the time derivative of the functional $\mathcal{H}_\alpha$:

$$\begin{aligned} \frac{d\mathcal{H}_\alpha}{dt} = & \int_{\Omega} U_T \left(\log \frac{n + \alpha}{n^e + \alpha} \nabla \cdot \mathbf{J}_n - \log \frac{p + \alpha}{p^e + \alpha} \nabla \cdot \mathbf{J}_p \right) d\mathbf{x} \\ & - \int_{\Omega} q U_T R \log \frac{(n + \alpha)(p + \alpha)}{(n^e + \alpha)(p^e + \alpha)} d\mathbf{x} + \int_{\Omega} \epsilon_s \nabla(\psi - \psi^e) \cdot \partial_t \nabla \psi d\mathbf{x} \\ & + \mathbf{y}^\top \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}(t)) - \mathbf{u}_D^\top \mathbf{J}_D. \end{aligned} \quad (3.34)$$

We integrate by parts the first term on the right-hand side of (3.34), and using the definitions (3.31) and (3.32) we obtain:

$$\begin{aligned} & \int_{\Omega} U_T \left(\log \frac{n + \alpha}{n^e + \alpha} \nabla \cdot \mathbf{J}_n - \log \frac{p + \alpha}{p^e + \alpha} \nabla \cdot \mathbf{J}_p \right) d\mathbf{x} \\ & = \int_{\Omega} (\mathbf{J}_n \cdot \nabla(\psi_n - \psi_n^e) + \mathbf{J}_p \cdot \nabla(\psi_p - \psi_p^e)) d\mathbf{x} - \int_{\Omega} (\mathbf{J}_n + \mathbf{J}_p) \cdot \nabla(\psi - \psi^e) d\mathbf{x}. \end{aligned}$$

Inserting this in (3.34) we get

$$\begin{aligned} \frac{d\mathcal{H}_\alpha}{dt} &= \int_{\Omega} (\mathbf{J}_n \cdot \nabla(\psi_n - \psi_n^e) + \mathbf{J}_p \cdot \nabla(\psi_p - \psi_p^e)) \, d\mathbf{x} - \int_{\Omega} qU_T R \log \frac{(n + \alpha)(p + \alpha)}{(n^e + \alpha)(p^e + \alpha)} \, d\mathbf{x} \\ &\quad - \int_{\Omega} (-\epsilon_s \partial_t \nabla \psi + \mathbf{J}_n + \mathbf{J}_p) \cdot \nabla(\psi - \psi^e) \, d\mathbf{x} + \mathbf{y}^\top \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}(t)) - \mathbf{u}_D^\top \mathbf{J}_D. \end{aligned}$$

Integrating by parts the last integral and using (2.17) and the definition of $\mathbf{J}_D$, we obtain

$$\begin{aligned} \frac{d\mathcal{H}_\alpha}{dt} &= \int_{\Omega} (\nabla(\psi_n - \psi_n^e) \cdot \mathbf{J}_n + \nabla(\psi_p - \psi_p^e) \cdot \mathbf{J}_p) \, d\mathbf{x} \\ &\quad - \int_{\Omega} qU_T R \log \frac{(n + \alpha)(p + \alpha)}{(n^e + \alpha)(p^e + \alpha)} \, d\mathbf{x} + \mathbf{y}^\top \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}(t)). \end{aligned} \quad (3.35)$$

Integrating on time, we get the thesis. ■

Next, we prove the a priori bounds for the solution.

Lemma 2 *Under the assumptions (A1)–(A7), there exist constants C_1, C_2 independent of t such that for any solution $(n, p, \psi, \mathbf{x})$ of the system (3.1) it holds that*

$$\|n(\cdot, t)\|_1 + \|p(\cdot, t)\|_1 + \|\nabla \psi(\cdot, t)\|_2^2 + |\mathbf{P}\mathbf{x}(t)|^2 \leq C_1 e^{C_2 t}, \quad t \geq 0. \quad (3.36)$$

Proof. We estimate all the terms on the right-hand side of (3.33). We start observing that, thanks to (3.31), we have $n = n_i \exp\left(\frac{\psi - \psi_n}{U_T}\right) - \alpha$, which leads to

$$\nabla n = n_i \exp\left(\frac{\psi - \psi_n}{U_T}\right) \frac{1}{U_T} \nabla(\psi - \psi_n) = \frac{n + \alpha}{U_T} \nabla(\psi - \psi_n).$$

Then, the electron current can be written as

$$\mathbf{J}_n = q\mu_n(U_T \nabla n - n \nabla \psi) = q\mu_n(-(n + \alpha) \nabla \psi_n + \alpha \nabla \psi), \quad (3.37)$$

which implies that

$$\begin{aligned} \int_{\Omega} \mathbf{J}_n \cdot \nabla(\psi_n - \psi_n^e) \, d\mathbf{x} &= \int_{\Omega} q\mu_n(-(n + \alpha) \nabla \psi_n + \alpha \nabla \psi) \cdot \nabla(\psi_n - \psi_n^e) \, d\mathbf{x} \\ &= \int_{\Omega} q\mu_n \left(-(n + \alpha) |\nabla \psi_n|^2 + (n + \alpha) \nabla \psi_n \cdot \nabla \psi_n^e + \alpha \nabla \psi \cdot (\nabla \psi_n - \nabla \psi_n^e) \right) \, d\mathbf{x} \\ &\leq \int_{\Omega} q\mu_n \left(-(n + \alpha) |\nabla \psi_n|^2 + (n + \alpha) |\nabla \psi_n| |\nabla \psi_n^e| + \alpha |\nabla \psi| (|\nabla \psi_n| + |\nabla \psi_n^e|) \right) \, d\mathbf{x} \\ &\leq \int_{\Omega} q\mu_n \left((n + \alpha) |\nabla \psi_n^e|^2 + \frac{\alpha^2}{(n + \alpha)} |\nabla \psi|^2 \right) \, d\mathbf{x} \leq c(1 + \|n\|_1 + \|\psi\|_2^2). \end{aligned}$$

In a similar way, we can estimate

$$\int_{\Omega} \mathbf{J}_p \cdot \nabla(\psi_p - \psi_p^e) \, d\mathbf{x} \leq c(1 + \|p\|_1 + \|\psi\|_2^2).$$

For the recombination term, we have

$$\begin{aligned} & \int_{\Omega} q U_T R(n, p) \log \frac{(n + \alpha)(p + \alpha)}{(n^e + \alpha)(p^e + \alpha)} d\mathbf{x} \\ &= \int_{\Omega} q U_T r(n, p) (np - n_i^2) \log \frac{(n + \alpha)(p + \alpha)}{(n^e + \alpha)(p^e + \alpha)} d\mathbf{x}. \end{aligned}$$

Observing that $n^e p^e = n_i^2$, we have the identity:

$$np - n_i^2 = (n + \alpha)(p + \alpha) - (n^e + \alpha)(p^e + \alpha) - \alpha(n + p - n^e - p^e), \quad (3.38)$$

which leads to

$$\begin{aligned} & - \int_{\Omega} q U_T R \log \frac{(n + \alpha)(p + \alpha)}{(n^e + \alpha)(p^e + \alpha)} d\mathbf{x} \\ &= -q U_T \int_{\Omega} r \left((n + \alpha)(p + \alpha) - (n^e + \alpha)(p^e + \alpha) \right) \log \frac{(n + \alpha)(p + \alpha)}{(n^e + \alpha)(p^e + \alpha)} d\mathbf{x} \\ &\quad + q U_T \int_{\Omega} \alpha r (n + p - n^e - p^e) \log \frac{(n + \alpha)(p + \alpha)}{(n^e + \alpha)(p^e + \alpha)} d\mathbf{x} \\ &\leq q U_T \int_{\Omega} \alpha r (n + p - n^e - p^e) \log \frac{(n + \alpha)(p + \alpha)}{(n^e + \alpha)(p^e + \alpha)} d\mathbf{x} \\ &\leq c \int_{\Omega} \left| r \log \frac{(n + \alpha)(p + \alpha)}{(n^e + \alpha)(p^e + \alpha)} \right| |n + p + n^e + p^e| d\mathbf{x} \\ &\leq c(1 + \|n\|_1 + \|p\|_1). \end{aligned}$$

The last inequality follows from assumption (A5), and from the inequality

$$\left| \frac{\log(z + a)}{z + a} \right| \leq \max \left(\frac{1}{e}, \left| \frac{\log a}{a} \right| \right), \quad \forall z \geq 0, a > 0.$$

Finally, for the last term, recalling (3.6c), we have immediately

$$\mathbf{y}^\top \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}(t)) \leq c(1 + |\mathbf{y}|^2),$$

where $\mathbf{y} = \mathbf{P}\mathbf{x}$.

Combining the previous estimates, we get

$$\mathcal{H}_\alpha(t) \leq c_1 + c_2 \int_0^t \left(1 + \|n\|_1 + \|p\|_1 + \|\nabla \psi\|_2^2 + |\mathbf{P}\mathbf{x}|^2 \right) ds. \quad (3.39)$$

The thesis follows after noting that

$$\mathcal{H}_\alpha(t) \geq c \left(1 + \|n\|_1 + \|p\|_1 + \|\nabla \psi\|_2^2 + |\mathbf{P}\mathbf{x}|^2 \right),$$

with the help of Gronwall's lemma. ■

Lemma 3 *Under the assumptions (A1)–(A7), there exists a constant $c = c(T)$ such that for all $t \leq T$ we have*

$$\|n(\cdot, t)\|_2^2 + \|p(\cdot, t)\|_2^2 + \int_0^t \left(\|\nabla n(\cdot, s)\|_2^2 + \|\nabla p(\cdot, s)\|_2^2 \right) d\tau \leq c. \quad (3.40)$$

Proof. We consider the weak formulation (3.9a) with $\xi_n = n - n^e$ and integrate with respect to time

$$\begin{aligned} & \frac{1}{2} \left(\|n(\cdot, t) - n^e\|_2^2 - \|n(\cdot, 0) - n^e\|_2^2 \right) + D_n \int_0^t \int_{\Omega} |\nabla(n - n^e)|^2 dx d\tau \\ &= -D_n \int_0^t \int_{\Omega} \nabla n^e \cdot \nabla(n - n^e) dx d\tau + \int_0^t (\mu_n n \nabla \psi, \nabla(n - n^e)) d\tau - \int_0^t (R, n - n^e) d\tau. \end{aligned}$$

We have the estimate

$$- \int_0^t \int_{\Omega} \nabla n^e \cdot \nabla(n - n^e) dx d\tau \leq \frac{1}{2} \int_0^t \left(\|\nabla n(\cdot, \tau)\|_2^2 + \|\nabla n^e\|_2^2 \right) d\tau.$$

Furthermore, integrating by parts and using the Poisson equation, we obtain

$$\begin{aligned} & \int_0^t \int_{\Omega} n \nabla \psi \cdot \nabla(n - n^e) dx d\tau \\ &= - \int_0^t \int_{\Omega} (n - n^e)(\nabla \cdot \nabla \psi)(n - n^e) dx d\tau - \int_0^t \int_{\Omega} n^e \nabla \psi \cdot \nabla(n - n^e) dx d\tau \\ &\leq \frac{q}{2\epsilon_s} \int_0^t \int_{\Omega} (N - n + p)(n - n^e)^2 dx d\tau + \int_0^t \left(c \|\nabla \psi\|_2^2 + \frac{U_T}{4} \|\nabla(n - n^e)\|_2^2 \right) d\tau \\ &\leq \frac{q}{2\epsilon_s} \int_0^t \int_{\Omega} n^2(p - n) dx d\tau + \int_0^t c \|\nabla \psi\|_2^2 d\tau + \frac{U_T}{4} \int_0^t \|\nabla(n - n^e)\|_2^2 d\tau \\ &\leq c(T) \left\{ 1 + \|n\|_2^2 + \|p\|_2^2 \right\} + \frac{q}{2\epsilon_s} \int_0^t \int_{\Omega} n^2(p - n) dx d\tau + \frac{U_T}{4} \int_0^t \|\nabla(n - n^e)\|_2^2 d\tau. \end{aligned}$$

We have used Lemma 2 to estimate all linear terms in n and p and the L^2 -norm of $\nabla \psi$. Using assumption (A5), we have

$$|(R, n - n^e)| \leq \int_{\Omega} |r(n, p)(np - n_i^2)(n - n^e)| dx \leq c(1 + \|n\|_2^2 + \|p\|_2^2)$$

Finally, we get the estimate

$$\begin{aligned} & \|n(t) - n^e\|_2^2 + \frac{D_n}{2} \int_0^t \int_{\Omega} |\nabla(n - n^e)|^2 dx d\tau \\ &\leq c(T) \left\{ 1 + \int_0^t (\|n\|_2^2 + \|p\|_2^2) d\tau \right\} + \frac{q}{\epsilon_s} \int_0^t \int_{\Omega} n^2(p - n) dx d\tau, \end{aligned}$$

which leads to

$$\|n(t)\|_2^2 + \int_0^t \|\nabla n\|_2^2 d\tau \leq c(T) \left\{ 1 + \int_0^t (\|n\|_2^2 + \|p\|_2^2) d\tau \right\} + c_n \int_0^t \int_{\Omega} n^2(p - n) dx d\tau,$$

for some positive constant c_n . Similarly for p we have

$$\|p(t)\|_2^2 + \int_0^t \|\nabla p\|_2^2 d\tau \leq c(T) \left\{ 1 + \int_0^t (\|n\|_2^2 + \|p\|_2^2) d\tau \right\} - c_p \int_0^t \int_{\Omega} p^2(p-n) dx d\tau,$$

for some positive constant c_p . Summing up we get

$$\begin{aligned} \|n(t)\|_2^2 + \|p(t)\|_2^2 + \int_0^t (\|\nabla n\|_2^2 + \|\nabla p\|_2^2) d\tau \\ \leq c(T) \left\{ 1 + \int_0^t (\|n\|_2^2 + \|p\|_2^2) d\tau \right\} + c_1 \left\{ \int_0^t \int_{\Omega} (n^2 - p^2)(p-n) dx d\tau \right\} \\ \leq c(T) \left\{ 1 + \int_0^t (\|n\|_2^2 + \|p\|_2^2) d\tau \right\}, \end{aligned}$$

in which we applied $(n^2 - p^2)(p-n) = -(n+p)(n-p)^2 \leq 0$. Using Gronwall's lemma, we get the thesis. $\blacksquare$

Combining the results in the previous Lemmas we have:

Lemma 4 *There exists a constant $c = c(T)$ such that for all $t \leq T$ we have*

$$\|n(\cdot, t)\|_2^2 + \|p(\cdot, t)\|_2^2 + |\mathbf{P}\mathbf{x}(t)|^2 + \int_0^t (\|\nabla n(\cdot, \tau)\|_2^2 + \|\nabla p(\cdot, \tau)\|_2^2) d\tau \leq c. \quad (3.41)$$

The main result of this paper is the following global existence and uniqueness theorem.

Theorem 2 *[Global existence and uniqueness] Under the assumptions (A1)–(A7), problem (3.1) admits a unique solution on the time interval $[0, T]$ for any $T \in (0, \infty)$.*

Proof. The local, unique solution established by Theorem 1 can be prolonged up to an arbitrary time T by using the results in Lemmas 2, 3 and 4. $\blacksquare$

4 Asymptotic behaviour of the solution

In this section, we study the asymptotic behaviour of the solution to (3.1) as t tends to infinity. For a single device without coupling with an electric circuit, it is known that all solutions relax to equilibrium [27] if the applied potential vanishes. Therefore, in the coupled system, we need to demand appropriate topological conditions on the electric network to enable such a result. We derive sufficient conditions on the network topology during the development of this section.

Theorem 3 (Asymptotic behaviour of the solution) *Under the assumptions (A1)–(A7), let the network system satisfy the additional topological conditions*

$$\ker \mathbf{A}_R^\top \subset \ker \mathbf{A}_C^\top \subset \ker \mathbf{A}_L^\top. \quad (4.1)$$

Moreover, let the sources satisfy the decay conditions

$$\lim_{t \rightarrow \infty} |\mathbf{s}(t)|^2 = 0, \quad \int_0^t |\mathbf{s}(\tau)|^2 d\tau \leq K \text{ for all } t \geq 0. \quad (4.2)$$

Then, as time tends to infinity, the solution of the problem (3.1) approaches the equilibrium state, meaning that there exists $\bar{\mathbf{i}}_L \in \ker \mathbf{A}_L \subset \mathbb{R}^{n_L}$ such that

$$\begin{aligned} n &\rightarrow n^e, & p &\rightarrow p^e & & \text{in } L^1, \\ \psi &\rightarrow \psi^e, & & & & \text{in } H^1, \\ \mathbf{u} &\rightarrow 0, & \mathbf{i}_V &\rightarrow 0, & \mathbf{i}_L &\rightarrow \bar{\mathbf{i}}_L \quad \text{in Euclidean norm.} \end{aligned} \tag{4.3}$$

Proof. The proof follows along the line of Gajewski's analogous result in [27], modified to take into account the coupling of the device with the circuit.

Summing and subtracting the term $q\mu_n(n + \alpha)\nabla\psi_n^e$ in the definition of the current densities $\mathbf{J}_n$ rewritten as in (3.37), we get

$$\mathbf{J}_n = -q\mu_n(n + \alpha)\nabla(\psi_n - \psi_n^e) - q\mu_n\alpha\frac{n - n^e}{n^e + \alpha}\nabla\psi^e + q\mu_n\alpha\nabla(\psi - \psi^e).$$

Similarly, for $\mathbf{J}_p$ we get

$$\mathbf{J}_p = -q\mu_p(p + \alpha)\nabla(\psi_p - \psi_p^e) - q\mu_p\alpha\frac{p - p^e}{p^e + \alpha}\nabla\psi^e + q\mu_p\alpha\nabla(\psi - \psi^e).$$

In the above identities we have used (3.32) to obtain $\nabla\psi_n^e = \frac{\alpha}{n^e + \alpha}\nabla\psi^e$ and $\nabla\psi_p^e = \frac{\alpha}{p^e + \alpha}\nabla\psi^e$. Then, for an arbitrary time $t_1 \geq t_0 \geq 0$, we find

$$\begin{aligned} \int_{t_0}^{t_1} \int_{\Omega} \mathbf{J}_n \cdot \nabla(\psi_n - \psi_n^e) \, d\mathbf{x} \, d\tau &= - \int_{t_0}^{t_1} \int_{\Omega} q\mu_n(n + \alpha) |\nabla(\psi_n - \psi_n^e)|^2 \, d\mathbf{x} \, d\tau \\ &\quad + \int_{t_0}^{t_1} \int_{\Omega} q\mu_n\alpha \left\{ \nabla(\psi - \psi^e) - \frac{n - n^e}{n^e + \alpha} \nabla\psi^e \right\} \cdot \nabla(\psi_n - \psi_n^e) \, d\mathbf{x} \, d\tau \\ &\leq -\frac{q}{2} \int_{t_0}^{t_1} \int_{\Omega} \mu_n(n + \alpha) |\nabla(\psi_n - \psi_n^e)|^2 \, d\mathbf{x} \, d\tau \\ &\quad + q\alpha \int_{t_0}^{t_1} \int_{\Omega} \frac{\mu_n}{2} \left| \nabla\psi - \nabla\psi^e - \frac{\nabla\psi^e(n - n^e)}{n^e + \alpha} \right|^2 \, d\mathbf{x} \, d\tau, \end{aligned} \tag{4.4}$$

and, similarly,

$$\begin{aligned} \int_{t_0}^{t_1} \int_{\Omega} \mathbf{J}_p \cdot \nabla(\psi_p - \psi_p^e) \, d\mathbf{x} \, d\tau &\leq -\frac{q}{2} \int_{t_0}^{t_1} \int_{\Omega} \mu_p(p + \alpha) |\nabla(\psi_p - \psi_p^e)|^2 \, d\mathbf{x} \, d\tau \\ &\quad + q\alpha \int_{t_0}^{t_1} \int_{\Omega} \frac{\mu_p}{2} \left| \nabla\psi - \nabla\psi^e - \frac{\nabla\psi^e(p - p^e)}{p^e + \alpha} \right|^2 \, d\mathbf{x} \, d\tau. \end{aligned} \tag{4.5}$$

Using (4.4) and (4.5) in Lemma 1, considering (A5) and (3.38), for arbitrary time

$t_1 \geq t_0 \geq 0$, we can write

$$\begin{aligned}
& \mathcal{H}_\alpha(t_1) + \frac{q}{2} \int_{t_0}^{t_1} \int_{\Omega} (\mu_n(n+\alpha)|\nabla(\psi_n - \psi_n^e)|^2 + \mu_p(p+\alpha)|\nabla(\psi_p - \psi_p^e)|^2) \, d\mathbf{x} \, d\tau \\
& + q \int_{t_0}^{t_1} \int_{\Omega} \frac{U_T r}{n_i^2} [(n+\alpha)(p+\alpha) - (n^e+\alpha)(p^e+\alpha)] \log \frac{(n+\alpha)(p+\alpha)}{(n^e+\alpha)(p^e+\alpha)} \, d\mathbf{x} \, d\tau \\
& \leq \mathcal{H}_\alpha(t_0) + \int_{t_0}^{t_1} \mathbf{y}^\top \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}) \, d\tau \\
& + \alpha q \int_{t_0}^{t_1} \int_{\Omega} \left[\frac{\mu_n}{2} \left| \nabla \psi - \nabla \psi^e - \frac{\nabla \psi^e (n - n^e)}{n^e + \alpha} \right|^2 + \frac{\mu_p}{2} \left| \nabla \psi - \nabla \psi^e - \frac{\nabla \psi^e (p - p^e)}{p^e + \alpha} \right|^2 \right. \\
& \left. + \frac{U_T r}{n_i^2} (n + p - n^e - p^e) \log \frac{(n + \alpha)(p + \alpha)}{(n^e + \alpha)(p^e + \alpha)} \right] \, d\mathbf{x} \, d\tau,
\end{aligned} \tag{4.6}$$

with $\mathbf{y} = \mathbf{P}\mathbf{x}$. The last integral term in (4.6) is bounded, so we can pass to the limit for $\alpha \rightarrow 0$ and find

$$\begin{aligned}
& \mathcal{H}(t_1) + \frac{q}{2} \int_{t_0}^{t_1} \int_{\Omega} (\mu_n n |\nabla(\phi_n - \phi_n^e)|^2 + \mu_p p |\nabla(\phi_p - \phi_p^e)|^2) \, d\mathbf{x} \, d\tau \\
& \leq \mathcal{H}(t_0) + \int_{t_0}^{t_1} \mathbf{y}^\top \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}) \, d\tau
\end{aligned} \tag{4.7}$$

where ϕ_n, ϕ_n^e, ϕ_p and ϕ_p^e are the physical quasi-Fermi levels,

$$\phi_n = \psi - U_T \log \frac{n}{n_i}, \quad \phi_p = \psi + U_T \log \frac{p}{n_i}, \tag{4.8}$$

$$\phi_n^e = \psi^e - U_T \log \frac{n^e}{n_i}, \quad \phi_p^e = \psi^e + U_T \log \frac{p^e}{n_i}, \tag{4.9}$$

and

$$\begin{aligned}
\mathcal{H}(t) &= \int_{\Omega} q U_T \left[n \left(\log \frac{n}{n^e} - 1 \right) + n^e + p \left(\log \frac{p}{p^e} - 1 \right) + p^e \right] \, d\mathbf{x} \\
&+ \int_{\Omega} \frac{\epsilon_s}{2} |\mathbf{E} - \mathbf{E}^e|^2 \, d\mathbf{x} + \frac{1}{2} \mathbf{u}^\top \mathbf{A}_C \mathbf{C} \mathbf{A}_C^\top \mathbf{u} + \frac{1}{2} \mathbf{i}_L^\top \mathbf{L} \mathbf{i}_L \\
&= \mathcal{H}^*(t) + \frac{1}{2} \mathbf{u}^\top(t) \mathbf{A}_C \mathbf{C} \mathbf{A}_C^\top \mathbf{u}(t) + \frac{1}{2} \mathbf{i}_L^\top(t) \mathbf{L} \mathbf{i}_L(t).
\end{aligned} \tag{4.10}$$

Using Definition 1, and definition (2.31) of $\mathbf{E}_1$ and $\mathbf{A}_1$, for $\mathbf{y} = \mathbf{P}\mathbf{x}$, $\mathbf{z} = \mathbf{Q}\mathbf{x}$, we can write

$$\begin{aligned}
& \mathbf{y}^\top \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}) \\
&= \mathbf{y}^\top \mathbf{E}_1 (\mathbf{P} + \mathbf{Q}) \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}) - \mathbf{y}^\top \mathbf{E}_1 \mathbf{Q} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}) \\
&= \mathbf{y}^\top (\mathbf{A}_1 \mathbf{y} + \mathbf{s}) - \mathbf{y}^\top \mathbf{E}_1 \mathbf{z} = \mathbf{y}^\top (\mathbf{A} \mathbf{y} + \mathbf{s}) + \mathbf{y}^\top \mathbf{A} \mathbf{z}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
-\mathbf{z}^\top \mathbf{A} \mathbf{z} &= \mathbf{z}^\top \mathbf{E}_1 \mathbf{z} = \mathbf{z}^\top \mathbf{E}_1 \mathbf{Q} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}) \\
&= \mathbf{z}^\top \mathbf{E}_1 (\mathbf{P} + \mathbf{Q}) \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}) - \mathbf{z}^\top \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}) \\
&= \mathbf{z}^\top (\mathbf{A}_1 \mathbf{y} + \mathbf{s}) - \mathbf{z}^\top \mathbf{E}_1 \mathbf{y} = \mathbf{z}^\top (\mathbf{A} \mathbf{y} + \mathbf{s}),
\end{aligned}$$

since $\mathbf{Q}^\top \mathbf{E}_1 \mathbf{P} = \mathbf{Q}^\top (\mathbf{E} - \mathbf{A} \mathbf{Q}) \mathbf{P} = \mathbf{O}$. Summing up, we obtain the identity

$$\begin{aligned} \mathbf{y}^\top \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}) &= (\mathbf{y} + \mathbf{z})^\top [\mathbf{A}(\mathbf{y} + \mathbf{z}) + \mathbf{s}] \\ &= -\mathbf{u}^\top \mathbf{A}_R \mathbf{G} \mathbf{A}_R^\top \mathbf{u} + \mathbf{i}_V^\top \mathbf{v}_V + \mathbf{u}^\top \mathbf{A}_I \mathbf{i}_I. \end{aligned}$$

Next, recalling the definition (2.32) of $\mathbf{Q}$ and $\mathbf{P}$, we write

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}' \\ \mathbf{y}'' \\ 0 \end{pmatrix} := \begin{pmatrix} \mathbf{P}_{CD} \mathbf{u} \\ \mathbf{i}_L \\ 0 \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} \mathbf{z}' \\ 0 \\ \mathbf{z}'' \end{pmatrix} := \begin{pmatrix} \mathbf{Q}_{CD} \mathbf{u} \\ 0 \\ \mathbf{i}_V \end{pmatrix}.$$

The condition $\ker \mathbf{A}_R^\top \subset \ker \mathbf{A}_C^\top$, and the positive-definiteness of $\mathbf{G}$, imply

$$\mathbf{u}^\top \mathbf{A}_R \mathbf{G} \mathbf{A}_R^\top \mathbf{u} \geq c_G |\mathbf{P}_R \mathbf{u}|^2 \geq c_G |\mathbf{P}_C \mathbf{u}|^2 = c_G |\mathbf{y}'|^2.$$

Also, we have

$$\mathbf{i}_V^\top \mathbf{v}_V + \mathbf{u}^\top \mathbf{A}_I \mathbf{i}_I = \mathbf{y}'^\top \mathbf{A}_I \mathbf{i}_I + \mathbf{z}'^\top \mathbf{A}_I \mathbf{i}_I + \mathbf{z}_2^\top \mathbf{v}_V \leq |\mathbf{y}'| |\mathbf{s}| + |\mathbf{z}| |\mathbf{s}|.$$

Recalling the expression (2.39) of $\mathbf{z}$ in terms of $\mathbf{y}$, and (2.37), we find

$$\mathbf{z} = \mathbf{Q} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}) = \bar{\mathbf{E}}_1^{-1} \mathbf{Q}^\top (\mathbf{A}_1 \mathbf{y} + \mathbf{s}).$$

Using the condition $\ker \mathbf{A}_C^\top \subset \ker \mathbf{A}_L^\top$, we can compute explicitly

$$\mathbf{Q}^\top \mathbf{A}_1 \mathbf{y} = \begin{pmatrix} -\mathbf{Q}_{CD}^\top \mathbf{A}_R \mathbf{G} \mathbf{A}_R^\top \\ \mathbf{O} \\ \mathbf{A}_V^\top \end{pmatrix} \mathbf{P}_{CD}^\top \mathbf{u} = \mathbf{M}_1 \mathbf{y}',$$

which implies

$$|\mathbf{z}| \leq c(|\mathbf{y}'| + |\mathbf{s}|). \quad (4.11)$$

In conclusion, we find

$$\begin{aligned} \mathbf{y}^\top \mathbf{E}_1 \mathbf{P} \mathbf{E}_1^{-1} (\mathbf{A}_1 \mathbf{y} + \mathbf{s}) &\leq -c_G |\mathbf{y}'|^2 + |\mathbf{y}'| |\mathbf{s}| + c(|\mathbf{y}'| + |\mathbf{s}|) |\mathbf{s}| \\ &\leq -\frac{c_G}{2} |\mathbf{y}'|^2 + c |\mathbf{s}|^2, \end{aligned} \quad (4.12)$$

where we have used the weighted Schwarz's inequality. Inserting (4.12) in (4.7), we obtain

$$\begin{aligned} \mathcal{H}(t_1) + \frac{q}{2} \int_{t_0}^{t_1} \int_{\Omega} (\mu_n n |\nabla(\phi_n - \phi_n^e)|^2 + \mu_p p |\nabla(\phi_p - \phi_p^e)|^2) \, d\mathbf{x} \, d\tau \\ + \frac{c_G}{2} \int_{t_0}^{t_1} |\mathbf{y}'|^2 \, d\tau \leq \mathcal{H}(t_0) + c \int_{t_0}^{t_1} |\mathbf{s}(\tau)|^2 \, d\tau. \end{aligned} \quad (4.13)$$

which implies for all $t_1 \geq t_0 \geq 0$ that

$$\mathcal{H}(t_1) - c \int_0^{t_1} |\mathbf{s}(\tau)|^2 \, d\tau \leq \mathcal{H}(t_0) - c \int_0^{t_0} |\mathbf{s}(\tau)|^2 \, d\tau, \quad (4.14)$$

and, using (4.2), that

$$\begin{aligned} \mathcal{H}(t_1) + \frac{q}{2} \int_0^{t_1} \int_{\Omega} (\mu_n n |\nabla(\phi_n - \phi_n^e)|^2 + \mu_p p |\nabla(\phi_p - \phi_p^e)|^2) \, d\mathbf{x} \, d\tau \\ + \frac{c_G}{2} \int_0^{t_1} |\mathbf{y}'|^2 \, d\tau \leq \mathcal{H}(0) + cK. \end{aligned} \quad (4.15)$$

This means that there exists a sequence $t_j \rightarrow \infty$ such that

$$\lim_{j \rightarrow \infty} \left[\int_{\Omega} (n |\nabla \phi_n|^2 + p |\nabla \phi_p|^2) \, d\mathbf{x} \right]_{t=t_j} = 0. \quad (4.16)$$

We show that (4.16) yields a bound on the norm of $\nabla \sqrt{n}$ and $\nabla \sqrt{p}$, thanks to the identity

$$\begin{aligned} \int_{\Omega} (n |\nabla \phi_n|^2 + p |\nabla \phi_p|^2) \, d\mathbf{x} &= \int_{\Omega} \left(n \left| -\nabla \psi - U_T \frac{\nabla n}{n} \right|^2 + p \left| -\nabla \psi + U_T \frac{\nabla p}{p} \right|^2 \right) \, d\mathbf{x} \\ &= \int_{\Omega} \left((n+p) |\nabla \psi|^2 + 2U_T \nabla \psi \cdot \nabla(n-p) + 4U_T^2 (|\nabla \sqrt{n}|^2 + |\nabla \sqrt{p}|^2) \right) \, d\mathbf{x}. \end{aligned} \quad (4.17)$$

Rearranging the terms in (4.17), we find, for $t = t_j$,

$$\begin{aligned} 4U_T^2 (\|\nabla \sqrt{n}\|^2 + \|\nabla \sqrt{p}\|^2) + \int_{\Omega} (n+p) |\nabla \psi|^2 \, d\mathbf{x} \\ = \int_{\Omega} (n |\nabla \phi_n|^2 + p |\nabla \phi_p|^2) \, d\mathbf{x} - \int_{\Omega} 2U_T \nabla \psi \cdot \nabla(n-p) \, d\mathbf{x} \leq c, \end{aligned} \quad (4.18)$$

since, integrating by parts, considering the boundary conditions (3.1f)-(3.1h) and the Poisson equation for the electrostatic potential, we have

$$\begin{aligned} - \int_{\Omega} \nabla \psi \cdot \nabla(n-p) \, d\mathbf{x} &= \frac{q}{\epsilon_s} \int_{\Omega} N(n-p) \, d\mathbf{x} - \frac{q}{\epsilon_s} \int_{\Omega} (n-p)^2 \, d\mathbf{x} + \int_{\Gamma_D} (n-p) \nabla \psi \cdot \boldsymbol{\nu} \, d\sigma \\ &\leq \frac{q}{2\epsilon_s} \|N\|^2 - \frac{q}{2\epsilon_s} \|n-p\|^2 + \frac{1}{4\delta} \int_{\Gamma_D} |n_D - p_D|^2 \, d\sigma + \delta \|\nabla \psi\|_{L^2(\partial\Omega)}^2 \\ &\leq \frac{3q}{4\epsilon_s} \|N\|^2 + \frac{1}{4\delta} \int_{\Gamma_D} |n_D - p_D|^2 \, d\sigma. \end{aligned}$$

Here, we have used $\|\nabla \psi\|_{L^2(\partial\Omega)}^2 \leq c \|\psi\|_{H^2}^2 \leq c(\|N\|^2 + \|n-p\|^2)$, and chosen δ small enough, so that $c\delta \leq \frac{q}{4\epsilon_s}$. Then from (4.15) we can write

$$\|\sqrt{n(t_j)}\|_{1,2} + \|\sqrt{p(t_j)}\|_{1,2} + \|\psi(t_j)\|_{1,2} \leq c. \quad (4.19)$$

Following [27], we use the compactness of the embeddings of W_2^1 into $L_{2\beta}$ with $\frac{3}{2} < \beta < 3$ and $L_{2\gamma}(\partial\Omega)$ with $1 < \gamma < 2$, to assert the existence of two functions $\bar{n}$ and $\bar{p}$ belonging to L_{β} and $L_{\gamma}(\partial\Omega)$ such that $\sqrt{\bar{n}}, \sqrt{\bar{p}} \in W_2^1$ and

$$\lim_{j \rightarrow \infty} n(t_j) = \bar{n}, \quad \lim_{j \rightarrow \infty} p(t_j) = \bar{p} \quad \text{in} \quad L_{\beta} \quad \text{and} \quad L_{\gamma}(\partial\Omega). \quad (4.20)$$

Next, we prove the convergence of $\mathbf{y}$ and $\mathbf{z}$. Using (4.15), we can write

$$\frac{1}{2}\mathbf{u}^\top(t)\mathbf{A}_C\mathbf{C}\mathbf{A}_C^\top\mathbf{u}(t) + \frac{1}{2}\mathbf{i}_L^\top(t)\mathbf{L}\mathbf{i}_L(t) + \frac{c_G}{2}\int_0^t |\mathbf{y}'|^2 d\tau \leq \mathcal{H}(0) + cK. \quad (4.21)$$

Since $\mathbf{C}$ is positive definite, and $\mathbf{y}'$ does not belong to the nullspace of $\mathbf{A}_C^\top$, we have $\mathbf{u}^\top\mathbf{A}_C\mathbf{C}\mathbf{A}_C^\top\mathbf{u} \geq c_C|\mathbf{y}'|^2$, which yields

$$\frac{c_C}{2}|\mathbf{y}'|^2 + \frac{c_G}{2}\int_0^t |\mathbf{y}'|^2 d\tau \leq \mathcal{H}(0) + cK.$$

Gronwal's lemma implies that $\mathbf{y}'(t) \rightarrow 0$, with asymptotic decay as $t \rightarrow \infty$. Then, recalling assumption (4.2), estimate (4.11) implies that $\mathbf{z}(t) \rightarrow 0$ as $t \rightarrow \infty$. In particular, we have shown that $\mathbf{u} = \mathbf{y}' + \mathbf{z}'$ tends to zero, that is,

$$\lim_{j \rightarrow \infty} \mathbf{u}(t_j) = 0. \quad (4.22)$$

Finally, (4.21) implies the boundedness of $\mathbf{y}'' = \mathbf{i}_L$, due to the positive definiteness of $\mathbf{L}$, which implies the convergence of $\mathbf{i}_L(t) \rightarrow \bar{\mathbf{i}}_L$.

From (4.20) and (4.22), recalling the definition (2.11) of $\mathcal{L}$, it follows that

$$\lim_{j \rightarrow \infty} \psi(t_j) = \bar{\psi} = q\mathcal{L}(\bar{p} - \bar{n} + N) \quad \text{in} \quad W_\beta^2, \quad (4.23)$$

and, since $W_\beta^2 \subset C$, also that

$$\lim_{j \rightarrow \infty} n(t_j) \exp\left(-\frac{\psi(t_j)}{U_T}\right) = \bar{n}e^{-\bar{\psi}/U_T}, \quad \lim_{j \rightarrow \infty} p(t_j) \exp\left(\frac{\psi(t_j)}{U_T}\right) = \bar{p}e^{\bar{\psi}/U_T} \quad \text{in } L_\beta.$$

Moreover, recalling (4.8), we have the identities

$$n \exp\left(-\frac{\psi}{U_T}\right) = n_i \exp\left(-\frac{\phi_n}{U_T}\right), \quad p \exp\left(\frac{\psi}{U_T}\right) = n_i \exp\left(\frac{\phi_p}{U_T}\right),$$

which imply

$$\nabla(ne^{-\psi/U_T}) = -\frac{n}{U_T}e^{-\psi/U_T}\nabla\phi_n, \quad \nabla(pe^{\psi/U_T}) = \frac{p}{U_T}e^{\psi/U_T}\nabla\phi_p.$$

Then, using (4.16) and (4.20), it is possible to show that $\bar{n}e^{-\bar{\psi}/U_T}$ and $\bar{p}e^{\bar{\psi}/U_T}$ are constant, and from the boundary conditions (3.1f) and (3.1g), written with $\mathbf{u}_D = 0$, because of (4.22), we find

$$\bar{n}e^{-\bar{\psi}/U_T} = n_De^{-\psi_D/U_T} = n_i, \quad \bar{p}e^{\bar{\psi}/U_T} = p_De^{\psi_D/U_T} = n_i.$$

This shows that

$$\bar{n} = n_i e^{\bar{\psi}/U_T}, \quad \bar{p} = n_i e^{-\bar{\psi}/U_T},$$

and, thanks to (4.23), $\bar{\psi}$ satisfies the defining equations for the equilibrium potential, with the same boundary conditions, thus

$$\bar{\psi} = \psi^e, \quad \bar{n} = n^e, \quad \bar{p} = p^e.$$

In particular, this implies that the limit current from the device to the circuit becomes zero as t goes to infinity. Thus, from the circuit equations (2.1), recalling that $\mathbf{y}'$ and $\mathbf{z}$ tend to zero as time increases, the limit value $\bar{\mathbf{i}}_L$ for $\mathbf{y}'' = \mathbf{i}_L$ satisfies the condition $\mathbf{A}_L \bar{\mathbf{i}}_L = 0$, that is, $\bar{\mathbf{i}}_L \in \ker \mathbf{A}_L$.

We can finally pass from the sequence $\{t_j\}$ to the continuous limit for $t \rightarrow \infty$, by recalling that the function $\mathcal{H}(t) - c \int_0^t |\mathbf{s}(\tau)|^2 d\tau$ is decreasing, as shown in (4.14). Then, we have

$$\lim_{t \rightarrow \infty} \left\{ \mathcal{H}(t) - c \int_0^t |\mathbf{s}(\tau)|^2 d\tau \right\} = \lim_{j \rightarrow \infty} \left\{ \mathcal{H}(t_j) - c \int_0^{t_j} |\mathbf{s}(\tau)|^2 d\tau \right\} = -c \lim_{j \rightarrow \infty} \int_0^{t_j} |\mathbf{s}(\tau)|^2 d\tau,$$

which yields

$$\lim_{t \rightarrow \infty} \mathcal{H}(t) = 0.$$

The estimates (4.3) follow as in [27]. ■

5 Conclusions

In this paper, we have investigated a coupled system of differential-algebraic equations for an electric network and partial differential equations for a semiconductor device. The semiconductor device is modeled by multi-dimensional, parabolic, drift-diffusion equations.

We have proposed a reasonable index concept for the resulting partial differential-algebraic equation, which extends the usual tractability index for differential-algebraic equations.

In the index-1 case, for the above-described multi-dimensional parabolic partial differential-algebraic equation, we have established existence and uniqueness of solutions up to an arbitrary final time.

Finally, under more restrictive assumptions on the topology of the electric network and the circuit sources, we have discussed the asymptotic behaviour of the solution for long times, proving that it relaxes to an equilibrium solution.

Conflicts of interest and funding

This work does not have any conflicts of interest. There are no funders to report for this submission.

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