Fractional Orlicz-Sobolev embeddings

FRACTIONAL ORLICZ-SOBOLEV EMBEDDINGS

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Abstract. The optimal Orlicz target space is exhibited for embeddings of fractional-order Orlicz–Sobolev spaces in $\mathbb{R}^n$. An improved embedding with an Orlicz–Lorentz target space, which is optimal in the broader class of all rearrangement-invariant spaces, is also established. Both spaces of order $s \in (0,1)$, and higher-order spaces are considered. Related Hardy type inequalities are proposed as well.

Résumé. L'espace d'arrivée d'Orlicz optimal pour les inclusions fractionnaires des espaces d'Orlicz-Sobolev dans $\mathbb{R}^n$ est exposé. Une inclusion améliorée avec un espace d'arrivée d'Orlicz-Lorentz, qui est optimal dans la large classe constituée de tous les espaces invariants par réarrangements est aussi établit. Les espaces d'ordre $s \in (0,1)$ ainsi que les espaces d'ordre plus élevé sont considérés. Les inégalités de Hardy associées à ces espaces sont également proposées.

1. Introduction

The present paper is aimed at offering optimal Sobolev–Poincaré type inequalities and related embeddings for fractional-order Orlicz-Sobolev spaces. These spaces extend the classical fractional Sobolev spaces introduced in [5, 52, 75]. Given a number $s \in (0,1)$ and a Young function $A : [0,\infty) \to [0,\infty]$, namely a convex function vanishing at 0, the fractional-order Orlicz-Sobolev space $W^{s,A}(\mathbb{R}^n)$ is defined via a seminorm $|\cdot|_{s,A,\mathbb{R}^n}$ built upon the functional defined as

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} \tag{1.1}$$

for a measurable function u in $\mathbb{R}^n$. The definition of the seminorm $|\cdot|_{s,A,\mathbb{R}^n}$ is given, via the functional (1.1), in analogy with the notion of Luxemburg norm in Orlicz spaces. The bases for a theory of the spaces $W^{s,A}(\mathbb{R}^n)$, motivated e.g. by the analysis of nonlinear fractional Laplacians with non-polynomial kernels, have recently been laid in [40, 47] under the Δ_2-condition and a sublinear growth condition near zero on the Young function A. Neither of these additional assumptions will be imposed throughout.

The standard Gagliardo functional and the associated seminorm $|\cdot|_{s,p,\mathbb{R}^n}$, underlying the notion of the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ for $p \in [1,\infty)$, are recovered by the choice $A(t) = t^p$. A renewed interest in the area around fractional Sobolev spaces has flourished in the last two decades. This has been favoured by a myriad of investigations on nonlocal equations of elliptic and parabolic type, whose solutions naturally belong to the spaces $W^{s,p}(\mathbb{R}^n)$. A touch of recent contributions in this connection is furnished by [8, 9, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 39, 42, 43, 48, 49, 50, 53, 54, 57, 58, 59, 62, 63, 65, 67, 68, 72, 73, 74, 77, 80]. Comprehensive treatments of the theory of fractional Sobolev spaces, as special instances of the more general Besov spaces, can be found e.g. in [11, 55]. A self-contained presentation of their basic properties is provided in [41]. Fine properties of functions in fractional Sobolev spaces in connection with potential theory are exposed in [61, Chapters 11 and 12].

Embeddings for the spaces $W^{s,p}(\mathbb{R}^n)$ into Lebesgue spaces are classical. In particular, if $1 \le p < \frac{n}{s}$, then there exists a constant C such that

$$\|u\|_{L^{\frac{np}{n-sp}}(\mathbb{R}^n)} \le C|u|_{s,p,\mathbb{R}^n} \tag{1.2}$$

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for every measurable function u decaying to 0 (in a suitable sense) near infinity.
An improved version of inequality (1.2) has been established in [51]. It asserts that, in fact,

$$\|u\|_{L^{\frac{np}{n-sp},p}(\mathbb{R}^n)} \leq C|u|_{s,p,\mathbb{R}^n}\,, \tag{1.3}$$

where the Lorentz space $L^{\frac{np}{n-sp},p}(\mathbb{R}^n) \subsetneq L^{\frac{np}{n-sp}}(\mathbb{R}^n)$.

Our main results amount to sharp counterparts of inequalities (1.2) and (1.3) for general seminorms $|\cdot|_{s,A,\mathbb{R}^n}$. Given any $s \in (0,1)$ and any Young function A, with subcritical growth corresponding to the assumption $p \leq \frac{n}{s}$ in the case of powers, we detect the optimal Orlicz target space $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ such that

$$\|u\|_{L^{A_{\frac{n}{s}}}(\mathbb{R}^n)} \leq C|u|_{s,A,\mathbb{R}^n} \tag{1.4}$$

for some constant C and every measurable function u decaying to 0 near infinity. An explicit formula for the Young function $A_{\frac{n}{s}}$ is provided, which only depends on A and on the ratio $\frac{n}{s}$. Here, and in what follows, the expression "optimal target space" referred to an embedding or inequality means "smallest possible" within a specified family, in the sense that if the embedding also holds with the optimal target space in question replaced by another space from the same family, then the former is continuously embedded into the latter.
Inequality (1.4) is derived as a consequence of a stronger inequality

$$\|u\|_{L(\widehat{A},\frac{n}{s})(\mathbb{R}^n)} \leq C|u|_{s,A,\mathbb{R}^n}\,, \tag{1.5}$$

where $L(\widehat{A},\frac{n}{s})(\mathbb{R}^n)$ – a space of Orlicz-Lorentz type depending only on A and $\frac{n}{s}$ – is optimal in the larger class of all rearrangement-invariant spaces.
In particular, these results reproduce inequalities (1.2) and (1.3) when $A(t) = t^p$. The latter also provides new information about (1.3), and tells us that the space $L^{\frac{np}{n-sp},p}(\mathbb{R}^n)$ is indeed optimal in (1.3) among all rearrangement-invariant spaces. Let us mention that embeddings for the spaces $W^{s,A}(\mathbb{R}^n)$, under additional technical assumptions on A and into non-optimal (in general) target spaces have recently appeared in [7]. Embeddings for fractional Orlicz-Sobolev spaces modeled on a different definition can be found in [56].

Optimal inequalities for Orlicz-Sobolev spaces of fractional-order $s > 1$ are also presented. As customary, these spaces are defined on replacing the function u in (1.1) by $\nabla^{[s]}u$, the vector of all weak derivatives of u whose order is the integer part $[s]$ of s. The inequalities in question parallel (1.4) and (1.5). They extend inequalities (1.4) and (1.5) to any $s \in (0,n) \setminus \mathbb{N}$, and take the form

$$\|u\|_{L^{A_{\frac{n}{s}}}(\mathbb{R}^n)} \leq C\big|\nabla^{[s]}u\big|_{s,A,\mathbb{R}^n}\,, \tag{1.6}$$

and

$$\|u\|_{L(\widehat{A},\frac{n}{s})(\mathbb{R}^n)} \leq C\big|\nabla^{[s]}u\big|_{s,A,\mathbb{R}^n}\,, \tag{1.7}$$

respectively, for functions u all of whose derivatives up to the order $[s]$ decay to 0 near infinity. The sharp spaces $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ and $L(\widehat{A},\frac{n}{s})(\mathbb{R}^n)$ appearing in (1.6) and (1.7) are defined exactly via the same formulas as in the case when $s \in (0,1)$, save that now are applied for $s \in (1,n) \setminus \mathbb{N}$. Our conclusions could thus be formulated via statements simultaneously covering the cases when $s \in (0,1)$ and $s \in (1,n) \setminus \mathbb{N}$. We prefer to enucleate the results for $s \in (0,1)$ in a separate section for ease of presentation, and also because those for $s \in (1,n) \setminus \mathbb{N}$ call for a combination of the former and of inequalities for integer-order Orlicz-Sobolev spaces.

The integer-order Orlicz-Sobolev embeddings have been established in [31, 32, 35, 36]. Importantly, these embeddings are exactly matched by the fractional-order embeddings announced above, although the latter rely on a substantially different approach. Indeed, applying our formulas for the optimal spaces $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ and $L(\widehat{A},\frac{n}{s})(\mathbb{R}^n)$ with $s \in \mathbb{N}$ recovers the optimal Orlicz and the optimal rearrangement-invariant space, respectively, in the Orlicz-Sobolev inequality of integer-order s.

Closely related fractional-order Hardy type inequalities in $\mathbb{R}^n$ are proposed as well. In fact, a crucial step in our approach is a Hardy inequality of order $s \in (0,1)$, which extends to the Orlicz realm a result from [63]. The Hardy inequality for $s \in (1,n) \setminus \mathbb{N}$ is, by contrast, deduced as a consequence of inequality (1.7).

The material is organized as follows. Section 2 and Section 3 are devoted to notations, definitions and necessary background from the theory of Orlicz and rearrangement-invariant spaces, and the theory of integer and fractional-order Orlicz-Sobolev spaces, respectively. Several sharp one-dimensional Hardy type inequalities in Orlicz and rearrangement-invariant spaces of critical use in the proofs of our main results are collected in Section 4. Some of them are known, but others are new. The main results are exposed in Sections 5–7. Section 5 deals with the Hardy inequality for fractional Orlicz-Sobolev spaces on $\mathbb{R}^n$ of order $s \in (0, 1)$. Orlicz-Sobolev embeddings for spaces of order in the same range are presented in Section 6, whereas Section 7 is concerned with embeddings of arbitrary order.
For the readers convenience, we list at the end of the paper the main symbols employed throughout, with a reference to the equation where they are introduced.

2. Orlicz spaces and rearrangement-invariant spaces

A function $A : [0,\infty) \to [0,\infty]$ is called a Young function if it has the form

$$A(t) = \int_0^t a(\tau)d\tau \quad \text{for } t \geq 0 \tag{2.1}$$

for some non-decreasing, left-continuous function $a : [0,\infty) \to [0,\infty]$ which is neither identically equal to 0 nor to ∞. Clearly, any convex (non trivial) function from $[0,\infty)$ into $[0,\infty]$, which is left-continuous and vanishes at 0, is a Young function.
Note that, if $k \geq 1$, then

$$k\,A(t) \leq A(kt) \quad \text{for } t \geq 0. \tag{2.2}$$

The Young conjugate $\widetilde{A}$ of A is defined by

$$\widetilde{A}(t) = \sup\{\tau t - A(\tau) :\, \tau \geq 0\} \quad \text{for } t \geq 0. \tag{2.3}$$

The following representation formula for $\widetilde{A}$ holds:

$$\widetilde{A}(t) = \int_0^t a^{-1}(\tau)d\tau \quad \text{for } t \geq 0. \tag{2.4}$$

Here, a^{-1} denotes the left-continuous inverse of the function a appearing in (2.1).
A Young function A is said to dominate another Young function B if there exists a positive constant C such that

$$B(t) \leq A(Ct) \quad \text{for } t \geq 0. \tag{2.5}$$

The functions A and B are called equivalent if they dominate each other.
The growth of a Young function A can be compared with that of a power function via its Matuszewska-Orlicz indices. Recall that the upper Matuszewska-Orlicz index $I(A)$ of a finite-valued Young function A is defined as

$$I(A) = \lim_{\lambda\to\infty} \frac{\log\left(\sup_{t>0} \frac{A(\lambda t)}{A(t)}\right)}{\log \lambda}. \tag{2.6}$$

Let us set

$$\mathcal{M}(\mathbb{R}^n) = \{u : \mathbb{R}^n \to \mathbb{R} : u \text{ is measurable}\}\,, \tag{2.7}$$

and

$$\mathcal{M}_+(\mathbb{R}^n) = \{u \in \mathcal{M}(\mathbb{R}^n) : u \geq 0\}\,. \tag{2.8}$$

The notation $\mathcal{M}_d(\mathbb{R}^n)$ is adopted for the subset of $\mathcal{M}(\mathbb{R}^n)$ of those functions u that decay near infinity, in the sense that all their level sets $\{|u| > t\}$ have finite Lebesgue measure for $t > 0$. Namely,

$$\mathcal{M}_d(\mathbb{R}^n) = \{u \in \mathcal{M}(\mathbb{R}^n) : |\{|u| > t\}| < \infty \text{ for every } t > 0\}\,, \tag{2.9}$$

where $|E|$ stands for the Lebesgue measure of a set $E \subset \mathbb{R}^n$.

The Orlicz space $L^A(\mathbb{R}^n)$ associated with a Young function A is the Banach function space of those functions $u \in \mathcal{M}(\mathbb{R}^n)$ for which the Luxemburg norm

$$\|u\|_{L^A(\mathbb{R}^n)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} A\left(\frac{|u|}{\lambda}\right) dx \le 1\right\} \tag{2.10}$$

is finite. In particular, $L^A(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ if $A(t) = t^p$ for some $p \in [1,\infty)$, and $L^A(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ if $A(t) = 0$ for $t \in [0,1]$ and $A(t) = \infty$ for $t > 1$.
The Hölder type inequality

$$\int_{\mathbb{R}^n} |uv|\, dx \le 2\|u\|_{L^A(\mathbb{R}^n)}\|v\|_{L^{\widetilde{A}}(\mathbb{R}^n)} \tag{2.11}$$

holds for every $u \in L^A(\mathbb{R}^n)$ and $v \in L^{\widetilde{A}}(\mathbb{R}^n)$.
If A dominates B, then

$$\|u\|_{L^B(\mathbb{R}^n)} \le C\|u\|_{L^A(\mathbb{R}^n)} \tag{2.12}$$

for every $u \in L^A(\mathbb{R}^n)$, where C is the same constant as in (2.5). Thus, if A is equivalent to B, then $L^A(\mathbb{R}^n) = L^B(\mathbb{R}^n)$, up to equivalent norms.

The Orlicz spaces are members of the more general class of rearrangement-invariant spaces, whose definition is based upon that of decreasing rearrangement of a function.
The *decreasing rearrangement* u^* of a function $u \in \mathcal{M}(\mathbb{R}^n)$ is the (unique) non-increasing, right-continuous function from $[0,\infty)$ into $[0,\infty]$ which is equidistributed with u. In formulas,

$$u^*(r) = \inf\{t \ge 0 : |\{x \in \mathbb{R}^n : |u(x)| > t\}| \le r\} \quad \text{for } r \ge 0. \tag{2.13}$$

Moreover, we define the function $u^{**} : (0,\infty) \to [0,\infty]$ as

$$u^{**}(r) = \frac{1}{r}\int_0^r u^*(\varrho)d\varrho \quad \text{for } r > 0. \tag{2.14}$$

Notice that $u^* \le u^{**}$. The *Hardy-Littlewood inequality* states that

$$\int_{\mathbb{R}^n} |uv|\, dx \le \int_0^\infty u^* v^*\, dr \tag{2.15}$$

for all functions $u, v \in \mathcal{M}(\mathbb{R}^n)$. As a consequence, one also has that

$$\int_{\mathbb{R}^n} A(|uv|)\, dx \le \int_0^\infty A(u^* v^*)\, dr \tag{2.16}$$

for every Young function A.
A Banach function space $X(\mathbb{R}^n)$, in the sense of Luxemburg [10, Chapter 1, Section 1], is called a *rearrangement-invariant space* if

$$\|u\|_{X(\mathbb{R}^n)} = \|v\|_{X(\mathbb{R}^n)} \quad \text{whenever } u^* = v^*. \tag{2.17}$$

The *associate space* $X'(\mathbb{R}^n)$ of $X(\mathbb{R}^n)$ is the rearrangement-invariant space of all functions in $\mathcal{M}(\mathbb{R}^n)$ for which the norm

$$\|v\|_{X'(\mathbb{R}^n)} = \sup_{u \ne 0} \frac{\int_{\mathbb{R}^n} |uv| dx}{\|u\|_{X(\mathbb{R}^n)}} \tag{2.18}$$

is finite. Notice that, given two rearrangement-invariant spaces $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$,

$$X(\mathbb{R}^n) \to Y(\mathbb{R}^n) \quad \text{if and only if} \quad Y'(\mathbb{R}^n) \to X'(\mathbb{R}^n) \tag{2.19}$$

with the same embedding constants. Here, and in what follows, the arrow " $\to$ " stands for continuous embedding.
If $X(\mathbb{R}^n) = L^A(\mathbb{R}^n)$ for some Young function A, then

$$(L^A)'(\mathbb{R}^n) = L^{\widetilde{A}}(\mathbb{R}^n)\,, \tag{2.20}$$

up to equivalent norms, with absolute equivalence constants.
The *representation space* $\overline{X}(0,\infty)$ of a rearrangement-invariant space $X(\mathbb{R}^n)$ is the unique rearrangement-invariant space on $(0,\infty)$ satisfying

$$\|u\|_{X(\mathbb{R}^n)} = \|u^*\|_{\overline{X}(0,\infty)} \tag{2.21}$$

for every $u \in X(\mathbb{R}^n)$.
Given any $\lambda > 0$, the *dilation operator* E_λ, defined at $f \in \mathcal{M}(0,\infty)$ by

$$(E_\lambda f)(t) = f(t/\lambda) \quad \text{for } t > 0, \tag{2.22}$$

is bounded on any rearrangement-invariant space $X(0,\infty)$, with norm not exceeding $\max\{1, 1/\lambda\}$.

The Orlicz-Lorentz spaces are a family of function spaces that extends that of the Orlicz spaces. Given a Young function A and a number $q \in \mathbb{R}$, we denote by $L(A,q)(\mathbb{R}^n)$ the Orlicz-Lorentz space of all functions $u \in \mathcal{M}(\mathbb{R}^n)$ for which the quantity

$$\|u\|_{L(A,q)(\mathbb{R}^n)} = \|r^{-\frac{1}{q}} u^*(r)\|_{L^A(0,\infty)} \tag{2.23}$$

is finite. Under suitable assumptions on A and q, this quantity is a norm, and $L(A,q)(\mathbb{R}^n)$, equipped with this norm, is a (non-trivial) rearrangement-invariant space. This is certainly the case when $q > 1$ and

$$\int^\infty \frac{A(t)}{t^{1+q}}\, dt < \infty\,, \tag{2.24}$$

see [35, Proposition 2.1].
The spaces $L(A,q)(\mathbb{R}^n)$ come into play in the description of the associate spaces of another closely related family of Orlicz-Lorentz type spaces. They are denoted by $L[A,q](\mathbb{R}^n)$, and consist of all functions $u \in \mathcal{M}(\mathbb{R}^n)$ that make the functional

$$\|u\|_{L[A,q](\mathbb{R}^n)} = \|r^{-\frac{1}{q}} u^{**}(r)\|_{L^A(0,\infty)} \tag{2.25}$$

finite. One can verify that, if $q < -1$, then this functional is a rearrangement-invariant norm that renders $L[A,q](\mathbb{R}^n)$ a rearrangement-invariant space provided that

$$\int_0 \frac{A(t)}{t^{1+(-q)'}}\, dt < \infty\,, \tag{2.26}$$

where $(-q)' = \frac{q}{q+1}$, the Hölder conjugate of $-q$.
In particular, if the function A is a power, then the space $L(A,q)(\mathbb{R}^n)$ agrees, up to equivalent norms, with a customary Lorentz space. Recall that the Lorentz space $L^{\sigma,p}(\mathbb{R}^n)$, with $\sigma, p \in (0,\infty]$, is the space of those functions $u \in \mathcal{M}(\mathbb{R}^n)$ for which the quantity

$$\|u\|_{L^{\sigma,p}(\mathbb{R}^n)} = \left\| r^{\frac{1}{\sigma}-\frac{1}{p}} u^*(r) \right\|_{L^p(0,\infty)} \tag{2.27}$$

is finite. Here, and in what follows, we use the convention that $\frac{1}{\infty} = 0$. If either $1 < \sigma < \infty$ and $1 \le p \le \infty$, or $\sigma = p = 1$, or $\sigma = p = \infty$, then the functional $\|\cdot\|_{L^{\sigma,p}(\mathbb{R}^n)}$ is equivalent to a rearrangement-invariant norm.

3. Fractional Orlicz-Sobolev spaces

Given $m \in \mathbb{N}$ and a Young function A, we denote by $V^{m,A}(\mathbb{R}^n)$ the homogeneous Orlicz-Sobolev space given by

$$V^{m,A}(\mathbb{R}^n) = \{u \in W^{m,1}_{\mathrm{loc}}(\mathbb{R}^n) : |\nabla^m u| \in L^A(\mathbb{R}^n)\}. \tag{3.1}$$

Here, $\nabla^m u$ denotes the vector of all weak derivatives of u of order m. If $m = 1$, we also simply write ∇u instead of $\nabla^1 u$. The notation $W^{m,A}(\mathbb{R}^n)$ is adopted for the classical Orlicz-Sobolev space defined by

$$W^{m,A}(\mathbb{R}^n) = \{u \in V^{m,A}(\mathbb{R}^n) : |\nabla^k u| \in L^A(\mathbb{R}^n),\, k = 0, 1, \dots, m-1\}\,, \tag{3.2}$$

where $\nabla^0 u$ has to be interpreted as u. The space $W^{m,A}(\mathbb{R}^n)$ is a Banach space equipped with the norm

$$\|u\|_{W^{m,A}(\mathbb{R}^n)} = \sum_{k=0}^{m} \|\nabla^k u\|_{L^A(\mathbb{R}^n)} .$$

Now, let $s \in (0,1)$. The seminorm $|u|_{s,A,\mathbb{R}^n}$ of a function $u \in \mathcal{M}(\mathbb{R}^n)$ is given by

$$|u|_{s,A,\mathbb{R}^n} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{\lambda |x-y|^s} \right) \frac{dx\,dy}{|x-y|^n} \leq 1 \right\} . \tag{3.3}$$

The homogeneous fractional Orlicz-Sobolev space $V^{s,A}(\mathbb{R}^n)$ is defined as

$$V^{s,A}(\mathbb{R}^n) = \big\{ u \in \mathcal{M}(\mathbb{R}^n) : |u|_{s,A,\mathbb{R}^n} < \infty \big\} . \tag{3.4}$$

The definitions of the seminorm $|u|_{s,A,\mathbb{R}^n}$ and of the space $V^{s,A}(\mathbb{R}^n)$ carry over to vector-valued functions u just by replacing the absolute value of $u(x)-u(y)$ by the norm of the same expression on the right-hand side of equation (3.3).
The subspace $V^{s,A}(\mathbb{R}^n) \cap \mathcal{M}_d(\mathbb{R}^n)$ of those functions in $V^{s,A}(\mathbb{R}^n)$ that decay near infinity is denoted by $V^{s,A}_d(\mathbb{R}^n)$. Thus,

$$V^{s,A}_d(\mathbb{R}^n) = \{ u \in V^{s,A}(\mathbb{R}^n) : |\{|u| > t\}| < \infty \text{ for every } t > 0 \} . \tag{3.5}$$

The definition of $V^{s,A}(\mathbb{R}^n)$ is extended to all $s \in (0,\infty) \setminus \mathbb{N}$ in a customary way. Denote by $[s]$ the integer part of s, and set $\{s\} = s - [s]$, the fractional part of s. Then we set

$$V^{s,A}(\mathbb{R}^n) = \{ u \in W^{[s],1}_{\rm loc}(\mathbb{R}^n) : \, \nabla^{[s]} u \in V^{\{s\},A}(\mathbb{R}^n) \} . \tag{3.6}$$

In analogy with (3.5), we extend definition (3.5) to every $s \in (0,\infty) \setminus \mathbb{N}$ on setting

$$V^{s,A}_d(\mathbb{R}^n) = \{ u \in V^{s,A}(\mathbb{R}^n) : |\{|\nabla^k u| > t\}| < \infty \text{ for } k = 0, 1, \dots, [s], \text{ and for every } t > 0 \} . \tag{3.7}$$

The functional $\left|\nabla^{[s]} u\right|_{\{s\},A,\mathbb{R}^n}$ defines a norm on the space $V^{s,A}_d(\mathbb{R}^n)$.
The fractional-order Orlicz-Sobolev space $W^{s,A}(\mathbb{R}^n)$ is defined, for $s \in (0,\infty) \setminus \mathbb{N}$, as

$$W^{s,A}(\mathbb{R}^n) = \{ u \in V^{s,A}(\mathbb{R}^n) : \, u \in W^{[s],A}(\mathbb{R}^n) \}, \tag{3.8}$$

and is a Banach space equipped with the norm

$$\|u\|_{W^{s,A}(\mathbb{R}^n)} = \|u\|_{W^{[s],A}(\mathbb{R}^n)} + \left|\nabla^{[s]} u\right|_{\{s\},A,\mathbb{R}^n} .$$

Clearly, $W^{s,A}(\mathbb{R}^n) \to V^{s,A}_d(\mathbb{R}^n)$. The space $V^{s,A}_d(\mathbb{R}^n)$ naturally arises as a natural maximal domain space for various embeddings of ours to hold.

For the sake of completeness, let us recall that inclusion relations hold between integer-order and fractional-order Orlicz-Sobolev spaces. If $s \in (0,1)$ and A is a Young function, then

$$W^{1,A}(\mathbb{R}^n) \to W^{s,A}(\mathbb{R}^n). \tag{3.9}$$

Moreover, denote by $A_\circ$ the Young function defined as

$$A_\circ(t) = \int_0^t \int_{\mathbb{S}^{n-1}} A(|x_1|\tau) \, d\mathcal{H}^{n-1}(x) \frac{d\tau}{\tau} \quad \text{for } t \geq 0, \tag{3.10}$$

where $x = (x_1, \dots, x_n)$, $\mathbb{S}^{n-1}$ stands for the $(n-1)$–dimensional unit sphere in $\mathbb{R}^n$, and $\mathcal{H}^{n-1}$ for the $(n-1)$–dimensional Hausdorff measure. Then the function $A_\circ$ is equivalent to A, and if $u \in W^{1,A}(\mathbb{R}^n)$, then there exists $\lambda_0 > 0$ such that

$$\lim_{s \to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{\lambda |x-y|^s} \right) \frac{dx\,dy}{|x-y|^n} = \int_{\mathbb{R}^n} A_\circ\left(\frac{|\nabla u|}{\lambda} \right) \, dx \tag{3.11}$$

for every $\lambda \geq \lambda_0$.
If, in particular, u belongs to the subspace of $W^{1,A}(\mathbb{R}^n)$ of those functions such that

$$\int_{\mathbb{R}^n} A\Big(\frac{|u|}{\lambda}\Big)\,dx + \int_{\mathbb{R}^n} A\Big(\frac{|\nabla u|}{\lambda}\Big)\,dx < \infty \tag{3.12}$$

for every $\lambda > 0$, then equation (3.11) also holds for every $\lambda > 0$. Recall that the subspace of functions u fulfilling (3.12) agrees with the closure of $C_0^\infty(\mathbb{R}^n)$ in $W^{1,A}(\mathbb{R}^n)$. It coincides with the whole of $W^{1,A}(\mathbb{R}^n)$ if and only if A fulfills the so called Δ_2-condition.
In the classical case when $A(t) = t^p$ for some $p \geq 1$, embedding (3.9) and equation (3.11) have been established in [12]. For functions A satisfying, together with their conjugate $\widetilde{A}$, the Δ_2-condition, and with the function $A_\circ$ in a somewhat implicit form, they are proved in [47]. The present general version can be found in [3].
As a consequence of embedding (3.9), one also has that

$$W^{[s]+1,A}(\mathbb{R}^n) \to W^{s,A}(\mathbb{R}^n) \tag{3.13}$$

for every $s \in (0,\infty) \setminus \mathbb{N}$.
Let us also mention that the limit as $s \to 0^+$ of (a normalized version of) the functional in (1.1) has been characterized as well. This is the subject of the paper [2], which extends a result of [63] to the Orlicz setting.

We conclude this section with a fractional-order Pólya–Szegő principle on the decrease of the functional (1.1) under symmetric rearrangement of functions u. Recall that the symmetric rearrangement $u^\bigstar$ of a function $u \in \mathcal{M}_d(\mathbb{R}^n)$ is defined as

$$u^\bigstar(x) = u^*(\omega_n |x|^n) \quad \text{for } x \in \mathbb{R}^n, \tag{3.14}$$

where ω_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Thus, $u^\bigstar$ is radially decreasing about 0 and is equidistributed with u. The fractional Pólya–Szegő principle goes back to [4, 6] in the case when A is a power. The result for Young functions A satisfying the Δ_2-condition and functions $u \in W^{s,A}(\mathbb{R}^n)$ is the subject of [40]. The general version stated in Theorem 3.1 below can be proved via the same route. The necessary variants are sketched after its statement.

Theorem 3.1. **[Fractional Pólya–Szegő principle]** *Let $s \in (0,1)$ and let A be a Young function. Assume that $u \in \mathcal{M}_d(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} \geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u^\bigstar(x)-u^\bigstar(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n}. \tag{3.15}$$

Sketch of proof. The proof follows along the same lines as that of [40, Theorem 3.7]. One step of the proof of that theorem requires that u be approximated by a subsequence of polarizations of u that converges to $u^\bigstar$ a.e. in $\mathbb{R}^n$. This is guaranteed if u is just nonnegative and belongs to the space $\mathcal{M}_d(\mathbb{R}^n)$. Actually, as observed in [76, Section 4.1], under these assumptions, there exists a sequence of polarizations of u (with respect to a sequence of hyperplanes independent of u) that converges to $u^\bigstar$ in measure. The assumption that u be nonnegative is not a restriction, since

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} \geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{||u(x)|-|u(y)||}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n}, \tag{3.16}$$

and $|u|^\bigstar = u^\bigstar$. □

4. One-dimensional Hardy type inequalities in Orlicz spaces

The results recalled in the first part of this section concern optimal target norms in inequalities for the integral operator T_s defined, for $n \in \mathbb{N}$ and $s \in (0,n)$ as

$$T_s f(r) = \int_r^\infty \varrho^{-1+\frac{s}{n}} f(\varrho)\,d\varrho \quad \text{for } r \geq 0, \tag{4.1}$$

for $f \in \mathcal{M}_+(0,\infty)$.

We begin with the optimal Orlicz target space corresponding to an Orlicz domain space $L^A(0,\infty)$, where A is a Young function such that

$$\int^\infty \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt = \infty \tag{4.2}$$

and

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt < \infty. \tag{4.3}$$

Such an Orlicz target is defined in terms of the Young function $A_{\frac{n}{s}}$ given by

$$A_{\frac{n}{s}}(t) = A(H^{-1}(t)) \quad \text{for } t \geq 0, \tag{4.4}$$

where

$$H(t) = \left(\int_0^t \left(\frac{\tau}{A(\tau)}\right)^{\frac{s}{n-s}} d\tau\right)^{\frac{n-s}{n}} \quad \text{for } t \geq 0. \tag{4.5}$$

Theorem A. *Let $n \in \mathbb{N}$ and $s \in (0,n)$. Assume that A is a Young function fulfilling conditions (4.2) and (4.3). Let $A_{\frac{n}{s}}$ be the Young function defined by (4.4). Then there exists a constant $C = C(\frac{n}{s})$ such that*

$$\left\| \int_r^\infty f(\varrho)\varrho^{-1+\frac{s}{n}}\, d\varrho \right\|_{L^{A_{\frac{n}{s}}}(0,\infty)} \leq C\|f\|_{L^A(0,\infty)} \tag{4.6}$$

for every function $f \in L^A(0,\infty)$. Moreover, $L^{A_{\frac{n}{s}}}(0,\infty)$ is the optimal Orlicz target space in (4.6).

Inequality (4.6) is equivalent to [32, inequality (2.7)], with n replaced by n/s. The optimality of the space $L^{A_{\frac{n}{s}}}(0,\infty)$ follows from [31, Lemma 1], where such an optimality is proved with $A_{\frac{n}{s}}$ replaced by an equivalent Young function. Such an equivalence is shown in [34, Lemma 2].

We next focus on an inequality parallel to (4.6), but with a target space which is optimal among all rearrangement-invariant spaces. Let n, s and A be as in Theorem A. Denote by $\widehat{A}$ the Young function given by

$$\widehat{A}(t) = \int_0^t \widehat{a}(\tau)\, d\tau \quad \text{for } t \geq 0, \tag{4.7}$$

where

$$\widehat{a}^{\,-1}(r) = \left(\int_{a^{-1}(r)}^\infty \left(\int_0^t \left(\frac{1}{a(\varrho)}\right)^{\frac{s}{n-s}} d\varrho\right)^{-\frac{n}{s}} \frac{dt}{a(t)^{\frac{n}{n-s}}}\right)^{\frac{s}{s-n}} \quad \text{for } r \geq 0. \tag{4.8}$$

Note that the function $\widehat{A}$ depends on both A and the ratio $\frac{n}{s}$. The notation $\widehat{A}_{\frac{n}{s}}$ will also be employed for $\widehat{A}$ when needed, in order to make the latter dependence explicit.

Let $L(\widehat{A}, \frac{n}{s})(0,\infty)$ be the Orlicz-Lorentz space equipped with the norm defined as in (2.23), namely as

$$\|f\|_{L(\widehat{A},\frac{n}{s})(0,\infty)} = \|r^{-\frac{s}{n}} f^*(r)\|_{L^{\widehat{A}}(0,\infty)} \tag{4.9}$$

for $f \in \mathcal{M}(0,\infty)$. Assumption (4.3) ensures that condition (2.24) is certainly fulfilled with A replaced by $\widehat{A}$ and q by $\frac{n}{s}$. This is a consequence of [35, Propositions 2.1 and 2.2]. Thus, $L(\widehat{A}, \frac{n}{s})(0,\infty)$ is actually a rearrangement-invariant space.

By [35, Lemma 2.3], assumption (4.3) is equivalent to condition (2.26), with A replaced by $\widetilde{A}$ and q by $-\frac{n}{s}$. Hence, the space $L[\widetilde{A}, -\frac{n}{s}](0,\infty)$, endowed with the norm defined as in (2.25) by

$$\|f\|_{L[\widetilde{A},-\frac{n}{s}](0,\infty)} = \|r^{\frac{s}{n}} f^{**}(r)\|_{L^{\widehat{A}}(0,\infty)} \tag{4.10}$$

for $f \in \mathcal{M}(0,\infty)$, is also a rearrangement-invariant space. Moreover, one has that

$$L[\widetilde{A}, -\tfrac{n}{s}](0,\infty) = L(\widehat{A}, \tfrac{n}{s})'(0,\infty), \tag{4.11}$$

up to equivalent norms, where $L(\widehat{A}, \frac{n}{s})'(0,\infty)$ denotes the associate space of $L(\widehat{A}, \frac{n}{s})(0,\infty)$. Property (4.11) is stated and established in [37, Lemma 4.5] in the case when the interval $(0,\infty)$ is replaced by an interval of the form $(0, L)$ with $L > 0$; the proof for $(0,\infty)$ is completely analogous.

Theorem B. *Let $n \in \mathbb{N}$ and $s \in (0,n)$. Assume that A is a Young function fulfilling conditions* (4.2) *and* (4.3)*, and let $\widehat{A}$ be the Young function defined by* (4.7)*. Then there exists a constant $C = C(\frac{n}{s})$ such that*

$$\left\| \int_r^\infty f(\varrho)\varrho^{-1+\frac{s}{n}}\, d\varrho \right\|_{L(\widehat{A}, \frac{n}{s})(0,\infty)} \le C\|f\|_{L^A(0,\infty)} \tag{4.12}$$

for every function $f \in L^A(0,\infty)$. Moreover, $L(\widehat{A}, \frac{n}{s})(0,\infty)$ is the optimal rearrangement-invariant target space in (4.12)*.*

Inequality (4.12) agrees with inequality [35, inequality (3.1)], with n replaced with n/s. The optimality of the space $L(\widehat{A}, \frac{n}{s})(0,\infty)$ follows from [35, inequalities (4.6)–(4.8)].

Let us notice that the function A always dominates $\widehat{A}$. Moreover, A is equivalent to $\widehat{A}$ if and only if $A(t)$ grows less than the power $t^{\frac{n}{s}}$ in the sense that its Matuszewska-Orlicz index $I(A)$, defined by (2.6), is smaller than $\frac{n}{s}$. These assertions are proved in [35, Propositions 5.1 and 5.2], and collected in the following proposition.

Proposition C. *Let $n \in \mathbb{N}$ and $s \in (0,n)$. Assume that A is a Young function fulfilling conditions* (4.2) *and* (4.3)*, and let $\widehat{A}$ be the Young function defined by* (4.7)*. Then there exists a constant $c = c(n/s)$ such that*

$$\widehat{A}(t) \le A(ct) \quad \text{for } t > 0. \tag{4.13}$$

Moreover, the function $\widehat{A}$ is equivalent to A if and only if $I(A) < \frac{n}{s}$.

As a consequence of Proposition C, the space $L(\widehat{A}, \frac{n}{s})(0,\infty)$ reduces to $L(A, \frac{n}{s})(0,\infty)$ if $A(t)$ is subcritical with respect to $t^{\frac{n}{s}}$ in the sense of Matuszewska-Orlicz indices.

Proposition D. *Let $n \in \mathbb{N}$, $s \in (0,n)$. Assume that A is a Young function fulfilling conditions* (4.2) *and* (4.3)*. If $I(A) < \frac{n}{s}$, then $L(\widehat{A}, \frac{n}{s})(0,\infty) = L(A, \frac{n}{s})(0,\infty)$, up to equivalent norms.*

The embedding of the space $L(\widehat{A}, \frac{n}{s})(0,\infty)$ into $L^{A_{\frac{n}{s}}}(0,\infty)$ is a trivial consequence of the optimality of the former in inequality (4.12) in the class of all rearrangement-invariant spaces, which includes, in particular, the Orlicz spaces. This fact is stated in Proposition 4.1 below. A direct proof of this proposition is however given, which shows that the norm of the embedding in question is independent of A, a piece of information of use in the proofs of our main results.

Proposition 4.1. *Let $n \in \mathbb{N}$ and $s \in (0,n)$. Assume that A is a Young function fulfilling conditions* (4.2) *and* (4.3)*. Let $A_{\frac{n}{s}}$ and $\widehat{A}$ be the Young functions defined as in* (4.4) *and* (4.7)*, respectively. Then*

$$L(\widehat{A}, \tfrac{n}{s})(0,\infty) \to L^{A_{\frac{n}{s}}}(0,\infty). \tag{4.14}$$

Moreover, the norm of embedding (4.14) *depends only on $\frac{n}{s}$.*

Proof. Denote by $L[\widetilde{A}, -\frac{n}{s}](0,\infty)$ the Orlicz–Lorentz space endowed with the norm defined as in (2.25). Thanks to equation (4.11) and to property (2.19), embedding (4.14) is equivalent to

$$\big(L^{A_{\frac{n}{s}}}\big)'(0,\infty) \to L(\widehat{A}, \tfrac{n}{s})'(0,\infty), \tag{4.15}$$

with the same embedding constants. Also, by property (2.20),

$$\big(L^{A_{\frac{n}{s}}}\big)'(0,\infty) = L^{\widetilde{A_{\frac{n}{s}}}}(0,\infty), \tag{4.16}$$

up to equivalent norms, with absolute equivalence constants. Owing to equations by (4.11) –(4.16), embedding (4.14) will follow if we show that

$$L^{\widetilde{A_{\frac{n}{s}}}}(0,\infty) \to L[\widetilde{A}, -\tfrac{n}{s}](0,\infty). \tag{4.17}$$

Embedding (4.17) is equivalent to the inequality

$$\|r^{\frac{s}{n}} f^{**}(r)\|_{L^{\widetilde{A}}(0,\infty)} \le C\|f^*\|_{L^{\widetilde{A_{\frac{n}{s}}}}(0,\infty)} \tag{4.18}$$

for some constant C and for every function $f \in \mathcal{M}_+(0,\infty)$. Moreover, the norm of embedding (4.17) equals the optimal constant C in inequality (4.18). Inequality (4.18) is in its turn equivalent to the inequality

$$\left\|\int_r^\infty g(\varrho)\varrho^{-1+\frac{s}{n}}\,d\varrho\right\|_{L^{A_{\frac{n}{s}}}(0,\infty)} \le C'\|g\|_{L^A(0,\infty)} \tag{4.19}$$

for every function $g \in \mathcal{M}_+(0,\infty)$, and for some constant C' equivalent to C, up to absolute multiplicative constants. Indeed,

$$\begin{aligned}
\sup_{f\in L^{\widetilde{A_{\frac{n}{s}}}}(0,\infty)} \frac{\|r^{\frac{s}{n}} f^{**}(r)\|_{L^{\widetilde{A}}(0,\infty)}}{\|f^*\|_{L^{\widetilde{A_{\frac{n}{s}}}}(0,\infty)}} &\approx \sup_{f\in L^{\widetilde{A_{\frac{n}{s}}}}(0,\infty)} \sup_{g\in L^A(0,\infty)} \frac{\int_0^\infty r^{\frac{s}{n}} f^{**}(r) g(r)\,dr}{\|g\|_{L^A(0,\infty)}\|f^*\|_{L^{\widetilde{A_{\frac{n}{s}}}}(0,\infty)}} \\
= \sup_{g\in L^A(0,\infty)} \sup_{f\in L^{\widetilde{A_{\frac{n}{s}}}}(0,\infty)} \frac{\int_0^\infty f^*(r)\int_r^\infty \varrho^{-1+\frac{s}{n}} g(\varrho)\,d\varrho\,dr}{\|g\|_{L^A(0,\infty)}\|f^*\|_{L^{\widetilde{A_{\frac{n}{s}}}}(0,\infty)}} &\approx \sup_{g\in L^A(0,\infty)} \frac{\left\|\int_r^\infty g(\varrho)\varrho^{-1+\frac{s}{n}}\,d\varrho\right\|_{L^{A_{\frac{n}{s}}}(0,\infty)}}{\|g\|_{L^A(0,\infty)}},
\end{aligned}$$

where the relations "$\approx$" hold up to absolute multiplicative constants. Inequality (4.19) is nothing but (4.6), and hence the latter holds with a constant C' depending only on $\frac{n}{s}$. □

A characterization of the optimal rearrangement-invariant target space $X_s(0,\infty)$, corresponding to any given rearrangement-invariant domain space $X(0,\infty)$, for the operator T_s defined by (4.1) is contained in the next result. Notice that, in view of Theorem B, if $X(0,\infty) = L^A(0,\infty)$, then $X_s(0,\infty) = L(\widehat{A}, \frac{n}{s})(0,\infty)$.

Theorem E. *Let $s \in (0,n)$. Assume that $\|\cdot\|_{X(0,\infty)}$ is a rearrangement-invariant norm satisfying*

$$\|(1+r)^{-1+\frac{s}{n}}\|_{X'(0,\infty)} < \infty. \tag{4.20}$$

(i) Define the functional $\|\cdot\|_{Z(0,\infty)}$ as

$$\|f\|_{Z(0,\infty)} = \left\|r^{\frac{s}{n}} f^{**}(r)\right\|_{X'(0,\infty)} \tag{4.21}$$

for $f \in \mathcal{M}(0,\infty)$. Then $\|\cdot\|_{Z(0,\infty)}$ is a rearrangement-invariant norm on $(0,\infty)$. Denote by $\|\cdot\|_{X_s(0,\infty)}$ the norm defined by

$$\|f\|_{X_s(0,\infty)} = \|f\|_{Z'(0,\infty)} \tag{4.22}$$

for $f \in \mathcal{M}(0,\infty)$. Then

$$\left\|\int_r^\infty \varrho^{-1+\frac{s}{n}} f(\varrho)\,d\varrho\right\|_{X_s(0,\infty)} \le C\|f\|_{X(0,\infty)} \tag{4.23}$$

for every $f \in X(0,\infty)$. Moreover, $X_s(0,\infty)$ is the optimal rearrangement-invariant target space in (4.23).
(ii) Let $s_1, s_2 > 0$ be such that $s_1 + s_2 < n$. Then

$$(X_{s_1})_{s_2}(0,\infty) = X_{s_1+s_2}(0,\infty). \tag{4.24}$$

In the case when the interval $(0,\infty)$ is replaced by any bounded interval $(0,L)$ with $L > 0$, Part (i) of Theorem E is established in [45, Theorem 4.5] and Part (ii) in [38, Theorem 3.5]. The proof for $(0,\infty)$ follows along the same lines; the result of Part (i) is stated in [46, Theorem 4.4] and that of Part (ii) in [64, Proposition 4.3].

Remark 4.2. Condition (4.20) is indispensable. Indeed, if (4.20) fails, then the functional $\|\cdot\|_{Z(0,\infty)}$ given by (4.21) is trivial in the sense that $\|f\|_{Z(0,\infty)} < \infty$ only if $f = 0$. Furthermore, the inequality

$$\left\| \int_r^\infty \varrho^{-1+\frac{s}{n}} f(\varrho)\, d\varrho \right\|_{Y(0,\infty)} \le C\|f\|_{X(0,\infty)} \tag{4.25}$$

does not hold for any rearrangement-invariant space $Y(0,\infty)$.

The last two results of this section are new. They amount to Hardy type inequalities in integral form in Orlicz spaces. Of course, they can be equivalently stated in the form of the boundedness of suitable integral operators in the relevant Orlicz spaces.

Lemma 4.3. *Let $s \in (0,1)$. Assume that A is any Young function. Then*

$$\int_0^\infty A\left(r^{-s}\int_0^r f(\varrho)\, d\varrho\right)\frac{dr}{r} \le \int_0^\infty A\left(\frac{1}{s} r^{1-s} f(r)\right)\frac{dr}{r} \tag{4.26}$$

for every function $f \in \mathcal{M}_+(0,\infty)$.

Proof. The change of variables $r = e^{-\xi}$ and $\varrho = e^{-\eta}$ turns inequality (4.26) into

$$\int_{-\infty}^\infty A\left(e^{s\xi}\int_\xi^\infty f(e^{-\eta})e^{-\eta}\, d\eta\right) d\xi \le \int_{-\infty}^\infty A\left(\frac{1}{s}e^{(s-1)\xi} f(e^{-\xi})\right) d\xi.$$

On setting

$$g(\xi) = e^{(s-1)\xi} f(e^{-\xi}) \quad \text{for } \xi \in (-\infty,\infty),$$

the last inequality can be rewritten as

$$\int_{-\infty}^\infty A\left(e^{s\xi}\int_\xi^\infty g(\eta)e^{-s\eta}\, d\eta\right) d\xi \le \int_{-\infty}^\infty A\left(\frac{1}{s} g(\xi)\right) d\xi. \tag{4.27}$$

Define the operator

$$Rg(\xi) = e^{s\xi}\int_\xi^\infty g(\eta)e^{-s\eta}\, d\eta \quad \text{for } \xi \in (-\infty,\infty),$$

for $g \in \mathcal{M}_+(-\infty,\infty)$. We claim that $R\colon L^\infty(-\infty,\infty) \to L^\infty(-\infty,\infty)$, and

$$\|R\|_{L^\infty \to L^\infty} \le \frac{1}{s}. \tag{4.28}$$

Indeed,

$$\begin{aligned}\|Rg\|_{L^\infty(-\infty,\infty)} &= \sup_{t\in(-\infty,\infty)} e^{s\xi}\int_\xi^\infty g(\eta)e^{-s\eta}\, d\eta \\ &\le \|g\|_{L^\infty(-\infty,\infty)} \sup_{\xi\in(-\infty,\infty)} e^{s\xi}\left[\frac{e^{-\eta s}}{-s}\right]_{\eta=\xi}^{\eta=\infty} = \frac{1}{s}\|g\|_{L^\infty(-\infty,\infty)}\end{aligned}$$

for $g \in L^\infty(-\infty,\infty)$. Moreover, $R\colon L^1(-\infty,\infty) \to L^1(-\infty,\infty)$, and

$$\|R\|_{L^1 \to L^1} \le \frac{1}{s}, \tag{4.29}$$

inasmuch as

$$\begin{aligned}\|Rg\|_{L^1(-\infty,\infty)} &= \int_{-\infty}^\infty e^{s\xi}\int_\xi^\infty g(\eta)e^{-s\eta}\, d\eta\, d\xi = \int_{-\infty}^\infty g(\eta)e^{-\eta s}\int_{-\infty}^\eta e^{s\xi}\, d\xi\, d\eta \\ &\le \frac{1}{s}\int_{-\infty}^\infty g(\eta)\, d\eta = \frac{1}{s}\|g\|_{L^1(-\infty,\infty)}\end{aligned}$$

for $g \in L^1(-\infty,\infty)$. From (4.28)–(4.29), via an interpolation theorem by Calderón [10, Chapter 3, Theorem 2.12], one can infer that

$$R\colon L^A(-\infty,\infty) \to L^A(-\infty,\infty) \tag{4.30}$$

with

$$\|R\|_{L^A\to L^A} \le \frac{1}{s}. \tag{4.31}$$

Notice that in deducing (4.30) and (4.31) one makes use of the fact that $L^A(-\infty,\infty)$ is a rearrangement-invariant space, and hence an exact interpolation space between $L^1(-\infty,\infty)$ and $L^\infty(-\infty,\infty)$ (see [10, Chapter 3, Theorem 2.12]). Therefore,

$$\|Rg\|_{L^A(-\infty,\infty)} \le \left\|\frac{1}{s}g\right\|_{L^A(-\infty,\infty)} \tag{4.32}$$

for every $g \in L^A(-\infty,\infty)$. An application of inequality (4.32) with A replaced by A_M, defined by $A_M(t) = \frac{A(t)}{M}$ for $t > 0$, where

$$M = \int_{-\infty}^{\infty} A\Big(\frac{1}{s}g(r)\Big)\,dr,$$

yields (4.27) via the definition of Luxemburg norm. □

Lemma 4.4. *Let $n \in \mathbb{N}$ and $s \in (0,n)$. Assume that A is a Young function fulfilling conditions (4.2) and (4.3), and let $\widehat{A}$ be the Young function defined by (4.7). Then there exists a constant $C = C(n/s)$ such that*

$$\int_0^\infty \widehat{A}\left(r^{-s}\int_r^\infty f(\varrho)\,d\varrho\right) r^{n-1}\,dr \le \int_0^\infty A\left(Cr^{1-s}f(r)\right) r^{n-1}\,dr \tag{4.33}$$

for every function $f \in \mathcal{M}_+(0,\infty)$. Moreover, $\lim_{s\to s_0} C(n/s) < \infty$ for any $s_0 \in [0,n)$.

Proof. The change of variables $t = r^n$ and $\tau = \varrho^n$ turns inequality (4.33) into

$$\int_0^\infty \widehat{A}\left(\frac{\xi^{-\frac{s}{n}}}{n}\int_\xi^\infty f(\eta^{\frac{1}{n}})\eta^{-\frac{1}{n'}}\,d\eta\right) d\xi \le \int_0^\infty A\left(C\xi^{\frac{1-s}{n}}f(\xi^{\frac{1}{n}})\right) d\xi. \tag{4.34}$$

On setting

$$g(\xi) = \xi^{\frac{1-s}{n}}f(\xi^{\frac{1}{n}}) \quad \text{for } \xi > 0,$$

inequality (4.34) reads

$$\int_0^\infty \widehat{A}\left(\frac{\xi^{-\frac{s}{n}}}{n}\int_\xi^\infty g(\eta)\eta^{-1+\frac{s}{n}}\,d\eta\right) d\xi \le \int_0^\infty A(Cg(\xi))\,d\xi \tag{4.35}$$

for $g \in \mathcal{M}_+(0,\infty)$. Denote by S the operator defined as

$$Sg(\xi) = \xi^{-\frac{s}{n}}\int_\xi^\infty g(\eta)\eta^{-1+\frac{s}{n}}\,d\eta \quad \text{for } \xi > 0,$$

for $g \in \mathcal{M}_+(0,\infty)$. We have that $S\colon L^1(0,\infty) \to L^1(0,\infty)$, with

$$\|S\|_{L^1\to L^1} \le \frac{n}{n-s}, \tag{4.36}$$

inasmuch as

$$\|Sg\|_{L^1(0,\infty)} = \int_0^\infty \xi^{-\frac{s}{n}}\int_\xi^\infty g(\eta)\eta^{-1+\frac{s}{n}}\,d\eta\,d\xi = \int_0^\infty g(\eta)\eta^{-1+\frac{s}{n}}\int_0^\eta \xi^{-\frac{s}{n}}\,d\xi\,d\eta = \frac{n}{n-s}\|g\|_{L^1(0,\infty)}$$

for $g \in L^1(0,\infty)$. Also, $S\colon L^{\frac{n}{s},1}(0,\infty) \to L^{\frac{n}{s},\infty}(0,\infty)$, with

$$\|S\|_{L^{\frac{n}{s},1}\to L^{\frac{n}{s},\infty}} \le 1. \tag{4.37}$$

Actually,

$$\|Sg\|_{L^{\frac{n}{s},\infty}(0,\infty)} = \sup_{\xi\in(0,\infty)} \xi^{\frac{s}{n}}(Sg)^*(\xi) = \sup_{\xi\in(0,\infty)} \xi^{\frac{s}{n}}Sg(\xi) = \int_0^\infty g(\eta)\eta^{-1+\frac{s}{n}}\,d\eta \le \|g\|_{L^{\frac{n}{s},1}(0,\infty)}$$

for $g \in L^{\frac{n}{s},1}(0,\infty)$, where the second equality follows from the fact that $(Sg)^* = Sg$, for Sg is a decreasing function, and the inequality follows from the Hardy–Littlewood inequality, since the function $\eta \mapsto \eta^{-1+\frac{s}{n}}$ is decreasing. Owing to the boundedness properties (4.36) and (4.37) of the operator S, inequality (4.35) follows from [35, Theorem 3.1]. The finiteness of the limit of the constant $C(n/s)$ can be checked via a close inspection of the proof of that theorem. □

5. A fractional Hardy type inequality: case $s \in (0,1)$

This section is devoted to our Hardy type inequality in fractional Orlicz-Sobolev spaces associated with Young functions A. Notice that, in contrast with the case of classical Sobolev and fractional Sobolev spaces corresponding to the case when $A(t) = t^p$, the Young function $\widehat{A}$ appearing on the left-hand side of the relevant inequality is not always equivalent to A. This is due to the generality of conditions (4.2) and (4.3), that extend the assumption $1 \le p < \frac{n}{s}$ required in the classical case, and allow for Young functions $A(t)$ whose growth can be very close to that of the critical power $t^{\frac{n}{s}}$. Apart from its own interest, this Hardy type inequality is crucial in our approach to the other main results of the paper.

Theorem 5.1. **[Fractional Orlicz–Hardy inequality]** *Let $n \in \mathbb{N}$ and $s \in (0,1)$. Assume that A is a Young function satisfying conditions (4.2) and (4.3) and let $\widehat{A}$ be the Young function given by (4.7). Then, there exists a constant $C = C(n,s)$ such that*

$$\left\| \frac{|u(x)|}{|x|^s} \right\|_{L^{\widehat{A}}(\mathbb{R}^n)} \le C|u|_{s,A,\mathbb{R}^n} \tag{5.1}$$

for every function $u \in V^{s,A}_d(\mathbb{R}^n)$. Moreover, $\lim_{s\to 1^-} C(n,s) < \infty$. Also,

$$\int_{\mathbb{R}^n} \widehat{A}\left(\frac{|u(x)|}{|x|^s}\right) dx \le (1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(C\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} \tag{5.2}$$

for every function $u \in \mathcal{M}_d(\mathbb{R}^n)$.

The following example provides us with an application of Theorem 5.1 to a Young function whose behaviour near zero and near infinity is of power-logarithmic type. Although quite simple, this model Young function enables us to recover the results known until now and to exhibit genuinely new inequalities. This model function will also be called into play in order to illustrate the results of the next sections.

Example 5.2. Consider a Young function A such that

$$A(t) \text{ is equivalent to } \begin{cases} t^{p_0}(\log\frac{1}{t})^{\alpha_0} & \text{near zero} \\ t^{p}(\log t)^{\alpha} & \text{near infinity,} \end{cases} \tag{5.3}$$

where either $p_0 > 1$ and $\alpha_0 \in \mathbb{R}$, or $p_0 = 1$ and $\alpha_0 \le 0$, and either $p > 1$ and $\alpha \in \mathbb{R}$ or $p = 1$ and $\alpha \ge 0$. Here, equivalence is meant in the sense of Young functions.

Let $n \in \mathbb{N}$ and $s \in (0,1)$. The function A satisfies assumption (4.2) if

$$\text{either } 1 \le p < \frac{n}{s} \text{ and } \alpha \text{ is as above, or } p = \frac{n}{s} \text{ and } \alpha \le \frac{n}{s} - 1, \tag{5.4}$$

and satisfies assumption (4.3) if

$$\text{either } 1 \le p_0 < \frac{n}{s} \text{ and } \alpha_0 \text{ is as above, or } p_0 = \frac{n}{s} \text{ and } \alpha_0 > \frac{n}{s} - 1. \tag{5.5}$$

Theorem 5.1 tells us that, under assumptions (5.4) and (5.5), inequalities (5.1) and (5.2) hold, where

$$\widehat{A}(t) \text{ is equivalent to } \begin{cases} t^{p_0}(\log\frac{1}{t})^{\alpha_0} & \text{if } 1 \le p_0 < \frac{n}{s} \\ t^{\frac{n}{s}}(\log\frac{1}{t})^{\alpha_0 - \frac{n}{s}} & \text{if } p_0 = \frac{n}{s} \text{ and } \alpha_0 > \frac{n}{s} - 1 \end{cases} \quad \text{near zero,} \tag{5.6}$$

and

$$
\widehat{A}(t) \text{ is equivalent to } \begin{cases} t^p(\log t)^\alpha & \text{if } 1 \le p < \frac{n}{s} \\ t^{\frac{n}{s}}(\log t)^{\alpha-\frac{n}{s}} & \text{if } p = \frac{n}{s} \text{ and } \alpha < \frac{n}{s}-1 \\ t^{\frac{n}{s}}(\log t)^{-1}(\log(\log t))^{-\frac{n}{s}} & \text{if } p = \frac{n}{s} \text{ and } \alpha = \frac{n}{s}-1 \end{cases} \quad \text{near infinity.} \tag{5.7}
$$

In particular, the choices $p_0 = p < \frac{n}{s}$ and $\alpha_0 = \alpha = 0$ yield $\widehat{A}(t) = t^p$, and inequalities (5.1) and (5.2) recover (apart from the specific form of the constant involved) [63, Inequality (3)].

The next preliminary results will be of use in the proof of Theorem 5.1.

Proposition 5.3. *Let $s \in (0,1)$ and let A be a Young function. Assume that $u \in V^{s,A}(\mathbb{R}^n)$ and $|\{|u| > 0\}| < \infty$. Then there exists a constant $C = C(n, s, |\{|u| > 0\}|)$ such that*

$$
\|u\|_{L^A(\mathbb{R}^n)} \le C|u|_{s,A,\mathbb{R}^n}. \tag{5.8}
$$

Proof. Let us set $U = \{u^\bigstar > 0\}$ and $d = \mathrm{diam}(U)$. By Theorem 3.1, given any $\lambda > 0$, we have that

$$
\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{\lambda|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} &\ge \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u^\bigstar(x)-u^\bigstar(y)|}{\lambda|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} \\
&\ge \int_U \int_{\{y\in\mathbb{R}^n:2(d+1)\ge|x-y|\ge d+1\}} A\left(\frac{u^\bigstar(x)}{\lambda|x-y|^s}\right)\frac{dy\,dx}{|x-y|^n} \\
&= \int_U \int_{\{y\in\mathbb{R}^n:2(d+1)\ge|x-y|\ge d+1\}} A\left(\frac{(2(d+1))^s}{\lambda|x-y|^s}\frac{u^\bigstar(x)}{(2(d+1))^s}\right)\frac{dy\,dx}{|x-y|^n} \\
&\ge \int_U A\left(\frac{u^\bigstar(x)}{\lambda(2(d+1))^s}\right)\int_{\{y\in\mathbb{R}^n:2(d+1)\ge|x-y|\ge d+1\}} \frac{(2(d+1))^s}{|x-y|^{s+n}}\,dy\,dx \\
&= \frac{c(2^s-1)}{s}\int_U A\left(\frac{u^\bigstar(x)}{\lambda(2(d+1))^s}\right)dx \\
&= \frac{c(2^s-1)}{s}\int_{\mathbb{R}^n} A\left(\frac{|u(x)|}{\lambda(2(d+1))^s}\right)dx
\end{aligned} \tag{5.9}
$$

for some positive constant $c = c(n)$. Note that the third inequality holds, owing to property (2.2). Hence, inequality (5.8) follows. □

Proposition 5.4. *Let $s \in (0,1)$ and let A be a Young function. Assume that $u \in V_d^{s,A}(\mathbb{R}^n)$. Then*

$$
\int_E |u|\,dx < \infty \tag{5.10}
$$

for every set $E \subset \mathbb{R}^n$ with $|E| < \infty$. In particular, $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Define the function $\overline{u} : \mathbb{R}^n \to [0,\infty)$ as

$$
\overline{u} = (|u| - 1)\chi_{\{|u|>1\}}.
$$

One can verify that

$$
|u(x) - u(y)| \ge |\overline{u}(x) - \overline{u}(y)| \quad \text{for } x, y \in \mathbb{R}^n.
$$

Thus, $\overline{u} \in V^{s,A}(\mathbb{R}^n)$, and $|\{\overline{u} > 0\}| < \infty$. By Proposition 5.3, $\overline{u} \in L^A(\mathbb{R}^n)$, and since $|\{\overline{u} > 0\}| < \infty$ one has that $\overline{u} \in L^1(\mathbb{R}^n)$ as well. On the other hand, $|u| \le \overline{u} + 1$, whence

$$
\int_E |u|\,dx \le \int_E \overline{u}\,dx + |E| < \infty
$$

for any set $E \subset \mathbb{R}^n$ with $|E| < \infty$. Hence, (5.10) follows. □

Our proof of Theorem 5.1 makes use of a classical approach in the theory of fractional Sobolev norms of functions in $\mathbb{R}^n$, which consists in an extension of the relevant functions to $\mathbb{R}^{n+1}$. In particular, we follow the outline of the proof of [63, Theorem 2]. However, specific ad hoc Orlicz space techniques and sharp one-dimensional Hardy type inequalities in Orlicz spaces, presented in Section 4, have to be exploited in the present setting.

Proof of Theorem 5.1. Let $u \in \mathcal{M}_d(\mathbb{R}^n)$. If the right-hand side of inequality (5.2) is infinite for a certain constant C to be specified later, then the conclusion is trivially true. We may thus assume that it is finite. Hence, in particular, $u \in V^{s,A}_d(\mathbb{R}^n)$. Owing to Theorem 3.1, the integral on the right-hand side of inequality (5.2) is still finite if u is replaced by $u^\bigstar$. By Proposition 5.4, applied with u replaced by $u^\bigstar$, we have that $u^\bigstar \in L^1_{\rm loc}(\mathbb{R}^n)$. Consequently,

$$\int_{\{|u|>1\}} |u|\,dx = \int_{\{u^\bigstar>1\}} u^\bigstar\,dx < \infty. \tag{5.11}$$

In particular, $u \in L^1_{\rm loc}(\mathbb{R}^n)$. Let $\psi : \mathbb{R}^n \to [0,\infty)$ be the function defined as

$$\psi(y) = \frac{(n+1)}{\omega_n}(1-|y|)\chi_{\{|y|\le 1\}} \quad \text{for } y \in \mathbb{R}^n.$$

The function $U : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$, given by

$$U(x,t) = \int_{\mathbb{R}^n} \psi(y)u(x+ty)\,dy \quad \text{for } (x,t) \in \mathbb{R}^n \times [0,\infty), \tag{5.12}$$

is thus well defined. Classically, one has that

$$|\nabla U(x,t)| \le \frac{(n+1)(n+2)}{t\,\omega_n}\int_{\{|y|<1\}} |u(x+ty)-u(x)|\,dy \quad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0,\infty). \tag{5.13}$$

On setting $K = (n+1)(n+2)$, one has that

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} t^{-1} A\left(t^{1-s}|\nabla U(x,t)|\right) dx\,dt \\
&\le \int_0^\infty \int_{\mathbb{R}^n} t^{-1} A\left(\frac{K}{t^s\omega_n}\int_{\{|y|<1\}} |u(x+ty)-u(x)|\,dy\right) dx\,dt \\
&\le \int_0^\infty \int_{\mathbb{R}^n} t^{-1}\frac{1}{\omega_n}\int_{\{|y|<1\}} A\left(\frac{K|u(x+ty)-u(x)|}{t^s}\right) dy\,dx\,dt \\
&= \frac{1}{\omega_n}\int_0^\infty t^{-1}\int_{\{|y|<1\}}\int_{\mathbb{R}^n} A\left(\frac{K|u(x+ty)-u(x)|}{t^s}\right) dx\,dy\,dt \\
&= \frac{1}{\omega_n}\int_0^\infty t^{-1}\int_{\{|z|<t\}}\int_{\mathbb{R}^n} A\left(\frac{K|u(x+z)-u(x)|}{t^s}\right) \frac{dx}{t^n}\,dz\,dt \\
&= \frac{1}{\omega_n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{|z|}^\infty t^{-1-n} A\left(\frac{K|u(x+z)-u(x)|}{t^s}\right) dt\,dz\,dx
\end{aligned} \tag{5.14}$$

where the first inequality holds by (5.13), the second inequality by Jensen's inequality (since $\omega_n = |\{|y|<1\}|$), the first equality by Fubini's theorem, the second equality by the change of variables $z = ty$, and the last one by Fubini's theorem again. From (5.14) one obtains that

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^n} t^{-1} A\left(t^{1-s}|\nabla U(x,t)|\right) dx\,dt &\le \frac{1}{\omega_n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{|z|}^\infty t^{-1-n} A\left(\frac{K|u(x+z)-u(x)|}{t^s}\right) dt\,dz\,dx \\
&\le \frac{1}{\omega_n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{|z|}^\infty t^{-1-n} A\left(\frac{K|u(x+z)-u(x)|}{|z|^s}\right) dt\,dz\,dx
\end{aligned} \tag{5.15}$$

$$= \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(K \frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{dx\,dy}{|x-y|^n}.$$

Next,

$$\frac{|u(x)|}{|x|} = \frac{|U(x,0)|}{|x|} \le \frac{|U(x,t)|}{|x|} + \frac{1}{|x|}\int_0^t \left|\frac{\partial U}{\partial \tau}(x,\tau)\right| d\tau \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty). \tag{5.16}$$

Observe that if G is a Young function, then

$$\int_0^r \tau^{-1} G\left(\tau^{1-s}\eta\right) d\tau = \frac{1}{1-s}\int_0^{r^{1-s}\eta} \frac{G(\tau)}{\tau}\, d\tau \ge \frac{1}{1-s} G\left(r^{1-s}\eta/2\right) \quad \text{for } r, \eta > 0, \tag{5.17}$$

where the last inequality holds since the function $G(\tau)/\tau$ is non-decreasing. Equation (5.17), applied with $G = \widehat{A}$, $r = |x|$, $\eta = \frac{|u(x)|}{|x|}$, and equation (5.16) yield

$$\begin{aligned} \widehat{A}\left(\frac{|u(x)|}{2|x|^s}\right) &\le (1-s)\int_0^{|x|} t^{-1}\widehat{A}\left(t^{1-s}\frac{|u(x)|}{|x|}\right) dt \\ &\le (1-s)\int_0^{|x|} t^{-1}\widehat{A}\left(t^{1-s}\frac{|U(x,t)|}{|x|} + \frac{t^{1-s}}{|x|}\int_0^t \left|\frac{\partial U}{\partial \tau}(x,\tau)\right| d\tau\right) dt \\ &\le (1-s)\int_0^{|x|} t^{-1}\widehat{A}\left(2t^{1-s}\frac{|U(x,t)|}{|x|}\right) dt \\ &\quad + (1-s)\int_0^{|x|} t^{-1}\widehat{A}\left(2t^{-s}\int_0^t \left|\frac{\partial U}{\partial \tau}(x,\tau)\right| d\tau\right) dt \quad \text{for } x \in \mathbb{R}^n. \end{aligned} \tag{5.18}$$

Thus, owing to Lemma 4.3, applied with A replaced by $\widehat{A}$, and Proposition C

$$\begin{aligned} \widehat{A}\left(\frac{|u(x)|}{2|x|^s}\right) &\le (1-s)\int_0^{|x|} t^{-1}\widehat{A}\left(2t^{1-s}\frac{|U(x,t)|}{|x|}\right) dt + (1-s)\int_0^\infty t^{-1}\widehat{A}\left(\frac{2}{s}t^{1-s}\left|\frac{\partial U(x,t)}{\partial t}\right|\right) dt \\ &\le (1-s)\int_0^{|x|} t^{-1}\widehat{A}\left(2t^{1-s}\frac{|U(x,t)|}{|x|}\right) dt + (1-s)\int_0^\infty t^{-1}A\left(\frac{C}{s}t^{1-s}|\nabla U(x,t)|\right) dt \end{aligned} \tag{5.19}$$

for $x \in \mathbb{R}^n$, and for some constant $C = C(n,s)$. Hence, by (5.19),

$$\begin{aligned} \int_{\mathbb{R}^n} \widehat{A}\left(\frac{|u(x)|}{2|x|^s}\right) dx &\le (1-s)\int_{\mathbb{R}^n}\int_0^{|x|} t^{-1}\widehat{A}\left(2t^{1-s}\frac{|U(x,t)|}{|x|}\right) dt\,dx \\ &\quad + (1-s)\int_{\mathbb{R}^n}\int_0^\infty t^{-1}A\left(\frac{C}{s}t^{1-s}|\nabla U(x,t)|\right) dt\,dx. \end{aligned} \tag{5.20}$$

We now make use of polar coordinates $(\varrho, \theta, \varphi)$ in $\mathbb{R}^n \times (0,\infty)$, with $\varrho \in (0,\infty)$, $\theta \in (0, \frac{\pi}{2})$, $\varphi \in Q$, where Q is a parallelepiped in $\mathbb{R}^{n-1}$. In particular, $\varrho = \sqrt{|x|^2 + t^2}$ and $\cos\theta = t/\varrho$. From Lemma 4.4 one infers that

$$\begin{aligned} \int_{\mathbb{R}^n}\int_0^{|x|} t^{-1}\widehat{A}\left(\frac{t^{1-s}}{\sqrt{2}}\frac{|U(x,t)|}{|x|}\right) dt\,dx &\le \int_{\mathbb{R}^n}\int_0^\infty t^{-1}\widehat{A}\left(\frac{t^{1-s}}{\sqrt{|x|^2+t^2}}|U(x,t)|\right) dt\,dx \\ &= \int_Q \int_0^{\frac{\pi}{2}} \int_0^\infty (\cos\theta)^{-1}\varrho^{-1}\widehat{A}\left((\cos\theta)^{1-s}\varrho^{-s}|\overline{U}(\varrho,\theta,\varphi)|\right)\varrho^n\, J(\theta,\varphi) d\varrho\, d\theta\, d\varphi \\ &\le \int_Q \int_0^{\frac{\pi}{2}} \int_0^\infty (\cos\theta)^{-1}\varrho^{-1} A\left(C(\cos\theta)^{1-s}\varrho^{1-s}\left|\frac{\partial \overline{U}}{\partial \varrho}(\varrho,\theta,\varphi)\right|\right)\varrho^n\, J(\theta,\varphi)\, d\varrho\, d\theta\, d\varphi \\ &\le \int_0^\infty \int_{\mathbb{R}^n} t^{-1} A\left(Ct^{1-s}|\nabla U(x,t)|\right) dt\,dx \end{aligned} \tag{5.21}$$

for some constant $C = C(n, s)$. Here, $\overline{U}$ denotes the expression of U in polar coordinates $(\varrho, \varphi, \theta)$, and $J(\theta, \varphi)$ stands for the Jacobian of the change of coordinates. Also, observe that the present application of Lemma 4.4 relies upon the equality

$$\overline{U}(\varrho, \theta, \varphi) = \int_{\varrho}^{\infty} \frac{\partial \overline{U}}{\partial r}(r, \theta, \varphi)\, dr \quad \text{for } r > 0, \tag{5.22}$$

and for a.e. (θ, φ). Equality (5.22) holds provided that

$$\lim_{\varrho \to \infty} \overline{U}(\varrho, \theta, \varphi) = 0 \quad \text{for } (\theta, \varphi) \in (0, \tfrac{\pi}{2}) \times Q. \tag{5.23}$$

Equation (5.23) can be verified as follows. We have that

$$|U(x,t)| = \frac{1}{t^n}\left| \int_{\mathbb{R}^n} \psi\left(\frac{y-x}{t}\right) u(y)\, dy \right| \le \frac{C}{t^n} \int_{|y-x|<t} |u(y)|\, dy \quad \text{for } (x,t) \in \mathbb{R}^n \times (0, \infty), \tag{5.24}$$

for some constant $C = C(n)$. Thus, for each (θ, φ), there exists a constant $C = C(n, \theta)$ such that, for every $\varrho > 0$, there exists a ball $B_\varrho \subset \mathbb{R}^n$ of radius ϱ satisfying

$$|\overline{U}(\varrho, \theta, \varphi)| \le \frac{C}{|B_\varrho|} \int_{B_\varrho} |u(y)|\, dy. \tag{5.25}$$

Set

$$\mu(\tau) = |\{|u| > \tau\}| \quad \text{for } \tau > 0,$$

and observe that $\mu(\tau) < \infty$ for every $\tau > 0$, since $u \in \mathcal{M}_d(\mathbb{R}^n)$. Define the non-decreasing function $g : (0, \infty) \to [0, \infty)$ as

$$g(\tau) = \begin{cases} \frac{1}{\mu(\tau)} & \text{if } \tau \in (0, 1] \\ \frac{1}{\mu(1)} & \text{if } \tau \in (1, \infty). \end{cases}$$

Then the function $G : [0, \infty) \to [0, \infty)$, given by

$$G(\tau) = \int_0^\tau g(r)\, dr \quad \text{for } \tau \ge 0\,,$$

is a Young function such that $G(\tau) > 0$ for $\tau > 0$. Consequently,

$$\lim_{\tau \to 0} \frac{\widetilde{G}(\tau)}{\tau} = 0. \tag{5.26}$$

Now, we claim that $u \in L^G(\mathbb{R}^n)$. To verify this claim, note that

$$\int_{\mathbb{R}^n} G(|u|)\, dx = \int_0^\infty g(\tau)\mu(\tau)\, d\tau = \int_0^1 d\tau + \frac{1}{\mu(1)} \int_1^\infty \mu(\tau)\, d\tau \le 1 + \frac{1}{\mu(1)} \int_{\{|u|>1\}} |u(y)|\, dy < \infty, \tag{5.27}$$

where the last inequality holds by property (5.10). Thus, owing to equations (5.25), (2.11) and (5.26),

$$\begin{aligned} \lim_{\varrho \to \infty} |U(\varrho, \theta, \varphi)| &\le \lim_{\varrho \to \infty} \frac{C}{|B_\varrho|} \int_{B_\varrho} |u(y)|\, dy \le \lim_{\varrho \to \infty} \frac{2C}{|B_\varrho|} \|u\|_{L^G(\mathbb{R}^n)} \|1\|_{L^{\widetilde{G}}(B_\varrho)} \\ &= \lim_{\varrho \to \infty} \frac{2C}{|B_\varrho|} \|u\|_{L^G(\mathbb{R}^n)} \frac{1}{\widetilde{G}^{-1}(1/|B_\varrho|)} = 0, \end{aligned} \tag{5.28}$$

whence (5.23) follows. Inequalities (5.20) and (5.21) imply that

$$\int_{\mathbb{R}^n} \widehat{A}\left(\frac{|u(x)|}{|x|^s}\right) dx \le (1-s) \int_{\mathbb{R}^n} \int_0^\infty t^{-1} A\left(Ct^{1-s} |\nabla U(x,t)|\right) dt\, dx \tag{5.29}$$

for some constant $C = C(n, s)$, with the property that $\lim_{s \to 1^-} C(n, s) < \infty$. Inequality (5.2) is a consequence of equations (5.15) and (5.29).
Inequality (5.1) can be deduced on applying inequality (5.2) with u replaced by u/λ for any $\lambda > 0$. $\square$

6. Sobolev embeddings: case $s \in (0,1)$

The Orlicz-Sobolev embedding for the space $V_d^{s,A}(\mathbb{R}^n)$ of order $s \in (0,1)$, with optimal Orlicz target space, reads as follows.

Theorem 6.1. [Optimal Orlicz target space] *Let $n \in \mathbb{N}$ and $s \in (0,1)$. Assume that A is a Young function satisfying conditions (4.2) and (4.3), and let $A_{\frac{n}{s}}$ be the Young function defined as in (4.4). Then,*

$$V_d^{s,A}(\mathbb{R}^n) \to L^{A_{\frac{n}{s}}}(\mathbb{R}^n), \tag{6.1}$$

and there exists a constant $C = C(n,s)$ such that

$$\|u\|_{L^{A_{\frac{n}{s}}}(\mathbb{R}^n)} \le C|u|_{s,A,\mathbb{R}^n} \tag{6.2}$$

for every function $u \in V_d^{s,A}(\mathbb{R}^n)$. Moreover, $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ is the optimal target space in inequality (6.2) among all Orlicz spaces.

A counterpart of embedding (6.1), with an improved target space which is optimal in the broader class of all rearrangement-invariant spaces, is stated in the next result.

Theorem 6.2. [Optimal rearrangement-invariant target space] *Assume that $n \in \mathbb{N}$, $s \in (0,1)$ and A are as in Theorem 6.1. Let $\widehat{A}$ be the Young function given by (4.7) and let $L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n)$ be the Orlicz-Lorentz space defined as in (2.23). Then*

$$V_d^{s,A}(\mathbb{R}^n) \to L(\widehat{A}, \tfrac{n}{s})(\mathbb{R}^n), \tag{6.3}$$

and there exists a constant $C = C(n,s)$ such that

$$\|u\|_{L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n)} \le C|u|_{s,A,\mathbb{R}^n} \tag{6.4}$$

for every function $u \in V_d^{s,A}(\mathbb{R}^n)$. Moreover, $L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n)$ is the optimal target space in inequality (6.4) among all rearrangement-invariant spaces.

We emphasize that assumption (4.3) on the Young function A, appearing in Theorems 6.1 and 6.2, is necessary for an embedding of the space $V_d^{s,A}(\mathbb{R}^n)$ to hold into any rearrangement-invariant space. This is the content of the following proposition.

Proposition 6.3. *Let $n \in \mathbb{N}$ and $s \in (0,1)$, and let A be a Young function. Assume that*

$$V_d^{s,A}(\mathbb{R}^n) \to Y(\mathbb{R}^n) \tag{6.5}$$

for some rearrangement-invariant space $Y(\mathbb{R}^n)$. Then condition (4.3) is fulfilled.

Example 6.4. Let A be a Young function as in (5.3). Assume that the parameters p, p_0, α and α_0 fulfill conditions (5.4) and (5.5). Theorem 6.1 then tells us that embedding (6.1) and inequality (6.2) hold, where

$$A_{\frac{n}{s}}(t) \text{ is equivalent to } \begin{cases} t^{\frac{np_0}{n-sp_0}} (\log \frac{1}{t})^{\frac{n\alpha_0}{n-sp_0}} & \text{if } 1 \le p_0 < \frac{n}{s} \\ e^{-t^{-\frac{n}{s(\alpha_0+1)-n}}} & \text{if } p_0 = \frac{n}{s} \text{ and } \alpha_0 > \frac{n}{s} - 1 \end{cases} \quad \text{near zero,} \tag{6.6}$$

and

$$A_{\frac{n}{s}}(t) \text{ is equivalent to } \begin{cases} t^{\frac{np}{n-sp}} (\log t)^{\frac{n\alpha}{n-sp}} & \text{if } 1 \le p < \frac{n}{s} \\ e^{t^{\frac{n}{n-(\alpha+1)s}}} & \text{if } p = \frac{n}{s} \text{ and } \alpha < \frac{n}{s} - 1 \\ e^{e^{t^{\frac{n}{n-s}}}} & \text{if } p = \frac{n}{s} \text{ and } \alpha = \frac{n}{s} - 1 \end{cases} \quad \text{near infinity.} \tag{6.7}$$

Moreover, the target space in the resultant embedding and inequality is optimal among all Orlicz spaces.
In the special case when

$$p = p_0 < \tfrac{n}{s} \quad \text{and} \quad \alpha = \alpha_0 = 0, \tag{6.8}$$

this recovers inequality (1.2) for the classical fractional space $W^{s,p}(\mathbb{R}^n)$. In the borderline situation corresponding to

$$p = p_0 = \tfrac{n}{s}, \quad \alpha = 0 \quad \text{and} \quad \alpha_0 > \tfrac{n}{s} - 1, \tag{6.9}$$

a fractional embedding of Pohozhaev-Trudinger-Yudovich type [71, 78, 79] is established – see also the recent paper [68] in this connection.

On the other hand, Theorem 6.2 provides us with the optimal embedding (6.3) and inequality (6.4), with a Young function $\widehat{A}$ whose behaviour is described in (5.6) and (5.7). The specific choices (6.8) yield inequality (1.3) – a fractional extension of results of [66, 69] – since the Orlicz-Lorentz target space (6.3) coincides with the standard Lorentz space $L^{\frac{np}{n-p},p}(\mathbb{R}^n)$ in this case. Also, when the parameters p, p_0, α, α_0 are as in (6.9), inequality (6.4) takes the form of a fractional version of a result which, in the integer-order case, follows from a capacitary inequality of [60] – see [61, Inequality (2.3.14)].

Lemma 6.5 below is critical in the proof of the optimality of the target spaces in Theorems 6.1 and 6.2, and in the proof of Proposition 6.3.

Lemma 6.5. *Let $n \in \mathbb{N}$ and $s \in (0,1)$. Let A be a Young function and let $Y(\mathbb{R}^n)$ be a rearrangement-invariant space. Assume that there exists a constant C such that*

$$\|u\|_{Y(\mathbb{R}^n)} \le C|u|_{s,A,\mathbb{R}^n} \tag{6.10}$$

for every function $u \in V_d^{s,A}(\mathbb{R}^n)$. Then there exists a constant C' such that

$$\left\| \int_r^\infty f(\varrho)\varrho^{-1+\frac{s}{n}}\, d\varrho \right\|_{\overline{Y}(0,\infty)} \le C'\|f\|_{L^A(0,\infty)} \tag{6.11}$$

for every function $f \in L^A(0,\infty)$.

Remark 6.6. Under the assumption that the function A fulfills conditions (4.2)–(4.3), a converse of Lemma 6.5 also holds. Namely, inequality (6.11) is a sufficient condition for inequality (6.10). To verify this assertion, recall from Theorem B that the space $L(\widehat{A}, \frac{n}{s})(0,\infty)$ is optimal in inequality (6.4). Thereby, if inequality (6.11) holds, then $L(\widehat{A}, \frac{n}{s})(0,\infty) \to Y(0,\infty)$. Hence, $L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n) \to Y(\mathbb{R}^n)$ as well, and inequality (6.10) is thus a consequence of (6.4).

Proof of Lemma 6.5. In what follows, the relation $\lesssim$ between two expressions will be used to denote that the former is bounded by the latter, up to a positive constant depending only on n and s. The relation $\approx$ means that the two expressions are bounded by each other, up to positive constants depending only on n and s.

Assume that inequality (6.10) holds. Owing to [70, Theorem 1.1], inequality (6.11) holds for every function $f \in \mathcal{M}_+(0,\infty)$ if and only if it just holds for every non-increasing function $f : (0,\infty) \to [0,\infty)$. It thus suffices to prove inequality (6.11) for this class of functions f. Given any f of this kind, define the function $u : \mathbb{R}^n \to [0,\infty)$ as

$$u(x) = \int_{\omega_n|x|^n}^\infty f(r)r^{-1+\frac{s}{n}}\, dr \quad \text{for } x \in \mathbb{R}^n.$$

Let $x, y \in \mathbb{R}^n$. Suppose first that $|y| \ge 2|x|$. Then

$$\begin{aligned}\frac{|u(x)-u(y)|}{|x-y|^s} &= \frac{\left|\int_{\omega_n|x|^n}^\infty f(r)r^{-1+\frac{s}{n}}\, dr - \int_{\omega_n|y|^n}^\infty f(r)r^{-1+\frac{s}{n}}\, dr\right|}{|x-y|^s} \\ &= \frac{\int_{\omega_n|x|^n}^{\omega_n|y|^n} f(r)r^{-1+\frac{s}{n}}\, dr}{|x-y|^s} \lesssim \frac{\int_{\omega_n|x|^n}^{\omega_n|y|^n} f(r)r^{-1+\frac{s}{n}}\, dr}{|y|^s}.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^n}\int_{|y|\ge 2|x|} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} \lesssim \int_{\mathbb{R}^n}\int_{\{|y|\ge 2|x|\}} A\left(C\frac{\int_{\omega_n|x|^n}^{\omega_n|y|^n} f(r)r^{-1+\frac{s}{n}}\, dr}{|y|^s}\right)\frac{dy}{|y|^n}\, dx \tag{6.12}$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{R}^n}\int_{\{|y|\geq 2|x|\}}\int_{\omega_n|x|^n}^{\omega_n|y|^n} A\left(C'f(r)\right) r^{-1+\frac{s}{n}}\,dr\frac{dy}{|y|^{n+s}}\,dx\\
&\lesssim \int_0^\infty A\left(C'f(r)\right) r^{-1+\frac{s}{n}}\int_{\{\omega_n|x|^n<r\}}\int_{\{|y|\geq 2|x|\}}\frac{dy}{|y|^{n+s}}\,dx\,dr\\
&\lesssim \int_0^\infty A\left(C'f(r)\right) r^{-1+\frac{s}{n}}\int_{\{\omega_n|x|^n<r\}}|x|^{-s}\,dx\,dr\\
&\lesssim \int_0^\infty A\left(C'f(r)\right) r^{-1+\frac{s}{n}}r^{1-\frac{s}{n}}\,dr \approx \int_0^\infty A\left(C'f(r)\right)\,dr,
\end{aligned}
$$

for some positive constants C and C' depending on n and s. In particular, the second inequality in chain (6.12) relies upon Jensen's inequality.

Now assume that $|x|\leq |y|\leq 2|x|$. Thereby,

$$
\begin{aligned}
\frac{|u(x)-u(y)|}{|x-y|^s} &\leq \frac{\int_{\omega_n|x|^n}^{\omega_n|y|^n} f(r)r^{-1+\frac{s}{n}}\,dr}{|x-y|^s} \lesssim f(\omega_n|x|^n)\frac{|y|^s-|x|^s}{|x-y|^s} \lesssim f\left(\frac{\omega_n}{2^n}|y|^n\right)\frac{|y|^s-|x|^s}{|x-y|^s}\\
&\lesssim f\left(\frac{\omega_n}{2^n}|y|^n\right)\frac{|y|-|x|}{|y|^{1-s}|x-y|^s} \lesssim f\left(\frac{\omega_n}{2^n}|y|^n\right)\frac{|x-y|}{|y|^{1-s}|x-y|^s} \lesssim f\left(\frac{\omega_n}{2^n}|y|^n\right)|x-y|^{1-s}|y|^{s-1}.
\end{aligned}
$$

Note that

$$
|x-y|^{1-s}|y|^{s-1} \leq \left(|x|^{1-s}+|y|^{1-s}\right)|y|^{s-1}\leq 2.
$$

Thus, there exists a constant $C=C(n,s)$ such that

$$
A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \leq A\left(2Cf\left(\frac{\omega_n}{2^n}|y|^n\right)\frac{|x-y|^{1-s}|y|^{s-1}}{2}\right) \lesssim |x-y|^{1-s}|y|^{s-1}A\left(2Cf\left(\frac{\omega_n}{2^n}|y|^n\right)\right),
$$

where the last inequality holds thanks to property (2.2). Therefore,

$$
\begin{aligned}
(6.13)\qquad &\int_{\mathbb{R}^n}\int_{\{|x|\leq|y|\leq 2|x|\}} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n}\\
&\lesssim \int_{\mathbb{R}^n}\int_{\{|x|\leq|y|\leq 2|x|\}} A\left(2Cf\left(\frac{\omega_n}{2^n}|y|^n\right)\right)|y|^{s-1}|x-y|^{1-s-n}\,dy\,dx\\
&\lesssim \int_{\mathbb{R}^n} A\left(2Cf\left(\frac{\omega_n}{2^n}|y|^n\right)\right)|y|^{s-1}\int_{\{|x|\leq|y|\}}|x-y|^{1-s-n}\,dx\,dy\\
&\lesssim \int_{\mathbb{R}^n} A\left(2Cf\left(\frac{\omega_n}{2^n}|y|^n\right)\right)|y|^{s-1}\int_{\{x\in\mathbb{R}^n:|x-y|\leq 2|y|\}}|x-y|^{1-s-n}\,dx\,dy\\
&\lesssim \int_{\mathbb{R}^n} A\left(2Cf\left(\frac{\omega_n}{2^n}|y|^n\right)\right)|y|^{s-1}|y|^{1-s}\,dy = \int_{\mathbb{R}^n} A\left(2Cf\left(\frac{\omega_n}{2^n}|y|^n\right)\right)\,dy\\
&= \int_0^\infty A\left(2Cf\left(\frac{r}{2^n}\right)\right)\,dr \approx \int_0^\infty A\left(2Cf\left(r\right)\right)\,dr.
\end{aligned}
$$

Coupling inequality (6.12) with (6.13) yields

$$
\int_{\mathbb{R}^n}\int_{\{|y|\geq|x|\}} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} \lesssim \int_0^\infty A\left(Cf\left(r\right)\right)\,dr
$$

for some positive constant $C=C(n,s)$. Exchanging the roles of x and y tells us that

$$
\int_{\mathbb{R}^n}\int_{\{|x|\geq|y|\}} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} \lesssim \int_0^\infty A\left(Cf\left(r\right)\right)\,dr.
$$

Altogether,

$$
(6.14)\qquad \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} \lesssim \int_0^\infty A\left(Cf\left(r\right)\right)\,dr
$$

for some constant $C = C(n, s)$. On replacing f by f/λ for any $\lambda > 0$ in inequality (6.14) one deduces that

$$|u|_{s,A,\mathbb{R}^n} \le C\|f\|_{L^A(0,\infty)} \tag{6.15}$$

for some positive constant $C = C(n, s)$. On the other hand,

$$u^*(r) = \int_r^\infty f(\varrho)\varrho^{-1+\frac{s}{n}}\, d\varrho \quad \text{for } r > 0, \tag{6.16}$$

and

$$\|u\|_{Y(\mathbb{R}^n)} = \|u^*\|_{\overline{Y}(0,\infty)}. \tag{6.17}$$

Inequality (6.11) follows from equations (6.15)–(6.17). □

Proof of Theorem 6.2. Inequality (3.15) ensures that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} \ge \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u^\bigstar(x)-u^\bigstar(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n}. \tag{6.18}$$

Since

$$\int_{\mathbb{R}^n} \widehat{A}\left(\frac{|u^\bigstar(x)|}{|x|^s}\right)dx = \int_0^\infty \widehat{A}\left(\frac{\omega_n^{\frac{s}{n}}|u^*(r)|}{r^{\frac{s}{n}}}\right)dr, \tag{6.19}$$

an application of inequality (5.2) to the function $u^\bigstar$ yields

$$\int_0^\infty \widehat{A}\left(C\frac{|u^*(r)|}{r^{\frac{s}{n}}}\right)dr \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u^\bigstar(x)-u^\bigstar(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} \tag{6.20}$$

for a suitable positive constant $C = C(n, s)$. From inequalities (6.18) and (6.20) we deduce that

$$\int_0^\infty \widehat{A}\left(C\frac{|u^*(r)|}{r^{\frac{s}{n}}}\right)dr \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n}. \tag{6.21}$$

Inequality (6.21) is a version of (6.4) in integral form. Inequality (6.4) follows on applying (6.21) with u replaced by u/λ for any $\lambda > 0$.
It remains to prove that the target space $L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n)$ is optimal in inequality (6.4). To this purpose, assume that $Y(\mathbb{R}^n)$ is a rearrangement-invariant space which renders inequality (6.4) true. Thus, by Lemma 6.5, inequality (6.11) holds. The conclusion then follows via Theorem B. □

Proof of Theorem 6.1. Inequality (6.2) can be deduced via Theorem 6.2 and Proposition 4.1. The optimality of the space $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ is a consequence of Lemma 6.5 and of Theorem A. □

Proof of Proposition 6.3. Assume that embedding (6.5) holds for some rearrangement-invariant space $Y(\mathbb{R}^n)$, namely that inequality (6.10) holds. Then, by Proposition 6.5, inequality (6.11) holds as well. A necessary condition for one-dimensional Hardy type inequalities – a dual version of [44, Lemma 1], see e.g. [33, Lemma 2] – implies that

$$\sup_{r>0} \|1\|_{\overline{Y}(0,r)}\|\varrho^{-1+\frac{s}{n}}\|_{L^{\widetilde{A}}(r,\infty)} < \infty. \tag{6.22}$$

Computations show that the second norm on the left-hand side of equation (6.22) is finite for any $r > 0$ if and only if

$$\int_0 \frac{\widetilde{A}(t)}{t^{1+\frac{n}{n-s}}}\,dt < \infty, \tag{6.23}$$

see e.g. [30, Lemma 3]. Condition (6.23) is in its turn equivalent to (6.5) – see [35, Lemma 2.3]. □

7. Sobolev embeddings: case $s > 1$

The results of the previous section are extended here to any fractional-order power $s \in (0, n)$. The optimal Orlicz target space for embeddings of the space $V_d^{s,A}(\mathbb{R}^n)$ is exhibited in the following theorem.

Theorem 7.1. [Higher-order optimal Orlicz target space] *Let $n \in \mathbb{N}$ and $s \in (0, n) \setminus \mathbb{N}$. Assume that A is a Young function fulfilling conditions (4.2) and (4.3), and let $A_{\frac{n}{s}}$ be the Young function defined as in (4.4)–(4.5). Then*

$$V_d^{s,A}(\mathbb{R}^n) \to L^{A_{\frac{n}{s}}}(\mathbb{R}^n), \tag{7.1}$$

and there exists a constant C such that

$$\|u\|_{L^{A_{\frac{n}{s}}}(\mathbb{R}^n)} \le C\big|\nabla^{[s]}u\big|_{\{s\},A,\mathbb{R}^n} \tag{7.2}$$

for every function $u \in V_d^{s,A}(\mathbb{R}^n)$. Moreover, $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ is the optimal target space in inequality (7.2) among all Orlicz spaces.

The next result enhances Theorem 7.1 and provides us with the optimal rearrangement-invariant target space for embeddings of $V_d^{s,A}(\mathbb{R}^n)$.

Theorem 7.2. [Higher-order optimal rearrangement-invariant target space] *Let n, s and A be as in Theorem 7.1. Let $\widehat{A}$ be the Young function defined as in (4.7)–(4.8), and let $L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n)$ be the Orlicz-Lorentz space equipped with the norm given by (4.9). Then*

$$V_d^{s,A}(\mathbb{R}^n) \to L(\widehat{A}, \tfrac{n}{s})(\mathbb{R}^n), \tag{7.3}$$

and there exists a constant C such that

$$\|u\|_{L(\widehat{A},\frac{n}{s})(\mathbb{R}^n)} \le C\big|\nabla^{[s]}u\big|_{\{s\},A,\mathbb{R}^n} \tag{7.4}$$

for every function $u \in V_d^{s,A}(\mathbb{R}^n)$ Moreover, $L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n)$ is the optimal target space in inequality (7.4) among all rearrangement-invariant spaces.

Remark 7.3. Assumption (4.3) on the Young function A in Theorems 7.1 and 7.2 is necessary for an embedding of the form

$$V_d^{s,A}(\mathbb{R}^n) \to Y(\mathbb{R}^n)$$

to hold for some rearrangement-invariant space, also if $s \in (0, n) \setminus \mathbb{N}$. Indeed, Proposition 6.3 continues to hold for s in this range, with a completely analogous proof which makes use of Lemma 7.6 below, a higher-order analogue of Lemma 6.5.

As a consequence of Theorem 7.2, we can derive a higher-order version of the Hardy type inequality (5.1).

Theorem 7.4. [Higher-order fractional Orlicz–Hardy inequality] *Let n, s, A and $\widehat{A}$ be as in Theorem 7.2. Then there exists a constant C such that*

$$\left\|\frac{|u(x)|}{|x|^s}\right\|_{L^{\widehat{A}}(\mathbb{R}^n)} \le C\big|\nabla^{[s]}u\big|_{\{s\},A,\mathbb{R}^n} \tag{7.5}$$

for every function $u \in V_d^{s,A}(\mathbb{R}^n)$.

Example 7.5. Assume that A is a Young function as in (5.3), with parameters p, p_0, α and α_0 fulfilling conditions (5.4) and (5.5). Then Theorem 7.1 yields embedding (7.1) and inequality (7.2) for every $s \in (0,n)\setminus\mathbb{N}$, where the Young function $A_{\frac{n}{s}}$ obeys equations (6.6) and (6.7). For the same range of exponents s, Theorem 7.2 tells us that embedding (7.3) and inequality (7.4) hold, where the Young function $\widehat{A}$ fulfills (5.6) and (5.7). Moreover, the target spaces in the relevant embeddings and inequalities are optimal in their respective classes. The same kind of Young function $\widehat{A}$ renders the Hardy inequality (7.5) true.
The special choices of the parameters p, p_0, α, α_0 as in (6.8) or (6.9) produce higher-order versions of inequalities

(1.2) and (1.3), or of their limiting versions in the spirit of Pohozhaev-Trudinger-Yudovich and Brezis-Wainger-Hansson, respectively.

The proof of the optimality of the target spaces in Theorems 7.1 and 7.2 relies upon the following lemma.

Lemma 7.6. *Let $n \in \mathbb{N}$ and $s \in (0,n) \setminus \mathbb{N}$. Let A be a Young function and let $Y(\mathbb{R}^n)$ be a rearrangement-invariant space. Assume that there exists a constant C such that*

$$\|u\|_{Y(\mathbb{R}^n)} \leq C\big|\nabla^{[s]}u\big|_{\{s\},A,\mathbb{R}^n} \tag{7.6}$$

for every function $u \in V_d^{s,A}(\mathbb{R}^n)$. Then there exists a constant C' such that

$$\bigg\|\int_r^\infty f(\varrho)\varrho^{-1+\frac{s}{n}}\,d\varrho\bigg\|_{\overline{Y}(0,\infty)} \leq C'\|f\|_{L^A(0,\infty)} \tag{7.7}$$

for every function $f \in L^A(0,\infty)$.

Remark 7.7. The implication of Lemma 7.6 can be reversed if the function A satisfies conditions (4.2)–(4.3). This has been pointed out in Remark 6.6 for the case when $s \in (0,1)$. The argument supporting this assertion is completely analogous to that presented in that remark.

An algebraic inequality to be used in the proof of Lemma 7.6 is the subject of the next result.

Lemma 7.8. *Let $x,\ y \in \mathbb{R}^n$, $n \geq 1$ be such that $0 < |x| \leq |y| \leq 2|x|$. Let $i \in \mathbb{N} \cup \{0\}$ and let $\beta \in \mathbb{R}$ be such that $i \leq n$ and $\beta + i \geq 0$. Assume that $\alpha_1, \alpha_2, \dots, \alpha_i \in \{1,2,\dots,n\}$. Then*

$$|x_{\alpha_1}\cdots x_{\alpha_i}|x|^\beta - y_{\alpha_1}\cdots y_{\alpha_i}|y|^\beta| \lesssim |x-y||x|^{\beta+i-1}, \tag{7.8}$$

up to a multiplicative constant depending on i and β. Here, the products $x_{\alpha_1}\cdots x_{\alpha_i}$ and $y_{\alpha_1}\cdots y_{\alpha_i}$ have to be interpreted as 1 if $i = 0$.

Proof. Fix x and y as in the statement. If $i = 0$, then inequality (7.8) reads

$$||x|^\beta - |y|^\beta| \lesssim |x-y||x|^{\beta-1}. \tag{7.9}$$

This inequality holds, for instance, as a consequence of the mean value theorem for functions of several variables. Let us now assume that $i \geq 1$. Thanks to homogeneity of inequality (7.8), we may assume, without loss of generality, that $|x| = 1$, and hence that $1 \leq |y| \leq 2$. Thus,

$$\begin{aligned} |x_{\alpha_1}\cdots x_{\alpha_i}|x|^\beta - y_{\alpha_1}\cdots y_{\alpha_i}|y|^\beta| &= |x_{\alpha_1}\cdots x_{\alpha_i} - y_{\alpha_1}\cdots y_{\alpha_i}|y|^\beta| \\ &\leq |x_{\alpha_1}\cdots x_{\alpha_i} - y_{\alpha_1}\cdots y_{\alpha_i}| + |y_{\alpha_1}\cdots y_{\alpha_i} - y_{\alpha_1}\cdots y_{\alpha_i}|y|^\beta|. \end{aligned} \tag{7.10}$$

One has that

$$|x_{\alpha_1}\cdots x_{\alpha_i} - y_{\alpha_1}\cdots y_{\alpha_i}| \leq \Big|\sum_{j=1}^i x_{\alpha_1}\cdots x_{\alpha_{j-1}}(x_{\alpha_j} - y_{\alpha_j})y_{\alpha_{j+1}}\cdots y_{\alpha_i}\Big| \leq 2^n\sum_{j=1}^i |x_{\alpha_j} - y_{\alpha_j}| \lesssim |x-y|. \tag{7.11}$$

Here, the product $x_{\alpha_1}\cdots x_{\alpha_{j-1}}$ has to be understood as 1 if $j=1$, and the product $y_{\alpha_{j+1}}\cdots y_{\alpha_i}$ as 1 if $j=i$. Furthermore,

$$|y_{\alpha_1}\cdots y_{\alpha_i} - y_{\alpha_1}\cdots y_{\alpha_i}|y|^\beta| \leq |y_{\alpha_1}\cdots y_{\alpha_i}||1-|y|^\beta| \leq 2^n||x|^\beta - |y|^\beta| \lesssim |x-y|, \tag{7.12}$$

where the last inequality follows from (7.9). Inequality (7.8) is a consequence of (7.10)–(7.12). □

Proof of Lemma 7.6. We focus on the case when $s \in (1,\infty) \setminus \mathbb{N}$, and hence $n \geq 2$, since the result for $s \in (0,1)$ has already been proved in Lemma 6.5. Assume that inequality (7.6) holds. By the reason explained in the proof of Lemma 6.5 in connection with inequality (6.11), it suffices to prove inequality (7.7) for every non-increasing function $f : (0,\infty) \to [0,\infty)$. Moreover, on replacing, if necessary, f by $f\chi_{(0,L)}$ for $L > 0$, and letting $L \to \infty$, we may also assume that f has a bounded support. Passage to the limit as $L \to \infty$ in inequality (7.7) applied with f replaced by $f\chi_{(0,L)}$ is legitimate owing to the Fatou property of rearrangement-invariant norms [10, Chapter 1, Definition 1.1]. Denote, for simplicity, $[s]=m$. Given any function f as above, define the function $u:\mathbb{R}^n\to[0,\infty)$ as

$$u(x)=\int_{\omega_n|x|^n}^\infty\int_{r_1}^\infty\cdots\int_{r_m}^\infty f(r_{m+1})r_{m+1}^{-m-1+\frac{s}{n}}\,dr_{m+1}\dots dr_1\quad\text{for }x\in\mathbb{R}^n. \tag{7.13}$$

It is easily verified that u is m-times weakly differentiable, and that $|\{|\nabla^k u|>t\}|<\infty$ for every $k=0,1,\dots,m$ and every $t>0$. From Fubini's theorem, one can deduce that

$$u(x)=\frac{1}{m!}\int_{\omega_n|x|^n}^\infty f(r)r^{-m-1+\frac{s}{n}}(r-\omega_n|x|^n)^m\,dr\gtrsim\int_{2\omega_n|x|^n}^\infty f(r)r^{-1+\frac{s}{n}}\,dr\quad\text{for }x\in\mathbb{R}^n. \tag{7.14}$$

Throughout this proof, the relations $\gtrsim$, $\lesssim$ and $\approx$ hold up to multiplicative constants depending on n, m and s. The same dependence concerns all constants appearing explicitly. Inequality (7.14), combined with the boundedness of the dilation operator on rearrangement-invariant spaces, implies that

$$\|u\|_{Y(\mathbb{R}^n)}\gtrsim\left\|\int_{2\omega_n|x|^n}^\infty f(\varrho)\varrho^{-1+\frac{s}{n}}\,d\varrho\right\|_{Y(\mathbb{R}^n)}=\left\|\int_{2r}^\infty f(\varrho)\varrho^{-1+\frac{s}{n}}\,d\varrho\right\|_{\overline{Y}(0,\infty)}\gtrsim\left\|\int_{r}^\infty f(\varrho)\varrho^{-1+\frac{s}{n}}\,d\varrho\right\|_{\overline{Y}(0,\infty)} \tag{7.15}$$

for any rearrangement-invariant space $Y(\mathbb{R}^n)$. Note that the last inequality holds owing to the boundedness of the dilation operator defined as in (2.22).

Let $g:(0,\infty)\to[0,\infty)$ be the function defined as

$$g(r)=\frac{1}{m!}\int_{\omega_n r^n}^\infty f(\varrho)\varrho^{-m-1+\frac{s}{n}}(\varrho-\omega_n r^n)^m\,d\varrho\quad\text{for }r>0.$$

One can verify (see [1, Proof of Theorem 3.1]) that any m-th order derivative of u is a linear combination of terms of the form

$$\frac{x_{\alpha_1}\dots x_{\alpha_i}g^{(k)}(|x|)}{|x|^{m-k+i}},$$

where $i=0,1,\dots,m$, $k=1,\dots,m$ and $\alpha_1,\dots,\alpha_i\in\{1,\dots,n\}$. Furthermore, $g^{(k)}(r)$ is a linear combination of functions of the form

$$r^{jn-k}\int_{\omega_n r^n}^\infty\int_{r_{j+1}}^\infty\cdots\int_{r_m}^\infty f(r_{m+1})r_{m+1}^{-m-1+\frac{s}{n}}\,dr_{m+1}\dots dr_{j+1},$$

where $j=1,\dots,k$. Note that, if $j=k=m$, then the last expression has to be interpreted as

$$r^{mn-m}\int_{\omega_n r^n}^\infty f(r_{m+1})r_{m+1}^{-m-1+\frac{s}{n}}\,dr_{m+1}.$$

Altogether, we deduce that any m-th order derivative of u agrees with a linear combination of terms of the form

$$x_{\alpha_1}\dots x_{\alpha_i}|x|^{jn-m-i}\int_{\omega_n|x|^n}^\infty\int_{r_{j+1}}^\infty\cdots\int_{r_m}^\infty f(r_{m+1})r_{m+1}^{-m-1+\frac{s}{n}}\,dr_{m+1}\dots dr_{j+1},$$

where $i=0,1,\dots,m$, $j=1,\dots,m$ and $\alpha_1,\dots,\alpha_i\in\{1,\dots,n\}$.

For a.e. $x,y\in\mathbb{R}^n$ we have what follows. Assume first that $|y|\ge2|x|$. Then

$$\begin{aligned}|\nabla^m u(x)-\nabla^m u(y)|&\le|\nabla^m u(x)|+|\nabla^m u(y)|\\&\lesssim\sum_{j=1}^m|x|^{jn-m}\int_{\omega_n|x|^n}^\infty\int_{r_{j+1}}^\infty\cdots\int_{r_m}^\infty f(r_{m+1})r_{m+1}^{-m-1+\frac{s}{n}}\,dr_{m+1}\dots dr_{j+1}\\&+\sum_{j=1}^m|y|^{jn-m}\int_{\omega_n|y|^n}^\infty\int_{r_{j+1}}^\infty\cdots\int_{r_m}^\infty f(r_{m+1})r_{m+1}^{-m-1+\frac{s}{n}}\,dr_{m+1}\dots dr_{j+1}\\&\lesssim\sum_{j=1}^m|x|^{jn-m}\int_{\omega_n|x|^n}^\infty f(r)r^{-j-1+\frac{s}{n}}\,dr+\sum_{j=1}^m|y|^{jn-m}\int_{\omega_n|y|^n}^\infty f(r)r^{-j-1+\frac{s}{n}}\,dr\end{aligned} \tag{7.16}$$

$$\lesssim |x|^{s-m} f(\omega_n |x|^n) + |y|^{s-m} f(\omega_n |y|^n),$$

where the third inequality follows via an analogue of equation (7.14), and the last one thanks to the monotonicity of f. Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_{|y|\geq 2|x|} A\left(\frac{|\nabla^m u(x) - \nabla^m u(y)|}{|x-y|^{s-m}}\right) \frac{dx\,dy}{|x-y|^n} \\
&\quad \lesssim \int_{\mathbb{R}^n} \int_{|y|\geq 2|x|} A\left(C|x|^{s-m}|y|^{m-s} f(\omega_n|x|^n)\right) \frac{dy}{|y|^n}\,dx + \int_{\mathbb{R}^n}\int_{|y|\geq 2|x|} A\left(Cf(\omega_n|y|^n)\right)\,dx \frac{dy}{|y|^n} \\
&\quad \lesssim \int_{\mathbb{R}^n} \int_{2|x|}^{\infty} A\left(C' r^{m-s}|x|^{s-m} f(\omega_n|x|^n)\right) \frac{dr}{r}\,dx + \int_{\mathbb{R}^n} A\left(C' f(\omega_n|y|^n)\right)\,dy
\end{aligned}$$

for some constants C and C'. The change of variables $t = C' r^{m-s}|x|^{s-m} f(\omega_n|x|^n)$ yields

$$\int_{2|x|}^{\infty} A\left(C' r^{m-s}|x|^{s-m} f(\omega_n|x|^n)\right) \frac{dr}{r}\,dx \approx \int_0^{C' f(\omega_n|x|^n)} A(t)\frac{dt}{t} \leq A\left(C' f(\omega_n|x|^n)\right),$$

where the last inequality holds due to the monotonicity of the function $t \mapsto \frac{A(t)}{t}$. Therefore,

$$\int_{\mathbb{R}^n} \int_{|y|\geq 2|x|} A\left(\frac{|\nabla^m u(x) - \nabla^m u(y)|}{|x-y|^{s-m}}\right) \frac{dx\,dy}{|x-y|^n} \lesssim \int_{\mathbb{R}^n} A\left(Cf(\omega_n|x|^n)\right)\,dx = \int_0^{\infty} A\left(Cf(r)\right)\,dr \tag{7.17}$$

for some constant C.

Let us now assume that $|x| \leq |y| \leq 2|x|$. Then

$$\begin{aligned}
&|\nabla^m u(x) - \nabla^m u(y)| \\
&\lesssim \sum_{i=0}^{m} \sum_{(\alpha_1,\dots,\alpha_i)\in\{1,\dots,n\}^i} \sum_{j=1}^{m} \Bigg| x_{\alpha_1}\cdots x_{\alpha_i}|x|^{jn-m-i} \int_{\omega_n|x|^n}^{\infty} \int_{r_{j+1}}^{\infty} \cdots \int_{r_m}^{\infty} f(r_{m+1}) r_{m+1}^{-m-1+\frac{s}{n}}\,dr_{m+1}\dots dr_{j+1} \\
&\quad - y_{\alpha_1}\cdots y_{\alpha_i}|y|^{jn-m-i} \int_{\omega_n|y|^n}^{\infty} \int_{r_{j+1}}^{\infty} \cdots \int_{r_m}^{\infty} f(r_{m+1}) r_{m+1}^{-m-1+\frac{s}{n}}\,dr_{m+1}\dots dr_{j+1} \Bigg| \\
&\lesssim \sum_{i=0}^{m} \sum_{(\alpha_1,\dots,\alpha_i)\in\{1,\dots,n\}^i} \sum_{j=1}^{m} \Bigg| x_{\alpha_1}\cdots x_{\alpha_i}|x|^{jn-m-i} \int_{\omega_n|x|^n}^{\omega_n|y|^n} \int_{r_{j+1}}^{\infty} \cdots \int_{r_m}^{\infty} f(r_{m+1}) r_{m+1}^{-m-1+\frac{s}{n}}\,dr_{m+1}\dots dr_{j+1} \Bigg| \\
&\quad + \sum_{i=0}^{m} \sum_{(\alpha_1,\dots,\alpha_i)\in\{1,\dots,n\}^i} \sum_{j=1}^{m} \left| x_{\alpha_1}\cdots x_{\alpha_i}|x|^{jn-m-i} - y_{\alpha_1}\cdots y_{\alpha_i}|y|^{jn-m-i}\right| \\
&\qquad\qquad \times \int_{\omega_n|y|^n}^{\infty} \int_{r_{j+1}}^{\infty} \cdots \int_{r_m}^{\infty} f(r_{m+1}) r_{m+1}^{-m-1+\frac{s}{n}}\,dr_{m+1}\dots dr_{j+1} \\
&\lesssim \sum_{j=1}^{m} |x|^{jn-m} \int_{\omega_n|x|^n}^{\omega_n|y|^n} \int_{r_{j+1}}^{\infty} \cdots \int_{r_m}^{\infty} f(r_{m+1}) r_{m+1}^{-m-1+\frac{s}{n}}\,dr_{m+1}\dots dr_{j+1} \\
&\quad + \sum_{j=1}^{m} |x-y||x|^{jn-m-1} \int_{\omega_n|y|^n}^{\infty} \int_{r_{j+1}}^{\infty} \cdots \int_{r_m}^{\infty} f(r_{m+1}) r_{m+1}^{-m-1+\frac{s}{n}}\,dr_{m+1}\dots dr_{j+1} \\
&\lesssim \sum_{j=1}^{m} |x|^{jn-m} \int_{\omega_n|x|^n}^{\omega_n|y|^n} \int_{r}^{\infty} f(\varrho)\varrho^{-j-2+\frac{s}{n}}\,d\varrho\,dr + \sum_{j=1}^{m} |x-y||x|^{jn-m-1} \int_{\omega_n|y|^n}^{\infty} f(r) r^{-j-1+\frac{s}{n}}\,dr \\
&\lesssim \sum_{j=1}^{m} |x|^{jn-m}(|y|^n - |x|^n) \int_{\omega_n|x|^n}^{\infty} f(r) r^{-j-2+\frac{s}{n}}\,dr + \sum_{j=1}^{m} |x-y||x|^{jn-m-1} \int_{\omega_n|x|^n}^{\infty} f(r) r^{-j-1+\frac{s}{n}}\,dr \\
&\lesssim |y-x||x|^{-m-1+s} f(\omega_n|x|^n).
\end{aligned}$$

Observe that the third inequality holds owing to Lemma 7.8, the fourth one by an analogue of equation (7.14), and the last one since f is non-increasing. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{|x|\le|y|\le 2|x|} A\left(\frac{|\nabla^m u(x)-\nabla^m u(y)|}{|x-y|^{s-m}}\right)\frac{dx\,dy}{|x-y|^n}\\
&\quad\lesssim \int_{\mathbb{R}^n}\int_{|x|\le|y|\le 2|x|} A\left(C|x|^{s-m-1}|y-x|^{m-s+1}f(\omega_n|x|^n)\right)\frac{dy}{|y-x|^n}\,dx\\
&\quad= \int_{\mathbb{R}^n}\int_{|x|\le|z+x|\le 2|x|} A\left(C|x|^{s-m-1}|z|^{m-s+1}f(\omega_n|x|^n)\right)\frac{dz}{|z|^n}\,dx\\
&\quad\le \int_{\mathbb{R}^n}\int_{|z|\le 3|x|} A\left(C|x|^{s-m-1}|z|^{m-s+1}f(\omega_n|x|^n)\right)\frac{dz}{|z|^n}\,dx\\
&\quad\lesssim \int_{\mathbb{R}^n}\int_0^{3|x|} A\left(C|x|^{s-m-1}r^{m-s+1}f(\omega_n|x|^n)\right)\frac{dr}{r}\,dx
\end{aligned}$$

for some positive constant C. The change of variables $t=C|x|^{s-m-1}f(\omega_n|x|^n)r^{m-s+1}$ yields

$$\int_0^{3|x|} A\left(C|x|^{s-m-1}r^{m-s+1}f(\omega_n|x|^n)\right)\frac{dr}{r}\approx\int_0^{Cf(\omega_n|x|^n)} A(t)\frac{dt}{t}\le A(Cf(\omega_n|x|^n)).$$

Thereby,

$$\int_{\mathbb{R}^n}\int_{|x|\le|y|\le 2|x|} A\left(\frac{|\nabla^m u(x)-\nabla^m u(y)|}{|x-y|^{s-m}}\right)\frac{dx\,dy}{|x-y|^n}\lesssim\int_{\mathbb{R}^n} A\left(Cf(\omega_n|x|^n)\right)dx=\int_0^\infty A\left(Cf(r)\right)dr. \tag{7.18}$$

Coupling equations (7.17) and (7.18) tells us that

$$\int_{\mathbb{R}^n}\int_{|x|\le|y|} A\left(\frac{|\nabla^m u(x)-\nabla^m u(y)|}{|x-y|^{s-m}}\right)\frac{dx\,dy}{|x-y|^n}\lesssim\int_0^\infty A\left(Cf(r)\right)dr, \tag{7.19}$$

for some positive constant C. Adding inequality (7.19) to a parallel inequality obtained by exchanging the roles of x and y, and applying the resultant inequality with f replaced by f/λ for any $\lambda>0$ yield

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|\nabla^m u(x)-\nabla^m u(y)|}{\lambda|x-y|^{s-m}}\right)\frac{dx\,dy}{|x-y|^n}\lesssim\int_0^\infty A\left(\frac{C}{\lambda}f(r)\right)dr$$

for some constant C. Now recall that $m=[s]$ and $s-m=\{s\}$ to infer that

$$\left|\nabla^{[s]}u\right|_{\{s\},A,\mathbb{R}^n}\lesssim\|f\|_{L^A(0,\infty)}. \tag{7.20}$$

Combining inequalities (7.15) and (7.20) shows that (7.6) implies (7.7). □

We have now all the preliminaries at our disposal to accomplish the proof of Theorem 7.2.

Proof of Theorem 7.2. In this proof we need to make use of the function $\widehat{A}$, defined as in (4.7)–(4.8), also with s replaced by $\{s\}$. For clarity of notation, we shall denote by $\widehat{A}_s$ and $\widehat{A}_{\{s\}}$ the functions defined by (4.7)–(4.8) with s and $\{s\}$, respectively. Theorem 6.2, applied with u replaced by $\nabla^{[s]}u$, and with s replaced by $\{s\}$, tells us that

$$\|\nabla^{[s]}u\|_{L(\widehat{A}_{\{s\}},\frac{n}{\{s\}})(\mathbb{R}^n)}\le C\left|\nabla^{[s]}u\right|_{s,A,\mathbb{R}^n} \tag{7.21}$$

for some constant C and for every function $u\in V^{s,A}_d(\mathbb{R}^n)$. Inequality (7.4) will thus follow if we show that

$$\|u\|_{L(\widehat{A}_s,\frac{n}{s})(\mathbb{R}^n)}\le C\|\nabla^{[s]}u\|_{L(\widehat{A}_{\{s\}},\frac{n}{\{s\}})(\mathbb{R}^n)} \tag{7.22}$$

for some constant C and for every function $u\in V^{s,A}_d(\mathbb{R}^n)$. In order to prove inequality (7.22), we make use of Theorems B and E. By Theorem B,

$$\|T_{\{s\}}f\|_{L(\widehat{A}_{\{s\}},\frac{n}{\{s\}})(0,\infty)}\le C\|f\|_{L^A(0,\infty)}, \tag{7.23}$$

for some constant C and every function $f \in L^A(0,\infty)$, where $T_{\{s\}}$ is the operator defined as in (4.1). Furthermore, $L(\widehat{A}_{\{s\}}, \frac{n}{\{s\}})(0,\infty)$ is the optimal rearrangement-invariant target space in (7.23). The same result also tells us that

$$\|T_s f\|_{L(\widehat{A}_s, \frac{n}{s})(0,\infty)} \leq C\|f\|_{L^A(0,\infty)}, \tag{7.24}$$

for some constant C and every function $f \in L^A(0,\infty)$, and that $L(\widehat{A}, \frac{n}{s})(0,\infty)$ is the optimal rearrangement-invariant target space in (7.24). Denote by $X_{[s]}(0,\infty)$ the optimal rearrangement-invariant target space in the inequality

$$\|T_{[s]} f\|_{X_{[s]}(0,\infty)} \leq C\|f\|_{L(\widehat{A}_{\{s\}}, \frac{n}{\{s\}})(0,\infty)} \tag{7.25}$$

for some constant C and for every function $f \in L(\widehat{A}_{\{s\}}, \frac{n}{\{s\}})(0,\infty)$. Hence, since $s = [s] + \{s\}$, from Theorem E, part (ii), one can deduce that

$$X_{[s]}(0,\infty) = L(\widehat{A}_s, \tfrac{n}{s})(0,\infty). \tag{7.26}$$

Therefore,

$$\|T_{[s]} f\|_{L(\widehat{A}_s, \frac{n}{s})(0,\infty)} \leq C\|f\|_{L(\widehat{A}_{\{s\}}, \frac{n}{\{s\}})(0,\infty)} \tag{7.27}$$

for some constant C and for every function $f \in L(\widehat{A}_{\{s\}}, \frac{n}{\{s\}})(0,\infty)$. Owing to the reduction principle for integer-order Sobolev inequalities in the version of [64, Theorem 3.3], inequality (7.27) implies inequality (7.22).

The optimality of the space $L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n)$ in inequality (7.4) follows from Lemma 7.6 and Theorem B. □

Proof of Theorem 7.1. Inequality (7.2) follows from Theorem 7.2 and Proposition 4.1. The optimality of the space $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ is a consequence of Lemma 7.6 and Theorem A. □

Proof of Theorem 7.4. Property (2.16) ensures that

$$\left\| \frac{|u(x)|}{|x|^s} \right\|_{L^{\widehat{A}}(\mathbb{R}^n)} \leq \|\omega_n^{\frac{s}{n}} r^{-\frac{s}{n}} u^*(r)\|_{L^{\widehat{A}}(0,\infty)} = \omega_n^{\frac{s}{n}} \|u\|_{L(\widehat{A}, \frac{n}{s})(0\infty)}. \tag{7.28}$$

Coupling inequality (7.28) with (7.4) yields (7.5). □

List of symbols

Function spaces

$V^{m,A}(\mathbb{R}^n)$ (3.1)
$W^{m,A}(\mathbb{R}^n)$ (3.2)
$V^{s,A}(\mathbb{R}^n)$ (3.4) and (3.6)
$V_d^{s,A}(\mathbb{R}^n)$ (3.5) and (3.7)
$W^{s,A}(\mathbb{R}^n)$ (3.8)

Function norms and seminorms

$\|\cdot\|_{L^{\widehat{A}}(\mathbb{R}^n)}$ (2.10)
$\|\cdot\|_{L(A,q)(\mathbb{R}^n)}$ (2.23)
$\|\cdot\|_{L[A,q](\mathbb{R}^n)}$ (2.25)
$\|\cdot\|_{L^{\sigma,p}(\mathbb{R}^n)}$ (2.27)
$\|\cdot\|_{L(\widehat{A}, \frac{n}{s})(0,\infty)}$ (4.9)
$\|\cdot\|_{L[\widetilde{A}, -\frac{n}{s}](0,\infty)}$ (4.10)
$\|\cdot\|_{X'(\mathbb{R}^n)}$ (2.18)
$\|\cdot\|_{\overline{X}(0,\infty)}$ (2.21)

Operators and operations

Miscellaneous

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