



UNIVERSITÀ
DEGLI STUDI
FIRENZE

FLORE

Repository istituzionale dell'Università degli Studi di Firenze

On the primary coverings of finite solvable and symmetric groups

Questa è la Versione finale referata (Post print/Accepted manuscript) della seguente pubblicazione:

Original Citation:

On the primary coverings of finite solvable and symmetric groups / Fumagalli F.; Garonzi M.. - In: JOURNAL OF GROUP THEORY. - ISSN 1433-5883. - STAMPA. - 24:(2021), pp. 1189-1211. [10.1515/jgth-2020-0056]

Availability:

The webpage <https://hdl.handle.net/2158/1250969> of the repository was last updated on 2021-12-07T15:10:20Z

Published version:

DOI: 10.1515/jgth-2020-0056

Terms of use:

Open Access

La pubblicazione è resa disponibile sotto le norme e i termini della licenza di deposito, secondo quanto stabilito dalla Policy per l'accesso aperto dell'Università degli Studi di Firenze (<https://www.sba.unifi.it/upload/policy-oa-2016-1.pdf>)

Publisher copyright claim:

La data sopra indicata si riferisce all'ultimo aggiornamento della scheda del Repository FloRe - The above-mentioned date refers to the last update of the record in the Institutional Repository FloRe

(Article begins on next page)

On the primary coverings of finite solvable and symmetric groups

Francesco Fumagalli and Martino Garonzi*

Communicated by Evgeny Vdovin

Abstract. A primary covering of a finite group G is a family of proper subgroups of G whose union contains the set of elements of G having order a prime power. We denote by $\sigma_0(G)$ the smallest size of a primary covering of G and call it the primary covering number of G . We study this number and compare it with its analogue $\sigma(G)$, the covering number, for the classes of groups G that are solvable and symmetric.

1 Introduction

A *covering* of a finite group G is a family of proper subgroups of G whose union is equal to G . The *covering number* of G is defined as the minimal size of a covering of G , and it is denoted by $\sigma(G)$ (by Cohn [1]). A group admits a covering unless it is cyclic, in which case $\sigma(G)$ is generally set to be ∞ (with the convention that $n < \infty$ for every integer n). The covering number has been studied by many authors, and in particular, $\sigma(G)$ was determined when G is solvable by Tomkinson [15, Theorem 2.2] and when G is symmetric by Maróti [11, Theorem] (see also [9, 12, 14]).

Given a subset Π of G , we may be interested in the minimal number of proper subgroups of G whose union contains Π . In this paper, we focus on the set of primary elements, complementing the work done in [4]. A *primary element* of G is an element of G whose order is some prime power. We define G_0 to be the set of primary elements of G and a *primary covering* of G to be a family of proper subgroups of G whose union contains G_0 . We set $\sigma_0(G)$ to be the smallest size of a primary covering of G and call it the *primary covering number* of G . Observe that G admits primary coverings if and only if G is not a cyclic p -group for any prime p , so in this case, we define $\sigma_0(G) = \infty$, with the convention that $n < \infty$ for every integer n . Clearly, we always have $\sigma_0(G) \leq \sigma(G)$. Moreover, a deep

M. Garonzi was supported by Fundação de Apoio à Pesquisa do Distrito Federal (FAPDF) – demanda espontânea 03/2016, and by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) – Grant numbers 302134/2018-2, 422202/2018-5.

result [3, Theorem 1] shows that a primary covering of any finite group is never a unique conjugacy class of a proper subgroup.

In this paper, we study $\sigma_0(G)$ when G is solvable and when G is a symmetric group S_n . Our main result about solvable groups is the following.

Theorem 1. *Let G be a finite solvable group which is not a cyclic p -group for every prime p . If G/G' is not a p -group, then $\sigma_0(G) = 2$. Otherwise, $\sigma_0(G) = \sigma(G)$.*

Our results on the primary covering number for S_n can be summarized in the following statement.

Theorem 2. *The following hold for $n \geq 3$.*

- (1) $\sigma_0(S_3) = 4$ and $\sigma_0(S_6) = 7$ (see Lemmas 8 and 11).
- (2) If $n = 2^a$ for some $a > 1$, then $\sigma_0(S_n) = 1 + \frac{1}{2} \binom{n}{n/2}$ (see Proposition 1).
- (3) If $n \neq 3^\varepsilon 2^a$ for $\varepsilon \in \{0, 1\}$ and $a > 1$, then $\sigma_0(S_n) = 1 + \binom{n}{n_2}$, where n_2 denotes the maximum power of 2 that divides n (Proposition 4 and Lemma 10).
- (4) If $n = 3 \cdot 2^a$, with $a \geq 2$, then $c_1 \leq \sigma_0(S_n) \leq c_2$, where

$$c_1 = \begin{cases} 117 & \text{if } a = 2, \\ 1 + \binom{n-1}{2^{a-1}} & \text{if } a \geq 3, \end{cases}$$

$$c_2 = 2 + \binom{n-1}{2^a-1} + \sum_{i=2}^{2^{a+1}} \binom{n-i}{2^{a-1}-1}$$

(see Proposition 5).

As a comparison with the previous paper [4], observe the following. Denoting by $\gamma_0(G)$ the normal primary covering number of a finite group G , that is the smallest natural number such that

$$G_0 \subseteq \bigcup_{i=1}^{\gamma_0(G)} \bigcup_{g \in G} H_i^g$$

for some proper – pairwise non-conjugate – subgroups H_i of G , by [4, Propositions 1 and 5], we have that if G is a group whose order is not a prime power and G is either solvable or symmetric, then $\gamma_0(G) = 2$.

2 Solvable groups

In this section, we determine the primary covering number for every finite solvable group G . For the basic results, as well as the notation, concerning solvable groups, we refer the reader to [13, Chapter 5]. We start with the following trivial observations that hold in a general context.

Lemma 1. *Assume that G is a finite group.*

- (1) *If G/G' is not a p -group for some prime p , then $\sigma_0(G) = 2$.*
- (2) *If $N \trianglelefteq G$, then $\sigma_0(G) \leq \sigma_0(G/N)$. Moreover, if N is contained in the Frattini subgroup of G , then $\sigma_0(G) = \sigma_0(G/N)$.*
- (3) *If M is a maximal subgroup of G such that $\sigma_0(M) > \sigma_0(G)$, then M belongs to every minimal primary covering of G .*

Proof. (1) We trivially have that $G_0 \subseteq H \cup K$, where H and K are maximal subgroups containing G' of coprime indices.

(2) Since any primary covering of G/N lifts to a primary covering of G , we have $\sigma_0(G) \leq \sigma_0(G/N)$. Also, any subgroup in a primary covering of G can be replaced by a maximal subgroup containing it. Thus, when $N \leq \Phi(G)$, we have $\sigma_0(G) = \sigma_0(G/N)$.

(3) Let M be a maximal subgroup of G such that $\sigma_0(G) < \sigma_0(M)$, and let H_1, \dots, H_n be any primary covering of G of size $n = \sigma_0(G)$. Of course, the family $\{H_i \cap M\}_{i=1}^n$ covers M_0 (the set of primary elements of M), so since $\sigma_0(M) > n$, we deduce that there exists $i \in \{1, \dots, n\}$ such that $H_i \cap M = M$; in other words, $M \leq H_i$. Since M is a maximal subgroup of G , we deduce that $M = H_i$. \square

Lemma 2. *Let N be a complemented minimal normal subgroup of a solvable group G , and let b be the number of complements of N in G . If $b > 1$, then $b \geq |N|$.*

Proof. Let $E = \text{End}_G(N)$. By a result of Gaschütz [5, Satz 3], we know that $b = |N|^\varepsilon |E|^{\beta-1}$, where ε is 0 or 1 according to whether N is central or not in G , and β is the number of non-Frattini chief factors G -isomorphic to N in a chief series of G starting with N . It follows that $b \geq |N|$ if N is not central, and if N is central, then $|N| = p$ is a prime number and E is the field with p elements, so that $\beta \neq 1$ (since $b > 1$ by hypothesis); hence $b \geq |E| = p = |N|$. \square

Recall Tomkinson's result [15, Theorem 2.2] which states that if G is a finite solvable group, then $\sigma(G) = q + 1$, where q is the order of the smallest chief factor of G with more than one complement.

Proof of Theorem 1. Let G be a non-cyclic finite solvable group. If $|G/G'|$ is not a prime power, then by Lemma 1 (1), we have $\sigma_0(G) = 2$. Thus assume that G/G' is a p -group for some prime p , and define α to be the smallest order of a chief factor of G admitting more than one complement (this is well-defined because G is not a cyclic p -group). We need to show that $\sigma_0(G) = 1 + \alpha$. By the aforementioned result of Tomkinson, $\sigma_0(G) \leq \sigma(G) = 1 + \alpha$. Let N be a normal subgroup of G such that $\sigma_0(G) = \sigma_0(G/N)$ with $|G/N|$ minimal with this property. Let K/N be a minimal normal subgroup of G/N ; then $\sigma_0(G/N) < \sigma_0(G/K)$; hence by Lemma 1 (2), K/N admits a complement M/N in G/N which is a maximal subgroup. Since $\sigma_0(G/N) < \sigma_0(G/K) = \sigma_0(M/N)$, we deduce that M/N appears in every minimal primary covering of G/N ; hence all b complements of K/N in G/N appear in a fixed minimal primary covering of G/N . However, no element of K/N belongs to any complement of K/N ; hence $\sigma_0(G/N) \geq 1 + b$. If $b \neq 1$, then $b \geq |K/N|$ by Lemma 2, and this implies the result. Assume now $b = 1$. Then G/N is a direct product $K/N \times M/N$; hence K/N , being a central chief factor, is cyclic of prime order, and since it is an epimorphic image of the p -group G/G' , we deduce that $|K/N| = p$. Moreover, M/N is nontrivial because $G/N \not\cong C_p$ (since $\sigma_0(G) = \sigma_0(G/N)$), so M/N has nontrivial abelianization. Since G/G' is a p -group, it follows that M/N projects onto C_p . This C_p is therefore a chief factor above K/N which is G -isomorphic to K/N . In particular, the number of non-Frattini chief factors G -isomorphic to C_p is $\beta \geq 2$. This contradicts the formula in the proof of Lemma 2. \square

3 Symmetric groups

We introduce some notation that will be used frequently. Let n be a positive integer, and set $\Omega := \{1, 2, \dots, n\}$ on which the symmetric group S_n acts naturally. Given natural numbers $k \geq 1$ and $x_1 \geq x_2 \geq \dots \geq x_k \geq 1$ such that $\sum_{i=1}^k x_i = n$, we denote by (x_1, x_2, \dots, x_k) both the partition of n whose parts are precisely the x_i and the set of all permutations in S_n having as cyclic type this partition. In particular, (n) denotes the set of n -cycles of S_n .

The maximal subgroups of S_n split in three different classes, according to their action on Ω : intransitive, imprimitive and primitive subgroups (see [2]).

Any intransitive maximal subgroup is the setwise stabilizer of a set of cardinality m for some $1 \leq m < n/2$. In particular, any such subgroup is conjugate to the stabilizer of the set $\{1, 2, \dots, m\} \subseteq \Omega$, which we denote with X_m . It is therefore isomorphic to $S_m \times S_{n-m}$, and its index is $\binom{n}{m}$. We set

$$\mathcal{X}_m = \{\text{conjugates of } X_m \simeq S_m \times S_{n-m}\}$$

and

$$\mathcal{X} = \bigcup_{1 \leq m < n/2} \mathcal{X}_m = \{\text{intransitive maximal subgroups of } S_n\}.$$

The imprimitive maximal subgroups of S_n are the stabilizers of partitions of Ω into equal-sized subsets. If d is any proper nontrivial divisor of n , we set W_d the stabilizer of the partition

$$\{\{1, 2, \dots, d\}, \{d+1, d+2, \dots, 2d\}, \dots, \{n-d+1, n-d+2, \dots, n\}\}.$$

Note that W_d is isomorphic to the wreath product $S_d \wr S_{n/d}$, and it has index

$$\frac{n!}{(d!)^{n/d} \cdot (n/d)!}.$$

Also, note that any imprimitive maximal subgroup of S_n is conjugate to W_d for some proper nontrivial divisor d of n . We set

$$\mathcal{W}_d = \{\text{conjugates of } W_d \simeq S_d \wr S_{n/d}\}$$

and

$$\mathcal{W} = \bigcup_{1 < d | n, d \neq n} \mathcal{W}_d = \{\text{imprimitive maximal subgroups of } S_n\}.$$

Finally, set

$$\mathcal{P} = \{\text{proper primitive maximal subgroups of } S_n\},$$

where proper means that both S_n and A_n are not members of \mathcal{P} .

Lemma 3. *If $a > b$ are positive integers, then $a!^b b! \geq b!^a a!$ with equality if and only if $b = 1$.*

Proof. If $b = 1$, we have equality; now assume $b \geq 2$ so that $a \geq 3$. Since the stated inequality is equivalent to

$$\frac{\ln(a!)}{a-1} \geq \frac{\ln(b!)}{b-1},$$

it is enough to prove that the function $\ln(a!)/(a-1)$ is increasing; hence we may assume $b = a-1$. So we need to prove that $a!^{a-2} \geq (a-1)!^{a-1}$ which is equivalent to $a^{a-1} \geq a!$, which is actually a strict inequality since $a \geq 3$. \square

The following lemmas are part of [10, Corollary 1.2 and Lemma 2.1].

Lemma 4 (Maróti [10, Lemma 2.1], On the orders of primitive groups). *Let $m > 1$ be an integer, and suppose $m = a_1 b_1 = a_2 b_2$ with a_1, a_2, b_1, b_2 positive integers at least 2, $b_1 \geq a_1$, $b_2 \geq a_2$, $a_1 \leq a_2$ and (consequently) $b_1 \geq b_2$. Then*

$$b_1!^{a_1} a_1! \geq b_2!^{a_2} a_2!$$

with equality if and only if $a_1 = a_2$ and $b_1 = b_2$. Moreover, if p is the smallest prime divisor of m and d is any divisor of m with $1 < d < m$, then

$$(m/p)!^p p! \geq (m/d)!^d d!$$

with equality if and only if $d = p$.

Proof. Observe that, since $b_2 \geq a_2 \geq 2$, we have $b_2^{b_2} \geq b_2! b_2 \geq b_2! a_2$.

$$\begin{aligned} b_1!^{a_1} a_1! &\geq b_2!^{a_1} (b_2 + 1)^{a_1} \cdots b_1^{a_1} a_1! \geq b_2!^{a_1} a_1! b_2^{a_1(b_1-b_2)} \\ &\geq b_2!^{a_1} a_1! (b_2! a_2)^{(a_1/b_2)(b_1-b_2)} = b_2!^{a_1} a_1! (b_2! a_2)^{(a_2-a_1)} \\ &\geq b_2!^{a_1} a_1! b_2!^{a_2-a_1} (a_1 + 1) \cdots a_2 = b_2!^{a_2} a_2!. \end{aligned}$$

If equality holds, then all of the above inequalities are equalities, and it is easy to deduce that $a_1 = a_2$ and consequently $b_1 = b_2$. To deduce the last statement, observe that it is trivial if $m = p$, so assume this is not the case so that $p^2 \leq m$. Choose $a_1 = p$, $b_1 = m/p$; then $a_1 \leq b_1$, and the inequality $a_1 \leq a_2$ will be true for every choice of a divisor $a_2 > 1$ of m by minimality of p . If $d^2 \leq m$, then choose $a_2 = d$, proving the strict inequality with equality if and only if $d = p$. If $d^2 > m$, then $d > m/d$; hence Lemma 3 implies that

$$d!^{m/d} (m/d)! \geq (m/d)!^d d!,$$

so it is enough to show that $(m/p)!^p p! \geq d!^{m/d} (m/d)!$, which follows from the above choosing $a_2 = m/d$, and again by Lemma 3, equality does not hold in this case since $m/d \neq 1$. \square

The orders of the different primitive and imprimitive maximal subgroups of S_n can be compared as in the following lemma.

Lemma 5. *Let $n \geq 2$; then the following hold.*

(1) *For every proper nontrivial divisor d of n , we have*

$$|W_{n/p}| = (n/p)!^p p! \geq (n/d)!^d d! = |W_d|,$$

where p is the smallest prime divisor of n , and equality holds if and only if $d = p$.

(2) If $P \in \mathcal{P}$, then $|P| < 3^n$, and when $n > 24$, then $|P| < 2^n$.

In particular, for $n \geq 12$, every $M \in \mathcal{W} \cup \mathcal{P}$ has order

$$|M| \leq 2(\lfloor n/2 \rfloor)!(n - \lfloor n/2 \rfloor)!$$

with equality if and only if $M \in \mathcal{W}_{n/2}$.

Proof. (1) follows from Lemma 4, and (2) is [10, Corollary 1.2].

We now prove the last part of the lemma. Let first $M \in \mathcal{P}$. By (2), the order of M is bounded above by 3^n . Recall Stirling's bound $k! > e(k/e)^k$, which holds for every $k \geq 2$ and can be proved by noting that

$$\frac{k^k}{k!} = \frac{k^{k-1}}{(k-1)!} = \prod_{i=1}^{k-1} \left(\frac{i+1}{i} \right)^i < \prod_{i=1}^{k-1} (e^{1/i})^i = e^{k-1}.$$

We deduce that

$$2(\lfloor n/2 \rfloor)!(n - \lfloor n/2 \rfloor)! > g(n) = \begin{cases} 2e^2 \left(\frac{n}{2e} \right)^n & \text{if } n \text{ is even,} \\ e(n+1) \left(\frac{n-1}{2e} \right)^{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Computation shows that, for every $n \geq 14$, the function $g(n)$ is at least 3^n . The cases $n = 12$ and $n = 13$ can be done by inspection, and the proof for the primitive case is completed.

Assume now that $M \in \mathcal{W}$; in particular, m is not a prime number. Then, by (1), we have $|M| \leq |W_{n/p}|$, where p is the smallest prime number that divides n . The result is therefore trivial if n is even. Let n be odd. We need to prove that

$$R = \frac{(n+1)((n-1)/2)!)^2}{(n/p)!^p p!} > 1.$$

Observe that, since $p^2 \leq n$, the map $x \mapsto x^{n-1}/n^x$ is an increasing function in the interval $1 \leq x \leq p$ so that $p^{n-1}/n^p \geq 3^{n-1}/n^3$. Using the inequalities $e(n/e)^n \leq n! \leq en(n/e)^n$, we see that

$$\begin{aligned} R &\geq \frac{e^2(n+1)\left(\frac{n-1}{2e}\right)^{n-1}}{epn^p\left(\frac{n}{ep}\right)^n} = 2e^2 \cdot \frac{n+1}{n-1} \cdot \frac{p^{n-1}}{n^p} \cdot \left(\frac{n-1}{2n}\right)^n \\ &> 2e^2 \cdot \frac{3^{n-1}}{n^3} \cdot \left(\frac{n-1}{2n}\right)^n = \frac{2e^2}{3n^3} \left(\frac{3(n-1)}{2n}\right)^n \geq 1 \end{aligned}$$

whenever $n \geq 22$. The case $12 \leq n \leq 21$ can be done by inspection. □

In the sequel, we will need the following result.

Lemma 6. *Let n be even and $W \in \mathcal{W}_{n/2}$. Assume that $n = \sum_{i=1}^k 2^{a_i}$ is a partition of n with $a_1 \geq a_2 \geq \dots \geq a_k \geq 1$ that does not contain subpartitions of $n/2$, and let Π be the conjugacy class of elements of S_n of type $(2^{a_1}, 2^{a_2}, \dots, 2^{a_k})$. Then*

$$|W \cap \Pi| = \frac{|\Pi|}{|S_n : W|} \cdot 2^{k-1}.$$

In particular, when n is a power of 2 and $\Pi = (n)$, the set of n -cycles, then

$$|W \cap \Pi| = \frac{|W|}{n} = (n/2)!(n/2 - 1)!.$$

Proof. Double counting the size of the set

$$\{(x, W) \mid x \in \Pi \cap W, W \in \mathcal{W}_{n/2}\},$$

we find that

$$|W \cap \Pi| = \frac{|\Pi|}{|G : W|} \cdot r,$$

where r is the number of elements of $\mathcal{W}_{n/2}$ containing a fixed element of Π . By the assumption that the partition defining Π does not contain partitions of $n/2$ as subpartitions, the elements of $\Pi \cap W$ move the two imprimitivity blocks of W ; hence $r = 2^{k-1}$ since no cycle can fix either block, and once we split up the elements in the 2^{a_1} -cycle, we have two choices for each 2^{a_j} -cycle, $2 \leq j \leq k$. \square

Moreover, we will make use of the following notation and terminology introduced in [11].

Definition 1. Let Π be a set of permutations of S_n . We will say that a collection $\mathcal{H} = \{H_1, \dots, H_m\}$ of m proper subgroups of S_n is *definitely unbeatable* on Π if the following three conditions hold:

- (1) $\Pi \subseteq \bigcup_{i=1}^m H_i$,
- (2) $\Pi \cap H_i \cap H_j = \emptyset$ for every $i \neq j$,
- (3) $|M \cap \Pi| \leq |H_i \cap \Pi|$ for every $1 \leq i \leq m$ and every proper subgroup M of S_n not belonging to \mathcal{H} .

If \mathcal{H} is definitely unbeatable on Π , then $|\mathcal{H}| = \sigma(\Pi)$, where $\sigma(\Pi)$ denotes the least integer m such that Π is a subset of the union of m proper subgroups of S_n . Moreover, we also say that \mathcal{H} is *strongly definitely unbeatable* on Π if the three conditions above hold and the third one always holds with strict inequalities. Note that, in the case when \mathcal{H} consists of maximal subgroups and it is strongly

definitely unbeatable on Π , then \mathcal{H} is the unique minimal cover of the elements of Π that uses only maximal subgroups (see also [14, Lemma 3.1]).

We start our considerations on the primary covering number of S_n by producing a general upper bound. Here and in the following, if p is any prime, we define n_p to be the p -part of n , that is the maximum power of p that divides n .

Lemma 7. *The following statements hold.*

- (1) *If n is a power of 2, then $\sigma_0(S_n) \leq 1 + \frac{1}{2} \binom{n}{n/2}$.*
- (2) *If n is not a power of 2, then $\sigma_0(S_n) \leq 1 + \binom{n}{n_2}$.*

Proof. Note that, in any case, the alternating groups A_n contain every permutation of odd order; therefore, in order to exhibit a primary covering for S_n , we may add to $\{A_n\}$ those subgroups that contain 2-elements (that are odd permutations).

(1) Assume $n = 2^a$ with $a \geq 2$. Then every 2-element of S_n stabilizes a 2-block partition of Ω , and therefore, every 2-element belongs to an imprimitive maximal subgroup of type $\mathcal{W}_{n/2}$. Since the number of such partitions (and subgroups) is $\frac{1}{2} \binom{n}{n/2}$, the lemma is proved in this case.

(2) The number of subsets of order n_2 of the set $\Omega = \{1, 2, \dots, n\}$ is $\binom{n}{n_2}$ which is an odd number (see for instance [7, Lemma 1.8]), and it is easy to prove by induction that every 2-element of S_n belongs to some stabilizer S_Δ with Δ a subset of Ω of cardinality n_2 ; in other words, if we write n as a sum of distinct powers of 2, then there exists a subsum that equals n_2 . Since the number of these stabilizers is exactly $\binom{n}{n_2}$, this point is also proved. \square

We already have enough ingredients to complete the proof in the case $n = 2^a$, with $a \geq 2$.

Proposition 1. *If $n = 2^a \geq 4$, then $\sigma_0(S_n) = 1 + \frac{1}{2} \binom{n}{n/2}$, and a minimal primary covering is given by $\{A_n\} \cup \mathcal{W}_{n/2}$.*

Proof. A direct inspection shows that $\sigma_0(S_4) = 4 = 1 + \frac{1}{2} \binom{4}{2}$. Thus, in the following, we assume $a > 2$.

By Lemma 7, we know that $\sigma_0(S_n) \leq 1 + \frac{1}{2} \binom{n}{n/2}$. Assume that H is a maximal subgroup of S_n . Then either $H \cap (n) = \emptyset$ or H is an imprimitive subgroup or, by [8, Theorem 3], the subgroup H satisfies $\text{PGL}_d(q) \leq H \leq \text{P}\Gamma\text{L}_d(q)$, where $n = (q^d - 1)/(q - 1)$ for some $d \geq 2$. Note that, in this last case, we necessarily have $d = 2$ and $q = 2^a - 1$ a Mersenne prime. To see this, observe that, from the equality $2^a = (q^d - 1)/(q - 1)$, one easily deduces that $d = 2$ so that $2^a = q + 1$; now q cannot be a square since $2^a - 1 \equiv 3 \pmod{4}$, and if $q = p^m$

is an odd power of the prime p , the usual factorization of

$$x^m + 1 = (x + 1)(x^{m-1} - x^{m-2} + \cdots + 1)$$

implies that q must be a prime. By Lemma 6 and the fact that the elements of order n in $\text{PGL}(2, q)$ are in number of $2^{a-2}(2^a - 1)(2^a - 2)$ (use [6, II, Satz 7.3]), we deduce in any case that

$$|H \cap (n)| \leq (n/2)! (n/2 - 1)!$$

with equality if and only if $H \in \mathcal{W}_{n/2}$. This shows that the set $\mathcal{W}_{n/2}$ is strongly definitely unbeatable on $\Pi = (n)$, and therefore, we obtain that

$$\sigma_0(S_n) \geq \frac{1}{2} \binom{n}{n/2}.$$

To complete this case, assume that $\sigma_0(S_n) = \frac{1}{2} \binom{n}{n/2}$. Then, by the above, the collection $\mathcal{W}_{n/2}$ must be the unique minimal primary covering for S_n . By Bertrand's postulate, there is a prime number p between $n/2$ and n ; we reach a contradiction by noting that p -cycles do not belong to imprimitive subgroups of type $\mathcal{W}_{n/2}$. \square

We assume now that $n \notin \{3^\varepsilon \cdot 2^a \mid \varepsilon \in \{0, 1\}, a \geq 0\}$. We deal separately with the case $n = 5$.

Lemma 8. *For $n = 5$, we have that $\sigma_0(S_5) = 1 + \binom{5}{1} = 6$ and $\{A_5\} \cup \mathcal{X}_1$ is the unique minimal primary covering of S_5 .*

Proof. We already know that $\{A_5\} \cup \mathcal{X}_1$ is a primary covering for S_5 , and therefore, $\sigma_0(S_5) \leq 6$. Assume by contradiction that \mathcal{C} is a primary covering of smaller cardinality. Inside S_5 , there are six subgroups of order 5; therefore, as $|\mathcal{C}| \leq 5$, there exists one element of \mathcal{C} containing at least two different Sylow 5-subgroups. But the only proper subgroup of S_5 containing more than one subgroup of order 5 is A_5 . Thus $A_5 \in \mathcal{C}$, and the remaining members of \mathcal{C} cover all of the odd 2-elements of S_5 , which are thirty 4-cycles and ten 2-cycles. Any maximal subgroup isomorphic to S_4 contains precisely six 4-cycles and six 2-cycles, any $S_3 \times S_2$ contains no 4-cycles and four 2-cycles and any Frobenius group $5 : 4$ contains ten 4-cycles and no 2-cycles. Therefore, if we assume that \mathcal{C} contains respectively a_1 subgroups in \mathcal{X}_1 (that is isomorphic to S_4), a_2 subgroups in \mathcal{X}_2 (that is isomorphic to $S_3 \times S_2$) and a_3 primitive subgroups isomorphic to $5 : 4$, we obtain the following system of Diophantine inequalities:

$$\begin{cases} a_1 + a_2 + a_3 \leq 4, \\ 3a_1 + 5a_3 \geq 15, \\ 6a_1 + 4a_2 \geq 10. \end{cases}$$

The only integer solution of this system is $(a_1, a_2, a_3) = (2, 0, 2)$, but then, if $\text{Stab}_{S_5}(i)$ and $\text{Stab}_{S_5}(j)$ are the two elements of \mathcal{X}_1 in \mathcal{C} , we have that the permutation (ij) is not covered by elements of \mathcal{C} , which is a contradiction. \square

Let $n \geq 7$ and $n \notin \{3^\varepsilon \cdot 2^a \mid \varepsilon \in \{0, 1\}, a \geq 0\}$, and write the 2-adic expansion of n as

$$n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_t},$$

where $a_1 > a_2 > \cdots > a_t \geq 0$ and $t \geq 2$. Note that $n_2 = 2^{a_t}$ and also that, when $t = 2$, then $a_1 \geq a_2 + 2$.

We define Π to be the following conjugacy class of permutations:

$$\Pi = \begin{cases} (2^{a_1}, 2^{a_2}, \dots, 2^{a_t}) & \text{if } n \not\equiv t \pmod{2}, \\ (2^{a_1-1}, 2^{a_1-1}, 2^{a_2}, \dots, 2^{a_t}) & \text{if } n \equiv t \pmod{2}. \end{cases}$$

The set Π consists of odd permutations, that is $A_n \cap \Pi = \emptyset$.

The computation of $\sigma_0(S_n)$ in this case depends on the following proposition.

Proposition 2. *Assume $n \notin \{3^\varepsilon \cdot 2^a \mid \varepsilon \in \{0, 1\}, a \geq 1\}$. If n is odd and $n \geq 15$ or if n is even, $n \geq 22$ and $n \neq 40$, the collection of subgroups \mathcal{X}_{n_2} is strongly definitely unbeatable on Π .*

For the proof of Proposition 2, we need the following number-theoretic result.

Lemma 9. *Using the above assumptions and notation, define*

$$s = \begin{cases} \sum_{i=1}^t a_i + a_1 - 1 & \text{if } n \equiv t \pmod{2} \\ \sum_{i=1}^t a_i & \text{if } n \not\equiv t \pmod{2} \end{cases}$$

and

$$f(n) = 2^{s+1} \frac{\binom{n}{n_2}}{\binom{n}{\lfloor n/2 \rfloor}}.$$

Then $f(n) < 1$ holds if and only if either n is odd and $n \geq 15$ or n is even, $n \geq 22$ and $n \neq 40$.

Proof. Since $a_i \leq a_1 - (i - 1)$ for every $i = 1, 2, \dots, t$, we have

$$s \leq \sum_{i=1}^t a_i + a_1 - 1 \leq ta_1 + a_t - 1 - \frac{(t-2)(t-1)}{2}.$$

The coefficient a_t satisfies $a_t \leq a_1 + 1 - t$, and therefore,

$$s \leq \begin{cases} ta_1 & \text{if } n \text{ is odd,} \\ (t+1)a_1 - 2 & \text{if } n \text{ is even,} \end{cases}$$

where we used the fact that $(t^2 - 3t + 2)/2 \geq 0$ and $(t^2 - t + 2)/2 \geq 2$ for every $t \geq 2$. Set $l = \log_2(n) \geq a_1$. Since $t \leq a_1 - a_t + 1$ and according to our assumptions $n \geq 15$ if n is odd and $n \geq 22$ if n is even, we have that $t \leq l + 1$ if n is odd and $t \leq l$ if n is even. Thus we obtain

$$s \leq \begin{cases} l^2 + l & \text{if } n \text{ is odd,} \\ l^2 + l - 2 & \text{if } n \text{ is even.} \end{cases} \quad (3.1)$$

By considering the binomial expansion of $(1+1)^n$, we have that $\binom{n}{\lfloor n/2 \rfloor} \geq \frac{2^n}{n+1}$, that is, since $l \leq \log_2(n+1) \leq l+1$,

$$\binom{n}{\lfloor n/2 \rfloor} \geq 2^{n-(l+1)}. \quad (3.2)$$

We distinguish now the different cases.

Case n odd. Then $\binom{n}{n_2} = n = 2^l$ and (3.1) and (3.2) imply that

$$f(n) \leq 2^{l^2+3l+2-n}.$$

We have that $l^2 + 3l + 2 - n < 0$ for every $n \geq 53$. The cases $15 \leq n \leq 51$ can be checked by direct computation. (Note that $f(n) > 1$ for $5 \leq n \leq 13$).

Case n even. Assume first that $t \geq 4$. Then

$$n \geq 2^{a_t}(1 + 2 + \dots + 2^{t-1}) = n_2(2^t - 1) \geq 15n_2;$$

hence $n_2 \leq \lfloor n/15 \rfloor$ and

$$\binom{n}{n_2} \leq \binom{n}{\lfloor n/15 \rfloor}.$$

Note that if k is any non-negative integer, then $k^k/k! < e^k$ by the well-known series expansion of the exponential function; therefore, if $k \leq n$, then

$$\binom{n}{k} = \frac{n}{k} \cdot \frac{n-1}{k} \cdot \dots \cdot \frac{n-k+2}{k} \cdot \frac{n-k+1}{k} \cdot \frac{k^k}{k!} < \left(\frac{ne}{k}\right)^k.$$

Using the fact that, for every real number x , we have $x - 1 < \lfloor x \rfloor \leq x$, we deduce that

$$\binom{n}{n_2} \leq \left(\frac{15en}{n-15}\right)^{n/15} = 2^{\frac{n}{15} \log_2(\frac{15en}{n-15})}. \quad (3.3)$$

Combining (3.1), (3.2) and (3.3), we have that

$$f(n) \leq 2^{l^2+2l-n(1-\frac{1}{15}\log_2(\frac{15en}{n-15}))}.$$

Note that $f(n) < 1$ if $l^2 + 2l - n(1 - \frac{1}{15}\log_2(\frac{15en}{n-15})) < 0$, which is true for every $n \geq 88$. The cases $22 \leq n \leq 86$, with $t \geq 4$, can be checked by a direct computation.

Let now $t = 2$ or $t = 3$. In both cases, we have that the value of s is bounded from above by $3l - 3$ because, when $t = 2$, then $a_1 \geq a_2 + 2$, and so

$$s = 2a_1 + a_2 - 1 \leq 3a_1 - 3 \leq 3l - 3,$$

and when $t = 3$, then

$$s = a_1 + a_2 + a_3 \leq a_1 + a_1 - 1 + a_1 - 2 \leq 3l - 3.$$

Moreover, when $t = 2$, since $a_1 \geq a_2 + 2$, we have that $n_2 \leq \lfloor n/5 \rfloor$. Therefore, by considerations analogous to the ones above, we have that $f(n) < 1$ if

$$4l - 1 - n\left(1 - \frac{1}{7}\log_2\left(\frac{7en}{n-7}\right)\right) < 0 \quad \text{when } t = 3, \quad (3.4)$$

$$4l - 1 - n\left(1 - \frac{1}{5}\log_2\left(\frac{5en}{n-5}\right)\right) < 0 \quad \text{when } t = 2. \quad (3.5)$$

Now, (3.4) holds for every $n \geq 62$, while (3.5) holds for every $n \geq 114$.

As above, the intermediate cases can be checked by computation, the only exceptions being all n even with $n \leq 20$ and $n = 40$. \square

Proof of Proposition 2. To prove that \mathcal{X}_{n_2} is strongly definitely unbeatable on Π , we need to show that the three conditions of Definition 1 are satisfied. Conditions (1) and (2) are straightforward (note that, when $t = 2$, we have that $a_1 - 1 > a_2$). We show condition (3), that is, for every $X_{n_2} \in \mathcal{X}_{n_2}$ and every maximal subgroup M of S_n , which is not the stabilizer of a n_2 -subset, the proportion

$$c(M) := \frac{|M \cap \Pi|}{|X_{n_2} \cap \Pi|} = \frac{|M|}{|X_{n_2}|} m_M < 1,$$

where m_M denotes the number of conjugates of M containing a fixed element of Π . The above expression for $c(M)$ was obtained via a double counting, as in the proof of Proposition 1, applied to compute both $|M \cap \Pi|$ and $|X_{n_2} \cap \Pi|$, keeping in mind that a given element of Π belongs to exactly one member of \mathcal{X}_{n_2} .

Assume first that M is an intransitive maximal subgroup not in \mathcal{X}_{n_2} . If we have $M \cap \Pi \neq \emptyset$, then M is necessarily the stabilizer of a union of disjoint 2-power

sized subsets, say A_i for some $i \geq 1$. Since $|M \cap \Pi| \leq |\text{Stab}(A_1) \cap \Pi|$, we can assume that M coincides with the stabilizer of a single subset of cardinality some power of 2, say 2^c , with $n_2 < 2^c < n/2$, since $2^{a_1} \geq n/2$, and therefore,

$$c \in \begin{cases} \{a_2, \dots, a_{t-1}\} & \text{when } n \not\equiv t \pmod{2}, \\ \{a_1 - 1, a_2, \dots, a_{t-1}\} & \text{when } n \equiv t \pmod{2}. \end{cases}$$

Now, except in the case $n \equiv t \pmod{2}$ and $c = a_1 - 1$, we have that $m_M = 1$ and

$$c(M) = \frac{\binom{n}{n_2}}{\binom{n}{2^c}} < 1$$

since $n_2 = 2^{a_t} < 2^c$. Otherwise, $m_M = 2$, and since $a_t \leq a_1 - 2$, we obtain

$$c(M) = 2 \cdot \frac{\binom{n}{2^{a_t}}}{\binom{n}{2^{a_1-1}}} \leq 2 \cdot \frac{\binom{n}{2^{a_1-2}}}{\binom{n}{2^{a_1-1}}} < 1.$$

Assume now that M is a primitive or an imprimitive maximal subgroup of S_n . Then, by Lemma 5,

$$|M \cap \Pi| \leq |M| \leq \frac{2n!}{\binom{n}{\lfloor n/2 \rfloor}},$$

while

$$|X_{n_2} \cap \Pi| = \frac{n_2!(n - n_2)!}{2^s},$$

where $s = \sum_{i=1}^t a_i$ if $n \not\equiv t \pmod{2}$ and $s = \sum_{i=1}^t a_i + a_1 - 1$ if $n \equiv t \pmod{2}$. Therefore,

$$c(M) = \frac{|M \cap \Pi|}{|X_{n_2} \cap \Pi|} \leq \frac{|M|}{|X_{n_2} \cap \Pi|} \leq 2^{s+1} \cdot \frac{\binom{n}{n_2}}{\binom{n}{\lfloor n/2 \rfloor}}$$

Lemma 9 proves that $c(M) < 1$ whenever n is odd and $n \geq 15$, or n is even and $n \geq 22$ and $n \neq 40$. \square

We now treat the case when $n \leq 20$ or $n = 40$.

Proposition 3. *For $n \in \{5, 7, 9, 10, 11, 13, 14, 18, 20, 40\}$, the collection of subgroups \mathcal{X}_{n_2} is strongly definitely unbeatable on Π if and only if $n \notin \{5, 10\}$.*

Proof. As in the proof of Proposition 2, conditions (1) and (2) of Definition 1 are straightforward to prove, so we limit ourselves to show that condition (3) holds. In the sequel, denote by M a maximal subgroup of S_n not in $\mathcal{X}_{n_2} \cup \{A_n\}$.

If M is intransitive, the inequality $|M \cap \Pi| \leq |X_{n_2} \cap \Pi|$ holds in all cases, with equality if and only if $M \in \mathcal{X}_{n_2}$ (see the proof of Proposition 2). So we may concentrate on the imprimitive and primitive maximal subgroups. We treat the various cases separately.

Case $n = 5$. Then $\Pi = (4, 1)$, and if we take M to be a primitive subgroup isomorphic to the Frobenius group $5 : 4$, we have that

$$|M \cap \Pi| = 10 > 6 = |X_{n_2} \cap \Pi|.$$

Therefore, \mathcal{X}_{n_2} is not unbeatable on Π .

Case $n = 7$. Then $\Pi = (2, 2, 2, 1)$ and $|X_{n_2} \cap \Pi| = 15$. There is only one class of transitive (primitive) maximal subgroups of $G = S_7$ (not containing A_7). They have order 42, and their intersection with Π has size 7.

Case $n = 9$. Then $\Pi = (8, 1)$ and $|X_{n_2} \cap \Pi| = 7!$, which is larger than 432, the maximal size of a primitive subgroup of S_9 not containing A_9 . Since subgroups in \mathcal{W}_3 have trivial intersection with Π , we have that \mathcal{X}_{n_2} is unbeatable on Π .

Case $n = 10$. Then $\Pi = (4, 4, 2)$, and if $M \in \mathcal{W}_5$, we have, by Lemma 6,

$$|M \cap \Pi| = 1\,800 > 1\,260 = |X_{n_2} \cap \Pi|,$$

which shows that \mathcal{X}_{n_2} is not unbeatable on Π .

Case $n = 11$. We have $\Pi = (4, 4, 2, 1)$ and $|X_{n_2} \cap \Pi| = 56\,700$, which is larger than 110, the maximal size of a primitive subgroup of S_{11} not containing A_{11} .

Case $n = 13$. Then $\Pi = (4, 4, 4, 1)$ and $|X_{n_2} \cap \Pi| \geq 1.24 \cdot 10^6$. This number is larger than the maximal size of a primitive subgroup of S_{13} not containing A_{13} , which is 156.

Case $n = 14$. Then $\Pi = (8, 4, 2)$ and $|X_{n_2} \cap \Pi| \geq 1.4 \cdot 10^7$. If M is a primitive maximal subgroup of S_{14} not containing A_{14} , then

$$|M \cap \Pi| < |M| < 3^{14} < |X_{n_2} \cap \Pi|.$$

Assume that M is imprimitive. If $M \in \mathcal{W}_2$, then

$$|M \cap \Pi| \leq |M| = 2^7 \cdot 7! = 645\,120 < |X_{n_2} \cap \Pi|.$$

If $M \in \mathcal{W}_7$ then by Lemma 6,

$$|M \cap \Pi| = \frac{|\Pi|}{|G : M|} \cdot 2^2 = 3\,175\,200 < |X_{n_2} \cap \Pi|.$$

Case $n = 18$. Then $\Pi = (8, 8, 2)$ and $|X_{n_2} \cap \Pi| = \frac{16!}{2^7} \geq 1.63 \cdot 10^{11}$. If M is a primitive maximal subgroup of S_{18} not containing A_{18} , then

$$|M \cap \Pi| < |M| < 3^{18} = 387\,420\,489 < |X_{n_2} \cap \Pi|.$$

If M is an imprimitive maximal subgroup of S_{18} , not in \mathcal{W}_9 , then

$$|M \cap \Pi| < |M| \leq (6!)^3 \cdot 3! \leq 2.24 \cdot 10^9 < |X_{n_2} \cap \Pi|.$$

If $M \in \mathcal{W}_9$, then, by Lemma 6,

$$|M \cap \Pi| = \frac{|\Pi|}{|G : M|} 2^2 \leq 4.12 \cdot 10^9 < |X_{n_2} \cap \Pi|.$$

Case $n = 20$. Then $\Pi = (8, 8, 4)$ and $|X_{n_2} \cap \Pi| = \frac{16!}{2^7} \cdot 3! \geq 9.8 \cdot 10^{11}$. If M is a primitive maximal subgroup of S_{20} not containing A_{20} , then

$$|M \cap \Pi| \leq 3^{20} \leq 3.49 \cdot 10^9 < |X_{n_2} \cap \Pi|.$$

If M is an imprimitive maximal subgroup of G , not in \mathcal{W}_{10} , then

$$|M \cap \Pi| \leq |M| \leq |S_5 \wr S_4| = (5!)^4 4! \leq 4.98 \cdot 10^9 < |X_{n_2} \cap \Pi|.$$

If $M \in \mathcal{W}_{10}$, then, by Lemma 6,

$$|M \cap \Pi| = \frac{|\Pi|}{|G : M|} 2^2 \leq 2.06 \cdot 10^{11} < |X_{n_2} \cap \Pi|.$$

Case $n = 40$. Then $\Pi = (16, 16, 8)$ and $|X_{n_2} \cap \Pi| = \frac{32!}{2^9} \cdot 7! \geq 2.59 \cdot 10^{36}$. If M is a primitive maximal subgroup of S_{40} not containing A_{40} , then

$$|M \cap \Pi| < |M| \leq 2^{40} \approx 1.10 \cdot 10^{12} < |X_{n_2} \cap \Pi|.$$

If M is an imprimitive maximal subgroup of G , not in \mathcal{W}_{20} , then

$$|M \cap \Pi| < |M| \leq |S_{10} \wr S_4| = 10!^4 \cdot 4! \leq 4.17 \cdot 10^{27} < |X_{n_2} \cap \Pi|.$$

If $M \in \mathcal{W}_{20}$, then, by Lemma 6,

$$|M \cap \Pi| = \frac{|\Pi|}{|G : M|} 2^2 \leq 1.16 \cdot 10^{34} < |X_{n_2} \cap \Pi|.$$

The proof is now complete. □

Lemma 10. For $n = 10$, we have that $\sigma_0(S_{10}) = 46$.

Proof. We already know that $\{A_{10}\} \cup \mathcal{X}_2$ is a primary covering, and therefore, $\sigma_0(S_{10}) \leq 46$. Assume by contradiction that \mathcal{C} is a primary covering of smaller cardinality. A_{10} belongs to \mathcal{C} because the maximal intersection of a maximal subgroup of S_{10} distinct from A_{10} with the set of the 72 576 elements of cycle structure $(5, 5)$ is 576, realized by the class of subgroups \mathcal{W}_5 , and

$$\frac{72\,576}{576} = 126 > 46 \geq \sigma_0(S_{10}).$$

To conclude, it is enough to show that at least 45 maximal subgroups are needed to cover the elements of cycle type $(4, 4, 2)$. We were able to do this using the programs [16, 17]. More specifically, this works as follows. Let \mathcal{M} be the set of maximal subgroups of $G = S_{10}$, and let C be the conjugacy class of elements of cycle structure $(4, 4, 2)$ in G . Let r_M be a variable for every $M \in \mathcal{M}$, and for every $x \in C$, define $\mathcal{M}_x := \{M \in \mathcal{M} : x \in M\}$. We found that

$$\min \left\{ \sum_{M \in \mathcal{M}} r_M \mid r_M \in \{0, 1\} \text{ for all } M \in \mathcal{M}, \sum_{M \in \mathcal{M}_x} r_M \geq 1 \text{ for all } x \in C \right\} = 45.$$

For this, [16] was used to compute the sets \mathcal{M}_x , and [17] was used to solve the optimization problem. \square

We can now complete this case.

Proposition 4. Assume that $n \neq 10$ and $n \neq 3^\varepsilon 2^a$ for every $\varepsilon \in \{0, 1\}$ and every $a \geq 0$. Then $\sigma_0(S_n) = 1 + \binom{n}{n_2}$.

Proof. The case $n = 5$ has been done in Lemma 8. By Propositions 2 and 3, we have that

$$\sigma_0(S_n) \geq \sigma(\Pi) = |\mathcal{X}_{n_2}| = \binom{n}{n_2}.$$

Moreover, if we had $\sigma_0(S_n) = \binom{n}{n_2}$, then, by the strongly definitely unbeatable property, \mathcal{X}_{n_2} would be a primary covering for S_n , which is impossible by [3, Theorem 1], or simply because this collection does not cover the primary elements acting fixed-point-freely. Therefore, $\sigma_0(S_n) > \binom{n}{n_2}$, and then Lemma 7 completes the proof. Note that, in this situation, a minimal primary covering is given by $\{A_n\} \cup \mathcal{X}_{n_2}$. \square

Finally, assume now that $n = 3 \cdot 2^a$ for some $a \geq 0$. The case $n = 3$ is trivial; thus assume that $a \geq 1$. We first deal with the case $n = 6$.

Lemma 11. *The primary covering number for S_6 is 7. A minimal primary covering is (a conjugate to) the following:*

$$\mathcal{C} = \{A_6, X_1, X_1^{(12)}, X_1^{(13)}, P_1, P_1^{(34)}, P_1^{(35)}\},$$

where $X_1 = \text{Stab}_{S_6}(\{1\}) \in \mathcal{X}_1$ and $P_1 = \langle (3465), (123)(456) \rangle$ belongs to the family \mathcal{P} of primitive maximal subgroups isomorphic to S_5 .

Proof. A direct check with GAP shows that \mathcal{C} is a covering for the set of primary elements of S_6 . Assume by contradiction that \mathcal{D} is a primary covering (consisting of maximal subgroups of G and) containing less than seven elements.

We first show that $A_6 \in \mathcal{D}$. If this is not the case, then the class Π_0 of 5-cycles should be covered by at most six maximal subgroups, which are either 1-point stabilizers, that is elements of \mathcal{X}_1 , or primitive maximal subgroups, that is elements of \mathcal{P} , and in both cases, they are all isomorphic to S_5 . Note that $|\Pi_0| = 6 \cdot 24$ and that, for every $S_1 \neq S_2 \in \mathcal{X}_1$ and every $P_1 \neq P_2 \in \mathcal{P}$, we have

- $|\Pi_0 \cap S_1| = |\Pi_0 \cap P_1| = 24$,
- $|\Pi_0 \cap S_1 \cap S_2| = |\Pi_0 \cap P_1 \cap P_2| = 0$,
- $|\Pi_0 \cap S_1 \cap P_1| = 4$

(the second equation is trivial for the 1-point stabilizers, and it holds for the members of \mathcal{P} too, since there is an outer involutory automorphism of S_6 that interchanges 1-point stabilizers with the members of \mathcal{P}).

By applying an inclusion/exclusion argument, there are only two ways to cover Π_0 with no more than six of these proper subgroups, either using all of the six elements of \mathcal{X}_1 , or all of the elements of \mathcal{P} . It follows that \mathcal{D} is either \mathcal{X}_1 or \mathcal{P} . In both cases, we have a contradiction since \mathcal{X}_1 does not cover the 2-elements of type $\Pi_1 = (2, 2, 2)$, while \mathcal{P} does not cover the 2-cycles. We proved therefore that $A_6 \in \mathcal{D}$.

We set $\mathcal{D}_1 = \mathcal{D} \setminus \{A_6\}$. The collection \mathcal{D}_1 consists of at most five subgroups, which should cover the set of odd 2-elements, that is the set $\Pi_1 \cup \Pi_2 \cup \Pi_3$, where $\Pi_1 = (2, 2, 2)$, $\Pi_2 = (4, 1, 1)$ and $\Pi_3 = (2, 1, 1, 1, 1)$. The following table shows the sizes of the intersections of these classes with the maximal subgroups (different from A_6).

Π_i	$ \Pi_i $	$ \Pi_i \cap \mathcal{X}_1 $	$ \Pi_i \cap \mathcal{X}_2 $	$ \Pi_i \cap \mathcal{W}_3 $	$ \Pi_i \cap \mathcal{W}_2 $	$ \Pi_i \cap \mathcal{P} $
$(2, 2, 2)$	15	0	3	6	7	10
$(4, 1, 1)$	90	30	6	0	6	30
$(2, 1, 1, 1, 1)$	15	10	7	6	3	0

We claim that, in order to cover the class Π_2 of 4-cycles, we need to take either at least three different elements of \mathcal{X}_1 or at least three different elements of \mathcal{P} . This comes from the fact that, for every $S_1 \neq S_2 \in \mathcal{X}_1$ and every $P_1 \neq P_2 \in \mathcal{P}$, the following holds:

- $|\Pi_2 \cap S_1 \cap S_2| = |\Pi_2 \cap P_1 \cap P_2| = 6$,
- $|\Pi_2 \cap S_1 \cap P_1| = 10$,
- $|\Pi_2 \cap S_1 \cap S_2 \cap P_1| = |\Pi_2 \cap S_1 \cap P_1 \cap P_2| = 2$,
- $|\Pi_2 \cap S_1 \cap S_2 \cap P_1 \cap P_2| \leq 2$;

hence

$$|\Pi_2 \cap (S_1 \cup S_2 \cup P_1 \cup P_2)| \leq 76.$$

Assume that \mathcal{D}_1 contains three different elements of \mathcal{P} ; then, by looking at the last line of the table, the class Π_3 should be covered using just two different subgroups, say $A, B \in \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{W}_3$. This is impossible since,

- if $A, B \in \mathcal{X}_1$, then $|\Pi_3 \cap A \cap B| = 6$,
- if $A \in \mathcal{X}_1$ and $B \in \mathcal{X}_2$, then $|\Pi_3 \cap A \cap B| \geq 4$,
- if $A \in \mathcal{X}_1$ and $B \in \mathcal{W}_3$, then $|\Pi_3 \cap A \cap B| = 4$.

The opposite case when \mathcal{D}_1 contains three different elements of \mathcal{X}_1 follows immediately by using the duality of the outer automorphism of order two of S_6 (or with similar arguments applied to the first line of the table). \square

Lemma 12. *Let a_1, \dots, a_r, a be integers with $0 \leq a_1 \leq \dots \leq a_r$, $r \geq 2$ and $a \geq 1$.*

(1) *If $\sum_{i=1}^r 2^{a_i} = 2^a$, then there exists a subset $J \subseteq \{1, \dots, r\}$ such that*

$$\sum_{i \in J} 2^{a_i} = 2^{a-1}.$$

(2) *If $\sum_{i=1}^r 2^{a_i} = 3 \cdot 2^a$, then one of the following occurs.*

- $r = 2, a_1 = a, a_2 = a + 1$.
- $r = 3, a_1 = a_2 = a_3 = a$.
- *There exist two disjoint subsets $J_1, J_2 \subseteq \{1, \dots, r\}$ such that*

$$\sum_{i \in J_1} 2^{a_i} = \sum_{i \in J_2} 2^{a_i} = 2^{a-1}.$$

Proof. (1) Assume that what we want to prove is false, and let r be minimal with the property that

$$\sum_{i=1}^r 2^{a_i} = 2^a$$

and there is no $J \subseteq \{1, \dots, r\}$ with $\sum_{i \in J} 2^{a_i} = 2^{a-1}$. This implies $r \geq 3$. Note that, since twice a power of 2 is a power of 2, we have $a_1 < \dots < a_r \leq a-2$; hence

$$2^a = \sum_{i=1}^r 2^{a_i} \leq \sum_{i=0}^{a-2} 2^i \leq 2^{a-1} - 1,$$

a contradiction.

(2) Assume that $\sum_{i=1}^r 2^{a_i} = 3 \cdot 2^a$. By item (1), to conclude, it is enough to show that there exists $J \subseteq \{1, \dots, r\}$ such that $\sum_{i \in J} 2^{a_i} = 2^a$, so suppose this is not the case, by contradiction. As for item (1), we may assume that $a_1 < \dots < a_r$. We know that $a_r \leq a+1$ and $a_{r-1} \neq a$; therefore,

$$3 \cdot 2^a = \sum_{i=1}^r 2^{a_i} \leq \sum_{i=1}^{r-1} 2^{a_i} + 2^{a+1} \leq \sum_{i=0}^{a-1} 2^i + 2^{a+1} \leq 2^a - 1 + 2^{a+1},$$

a contradiction. □

Proposition 5. Let $n = 2^{a+1} + 2^a = 3 \cdot 2^a$, with $a \geq 2$. Then $c_1 \leq \sigma_0(S_n) \leq c_2$, where

$$c_1 = \begin{cases} 117 & \text{if } a = 2, \\ 1 + \binom{n-1}{2^{a-1}} & \text{if } a \geq 3, \end{cases}$$

$$c_2 = 2 + \binom{n-1}{2^a-1} + \sum_{i=2}^{2^{a+1}} \binom{n-i}{2^{a-1}-1}$$

Proof. To prove the upper bound, we consider the collection

$$\mathcal{C}_2 = \bigcup_{i=0}^m \mathcal{M}_i,$$

where $m = 2^{a+1} + 1$ and

$$\mathcal{M}_0 = \{A_n\},$$

$$\mathcal{M}_1 = \{\text{Stab}_{S_n}(U) \mid 1 \in U, |U| = 2^a\},$$

$$\mathcal{M}_i = \{\text{Stab}_{S_n}(V) \mid i \in V \subset \{i, \dots, n\}, |V| = 2^{a-1}\} \text{ for } i = 2, \dots, 2^{a+1},$$

$$\mathcal{M}_m = \{\text{Stab}_{S_n}(\{m, \dots, n\})\}.$$

The primary elements of odd order as well as the 2-elements that are even permutations are covered by A_n . Let g be an odd 2-element. If $g \in (2^a, 2^a, 2^a)$, then g is covered by a unique subgroup in \mathcal{M}_1 . Otherwise, since g is an odd permutation, by Lemma 12, there are at least two disjoint subsets of cardinality 2^{a-1} on which g acts. If one of these two contains the point 1, then again g is covered by a unique element of \mathcal{M}_1 ; otherwise, g lies in the stabilizer of a subset of size 2^{a-1} and not containing 1. In this case, g is covered by a subgroup that lies in $\bigcup_{i=2}^m \mathcal{M}_i$. Since $|\mathcal{C}_2| = c_2$, the upper bound is proved.

We prove now that c_1 is a lower bound for $\sigma_0(S_n)$. Assume first that $a = 2$, that is $n = 12$. Arguing in a similar way to the cases $n = 5, 6$ or 10 , it is straightforward to prove that the alternating group A_{12} belongs to every minimal primary covering. Now, the maximal subgroups that contain most elements of type $\Pi_1 = (4, 4, 4)$ are the ones in \mathcal{W}_6 , each of which, by Lemma 6 contains exactly $8|\Pi_1|(6!)^2/12!$ such elements. Since $12!/8(6!)^2 = 115.5$, we conclude that a minimal primary covering has at least 117 elements.

Let $a \geq 3$. We prove that c_1 is a lower bound for $\sigma_0(S_n)$ by showing that the collection \mathcal{M}_1 is definitely unbeatable on $\Pi_1 = (2^a, 2^a, 2^a)$.

Conditions (1) and (2) of Definition 1 follow immediately. Let us prove condition (3). Note that the only maximal subgroups M having nontrivial intersection with Π_1 are either stabilizers of a set of cardinality 2^a , or imprimitive, or proper primitive maximal subgroups, that is elements of $\mathcal{W} \cup \mathcal{P}$. For such M , we define $c(M) := |M \cap \Pi_1|/|M_1 \cap \Pi_1|$ for $M_1 \in \mathcal{M}_1$, and we will prove that $c(M) \leq 1$.

In the first case, we have that

$$|M \cap \Pi_1| = |M_1 \cap \Pi_1| = \frac{(2^a!)(2^{a+1}!)}{2^{3a+1}}$$

for every $M_1 \in \mathcal{M}_1$ and every $M = \text{Stab}_{S_n}(U)$, $1 \notin U$, $|U| = 2^a$. Therefore, $c(M) = 1$.

Assume $M \in \mathcal{W}_{n/2}$. Then, by Lemma 6,

$$|M \cap \Pi_1| = \frac{4|\Pi_1||W_{n/2}|}{n!} = \frac{4}{3 \cdot 2^{3a}}((n/2)!)^2,$$

and therefore,

$$\begin{aligned} c(M) &= \frac{8}{3} \cdot \frac{((n/2)!)^2}{(2^{a+1}!)(2^a!)} = \frac{8}{3} \cdot \frac{((2^a + 2^{a-1})!)^2}{(2^{a+1}!)(2^a!)} \\ &= \frac{8}{3} \cdot \frac{(3t!)^2}{(4t)!(2t)!} = \frac{8}{3} \cdot \frac{\binom{6t}{2t}}{\binom{6t}{3t}}, \end{aligned}$$

where $t = 2^{a-1}$. Therefore, $c(M) < 1$ for every $t \geq 4$, that is for every $a \geq 3$.

Assume $M \in \mathcal{W}_d$, with $d \neq n/2$. Then we have $|M \cap \Pi_1| < |M| \leq |W_{n/3}|$ by Lemma 5; therefore,

$$\begin{aligned} c(M) &< \frac{|W_{n/3}|}{|M_1 \cap \Pi_1|} = \frac{(n/3!)^3 \cdot 6 \cdot 2^{3a+1}}{(2^a!)(2^{a+1}!)} \\ &< \frac{2^{3a+4}}{\binom{2^{a+1}}{2^a}} \leq \frac{2^{3a+4}(2^{a+1} + 1)}{2^{2a+1}} < \frac{2^{4a+6}}{2^{2a+1}} \leq 1 \end{aligned}$$

for every $a \geq 4$. The case $a = 3$ can be checked directly.

Finally, assume that $M \in \mathcal{P}$. If $a \geq 3$, then $|M| \leq 2^n$ by [10]; hence

$$c(M) < \frac{|M|}{|M_1 \cap \Pi_1|} < \frac{2^{n+3a+1}}{(2^{a+1}!)(2^a!)} < 1. \quad \square$$

Acknowledgments. The authors are grateful to the referee for the careful reading of the paper and for completing the calculations of $\sigma_0(S_{10})$ using [16, 17].

Bibliography

- [1] J. H. E. Cohn, On n -sum groups, *Math. Scand.* **75** (1944), 44–58.
- [2] J. D. Dixon and B. Mortimer, *Permutation Groups*, Grad. Texts in Math. 163, Springer, New York, 1996.
- [3] B. Fein, W. M. Kantor and M. Schacher, Relative Brauer groups. II, *J. Reine Angew. Math.* **328** (1981), 39–57.
- [4] F. Fumagalli, On the indices of maximal subgroups and the normal primary coverings of finite groups, *J. Group Theory* **22** (2019), no. 6, 1015–1034.
- [5] W. Gaschütz, Die Eulersche Funktion endlicher auflösbarer Gruppen, *Illinois J. Math.* **3** (1959), 469–476.
- [6] B. Huppert, *Endliche Gruppen. I*, Grundlehren Math. Wiss. 134, Springer, Berlin, 1967.
- [7] I. M. Isaacs, *Finite Group Theory*, Grad. Stud. Math. 92, American Mathematical Society, Providence, 2008.
- [8] G. A. Jones, Cyclic regular subgroups of primitive permutation groups, *J. Group Theory* **5** (2002), no. 4, 403–407.
- [9] L.-C. Kappe, D. Nikolova-Popova and E. Swartz, On the covering number of small symmetric groups and some sporadic simple groups, *Groups Complex. Cryptol.* **8** (2016), no. 2, 135–154.
- [10] A. Maróti, On the orders of primitive groups, *J. Algebra* **258** (2002), no. 2, 631–640.

- [11] A. Maróti, Covering the symmetric groups with proper subgroups, *J. Combin. Theory Ser. A* **110** (2005), no. 1, 97–111.
- [12] R. Oppenheim and E. Swartz, On the covering number of S_{14} , *Involve* **12** (2019), no. 1, 89–96.
- [13] D. J. S. Robinson, *A Course in the Theory of Groups*, 2nd ed., Grad. Texts in Math. 80, Springer, New York, 1996.
- [14] E. Swartz, On the covering number of symmetric groups having degree divisible by six, *Discrete Math.* **339** (2016), no. 11, 2593–2604.
- [15] M. J. Tomkinson, Groups as the union of proper subgroups, *Math. Scand.* **81** (1997), no. 2, 191–198.
- [16] The GAP Group. GAP – Groups, Algorithms, and Programming, Version 4.7.5, 2014.
- [17] Gurobi Optimizer Reference Manual, Gurobi Optimization, Inc., 2014, <http://www.gurobi.com>.

Received April 13, 2020; revised April 15, 2021.

Author information

Francesco Fumagalli, Dipartimento di Matematica e Informatica “Ulisse Dini”,
Università degli Studi di Firenze, viale Morgagni 67/A, 50134 Firenze, Italy.
E-mail: francesco.fumagalli@unifi.it

Corresponding author:

Martino Garonzi, Departamento de Matemática, Universidade de Brasília,
Campus Universitário Darcy Ribeiro, Brasília – DF 70910-900, Brazil.
E-mail: mgaronzi@gmail.com