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Structural stability without coercivity

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ABSTRACT

We study structural stability and local uniqueness of bang-bang-singular extremals in a fixed-free Mayer problem. The problem can be seen as an optimal control problem where the dynamics is control-affine. We prove our results under the same assumptions on the nominal problem that permit us to prove that the extremal is a strict strong local optimiser: suitable regularity assumptions and coercivity of a restriction of the *extended second variation*. We also assume that the condition that prevents the coercivity of the extended second variation is stable with respect to the parameter.

KEYWORDS

Structural stability; bang-bang-singular extremals; Hamiltonian methods

1. Introduction

An important requirement for any mathematical model of an actual phenomenon is the stability of the result with respect to small variations of data. Indeed any measurement is subject to some error, so for the model to be of any practical usefulness it is necessary that at least some structural property of the solution is not affected by the uncertainty on the data measurement.

Here we study the structural stability of a bang-bang-singular optimiser in a Mayer problem. Indeed we consider the following optimal control problem $(\mathbf{P}_r)$ depending on the parameter r varying in $\mathbb{R}^k$:

$$\begin{aligned} &\text{minimise } c^r(\xi(T^r)) \text{ subject to} \\ &\dot{\xi}(t) \in \mathcal{X}^r(\xi(t)), \quad \xi(0) = x_0^r, \end{aligned} \tag{1}$$

where the interval $[0, T^r]$ is fixed, x_0^r is a fixed point in $\mathbb{R}^n$, and $\mathcal{X}^r(x) \subset \mathbb{R}^n$ is the convex hull at each point $x \in \mathbb{R}^n$ of the smooth vector fields $X_1^r, \dots, X_m^r$.

We consider the problems $(\mathbf{P}_r)$ as perturbations of the nominal problem $(\mathbf{P}_0)$ corresponding to $r = 0$. More precisely we ask for the C^∞ regularity of the maps

$$r \mapsto x_0^r, \quad (r, x) \mapsto c^r(x), \quad (r, x) \mapsto X_j^r(x), \quad j = 1, \dots, m,$$

even though C^2 -regularity would suffice. As usual, solutions of (1) are meant in the Carathéodory sense. Moreover, when referring to $(\mathbf{P}_0)$ we shall omit the superscript to the data, for example T means T^0 .

In a recent paper [1] the authors studied the optimal control problem $(\mathbf{P}_0)$ and considered the case when such problem admits an extremal trajectory $\hat{\xi}$ which is the concatenation of two bang arcs and a partially singular one. Namely there exist $0 < \hat{\tau}_1 < \hat{\tau}_2 < T$, $\hat{v} \in L^\infty([\hat{\tau}_2, T], (0, 1))$ and $h_1, h_2, h_3 \in \{X_1, \dots, X_m\}$, such that

$$\dot{\hat{\xi}}(t) = \hat{f}_t(\hat{\xi}(t)) := \begin{cases} h_1(\hat{\xi}(t)), & t \in (0, \hat{\tau}_1), \\ h_2(\hat{\xi}(t)), & t \in (\hat{\tau}_1, \hat{\tau}_2), \\ h_2(\hat{\xi}(t)) + \hat{v}(t)(h_3 - h_2)(\hat{\xi}(t)), & t \in (\hat{\tau}_2, T). \end{cases} \quad (2)$$

Under some regularity assumptions and assuming the coercivity of a quadratic form related to a sub-problem of $(\mathbf{P}_0)$, it is shown that $\hat{\xi}$ is a strict strong local optimal trajectory for $(\mathbf{P}_0)$, according to the following definition:

Definition 1. An admissible trajectory $\xi^r : [0, T^r] \rightarrow \mathbb{R}^n$ is a *strict strong local minimiser* of problem $(\mathbf{P}_r)$ if it is a strict minimiser among the admissible trajectories of $(\mathbf{P}_r)$ which are close to ξ^r in the $C^0([0, T^r], \mathbb{R}^n)$ topology, regardless of their velocities.

Problem $(\mathbf{P}_0)$ can be written as an optimal control problem with a control-affine dynamics, see [1]. Hence its classical second variation is degenerate. In the same paper the authors construct the *extended second variation* J''_{ext} starting from a sub-problem of $(\mathbf{P}_0)$ and applying an invariant version of Goh's transformation.

Setting $f_d := h_3 - h_2$, the coercivity of J''_{ext} implies $L_{f_d}^2 c(\hat{\xi}(T)) > 0$. So that, if $L_{f_d} c \equiv 0$, J''_{ext} cannot be coercive. Nevertheless, in [1] it was proved that, if this is the case, then the coercivity assumption can be weakened and the strong local optimality result still holds. Hence the title of this paper: we aim at proving the structural stability of bang-bang-singular extremals in the case when this condition is preserved under perturbations of the data, i.e. setting $f_d^r := h_3^r - h_2^r$,

Assumption 1. We assume $L_{f_d^r} c^r \equiv 0 \quad \forall r$.

We point out that even if the bang-bang structure seems more generic, examples in the literature, such as the Van der Pol oscillator and the Rayleigh problem studied in [2,3], suggest that bang-bang-singular concatenations cannot be neglected. Moreover Assumption 1 is satisfied by Lagrange problems where the running cost does not depend on the control. We give more details in Section 3.1.

The proof of the structural stability result is based on the Hamiltonian approach, an approach already exploited in [4] where the extremal of the nominal problem is bang-bang with a double switching time and [5–7] where the authors consider the minimum time problem for different control structures.

We point out that we only ask for the same conditions required to prove the strong local optimality of $\hat{\xi}$. Namely we require that $\hat{\xi}$ satisfies Pontryagin Maximum Principle (a necessary condition for strong local optimality), the regularity of each of the three arcs and of their junction points (see Section 2.1), plus the coercivity of a suitable second variation related to a sub-problem of $(\mathbf{P}_0)$, see Section 2.2.

We construct, via the implicit function theorem, a bang-bang-singular extremal trajectory ξ^r for $(\mathbf{P}_r)$ which is also a strict strong local optimiser for $(\mathbf{P}_r)$. Using the implicit function theorem guarantees that ξ^r and its switching times depend smoothly

on the parameter r , so that a sensitivity analysis could also be carried out.

Here we consider the fixed-free case (i.e. the initial point of admissible trajectories is prescribed while no constraint is imposed on the final one) which implies less technicalities with respect to the generic constraints on end points and, nevertheless, it shows the underlying ideas and methods we use. Optimality results for generic constraints were obtained in [8] and, from the results therein, we could extend the structural stability result to that kind of problems.

For classical approaches to structural stability and sensitivity analysis, see [9] and the references therein. For more recent results, see [10–12].

1.1. Notation

For the sake of conciseness and clarity we use notation from differential geometry. Indeed, this notation helps to realise that the discussion is intrinsic and the result can be applied to problems on manifolds. In particular we denote the trivial cotangent bundle $(\mathbb{R}^n)^* \times \mathbb{R}^n$ as $T^*\mathbb{R}^n$ and we distinguish between points in $\mathbb{R}^n$, usually denoted as x , and tangent vectors to $\mathbb{R}^n$, denoted as δx .

If N is a smooth manifold of $\mathbb{R}^n$, we denote as $T_x N$ and $T_x^* N$ the tangent and cotangent spaces to N at a point $x \in N$, respectively.

Given a C^1 vector field f on N , we denote as $\exp tf(x)$ the flow at time t emanating from a point x at time 0, i.e. $\exp tf(x)$ is the solution to

$$\dot{\xi}(t) = f(\xi(t)), \quad \xi(0) = x.$$

If g is another C^1 vector field, then the Lie bracket between f and g is denoted as $[f, g]$, i.e. $[f, g](x) := Dg(x)f(x) - Df(x)g(x)$.

If $\alpha: N \rightarrow \mathbb{R}$ is a C^2 function, $d\alpha(x)$ is its differential at a point $x \in N$ and we denote $L_f \alpha(x) := \langle d\alpha(x), f(x) \rangle$, while $d\alpha_* \delta x$ is $(D^2 \alpha(x)(\delta x, \cdot), \delta x) \in T^*\mathbb{R}^n$ whenever the point x is clear from the context.

If G is a C^1 map from a manifold X to a manifold Y , its tangent map at a point $x \in X$ is denoted as G_* whenever x is clear from the context.

We also use some basic element of the theory of symplectic manifolds referred to the trivial cotangent bundle $T^*\mathbb{R}^n$, see e.g. [13].

Let $\pi: \ell = (p, x) \in T^*\mathbb{R}^n \mapsto x \in \mathbb{R}^n$ be the canonical projection, and denote by $\mathbf{s} := \sum_{i=1}^n p^i dx_i$ the canonical Liouville one-form on $T^*\mathbb{R}^n$. Given a time-dependent smooth Hamiltonian $H_t: T^*\mathbb{R}^n \rightarrow \mathbb{R}$ we can define a Hamiltonian vector field $\vec{H}_t$ by means of the canonical symplectic two-form $\sigma = d\mathbf{s} = \sum_{i=1}^n dp^i \wedge dx_i$. Indeed one sets $\sigma(V, \vec{H}_t(\ell)) = \langle dH_t(\ell), V \rangle$, for any $V \in T_\ell T^*\mathbb{R}^n$, i.e. $\vec{H}_t(\ell) = (-\partial_x H_t(\ell), \partial_p H_t(\ell))$, for any $\ell = (p, x) \in T^*\mathbb{R}^n$.

We recall that any vector field f on $\mathbb{R}^n$ defines a Hamiltonian

$$F: \ell = (p, x) \in T^*\mathbb{R}^n \mapsto \langle p, f(x) \rangle \in \mathbb{R}.$$

In particular we denote by H_1, H_2, H_3 and F_d the Hamiltonians associated with h_1, h_2, h_3 and f_d , respectively. Moreover H_{12} and H_{23} denote the Hamiltonians associated with the brackets $[h_1, h_2]$ and $[h_2, h_3]$, respectively. Analogously H_{232} and H_{323} are the Hamiltonians associated with $[h_2, [h_3, h_2]]$ and $[h_3, [h_2, h_3]]$, respectively. Finally $\hat{F}_t$ denotes the time-dependent Hamiltonian function obtained by lifting the reference vector field $\hat{f}_t$ and $\hat{\mathcal{F}}_t$ denotes its flow from time T , i.e. the flow associated with its

Hamiltonian vector field. Indeed, due to the construction required to carry on our proof, we deal with flows starting from time T and evolving backwards in time. In particular the flow $\widehat{S}_t$ of $\widehat{f}_t$ is a local diffeomorphism defined in a neighbourhood of $\widehat{\xi}(T)$. Finally, we recall that for any $\ell = (p, x) \in T^*\mathbb{R}^n$ we can define the maximised Hamiltonian associated to problem $(\mathbf{P}_0)$ as

$$H^{\max}(\ell) := \max \{ \langle p, Y \rangle : Y \in \mathcal{X}(x) \} = \max \{ \langle p, X_j(x) \rangle : j = 1, \dots, m \}.$$

We conclude this section recalling that, for any interval $I \subset \mathbb{R}$, the symbol $C_c^\infty(I)$ denotes the set of $C^\infty(\mathbb{R})$ functions which are compactly supported in I .

2. Assumptions on the nominal problem $(\mathbf{P}_0)$

In this section we state the assumptions on the nominal problem that ensure that $\widehat{\xi}$ is a strict strong local minimiser of $(\mathbf{P}_0)$. We refer the reader to [1] for details and for the relations between the sufficient conditions and the necessary ones.

The first assumption is the main necessary condition for optimality, i.e. Pontryagin Maximum Principle (PMP). The principle holds in its normal form since there is no constraint on the final point of admissible trajectories.

Assumption 2 (PMP). *There exists an absolutely continuous mapping $\widehat{\mu}: [0, T] \rightarrow (\mathbb{R}^n)^*$, such that*

$$\begin{aligned} \dot{\widehat{\mu}}(t) &= -\widehat{\mu}(t) D\widehat{f}_t(\widehat{\xi}(t)), & a.e. \ t \in [0, T], \\ \widehat{\mu}(T) &= -dc(\widehat{\xi}(T)), \\ \widehat{F}_t(\widehat{\mu}(t), \widehat{\xi}(t)) &= H^{\max}(\widehat{\mu}(t), (\widehat{\xi}(t))). \end{aligned}$$

The trajectory $\widehat{\xi}$ is called a state extremal while the couple $\widehat{\lambda}(t) := (\widehat{\mu}(t), \widehat{\xi}(t)) \in T^\mathbb{R}^n$ is called an extremal. Notice that*

$$\dot{\widehat{\lambda}}(t) = \overrightarrow{\widehat{F}}_t(\widehat{\lambda}(t)) \quad a.e. \ t \in [0, T].$$

We denote the terminal and switching points of $\widehat{\lambda}$ and $\widehat{\xi}$ as follows:

$$\begin{aligned} \widehat{\ell}_0 &:= \widehat{\lambda}(0), & \widehat{\ell}_1 &:= \widehat{\lambda}(\widehat{\tau}_1), & \widehat{\ell}_2 &:= \widehat{\lambda}(\widehat{\tau}_2), & \widehat{\ell}_T &:= \widehat{\lambda}(T), \\ \widehat{x}_0 &:= \widehat{\xi}(0) = \pi\widehat{\ell}_0, & \widehat{x}_1 &:= \widehat{\xi}(\widehat{\tau}_1) = \pi\widehat{\ell}_1, & \widehat{x}_2 &:= \widehat{\xi}(\widehat{\tau}_2) = \pi\widehat{\ell}_2, & \widehat{x}_T &:= \widehat{\xi}(T) = \pi\widehat{\ell}_T. \end{aligned}$$

2.1. Regularity conditions

In this section we state the regularity conditions given in [1, Section 3.1] in a form which is more suitable for our scopes. We also introduce some constants which we are using further on.

Assumption 3 (Strong generalized Legendre condition (SGLC)). *Set*

$$\mathbb{L}(\ell) := (H_{323} + H_{232})(\ell), \quad \ell \in T^*\mathbb{R}^n, \quad \text{and} \quad R(t) := \mathbb{L}(\widehat{\lambda}(t)).$$

Let $m_R := \min\{R(t) : t \in [\widehat{\tau}_2, T]\}$, $M_R := \max\{R(t) : t \in [\widehat{\tau}_2, T]\}$. We assume $m_R > 0$.

Remark 1. In [1] the authors prove that if (SGLC) holds, then $f_d(\hat{x}_T) \neq 0$, and that the singular control is given by $\hat{v}(t) = \frac{H_{232}}{\mathbb{L}}(\hat{\lambda}(t))$, hence $\hat{v} \in C^\infty([\hat{\tau}_2, T], [0, 1])$. We recall that when (SGLC) holds, a singular extremal is said to be a singular extremal of the first kind (see, e.g., [14, page 7]).

Assumption 4 (Regularity along the singular arc).

$$H_3(\hat{\lambda}(t)) = H_2(\hat{\lambda}(t)) > \langle \hat{\mu}(t), X_j(\hat{\xi}(t)) \rangle, \quad t \in [\hat{\tau}_2, T], \quad X_j \neq h_2, h_3,$$

$$\hat{v} \in C^\infty([\hat{\tau}_2, T], (0, 1))$$

so that $m_{2s} := \min \left\{ H_2(\hat{\lambda}(t)) - \langle \hat{\mu}, X_j(\hat{\xi}(t)) \rangle : t \in [\hat{\tau}_2, T], X_j \neq h_2, h_3 \right\} > 0$, and $\gamma_S := \min \{ \text{dist}(\hat{v}(t), \{0, 1\}) : t \in [\hat{\tau}_2, T] \} > 0$.

Assumption 5 (Regularity along the bang arcs).

$$H_1(\hat{\lambda}(t)) > \langle \hat{\mu}(t), X_j(\hat{\xi}(t)) \rangle, \quad t \in [0, \hat{\tau}_1), \quad X_j \neq h_1,$$

$$H_2(\hat{\lambda}(t)) > \langle \hat{\mu}(t), X_j(\hat{\xi}(t)) \rangle, \quad t \in (\hat{\tau}_1, \hat{\tau}_2), \quad X_j \neq h_2.$$

$$m_1 := \min \left\{ H_1(\hat{\lambda}(t)) - \langle \hat{\mu}(t), X_j(\hat{\xi}(t)) \rangle : t \in [0, \hat{\tau}_1], X_j \neq h_1, h_2 \right\} > 0,$$

$$m_{2b} := \min \left\{ H_2(\hat{\lambda}(t)) - \langle \hat{\mu}(t), X_j(\hat{\xi}(t)) \rangle : t \in [\hat{\tau}_1, \hat{\tau}_2], X_j \neq h_1, h_2, h_3 \right\} > 0.$$

Assumption 6 (Regularity at the switching times).

$$\gamma_{12} := H_{12}(\hat{\ell}_1) > 0, \quad H_{232}(\hat{\ell}_2) > 0.$$

Remark 2. • By Assumption 4

$$\gamma_S \leq \frac{H_{232}}{\mathbb{L}}(\hat{\lambda}(t)) \leq 1 - \gamma_S, \quad t \in [\hat{\tau}_2, T].$$

In particular $H_{232}(\hat{\lambda}(t)) < \mathbb{L}(\hat{\lambda}(t))$, thus $m_{323} := \min \{ H_{323}(\hat{\lambda}(t)) > 0 : t \in [\hat{\tau}_2, T] \} > 0$.

• Assumptions 4-5 mean that the value of $H^{\max}$ is reached by a vertex of the set of admissible velocities along the two bang arcs, while it is reached by an edge along the singular arc.

• The first inequality in Assumption 6 is the classical regularity condition for bang-bang concatenations, see e.g. [15]. The second inequality means that left-hand side second order derivative of $t \mapsto (H_2 - H_3) \circ \hat{\lambda}(t)$ is positive, see [16], so that this function is strictly monotone decreasing on a left-hand side neighbourhood of $\hat{\tau}_2$.

• Notice that $t \mapsto \hat{f}_t$ is discontinuous at $\hat{\tau}_2$.

2.2. The second variation

Problem $(\mathbf{P}_0)$ can be written as an optimal control problem with a control-affine dynamics, see [1]. Hence its classical second variation is degenerate. We are going to describe the *extended second variation* J''_{ext} constructed in [1]. Indeed in a neighbour-

hood of $\widehat{x}_T$, define the "push-forwards" vector fields

$$g_t := \widehat{S}_{t*}^{-1} f_d \circ \widehat{S}_t, \quad t \in [\widehat{\tau}_2, T], \quad k_i := \widehat{S}_{\widehat{\tau}_1*}^{-1} h_i \circ \widehat{S}_{\widehat{\tau}_1}, \quad i = 1, 2, \quad k := k_1 - k_2, \quad (3)$$

and recall the following equalities:

$$L_{[k_2, k_1]} c(\widehat{x}_T) = H_{12}(\widehat{\ell}_1), \quad \dot{g}_t = \widehat{S}_{t*}^{-1} [h_2, h_3] \circ \widehat{S}_t \quad \forall t \in [\widehat{\tau}_2, T].$$

Consider the linear system on $[\widehat{\tau}_2, T]$

$$\dot{\zeta}(t) = w(t) \dot{g}_t(\widehat{x}_T), \quad \zeta(\widehat{\tau}_2) = \varepsilon_0 k(\widehat{x}_T), \quad \zeta(T) = \delta x + \varepsilon_1 f_d(\widehat{x}_T) \quad (4)$$

and let $\delta e := (\delta x, \varepsilon_0, \varepsilon_1, w) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times L^2([\widehat{\tau}_2, T], \mathbb{R})$. The space of *extended admissible variations* is given by the Hilbert space

$$\mathcal{W}_{\text{ext}} := \{\delta e : \text{system (4) admits a solution}\}.$$

In [1] the extended second variation is obtained as the quadratic form on $\mathcal{W}_{\text{ext}}$

$$\begin{aligned} J''_{\text{ext}}[\delta e]^2 &= -\varepsilon_1 L_{\delta x} L_{f_d} c(\widehat{x}_T) - \frac{\varepsilon_1^2}{2} L_{f_d}^2 c(\widehat{x}_T) + \frac{\varepsilon_0^2}{2} \left(L_k^2 c(\widehat{x}_T) + H_{12}(\widehat{\ell}_1) \right) \\ &\quad + \frac{1}{2} \int_{\widehat{\tau}_2}^T (2w(t) L_{\zeta(t)} L_{\dot{g}_t} c(\widehat{x}_T) + w(t)^2 R(t)) \, dt. \end{aligned}$$

Under the assumption $L_{f_d} c \equiv 0$ we obtain $J''_{\text{ext}}[(-f_d(\widehat{x}_T), 0, 1, 0)]^2 = 0$, i.e. the extended second variation is not coercive on $\mathcal{W}_{\text{ext}}$.

We thus consider only the variations $\delta e \in \mathcal{W}_{\text{ext}}$ such that $\varepsilon_1 = 0$. Indeed, let $\mathcal{W} \subset \mathbb{R}^n \times \mathbb{R} \times L^2([\widehat{\tau}_2, T], \mathbb{R})$ be the Hilbert space of triplets $\delta e = (\delta x, \varepsilon_0, w)$ such that the following boundary value problem admits a solution:

$$\dot{\zeta}(t) = w(t) \dot{g}_t(\widehat{x}_T), \quad \zeta(\widehat{\tau}_2) = \varepsilon_0 k(\widehat{x}_T), \quad \zeta(T) = \delta x.$$

Set $J_0 := L_k^2 c(\widehat{x}_T) + H_{12}(\widehat{\ell}_1)$ and define

$$J[\delta e]^2 := \frac{\varepsilon_0^2}{2} J_0 + \frac{1}{2} \int_{\widehat{\tau}_2}^T (2w(t) L_{\zeta(t)} L_{\dot{g}_t} c(\widehat{x}_T) + w(t)^2 R(t)) \, dt.$$

We can now state our coercivity assumption.

Assumption 7. *The quadratic form J is coercive on $\mathcal{W}$.*

In [1] the following strong local optimality result is proved

Theorem 2.1. *Let $L_{f_d} c \equiv 0$. If $\widehat{\xi}$ is a bang-bang-singular trajectory which satisfies Assumptions 2-7, then $\widehat{\xi}$ is a strict strong locally optimal trajectory for $(\mathbf{P}_0)$.*

3. The main results

In this section we state the main results of this paper. Moreover, in Section 3.1, we show a class of Lagrange problems that trivially satisfy Assumption 1 when transformed into Mayer problems.

Theorem 3.1. *Let Assumption 1 hold. Assume that $\widehat{\xi}$ is a bang-bang-singular trajectory of the nominal problem $(\mathbf{P}_0)$ satisfying Assumptions 2–7.*

Then there exist $\rho > 0$, and a neighbourhood $\mathcal{U}$ of $\widehat{\ell}_T$ in $T^\mathbb{R}^n$ such that for any r , $|r| < \rho$, there exists a normal extremal $\lambda^r := (\mu^r, \xi^r)$ of $(\mathbf{P}_r)$ such that $\lambda^r(T^r) \in \mathcal{U}$, and λ^r satisfies Assumptions 2–7 so that ξ^r is a bang-bang-singular strict strong local optimal trajectory for $(\mathbf{P}_r)$. The switching times and the final point of λ^r depend smoothly on r .*

We point out that the smooth dependency of λ^r with respect to r allows to perform a sensitivity analysis with respect to the parameter r .

Theorem 3.2. *Let Assumption 1 hold. Assume that $\widehat{\xi}$ is a bang-bang-singular trajectory of the nominal problem $(\mathbf{P}_0)$ satisfying Assumptions 2–7.*

Then there exist $\rho > 0$ and a neighbourhood $\mathcal{O}$ of the graph of $\widehat{\lambda}$ in $\mathbb{R} \times T^\mathbb{R}^n$ such that if $|r| < \rho$, then the extremal λ^r defined in Theorem 3.1 is the only extremal of $(\mathbf{P}_r)$ whose graph is in $\mathcal{O}$.*

The proofs of the above theorems are quite technical and they are postponed to sections 5 and 6, respectively.

3.1. Lagrange problems

In this section we introduce Lagrange problems which naturally fit in our setting. Indeed, Lagrange problems whose running cost does not depend on the control, trivially satisfy Assumption 1 when transformed into a Mayer problem with an augmented state space. This class of problems includes examples studied in the literature, for instance the Van der Pol oscillator and the Rayleigh problem studied in [2,3]. They are Lagrange problems in $\mathbb{R}^2$ where the running cost is $|x|^2$ and for which the authors prove the existence of bang-bang-singular extremals. Moreover in [1,17] we proved that the bang-bang-singular trajectory provided by [2] for the Van der Pol oscillator is a strict strong local optimiser.

Consider the family of problems $(\mathbf{L}_r)$

$$\begin{aligned} & \text{minimise } \int_0^{T^r} \mathcal{L}^r(\xi(t)) dt \quad \text{subject to} \\ & \dot{\xi}(t) \in \mathcal{X}^r(\xi(t)), \quad \text{a.e. } t \in [0, T^r], \quad \xi(0) = x_0^r. \end{aligned}$$

As before, when we omit the superscript r , we are referring to the nominal problem $(\mathbf{L}_0)$ obtained for $r = 0$.

Let $y := (x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$. For $j = 1, \dots, m$ and $r \in \mathbb{R}^k$ define

$$Y_j^r(y) = Y_j^r(x, x_{n+1}) := \begin{pmatrix} X_j^r(x) \\ \mathcal{L}^r(x) \end{pmatrix}.$$

Problem $(\mathbf{L_r})$ can be written as the following Mayer problem $(\mathbf{P_r})$

$$\begin{aligned} & \text{minimise } C^r(\eta) = \eta_{n+1}(T^r) \text{ subject to} \\ & \dot{\eta}(t) \in \text{conv}\{Y_j^r(\eta(t)) : j = 1, \dots, m\}, \quad \text{a.e. } t \in [0, T^r], \quad \eta(0) = \begin{pmatrix} x_0^r \\ 0 \end{pmatrix}. \end{aligned}$$

A trajectory $\hat{\xi}$ for the nominal Lagrange problem has the bang-bang-singular structure as in equation (2) if and only if

$$\dot{\hat{\eta}}(t) = \begin{cases} \kappa_1(\hat{\eta}(t)), & t \in (0, \hat{\tau}_1), \\ \kappa_2(\hat{\eta}(t)), & t \in (\hat{\tau}_1, \hat{\tau}_2), \\ \kappa_2(\hat{\eta}(t)) + \hat{v}(t)\varphi_d(\hat{\eta}(t)), & t \in (\hat{\tau}_2, T). \end{cases}$$

where

$$\kappa_i^r(y) := \begin{pmatrix} h_i^r(x) \\ \mathcal{L}^r(x) \end{pmatrix}, \quad i = 1, 2, 3, \quad \varphi_d^r(y) = (\kappa_3^r - \kappa_2^r)(y) = \begin{pmatrix} (h_3^r - h_2^r)(x) \\ 0 \end{pmatrix} = \begin{pmatrix} f_d^r(x) \\ 0 \end{pmatrix}.$$

Assumption 1 for $(\mathbf{P_r})$ reads $L_{\varphi_d^r} x_{n+1} \equiv 0$ which is trivially satisfied.

We aim at rephrasing Assumptions 2-7 for problem $(\mathbf{P_0})$ directly in terms of the quantities appearing in problem $(\mathbf{L_0})$.

Assumption 2. *There exists an absolutely continuous mapping $\hat{\mu}: [0, T] \rightarrow (\mathbb{R}^n)^*$ such that*

$$\begin{aligned} \dot{\hat{\mu}}(t) &= d\mathcal{L}(\hat{\xi}(t)) - \hat{\mu}(t) D\hat{f}_t(\hat{\xi}(t)), \quad \text{a.e. } t \in [0, T], \\ \hat{\mu}(T) &= 0, \\ \hat{F}_t(\hat{\mu}(t), \hat{\xi}(t)) &= H^{\max}(\hat{\mu}(t), \hat{\xi}(t)), \quad t \in [0, T]. \end{aligned}$$

Assumption 3 (SGLC). *For any $t \in [\hat{\tau}_2, T]$*

$$R(t) := L_{f_d}^2(\hat{\xi}(t)) + (H_{323} + H_{232})(\hat{\lambda}(t)) > 0.$$

Assumption 4 (Regularity along the singular arc).

$$\begin{aligned} H_3(\hat{\lambda}(t)) &= H_2(\hat{\lambda}(t)) > \langle \hat{\mu}(t), X_j(\hat{\xi}(t)) \rangle, \quad t \in [\hat{\tau}_2, T], \quad X_j \neq h_2, h_3, \\ \hat{v} &\in C^\infty([\hat{\tau}_2, T], (0, 1)). \end{aligned}$$

Assumption 5 (Regularity along the bang arcs).

$$\begin{aligned} H_1(\hat{\lambda}(t)) &> \langle \hat{\mu}(t), X_j(\hat{\xi}(t)) \rangle, \quad t \in [0, \hat{\tau}_1], \quad X_j \neq h_1, \\ H_1(\hat{\lambda}(t)) &> \langle \hat{\mu}(t), X_j(\hat{\xi}(t)) \rangle, \quad t \in [0, \hat{\tau}_1], \quad X_j \neq h_1, h_2, \\ H_2(\hat{\lambda}(t)) &> \langle \hat{\mu}(t), X_j(\hat{\xi}(t)) \rangle, \quad t \in (\hat{\tau}_1, \hat{\tau}_2), \quad X_j \neq h_2, \\ H_2(\hat{\lambda}(t)) &> \langle \hat{\mu}(t), X_j(\hat{\xi}(t)) \rangle, \quad t \in [\hat{\tau}_1, \hat{\tau}_2], \quad X_j \neq h_1, h_2, h_3. \end{aligned}$$

Assumption 6 (Regularity at the switching points).

$$H_{12}(\widehat{\ell}_1) - L_{h_1-h_2}\mathcal{L}(\widehat{x}_1) > 0, \quad H_{232}(\widehat{\ell}_2) - (L_{h_2}L_{f_d}\mathcal{L}(\widehat{x}_2) + L_{h_{23}}\mathcal{L}(\widehat{x}_2)) > 0.$$

We now introduce some quantities and write the quadratic form J . Let $k := h_1 - h_2$ and $\widehat{\mathcal{L}}_t := \mathcal{L} \circ \widehat{S}_t$. Define

$$J_0 := H_{12}(\widehat{\ell}_1) - L_k\mathcal{L}(\widehat{x}_1) + \int_{\widehat{\tau}_2}^T L_k^2\widehat{\mathcal{L}}_s(\widehat{x}_T) ds.$$

Then J reads

$$\begin{aligned} J[\delta e]^2 &:= \frac{\varepsilon_0^2}{2} J_0 + \frac{1}{2} \int_{\widehat{\tau}_2}^T w(t)^2 R(t) + \\ &+ 2 w(t) L_{\zeta(t)} \left(L_{\dot{g}_t} \mathcal{L}(\widehat{x}_T) - L_{g_t} \widehat{\mathcal{L}}_t(\widehat{x}_T) + \int_t^T L_{\dot{g}_t} \widehat{\mathcal{L}}_s(\widehat{x}_T) ds \right) dt, \end{aligned}$$

where $\delta e = (\delta x, \varepsilon_0, w)$ is subject to the constraints

$$\dot{\zeta}(t) = w(t)\dot{g}_t(\widehat{x}_T), \quad \zeta(\widehat{\tau}_2) = \varepsilon_0 k(\widehat{x}_T), \quad \zeta(T) = \delta x.$$

Assumption 7. J is coercive.

4. Exploiting the assumptions on the nominal problem ($\mathbf{P}_0$)

The first step in applying our Hamiltonian approach to sufficient conditions for strong local optimality of trajectories having a singular arc is the construction of an over-maximised Hamiltonian, see [5,16,18,19]. Indeed, as shown in [16, Section 3.1], the presence of a singular arc does not permit to use the flow of the maximised Hamiltonian which can be used when this Hamiltonian is C^2 (see [13, Section 17.1]).

In [1], under the same regularity assumptions of this paper, we proved the existence of an appropriate over maximised Hamiltonian flow. Moreover, if

$$\Lambda := \{(d(-c)(x), x) : x \in \mathbb{R}^n\}$$

is the Lagrangian sub-manifold of the final transversality conditions, we proved that the coercivity assumption guarantees the invertibility of the projected over-maximised flow emanating from Λ . This invertibility property is crucial in order to obtain the strong local optimality result via Hamiltonian methods. We refer to [1] for details.

Here we recall the main tools that we constructed in there and that we are going to use for the structural stability result.

(SGLC) implies that there exists a neighbourhood $\mathcal{U}_s$ of the range of the singular arc $\widehat{\lambda}([\widehat{\tau}_2, T])$ in $T^*\mathbb{R}^n$ such that the sets

$$\Sigma := \{\ell \in \mathcal{U}_s : F_d(\ell) = 0\}, \quad \mathcal{S} := \{\ell \in \Sigma : H_{23}(\ell) = 0\}$$

are smooth simply connected manifolds of codimension 1 and 2, respectively. More precisely $\vec{H}_{23}$ is transverse to Σ in $\mathcal{U}_s$, while $\vec{F}_d$ is tangent to Σ and transverse to $\mathcal{S}$ in Σ . We recall that every singular arc evolves on $\mathcal{S}$.

Under Assumptions 2-6, there exists a tubular neighbourhood $\mathcal{O}$ of the graph of $\widehat{\lambda}$ and a piece-wise C^1 Hamiltonian function

$$H: (t, \ell) \in \mathcal{O} \mapsto H_t(\ell) \in \mathbb{R}$$

that satisfies the regularity and over-maximisation properties required to pursue our Hamiltonian approach to strong local optimality. The construction of H_t is based on the existence of the Hamiltonian $\widetilde{H}_2: \mathcal{U}_s \rightarrow \mathbb{R}$, defined and studied in [1, Section 4.1] and which satisfies the following properties

- $\widetilde{H}_2(\ell) \geq H_2(\ell)$ for any $\ell \in \Sigma$. Equality holds if and only if $\ell \in \mathcal{S}$,
- $\widetilde{H}_2$ is tangent to Σ and $\vec{\widetilde{H}}_2(\ell) = \vec{H}_2(\ell)$ for any $\ell \in \mathcal{S}$.

For the purposes of this paper we need to know the definition of H_t only for $t \in [\widehat{\tau}_2, T]$. Namely

$$H_t(\ell) = \widetilde{H}_2(\ell) + \widehat{v}(t)F_d(\ell) \quad (t, \ell) \in [\widehat{\tau}_2, T] \times \mathcal{U}_s.$$

The flow associated with this Hamiltonian and starting from time T is denoted as $\mathcal{H}_t$ and it is C^∞ in a neighbourhood $\mathcal{U}_T$ of $\widehat{\ell}_T$ in Σ . The crucial properties are the following

- $\widehat{\lambda}(t) = \mathcal{H}_t(\widehat{\ell}_T)$,
- $\mathcal{H}_t(\ell) \in \Sigma$ for $(t, \ell) \in [\widehat{\tau}_2, T] \times \mathcal{U}_T$,
- $\vec{F}_d$ is invariant with respect to $\mathcal{H}_t$ on Σ i.e. for any $(t, \ell) \in [\widehat{\tau}_2, T] \times \Sigma$

$$\vec{F}_d(\mathcal{H}_t(\ell)) = \mathcal{H}_{t*} \vec{F}_d(\ell). \quad (5)$$

Let $L_T := T_{\widehat{\ell}_T} \Lambda = \{d(-c)_* \delta x, : \delta x \in \mathbb{R}^n\}$ be the tangent space to Λ at $\widehat{\ell}_T$ and let k be as in (3). Define $\widetilde{k} := \widehat{S}_{\widehat{\tau}_2*} k \circ \widehat{S}_{\widehat{\tau}_2}^{-1}$ and let $\widetilde{K} = K \circ \widehat{\mathcal{F}}_{\widehat{\tau}_2}^{-1}$ be the associated Hamiltonian.

As a consequence of the coercivity Assumption 7 we obtain the existence of a function $\alpha_2: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\mathcal{H}_{\widehat{\tau}_2*} L_T = \{d\alpha_2* \delta x : \delta x \in \mathbb{R}^n\}, \quad \mathcal{H}_{\widehat{\tau}_2*} \delta \ell = d\alpha_2*(\pi_* \mathcal{H}_{\widehat{\tau}_2*} \delta \ell), \quad \forall \delta \ell \in L_T, \quad (6)$$

$$d\alpha_2(\widehat{x}_2) = \widehat{\mu}(\widehat{\tau}_2), \quad H_{12}(\widehat{\ell}_1) - L_k^2 \alpha_2(\widehat{x}_2) > 0, \quad (7)$$

see [1, Section 5.4].

5. Proof of the main theorem, Theorem 3.1

The first step in proving Theorem 3.1 is finding a bang-bang-singular extremal for problem $(\mathbf{P}_r)$. The proof is carried out by using the Implicit Function Theorem. Afterwards we prove that the extremal satisfies the regularity Assumptions 3-6 and finally we prove that the second variation of such extremal is coercive.

5.1. Preliminaries

In order to prove our structural stability result, we take advantage of the Hamiltonian flow associated to singular extremals of the first kind. Indeed, differentiating twice the

equality $F_d(\widehat{\lambda}(t)) \equiv 0$, SGLC allows to define in a neighborhood of the range of the singular arc of $\widehat{\lambda}$ the *feedback control of singular extremals of the first kind*

$$u_S(\ell) := \frac{H_{232}}{\mathbb{L}}(\ell).$$

Associated to this control we can consider the *feedback Hamiltonian of singular extremals of the first kind*

$$F^S := H_2 + u_S F_d.$$

which satisfies the following properties

Proposition 5.1. • $\overrightarrow{F^S}$ is tangent to $\mathcal{S}$.

- For any $t \in [\widehat{\tau}_2, T]$, $F^S(\widehat{\lambda}(t)) = \widehat{F}_t(\widehat{\lambda}(t))$ and $\overrightarrow{F^S}(\widehat{\lambda}(t)) = \overrightarrow{\widehat{F}_t}(\widehat{\lambda}(t))$.
- For any $\delta\ell \in T_{\widehat{\ell}_T} \mathcal{S}$, let $a_2(\delta\ell) := \int_T^{\widehat{\tau}_2} \langle du_S(\widehat{\lambda}(s)), \mathcal{F}_{s*}^S \delta\ell \rangle ds$. Then

$$\mathcal{F}_{\widehat{\tau}_2*}^S \delta\ell = \mathcal{H}_{\widehat{\tau}_2*} \delta\ell + a_2(\delta\ell) \overrightarrow{F_d}(\widehat{\ell}_2). \quad (8)$$

Proof. The proof is analogous to the one of [5, Lemmata 4.1 and 4.2]. $\square$

We extend all the definitions given for the quantities involved in the nominal problem $(\mathbf{P}_0)$ in the obvious way to the analogous quantities for problem $(\mathbf{P}_r)$. In particular

$$\Sigma^r := \{\ell \in T^* \mathbb{R}^n : F_d^r(\ell) = 0\}, \quad \mathcal{S}^r := \{\ell \in \Sigma^r : H_{23}^r(\ell) = 0\}.$$

Moreover $F^{S,r} := H_2^r + \frac{H_{232}^r}{\mathbb{L}^r} F_d^r$ is obviously tangent to $\mathcal{S}^r$.

5.2. Finding the extremal

We are looking for an extremal of $(\mathbf{P}_r)$ which joins the Lagrangian manifolds defining the transversality conditions of the fixed-free case

$$\Lambda_0^r := \{x_0^r\} \times (\mathbb{R}^n)^*, \quad \Lambda_{T^r}^r := \{\ell = (p, x) : x \in \mathbb{R}^n, p = -dc^r(x)\} \subset \Sigma^r.$$

We point out that the inclusion $\Lambda_{T^r}^r \subset \Sigma^r$ is a direct consequence of Assumption 1.

Since we are looking for a bang-bang-singular extremal, the unknowns of our problem are $\ell \in \Lambda_{T^r}^r \cap \mathcal{S}^r$, and the switching times $\tau_1, \tau_2, 0 < \tau_1 < \tau_2 < T^r$, while the equations are given by

$$\pi \exp \left(-\tau_1 \overrightarrow{H_1^r} \right) \circ \exp \left((\tau_1 - \tau_2) \overrightarrow{H_2^r} \right) \circ \exp \left((\tau_2 - T^r) \overrightarrow{F^{S,r}} \right) (\ell) = x_0^r, \quad (9)$$

$$(H_1^r - H_2^r) \circ \exp \left((\tau_1 - \tau_2) \overrightarrow{H_2^r} \right) \circ \exp \left((\tau_2 - T^r) \overrightarrow{F^{S,r}} \right) (\ell) = 0. \quad (10)$$

The unknowns for our problem are $n + 1$ as well as equations (9)-(10). We use the Implicit Function Theorem to prove the following lemma.

Lemma 5.2 (Existence of a bang-bang-singular extremal). *Under the assumptions of Theorem 3.1 there exist $\rho > 0$, $\varepsilon > 0$ and a neighbourhood $\mathcal{U}$ of $\widehat{\ell}_T$ in $T^*\mathbb{R}^n$ such that for any r , $|r| < \rho$, there exists a unique normal bang-bang-singular extremal $\lambda^r := (\mu^r, \xi^r)$ of $(\mathbf{P}_r)$ with the following properties:*

- (1) $\lambda^r(T^r) \in \mathcal{U} \cap \Lambda_{T^r}^r$, $|\tau_1^r - \widehat{\tau}_1| < \varepsilon$, $|\tau_2^r - \widehat{\tau}_2| < \varepsilon$, $|T^r - T| < \varepsilon$;
- (2) the switching times τ_1^r and τ_2^r and the final point $\ell^r := \lambda^r(T^r)$ depend smoothly on r .

Proof. We start by linearising equations (9)–(10) with respect to ℓ , τ_1 and τ_2 at $(r, \ell, \tau_1, \tau_2) = (0, \widehat{\ell}_T, \widehat{\tau}_1, \widehat{\tau}_2)$. For $\delta\ell \in T_{\widehat{\ell}_T} \mathcal{S} \cap T_{\widehat{\ell}_T} \Lambda$, we obtain

$$\pi_* \exp(-\widehat{\tau}_1 \vec{H}_1)_* \exp((\widehat{\tau}_1 - \widehat{\tau}_2) \vec{H}_2)_* (-\delta\tau_1 \vec{K}(\widehat{\ell}_2) + \delta\tau_2 u_{\mathcal{S}} \vec{F}_d(\widehat{\ell}_2) + \mathcal{F}_{\widehat{\tau}_2}^S \delta\ell) = 0, \quad (11)$$

$$-\delta\tau_1 H_{12}(\widehat{\ell}_1) + \sigma \left(\delta\tau_2 u_{\mathcal{S}} \vec{F}_d(\widehat{\ell}_2) + \mathcal{F}_{\widehat{\tau}_2}^S \delta\ell, \vec{K} \right) (\widehat{\ell}_2) = 0. \quad (12)$$

If we show that equations (11)–(12) are satisfied only by $\delta\ell = 0$, $\delta\tau_1 = \delta\tau_2 = 0$ then, by the Implicit Function Theorem, we get a unique λ^r .

Since the Hamiltonians H_1 and H_2 are obtained by lifting the vector fields h_1 and h_2 , respectively, equations (11)–(12) are equivalent to

$$\pi_* \left(-\delta\tau_1 \vec{K}(\widehat{\ell}_2) + \delta\tau_2 u_{\mathcal{S}} \vec{F}_d(\widehat{\ell}_2) + \mathcal{F}_{\widehat{\tau}_2}^S \delta\ell \right) = 0, \quad (13)$$

$$-\delta\tau_1 H_{12}(\widehat{\ell}_1) + \sigma \left(\delta\tau_2 u_{\mathcal{S}} \vec{F}_d(\widehat{\ell}_2) + \mathcal{F}_{\widehat{\tau}_2}^S \delta\ell, \vec{K}(\widehat{\ell}_2) \right) = 0. \quad (14)$$

Taking into account (8) and (5) we can compute

$$\begin{aligned} \delta\tau_2 u_{\mathcal{S}} \vec{F}_d(\widehat{\ell}_2) + \mathcal{F}_{\widehat{\tau}_2}^S \delta\ell &= \delta\tau_2 u_{\mathcal{S}} \vec{F}_d(\widehat{\ell}_2) + \mathcal{H}_{\widehat{\tau}_2} \delta\ell + a_2(\delta\ell) \vec{F}_d(\widehat{\ell}_2) = \\ &= \mathcal{H}_{\widehat{\tau}_2} \left(\delta\ell + \left(\delta\tau_2 u_{\mathcal{S}}(\widehat{\ell}_2) + a_2(\delta\ell) \right) \vec{F}_d(\widehat{\ell}_T) \right). \end{aligned}$$

Set $\delta a_2 := \delta\tau_2 u_{\mathcal{S}}(\widehat{\ell}_2) + a_2(\delta\ell)$ and notice that the identity $L_{f_d} c \equiv 0$ implies $\vec{F}_d(\widehat{\ell}_T) = d(-c)_* f_d(\widehat{x}_T) \in L_T$. Therefore $\delta\ell_T := \delta\ell + \delta a_2 \vec{F}_d(\widehat{\ell}_T) = d(-c)_* \pi_* \delta\ell_T$. Thus, from (13), we get $(\pi \mathcal{H}_{\widehat{\tau}_2})_* \delta\ell_T = \delta\tau_1 \vec{k}(\widehat{x}_2)$. Moreover equation (6) gives

$$\mathcal{H}_{\widehat{\tau}_2} \delta\ell_T = d\alpha_{2*} (\pi \mathcal{H}_{\widehat{\tau}_2})_* \delta\ell_T = \delta\tau_1 d\alpha_{2*} \vec{k}(\widehat{x}_2). \quad (15)$$

Hence, from (14), we get

$$0 = -\delta\tau_1 H_{12}(\widehat{\ell}_1) + \sigma \left(\delta\tau_1 d\alpha_{2*} \vec{k}(\widehat{x}_2), \vec{K}(\widehat{\ell}_2) \right) = -\delta\tau_1 \left(H_{12}(\widehat{\ell}_1) - L_k^2 \alpha_2(\widehat{x}_2) \right).$$

From inequality (7) we obtain $\delta\tau_1 = 0$. Hence, by (15), we get $\delta\ell_T = 0$. Since $\delta\ell_T = \delta\ell + \delta a_2 \vec{F}_d(\widehat{\ell}_T)$ and $\vec{F}_d$ is transverse to $\mathcal{S}$ in Σ , we have $\delta\ell = 0$ so that $a_2(\delta\ell) = 0$. Finally $\delta\tau_2 = 0$ by Assumption 4.

We define

$$\lambda^r(t) := \begin{cases} \exp(t - T^r) \overrightarrow{F^{S,r}}(\ell^r) & t \in [\tau_2^r, T^r], \\ \exp(t - \tau_2^r) \overrightarrow{H}_2^r \circ \exp(\tau_2^r - T^r) \overrightarrow{F^{S,r}}(\ell^r) & t \in [\tau_1^r, \tau_2^r], \\ \exp(t - \tau_1^r) \overrightarrow{H}_1^r \circ \exp(\tau_1^r - \tau_2^r) \overrightarrow{H}_2^r \circ \exp(\tau_2^r - T^r) \overrightarrow{F^{S,r}}(\ell^r) & t \in [0, \tau_1^r]. \end{cases}$$

In the following Section 5.3 we show that λ^r is an extremal of $(\mathbf{P}_r)$ as well as its regularity. $\square$

5.3. Proof of the extremality and the regularity of λ^r

The Hamiltonian vector field $\overrightarrow{F^{S,r}}$ is tangent to $\mathcal{S}^r$, hence λ^r is an extremal on the interval $[\tau_2^r, T^r]$.

By continuity and possibly restricting ρ , SGLC and the regularity at the switching times hold true, i.e.

$$\begin{aligned} H_{12}^r(\lambda^r(\tau_1^r)) &> 0, \quad H_{232}^r(\lambda^r(\tau_2^r)) > 0, \\ R^r(t) := \mathbb{L}^r(\lambda^r(t)) &> 0, \quad \forall t \in [\tau_2^r, T^r]. \end{aligned} \tag{16}$$

Moreover we also have $v^r(t) := \frac{H_{232}^r}{\mathbb{L}^r}(\lambda^r(t)) \in (0, 1)$.

5.3.1. Regularity along the singular arc

We have to show that for any $X_j^r \neq h_2^r, h_3^r$, and for any $t \in [\tau_2^r, T^r]$

$$H_2^r(\lambda^r(t)) > \langle \mu^r(t), X_j^r(\xi^r(t)) \rangle. \tag{17}$$

We prove this claim by a contradiction argument. If (17) does not hold, then there exists a sequence $r_n \rightarrow 0$ such that for any $n \in \mathbb{N}$, there exist $t_{r_n} \in [\tau_2^{r_n}, T^{r_n}]$, and $X_{j_{r_n}}^{r_n} \neq h_2^{r_n}, h_3^{r_n}$ such that

$$H_2^{r_n}(\lambda^{r_n}(t_{r_n})) \leq \langle \mu^{r_n}(t_{r_n}), X_{j_{r_n}}^{r_n}(\xi^{r_n}(t_{r_n})) \rangle. \tag{18}$$

Up to a sub-sequence we can assume that the index j_{r_n} does not depend on n . Denote its constant value as $\bar{j}$. Moreover $t_{r_n} \in [\tau_2^{r_n}, T^{r_n}]$, so that, again up to a sub-sequence, we can assume that t_{r_n} converges to some $\bar{t} \in [\widehat{\tau}_2, T]$. Since ℓ^{r_n} converges to $\widehat{\ell}_T$ as $n \rightarrow \infty$ we get that $\lambda^{r_n}(t_{r_n})$ converges to $\widehat{\lambda}(\bar{t})$. Thus, as $n \rightarrow \infty$ in (18) we obtain

$$H_2(\widehat{\lambda}(\bar{t})) \leq \langle \widehat{\mu}(\bar{t}), X_{\bar{j}}(\widehat{\xi}(\bar{t})) \rangle,$$

a contradiction to Assumption 4.

5.3.2. Regularity along the second bang arc

We have to show that for any $t \in (\tau_1^r, \tau_2^r)$

$$H_2^r(\lambda^r(t)) > \langle \mu^r(t), X_j^r(\xi^r(t)) \rangle \quad X_j^r \neq h_2^r.$$

Indeed, proceeding as in the previous section we can show that for any $t \in [\tau_1^r, \tau_2^r]$

$$H_2^r(\lambda^r(t)) > \langle \mu^r(t), X_j^r(\xi^r(t)) \rangle \quad X_j^r \neq h_1^r, h_2^r, h_3^r.$$

A Taylor expansion at $t = \tau_2^r$ and τ_1^r , respectively, give

$$(H_2^r - H_3^r)(\lambda^r(t)) = \frac{(t - \tau_2^r)^2}{2} H_{232}^r(\lambda^r(\tau_2^r)) + o((t - \tau_2^r)^2), \quad (19)$$

$$(H_2^r - H_1^r)(\lambda^r(t)) = (t - \tau_1^r) H_{12}^r(\lambda^r(\tau_1^r)) + o(t - \tau_1^r). \quad (20)$$

By (16), $H_{232}^r(\lambda^r(\tau_2^r))$, $H_{12}^r(\lambda^r(\tau_1^r))$ are strictly positive. If $\varepsilon > 0$ is as in Lemma 5.2, then there exists $\tilde{\varepsilon} \in (0, \varepsilon)$ such that equations (19)-(20) give

$$\begin{aligned} (H_2^r - H_3^r)(\lambda^r(t)) &> 0 \quad \forall t \in (\tau_2^r - \tilde{\varepsilon}, \tau_2^r), \\ (H_2^r - H_1^r)(\lambda^r(t)) &> 0 \quad \forall t \in (\tau_1^r, \tau_1^r + \tilde{\varepsilon}). \end{aligned}$$

Possibly restricting ρ , we obtain $(H_2^r - H_3^r)(\lambda^r(t)) > 0$, for any $t \in (\tau_1^r, \tau_2^r - \tilde{\varepsilon})$ and $(H_2^r - H_1^r)(\lambda^r(t)) > 0$, for any $t \in (\tau_1^r + \tilde{\varepsilon}, \tau_2^r)$, which proves the claim.

5.3.3. Regularity along the first bang arc

We have to show that for any $t \in [0, \tau_1^r]$

$$H_1^r(\lambda^r(t)) > \langle \mu^r(t), X_j^r(\xi^r(t)) \rangle \quad X_j^r \neq h_1^r.$$

The proof follows the same lines of the previous section.

5.4. Coercivity of the second variation J^r

In order to conclude the proof of Theorem 3.1 we only need to prove that J^r is coercive. Namely, let $x_{T^r} := \xi^r(T^r)$ and consider the differential system

$$\dot{\zeta}(t) = w(t) \dot{g}_t^r(x_{T^r}), \quad t \in [\tau_2^r, T^r], \quad \zeta(T^r) = \delta x, \quad \zeta(\tau_2^r) = \varepsilon_0 k^r(x_{T^{r_j}}) \quad (21)$$

the space of admissible variations is given by

$$\mathcal{W}^r = \{ \delta e = (\delta x, \varepsilon_0, w) \in \mathbb{R}^n \times \mathbb{R} \times L^2([\tau_2^r, T^r]) \text{ s.t. system (21) admits a solution} \}.$$

Let $J_0^r := L_{k^r}^2 c^r(x^r) + H_{12}(\lambda^r(\tau_1^r))$. We need to prove the coercivity on $\mathcal{W}^r$ of the quadratic form

$$J^r[\delta e]^2 := \frac{\varepsilon_0^2}{2} J_0^r + \frac{1}{2} \int_{\tau_2^r}^{T^r} (2w(t) L_{\zeta(t)} L_{\dot{g}_t^r} c^r(x^r) + w(t)^2 R^r(t)) \, dt.$$

Since J^r is a Legendre form, it suffices to prove that J^r is a positive quadratic form, see [20, Theorem 11.1]. We prove the claim with a contradiction argument:

*Assume there is a sequence $\{r_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^k$, $r_j \rightarrow 0$ and that there exists $\{\delta e_j\}_{j \in \mathbb{N}}$, $\delta e_j := (\delta x_j, \varepsilon_{0,j}, w_j) \in \mathcal{W}^{r_j}$ such that $\delta e_j \neq 0$, $J^{r_j}[\delta e_j]^2 \leq 0$.
Then J is not coercive on $\mathcal{W}$.*

Let $\varepsilon > 0$ be given by Lemma 5.2. Then $[\tau_2^{r_j}, T^{r_j}] \subset I := [\widehat{\tau}_2 - \varepsilon, T + \varepsilon]$. Thus we can prolong to the interval I , the function $t \rightarrow w_j(t)$ as zero and the functions

$$t \rightarrow \dot{g}_t^{r_j}(x_{T^{r_j}}), \quad t \rightarrow \dot{g}_t(\widehat{x}_T), \quad t \rightarrow R^j(t), \quad t \rightarrow R(t)$$

as constants. Hence $\dot{g}_t^{r_j}(x_{T^{r_j}})$ and $R^j(t)$ converge uniformly on I to $\dot{g}_t(\widehat{x}_T)$ and $R(t)$, respectively.

For any δe_j , let $\|\delta e_j\| := |\delta x_j| + |\varepsilon_{0,j}| + \|w_j\|_2$, where $\|w_j\|_2$ is the L^2 -norm of w_j . Without any loss of generality we can assume $\|\delta e_j\| = 1$ for any $j \in \mathbb{N}$ so that, up to a sub-sequence,

$$\delta x_j \rightarrow \overline{\delta x} \in \mathbb{R}^n, \quad \varepsilon_{0,j} \rightarrow \overline{\varepsilon}_0, \quad w_j \rightharpoonup \overline{w}.$$

If $\overline{\delta e} := (\overline{\delta x}, \overline{\varepsilon}_0, \overline{w})$, then $\|\overline{\delta e}\| \leq 1$.

5.4.1. $\overline{w}(t) = 0$ a.e. $t \in [\widehat{\tau}_2 - \varepsilon, \widehat{\tau}_2] \cup [T, T + \varepsilon]$

We prove that $\overline{w}(t) = 0$ a.e. $t \in [\widehat{\tau}_2 - \varepsilon, \widehat{\tau}_2]$. For $t \in [T, T + \varepsilon]$ the proof follows the same lines. Let $\varphi \in C_c^\infty([\widehat{\tau}_2 - \varepsilon, \widehat{\tau}_2])$, hence

$$\lim_{j \rightarrow \infty} \int_I w_j(t) \varphi(t) dt = \int_I \overline{w}(t) \varphi(t) dt. \quad (22)$$

Let $I_j := [\min\{\tau_2^{r_j}, \widehat{\tau}_2\}, \max\{\tau_2^{r_j}, \widehat{\tau}_2\}]$. We have

$$\begin{aligned} \left| \int_I w_j(t) \varphi(t) dt \right| &= \left| \int_{I_j} w_j(t) \varphi(t) dt \right| \leq \\ &\leq \|w_j\|_2 \left(\int_{I_j} \varphi^2(t) dt \right)^{\frac{1}{2}} \leq \|\varphi\|_\infty |\tau_2^{r_j} - \widehat{\tau}_2|^{\frac{1}{2}}. \end{aligned} \quad (23)$$

Comparing (22) and (23) we get

$$\int_I \overline{w}(t) \varphi(t) dt = 0 \quad \forall \varphi \in C_c^\infty([\widehat{\tau}_2 - \varepsilon, \widehat{\tau}_2]).$$

Since $C_c^\infty([\widehat{\tau}_2 - \varepsilon, \widehat{\tau}_2])$ is a dense subspace of $L^2([\widehat{\tau}_2 - \varepsilon, \widehat{\tau}_2])$, see [21, Corollary 4.23], we get the claim.

5.4.2. $\overline{\delta e} \in \mathcal{W}$

By construction $\delta x_j - \varepsilon_{0,j} k^{r_j}(x_{T^{r_j}}) - \int_{\tau_2^{r_j}}^{T^{r_j}} w_j(t) \dot{g}_t^{r_j}(x_{T^{r_j}}) dt = 0$, moreover

$$\begin{aligned} \left| \int_{\tau_2^{r_j}}^{T^{r_j}} w_j(t) \dot{g}_t^{r_j}(x_{T^{r_j}}) dt - \int_{\widehat{\tau}_2}^T \overline{w}(t) \dot{g}_t(\widehat{x}_T) dt \right| &\leq \\ &\leq \left| \int_I w_j(t) (\dot{g}_t^{r_j}(x_{T^{r_j}}) - \dot{g}_t(\widehat{x}_T)) dt \right| + \left| \int_I (w_j(t) - \overline{w}(t)) \dot{g}_t(\widehat{x}_T) dt \right| \leq \end{aligned}$$

$$\leq \|w_j\|_2 \|\dot{g}_t^{r_j}(x_{T^{r_j}}) - \dot{g}_t(\widehat{x}_T)\|_2 + \left| \int_I (w_j(t) - \overline{w}(t)) \dot{g}_t(\widehat{x}_T) dt \right|.$$

Since both addenda converge to zero, we get the claim.

5.4.3. Convergence properties

Clearly $J_0^{r_j}$ converges to J_0 . Define

$$\zeta_j(t) = \delta x_j - \int_t^{T^{r_j}} w_j(t) \dot{g}_t^{r_j}(x_{T^{r_j}}) dt \quad \text{and} \quad \bar{\zeta}(t) = \overline{\delta x} - \int_t^T \overline{w}(t) \dot{g}_t(\widehat{x}_T) dt,$$

following the same lines of section 5.4.1, we obtain that ζ_j converges uniformly to $\bar{\zeta}$ in I and that $\int_I w_j(t) L_{\zeta_j(t)} L_{\dot{g}_t^{r_j}} c^{r_j}(x_{T^{r_j}}) dt$ converges to $\int_I \overline{w}(t) L_{\bar{\zeta}(t)} L_{\dot{g}_t} c(\widehat{x}_T) dt$.

5.4.4. J is not coercive

Consider $\int_I w_j^2(t) R^{r_j}(t) dt$, replacing $w_j^2 - \overline{w}^2 = 2\overline{w}(w_j - \overline{w}) + (w_j - \overline{w})^2$ we have

$$\begin{aligned} \int_I w_j^2(t) R^{r_j}(t) dt - \int_I \overline{w}^2(t) R(t) dt &= \int_I w_j^2(t) (R^{r_j}(t) - R(t)) dt + \\ &+ \int_I 2\overline{w}(t) (w_j(t) - \overline{w}(t)) R(t) dt + \int_I (w_j(t) - \overline{w}(t))^2 R(t) dt \geq \\ &\geq \int_I (w_j(t) - \overline{w}(t))^2 R(t) dt - \|R^j - R\|_{C^0} \|w_j\|_2^2 - \left| \int_I 2(w_j(t) - \overline{w}(t)) \overline{w}(t) R(t) dt \right|. \end{aligned}$$

The two final addenda converge to zero as $j \rightarrow \infty$. Taking into account the convergence results of the previous sections we thus obtain

$$0 \geq \liminf_{j \rightarrow \infty} J^{r_j}[\delta e_j]^2 \geq J[\overline{\delta e}]^2 + \liminf_{j \rightarrow \infty} \frac{1}{2} \int_I (w_j(t) - \overline{w}(t))^2 R(t) dt. \quad (24)$$

If $\overline{\delta e} \neq (0, 0, 0)$ we immediately get that J is not coercive on $\mathcal{W}$. If $\overline{\delta e} = (0, 0, 0)$ then, for large enough j 's we have $\|w_j\|_2 \geq \frac{1}{2}$. Hence from (24) we get

$$0 \geq \frac{1}{2} \liminf_{j \rightarrow \infty} \|w_j\|_2^2 \min_{[\widehat{\tau}_2, T]} R(t) \geq \frac{1}{8} \min_{[\widehat{\tau}_2, T]} R(t)$$

which contradicts (SGLC). Since this condition is necessary for the coercivity of J , we get the claim.

6. Proof of Theorem 3.2

The proof is quite technical. We split the construction of $\mathcal{O}$ and prove the uniqueness result in several steps. The leading idea is that if λ^r is an extremal of $(\mathbf{P}_r)$ whose

graph is in $\mathcal{O}$, then it satisfies the properties of Lemma 5.2.

6.1. Defining $\mathcal{O}$

Let ε be as in Lemma 5.2 and $m := \min\{m_1, m_{2s}, m_{2b}, \gamma_S, \gamma_{12}, m_{323}, m_R\}$ (these quantities are defined in Section 2). For $t \in (T, T + \varepsilon]$ we set $\hat{\lambda}(t) \equiv \hat{\lambda}(T)$.

We are going to define a tubular neighbourhood of the graph of $\hat{\lambda}$ of the following form

$$\mathcal{O} = \{(t, \ell) : t \in [0, T + \bar{\varepsilon}], \quad \ell \in B(\hat{\lambda}(t), \delta)\},$$

where $B(\hat{\lambda}(t), \delta)$ is the ball in $T^*\mathbb{R}^n$ centred at $\hat{\lambda}(t)$ and of radius δ .

6.1.1. Defining the appropriate $\bar{\varepsilon}$

Let

$$\vartheta := \min \left\{ \frac{m^2}{12}(T - \hat{\tau}_2), \varepsilon \left(\frac{6}{m^2} + \sqrt{\frac{36}{m^4} + \frac{12}{m^2}} \right)^{-1} \right\},$$

by the regularity assumptions on $\hat{\lambda}$ there exists $\bar{\varepsilon} \in (0, \varepsilon)$ such that

$$\begin{aligned} H_1(\hat{\lambda}(t)) - \langle \hat{\mu}(t), X_j(\hat{\xi}(t)) \rangle &> \frac{7}{8}m, & t \in [0, \hat{\tau}_1 + \bar{\varepsilon}], & X_j \neq h_1, h_2, \\ H_2(\hat{\lambda}(t)) - \langle \hat{\mu}(t), X_j(\hat{\xi}(t)) \rangle &> \frac{7}{8}m, & t \in [\hat{\tau}_1 - \bar{\varepsilon}, T + \bar{\varepsilon}], & X_j \neq h_1, h_2, h_3, \\ H_{12}(\hat{\lambda}(t)) &> \frac{7}{8}m, & t \in [\hat{\tau}_1 - \bar{\varepsilon}, \hat{\tau}_1 + \bar{\varepsilon}], & \end{aligned}$$

and, for any $t \in [\hat{\tau}_2 - \bar{\varepsilon}, T + \bar{\varepsilon}]$,

$$\begin{aligned} \frac{7}{8}m < \mathbb{L}(\hat{\lambda}(t)) < \frac{9}{8}M_R, & \quad \frac{7}{8}m \leq \frac{H_{232}}{\mathbb{L}}(\hat{\lambda}(t)) \leq 1 - \frac{7}{8}m, \\ H_{323}(\hat{\lambda}(t)) \geq \frac{7}{8}m, & \quad |H_{23}(\hat{\lambda}(t))| \leq \vartheta, \quad |F_d(\hat{\lambda}(t))| \leq \vartheta^2. \end{aligned}$$

6.1.2. Defining the appropriate δ

Define

$$\bar{m} = \min \left\{ \min_{t \in [0, \hat{\tau}_1 - \bar{\varepsilon}]} (H_1 - H_2)(\hat{\lambda}(t)), \min_{t \in [\hat{\tau}_1 + \bar{\varepsilon}, \hat{\tau}_2]} (H_2 - H_1)(\hat{\lambda}(t)) \right\}$$

so that, by Assumption 5, $\bar{m} > 0$. By continuity, there exists $\delta > 0$ such that for any $\ell = (p, x) \in B(\hat{\lambda}(t), \delta)$

$$\begin{aligned} H_1(\ell) - \langle p, X_j(x) \rangle &> \frac{3}{4}m, & t \in [0, \hat{\tau}_1 + \bar{\varepsilon}], & X_j \neq h_1, h_2, \\ (H_1 - H_2)(\ell) &> \frac{3}{4}\bar{m}, & t \in [0, \hat{\tau}_1 - \bar{\varepsilon}], & \end{aligned}$$

$$\begin{aligned}
H_2(\ell) - \langle p, X_j(x) \rangle &> \frac{3}{4}m, & t \in [\widehat{\tau}_1 - \bar{\varepsilon}, T + \bar{\varepsilon}], & X_j \neq h_1, h_2, h_3 \\
(H_2 - H_1)(\ell) &> \frac{3}{4}\bar{m}, & t \in [\widehat{\tau}_1 + \bar{\varepsilon}, \widehat{\tau}_2], & \\
H_{12}(\ell) &> \frac{3}{4}m, & t \in [\widehat{\tau}_1 - \bar{\varepsilon}, \widehat{\tau}_1 + \bar{\varepsilon}], &
\end{aligned}$$

and, for any $t \in [\widehat{\tau}_2 - \bar{\varepsilon}, T + \bar{\varepsilon}]$ and for any $\ell \in B(\widehat{\lambda}(t), \delta)$,

$$\begin{aligned}
\frac{3}{4}m < \mathbb{L}(\ell) &< \frac{5}{4}M_R, & \frac{3}{4}m \leq \frac{H_{232}}{\mathbb{L}}(\ell) &\leq 1 - m, \\
H_{323}(\ell) &\geq \frac{3}{4}m, & |H_{23}(\ell)| &\leq \frac{5}{4}\vartheta, & |F_d(\ell)| &\leq \frac{5}{4}\vartheta^2.
\end{aligned}$$

6.2. Defining the appropriate ρ

By continuity with respect to r , there exists $\rho > 0$ such that for any r with $|r| < \rho$ we have $|T^r - T| < \bar{\varepsilon}$ and, for any $\ell = (p, x) \in B(\widehat{\lambda}(t), \delta)$,

$$H_1^r(\ell) - \langle p, X_j^r(x) \rangle > \frac{m}{2}, \quad t \in [0, \widehat{\tau}_1 + \bar{\varepsilon}], \quad X_j \neq h_1, h_2, \quad (25)$$

$$(H_1^r - H_2^r)(\ell) > \frac{\bar{m}}{2}, \quad t \in [0, \widehat{\tau}_1 - \bar{\varepsilon}], \quad (26)$$

$$H_2^r(\ell) - \langle p, X_j^r(x) \rangle > \frac{m}{2}, \quad t \in [\widehat{\tau}_1 - \bar{\varepsilon}, T + \bar{\varepsilon}], \quad X_j \neq h_1, h_2, h_3, \quad (27)$$

$$(H_2^r - H_1^r)(\ell) > \frac{\bar{m}}{2}, \quad t \in [\widehat{\tau}_1 + \bar{\varepsilon}, \widehat{\tau}_2], \quad (28)$$

$$H_{12}^r(\ell) > \frac{m}{2}, \quad t \in [\widehat{\tau}_1 - \bar{\varepsilon}, \widehat{\tau}_1 + \bar{\varepsilon}].$$

Moreover, for any $t \in [\widehat{\tau}_2 - \bar{\varepsilon}, T + \bar{\varepsilon}]$, $\ell \in B(\widehat{\lambda}(t), \delta)$ and r such that $|r| < \rho$,

$$\begin{aligned}
\frac{m}{2} < \mathbb{L}^r(\ell) &< \frac{3}{2}M_R, & \frac{m}{2} \leq \frac{H_{232}^r}{\mathbb{L}^r}(\ell) &\leq 1 - \frac{m}{2}, \\
H_{323}^r(\ell) &\geq \frac{m}{2}, & |H_{23}^r(\ell)| &\leq \frac{3}{2}\vartheta, & |F_d^r(\ell)| &\leq \frac{3}{2}\vartheta^2.
\end{aligned} \quad (29)$$

6.3. Uniqueness of the extremal in $\mathcal{O}$

Assume $\widetilde{\lambda}: [0, T^r] \rightarrow T^*\mathbb{R}^n$ is an extremal of $(\mathbf{P}_r)$ for some r such that $|r| < \rho$ and assume that the graph of $\widetilde{\lambda}$ is in $\mathcal{O}$. We prove that $\widetilde{\lambda}$ satisfies the properties described in Lemma 5.2 so that $\widetilde{\lambda} = \lambda^r$.

Since $\widetilde{\lambda}(t) \in B(\widehat{\lambda}(t), \delta)$, thanks to (25)-(26)-(27)-(28),

$$\begin{aligned}
\dot{\widetilde{\lambda}}(t) &= \overrightarrow{H_1^r}(\widetilde{\lambda}(t)), & t \in [0, \widehat{\tau}_1 - \bar{\varepsilon}], \\
\dot{\widetilde{\lambda}}(t) &= \overrightarrow{H_2^r}(\widetilde{\lambda}(t)), & t \in [\widehat{\tau}_1 + \bar{\varepsilon}, \widehat{\tau}_2 - \bar{\varepsilon}],
\end{aligned}$$

and there exists some measurable function $\widetilde{u}$ taking values in $[0, 1]$ such that

$$\dot{\widetilde{\lambda}}(t) = \widetilde{u}(t)\overrightarrow{H_1^r}(\widetilde{\lambda}(t)) + (1 - \widetilde{u}(t))\overrightarrow{H_2^r}(\widetilde{\lambda}(t)), \quad t \in [\widehat{\tau}_1 - \bar{\varepsilon}, \widehat{\tau}_1 + \bar{\varepsilon}],$$

The function $\varphi: t \in [\hat{\tau}_1 - \bar{\varepsilon}, \hat{\tau}_1 + \bar{\varepsilon}] \mapsto (H_2^r - H_1^r)(\tilde{\lambda}(t))$ is strictly monotone. Indeed

$$\dot{\varphi}(t) = \sigma \left(\dot{\tilde{\lambda}}(t), (\overrightarrow{H_2^r} - \overrightarrow{H_1^r})(\tilde{\lambda}(t)) \right) = H_{12}^r(\tilde{\lambda}(t)) \geq \frac{m}{2} > 0.$$

Since $\varphi(\hat{\tau}_1 - \bar{\varepsilon}) < 0 < \varphi(\hat{\tau}_1 + \bar{\varepsilon})$, there exists one and only one $\tilde{\tau}_1 \in (\hat{\tau}_1 - \bar{\varepsilon}, \hat{\tau}_1 + \bar{\varepsilon})$ such that φ vanishes and changes sign at $\tilde{\tau}_1$. Hence, by PMP,

$$\dot{\tilde{\lambda}}(t) = \overrightarrow{H_1^r}(\tilde{\lambda}(t)), \quad t \in [0, \tilde{\tau}_1], \quad \dot{\tilde{\lambda}}(t) = \overrightarrow{H_2^r}(\tilde{\lambda}(t)), \quad t \in [\tilde{\tau}_1, \hat{\tau}_2 - \bar{\varepsilon}].$$

Thanks to (27)-(28), for $t \in [\hat{\tau}_2 - \bar{\varepsilon}, T^r]$, the maximised Hamiltonian in a neighbourhood of $\tilde{\lambda}(t)$ is either H_2^r or H_3^r . Moreover, since $\tilde{\lambda}$ is an extremal of $(\mathbf{P}_r)$, $F_d^r(\tilde{\lambda}(T^r)) = 0$.

Assume, by contradiction, $F_d^r(\tilde{\lambda}(t)) < 0$ for any $t \in [\hat{\tau}_2 - \bar{\varepsilon}, T^r]$. Then, by (29)

$$\begin{aligned} 0 = F_d^r(\tilde{\lambda}(T^r)) &= F_d^r(\tilde{\lambda}(\hat{\tau}_2 - \bar{\varepsilon})) + \int_{\hat{\tau}_2 - \bar{\varepsilon}}^{T^r} \sigma \left(\dot{\tilde{\lambda}}(t), (\overrightarrow{H_3^r} - \overrightarrow{H_2^r})(\tilde{\lambda}(t)) \right) dt < \\ &< \int_{\hat{\tau}_2 - \bar{\varepsilon}}^{T^r} H_{23}^r(\tilde{\lambda}(t)) dt = \int_{\hat{\tau}_2 - \bar{\varepsilon}}^{T^r} \left(H_{23}^r(\tilde{\lambda}(\hat{\tau}_2 - \bar{\varepsilon})) + \int_{\hat{\tau}_2 - \bar{\varepsilon}}^t \sigma \left(\dot{\tilde{\lambda}}(s), \overrightarrow{H_{23}^r}(\tilde{\lambda}(s)) \right) ds \right) dt = \\ &= \int_{\hat{\tau}_2 - \bar{\varepsilon}}^{T^r} \left(H_{23}^r(\tilde{\lambda}(\hat{\tau}_2 - \bar{\varepsilon})) + \int_{\hat{\tau}_2 - \bar{\varepsilon}}^t -H_{232}^r(\tilde{\lambda}(s)) ds \right) dt < \\ &< \frac{3}{2} \theta (T^r - \hat{\tau}_2 + \bar{\varepsilon}) - \frac{m^2}{8} (T^r - \hat{\tau}_2 + \bar{\varepsilon})^2 = \\ &= (T^r - \hat{\tau}_2 + \bar{\varepsilon}) \left\{ \frac{3}{2} \theta - \frac{m^2}{8} (T^r - \hat{\tau}_2 + \bar{\varepsilon}) \right\} < \\ &< (T^r - \hat{\tau}_2 + \bar{\varepsilon}) \left\{ \frac{3}{2} \theta - \frac{m^2}{8} (T - \hat{\tau}_2) \right\} \leq 0, \end{aligned}$$

i.e. $0 < 0$, a contradiction. Let

$$\tilde{\tau}_2 := \min\{t \in [\hat{\tau}_2 - \bar{\varepsilon}, T^r] : F_d^r(\tilde{\lambda}(t)) = 0\}.$$

We have just shown that $\tilde{\tau}_2 < T^r$. We want to show that $\tilde{\tau}_2 \leq \hat{\tau}_2 + \varepsilon$. If $\tilde{\tau}_2 \leq \hat{\tau}_2$ we are done. Assume $\tilde{\tau}_2 > \hat{\tau}_2$ and let $s := \tilde{\tau}_2 - \hat{\tau}_2$. With a Taylor expansion at $t = \tilde{\tau}_2$ we get that there exists $\bar{t} \in (\hat{\tau}_2, \tilde{\tau}_2)$ such that

$$F_d^r(\tilde{\lambda}(\hat{\tau}_2)) = -s H_{23}^r(\tilde{\lambda}(\tilde{\tau}_2)) - \frac{s^2}{2} H_{232}^r(\tilde{\lambda}(\bar{t})).$$

This equation, in the unknown s , has one and only one positive root which is given by

$$\begin{aligned} s &= \frac{H_{23}^r(\tilde{\lambda}(\tilde{\tau}_2))}{H_{232}^r(\tilde{\lambda}(\bar{t}))} + \sqrt{\left(\frac{H_{23}^r(\tilde{\lambda}(\tilde{\tau}_2))}{H_{232}^r(\tilde{\lambda}(\bar{t}))} \right)^2 - \frac{2F_d^r(\tilde{\lambda}(\hat{\tau}_2))}{H_{232}^r(\tilde{\lambda}(\bar{t}))}} \leq \\ &\leq \frac{\frac{3\vartheta}{2}}{\frac{m^2}{4}} + \sqrt{\left(\frac{\frac{3\vartheta}{2}}{\frac{m^2}{4}} \right)^2 + \frac{3\vartheta^2}{m^2}} = \vartheta \left(\frac{6}{m^2} + \sqrt{\left(\frac{6}{m^2} \right)^2 + \frac{12}{m^2}} \right) \end{aligned}$$

$$= \frac{\varepsilon m}{2} \left(\frac{3}{m} + \sqrt{\frac{9}{m^2} + 3} \right)^{-1} \frac{2}{m} \left(\frac{3}{m} + \sqrt{\frac{9}{m^2} + 3} \right) = \varepsilon.$$

To conclude the proof we need to show that $F_d^r(\tilde{\lambda}(t)) = 0$ for any $t \in [\tilde{\tau}_2, T^r]$. Let

$$\mathcal{A} := \left\{ t \in (\tilde{\tau}_2, T^r) : F_d^r(\tilde{\lambda}(t)) \neq 0 \right\}.$$

$\mathcal{A}$ is an open set. Assume by contradiction that $\mathcal{A} \neq \emptyset$. Then it contains at least one maximal interval (t_1, t_2) , so that two cases may occur:

Case 1. $F_d^r(\tilde{\lambda}(t_1)) = F_d^r(\tilde{\lambda}(t_2)) = 0$, $F_d^r(\tilde{\lambda}(t)) < 0 \quad \forall t \in (t_1, t_2)$.

In this case, since $\tilde{\lambda}$ is an extremal, we have $\dot{\tilde{\lambda}}(t) = \overrightarrow{H_2^r}(\tilde{\lambda}(t))$ for any $t \in (t_1, t_2)$.

Moreover $H_{23}^r(\tilde{\lambda}(t_1)) = \frac{d}{dt} F_d^r(\tilde{\lambda}(t_1)) \leq 0$. Hence

$$\begin{aligned} 0 &= F_d^r(\tilde{\lambda}(t_2)) - F_d^r(\tilde{\lambda}(t_1)) = \int_{t_1}^{t_2} \sigma \left(\dot{\tilde{\lambda}}(t), (\overrightarrow{H_3^r} - \overrightarrow{H_2^r})(\tilde{\lambda}(t)) \right) dt = \int_{t_1}^{t_2} H_{23}^r(\tilde{\lambda}(t)) dt = \\ &= \int_{t_1}^{t_2} \left(H_{23}^r(\tilde{\lambda}(t_1)) - \int_{t_1}^t H_{232}^r(\tilde{\lambda}(s)) ds \right) dt \leq \frac{-m^2}{4} \frac{(t_2 - t_1)^2}{2} < 0, \end{aligned}$$

i.e. $0 < 0$, a contradiction.

Case 2. $F_d^r(\tilde{\lambda}(t_1)) = F_d^r(\tilde{\lambda}(t_2)) = 0$, $F_d^r(\tilde{\lambda}(t)) > 0 \quad \forall t \in (t_1, t_2)$.

In this case, since $\tilde{\lambda}$ is an extremal, we have $\dot{\tilde{\lambda}}(t) = \overrightarrow{H_3^r}(\tilde{\lambda}(t))$ for any $t \in (t_1, t_2)$.

Moreover $H_{23}^r(\tilde{\lambda}(t_1)) = \frac{d}{dt} F_d^r(\tilde{\lambda}(t_1)) \geq 0$. Hence

$$\begin{aligned} 0 &= F_d^r(\tilde{\lambda}(t_2)) - F_d^r(\tilde{\lambda}(t_1)) = \int_{t_1}^{t_2} \sigma \left(\dot{\tilde{\lambda}}(t), (\overrightarrow{H_3^r} - \overrightarrow{H_2^r})(\tilde{\lambda}(t)) \right) dt = \int_{t_1}^{t_2} H_{23}^r(\tilde{\lambda}(t)) dt = \\ &= \int_{t_1}^{t_2} \left(H_{23}^r(\tilde{\lambda}(t_1)) + \int_{t_1}^t H_{323}^r(\tilde{\lambda}(s)) ds \right) dt \geq \frac{m}{2} \frac{(t_2 - t_1)^2}{2} > 0. \end{aligned}$$

Again we have obtained a contradiction.

Summing up $\tilde{\lambda}$ satisfies all the properties of the extremal λ^r defined in Lemma 5.2. Therefore $\tilde{\lambda} \equiv \lambda^r$.

References

- [1] Poggiolini L, Stefani G. Strong local optimality for a bang-bang-singular extremal: The fixed-free case. SIAM Journal on Control and Optimization. 2018;56(3):2274–2294. Available from: <https://doi.org/10.1137/17M1140248>.
- [2] Vossen G. Switching time optimization for bang-bang and singular controls. Journal of Optimization Theory and Applications. 2010;144(2):409–429.
- [3] Maurer H, Osmolovskii N. Second-order conditions for optimal control problems with mixed control-state constraints and control appearing linearly. In: 52nd IEEE Conference on Decision and Control; 2013. p. 514–519.

- [4] Felgenhauer U, Poggiolini L, Stefani G. Optimality and stability result for bang–bang optimal controls with simple and double switch behaviour. *CONTROL AND CYBERNETICS*. 2009;38(4B):1305 – 1325.
- [5] Poggiolini L, Stefani G. Structural stability for bang–singular–bang extremals in the minimum time problem. *SIAM J Control Optim*. 2013;51(5):3511–3531. Available from: <http://dx.doi.org/10.1137/120895421>.
- [6] Poggiolini L, Stefani G. A case study in strong optimality and structural stability of bang-singular extremals. In: Stefani G, Boscain U, Gauthier JP, et al., editors. *Geometric control theory and sub-riemannian geometry*. (Springer INdAM Series; Vol. 5). Springer International Publishing; 2014. p. 333–350.
- [7] Poggiolini L. Structural stability of bang–bang trajectories with a double switching time in the minimum time problem. *SIAM JOURNAL ON CONTROL AND OPTIMIZATION*. 2017;55(6):3779–3798.
- [8] Poggiolini L, Stefani G. Strong local optimality for a bang–bang–singular extremal: General constraints. *Journal of Optimization Theory and Applications*. 2020;186.
- [9] Malanowski K. Stability and sensitivity analysis for optimal control problems with control-state constraints. In: *Dissertationes mathematicae*. Vol. CCCXCIV. Institute of Mathematics, Polish Academy of Sciences; 2001.
- [10] Felgenhauer U. Structural stability investigation of bang-singular-bang optimal controls. *Journal of Optimization Theory and Applications*. 2012;152:605–631. 10.1007/s10957-011-9925-0; Available from: <http://dx.doi.org/10.1007/s10957-011-9925-0>.
- [11] Aronna MS, Bonnans JF, Martinon P. A shooting algorithm for optimal control problems with singular arcs. *Journal of Optimization Theory and Applications*. 2013;158:419–459.
- [12] Bonnans JF, Hermant A. Stability and sensitivity analysis for optimal control problems with a first-order state constraint and application to continuation methods. *ESAIM Control Optim Calc Var*. 2008;8:825–863.
- [13] Agrachev AA, Sachkov YL. *Control theory from the geometric viewpoint*. Springer-Verlag; 2004.
- [14] Zelikin MI, Borisov VF. *Theory of chattering control*. Boston, Basel, Berlin: Birkhauser; 1994. *Systems & Control: Foundations & Applications*.
- [15] Agrachev AA, Stefani G, Zezza P. Strong optimality for a bang-bang trajectory. *SIAM J Control Optimization*. 2002;41(4):991–1014.
- [16] Poggiolini L, Stefani G. Bang-singular-bang extremals: sufficient optimality conditions. *Journal of Dynamical and Control Systems*. 2011;17:469–514. 10.1007/s10883-011-9127-y; Available from: <http://dx.doi.org/10.1007/s10883-011-9127-y>.
- [17] Poggiolini L, Stefani G. Strong local optimality for bang-bang-singular extremals in single input control problems. In: 20th IFAC World Congress Proceedings; (IFAC-PAPERSONLINE; Vol. 50); IFAC. Elsevier; 2017. p. 6128–6133.
- [18] Stefani G. Strong optimality of singular trajectories. In: Ancona F, Bressan A, Cannarsa P, et al., editors. *Geometric Control and Nonsmooth Analysis*; (Series on Advances in Mathematics for Applied Sciences; Vol. 76); Hackensack, NJ. World Scientific Publishing Co. Pte. Ltd.; 2008. p. 300–326. Pp. 361 ISBN: 978-981-277-606-8.
- [19] Chittaro FC, Stefani G. Minimum-time strong optimality of a singular arc: The multi-input non involutive case. *ESAIM: COCV*. 2016;22(3):786–810. Available from: <http://dx.doi.org/10.1051/cocv/2015026>.
- [20] Hestenes MR. Applications of the theory of quadratic forms in Hilbert space to calculus of variations. *Pacific J Math*. 1951;1:525–581.
- [21] Brezis H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. New York: Springer; 2011. Universitext.