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Existence for fractional (p, q) systems with critical and Hardy terms in $\mathbb{R}^N$

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Abstract

This paper deals with the existence of entire nontrivial solutions for fractional (p, q) systems with critical Hardy terms in $\mathbb{R}^N$. Existence of solutions for system $(\mathcal{S})$ is derived via the mountain pass lemma, whose application is pretty involved and requires some variants of the concentration–compactness principle of Lions and of the concentration–compactness principle at infinity of Chabrowski for vectorial fractional spaces, here proved. Indeed, the ”triple loss of compactness” in $(\mathcal{S})$, caused by the simultaneous presence of Hardy and critical terms in the whole $\mathbb{R}^N$, forces to study the exact behavior of the PS sequences, in the spirit of Lions. We also present a simplified version $(\tilde{\mathcal{S}})$ of the main system $(\mathcal{S})$, which is anyway interesting, since the existence proof for $(\tilde{\mathcal{S}})$ is simpler and more direct. Finally, both existence theorems improve or complement previous results for fractional (p, q) scalar and vectorial problems.

Keywords: Fractional partial differential equations, Variational methods for elliptic systems, Second order elliptic systems, Critical exponents, Integro–differential operators

2010 MSC: 35R11, 35J50, 35J47, 35B33, 47G20

1. Introduction

Recently, great attention has been devoted to the study of fractional elliptic problems with lack of compactness. These models arise quite naturally in many different applications, and we refer to the extensive paper [14] and the references cited therein for further details. This paper is devoted to the study of existence of solutions for the following fractional (p, q) system in $\mathbb{R}^N$, involving critical and Hardy terms,

$$\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^t u + |u|^{p-2}u + |u|^{q-2}u - \sigma \frac{|u|^{q-2}u}{|x|^{tq}} = \lambda H_u(x, u, v) + \frac{\alpha}{q_t^*} |v|^\beta |u|^{\alpha-2}u, \\ (-\Delta)_p^s v + (-\Delta)_q^t v + |v|^{p-2}v + |v|^{q-2}v - \sigma \frac{|v|^{q-2}v}{|x|^{tq}} = \lambda H_v(x, u, v) + \frac{\beta}{q_t^*} |u|^\alpha |v|^{\beta-2}v, \end{cases} \quad (\mathcal{S})$$

where $0 < s \leq t < 1 < p < q$, $N \geq 2$, $tq < N$. The exponents $\alpha > 1$ and $\beta > 1$ are such that $\alpha + \beta = q_t^*$, where $q_t^* = Nq/(N - tq)$ is a critical exponent, while $\lambda > 0$ and σ are real parameters. Finally $(-\Delta)_\varphi^s$, $0 < s < 1 < \varphi$, is the fractional (s, φ) -Laplacian operator which, up to normalization factors, is defined pointwise along any $\varphi \in C_c^\infty(\mathbb{R}^N)$ as

$$(-\Delta)_\varphi^s \varphi(x) = 2 \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{\varphi-2} (\varphi(x) - \varphi(y))}{|x - y|^{N+s\varphi}} dy$$

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for $x \in \mathbb{R}^N$. When $s = t = 1$, system $(\mathcal{S})$ reduces to a typical (p, q) elliptic system, which has been extensively studied by many authors, cf., e.g. [12, 15, 17, 24]. These problems arise from general two-component reaction–diffusion problems of the form

$$\begin{cases} u_t = \operatorname{div}(D(u)\nabla u) + f(x, u, v), \\ v_t = \operatorname{div}(D(v)\nabla v) + g(x, u, v), \end{cases}$$

and have a wide range of applications in physical and related sciences, e.g. in biophysics, plasma physics and chemical reaction design. We refer to [12] for further details about the physical relevancy of such problems. Here $\mathbf{w} = (u, v)^T$ is a vector of concentration variables, $R(\mathbf{w}) = (f(x, u, v), g(x, u, v))^T$ describes a local reaction term related to source and loss processes, while $\mathbf{D}$ is a diagonal diffusion coefficient matrix

$$\mathbf{D} = \begin{pmatrix} D(u) & 0 \\ 0 & D(v) \end{pmatrix},$$

where $D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$ and $D(v) = |\nabla v|^{p-2} + |\nabla v|^{q-2}$ have a power law dependency. Actually, two-component systems allow for a much larger range of possible phenomena than their one-component counterparts. The hypothesis that just a difference in diffusion coefficients of components could be enough to destabilize the homogeneous solution was put forward by Turing in [32] by studying the problem of biological morphogenesis.

In the fractional scenario, (p, q) critical problems set in bounded domains can be found, e.g., in [6, 21]. On the other hand, only few recent works deal with fractional (p, q) Laplacian problems in the whole $\mathbb{R}^N$. For instance, in [4] the author proves an existence result for a critical fractional (p, q) equation via the mountain pass theorem. In [18] the authors study critical and possibly degenerate Kirchhoff fractional systems, while in [2] concentration and multiplicity of solutions for a class of fractional problems have been established. Here, we investigate for the first time the interaction between critical Sobolev–Hardy nonlinearities and fractional (p, q) operators in the whole $\mathbb{R}^N$.

We assume that H_u, H_v are partial derivatives of a Carathéodory function H satisfying the subcritical growth conditions

(H₁) *There exist $\mathbf{m}$ and m , with $p < \mathbf{m} < m < q_t^*$, such that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ for which the inequality*

$$|\nabla H(x, u, v)| \leq \mathbf{m}\varepsilon|(u, v)|^{\mathbf{m}-1} + mC_\varepsilon|(u, v)|^{m-1} \quad \text{for any } (u, v) \in \mathbb{R}^2$$

holds for a.e. $x \in \mathbb{R}^N$, where $|(u, v)| = \sqrt{u^2 + v^2}$ and $\nabla H = (H_u, H_v)$.

(H₂) *There exists $\theta \in (p, q_t^*)$ such that the inequality*

$$0 \leq \theta H(x, u, v) \leq \nabla H(x, u, v) \cdot (u, v) \quad \text{for all } (u, v) \in \mathbb{R}^2$$

holds for a.e. $x \in \mathbb{R}^N$.

(H₃) *For a.e. $x \in \mathbb{R}^N$ it results $H(x, \cdot, \cdot) \in C^1(\mathbb{R}^2)$, $H(x, \cdot, \cdot) \geq 0$ in $\mathbb{R}^2$, $H_u(x, \cdot, 0) = 0$ in $\mathbb{R}$ and $H_v(x, 0, \cdot) = 0$ in $\mathbb{R}$. Moreover, for all $(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$ the function $H(\cdot, u, v)$ is strictly positive a.e. in $\mathbb{R}^N$.*

We emphasize that the subcritical term is p -superlinear, but q_t^* subcritical. This is interesting and new, being $0 < s \leq t < 1 < p < q$, with $tq < N$, so that $p < q < q_t^*$.

From here on, $\cdot$ denotes the natural inner product in any Euclidean space $\mathbb{R}^d$ for any dimension $d \geq 1$. The main results of the paper are based on the best constant $\mathcal{H}_{\mathfrak{s}, \wp} = \mathcal{H}_{\mathfrak{s}, \wp}(\mathfrak{s}, \wp, N)$ of the Hardy inequality

for fractional Sobolev spaces. The value of $\mathcal{H}_{\mathfrak{s},\varphi}$ has been determined in [20] and it is given for any $0 < \mathfrak{s} < 1$ and $\mathfrak{s}\varphi < N$ by

$$\mathcal{H}_{\mathfrak{s},\varphi} = \inf_{\substack{u \in D^{\mathfrak{s},\varphi}(\mathbb{R}^N) \\ u \neq 0}} \frac{[u]_{\mathfrak{s},\varphi}^\varphi}{\|u\|_{H_{\mathfrak{s},\varphi}}^\varphi}, \quad \text{where } \|u\|_{H_{\mathfrak{s},\varphi}} = \left(\int_{\mathbb{R}^N} |u|^\varphi \frac{dx}{|x|^{\mathfrak{s}\varphi}} \right)^{1/\varphi}, \quad (1.1)$$

and $D^{\mathfrak{s},\varphi}(\mathbb{R}^N)$ is the homogeneous Sobolev–Slobodeckii space, which is defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the Gagliardo seminorm

$$[\varphi]_{\mathfrak{s},\varphi} = \left(\iint_{\mathbb{R}^{2N}} \frac{|\varphi(x) - \varphi(y)|^\varphi}{|x - y|^{N+\mathfrak{s}\varphi}} dx dy \right)^{1/\varphi} \quad \text{along any } \varphi \in C_c^\infty(\mathbb{R}^N).$$

Hardy’s inequality plays a fundamental role in many physical contexts, such as molecular physics, quantum cosmology and linearization of combustion models. However, from a mathematical point of view, the main interest in considering Hardy potentials relies in their criticality. Indeed, the lack of compactness of the embedding $D^{\mathfrak{s},\varphi}(\mathbb{R}^N) \hookrightarrow L^\varphi(\mathbb{R}^N, |x|^{-\mathfrak{s}\varphi} dx)$, even locally in any neighborhood of zero, together with the critical terms and the unboundedness of the domain, causes a ”triple loss of compactness”.

As customary, the natural space where finding solutions of (S) is

$$W = [W^{s,p}(\mathbb{R}^N) \cap W^{t,q}(\mathbb{R}^N)]^2,$$

endowed with the norm

$$\|(u, v)\| = \|u\|_{s,p} + \|u\|_{t,q} + \|v\|_{s,p} + \|v\|_{t,q},$$

where $\|u\|_{\mathfrak{s},\varphi} = (\|u\|_\varphi^\varphi + [u]_{\mathfrak{s},\varphi}^\varphi)^{1/\varphi}$ for all $u \in W^{\mathfrak{s},\varphi}(\mathbb{R}^N)$, while $\|\cdot\|_\varphi$ denotes the canonical $L^\varphi(\mathbb{R}^N)$ norm for any $0 < \mathfrak{s} < 1 < \varphi$. We are now ready to state the main result of the paper, concerning the existence of solutions for system (S).

Theorem 1.1. *Suppose that (H_1) – (H_3) hold. Then, for any $\sigma \in (-\infty, \mathcal{H}_{t,q})$, there exists $\lambda_* = \lambda_*(\sigma, N, t, q, \theta) > 0$ such that system (S) admits at least one nontrivial solution $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ in W for all $\lambda \geq \lambda_*$. Moreover, each component of $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ is nontrivial and*

$$\lim_{\lambda \rightarrow \infty} \|(u_{\sigma,\lambda}, v_{\sigma,\lambda})\| = 0 \quad (1.2)$$

holds.

Theorem 1.1 extends in several directions previous results, not only for the fractional vectorial context, but also for the presence of the Hardy terms, cf. [4, 6, 15, 17]. Existence is obtained via the mountain pass lemma of Ambrosetti and Rabinowitz and follows somehow the ideas of [15]. Anyway, the presence of two fractional Laplacian operators and the triple lack of compactness in the vectorial setting make our analysis more delicate. In particular, in order to handle the critical Hardy Sobolev potentials, we first study the exact behavior of the weakly convergent sequences of $\mathcal{D} = D^{t,q}(\mathbb{R}^N) \times D^{t,q}(\mathbb{R}^N)$ in the space of measures, in the spirit of Lions.

When dealing with the study of critical problems, the concentration–compactness principle developed by Lions in [27, 28], *CC* principle briefly, has proved to be a fundamental tool. This method was originally introduced for dealing with local critical equation and since then it has been extensively developed in literature. In the fractional setting, the first extension of the *CC* method was obtained in [30] for $\varphi = 2$ and then in [29] for any $1 < \varphi < N/\mathfrak{s}$. We also refer to [4, 8, 14, 16] for similar results in this setting. Let us consider the following optimal constants, only related to the greater exponent q of the system

$$S_\sigma = \inf_{\substack{(u,v) \in \mathcal{D} \\ (u \neq 0) \wedge (v \neq 0)}} \frac{[u]_{t,q}^q + [v]_{t,q}^q - \sigma \left(\|u\|_{H_{t,q}}^q + \|v\|_{H_{t,q}}^q \right)}{\left(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \right)^{q/q_*}}, \quad \sigma \in (-\infty, \mathcal{H}_{t,q}), \quad S = S_0. \quad (1.3)$$

Let us also introduce, for notational simplicity, the fractional $(\mathfrak{s}, \wp)$ -gradient of a function $u \in D^{\mathfrak{s}, \wp}(\mathbb{R}^N)$ as

$$|D^{\mathfrak{s}} u|^{\wp}(x) = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{\wp}}{|x - y|^{N + \wp \mathfrak{s}}} dy.$$

Then, we are able to prove the following variant of the *CC* principle for fractional, critical-Hardy systems.

Theorem 1.2 (*CC principle for fractional systems*). *Let $\{(u_n, v_n)\}_n$ be a bounded sequence in $\mathcal{D}$ such that $(u_n, v_n) \rightharpoonup (u, v)$. Then, there exist three nonnegative Radon measures μ, ν and ω on $\mathbb{R}^N$, an at most countable set J , a family of points $\{x_j\}_{j \in J} \subset \mathbb{R}^N$, two families of nonnegative numbers $\{\mu_j\}_{j \in J}$ and $\{\nu_j\}_{j \in J}$ and three nonnegative numbers ν_0, μ_0, ω_0 , such that*

$$|u_n|^{\alpha} |v_n|^{\beta} dx \xrightarrow{*} \nu = |u|^{\alpha} |v|^{\beta} dx + \nu_0 \delta_0 + \sum_{j \in J} \nu_j \delta_{x_j} \quad \text{in } \mathcal{M}(\mathbb{R}^N), \quad (1.4)$$

$$(|D^t u_n|^q + |D^t v_n|^q) dx \xrightarrow{*} \mu \geq (|D^t u|^q + |D^t v|^q) dx + \mu_0 \delta_0 + \sum_{j \in J} \mu_j \delta_{x_j} \quad \text{in } \mathcal{M}(\mathbb{R}^N), \quad (1.5)$$

$$\frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx \xrightarrow{*} \omega = \frac{|u|^q + |v|^q}{|x|^{tq}} dx + \omega_0 \delta_0 \quad \text{in } \mathcal{M}(\mathbb{R}^N), \quad (1.6)$$

$$\nu_j^{q/q_t^*} \leq \frac{\mu_j}{S} \quad \text{for all } j \in J, \quad \nu_0^{q/q_t^*} \leq \frac{\mu_0 - \sigma \omega_0}{S_{\sigma}}, \quad (1.7)$$

where S_{σ} and S are optimal constants defined in (1.3) while δ_0, δ_{x_j} are the Dirac function at the points 0 and x_j of $\mathbb{R}^N$, respectively.

Theorem 1.2 extends Lemma 2.3 of [4], see also [8, 14, 16, 30]. The strategy is the same as the one in the seminal papers of Lions [27, 28], but there are several complications due to the nonlocal context. In order to overcome these difficulties, we employ Lemma 2.4 and Lemma 2.5. The scalar counterpart of Theorem 1.2 can be deduced as a particular case of our result when $u_n = v_n$, and it is still new because of the simultaneous presence of Hardy and critical terms in the nonlocal context.

The (second) *CC* principle, the so-called *limit case* [27, 28], is used to examine the behavior of weakly convergent sequences in Sobolev spaces in situations in which the lack of compactness occurs. The limit of this method is that it does not exclude a possible loss of compactness due to the "drifting towards infinity" of the mass, or – in other words – the concentration at infinity, cf. [14]. It is clear that in the whole $\mathbb{R}^N$ this phenomenon is really common and can be provided by translations. Just think, for example, that given a nontrivial smooth, compactly supported function u in $\mathbb{R}^N$, the sequence of functions $u_n(x) = u(x - ne_1)$ is not precompact. That also explains how the Palais–Smale sequences can fail to be strongly relatively compact. The aforementioned aspect becomes even more pronounced when dealing with fractional problems, since the nonlocal interaction could produce or send additional mass to infinity, and we refer to [14] for a wide range of examples and remarks.

The right condition to avoid this type of phenomena is the *tightness*: roughly speaking, if a sequence of measures is tight, then the amount of mass at infinity goes to zero uniformly. In order to prove that a sequence is tight one could employ the (first) *CC* principle, that is the *locally compact case* [25, 26], showing that the other two possibilities, namely *vanishing* and *dichotomy* cannot happen. However, the use of the first *CC* principle is often very technical and involved, see for example [31]. Another approach to prove tightness is supplied by [3], cf. Lemma 2.2 and 2.6, see also [14], cf. Lemma 4.2.4. In order to avoid these methods, we employ a variant of these two principles, that is the *concentration-compactness principle at infinity*. This variant was introduced in [7, 11] by Bianchi, Chabrowski and Szulkin, and has been employed in the nonlocal context in [4, 8]. In particular, we prove a suitable version of this principle in the order to deal with fractional, critical Hardy systems in the whole $\mathbb{R}^N$.

Theorem 1.3 (*CC principle at infinity for fractional systems*). *Let $\{(u_n, v_n)\}_n$ be a bounded sequence in $\mathcal{D}$ such that $(u_n, v_n) \rightharpoonup (u, v)$ and define*

$$\nu_{\infty} = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{B_R^c} |u_n|^{\alpha} |v_n|^{\beta} dx, \quad (1.8)$$

$$\mu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{B_R^c} (|D^t u_n|^q + |D^t v_n|^q) dx, \quad (1.9)$$

$$\omega_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{B_R^c} \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx. \quad (1.10)$$

Then,

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dx = \nu(\mathbb{R}^N) + \nu_\infty, \quad (1.11)$$

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|D^t u_n|^q + |D^t v_n|^q) dx = \mu(\mathbb{R}^N) + \mu_\infty, \quad (1.12)$$

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx = \omega(\mathbb{R}^N) + \omega_\infty, \quad (1.13)$$

$$\nu_\infty^{q/q_t^*} \leq \frac{\mu_\infty - \sigma \omega_\infty}{S_\sigma}, \quad (1.14)$$

where μ, ν, ω are the measures introduced in (1.4)–(1.6).

Due to the complexity of (S), we decided to present also a simplified version of the system, when $\sigma = 0$ and H does not depend on x , that is the system in $\mathbb{R}^N$

$$\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^t u + |u|^{p-2} u + |u|^{q-2} u = \lambda H_u(u, v) + \frac{\alpha}{q_t^*} |v|^\beta |u|^{\alpha-2} u, \\ (-\Delta)_p^s v + (-\Delta)_q^t v + |v|^{p-2} v + |v|^{q-2} v = \lambda H_v(u, v) + \frac{\beta}{q_t^*} |u|^\alpha |v|^{\beta-2} v. \end{cases} \quad (\tilde{S})$$

Clearly, the main assumptions required for (S) can be stated in a much easier way. Moreover, we are able to prove the following

Theorem 1.4. *Suppose that (H_1) – (H_3) hold with H independent of x . Then, there exists $\tilde{\lambda} > 0$ such that for all $\lambda \geq \tilde{\lambda}$, system (S) admits at least one nontrivial solution (u_λ, v_λ) in W for all. Moreover, each component of (u_λ, v_λ) is nontrivial and (1.2) holds.*

Theorem 1.4 can be seen as the vectorial analogue of Theorem 1.1 of [4]. However, the strategy on which is based the existence argument of Theorem 1.4 bypasses the proof of the Palais–Smale condition. Therefore, we are able to find solutions for system (S) for possibly different values of λ . Indeed, the critical value λ_* in Theorem 1.1 only appears in Lemma 4.4, where the application of the concentration compactness principles allows to find level sets c for which the $(PS)_c$ condition is valid. Since now we avoid proving the strong convergence of the PS sequences, the fact that the solution constructed in Theorem 1.4 is nontrivial is not automatic and relies on a deep result *à la Lions*, see Proposition 5.2, which has interesting consequences when dealing with autonomous systems. We thought to present also (S) since the proof argument of Theorem 1.4 is more direct and elegant, and Theorem 1.4 improves or complements previous results for (p, q) local and nonlocal problems such as, e.g., [4, 6, 15, 17].

The paper is organized as follows. In Section 2, we recall some fundamental definitions and properties related to the fractional Sobolev–Slobodeckii spaces and we give some useful lemmas. Section 3 is devoted to the proof of Theorems 1.2 and 1.3, while Section 4 deals with the proof of existence of nontrivial solutions for (S), that is Theorem 1.1. Finally, in Section 5 we prove Theorem 1.4, obtaining existence of nontrivial solutions for the simplified system (S).

2. Preliminaries

In this section, we recall some well known facts related to the fractional Sobolev–Slobodeckii spaces and we give some preliminary lemmas which will be used in Theorems 1.2 and 1.3. These results can be deduced

from [4] and [8]. However, following [8], we include a more direct proof of these facts in order to make the paper self contained.

Given $0 < \mathfrak{s} < 1$ and $1 < \wp < \infty$, the fractional Sobolev space $W^{\mathfrak{s},\wp}(\mathbb{R}^N)$ is defined as the set of functions $u : \mathbb{R}^N \rightarrow \mathbb{R}$ in $L^\wp(\mathbb{R}^N)$ such that $[u]_{\mathfrak{s},\wp} < \infty$, where

$$[u]_{\mathfrak{s},\wp} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^\wp}{|x - y|^{N+\wp\mathfrak{s}}} dx dy \right)^{1/\wp}.$$

This is a Banach space endowed with the norm $\|u\|_{W^{\mathfrak{s},\wp}} = (\|u\|_\wp^\wp + [u]_{\mathfrak{s},\wp}^\wp)^{1/\wp}$. Let us also define the homogeneous Sobolev–Slobodeckii space $D^{\mathfrak{s},\wp}(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $[\cdot]_{\mathfrak{s},\wp}$. Observe that the latter is indeed a norm on $C_c^\infty(\mathbb{R}^N)$. We endow $D^{\mathfrak{s},\wp}(\mathbb{R}^N)$ with the norm $\|\cdot\|_{D^{\mathfrak{s},\wp}(\mathbb{R}^N)} = [\cdot]_{\mathfrak{s},\wp}$, which turns it into a reflexive Banach space. For more details and a complete survey on the motivations for the homogeneous Sobolev–Slobodeckii spaces we refer to [9]. It is worth to emphasize that, in the case $\mathfrak{s}\wp < N$, then $D^{\mathfrak{s},\wp}(\mathbb{R}^N)$ is actually a function space. Indeed, if $\mathfrak{s}\wp < N$, the Sobolev embedding theorem (see e.g. Theorems 7.34 and 7.47 of [1] or Theorem 6.5 of [13]) asserts that $D^{\mathfrak{s},\wp}(\mathbb{R}^N) \subset L^{\wp^*}(\mathbb{R}^N)$ for $\wp^* = N\wp/(N - \mathfrak{s}\wp)$. In particular, there exists a constant $C = C(N, \wp, \mathfrak{s})$ such that

$$\|u\|_{\wp^*}^\wp \leq C[u]_{\mathfrak{s},\wp}^\wp \quad \text{for all } u \in D^{\mathfrak{s},\wp}(\mathbb{R}^N). \quad (2.1)$$

Thus, we can consider the best constant in this inequality, namely

$$C_{\wp^*} = \inf_{\substack{u \in D^{\mathfrak{s},\wp}(\mathbb{R}^N) \\ u \neq 0}} \frac{[u]_{\mathfrak{s},\wp}^\wp}{\|u\|_{\wp^*}^\wp}. \quad (2.2)$$

As a consequence, for any $r \in [\wp, \wp^*]$, we have the following continuous embedding

$$W^{\mathfrak{s},\wp}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N).$$

Let us also introduce, for notational simplicity, the fractional $(\mathfrak{s}, \wp)$ -gradient of a function $u \in D^{\mathfrak{s},\wp}(\mathbb{R}^N)$ as

$$|D^{\mathfrak{s}}u|^\wp(x) = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^\wp}{|x - y|^{N+\wp\mathfrak{s}}} dy$$

Observe that, by Tonelli's theorem, the $(\mathfrak{s}, \wp)$ -gradient of a function $u \in D^{\mathfrak{s},\wp}(\mathbb{R}^N)$ is well defined a.e. in $\mathbb{R}^N$ and $|D^{\mathfrak{s}}u|^\wp \in L^1(\mathbb{R}^N)$. The next lemma follows directly from Theorem 6.5 and Theorem 7.1 of [13].

Lemma 2.1. *Let $0 < s \leq t < 1 < p < q$, $N \geq 2$, $tq < N$. Then, the embedding*

$$W \hookrightarrow L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^N)$$

is continuous for all $r \in [p, q^]$, and*

$$\|(u, v)\|_r \leq 2^{r-1}(\|u\|_r + \|v\|_r) \leq C_r \|(u, v)\| \quad \text{for all } (u, v) \in W, \quad (2.3)$$

where C_r depends on r , N , p and q^ . Moreover, if $r \in [1, q^*)$, then the embedding*

$$W \hookrightarrow L_{\text{loc}}^r(\mathbb{R}^N) \times L_{\text{loc}}^r(\mathbb{R}^N)$$

is compact.

From now on we fix $0 < \mathfrak{s} < 1 < \wp$ with $\mathfrak{s}\wp < N$. First, we report the following scaling property of the fractional $(\mathfrak{s}, \wp)$ -gradient.

Lemma 2.2 (Lemma 2.1 of [8]). *Let $\varphi \in C_c^\infty(\mathbb{R}^N)$, $\varepsilon > 0$ and $x_0 \in \mathbb{R}^N$. Define $\varphi_\varepsilon = \varphi((x - x_0)/\varepsilon)$. Then,*

$$|D^{\mathfrak{s}}\varphi_\varepsilon(x)|^\varphi = \frac{1}{\varepsilon^{\mathfrak{s}\varphi}} \left| D^{\mathfrak{s}}\varphi\left(\frac{x - x_0}{\varepsilon}\right) \right|^\varphi.$$

Observe that in general –contrary to what happens for the conventional gradient– the nonlocal (t, q) -gradient of a compactly supported function does not have compact support. For this, we shall need the following lemma, which gives useful decay estimates as $|x| \rightarrow \infty$ of the fractional $(\mathfrak{s}, \varphi)$ -gradient of a $C_c^\infty(\mathbb{R}^N)$ function.

Lemma 2.3 (Lemma 2.2 of [8]). *Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be such that $0 \leq \varphi \leq 1$, $\text{supp } \varphi \subset B_1$ and $\|\nabla\varphi\|_\infty \leq 2$. Then, there exists a constant $C = C(N, \mathfrak{s}, \varphi)$ such that for any $x \in \mathbb{R}^N$*

$$|D^{\mathfrak{s}}\varphi(x)|^\varphi \leq C \min\{1, |x|^{-(N+\mathfrak{s}\varphi)}\}. \quad (2.4)$$

Using the previous estimates, we are able to prove the next result, which is a particular case of Lemma 2.4 of [8]. However, for completeness, we report the proof in this context.

Lemma 2.4. *Let $\varphi \in C_c^\infty(\mathbb{R}^N)$. Then, the following embedding is compact*

$$D^{\mathfrak{s}, \varphi}(\mathbb{R}^N) \hookrightarrow L^\varphi(\mathbb{R}^N, |D^{\mathfrak{s}}\varphi|^\varphi dx).$$

In particular, for any bounded sequence $(u_n)_n \subset D^{\mathfrak{s}, \varphi}(\mathbb{R}^N)$ there exists a subsequence $(u_{n_k})_k$ and a function $u \in D^{\mathfrak{s}, \varphi}(\mathbb{R}^N)$ such that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} |u_{n_k}(x) - u(x)|^\varphi |D^{\mathfrak{s}}\varphi(x)|^\varphi dx = 0.$$

Proof. From the reflexivity of $D^{\mathfrak{s}, \varphi}(\mathbb{R}^N)$ and the Rellich theorem, there exist $u \in D^{\mathfrak{s}, \varphi}(\mathbb{R}^N)$ and a subsequence, still denoted by $(u_n)_n$, such that

$$u_n \rightharpoonup u \text{ in } D^{\mathfrak{s}, \varphi}(\mathbb{R}^N), \quad u_n \rightarrow u \text{ in } L_{\text{loc}}^\varphi(\mathbb{R}^N). \quad (2.5)$$

Fix R such that $\text{supp } \varphi \subset B_R$ and compute

$$\int_{\mathbb{R}^N} |u_n(x) - u(x)|^\varphi |D^{\mathfrak{s}}\varphi(x)|^\varphi dx = \left(\int_{B_{2R}} + \int_{B_{2R}^c} \right) |u_n(x) - u(x)|^\varphi |D^{\mathfrak{s}}\varphi(x)|^\varphi dx.$$

Now, from (2.4) and (2.5)

$$\begin{aligned} \int_{B_{2R}} |u_n(x) - u(x)|^\varphi |D^{\mathfrak{s}}\varphi(x)|^\varphi dx &\leq \|D^{\mathfrak{s}}\varphi\|_\infty^\varphi \int_{B_{2R}} |u_n(x) - u(x)|^\varphi dx \\ &\leq C \int_{B_{2R}} |u_n(x) - u(x)|^\varphi dx = o_n(1) \end{aligned} \quad (2.6)$$

as $n \rightarrow \infty$. On the other hand, by Lemma 2.3, using (2.1) and the Hölder inequality with $r = \varphi_\mathfrak{s}^*/\varphi = N/(N - \mathfrak{s}\varphi)$ so that $r' = N/\mathfrak{s}\varphi$, we get

$$\begin{aligned} \int_{B_{2R}^c} |u_n(x) - u(x)|^\varphi |D^{\mathfrak{s}}\varphi(x)|^\varphi dx &\leq \|u_n - u\|_{\varphi_\mathfrak{s}^*}^\varphi \left(\int_{B_{2R}^c} |D^{\mathfrak{s}}\varphi(x)|^{N/\mathfrak{s}} dx \right)^{\mathfrak{s}\varphi/N} \\ &\leq 2^{\varphi-1} (\|u_n\|_{\varphi_\mathfrak{s}^*}^\varphi + \|u\|_{\varphi_\mathfrak{s}^*}^\varphi) C_2 \|\varphi\|_\infty^\varphi \left(\int_{B_{2R}^c} \frac{dx}{|x|^{N(1+N/\mathfrak{s}\varphi)}} \right)^{\mathfrak{s}\varphi/N} \\ &\leq \tilde{C} \left(\int_{B_{2R}^c} \frac{dx}{|x|^{N(1+N/\mathfrak{s}\varphi)}} \right)^{\mathfrak{s}\varphi/N} \end{aligned} \quad (2.7)$$

for some universal constant $\tilde{C}$. Now, for any $\delta > 0$ we can choose R sufficiently large so that

$$\text{supp } \varphi \subset B_R \quad \text{and} \quad \left(\int_{B_{2R}^c} \frac{dx}{|x|^{N(1+N/\mathfrak{s}\varphi)}} \right)^{\mathfrak{s}\varphi/N} < \delta/\tilde{C}.$$

Finally, by (2.6) and (2.7), we obtain

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n(x) - u(x)|^\varphi |D^\mathfrak{s} \varphi(x)|^\varphi dx \leq \delta$$

and sending $\delta \rightarrow 0^+$ we get the desired conclusion. $\square$

The following lemma extends to the vectorial case Lemma 2.3 of [4]. However, we give a more simple and direct proof based on the precise decay rate of $|D^\mathfrak{s} \varphi_\varepsilon|^\varphi$, first proved in [8], cf. (2.8). The main difficulties here, as we already noticed, are the fact that the nonlocal gradient $|D^\mathfrak{s} \varphi_\varepsilon|^\varphi$ does not have compact support and that the sequence $\{(u_n, v_n)\}_n \subset \mathcal{D}$ does not belong to $L^\varphi(\mathbb{R}^N) \times L^\varphi(\mathbb{R}^N)$. The proof uses the same strategy of Lemma 2.4, which is effective thanks to the decay estimates given by Lemma 2.2 and (2.4).

Lemma 2.5. *Let $\{(u_n, v_n)\}_n \subset \mathcal{D}$ be a bounded sequence and let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be such that $0 \leq \varphi \leq 1$, $\varphi(0) = 1$ and $\text{supp } \varphi \subset B_1$. Take $\varepsilon > 0$, fix $x_0 \in \mathbb{R}^N$ and put $\varphi_\varepsilon(x) = \varphi((x - x_0)/\varepsilon)$, $x \in \mathbb{R}^N$. Then*

$$\lim_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |D^\mathfrak{s} \varphi_{\varepsilon,j}|^\varphi (|u_n|^\varphi + |v_n|^\varphi) dy = 0.$$

Proof. By the scaling property in Lemma 2.2 and by (2.4) it follows

$$|D^\mathfrak{s} \varphi_\varepsilon(x)|^\varphi = \frac{1}{\varepsilon^{\mathfrak{s}\varphi}} \left| D^\mathfrak{s} \varphi \left(\frac{x - x_0}{\varepsilon} \right) \right|^\varphi \leq C \min\{\varepsilon^{-\mathfrak{s}\varphi}, \varepsilon^N |x - x_0|^{-(N+\mathfrak{s}\varphi)}\}. \quad (2.8)$$

From the reflexivity of $\mathcal{D}$ and the Rellich theorem, there exist $(u, v) \in \mathcal{D}$ and a subsequence, still denoted by $\{(u_n, v_n)\}_n$, such that

$$(u_n, v_n) \rightharpoonup (u, v) \text{ in } \mathcal{D}, \quad (u_n, v_n) \rightarrow (u, v) \text{ in } L_{\text{loc}}^\varphi(\mathbb{R}^N) \times L_{\text{loc}}^\varphi(\mathbb{R}^N). \quad (2.9)$$

Consider

$$\int_{\mathbb{R}^N} |D^\mathfrak{s} \varphi_\varepsilon|^\varphi (|u_n|^\varphi + |v_n|^\varphi) dx = \left(\int_{B_\varepsilon(x_0)} + \int_{B_\varepsilon^c(x_0)} \right) |D^\mathfrak{s} \varphi_\varepsilon|^\varphi (|u_n|^\varphi + |v_n|^\varphi) dx.$$

Let us first estimate the integral over $B_\varepsilon(x_0)$. By (2.8) and (2.9) we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{B_\varepsilon(x_0)} |D^\mathfrak{s} \varphi_\varepsilon|^\varphi (|u_n|^\varphi + |v_n|^\varphi) dx &\leq \limsup_{n \rightarrow \infty} C \varepsilon^{-\mathfrak{s}\varphi} \int_{B_\varepsilon(x_0)} (|u_n|^\varphi + |v_n|^\varphi) dx \\ &= C \varepsilon^{-\mathfrak{s}\varphi} \int_{B_\varepsilon(x_0)} (|u|^\varphi + |v|^\varphi) dx. \end{aligned} \quad (2.10)$$

Now we turn to the integral over $B_\varepsilon^c(x_0)$. Using again (2.1), (2.8) and the Hölder inequality with $r = \varphi^*/\varphi = N/(N - \mathfrak{s}\varphi)$ so that $r' = N/\mathfrak{s}\varphi$, we get

$$\begin{aligned} \int_{B_\varepsilon^c(x_0)} |D^\mathfrak{s} \varphi_\varepsilon|^\varphi (|u_n|^\varphi + |v_n|^\varphi) dx &\leq C \|(u_n, v_n)\|_{L^{\varphi^*}(\mathbb{R}^N; \mathbb{R}^2)}^\varphi \left(\int_{B_\varepsilon^c(x_0)} |D^\mathfrak{s} \varphi_\varepsilon(x)|^{N/\mathfrak{s}} dx \right)^{\mathfrak{s}\varphi/N} \\ &\leq C_1 \varepsilon^N \left(\int_{B_\varepsilon^c} \frac{dx}{|x|^{N(1+N/\mathfrak{s}\varphi)}} \right)^{\mathfrak{s}\varphi/N} \end{aligned}$$

$$\leq C_2 \varepsilon^N \left(|\varepsilon|^{N(1-N/\mathfrak{s}\varphi)} \right)^{\mathfrak{s}\varphi/N} = C_2 \varepsilon^{\mathfrak{s}\varphi},$$

where the constants C, C_1, C_2 are independent of n and ε . Thus, from what we obtained above, it follows that

$$\limsup_{n \rightarrow \infty} \int_{B_\varepsilon^c} |D^{\mathfrak{s}} \varphi_\varepsilon|^{\varphi} (|u_n|^{\varphi} + |v_n|^{\varphi}) dx \leq C_2 \varepsilon^{\mathfrak{s}\varphi}. \quad (2.11)$$

Finally, using (2.10) and (2.11), we conclude

$$\lim_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |D^{\mathfrak{s}} \varphi_{\varepsilon,j}|^{\varphi} (|u_n|^{\varphi} + |v_n|^{\varphi}) dx \leq \lim_{\varepsilon \rightarrow 0^+} \left(C \varepsilon^{-\mathfrak{s}\varphi} \int_{B_\varepsilon(x_0)} (|u|^{\varphi} + |v|^{\varphi}) dx + C_2 \varepsilon^{\mathfrak{s}\varphi} \right) = 0,$$

being $\mathfrak{s}\varphi < N$. $\square$

Finally, the next lemma concerns the behaviour of regular functions concentrating at infinity. For the proof we refer to [4] or [8].

Lemma 2.6. *Let $\{(u_n, v_n)\}_n \subset \mathcal{D}$ be a bounded sequence and let $\eta \in C^\infty(\mathbb{R}^N)$ be such that $0 \leq \eta \leq 1$, $\eta = 0$ in B_1 and $\eta = 1$ in B_2^c . Take $R > 0$ and put $\eta_R(x) = \eta(x/R)$, $x \in \mathbb{R}^N$. Then*

$$\lim_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |D^{\mathfrak{s}} \eta_R|^{\varphi} (|u_n|^{\varphi} + |v_n|^{\varphi}) dx = 0.$$

3. Concentration–compactness results

Throughout the section, we assume that $0 < t < 1 < q$ with $N \geq 2$, $tq < N$, and that $\alpha, \beta > 1$ are such that $\alpha + \beta = q_t^*$, where $q_t^* = Nq/(N - tq)$. Moreover, $\sigma \in (-\infty, \mathcal{H}_{t,q})$. Denote by $\mathcal{D} = D^{t,q}(\mathbb{R}^N) \times D^{t,q}(\mathbb{R}^N)$. Then, the optimal constants introduced in (1.3), namely

$$S = \inf_{\substack{(u,v) \in \mathcal{D} \\ (u \neq 0) \wedge (v \neq 0)}} \frac{[u]_{t,q}^q + [v]_{t,q}^q}{\left(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \right)^{q/q_t^*}},$$

$$S_\sigma = \inf_{\substack{(u,v) \in \mathcal{D} \\ (u \neq 0) \wedge (v \neq 0)}} \frac{[u]_{t,q}^q + [v]_{t,q}^q - \sigma \left(\|u\|_{H_{t,q}}^q + \|v\|_{H_{t,q}}^q \right)}{\left(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \right)^{q/q_t^*}}, \quad \sigma \in (-\infty, \mathcal{H}_{t,q}),$$

are well defined. Indeed, the Hölder inequality and (2.1) give

$$\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \leq \left(\int_{\mathbb{R}^N} |u|^{q_t^*} dx \right)^{\alpha/q_t^*} \left(\int_{\mathbb{R}^N} |v|^{q_t^*} dx \right)^{\beta/q_t^*} \leq C_{q_t^*}^{\alpha/q_t^*} [u]_{t,q}^\alpha [v]_{t,q}^\beta \quad (3.1)$$

for all $(u, v) \in \mathcal{D}$, since $\alpha, \beta > 1$ are such that $\alpha + \beta = q_t^*$. Therefore, (3.1) and the Young inequality yield

$$\begin{aligned} \left(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \right)^{q/q_t^*} &\leq C_{q_t^*}^{\alpha q/q_t^*} [u]_{t,q}^{\alpha q/q_t^*} [v]_{t,q}^{\beta q/q_t^*} \leq C_{q_t^*} \left(\frac{\alpha}{q_t^*} [u]_{t,q}^q + \frac{\beta}{q_t^*} [v]_{t,q}^q \right) \\ &\leq C_{q_t^*} ([u]_{t,q}^q + [v]_{t,q}^q) \end{aligned} \quad (3.2)$$

for all $(u, v) \in \mathcal{D}$. Hence $S \geq C_{q_t^*}^{-1} > 0$. and $S_\sigma \geq (1 - \sigma/\mathcal{H}_{t,q})/C_{q_t^*} > 0$ by (1.1), being $\sigma < \mathcal{H}_{t,q}$.

Since our goal is to study the behaviour of weakly convergent sequences in $\mathcal{D}$ in terms of measures, following [19] we recall some classical definitions related to the weak convergence of measures. Let $\mathcal{M}(\mathbb{R}^N)$ be the space of all real valued, finite, signed Radon measures on $\mathbb{R}^N$. We denote by $C_c(\mathbb{R}^N)$ the space of all

functions whose support is compact in $\mathbb{R}^N$ and by $C_0(\mathbb{R}^N)$ the completion of $C_c(\mathbb{R}^N)$, w.r.t. the supremum norm $\|\cdot\|_\infty$. From the Riesz representation theorem in $C_0(\mathbb{R}^N)$, see Theorem 1.200 in [19], it is known that

$$\mathcal{M}(\mathbb{R}^N) = (C_0(\mathbb{R}^N))'.$$

Thus, we say that a sequence of measures $(\mu_n)_n \subset \mathcal{M}(\mathbb{R}^N)$ *weak- $\ast$ converges* to a measure $\mu \in \mathcal{M}(\mathbb{R}^N)$, $\mu_n \xrightarrow{*} \mu$, if for every $\phi \in C_0(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} \phi d\mu_n \rightarrow \int_{\mathbb{R}^N} \phi d\mu \quad \text{as } n \rightarrow \infty.$$

Clearly, this notion of convergence is the usual weak star convergence with respect to $(C_0(\mathbb{R}^N))'$. A sequence $(\mu_n)_n$ of bounded Borel measures is tight if for all $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset X$ such that

$$\sup_n [\mu_n(\mathbb{R}^N \setminus K_\varepsilon)] < \varepsilon.$$

Roughly speaking, if a sequence of measures is tight, then the amount of mass at infinity goes to zero uniformly. Moreover, by Prokhorov Theorem, cf. Theorem 1.208 of [19], a bounded sequence of measures $(\mu_n)_n$ is relatively compact in duality with $C_b(\mathbb{R}^N)$ if and only if it is tight.

However, proving tightness is often very technical, and it is not necessary in order to get compactness of *PS* sequences, as explained in Remark 3.1. For this reason we employ a variant of these two principles, that is the concentration–compactness principle at infinity. This variant was introduced in [11, 7], and has been employed in the nonlocal context in [4, 8].

Let us first report the next Brezis–Lieb type lemma for vectorial critical terms, which is an extension of Lemma 2.1 in [22].

Lemma 3.1. *Let $\{(u_n, v_n)\}_n$ be a sequence in $\mathcal{D}$. Assume that $(u_n, v_n) \rightharpoonup (u, v)$ in $\mathcal{D}$ and $u_n \rightarrow u$, $v_n \rightarrow v$ a.e. in $\mathbb{R}^N$. Then,*

$$\lim_{n \rightarrow \infty} \left\{ \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dx - \int_{\mathbb{R}^N} |u_n - u|^\alpha |v_n - v|^\beta dx \right\} = \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx. \quad (3.3)$$

The proof of Lemma 3.1 is identical to the one in [22], just noting that the fact that $\{(u_n, v_n)\}_n$ is bounded in $\mathcal{D}$ implies that $(u_n)_n$ and $(v_n)_n$ are bounded in $L^{q_i^*}(\mathbb{R}^N)$ by (2.1).

Before proving Theorem 1.2 and Theorem 1.3, we observe the following simple but crucial fact. The next remark essentially says that if you are able to exclude both the concentration at finite points and the concentration at infinity, then you can turn the weak convergence into strong convergence.

Remark 3.1. If (1.4) holds and the coefficients are all zero, that is $\nu_j = \nu_\infty = 0$, $j \in J$, then up to a subsequence

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n - u|^\alpha |v_n - v|^\beta dx = 0. \quad (3.4)$$

Indeed, the fact that $\nu_j = 0$ for all $j \in J$ implies that $|u_n|^\alpha |v_n|^\beta dx \xrightarrow{*} \nu = |u|^\alpha |v|^\beta dx$ in $\mathcal{M}(\mathbb{R}^N)$. Moreover, if $\nu_\infty = 0$, then by (1.11) up to a subsequence

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dx = \nu(\mathbb{R}^N) + \nu_\infty = \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx.$$

Therefore, Lemma 3.1 yield (3.4). Similarly, if (1.6) holds and $\omega_0 = \omega_\infty = 0$, then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx = \omega(\mathbb{R}^N) + \omega_\infty = \int_{\mathbb{R}^N} \frac{|u|^q + |v|^q}{|x|^{tq}} dx$$

and so the celebrated Brézis and Lieb lemma of [10] gives as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \left\{ \|u_n\|_{H_{t,q}}^q - \|u_n - u\|_{H_{t,q}}^q \right\} = \|u\|_{H_{t,q}}^q, \quad \lim_{n \rightarrow \infty} \left\{ \|v_n\|_{H_{t,q}}^q - \|v_n - v\|_{H_{t,q}}^q \right\} = \|v\|_{H_{t,q}}^q.$$

Consequently

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{H_{t,q}}^q + \|v_n - v\|_{H_{t,q}}^q = 0. \quad (3.5)$$

Proof of Theorem 1.2. Since $\{(u_n, v_n)\}_n$ is a bounded sequence in $\mathcal{D}$, by (1.1) and (3.2), in virtue of Proposition 1.202 of [19], there exist three bounded nonnegative Radon measures μ , ν and ω on $\mathbb{R}^N$ such that, up to a subsequence, we have

$$\begin{aligned} |u_n|^\alpha |v_n|^\beta dx &\xrightarrow{*} \nu, \quad (|D^t u_n|^q + |D^t v_n|^q) dx \xrightarrow{*} \mu \\ \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx &\xrightarrow{*} \omega \quad \text{in } \mathcal{M}(\mathbb{R}^N). \end{aligned} \quad (3.6)$$

Step 1. Proof of (1.4): Consider $\{(u_n, v_n)\}_n$ and (u, v) as in the statement and divide the proof into two cases.

Case 1. $u = v = 0$. Fix $\varphi \in C_c^\infty(\mathbb{R}^N)$. Then, since clearly $(\varphi u_n, \varphi v_n) \in \mathcal{D}$ for all n we get by (1.3)

$$\begin{aligned} \left(\int_{\mathbb{R}^N} |\varphi|^{q_t^*} |u_n|^\alpha |v_n|^\beta dx \right)^{q/q_t^*} &\leq S^{-1} ([\varphi u_n]_{t,q}^q + [\varphi v_n]_{t,q}^q) \\ &= S^{-1} \left(\iint_{\mathbb{R}^{2N}} \frac{|(\varphi u_n)(x) - (\varphi u_n)(y)|^q}{|x - y|^{N+tq}} dx dy \right. \\ &\quad \left. + \iint_{\mathbb{R}^{2N}} \frac{|(\varphi v_n)(x) - (\varphi v_n)(y)|^q}{|x - y|^{N+tq}} dx dy \right) \end{aligned} \quad (3.7)$$

since $\alpha + \beta = q_t^*$. Now,

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|(\varphi u_n)(x) - (\varphi u_n)(y)|^q}{|x - y|^{N+tq}} dx dy \\ &\leq 2^{q-1} \left(\iint_{\mathbb{R}^{2N}} |\varphi(y)|^q \frac{|u_n(x) - u_n(y)|^q}{|x - y|^{N+tq}} dx dy + \iint_{\mathbb{R}^{2N}} |u_n(x)|^q \frac{|\varphi(x) - \varphi(y)|^q}{|x - y|^{N+tq}} dx dy \right) \end{aligned}$$

and, from Lemma 2.4, it follows that

$$\iint_{\mathbb{R}^{2N}} |u_n(x)|^q \frac{|\varphi(x) - \varphi(y)|^q}{|x - y|^{N+tq}} dx dy = o_n(1),$$

and similarly for v_n . Therefore, we get by (3.7)

$$\left(\int_{\mathbb{R}^N} |\varphi|^{q_t^*} |u_n|^\alpha |v_n|^\beta dx \right)^{q/q_t^*} \leq C \int_{\mathbb{R}^N} |\varphi|^q (|D^t u_n|^q + |D^t v_n|^q) dx + o_n(1) \quad (3.8)$$

where $C = 2^{p-1} S^{-1}$. Therefore, by (3.6) passing to the limit in (3.8), we have the following reverse Hölder inequality

$$\left(\int_{\mathbb{R}^N} |\varphi|^{q_t^*} d\nu \right)^{1/q_t^*} \leq C \left(\int_{\mathbb{R}^N} |\varphi|^q d\mu \right)^{1/q} \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N).$$

Thus, by Lemma I.2 of [27], we conclude that there exist an at most countable set J , a family of points $\{x_j\}_{j \in J} \subset \mathbb{R}^N$ and a family of nonnegative numbers $\{\nu_j\}_{j \in J}$ such that

$$\nu = \sum_{j \in J} \nu_j \delta_{x_j} + \nu_0 \delta_0. \quad (3.9)$$

This concludes the proof of (1.4) in Case 1.

Case 2. $(u, v) \neq (0, 0)$. Set $\tilde{u}_n = u_n - u$ and $\tilde{v}_n = v_n - v$. Clearly, $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (0, 0)$ in S and (3.7) still holds for $(\varphi \tilde{u}_n, \varphi \tilde{v}_n)$ for any $\varphi \in C_c^\infty(\mathbb{R}^N)$. Moreover, $n \mapsto |\tilde{u}_n|^\alpha |\tilde{u}_n|^\beta dx$ and $n \mapsto (|D^t \tilde{u}_n|^q + |D^t \tilde{v}_n|^q) dx$ are still bounded sequences of measures and so by Proposition 1.202 of [19], we can conclude that there exist two bounded nonnegative Radon measure $\tilde{\nu}$ and $\tilde{\mu}$ on $\mathbb{R}^N$, such that, up to a subsequence, still labelled $\{(\tilde{u}_n, \tilde{v}_n)\}_n$, we have

$$(|D^t \tilde{u}_n|^q + |D^t \tilde{v}_n|^q) dx \xrightarrow{*} \tilde{\mu}, \quad |\tilde{u}_n|^\alpha |\tilde{u}_n|^\beta dx \rightharpoonup \tilde{\nu} \text{ in } \mathcal{M}(\mathbb{R}^N). \quad (3.10)$$

Thus, from Case 1. there exist an at most countable set J , a family of points $\{x_j\}_{j \in J} \subset \mathbb{R}^N$ and a family of nonnegative numbers $\{\nu_j\}_{j \in J}$ such that

$$\tilde{\nu} = \sum_{j \in J} \nu_j \delta_{x_j} + \nu_0 \delta_0. \quad (3.11)$$

Furthermore, Lemma 3.1 implies that for all $\varphi \in C_c^\infty(\mathbb{R}^N)$

$$|u_n|^\alpha |v_n|^\beta dx - |u|^\alpha |v|^\beta dx = |u_n - u|^\alpha |v_n - v|^\beta dx - o(1), \quad (3.12)$$

where $o(1) \xrightarrow{*} 0$ in $\mathcal{M}(\mathbb{R}^N)$, since $\alpha + \beta = q_t^*$. Then, computing the limit in (3.12) using (3.6) and (3.10), we get $\tilde{\nu} = \nu - |u|^\alpha |v|^\beta dx$. Consequently, from (3.11), we obtain the claimed representation (1.4) of ν .

Let us now prove the first part of (1.7). Fix a test function $\varphi \in C_c^\infty(\mathbb{R}^N)$, such that $0 \leq \varphi \leq 1$, $\varphi(0) = 1$ and $\text{supp } \varphi = \overline{B_1}$. Take $\varepsilon > 0$ and put $\varphi_{\varepsilon,j}(x) = \varphi((x - x_j)/\varepsilon)$, $x \in \mathbb{R}^N$, for any fixed $j \in J$, where $\{x_j\}_j$ is introduced in (3.11). Fix $j \in J$. Then, for all $\delta > 0$ there exists $C_\delta > 0$ such that for all $\xi, \eta \in \mathbb{R}$

$$|\xi + \eta|^q \leq (1 + \delta)|\xi|^q + C_\delta |\eta|^q.$$

Consequently, (1.3) and the above inequality applied to $(\varphi_{\varepsilon,j} u_n, \varphi_{\varepsilon,j} v_n)$ yield

$$\begin{aligned} & S \left(\int_{\mathbb{R}^N} |\varphi_{\varepsilon,j}|^{q_t^*} |u_n|^\alpha |v_n|^\beta dx \right)^{q/q_t^*} \\ & \leq \iint_{\mathbb{R}^{2N}} \frac{|(\varphi_{\varepsilon,j} u_n)(x) - (\varphi_{\varepsilon,j} u_n)(y)|^q}{|x - y|^{N+q}} dx dy + \iint_{\mathbb{R}^{2N}} \frac{|(\varphi_{\varepsilon,j} v_n)(x) - (\varphi_{\varepsilon,j} v_n)(y)|^q}{|x - y|^{N+q}} dx dy \\ & \leq (1 + \delta) \int_{\mathbb{R}^N} |\varphi_{\varepsilon,j}|^q (|D^t u_n|^q + |D^t v_n|^q) dx + C_\delta \int_{\mathbb{R}^N} |D^t \varphi_{\varepsilon,j}|^q (|u_n|^q + |v_n|^q) dx \end{aligned} \quad (3.13)$$

for any $\delta > 0$. Now we want to pass to the limit in (3.13) as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0^+$. To this aim, let us observe first that from (3.6) and (1.4) we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\varphi_{\varepsilon,j}|^{q_t^*} |u_n|^\alpha |v_n|^\beta dx &= \lim_{\varepsilon \rightarrow 0^+} \int_{B_\varepsilon(x_j)} |\varphi_{\varepsilon,j}|^{q_t^*} d\nu \\ &= \lim_{\varepsilon \rightarrow 0^+} \left\{ \int_{B_\varepsilon(x_j)} |\varphi_{\varepsilon,j}|^{q_t^*} |u|^\alpha |v|^\beta dx + \nu_j \delta_{x_j}(\varphi_{\varepsilon,j}) \right\} \\ &= \nu_j \end{aligned} \quad (3.14)$$

since

$$\int_{B_\varepsilon(x_j)} |\varphi_{\varepsilon,j}|^{q_t^*} |u|^\alpha |v|^\beta dx \leq \int_{B_\varepsilon(x_j)} |u|^\alpha |v|^\beta dx = o(1)$$

as $\varepsilon \rightarrow 0^+$, being $0 \leq \varphi \leq 1$. On the other hand, (3.6) gives

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\varphi_{\varepsilon,j}|^q \left(|D^t u_n|^q + |D^t v_n|^q \right) dy = \int_{\mathbb{R}^N} |\varphi_{\varepsilon,j}|^q d\mu, \quad (3.15)$$

while Lemma 2.5 yields

$$\lim_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |D^t \varphi_{\varepsilon,j}|^q (|u_n|^q + |v_n|^q) dy = 0. \quad (3.16)$$

Then, combining (3.14)–(3.16) and letting $\varepsilon \rightarrow 0^+$ in (3.13) we get

$$S\nu_j^{q/q_t^*} \leq (1 + \delta)\mu_j, \quad \text{for any } j \in J \text{ and } \delta > 0$$

where $\mu_j = \lim_{\varepsilon \rightarrow 0^+} \mu(B_\varepsilon(x_j))$. Sending $\delta \rightarrow 0^+$ we finally obtain

$$S\nu_j^{q/q_t^*} \leq \mu_j, \quad j \in J.$$

Step 2. *Proof of (1.5):* Clearly,

$$\mu \geq \sum_{j \in J} \mu_j \delta_{\xi_j} + \mu_0 \delta_0.$$

On the other hand, by Lebesgue's differentiation theorem for measures (see for example [19]), in order to prove that $\mu \geq (|D^t u|^q + |D^t v|^q) dx$ it suffices to show that

$$\liminf_{r \rightarrow 0^+} \frac{\mu(B_r(x_0))}{\omega_N r^N} \geq (|D^t u|^q + |D^t v|^q)(x_0) \quad \text{for } \mathcal{L}^N \text{ a.e. } x_0 \in \mathbb{R}^N. \quad (3.17)$$

Clearly, since $(|D^t u|^q + |D^t v|^q) \in L_{\text{loc}}^1(\mathbb{R}^N)$, we know that

$$\lim_{r \rightarrow 0^+} \frac{1}{|B_r|} \int_{B_r(x_0)} (|D^t u|^q + |D^t v|^q) < \infty \quad \text{for a.e. } x_0 \in \mathbb{R}^N. \quad (3.18)$$

Fix then $x_0 \in \mathbb{R}^N$ such that (3.18) holds. Now, the functional $\Phi : \mathcal{D} \rightarrow \mathbb{R}$ defined as

$$\Phi(u, v) = \int_{B_r(x_0)} \int_{\mathbb{R}^N} \left(\frac{|u(x) - u(y)|^q}{|x - y|^{N+q}} + \frac{|v(x) - v(y)|^q}{|x - y|^{N+q}} \right) dy dx = \int_{B_r(x_0)} (|D^t u|^q + |D^t v|^q) dx$$

is convex and strongly continuous on $\mathcal{D}$. So, since $(u_n, v_n) \rightharpoonup (u, v)$ in $\mathcal{D}$, we have

$$\liminf_{n \rightarrow \infty} \int_{B_r(x_0)} (|D^t u_n|^q + |D^t v_n|^q) dx \geq \int_{B_r(x_0)} (|D^t u|^q + |D^t v|^q) dx.$$

Then, what we observed above and Proposition 1.203, Part (ii) of [19] give

$$\begin{aligned} \frac{\mu(B_r(x_0))}{\omega_N r^N} &\geq \limsup_{n \rightarrow \infty} \frac{\mu_n(B_r(x_0))}{\omega_N r^N} = \limsup_{n \rightarrow \infty} \frac{1}{\omega_N r^N} \int_{B_r(x_0)} (|D^t u_n|^q + |D^t v_n|^q) dx \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{\omega_N r^N} \int_{B_r(x_0)} (|D^t u_n|^q + |D^t v_n|^q) dx \geq \frac{1}{\omega_N r^N} \int_{B_r(x_0)} (|D^t u|^q + |D^t v|^q) dx \end{aligned}$$

where ω_N is the Lebesgue measure of the unit ball. Passing to the liminf as $r \rightarrow 0^+$ and using (3.18) we obtain (3.17). Finally, since $(|D^t u|^q + |D^t v|^q)dx$ is orthogonal to $\sum_{j \in J} \mu_j \delta_{x_j} + \mu_0 \delta_0$, we get the desired conclusion.

Step 3. Proof of (1.6): We proceed as in the first part diving the proof into two cases.

Case 1. $u = v = 0$. Fix $\varphi \in C_c^\infty(\mathbb{R}^N)$. Then, since clearly $(\varphi u_n, \varphi v_n) \in \mathcal{D}$ for all n we get by (1.1)

$$\begin{aligned} \mathcal{H}_{t,q} \left(\|\varphi u_n\|_{H_{t,q}}^q + \|\varphi v_n\|_{H_{t,q}}^q \right) &\leq [\varphi u_n]_{t,q}^q + [\varphi v_n]_{t,q}^q \\ &= \iint_{\mathbb{R}^{2N}} \frac{|(\varphi u_n)(x) - (\varphi u_n)(y)|^q}{|x - y|^{N+ tq}} dx dy \\ &\quad + \iint_{\mathbb{R}^{2N}} \frac{|(\varphi v_n)(x) - (\varphi v_n)(y)|^q}{|x - y|^{N+ tq}} dx dy. \end{aligned} \quad (3.19)$$

Now, exactly as before from Lemma 2.4 and (3.19) it follows that

$$\int_{\mathbb{R}^N} |\varphi|^q \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx \leq C \int_{\mathbb{R}^N} |\varphi|^q (|D^t u_n|^q + |D^t v_n|^q) dx + o_n(1) \quad (3.20)$$

where $C = 2^{q-1} \mathcal{H}_{t,q}^{-1}$. Therefore, by (3.6) passing to the limit in (3.20) we have that

$$\left(\int_{\mathbb{R}^N} |\varphi|^q d\omega \right)^{1/q} \leq C \left(\int_{\mathbb{R}^N} |\varphi|^q d\mu \right)^{1/q} \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N).$$

Thus, by Lemma I.2 of [27], we conclude that there exist an at most countable set J , a family of points $\{x_j\}_{j \in J} \subset \mathbb{R}^N$ and a family of nonnegative numbers $\{\omega_j\}_{j \in J}$, ω_0 such that

$$\omega = \omega_0 \delta_0 + \sum_{j \in J} \omega_j \delta_{x_j}. \quad (3.21)$$

Clearly, the set J determined in (3.21) is not necessary the same of the one obtained in the representation of ν . However, since the coefficients ν_j, μ_j, ω_j are allowed to be 0, we can replace these two set with their union (which is still at most countable). For this reason we keep the same notation J for the indexes set. In order to conclude the proof of (1.6) on Case 1, it remains to show that ω is concentrated at 0, namely that $\omega_j = 0$ for any $j \in J$. But this is obvious. Indeed, fix $\varphi \in C_c^\infty(\mathbb{R}^N)$, with $0 \notin \text{supp } \varphi$, so that $x \mapsto |\varphi(x)|^q |x|^{-tq}$ is in $L^\infty(\text{supp } \varphi)$. Then, by Lemma 2.1, $(u_n, v_n) \rightarrow (0, 0)$ in $L_{\text{loc}}^q(\mathbb{R}^N) \times L_{\text{loc}}^q(\mathbb{R}^N)$ and so we get

$$\begin{aligned} \int_{\mathbb{R}^N} |\varphi|^q \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx &= \int_{\text{supp } \varphi} |\varphi|^q \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} \\ &\leq C \int_{\text{supp } \varphi} (|u_n|^q + |v_n|^q) dx \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. This, combined with (3.6), gives $\int_{\mathbb{R}^N} |\varphi|^q d\omega = 0$, that is ω is a measure concentrated in 0. Hence $\omega = \omega_0 \delta_0$, and so (1.6) is proved in Case 1.

Case 2. $(u, v) \neq (0, 0)$. Set $\tilde{u}_n = u_n - u$ and $\tilde{v}_n = v_n - v$. Clearly, $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (0, 0)$ in $\mathcal{D}$ and (3.7) still holds for $(\varphi \tilde{u}_n, \varphi \tilde{v}_n)$ for any $\varphi \in C_c^\infty(\mathbb{R}^N)$. Moreover by Proposition 1.202 of [19] there exists a bounded nonnegative Radon measure $\tilde{\omega}$ on $\mathbb{R}^N$, such that, up to a subsequence, still labelled $\{(\tilde{u}_n, \tilde{v}_n)\}_n$, we have as $n \rightarrow \infty$

$$\frac{|\tilde{u}_n|^q + |\tilde{v}_n|^q}{|x|^{tq}} dx \xrightarrow{*} \tilde{\omega} \quad \text{in } \mathcal{M}(\mathbb{R}^N). \quad (3.22)$$

Thus, from Case 1, there exist an at most countable set J , a family of points $\{\xi_j\}_{j \in J} \subset \mathbb{R}^N$ and a family of nonnegative numbers $\{\omega_j\}_{j \in J}, \omega_0$ such that

$$\tilde{\omega} = \omega_0 \delta_0. \quad (3.23)$$

Now, the Rellich Theorem gives that both $(u_n)_n, (v_n)_n$ converge to u and v in $L^q_{\text{loc}}(\mathbb{R}^N)$, respectively and so, up to a subsequence we can assume

$$\begin{aligned} (u_n, v_n) &\rightarrow (u_{\sigma, \lambda}, v_{\sigma, \lambda}) \text{ a.e. in } \mathbb{R}^N, \\ |u_n| &\leq g_R, \quad |v_n| \leq g_R \text{ a.e. in } \mathbb{R}^N \end{aligned}$$

for some $g_R \in L^q(B_R)$ and all $R > 0$. Thus, for all $\varphi \in C_c^\infty(\mathbb{R}^N)$ by the Brézis–Lieb lemma we get

$$\lim_{n \rightarrow \infty} \|\varphi u_n\|_{H_{t,q}}^q - \|\varphi u\|_{H_{t,q}}^q = \lim_{n \rightarrow \infty} \|\varphi \tilde{u}_n\|_{H_{t,q}}^q,$$

and similarly in the v component. A combination of the above formulas yields that for all $\varphi \in C_c^\infty(\mathbb{R}^N)$

$$\frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx - \frac{|u|^q + |v|^q}{|x|^{tq}} dx = \frac{|u_n - u|^q + |v_n - v|^q}{|x|^{tq}} dx - o(1), \quad (3.24)$$

where $o(1) \xrightarrow{*} 0$ in $\mathcal{M}(\mathbb{R}^N)$. Then, computing the limit in (3.24) using (3.6) and (3.22), we get $\tilde{\omega} = \omega - (|u|^q + |v|^q)|x|^{-tq} dx$. Consequently, taking into account (3.23), we obtain (1.6).

It remains to prove that $\nu_0^{q/q_t^*} \leq (\mu_0 - \sigma\omega_0)/S_\sigma$. Fix $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$, $\varphi(0) = 1$ and $\text{supp } \varphi = \overline{B_1}$. Take $\varepsilon > 0$ and put $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$, $x \in \mathbb{R}^N$. Then, arguing as in (3.13), by (1.3) for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$\begin{aligned} S_\sigma \left(\int_{\mathbb{R}^N} |\varphi_\varepsilon|^{q_t^*} |u_n|^\alpha |v_n|^\beta dx \right)^{q/q_t^*} &= \iint_{\mathbb{R}^{2N}} \frac{|(\varphi_\varepsilon u_n)(x) - (\varphi_\varepsilon u_n)(y)|^q}{|x - y|^{N+tq}} dx dy \\ &\quad + \iint_{\mathbb{R}^{2N}} \frac{|(\varphi_\varepsilon v_n)(x) - (\varphi_\varepsilon v_n)(y)|^q}{|x - y|^{N+tq}} dx dy \\ &\quad - \sigma \int_{\mathbb{R}^N} |\varphi_\varepsilon|^q \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx \\ &\leq (1 + \delta) \int_{\mathbb{R}^N} |\varphi_\varepsilon|^q \left(|D^t u_n|^q + |D^t v_n|^q \right) dy \\ &\quad + C_\delta \int_{\mathbb{R}^N} |D^t \varphi_\varepsilon|^q (|u_n|^q + |v_n|^q) dy \\ &\quad - \sigma \int_{\mathbb{R}^N} |\varphi_\varepsilon|^q \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx \end{aligned} \quad (3.25)$$

since $\alpha + \beta = q_t^*$. Now, as in (3.14) we know that

$$\lim_{\varepsilon \rightarrow 0^+} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\varphi_\varepsilon|^{q_t^*} |u_n|^\alpha |v_n|^\beta dx = \nu_0, \quad (3.26)$$

while Lemma 2.5 implies

$$\lim_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |D^t \varphi_\varepsilon|^q (|u_n|^q + |v_n|^q) dy = 0. \quad (3.27)$$

Finally, from (1.6) and the fact that $\omega \geq \omega_0 \delta_0$ we get

$$\lim_{\varepsilon \rightarrow 0^+} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\varphi_\varepsilon|^q \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx = \lim_{\varepsilon \rightarrow 0^+} \int_{B_\varepsilon} |\varphi_\varepsilon|^q d\omega \geq \omega_0. \quad (3.28)$$

Hence, by (3.26)–(3.28) we obtain passing to the limit as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0^+$ in (3.25)

$$S_\sigma \nu_0^{q/q_t^*} \leq (1 + \delta)\mu_0 - \sigma\omega_0$$

for all $\delta > 0$. Finally, sending $\delta \rightarrow 0^+$ we obtain (1.7). This concludes the proof. $\square$

Let us now turn to the proof of the *CC* principle at infinity, that is Theorem 1.3. The strategy is similar to the one used before but instead of taking a function that concentrates at the finite points x_j , we consider the function η_R of Lemma 2.6, which is a function that in some sense "concentrates at infinity".

Proof of Theorem 1.3. Let us show (1.11)–(1.13). Let η_R be defined as in Lemma 2.6. We can write

$$\begin{aligned} \int_{\mathbb{R}^N} (|D^t u_n|^q + |D^t v_n|^q) dx \\ = \int_{\mathbb{R}^N} (|D^t u_n|^q + |D^t v_n|^q) |\eta_R|^q dx + \int_{\mathbb{R}^N} (|D^t u_n|^q + |D^t v_n|^q) (1 - |\eta_R|^q) dx. \end{aligned} \quad (3.29)$$

We first observe that

$$\int_{B_{2R}^c} (|D^t u_n|^q + |D^t v_n|^q) dx \leq \int_{\mathbb{R}^N} (|D^t u_n|^q + |D^t v_n|^q) |\eta_R|^q dx \leq \int_{B_R^c} (|D^t u_n|^q + |D^t v_n|^q) dx$$

and so by (1.9)

$$\mu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|D^t u_n|^q + |D^t v_n|^q) |\eta_R|^q dx. \quad (3.30)$$

On the other hand, since μ is finite and $1 - |\eta_R|^q$ has compact support and $\eta_R \rightarrow 0$ a.e. in $\mathbb{R}^N$, we have by the definition of μ and the dominated convergence theorem that

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} (|D^t u_n|^q + |D^t v_n|^q) (1 - |\eta_R|^q) dx = \lim_{R \rightarrow \infty} (1 - |\eta_R|^q) d\mu = \mu(\mathbb{R}^N). \quad (3.31)$$

Using (3.30) and (3.31) in (3.29) we obtain (1.12)

Arguing similarly for ν and ω , we can see that

$$\begin{aligned} \nu_\infty &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\eta_R|^{q_t^*} |u_n|^\alpha |v_n|^\beta dx, \\ \omega_\infty &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\eta_R|^q \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx \end{aligned} \quad (3.32)$$

and so (1.11)–(1.13) are proved in the same way.

In order to show (1.14), let us consider again the function η_R . Then, since $0 \leq \eta_R \leq 1$, by (1.3) applied to $(\eta_R u_n, \eta_R v_n) \in \mathcal{D}$, we get for all n and for any $\delta > 0$

$$\begin{aligned} S \left(\int_{\mathbb{R}^N} |\eta_R|^{q_t^*} |u_n|^\alpha |v_n|^\beta dx \right)^{q/q_t^*} &\leq ([\eta_R u_n]_{t,q}^q + [\eta_R v_n]_{t,q}^q) - \sigma \int_{\mathbb{R}^N} |\eta_R|^q \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx \\ &\leq (1 + \delta) \int_{\mathbb{R}^N} |\eta_R|^q (|D^t u_n|^q + |D^t v_n|^q) dx \\ &\quad + C_\delta \int_{\mathbb{R}^N} |D^t \eta_R|^q (|u_n|^q + |v_n|^q) dx \\ &\quad - \sigma \int_{\mathbb{R}^N} |\eta_R|^q \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx \end{aligned} \quad (3.33)$$

since $\alpha + \beta = q_t^*$. Now, from Lemma 2.6 we know that

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} |D^t \eta_R|^q (|u_n|^q + |v_n|^q) dx = 0. \quad (3.34)$$

Thus, using (3.30), (3.32) and (3.34) in (3.33) and sending $\delta \rightarrow 0^+$ we get the desired conclusion. This completes the proof of Theorem 1.3. $\square$

4. Existence for system (S)

In this section we assume, without further mentioning, that the structural assumptions required in Theorem 1.1 hold. We say that the couple $(u, v) \in W$ is a (weak) *solution* of problem (S) if

$$\begin{aligned} & \langle u, \varphi \rangle_{s,p} + \langle u, \varphi \rangle_{t,q} + \langle v, \psi \rangle_{s,p} + \langle v, \psi \rangle_{t,q} + \int_{\mathbb{R}^N} (|u|^{p-2} + |u|^{q-2}) u \varphi dx + \int_{\mathbb{R}^N} (|v|^{p-2} + |v|^{q-2}) v \psi dx \\ &= \sigma \int_{\mathbb{R}^N} \frac{|u|^{q-2} u \varphi + |v|^{q-2} v \psi}{|x|^{tq}} dx + \lambda \int_{\mathbb{R}^N} \{H_u(x, u, v) \varphi + H_v(x, u, v) \psi\} dx \\ &+ \frac{\alpha}{q_t^*} \int_{\mathbb{R}^N} |u|^{\alpha-2} u |v|^\beta \varphi dx + \frac{\beta}{q_t^*} \int_{\mathbb{R}^N} |u|^\alpha |v|^{\beta-2} v \psi dx \end{aligned}$$

for any $(\varphi, \psi) \in W$, where

$$\begin{aligned} \langle u, \varphi \rangle_{s,p} &= \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} dx dy, \\ \langle u, \varphi \rangle_{t,q} &= \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{q-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+tq}} dx dy \end{aligned}$$

and similarly for v . Clearly, the (weak) solutions of (S) are exactly the critical points of the Euler–Lagrange functional $I : W \rightarrow \mathbb{R}$ associated to (S) and given by

$$\begin{aligned} I(u, v) &= \frac{1}{p} \|u\|_{s,p}^p + \frac{1}{q} \|u\|_{t,q}^q + \frac{1}{p} \|v\|_{s,p}^p + \frac{1}{q} \|v\|_{t,q}^q - \lambda \int_{\mathbb{R}^N} H(x, u, v) dx \\ &- \frac{\sigma}{q} \int_{\mathbb{R}^N} \frac{|u|^q + |v|^q}{|x|^{tq}} dx - \frac{1}{q_t^*} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx, \end{aligned}$$

which is well defined and of class $C^1(W)$ by (H_1) , (H_3) and (2.1). Note that for any $\varepsilon > 0$ conditions (H_1) – (H_2) give the existence of $C_\varepsilon > 0$ such that for a.e. $x \in \mathbb{R}^N$

$$|H(x, u, v)| \leq \varepsilon |(u, v)|^m + C_\varepsilon |(u, v)|^m \quad \text{for all } (u, v) \in \mathbb{R}^2. \quad (4.1)$$

Furthermore, since $\alpha, \beta > 1$ are such that $\alpha + \beta = q_t^*$, then (3.2) yields

$$\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \leq C_{q_t^*}^{q_t^*/q} ([u]_{t,q}^q + [v]_{t,q}^q)^{q_t^*/q} \leq C_{q_t^*}^{q_t^*/q} \|(u, v)\|^{q_t^*} \quad (4.2)$$

for all $(u, v) \in W$.

The structural assumptions of Theorem 1.1 imply that the functional I possesses the geometric features of the mountain pass theorem of Ambrosetti and Rabinowitz. In particular, we have the following

Lemma 4.1 (Geometry of mountain pass). *For all $\sigma \in (-\infty, \mathcal{H}_{t,q})$ and $\lambda > 0$ we have*

(i) There exists a couple $(e_1, e_2) \in C_c^\infty(\mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N)$, independent of λ and σ^+ , with $e_1 \geq 0$, $e_2 \geq 0$ in $\mathbb{R}^N$ and $\int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta dx > 0$ such that

$$I(e_1, e_2) < 0, \quad \|(e_1, e_2)\| \geq 2;$$

(ii) There exist numbers $j = j(\sigma, \lambda) > 0$ and $\rho = \rho(\sigma, \lambda) \in (0, 1]$ such that

$$I(u, v) \geq j \quad \text{for any } (u, v) \in W \text{ with } \|(u, v)\| = \rho.$$

Proof. The proof of Lemma 4.1 is almost standard and very similar to the demonstrations which appear in Lemma 2.4 of [17]. Therefore, we only sketch it.

(i) Let $(u, v) \in C_c^\infty(\mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N)$ be such that $u \geq 0$ and $v \geq 0$ in $\mathbb{R}^N$, $\|(u, v)\| = 1$ and $\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx > 0$. Then, by (H_3) we have as $\tau \rightarrow \infty$

$$I(\tau u, \tau v) \leq \frac{\tau^p}{p} + \frac{\tau^q}{q} + \sigma^- \frac{t^q}{q} \left(\|u\|_{H_{t,q}}^q + \|v\|_{H_{t,q}}^q \right) - \frac{\tau^{q_t^*}}{q_t^*} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \rightarrow -\infty$$

since $1 < p < q < q_t^*$. Hence, taking $(e_1, e_2) = \tau_0(u, v)$, with $\tau_0 > 0$ sufficiently large, we obtain at once (i).

(ii) For the second part, fix any $(u, v) \in W$, with $\|(u, v)\| \leq 1$. Then view of (4.2), for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that, by (2.3) and (4.2), we have

$$\begin{aligned} I(u, v) &\geq \frac{1}{p} (\|u\|_{s,p}^p + \|v\|_{s,p}^p) + \frac{1}{q} (\|u\|_{t,q}^q + \|v\|_{t,q}^q) - \frac{\sigma^+}{q} (\|u\|_{H_{t,q}}^q + \|v\|_{H_{t,q}}^q) \\ &\quad - \lambda \varepsilon \|(u, v)\|_{\mathbf{m}}^m - \lambda C_\varepsilon \|(u, v)\|_m^m - \frac{1}{q_t^*} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \\ &\geq \frac{2^{1-p}}{p} \|(u, v)\|^p + \frac{2^{1-q}}{q} \|(u, v)\|^q - \frac{\sigma^+}{q \mathcal{H}_{t,q}} \|(u, v)\|^q \\ &\quad - \lambda \varepsilon C_{\mathbf{m}}^m \|(u, v)\|_{\mathbf{m}}^m - \lambda C_\varepsilon C_m^m \|(u, v)\|_m^m - \frac{1}{q_t^*} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \\ &\geq \left(\frac{2^{1-p}}{p} - \lambda \varepsilon C_{\mathbf{m}}^m \right) \|(u, v)\|^p + \frac{1}{q} \left(2^{1-q} - \frac{\sigma^+}{\mathcal{H}_{t,q}} \right) \|(u, v)\|^q - \lambda C_\varepsilon C_m^m \|(u, v)\|_m^m - \frac{C_{q_t^*}^{q_t^*}}{q_t^*} \|(u, v)\|^{q_t^*} \\ &= A \|(u, v)\|^p + B \|(u, v)\|^q - C \|(u, v)\|_m^m - D \|(u, v)\|^{q_t^*}, \end{aligned}$$

where

$$A = \frac{2^{1-p}}{p} - \lambda \varepsilon C_{\mathbf{m}}^m, \quad B = \frac{1}{q} \left(2^{1-q} - \frac{\sigma^+}{\mathcal{H}_{t,q}} \right), \quad C = \lambda C_\varepsilon C_m^m, \quad D = \frac{C_{q_t^*}^{q_t^*}}{q_t^*}.$$

Now, choosing $\varepsilon > 0$ so small that $A > 0$, we can conclude that there exists $\rho \in (0, 1]$ such that

$$\max_{\tau \in [0,1]} y(\tau) = y(\rho) > 0, \quad \text{where } y(\tau) = A \tau^p + B \tau^q - C \tau^m - D \tau^{q_t^*},$$

since $q, m \in (p, q_t^*)$. Consequently, $I(u, v) \geq y(\rho) = j$ for all $(u, v) \in W$, with $\|(u, v)\| = \rho$, as desired. $\square$

Thanks to the geometry given in Lemma 4.1, for any $\sigma \in (-\infty, \mathcal{H}_{t,q})$, we can introduce the special levels of I

$$c_{\sigma,\lambda} = \inf_{\gamma \in \Gamma} \max_{\tau \in [0,1]} I(\gamma(\tau)), \quad (4.3)$$

where

$$\Gamma = \{\gamma \in C([0,1], W) : \gamma(0) = (0,0), I(\gamma(1)) < 0\}.$$

Obviously, $c_{\sigma,\lambda} > 0$ thanks to Lemma 4.1, since in particular $\|(e_1, e_2)\| > \rho$. Let us also report a crucial asymptotic condition of the level $c_{\sigma,\lambda}$. We emphasize that this result was first observed in the local scalar case in [15], and then extended to the vectorial local case in [17], cf. Lemma 2.5, taking into account also Hardy terms. We also refer to [4] for the nonlocal case.

Lemma 4.2. *For any $\sigma \in (-\infty, \mathcal{H}_{t,q})$ we have*

$$\lim_{\lambda \rightarrow \infty} c_{\sigma,\lambda} = 0.$$

In particular, there exists $\lambda_ = \lambda_*(\sigma, N, t, q, \theta) > 0$ such that*

$$0 < c_{\sigma,\lambda} < \left(\frac{1}{\theta} - \frac{1}{q_t^*}\right) \min\{S^{N/tq}, S_\sigma^{N/tq}\} \quad \text{for all } \lambda \geq \lambda_*. \quad (4.4)$$

Proof. Fix $\sigma \in (-\infty, \mathcal{H}_{t,q})$ and $\lambda > 0$. Let (e_1, e_2) be the couple determined in Lemma 4.1. By the mountain pass geometry of the functional I , there exists $\tau_{\sigma,\lambda} > 0$ verifying $I(\tau_{\sigma,\lambda}e_1, \tau_{\sigma,\lambda}e_2) = \max_{\tau \geq 0} I(\tau e_1, \tau e_2)$. Therefore, $\langle I'(\tau_{\sigma,\lambda}e_1, \tau_{\sigma,\lambda}e_2), (e_1, e_2) \rangle = 0$, that is

$$\begin{aligned} & \tau_{\sigma,\lambda}^{p-1} \left(\|e_1\|_{s,p}^p + \|e_2\|_{s,p}^p \right) + \tau_{\sigma,\lambda}^{q-1} \left(\|e_1\|_{t,q}^q + \|e_2\|_{t,q}^q \right) - \tau_{\sigma,\lambda}^{q-1} \sigma \int_{\mathbb{R}^N} \frac{|e_1|^q + |e_2|^q}{|x|^{tq}} dx \\ &= \lambda \int_{\mathbb{R}^N} \nabla H(x, \tau_{\sigma,\lambda}e_1, \tau_{\sigma,\lambda}e_2) \cdot (e_1, e_2) dx + \tau_{\sigma,\lambda}^{q_t^*-1} \int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta dx \\ &\geq \tau_{\sigma,\lambda}^{q_t^*-1} \int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta dx \end{aligned} \quad (4.5)$$

by (H_2) , being $\lambda > 0$. This implies, since $\int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta dx > 0$ by Lemma 4.1 and $p < q < q_t^*$, that $\{\tau_{\sigma,\lambda}\}_\lambda$ is bounded in $\mathbb{R}$.

Fix now an arbitrary sequence $\{\lambda_n\}_n \subset \mathbb{R}^+$ such that $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Obviously $\{\tau_{\sigma,\lambda_n}\}_n$ is bounded. Thus, there exist a subsequence of $\{\lambda_n\}_n$, still denoted by $\{\lambda_n\}_n$ and a number $\tau_0 \geq 0$ such that $\tau_{\sigma,\lambda_n} \rightarrow \tau_0$ as $n \rightarrow \infty$. We claim that $\tau_0 = 0$. Otherwise, (H_1) and the dominated convergence theorem yield as $n \rightarrow \infty$

$$\int_{\mathbb{R}^N} (H_u(\tau_{\sigma,\lambda_n}e_1, \tau_{\sigma,\lambda_n}e_2)e_1 + H_v(\tau_{\sigma,\lambda_n}e_1, \tau_{\sigma,\lambda_n}e_2)e_2) dx \rightarrow \int_{\mathbb{R}^N} (H_u(\tau_0e_1, \tau_0e_2)e_1 + H_v(\tau_0e_1, \tau_0e_2)e_2) dx > 0,$$

by (H_3) . Consequently, recalling that $\lambda_n \rightarrow \infty$, we have as $n \rightarrow \infty$

$$\tau_{\sigma,\lambda_n}^{p-1} \left(\|e_1\|_{s,p}^p + \|e_2\|_{s,p}^p \right) + \tau_{\sigma,\lambda_n}^{q-1} \left(\|e_1\|_{t,q}^q + \|e_2\|_{t,q}^q \right) \rightarrow L \in (0, \infty)$$

while

$$\lambda_n \int_{\mathbb{R}^N} (H_u(\tau_{\sigma,\lambda_n}e_1, \tau_{\sigma,\lambda_n}e_2)e_1 + H_v(\tau_{\sigma,\lambda_n}e_1, \tau_{\sigma,\lambda_n}e_2)e_2) dx + \tau_{\sigma,\lambda_n}^{q_t^*-1} \int_{\mathbb{R}^N} |e_1|^\alpha |e_2|^\beta dx \rightarrow \infty,$$

which gives a contradiction in view of (4.5). Thus $\tau_0 = 0$ and so $\tau_{\sigma,\lambda} \rightarrow 0$ as $\lambda \rightarrow \infty$, because of the arbitrariness of the sequence $\{\lambda_n\}_n$.

Now the path $\gamma(\tau) = \tau(e_1, e_2)$, $\tau \in [0, 1]$, belongs to Γ and so by Lemma 4.1 we infer

$$\begin{aligned} 0 < c_{\sigma,\lambda} &\leq \max_{\tau \geq 0} I(\gamma(\tau)) = I(\tau_{\sigma,\lambda}e_1, \tau_{\sigma,\lambda}e_2) \\ &\leq \frac{\tau_{\sigma,\lambda}^p}{p} \left(\|e_1\|_{s,p}^p + \|e_2\|_{s,p}^p \right) + \frac{\tau_{\sigma,\lambda}^q}{q} \left(\|e_1\|_{t,q}^q + \|e_2\|_{t,q}^q \right) + \frac{\sigma^-}{q} \tau_{\sigma,\lambda}^q \left(\|e_1\|_{H_{t,q}}^q + \|e_2\|_{H_{t,q}}^q \right) \rightarrow 0 \end{aligned}$$

as $\lambda \rightarrow \infty$, since (e_1, e_2) does not depend on λ . In particular, there exists $\lambda_* = \lambda_*(\sigma, N, t, q, \theta) > 0$ sufficiently large such that (4.4) holds. $\square$

Our goal is now to prove that the Palais–Smale sequences at the levels $c_{\sigma,\lambda}$ weakly converge in W and that the weak limits are critical points of the functional I .

Lemma 4.3. *Let $\{(u_n, v_n)\}_n \subset W$ be a Palais–Smale sequence of I at the level $c_{\sigma,\lambda}$ for all $\sigma \in (-\infty, \mathcal{H}_{t,q})$ and $\lambda > 0$. Then,*

- (i) *up a subsequence, $(u_n, v_n) \rightharpoonup (u_{\sigma,\lambda}, v_{\sigma,\lambda})$ in W as $n \rightarrow \infty$,*
- (ii) *the weak limit $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ is a critical point of I .*

Proof. (i) Let $\{(u_n, v_n)\}_n \subset W$ be a Palais–Smale sequence of I at level $c_{\sigma,\lambda}$ for any $\lambda > 0$, that is

$$I(u_n, v_n) \rightarrow c_{\sigma,\lambda} \quad \text{and} \quad I'(u_n, v_n) \rightarrow 0 \quad \text{in } W' \text{ as } n \rightarrow \infty. \quad (4.6)$$

In order to prove our claim, we start showing that $\{(u_n, v_n)\}_n$ is bounded in W . Thanks to (4.6), there exists $d_{\sigma,\lambda} > 0$ such that as $n \rightarrow \infty$

$$\begin{aligned} c_{\sigma,\lambda} + d_{\sigma,\lambda} \|(u_n, v_n)\| + o_n(1) &\geq I(u_n, v_n) - \frac{1}{\theta} \langle I'(u_n, v_n), (u_n, v_n) \rangle \\ &= \left(\frac{1}{p} - \frac{1}{\theta} \right) \left(\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p \right) + \left(\frac{1}{q} - \frac{1}{\theta} \right) \left(\|u_n\|_{t,q}^q + \|v_n\|_{t,q}^q \right) \\ &\quad - \sigma \left(\frac{1}{q} - \frac{1}{\theta} \right) \left(\|u_n\|_{H_{t,q}}^q + \|v_n\|_{H_{t,q}}^q \right) \\ &\quad + \frac{\lambda}{\theta} \int_{\mathbb{R}^N} \left(\nabla H(x, u_n, v_n) \cdot (u_n, v_n) - \theta H(x, u_n, v_n) \right) dx \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{q_t^*} \right) \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dx \\ &\geq \left(\frac{1}{q} - \frac{1}{\theta} \right) \left(\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p + \|u_n\|_{t,q}^q + \|v_n\|_{t,q}^q \right), \\ &\quad - \frac{\sigma^+}{\mathcal{H}_{t,q}} \left(\frac{1}{q} - \frac{1}{\theta} \right) \left(\|u_n\|_{t,q}^q + \|v_n\|_{t,q}^q \right) \\ &\geq \ell \left(\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p + \|u_n\|_{t,q}^q + \|v_n\|_{t,q}^q \right) \end{aligned} \quad (4.7)$$

by (H_2) where $d_{\sigma,\lambda} = o_n(1)$ for every λ and σ and

$$\ell = \left(1 - \frac{\sigma^+}{\mathcal{H}_{t,q}} \right) \left(\frac{1}{q} - \frac{1}{\theta} \right) > 0, \quad (4.8)$$

since $q < \theta < q_t^*$.

Assume by contradiction that $\|(u_n, v_n)\| \rightarrow \infty$ as $n \rightarrow \infty$. Then, passing if necessary to a subsequence, still labelled by $\{(u_n, v_n)\}_n$, at least one component has at least one Sobolev norm diverging as n diverges. Thus, we could have either at least one component, with exactly one Sobolev norm divergent, say $\|\cdot\|_{s,p}$, or two components with different norms $\|\cdot\|_{s,p}$ and $\|\cdot\|_{t,q}$ divergent, respectively. In the first case,

$$\begin{aligned} 0 < \ell &\leq d_{\sigma,\lambda} \frac{\|u_n\|_{s,p} + \|v_n\|_{s,p} + \|u_n\|_{t,q} + \|v_n\|_{t,q}}{\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p + \|u_n\|_{t,q}^q + \|v_n\|_{t,q}^q} + o_n(1) \\ &\leq d_{\sigma,\lambda} \left(\frac{\|u_n\|_{s,p} + \|v_n\|_{s,p}}{\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p} + \frac{\|u_n\|_{t,q} + \|v_n\|_{t,q}}{\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p} \right) + o_n(1) \\ &\leq 2^{1/p'} d_{\sigma,\lambda} (\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p)^{-1/p'} + o_n(1) \end{aligned}$$

$$= o_n(1).$$

This gives the required contradiction and proves the claim. On the other hand, in the second case as $n \rightarrow \infty$

$$\begin{aligned} 0 < \ell &\leq d_{\sigma,\lambda} \frac{\|u_n\|_{s,p} + \|v_n\|_{s,p} + \|u_n\|_{t,q} + \|v_n\|_{t,q}}{\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p + \|u_n\|_{t,q}^q + \|v_n\|_{t,q}^q} + o_n(1) \\ &\leq d_{\sigma,\lambda} \left(\frac{\|u_n\|_{s,p} + \|v_n\|_{s,p}}{\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p} + \frac{\|u_n\|_{t,q} + \|v_n\|_{t,q}}{\|u_n\|_{t,q}^q + \|v_n\|_{t,q}^q} \right) + o_n(1) \\ &\leq 2^{1/p'} d_{\sigma,\lambda} (\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p)^{-1/p'} + 2^{1/q'} d_{\sigma,\lambda} (\|u_n\|_{t,q}^q + \|v_n\|_{t,q}^q)^{-1/q'} + o_n(1) \\ &= o_n(1). \end{aligned}$$

Consequently, the claim is proved in all the possible cases. Thus, since $\{(u_n, v_n)\}_n$ is bounded in the reflexive Banach space W , there exist $(u_{\sigma,\lambda}, v_{\sigma,\lambda}) \in W$ such that $(u_n, v_n) \rightharpoonup (u_{\sigma,\lambda}, v_{\sigma,\lambda})$ in W . For simplicity, in what follows we still denote by $\{(u_n, v_n)\}_n$ every subsequence extracted from the original sequence.

(ii) Proving now that the weak limit (u, v) is a weak solution of $(\mathcal{S})$, namely $\langle I'(u_n, v_n), (\phi, \psi) \rangle = 0$ for all $(\phi, \psi) \in W$ is somewhat standard. Indeed, given any couple $(\phi, \psi) \in C_c^\infty(\mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N)$, we know by (4.6) that

$$\begin{aligned} o_n(1) &= \langle I'(u_n, v_n), (\phi, \psi) \rangle = \langle u_n, \phi \rangle_{s,p} + \langle u_n, \phi \rangle_{t,q} + \langle v_n, \psi \rangle_{s,p} + \langle v_n, \psi \rangle_{t,q} \\ &\quad + \int_{\mathbb{R}^N} (|u_n|^{p-2} + |u_n|^{q-2}) u_n \phi dx + \int_{\mathbb{R}^N} (|v_n|^{p-2} + |v_n|^{q-2}) v_n \psi dx \\ &\quad - \sigma \int_{\mathbb{R}^N} \frac{|u|^{q-2} u \phi + |v|^{q-2} v \psi}{|x|^{tq}} dx \\ &\quad - \lambda \int_{\mathbb{R}^N} \{H_u(x, u_n, v_n) \phi + H_v(x, u_n, v_n) \psi\} dx \\ &\quad - \frac{\alpha}{q_t^*} \int_{\mathbb{R}^N} |u_n|^{\alpha-2} u_n |v_n|^\beta \phi dx - \frac{\beta}{q_t^*} \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^{\beta-2} v_n \psi dx. \end{aligned} \tag{4.9}$$

Now, from part (i), by (2.3), up to a subsequence we can assume that

$$\begin{aligned} (u_n, v_n) &\rightarrow (u_{\sigma,\lambda}, v_{\sigma,\lambda}) \text{ in } L_{\text{loc}}^r(\mathbb{R}^N) \times L_{\text{loc}}^r(\mathbb{R}^N) \text{ for all } r \in [1, q_t^*), \\ (u_n, v_n) &\rightarrow (u_{\sigma,\lambda}, v_{\sigma,\lambda}) \text{ a.e. in } \mathbb{R}^N, \\ |u_n| &\leq g_R, \quad |v_n| \leq g_R \text{ a.e. in } \mathbb{R}^N, \end{aligned} \tag{4.10}$$

for some $g_R \in L^q(B_R)$ and all $R > 0$. Furthermore, by (2.2) and the Hölder inequality, since $\alpha + \beta = q_t^*$, we obtain

$$\begin{aligned} \int_{\mathbb{H}^n} ||u_n|^{\alpha-1} |v_n|^\beta|^{\frac{q_t^*}{q_t^*-1}} dx &\leq \|u_n\|_{q_t^*}^{\frac{q_t^*(\alpha-1)}{q_t^*-1}} \|v_n\|_{q_t^*}^{\frac{q_t^*(\beta-1)}{q_t^*-1}} \leq C_{q_t^*}^{q_t^*/q} [u_n]_{t,q}^{\frac{q_t^*(\alpha-1)}{q_t^*-1}} [v_n]_{t,q}^{\frac{q_t^*(\beta-1)}{q_t^*-1}} \\ &\leq C_{q_t^*}^{q_t^*/q} \|(u_n, v_n)\|_{q_t^*}^{q_t^*} \leq C, \end{aligned}$$

where $C > 0$ is a suitable constant. Similarly,

$$\int_{\mathbb{R}^N} ||u_n|^\alpha |v_n|^{\beta-1}|^{\frac{q_t^*}{q_t^*-1}} dx \leq C.$$

Consequently, again up to a subsequence, we have

$$\begin{aligned} |u_n|^{\alpha-2} u_n |v_n|^\beta &\rightharpoonup |u_{\sigma,\lambda}|^{\alpha-2} u_{\sigma,\lambda} |v_{\sigma,\lambda}|^\beta \text{ in } L^{q_t^*/(q_t^*-1)}(\mathbb{R}^N), \\ |u_n|^\alpha |v_n|^{\beta-2} v_n &\rightharpoonup |u_{\sigma,\lambda}|^\alpha |v_{\sigma,\lambda}|^{\beta-2} v_{\sigma,\lambda} \text{ in } L^{q_t^*/(q_t^*-1)}(\mathbb{R}^N), \end{aligned} \tag{4.11}$$

thanks to (4.10). Consider now the sequences

$$U_n(x, y) = \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x - y|^{(N+sp)/p'}}, \quad V_n(x, y) = \frac{|v_n(x) - v_n(y)|^{p-2}(v_n(x) - v_n(y))}{|x - y|^{(N+sp)/p'}}$$

for $p' = p/(p-1)$ and put

$$U(x, y) = \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{(N+sp)/p'}}, \quad V(x, y) = \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x - y|^{(N+sp)/p'}}.$$

Then it is standard to see that $\{(U_n, V_n)\}_n$ is bounded in $L^{p'}(\mathbb{R}^{2N}; \mathbb{R}^2)$ and so there exist $\mathcal{U}, \mathcal{V}$ in $L^{p'}(\mathbb{R}^{2N})$ such that $U_n \rightharpoonup \mathcal{U}$ and $V_n \rightharpoonup \mathcal{V}$ weakly in $L^{p'}(\mathbb{R}^{2N})$. But $(U_n, V_n) \rightarrow (U, V)$ a.e. in $\mathbb{R}^{2N}$ and so $(\mathcal{U}, \mathcal{V}) = (U, V)$. Therefore,

$$\iint_{\mathbb{R}^{2N}} U_n(x, y) \Phi(x, y) dx dy \rightarrow \iint_{\mathbb{R}^{2N}} U(x, y) \Phi(x, y) dx dy \quad \text{for all } \Phi \in L^p(\mathbb{R}^{2N}),$$

and

$$\iint_{\mathbb{R}^{2N}} V_n(x, y) \Psi(x, y) dx dy \rightarrow \iint_{\mathbb{R}^{2N}} V(x, y) \Psi(x, y) dx dy \quad \text{for all } \Psi \in L^p(\mathbb{R}^{2N}).$$

This holds in particular if we take for any $\phi, \psi \in C_c^\infty(\mathbb{R}^N)$

$$\Phi(x, y) = \frac{\phi(x) - \phi(y)}{|x - y|^{(N+sp)/p}}, \quad \Psi(x, y) = \frac{\psi(x) - \psi(y)}{|x - y|^{(N+sp)/p}},$$

since obviously $\Phi, \Psi \in L^p(\mathbb{R}^{2N})$. Thus,

$$\begin{aligned} \langle u_n, \phi \rangle_{s,p} &= \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+ps}} dx dy \\ &\rightarrow \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+ps}} dx dy = \langle u, \phi \rangle_{s,p}. \end{aligned} \quad (4.12)$$

In a similar way we can see that

$$\langle v_n, \psi \rangle_{s,p} \rightarrow \langle v, \psi \rangle_{s,p}, \quad \langle u_n, \phi \rangle_{t,q} \rightarrow \langle u, \phi \rangle_{t,q}, \quad \langle v_n, \psi \rangle_{t,q} \rightarrow \langle v, \psi \rangle_{t,q}. \quad (4.13)$$

On the other hand, since H has standard subcritical growth we have by (H_1)

$$|H_u(x, u_n, v_n)\phi + H_v(x, u_n, v_n)\psi| \leq \mathfrak{m}|(u_n, v_n)|^{m-1}\phi + mC_1|(u_n, v_n)|^{m-1}\psi \leq \mathfrak{G} \in L^1(B_R),$$

and again the dominated convergence theorem yields as $n \rightarrow \infty$

$$\int_{\mathbb{R}^N} \{H_u(x, u_n, v_n)\phi + H_v(x, u_n, v_n)\psi\} dx \rightarrow \int_{\mathbb{R}^N} \{H_u(x, u_{\sigma,\lambda}, v_{\sigma,\lambda})\phi + H_v(x, u_{\sigma,\lambda}, v_{\sigma,\lambda})\psi\} dx. \quad (4.14)$$

Finally, as $n \rightarrow \infty$

$$\int_{\mathbb{R}^N} \frac{|u_n|^{q-2}u_n\phi + |v_n|^{q-2}v_n\psi}{|x|^{tq}} dx \rightarrow \int_{\mathbb{R}^N} \frac{|u|^{q-2}u\phi + |v|^{q-2}v\psi}{|x|^{tq}} dx, \quad (4.15)$$

since $|u_n|^{q-2}u_n \rightharpoonup |u_{\sigma,\lambda}|^{q-2}u_{\sigma,\lambda}$ in $L^{q'}(\mathbb{R}^N, |x|^{-tq})$ by Proposition A.8 of [5]. In conclusion, passing to the limit in (4.9) using (4.10)–(4.15) we obtain $\langle I'(u_{\sigma,\lambda}, v_{\sigma,\lambda}), (\phi, \psi) \rangle = 0$ for all $(\phi, \psi) \in C_c^\infty(\mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N)$. Consequently, from the density of $C_c^\infty(\mathbb{R}^N)$ in $W^{s,p}(\mathbb{R}^N)$ and in $W^{t,q}(\mathbb{R}^N)$ we obtain that $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ is a critical point of I for any $\sigma \in (-\infty, \mathcal{H}_{t,q})$ and for any $\lambda > 0$. $\square$

Finally, armed with Theorems 1.2 and 1.3, we are able to find special levels $c_{\sigma,\lambda}$ for which the the $(PS)_{c_{\sigma,\lambda}}$ conditions is valid. To this aim, the asymptotic property of the critical levels, cf. Lemma 4.2, and the relations between the singular coefficients of the measures, cf. (1.7) and (1.14), play a fundamental role. In particular, we first prove that for sufficiently large values of λ , the weak- $*$ limits of the critical sequences in W cannot have singular points. Then, we exclude the concentration at infinity, as well. These two facts together imply, thanks to Remark 3.1, the strong convergence of the related sequences and consequently the validity of the $(PS)_{c_{\sigma,\lambda}}$ condition.

Lemma 4.4 ($(PS)_{c_{\sigma,\lambda}}$ condition). *Let $\{(u_n, v_n)\}_n \subset W$ be a Palais-Smale sequence of I at the level $c_{\sigma,\lambda}$. Then, for any $\sigma \in (-\infty, \mathcal{H}_{t,q})$ and any $\lambda \geq \lambda_*$, up to a subsequence $(u_n, v_n) \rightarrow (u_{\sigma,\lambda}, v_{\sigma,\lambda})$ in W .*

Proof. Let $\{(u_n, v_n)\}_n \subset W$ be a Palais-Smale sequence of I at the level $c_{\sigma,\lambda}$. Now, since $\alpha + \beta = q_t^*$ and $\sigma < \mathcal{H}_{t,q}$, from Lemma 4.3 and (3.2) we can apply Theorem 1.2, obtaining the existence of three Radon measures μ, ν, ω , an at most countable set J , a family of points $\{x_j\}_{j \in J} \subset \mathbb{R}^N$, a number $\omega_0 \geq 0$ and two families of nonnegative numbers $\{\mu_j\}_{j \in J \cup \{0\}}$, $\{\nu_j\}_{j \in J \cup \{0\}}$ such that

$$\begin{aligned} |u_n|^\alpha |v_n|^\beta dx &\xrightarrow{*} \nu = |u|^\alpha |v|^\beta dx + \nu_0 \delta_0 + \sum_{j \in J} \nu_j \delta_{x_j}, \\ (|D^t u_n|^q + |D^t v_n|^q) dx &\xrightarrow{*} \mu \geq (|D^t u|^q + |D^t v|^q) dx + \mu_0 \delta_0 + \sum_{j \in J} \mu_j \delta_{x_j} \\ \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx &\xrightarrow{*} \omega = \frac{|u|^q + |v|^q}{|x|^{tq}} dx + \omega_0 \delta_0, \\ \nu_0^{q/q_t^*} &\leq \frac{\mu_0 - \sigma \omega_0}{S_\sigma}, \quad \nu_j^{q/q_t^*} \leq \frac{\mu_j}{S} \text{ for all } j \in J, \end{aligned} \tag{4.16}$$

where S_σ and S are the constants defined in (1.3) while δ_0, δ_{x_j} are the Dirac function at the points 0 and x_j of $\mathbb{R}^N$, respectively. Moreover, from (4.7) and Lemma 4.3 we obtain the crucial formula

$$c_{\sigma,\lambda} + o(1) \geq \ell \left(\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p + \|u_n\|_{t,q}^q + \|v_n\|_{t,q}^q \right) + \left(\frac{1}{\theta} - \frac{1}{q_t^*} \right) \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dx \tag{4.17}$$

as $n \rightarrow \infty$, where ℓ is defined in (4.8).

In order to show the strong convergence, we split the proof into several steps.

Step 1. No concentration at finite points: Let us first show that $\nu_j = 0$ for all $j \in J \cup \{0\}$. Assume by contradiction that $\nu_j > 0$ for some $j \in J \cup \{0\}$, that is x_j is a singular point of the measure ν . Fix a test function $\varphi \in C_0^\infty(\mathbb{R}^N)$, such that $0 \leq \varphi \leq 1$, $\varphi(0) = 1$ and $\text{supp } \varphi \subset B_1$. Take $\varepsilon > 0$ and put $\varphi_{\varepsilon,j}(x) = \varphi((x - x_j)/\varepsilon)$, $x \in \mathbb{R}^N$. Then $\varphi_{\varepsilon,j}(u_n, v_n) \in W$ and so $\langle I'(u_n, v_n), \varphi_{\varepsilon,j}(u_n, v_n) \rangle = o(1)$ as $n \rightarrow \infty$. Therefore,

$$\begin{aligned} o_n(1) &= \langle u_n, \varphi_{\varepsilon,j} u_n \rangle_{s,p} + \langle u_n, \varphi_{\varepsilon,j} u_n \rangle_{t,q} + \langle v_n, \varphi_{\varepsilon,j} v_n \rangle_{s,p} + \langle v_n, \varphi_{\varepsilon,j} v_n \rangle_{t,q} \\ &\quad + \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} (|u_n|^p + |u_n|^q) dx + \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} (|v_n|^p + |v_n|^q) dx - \sigma \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx \\ &\quad - \lambda \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} \{ H_u(x, u_n, v_n) u_n + H_v(x, u_n, v_n) v_n \} dx - \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} |u_n|^\alpha |v_n|^\beta dx \end{aligned} \tag{4.18}$$

as $n \rightarrow \infty$. Now, observe that

$$\begin{aligned}\langle u_n, \varphi_{\varepsilon,j} u_n \rangle_{s,p} &= \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y))}{|x - y|^{N+ps}} (\varphi_{\varepsilon,j}(x) u_n(x) - \varphi_{\varepsilon,j}(y) u_n(y)) dx dy, \\ &= \iint_{\mathbb{R}^{2N}} \varphi_{\varepsilon,j}(x) \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+ps}} dx dy \\ &\quad + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y))}{|x - y|^{N+ps}} u_n(y) (\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y)) dx dy\end{aligned}$$

where, using the Hölder inequality

$$\begin{aligned}&\left| \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y))}{|x - y|^{N+ps}} u_n(y) (\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y)) dx dy \right| \\ &\leq \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-1}}{|x - y|^{(N+ps)/p'}} |u_n(y)| \frac{|\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y)|}{|x - y|^{(N+ps)/p}} dx dy \\ &\leq [u_n]_{s,p}^{p-1} \left(\iint_{\mathbb{R}^{2N}} \frac{|\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y)|^p}{|x - y|^{N+ps}} |u_n(y)|^p dx dy \right)^{1/p} \\ &\leq C \left(\iint_{\mathbb{R}^{2N}} \frac{|\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y)|^p}{|x - y|^{N+ps}} |u_n(y)|^p dx dy \right)^{1/p}.\end{aligned}$$

Thus, Lemma 2.5 yields

$$\lim_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y))}{|x - y|^{N+ps}} u_n(y) (\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y)) dx dy = 0. \quad (4.19)$$

Arguing similarly, we can deduce the same for $\langle u_n, \varphi_{\varepsilon,j} u_n \rangle_{t,q}$, $\langle v_n, \varphi_{\varepsilon,j} v_n \rangle_{s,p}$ and $\langle v_n, \varphi_{\varepsilon,j} v_n \rangle_{t,q}$. On the other hand, from (4.10) it follows that

$$\lim_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} (|u_n|^p + |u_n|^q) dx = 0 \quad (4.20)$$

and similarly in the v component. Finally, using the subcritical growth of H , that is (H_1) , and again (4.10) we obtain

$$\begin{aligned}0 &\leq \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} \{H_u(x, u_n, v_n) u_n + H_v(x, u_n, v_n) v_n\} dx \\ &\leq \int_{B_\varepsilon(x_j)} (\mathfrak{m} |(u_n, v_n)|^{\mathfrak{m}} + m C_1 |(u_n, v_n)|^m) dx \\ &\rightarrow \int_{B_\varepsilon(x_j)} (\mathfrak{m} |(u_{\sigma,\lambda}, v_{\sigma,\lambda})|^{\mathfrak{m}} + m C_1 |(u_{\sigma,\lambda}, v_{\sigma,\lambda})|^m) dx\end{aligned}$$

and so

$$\lim_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} \{H_u(x, u_n, v_n) u_n + H_v(x, u_n, v_n) v_n\} dx = 0. \quad (4.21)$$

In conclusion, passing to the limit in (4.18) using (4.16) and (4.19)–(4.21) we obtain the crucial inequality for all $j \in J \cup \{0\}$

$$\int_{\mathbb{R}^N} \varphi_{\varepsilon,j} d\mu + o_\varepsilon(1) \leq \sigma \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} d\omega + \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} d\nu \quad \text{as } \varepsilon \rightarrow 0^+. \quad (4.22)$$

Consider first $j \in J$. Now fix $\lambda \geq \lambda_*$ so that by Lemma 4.2

$$0 < c_{\sigma,\lambda} < \left(\frac{1}{\theta} - \frac{1}{q_t^*}\right) \min\{S_\sigma^{N/tq}, S^{N/tq}\}. \quad (4.23)$$

Thus, (4.16) and (4.22) imply $S\nu_j^{q/q_t^*} \leq \mu_j \leq \nu_j$ and, since by contradiction $\nu_j > 0$, this yields $\nu_j \geq S^{N/tq}$. Moreover, by (4.16) and (4.17) we know that

$$c_{\sigma,\lambda} + o(1) \geq \left(\frac{1}{\theta} - \frac{1}{q_t^*}\right) \int_{\mathbb{R}^N} \varphi_{\varepsilon,j} d\nu$$

as $n \rightarrow \infty$ since $\varphi_{\varepsilon,j} \leq 1$ and so, sending $n \rightarrow \infty$ and $\varepsilon \rightarrow 0^+$, we have for all $\lambda \geq \lambda_*$

$$c_{\sigma,\lambda} \geq \left(\frac{1}{\theta} - \frac{1}{q_t^*}\right) \nu_j \geq \left(\frac{1}{\theta} - \frac{1}{q_t^*}\right) S^{N/tq} > 0,$$

where λ_* is obtained in Lemma 4.2. This is an obvious contradiction by (4.23). Hence, $\nu_j = \mu_j = 0$ for all $j \in J$ and for all $\lambda \geq \lambda_*$, as desired.

Step 2. No concentration at 0: Similarly, we prove that $\nu_0 = 0$. Indeed, taking as test function $\varphi_\varepsilon = \varphi_{\varepsilon,0}$, namely when the center of the ball is 0, then (4.16) and (4.22) give $S_\sigma \nu_0^{q/q_t^*} + \sigma \omega_0 \leq \mu_0 \leq \sigma \omega_0 + \nu_0$. Assume by contradiction that $\nu_0 \neq 0$. Then, $\nu_0 \geq S_\sigma^{N/tq}$. As above, by (4.17) we obtain as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0^+$

$$c_{\sigma,\lambda} \geq \left(\frac{1}{\theta} - \frac{1}{q_t^*}\right) \nu_j \geq \left(\frac{1}{\theta} - \frac{1}{q_t^*}\right) S_\sigma^{N/tq} > 0,$$

which is again a contradiction by (4.23). Therefore, $\nu_0 = 0$ and so $\mu_0 = \sigma \omega_0$ for all $\lambda \geq \lambda_*$.

In summary, we have shown that there exists $\lambda_* = \lambda_*(\sigma, N, t, q, \theta)$ such that for all $\lambda \geq \lambda_*$

$$|u_n|^\alpha |v_n|^\beta dx \xrightarrow{*} \nu = |u_{\sigma,\lambda}|^\alpha |v_{\sigma,\lambda}|^\beta dx \quad \text{in } \mathcal{M}(\mathbb{R}^N) \quad (4.24)$$

as $n \rightarrow \infty$, by (4.16). In particular, for all $\phi \in C_0^\infty(\mathbb{R}^N)$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \phi |u_n|^\alpha |v_n|^\beta dx = \int_{\mathbb{R}^N} \phi |u_{\sigma,\lambda}|^\alpha |v_{\sigma,\lambda}|^\beta dx.$$

We prove now that $\omega_0 = 0$. By (1.1) we get for all $\delta > 0$ the existence of a constant C_δ such that

$$\begin{aligned} \mathcal{H}_{t,q} \int_{\mathbb{R}^N} \frac{|\varphi_\varepsilon u_n|^q + |\varphi_\varepsilon v_n|^q}{|x|^{tq}} dx &\leq [\varphi_\varepsilon u_n]_{t,q} + [\varphi_\varepsilon v_n]_{t,q} \\ &\leq (1 + \delta) \int_{\mathbb{R}^N} |\varphi_\varepsilon|^q (|D^t u_n|^q + |D^t v_n|^q) dx \\ &\quad + C_\delta \int_{\mathbb{R}^N} |D^t \varphi_\varepsilon|^q (|u_n|^q + |v_n|^q) dx \end{aligned} \quad (4.25)$$

where by Lemma 2.5

$$\lim_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |D^t \varphi_\varepsilon|^q (|u_n|^q + |v_n|^q) dx = 0.$$

Therefore, by (4.16) and (4.32) we have for all $\delta > 0$

$$\mathcal{H}_{t,q} \int_{\mathbb{R}^N} |\varphi_\varepsilon|^q d\omega + o(1) \leq (1 + \delta) \int_{\mathbb{R}^N} |\varphi_\varepsilon|^q d\mu,$$

which yields as $\delta \rightarrow 0^+$ that $\mathcal{H}_{t,q}\omega_0 \leq \mu_0$. Thus, assuming by contradiction that $\omega_0 > 0$, we obtain by

$$\mathcal{H}_{t,q}\omega_0 \leq \mu_0 = \sigma\omega_0 < \mathcal{H}_{t,q}\omega_0$$

which is obviously impossible. Hence $\omega_0 = 0$ for all $\lambda \geq \lambda_*$ and so by (4.16)

$$\frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx \xrightarrow{*} \omega = \frac{|u_{\sigma,\lambda}|^q + |v_{\sigma,\lambda}|^q}{|x|^{tq}} dx \quad \text{in } \mathcal{M}(\mathbb{R}^N). \quad (4.26)$$

In particular, for all $\phi \in C_0^\infty(\mathbb{R}^N)$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \phi \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx = \int_{\mathbb{R}^N} \phi \frac{|u_{\sigma,\lambda}|^q + |v_{\sigma,\lambda}|^q}{|x|^{tq}} dx.$$

As a consequence, also $\mu_0 = 0$

Step 3. No concentration at infinity: Using the notations of Theorem 1.3, in view of Remark 3.1 if we manage to prove that also $\nu_\infty = \omega_\infty = 0$, then by (4.24) and (4.26) we can conclude that the corresponding convergences are strong. We argue as before, but instead of taking a function that concentrates at the finite points x_j , we consider the function η_R of Lemma 2.6, that is a function which in some sense "concentrates at infinity". Clearly, $(\eta_R u_n, \eta_R v_n) \in W$ and so $\langle I'(u_n, v_n), (\eta_R u_n, \eta_R v_n) \rangle = o(1)$ as $n \rightarrow \infty$. Therefore, as $n \rightarrow \infty$

$$\begin{aligned} o_n(1) &= \langle u_n, \eta_R u_n \rangle_{s,p} + \langle u_n, \eta_R u_n \rangle_{t,q} + \langle v_n, \eta_R v_n \rangle_{s,p} + \langle v_n, \eta_R v_n \rangle_{t,q} \\ &\quad + \int_{\mathbb{R}^N} \eta_R (|u_n|^p + |u_n|^q) dx + \int_{\mathbb{R}^N} \eta_R (|v_n|^p + |v_n|^q) dx - \sigma \int_{\mathbb{R}^N} \eta_R \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx \\ &\quad - \lambda \int_{\mathbb{R}^N} \eta_R \{H_u(x, u_n, v_n) u_n + H_v(x, u_n, v_n) v_n\} dx - \int_{\mathbb{R}^N} \eta_R |u_n|^\alpha |v_n|^\beta dx \\ &\geq \langle u_n, \eta_R u_n \rangle_{t,q} + \langle v_n, \eta_R v_n \rangle_{t,q} - \sigma \int_{\mathbb{R}^N} \eta_R \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx \\ &\quad - \lambda \int_{\mathbb{R}^N} \eta_R \{H_u(x, u_n, v_n) u_n + H_v(x, u_n, v_n) v_n\} dx - \int_{\mathbb{R}^N} \eta_R |u_n|^\alpha |v_n|^\beta dx. \end{aligned} \quad (4.27)$$

Again we have

$$\begin{aligned} \langle u_n, \eta_R u_n \rangle_{t,q} &= \iint_{\mathbb{R}^{2N}} \eta_R(x) \frac{|u_n(x) - u_n(y)|^q}{|x - y|^{N+ tq}} dx dy \\ &\quad + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2} (u_n(x) - u_n(y))}{|x - y|^{N+ tq}} u_n(y) (\eta_R(x) - \eta_R(y)) dx dy \end{aligned}$$

and similarly for v . Thus, by (4.27)

$$\begin{aligned} \int_{\mathbb{R}^N} \eta_R (|D^t u_n|^q + |D^t v_n|^q) dx &\leq - \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2} (u_n(x) - u_n(y))}{|x - y|^{N+ tq}} u_n(y) (\eta_R(x) - \eta_R(y)) dx dy \\ &\quad - \iint_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{q-2} (v_n(x) - v_n(y))}{|x - y|^{N+ tq}} v_n(y) (\eta_R(x) - \eta_R(y)) dx dy \\ &\quad + \lambda \int_{\mathbb{R}^N} \eta_R \{H_u(x, u_n, v_n) u_n + H_v(x, u_n, v_n) v_n\} dx \\ &\quad + \sigma \int_{\mathbb{R}^N} \eta_R \frac{|u_n|^q + |v_n|^q}{|x|^{tq}} dx + \int_{\mathbb{R}^N} \eta_R |u_n|^\alpha |v_n|^\beta dx + o_n(1). \end{aligned} \quad (4.28)$$

As before but using Lemma 2.6 instead of Lemma 2.5 we get

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{q-2} (u_n(x) - u_n(y))}{|x - y|^{N+q}} u_n(y) (\eta_R(x) - \eta_R(y)) dx dy = 0. \quad (4.29)$$

By Theorem 2.8 of [17], up to a subsequence as $n \rightarrow \infty$

$$(u_n, v_n) \rightarrow (u_{\sigma, \lambda}, v_{\sigma, \lambda}) \text{ strongly in } L^r(\mathbb{R}^N) \times L^r(\mathbb{R}^N) \text{ for any } r \in (p, q_t^*). \quad (4.30)$$

Therefore, (H_1) with $\varepsilon = 1$, together with the Hölder inequality and (4.30) give as $n \rightarrow \infty$

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}^N} \eta_R \{ H_u(x, u_n, v_n) u_n + H_v(x, u_n, v_n) v_n \} dx \right| \\ & \leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \eta_R \left(|(u_n, v_n)|^m + m C_1 |(u_n, v_n)|^m \right) dx \\ & \leq m \limsup_{n \rightarrow \infty} \int_{B_R^c} |(u_n, v_n)|^m dx + m C_1 \limsup_{n \rightarrow \infty} \int_{B_R^c} |(u_n, v_n)|^m dx \\ & = m \int_{B_R^c} |(u_{\sigma, \lambda}, v_{\sigma, \lambda})|^m dx + m C_1 \int_{B_R^c} |(u_{\sigma, \lambda}, v_{\sigma, \lambda})|^m dx. \end{aligned}$$

Consequently,

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \eta_R \{ H_u(x, u_n, v_n) u_n + H_v(x, u_n, v_n) v_n \} dx = 0. \quad (4.31)$$

Then, recalling (3.30) and (3.32) and using (4.28)–(4.31) we obtain $\mu_\infty \leq \sigma \omega_\infty + \nu_\infty$, which together with (1.14) give $S_\sigma \nu_\infty^{q/q_t^*} + \sigma \omega_\infty \leq \mu_\infty \leq \sigma \omega_\infty + \nu_\infty$. Assume by contradiction that $\nu_\infty \neq 0$. Then, $\nu_\infty \geq S_\sigma^{N/tq}$. This yield, using (4.17), (1.11) and (4.24)

$$\begin{aligned} c_{\sigma, \lambda} &= \lim_{n \rightarrow \infty} \left\{ I(u_n, v_n) - \frac{1}{\theta} \langle I'(u_n, v_n), (u_n, v_n) \rangle \right\} \\ &\geq \limsup_{n \rightarrow \infty} \left(\frac{1}{\theta} - \frac{1}{q_t^*} \right) \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dx \\ &= \left(\frac{1}{\theta} - \frac{1}{q_t^*} \right) \left\{ \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx + \nu_\infty \right\} \\ &\geq \left(\frac{1}{\theta} - \frac{1}{q_t^*} \right) S_\sigma^{N/tq} \end{aligned}$$

which contradicts again (4.23). Thus $\nu_\infty = 0$ and so $\mu_\infty = \sigma \omega_\infty$.

It remain to prove that $\omega_\infty = 0$, which in turn implies $\mu_\infty = 0$. But this is analogous to the proof of $\omega_0 = 0$, since (1.1) applied to $(\eta_R u_n, \eta_R v_n)$ yields

$$\begin{aligned} \mathcal{H}_{t, q} \int_{\mathbb{R}^N} \frac{|\eta_R u_n|^q + |\eta_R v_n|^q}{|x|^{tq}} dx &\leq [\eta_R u_n]_{t, q} + [\eta_R v_n]_{t, q} \\ &\leq (1 + \delta) \int_{\mathbb{R}^N} |\eta_R|^q (|D^t u_n|^q + |D^t v_n|^q) dx \\ &\quad + C_\delta \int_{\mathbb{R}^N} |D^t \eta_R|^q (|u_n|^q + |v_n|^q) dx \end{aligned} \quad (4.32)$$

where by Lemma 2.6

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |D^t \eta_R|^q (|u_n|^q + |v_n|^q) dx = 0.$$

So, assuming by contradiction that $\omega_0 > 0$, we have again

$$\mathcal{H}_{t,q}\omega_\infty \leq \mu_\infty = \sigma\omega_\infty < \mathcal{H}_{t,q}\omega_\infty$$

which is absurd.

Step 4. Strong convergence: By (4.24), (4.26) for all $\lambda \geq \lambda_*$ in virtue of Remark 3.1 we obtain up to a subsequence

$$\int_{\mathbb{R}^N} |u_n - u|^\alpha |v_n - v|^\beta dx \rightarrow 0, \quad \|u_n - u_{\sigma,\lambda}\|_{H_{t,q}} + \|v_n - v_{\sigma,\lambda}\|_{H_{t,q}} \rightarrow 0 \quad (4.33)$$

as $n \rightarrow \infty$, since $\nu_\infty = \omega_\infty = 0$. Note that we can not conclude the same for μ , since (1.5) does not hold in general with the equality. Then, by (4.16) and Lemma 4.3, part (ii) as $n \rightarrow \infty$

$$o_n(1) = \langle I'(u_n, v_n), (u_n, v_n) \rangle, \quad o_n(1) = \langle I'(u, v), (u, v) \rangle.$$

Subtracting the two formulas we obtain

$$\begin{aligned} o_n(1) &= \|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p + \|u_n\|_{t,q}^q + \|v_n\|_{t,q}^q - \|u_{\sigma,\lambda}\|_{s,p}^p - \|v_{\sigma,\lambda}\|_{s,p}^p - \|u_{\sigma,\lambda}\|_{t,q}^q - \|v_{\sigma,\lambda}\|_{t,q}^q \\ &\quad - \sigma (\|u_n\|_{H_{t,q}} + \|v_n\|_{H_{t,q}} - \|u_{\sigma,\lambda}\|_{H_{t,q}} - \|v_{\sigma,\lambda}\|_{H_{t,q}}) \\ &\quad - \lambda \int_{\mathbb{R}^N} \left\{ \nabla H(x, u_n, v_n) \cdot (u_n, v_n) - \nabla H(x, u_{\sigma,\lambda}, v_{\sigma,\lambda}) \cdot (u_{\sigma,\lambda}, v_{\sigma,\lambda}) \right\} dx \\ &\quad - \left(\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dx - \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \right). \end{aligned} \quad (4.34)$$

Now, adding and subtracting the term $\nabla H(x, u_n, v_n) \cdot (u_{\sigma,\lambda}, v_{\sigma,\lambda})$ we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \left\{ \nabla H(x, u_n, v_n) \cdot (u_n, v_n) - \nabla H(x, u_{\sigma,\lambda}, v_{\sigma,\lambda}) \cdot (u_{\sigma,\lambda}, v_{\sigma,\lambda}) \right\} dx \\ &= \int_{\mathbb{R}^N} \nabla H(x, u_n, v_n) \cdot (u_n - u_{\sigma,\lambda}, v_n - v_{\sigma,\lambda}) dx \\ &\quad + \int_{\mathbb{R}^N} \left\{ \nabla H(x, u_n, v_n) - \nabla H(x, u_{\sigma,\lambda}, v_{\sigma,\lambda}) \right\} \cdot (u_{\sigma,\lambda}, v_{\sigma,\lambda}) dx \end{aligned} \quad (4.35)$$

where, by (4.30), (H_1) with $\varepsilon = 1$ and the Hölder inequality, it is easy to see that

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla H(x, u_n, v_n) \cdot (u_n - u_{\sigma,\lambda}, v_n - v_{\sigma,\lambda}) dx \\ &= \left| \int_{\mathbb{R}^N} [H_u(x, u_n, v_n)(u_n - u_{\sigma,\lambda}) + H_v(x, u_n, v_n)(v_n - v_{\sigma,\lambda})] dx \right| \\ &\leq \int_{\mathbb{R}^N} [\mathfrak{m} |(u_n, v_n)|^{\mathfrak{m}-1} |(u_n, v_n) - (u_{\sigma,\lambda}, v_{\sigma,\lambda})| \\ &\quad + m C_1 |(u_n, v_n)|^{m-1} |(u_n, v_n) - (u_{\sigma,\lambda}, v_{\sigma,\lambda})|] dx \\ &\leq C (\|(u_n, v_n) - (u_{\sigma,\lambda}, v_{\sigma,\lambda})\|_{\mathfrak{m}} + \|(u_n, v_n) - (u_{\sigma,\lambda}, v_{\sigma,\lambda})\|_m) \rightarrow 0, \end{aligned} \quad (4.36)$$

since $q < \mathfrak{m} < m < q_t^*$, while the dominated convergence theorem implies, since (H_1) holds, that

$$\int_{\mathbb{R}^N} \left\{ \nabla H(x, u_n, v_n) - \nabla H(x, u_{\sigma,\lambda}, v_{\sigma,\lambda}) \right\} \cdot (u_{\sigma,\lambda}, v_{\sigma,\lambda}) dx \rightarrow 0 \quad (4.37)$$

as $n \rightarrow \infty$. Therefore, (4.35)–(4.37) imply as $n \rightarrow \infty$

$$\int_{\mathbb{R}^N} \left\{ \nabla H(x, u_n, v_n) \cdot (u_n, v_n) - \nabla H(x, u_{\sigma,\lambda}, v_{\sigma,\lambda}) \cdot (u_{\sigma,\lambda}, v_{\sigma,\lambda}) \right\} dx \rightarrow 0. \quad (4.38)$$

Then, (4.34) together with (4.33) and (4.38) yields

$$\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p + \|u_n\|_{t,q}^q + \|v_n\|_{t,q}^q \rightarrow \|u_{\sigma,\lambda}\|_{s,p}^p + \|v_{\sigma,\lambda}\|_{s,p}^p + \|u_{\sigma,\lambda}\|_{t,q}^q + \|v_{\sigma,\lambda}\|_{t,q}^q. \quad (4.39)$$

Finally, from an application of the Brézis and Lieb lemma [10] to the functions

$$|u_n(x) - u_n(y)| |x - y|^{-(N+sp/p)}, |v_n(x) - v_n(y)| |x - y|^{-(N+sp/p)} \in L^p(\mathbb{R}^{2N})$$

and

$$|u_n(x) - u_n(y)| |x - y|^{-(N+tq/q)}, |v_n(x) - v_n(y)| |x - y|^{-(N+tq/q)} \in L^q(\mathbb{R}^{2N})$$

we can conclude, thanks to (4.39), that

$$\|u_n - u_{\sigma,\lambda}\|_{s,p}^p + \|v_n - v_{\sigma,\lambda}\|_{s,p}^p + \|u_n - u_{\sigma,\lambda}\|_{t,q}^q + \|v_n - v_{\sigma,\lambda}\|_{t,q}^q \rightarrow 0$$

as $n \rightarrow \infty$. Consequently, $(u_n, v_n) \rightarrow (u_{\sigma,\lambda}, v_{\sigma,\lambda})$ in W as $n \rightarrow \infty$. $\square$

Proof of Theorem 1.1. First, Lemma 4.1 guarantees that for any $\sigma \in (-\infty, \mathcal{H}_{t,q})$ and any $\lambda > 0$ the functional I has the geometry of the mountain pass lemma. Thus, I admits a Palais–Smale sequence $\{(u_n, v_n)\}_n$ at level $c_{\sigma,\lambda}$ which, in virtue of Lemma 4.3 part (i), up to a subsequence, weakly converges to some limit $(u_{\sigma,\lambda}, v_{\sigma,\lambda}) \in W$. Moreover, by Lemma 4.3 part (ii), the weak limit $(u_{\sigma,\lambda}, v_{\sigma,\lambda}) \in W$ is a critical point of I . Now, for all $\sigma \in (-\infty, \mathcal{H}_{t,q})$ and all $\lambda \geq \lambda_*$ the functional I satisfies the $(PS)_{c_{\sigma,\lambda}}$ condition, as asserted in Lemma 4.4. Therefore, up to a subsequence, $(u_n, v_n) \rightarrow (u_{\sigma,\lambda}, v_{\sigma,\lambda})$ in W as $n \rightarrow \infty$ and so $(u_{\sigma,\lambda}, v_{\sigma,\lambda})$ is a nontrivial solution of system $\mathcal{S}$.

In order to prove the asymptotic behavior (1.2), let us recall that (4.17) yields

$$c_{\sigma,\lambda} \geq \ell(\|u_{\sigma,\lambda}\|_{s,p}^p + \|v_{\sigma,\lambda}\|_{s,p}^p + \|u_{\sigma,\lambda}\|_{t,q}^q + \|v_{\sigma,\lambda}\|_{t,q}^q) + \left(\frac{1}{\theta} - \frac{1}{q_t^*}\right) \int_{\mathbb{R}^N} |u_{\sigma,\lambda}|^\alpha |v_{\sigma,\lambda}|^\beta dx. \quad (4.40)$$

First we assert that as $\lambda \rightarrow \infty$

$$\imath_{\sigma,\lambda} = \|u_{\sigma,\lambda}\|_{s,p}^p + \|v_{\sigma,\lambda}\|_{s,p}^p + \|u_{\sigma,\lambda}\|_{t,q}^q + \|v_{\sigma,\lambda}\|_{t,q}^q \rightarrow 0. \quad (4.41)$$

Otherwise, $\limsup_{\lambda \rightarrow \infty} \imath_{\sigma,\lambda} = \imath_\sigma > 0$. Hence there is a sequence $j \mapsto \lambda_j \uparrow \infty$ such that $\imath_{\sigma,\lambda_j} \rightarrow \imath_\sigma$ as $j \rightarrow \infty$. Then, letting $j \rightarrow \infty$ we get from (4.40) and Lemma 4.2 that

$$0 \geq \ell \imath_\sigma > 0.$$

This contradiction proves the assertion (4.41). Therefore (1.1), (2.3) and (4.41) imply that

$$\lim_{\lambda \rightarrow \infty} \|u_{\sigma,\lambda}\|_{H_\varphi} = \lim_{\lambda \rightarrow \infty} \|v_{\sigma,\lambda}\|_{H_\varphi} = \lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}^N} |u_{\sigma,\lambda}|^\alpha |v_{\sigma,\lambda}|^\beta dx = \lim_{\lambda \rightarrow \infty} \|(u_{\sigma,\lambda}, v_{\sigma,\lambda})\| = 0. \quad (4.42)$$

This proves (1.2).

Finally, the fact that the constructed solution has both components nontrivial follows as in Lemma 2.3 of [17]. Indeed, assume by contradiction that $u_{\sigma,\lambda} \neq 0$, but $v_{\sigma,\lambda} = 0$ a.e. in $\mathbb{R}^N$. Then, taking $(\varphi, \psi) = (u_{\sigma,\lambda}, 0) \in W$ in the definition of solution, we have by (H_3) and (1.1)

$$\|u_{\sigma,\lambda}\|_{t,q}^q \leq \|u_{\sigma,\lambda}\|_{s,p}^p + \|u_{\sigma,\lambda}\|_{t,q}^q = \sigma \|u_{\sigma,\lambda}\|_{\mathcal{H}_{t,q}}^q \leq \frac{\sigma^+}{\mathcal{H}_{t,q}} \|u_{\sigma,\lambda}\|_{t,q}^q,$$

which implies at once the contradiction that $u_{\sigma,\lambda} = 0$ a.e. in $\mathbb{R}^N$, whenever $\sigma < \mathcal{H}_{t,q}$. Similarly, the case $u_{\sigma,\lambda} = 0$ a.e. in $\mathbb{R}^N$ and $v_{\sigma,\lambda} \neq 0$ cannot occur. This concludes the proof. $\square$

5. Existence for system $(\tilde{\mathcal{S}})$

Let us now deal with the existence of solutions for system $(\tilde{\mathcal{S}})$, that is in the case in which $\sigma = 0$ and H is independent of x . Clearly, the Euler–Lagrange functional $\tilde{I} : W \rightarrow \mathbb{R}$ associated to $(\tilde{\mathcal{S}})$ is given by

$$\tilde{I}(u, v) = \frac{1}{p} \|u\|_{s,p}^p + \frac{1}{q} \|u\|_{t,q}^q + \frac{1}{p} \|v\|_{s,p}^p + \frac{1}{q} \|v\|_{t,q}^q - \lambda \int_{\mathbb{R}^N} H(u, v) dx - \frac{1}{q_t^*} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx.$$

Of course, since $\tilde{I}$ is obtained as a particular case of I , it is well defined, of class $C^1(W)$ and the weak solutions of $(\tilde{\mathcal{S}})$ are exactly its critical points. Moreover, Lemmas 4.1–4.3 still hold with much simpler proofs.

Thus, we can introduce, thanks to the geometry of the mountain pass lemma, the special levels of $\tilde{I}$

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{\tau \in [0,1]} \tilde{I}(\gamma(\tau)),$$

where

$$\Gamma = \left\{ \gamma \in C([0,1], W) : \gamma(0) = (0,0), \tilde{I}(\gamma(1)) < 0 \right\}.$$

Obviously, $c_\lambda > 0$ and by Lemma 4.2

$$\lim_{\lambda \rightarrow \infty} c_\lambda = 0.$$

So far, since Lemma 4.3 holds in particular when $\sigma = 0$, we know that the PS sequences of $\tilde{I}$ at levels c_λ weakly converge in W to some couple (u_λ, v_λ) , which is a critical point of $\tilde{I}$. However, we can not conclude yet that (u_λ, v_λ) is nontrivial. Instead of proving the $(PS)_{c_\lambda}$ condition, in this case we employ a result of independent interest, which implies useful consequences when the main functions in $(\tilde{\mathcal{S}})$ are independent of x .

Let us first give a lemma by Lions [26] which gives conditions guaranteeing strong L^r convergence of bounded sequence of W . We report the proof of [33], Lemma 1.21.

Lemma 5.1 (Lions, [26]). *Let $\{(u_n, v_n)\}_n$ be a bounded sequence in W . If there exists $R > 0$ such that*

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} (|u_n|^p + |v_n|^p) dx = 0,$$

then for any $r \in (p, q_t^)$, $u_n \rightarrow 0$ and $v_n \rightarrow 0$ in $L^r(\mathbb{R}^N)$ as $n \rightarrow \infty$.*

Proof. By Lemma 2.1, the sequences $\{u_n\}_n$ and $\{v_n\}_n$ are bounded in $L^{q_t^*}(\mathbb{R}^N)$. Let $r \in (p, q_t^*)$ and consider $\vartheta \in (0, 1)$ defined by

$$\frac{1}{r} = \frac{1-\vartheta}{p} + \frac{\vartheta}{q_t^*}.$$

Then, applying the Hölder inequality we can see that

$$\|u_n\|_{L^r(B_R(y))} dx \leq \|u_n\|_{L^p(B_R(y))}^{1-\vartheta} \|u_n\|_{L^{q_t^*}(B_R(y))}^\vartheta.$$

Covering $\mathbb{R}^N$ with balls of radius R in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls we obtain

$$\int_{\mathbb{R}^N} |u_n|^r dx \leq (N+1)C \sup_{y \in \mathbb{R}^N} \left(\int_{B_R(y)} |u_n|^p dx \right)^{r(1-\vartheta)/p} \rightarrow 0$$

as $n \rightarrow \infty$. The same holds for v_n . □

We shall apply Lemma 5.1 in the next proposition *à la Lions*, which gives two alternatives for the Palais–Smale sequences of $\tilde{I}$. Proposition 5.2 is a variant of the result in [26] for vectorial, fractional (p, q) problems. We refer also to Proposition 2.5 of [15] for the scalar local case and Proposition 2.8 of [17] for the vectorial, local case.

Proposition 5.2. *There exists $\tilde{\lambda} > 0$ such that for any $\lambda \geq \tilde{\lambda}$ the following property holds. If $\{(u_n, v_n)\}_n \subset W$ is a Palais–Smale sequence of $\tilde{I}$ at level c_λ such that $(u_n, v_n) \rightarrow (0, 0)$ in W as $n \rightarrow \infty$. Then, either*

- (i) $(u_n, v_n) \rightarrow (0, 0)$ in W , or
- (ii) *there exists $R > 0$ and a sequence $(y_n)_n \in \mathbb{R}^N$ such that*

$$\limsup_{n \rightarrow \infty} \int_{B_R(y_n)} (|u_n|^p + |v_n|^p) dx > 0.$$

Moreover, $(y_n)_n$ is not bounded in $\mathbb{R}^N$.

Proof. If (ii) does not occur, then

$$\limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} (|u_n|^p + |v_n|^p) dx = 0.$$

Therefore, by Lemma 5.1 up to a subsequence $u_n \rightarrow 0$ and $v_n \rightarrow 0$ in $L^r(\mathbb{R}^N)$ as $n \rightarrow \infty$ for all $r \in (p, q_t^*)$. Consequently, by (H_1) and (4.35) as $n \rightarrow \infty$

$$0 \leq \int_{\mathbb{R}^N} [H_u(u_n, v_n)u_n + H_v(u_n, v_n)v_n] dx \leq \int_{\mathbb{R}^N} (\mathfrak{m}|(u_n, v_n)|^{\mathfrak{m}} + m C_1 |(u_n, v_n)|^m) dx \rightarrow 0,$$

since $1 < p < \mathfrak{m} < m < q_t^*$. Now, $\{(u_n, v_n)\}_n \subset W$ is a Palais–Smale sequence of $\tilde{I}$ at level c_λ , so that, as $n \rightarrow \infty$

$$\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p + \|u_n\|_{t,q}^q + \|v_n\|_{t,q}^q = \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dx + o_n(1). \quad (5.1)$$

Since in particular $\{(u_n, v_n)\}_n$ is bounded in W , up to a subsequence we have

$$\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p + \|u_n\|_{t,q}^q + \|v_n\|_{t,q}^q \rightarrow L_\lambda. \quad (5.2)$$

Therefore, by (5.1) we also have as $n \rightarrow \infty$

$$\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dx \rightarrow L_\lambda. \quad (5.3)$$

We claim that $L_\lambda \rightarrow 0$ as $\lambda \rightarrow \infty$. Otherwise, there exists $M > 0$ independent of λ such that $L_\lambda \geq M$ for all $\lambda > 0$. In this case, (4.17) yields

$$c_\lambda + o_n(1) \geq \ell \left(\|u_n\|_{s,p}^p + \|v_n\|_{s,p}^p + \|u_n\|_{t,q}^q + \|v_n\|_{t,q}^q \right) + \left(\frac{1}{\theta} - \frac{1}{q_t^*} \right) \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dx \quad (5.4)$$

and so for any λ it results

$$c_\lambda \geq \ell L_\lambda + \left(\frac{1}{\theta} - \frac{1}{q_t^*} \right) L_\lambda = \left(\ell + \frac{1}{\theta} - \frac{1}{q_t^*} \right) L_\lambda \geq \left(\ell + \frac{1}{\theta} - \frac{1}{q_t^*} \right) M.$$

Sending $\lambda \rightarrow \infty$ by Lemma 4.2 we obtain

$$0 \geq \left(\ell + \frac{1}{\theta} - \frac{1}{q_t^*} \right) M > 0,$$

which is the desired contradiction. Therefore, $L_\lambda \rightarrow 0$ as $\lambda \rightarrow \infty$. In particular, there exists $\tilde{\lambda} > 0$ such that for any $\lambda \geq \tilde{\lambda}$ we have $L_\lambda < (1/S)^{tq/N}$, where S is introduced in (1.3). Assume now by contradiction that there exists $\lambda_0 \geq \tilde{\lambda}$ such that $L_{\lambda_0} > 0$. From (1.3), (5.2) and (5.3) it results $(L_{\lambda_0})^{q/q_t^*} \leq SL_{\lambda_0}$, which implies $L_{\lambda_0} \geq (1/S)^{tq/N}$, since $L_{\lambda_0} > 0$. This is clearly impossible and therefore $L_\lambda = 0$ for all $\lambda \geq \tilde{\lambda}$, which in turn implies $\|(u_n, v_n)\| \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda \geq \tilde{\lambda}$, as required. In conclusion, (i) holds.

Assume now that (ii) is verified and suppose by contradiction that $(y_n)_n$ is bounded in $\mathbb{R}^N$. Consequently, there exists $M > 0$ so large that $B_R(y_n) \subset B_M$ for all n . Now, $(u_n, v_n) \rightarrow (0, 0)$ in $L_{\text{loc}}^r(\mathbb{R}^N) \times L_{\text{loc}}^r(\mathbb{R}^N)$ for all $r \in [1, q_t^*)$ by Lemma 5.1. Therefore,

$$0 = \lim_{n \rightarrow \infty} \int_{B_M} (|u_n|^p + |v_n|^p) dx \geq \limsup_{n \rightarrow \infty} \int_{B_R(y_n)} (|u_n|^p + |v_n|^p) dx > 0,$$

which gives the required contradiction. In conclusion, $(y_n)_n$ is not bounded in $\mathbb{R}^N$. $\square$

Let us now conclude the section with the proof of Theorem 1.4 thanks to Proposition 5.2.

Proof of Theorem 1.4. As already pointed out, it remains to prove that we can construct a nontrivial solution of $(\tilde{\mathcal{S}})$. This is possible exploiting the invariance for translations of the autonomous system $(\tilde{\mathcal{S}})$. Fix $\lambda \geq \tilde{\lambda}$. If $(u_\lambda, v_\lambda) = (0, 0)$ then of course $\{(u_n, v_n)\}_n$ cannot converge strongly to $(0, 0)$ in W , since otherwise $I'(u_\lambda, v_\lambda) = 0$ and $0 = I(u_\lambda, v_\lambda) = c_\lambda > 0$ by Lemma 4.3. Therefore, we are in the alternative (ii) of Proposition 5.2. Thus, there exist $R > 0$ and a sequence $(y_n)_n \in \mathbb{R}^N$ such that

$$\limsup_{n \rightarrow \infty} \int_{B_R(y_n)} (|u_n|^p + |v_n|^p) dx > 0. \quad (5.5)$$

Let us see that we can construct a nontrivial solution as follows. Define a new sequence $\{(\tilde{u}_n, \tilde{v}_n)\}_n$ by $\tilde{u}_n = u_n(\cdot + y_n)$, $\tilde{v}_n = v_n(\cdot + y_n)$. Then,

$$I(\tilde{u}_n, \tilde{v}_n) = I(u_n, v_n) \rightarrow c_\lambda, \quad I'(\tilde{u}_n, \tilde{v}_n) \rightarrow 0. \quad (5.6)$$

Indeed, for all $(\varphi, \psi) \in W$, with $\|(\varphi, \psi)\| = 1$, putting $\varphi_n(z) = \varphi(z - y_n)$ and $\psi_n(z) = \psi(z - y_n)$, $z \in \mathbb{R}^N$, we have

$$|\langle I'(u_n, v_n), (\varphi_n, \psi_n) \rangle| \leq \|I'(u_n, v_n)\|_{W'} \|(\varphi_n, \psi_n)\| = \|I'(u_n, v_n)\|_{W'}$$

since $1 = \|(\varphi, \psi)\| = \|(\varphi_n, \psi_n)\|$. On the other hand, with a simple change of variable it is easy to see that

$$\langle I'(\tilde{u}_n, \tilde{v}_n), (\varphi, \psi) \rangle = \langle I'(u_n, v_n), (\varphi_n, \psi_n) \rangle.$$

Therefore, as $n \rightarrow \infty$

$$\|I'(\tilde{u}_n, \tilde{v}_n)\|_{W'} = \sup_{\substack{(\varphi, \psi) \in W \\ \|(\varphi, \psi)\| = 1}} |\langle I'(\tilde{u}_n, \tilde{v}_n), (\varphi, \psi) \rangle| \leq \|I'(u_n, v_n)\|_{W'} = o_n(1).$$

Consequently, (5.6) holds and so $\{(\tilde{u}_n, \tilde{v}_n)\}_n$ is a Palais–Smale sequence at level c_λ . Thus, by Lemma 4.3 applied to the new sequence it follows that $\{(\tilde{u}_n, \tilde{v}_n)\}_n$ weakly converges to some $(\tilde{u}_\lambda, \tilde{v}_\lambda)$ in W and $(\tilde{u}_\lambda, \tilde{v}_\lambda)$ is a critical point of $\tilde{I}$. Furthermore, by (5.5) it results

$$0 < \limsup_{n \rightarrow \infty} \int_{B_R(y_n)} (|u_n|^p + |v_n|^p) dx = \lim_{n \rightarrow \infty} \int_{B_R} (|\tilde{u}_n|^p + |\tilde{v}_n|^p) d\tilde{x} = \int_{B_R} (|\tilde{u}_\lambda|^p + |\tilde{v}_\lambda|^p) d\tilde{x}.$$

Hence, $(\tilde{u}_\lambda, \tilde{v}_\lambda) \neq (0, 0)$ is the desired nontrivial solution of system $(\tilde{\mathcal{S}})$.

Finally, we can prove as in Theorem 1.1 that both components of $(\tilde{u}_\lambda, \tilde{v}_\lambda)$ are nontrivial and that (1.2) holds. This completes the proof. $\square$

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