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# Trace and Künneth formulas for singularity categories and applications

Bertrand Toën\* and Gabriele Vezzosi

September 2021

## Abstract

We present an  $\ell$ -adic trace formula for saturated and admissible dg-categories over a base monoidal dg-category. Moreover, we prove Künneth formulas for dg-category of singularities, and for inertia-invariant vanishing cycles. As an application, we prove a categorical version of Bloch's Conductor Conjecture (originally stated by Spencer Bloch in 1985), under the additional hypothesis that the monodromy action of the inertia group is unipotent.

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*Key words:* Trace formulas, Non-commutative geometry, Vanishing cycles, Bloch's Conductor conjecture.

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# 1 Introduction

Motivated by applications to arithmetic geometry, the first one being presented in this paper, we develop methods of derived and non-commutative geometry over a mixed-characteristic base. More precisely, we prove and investigate a trace formula for dg-categories, Künneth formulas for dg-categories of singularities and for inertia-invariant vanishing cycles, and we give an application of these results to a categorical version of Bloch’s Conductor Conjecture (BCC) in the presence of unipotent monodromy.

*Derived geometry* is a generalization of algebraic geometry where commutative rings are replaced by objects with a richer homotopical structure (such as simplicial commutative rings or commutative differential graded algebras over a base ring of characteristic zero); they allow for more flexibility in many constructions of moduli spaces. *Non-commutative geometry*, as we intend it here, aims at replacing an algebro-geometric object (a variety, a scheme, a stack, etc.) with some appropriately chosen differential graded (dg) category associated to it (e.g. the dg-category of coherent or perfect complexes on the given scheme or stack), and studying this dg-category using the methods of  $\infty$ -categories, in order to extract new information or just known information in a different way. Choosing the associated dg-category very much depends, of course, on the kind of problem one wants to tackle. As an example, if one is interested in studying the singularities of a (quasi-compact and quasi-separated) scheme  $Y$ , a natural dg-category to be associated to  $Y$  is the so-called *dg-category of singularities*  $\text{Sing}(Y) := \text{Coh}^b(Y)/\text{Perf}(Y)$ , which is trivial iff  $Y$  is regular. This point of view naturally leads to developing methods to extract interesting invariants in order to study certain classes of dg-categories for their own sake, even if they do not come from a specified algebro-geometric object. Such dg-categories are thus considered as non-commutative spaces.

Though both derived and non-commutative geometry have been mainly (though not exclusively) applied so far over a base field (possibly of positive characteristic), their flexibility is definitely wider. In particular, one might hope, to apply derived geometrical and non-commutative methods to open questions in arithmetic geometry. As a recent example of particular interest for us, in [BRTV] the authors work over an arbitrary excellent strictly Henselian ring  $A$  and exhibit a  $\ell$ -adic realization functor  $r_\ell$  from the Morita  $\infty$ -category of dg-categories over  $A$  to the derived  $\infty$ -category of  $\ell$ -adic sheaves over  $S = \text{Spec } A$ . The construction of  $r_\ell$  involves derived geometry, and it is obviously a part of non-commutative geometry over the base  $A$ , as outlined above: by its very definition,  $r_\ell$  is an invariant of non-commutative spaces over  $A$ . Moreover, as shown in [BRTV],  $r_\ell$  is an arithmetically interesting invariant, namely, if  $X$  is regular and  $X \rightarrow S$  proper and flat, then the  $\ell$ -adic realization of the dg-category of singularities  $\text{Sing}(X_s)$  of the special fiber  $X_s$  recovers the inertia invariant part of vanishing cohomology of

$X/S$  (suitably 2-periodized). This result admits a straightforward sheafified generalization (see below Remark 2.2.2).

For  $S$  the spectrum of a discrete valuation ring (with perfect residue field  $k$ , and fraction field  $K$ ), and  $X$  a regular scheme endowed with a generically smooth, flat and proper map to  $S$ , the proper base-change formula for  $\ell$ -adic cohomology, say with  $\mathbb{Q}_\ell$ -coefficients, implies that the  $\ell$ -adic Euler characteristic  $\chi(X_k, \nu_{X/S}[-1])$  of the  $(-1)$ -shifted vanishing cycles  $\nu_{X/S}[-1]$  equals to the change  $\delta_{X/S} := \chi(X_{\bar{k}}, \mathbb{Q}_\ell) - \chi(X_{\bar{K}}, \mathbb{Q}_\ell)$  of the  $\ell$ -adic characteristics of  $X/S$ , passing from the (possibly singular) special fiber  $X_k$  to the smooth geometric generic fiber  $X_{\bar{K}}$ . Now, *Bloch's conductor conjecture* (BCC, [Bl]) can be seen as a way to express  $\delta_{X/S}$  as an *intersection theoretic* term (the so-called Bloch's intersection number i.e. the self-intersection  $[\Delta_X, \Delta_X]_S$  of the diagonal  $\Delta_X$ ) plus an *arithmetic* correction (the so-called Swan conductor  $Sw(X_{\bar{K}})$  of the geometric generic fiber  $X_{\bar{K}}$ ) coming from wild ramification of  $X/S$ :

$$\delta_{X/S} = [\Delta_X, \Delta_X]_S + Sw(X_{\bar{K}}).$$

BCC is a kind of arithmetic version of Gauss-Bonnet formula, it seeks to describe the change in the topology of the family  $X/S$  in terms of intersection theory and of further arithmetic data. If  $X/S$  is of relative dimension 0, BCC is equivalent to the well-known classical conductor-discriminant formula (see e.g. [Neu, VII.11.9]), while in relative dimension 1, BCC was proved by S. Bloch himself in [Bl]. For higher relative dimensions, BCC is still open in its full generality, though there are many important partial positive results (see, e.g. [Ka-Sa, Sai]).

Let us now consider the special case of BCC where the inertia group  $I := \text{Gal}(\bar{K}/K^{\text{unr}}) \subseteq \text{Gal}(\bar{K}/K)$  acts *unipotently* on  $H^*(X_{\bar{K}}, \mathbb{Q}_\ell)$ . Note that, since we can always reduce to the case where  $k = \bar{k}$ , i.e. the residue field of  $A$  is separably closed, the reader may safely suppose that  $I = \text{Gal}(\bar{K}/K)$  if it helps. Since the inertia action is supposed to be unipotent, the Swan conductor vanishes, and moreover taking  $I$ -invariants does not modify  $\chi(X_k, \nu_{X/S}[-1])$ , therefore the main result of [BRTV] quoted above, tells us that we may express  $\delta_{X/S}$  in non-commutative terms via  $r_\ell(\text{Sing}(X_k))$ . Thus, it is also possible to conceive that BCC can be approached, at least in the case of unipotent inertia action, using non-commutative geometry, i.e. that, first of all, all the terms in BCC have a non-commutative interpretation in terms of  $\text{Sing}(X_k)$ , and finally that BCC equality itself is a consequence of an equality between invariants of the non-commutative space  $\text{Sing}(X_k)$ . The main idea of this paper is that, indeed, there is such a non-commutative interpretation of the unipotent case of BCC, via a *non-commutative trace formula*.

More precisely, we first prove the following general non-commutative trace formula. Let  $B$  is a monoidal dg-category over  $A$  ( $A$  being our base discrete valuation ring, as above), and  $T$  a dg-category on which  $B$  acts on the left (making  $T$  into a left  $B$ -module). Suppose  $T$  is *smooth and proper*<sup>1</sup> over  $B$ , and is furthermore  *$r_\ell$ -admissible over  $B$*  (i.e the canonical map  $r_\ell(T^{\text{op}}) \otimes_{r_\ell(B)} r_\ell(T) \rightarrow r_\ell(T^{\text{op}} \otimes_B T)$  is an isomorphism in the derived category of  $\ell$ -adic sheaves over  $S = \text{Spec } A$ ), and consider an arbitrary endomorphism  $f : T \rightarrow T$  of left  $B$ -modules. By our smooth and properness hypothesis, we may consider both the trace of  $f$  over  $B$ , and the trace of  $r_\ell(f) : r_\ell(T) \rightarrow r_\ell(T)$  over  $r_\ell(B)$ . In the additional hypothesis of  $r_\ell$ -admissibility of  $T$  over  $B$ , these two traces may indeed be compared (see 2.4.9 in the main text, and Theorem A later in this Introduction) via a  $\ell$ -adic version of the Chern character  $Ch_\ell$  (Definition 2.3.1 in the main text), yielding our *non-commutative trace formula*

$$Ch_\ell([Tr_B(f)]) = Tr_{r_\ell(B)}(r_\ell(f)).$$

---

<sup>1</sup>Since  $B$  is only monoidal but not necessarily symmetric monoidal, the definition of smooth and proper dg-category over  $B$  requires a bit of care.

Now, we would like to apply this trace formula to  $T = \mathbf{Sing}(X_k)$  (in the above notations), and relate it to BCC. However, since the  $\ell$ -adic realization of any dg-category over  $A$  is always a  $\mathbb{Q}_\ell(\beta) := \bigoplus_{n \in \mathbb{Z}} \mathbb{Q}_\ell(2n)$ -module, there is no chance for the trace formula to make sense with  $B := A$ , simply because  $T = \mathbf{Sing}(X_k)$  will not be smooth and proper over  $A$  itself. However, if we take  $B := \mathcal{B} = \mathbf{Sing}(\mathrm{Spec}(k \otimes_A^L k))$ ,  $\mathbf{Sing}(X_k)$  is indeed a dg-category which is smooth and proper over the monoidal dg-category  $\mathcal{B}$ . Note that  $\mathrm{Spec}(k \otimes_A^L k)$  is here a *derived scheme* representing the derived fiber product  $s \times_S s$ ,  $s \rightarrow S$  being the closed point, and the dg-category  $\mathcal{B}$  inherits its *convolution* (non-symmetric) *monoidal structure* from the fact that  $s \times_S s$  is, in fact, a *derived groupoid*. Therefore we see that, in order for  $\mathcal{B}$  to exist, and thus the trace formula to be applicable to  $T = \mathbf{Sing}(X_k)$ , it is necessary to work inside *derived geometry*.

In order to prove that  $\mathbf{Sing}(X_k)$  is smooth and proper over  $\mathcal{B}$ , and that it is furthermore  $r_\ell$ -admissible over  $\mathcal{B}$ , we use two general *Künneth type formulas*, one for inertia invariant vanishing cycles (Theorem 3.4.2 in the main text, and Theorem B later in this Introduction), and the other for dg-categories of singularities (Theorem 4.2.1 in the main text, and Theorem C later in this Introduction). We think these Künneth formulas are interesting in their own right and have a wider range of applications than the one shown in this paper.

The combination of these two Künneth formulas implies that  $\mathbf{Sing}(X_k)$  is smooth and proper over  $\mathcal{B}$ ; note that this was already known when  $A$  is a field of characteristic 0 but it was unknown in general, and we find it quite surprising in arbitrary mixed characteristic. Again by these Künneth formulas, we show that  $\mathbf{Sing}(X_k)$  is also  $r_\ell$ -admissible over  $\mathcal{B}$ , if we further assume that the inertia action is unipotent. It is important to remark that, in the non unipotent case,  $r_\ell$ -admissibility might indeed fail.

Once we have established that  $\mathbf{Sing}(X_k)$  is smooth, proper, and  $r_\ell$ -admissible over  $\mathcal{B}$ , we may apply our trace formula to  $f = \mathrm{id} : \mathbf{Sing}(X_k) \rightarrow \mathbf{Sing}(X_k)$ :

$$\mathrm{Ch}_\ell([Tr_{\mathcal{B}}(\mathrm{id}_{\mathbf{Sing}(X_k)})]) = Tr_{r_\ell(\mathcal{B})}(r_\ell(\mathrm{id}_{\mathbf{Sing}(X_k)})).$$

Now, the r.h.s. of this equality is easily seen to coincide with  $\delta_{X/S}$ , while the l.h.s. can be interpreted as the self-intersection class of the diagonal  $\Delta_{X/S}$  in non-commutative geometry, so it is reasonable to denote it by  $[\Delta_X, \Delta_X]_S^{\mathrm{cat}}$ , and rewrite the trace formula as

$$[\Delta_X, \Delta_X]_S^{\mathrm{cat}} = \delta_{X/S}.$$

The above formula is our *categorical* or *non-commutative version of BCC*, for unipotent inertia action (Theorem 5.2.2 in the main text, and Theorem D later in this Introduction). We believe that  $[\Delta_X, \Delta_X]_S^{\mathrm{cat}}$  is equal to Bloch's number  $[\Delta_X, \Delta_X]_S$  (even when the inertia action is not unipotent), so that the above formula should indeed prove BCC in the case of unipotent inertia action. The comparison between our categorical Bloch's class  $[\Delta_X, \Delta_X]_S^{\mathrm{cat}}$  and the original Bloch's number  $[\Delta_X, \Delta_X]_S$  will be investigated in a forthcoming paper.

We hope the example of this paper might encourage both arithmetic geometers in using methods of derived and non-commutative geometry, and experts in homotopical and higher categorical methods in applying their tools to questions in arithmetic geometry.

We give now a more detailed description of the content of this paper. We present four main results. As a first step, in Section 2, we prove a quite general *trace formula for dg-categories* (or non-commutative schemes). This is done by using the non-commutative  $\ell$ -adic realization functor  $r_\ell$  recently introduced in [BRTV] as a functor from dg-categories over an excellent discrete valuation ring  $A$  to  $\ell$ -adic sheaves

on  $\text{Spec } A$ . Via  $r_\ell$ , we first introduce a  $\ell$ -adic version of the Chern character for dg-categories over  $A$ , as a lax-monoidal natural transformation  $Ch_\ell : \mathbf{HK} \rightarrow |r_\ell|$ , where  $\mathbf{HK}$  is the non-connective, homotopy invariant  $K$ -theory functor on dg-categories, and  $|r_\ell|$  is the Eilenberg-Mac Lane construction applied to the derived global sections of the  $\ell$ -adic realization functor  $r_\ell$ . We prove the following result (Theorem 2.4.9 in the paper).

**Theorem A (Trace formula for dg-categories).** *Let  $\mathcal{B}$  be a monoidal dg-category over  $A$ . For  $T$  a dg-category which is a (left)  $\mathcal{B}$ -module, satisfying both a version of smoothness and properness over  $\mathcal{B}$  and an admissibility property<sup>2</sup> with respect to  $r_\ell$ , a trace formula holds*

$$Ch_\ell([\mathbf{HH}(T/\mathcal{B}; f)]) = tr_{r_\ell(\mathcal{B})}(r_\ell(f)) \tag{1}$$

for any endomorphism  $f : T \rightarrow T$  of  $\mathcal{B}$ -modules.

Here  $[\mathbf{HH}(T/\mathcal{B}; f)]$  is the class in  $\mathbf{HK}_0(\mathbf{HH}(\mathcal{B}/A))$  induced by the endomorphism  $f$ , and  $\mathbf{HK}_0(\mathbf{HH}(\mathcal{B}/A))$  is the 0-th  $K$ -theory of Hochschild homology of  $\mathcal{B}$  over  $A$ . The r.h.s.  $tr_{r_\ell(\mathcal{B})}(r_\ell(f))$  is the trace<sup>3</sup> of the endomorphism  $r_\ell(f)$ , and the trace formula (1) is an equality in  $\pi_0(|r_\ell(\mathbf{HH}(\mathcal{B}/A))|) \simeq H^0(S_{\text{ét}}, r_\ell(\mathbf{HH}(\mathcal{B}/A)))$ . Note that the extra generality of working over  $\mathcal{B}$  (instead of over  $A$ , or over an  $E_\infty$ -algebra over  $A$ ) is actually needed for applications, as shown in Section 5: the  $\ell$ -adic realization of a dg-category is always 2-periodic, i.e. it is a module over  $\mathbb{Q}_\ell(\beta) := \bigoplus_{n \in \mathbb{Z}} \mathbb{Q}_\ell(2n)$ , so it has no chance of being smooth and proper over  $A$  itself, unless it is trivial; in particular, no reasonable trace formula is available over  $A$ .

Our second main result (Section 3) is a *Künneth type formula for inertia-invariant vanishing cycles*. Here we push forward the investigation begun in [BRTV] about the relation between vanishing cycles and the  $\ell$ -adic realization of the dg-category of singularities. For  $X$  and  $Y$  regular schemes, endowed with a flat, proper and generically smooth map to the strictly henselian excellent trait  $S = \text{Spec } A$ , we define an *inertia-invariant convolution product*  $(E \otimes F)^I$ , for  $E$  an  $\ell$ -adic sheaf on the special fiber  $X_s$ , and  $F$  an  $\ell$ -adic sheaf on the special fiber  $Y_s$  ( $I := \text{Gal}(\bar{K}/K)$  denoting the inertia group in this situation). If we denote by  $\nu_X$  (respectively,  $\nu_Y$ ) the complex of vanishing cycles for  $X/S$  (respectively,  $Y/S$ ), we prove the following result (Theorem 3.4.2 in the paper).

**Theorem B (Künneth formula for inertia-invariant vanishing cycles).** *We have equivalences*

$$(\nu_X \otimes \nu_Y)^I \otimes \mathbb{Q}_\ell(\beta) \simeq r_\ell(\text{Sing}(X \times_S Y)) \simeq \text{Cofib}(\eta_{X \times_S Y} : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_{X \times_S Y} \otimes \mathbb{Q}_\ell(\beta)).$$

where  $\text{Sing}(X \times_S Y) = \text{Coh}^b(X \times_S Y)/\text{Perf}(X \times_S Y)$  is the dg-category of singularities of the fiber product  $X \times_S Y$ , and  $\eta_{X \times_S Y}$  is the (2-periodized)  $\ell$ -adic fundamental class.

This is our Künneth formula for inertia-invariant vanishing cycles, and it seems to be a new result in the theory of vanishing cycles, especially in the mixed characteristic case. Note that it might also be viewed as a Thom-Sebastiani formula for inertia-invariant vanishing cycles but we would like to stress that it is not a consequence of the usual Thom-Sebastiani formula for vanishing cycles (see [II2]), and that it holds only if inertia invariants are taken into account in defining the convolution  $(E \otimes F)^I$ <sup>4</sup>. We

<sup>2</sup>The  $\ell$ -adic realization functor is only lax-monoidal, and we say that  $T$  is  $r_\ell$ -admissible over  $\mathcal{B}$  if  $r_\ell$  behaves like a symmetric monoidal functor for  $T$  over  $\mathcal{B}$  (see Definition 2.4.8).

<sup>3</sup>Or rather its image under the canonical map  $\alpha : H^0(S_{\text{ét}}, \mathbf{HH}(r_\ell(\mathcal{B})/r_\ell(A))) \rightarrow H^0(S_{\text{ét}}, r_\ell(\mathbf{HH}(\mathcal{B}/A)))$ .

<sup>4</sup>In particular, only  $(E \otimes F)^I$ , and not  $(E \otimes F)$ , makes sense in our context.

also discuss appropriate conditions ensuring that  $(\nu_X \otimes \nu_Y)^I$  is equivalent to  $(\nu_X \boxtimes \nu_Y)^I$  (Corollary 3.4.5).

The third main result of this paper is a *Künneth type formula for dg-categories of singularities* (Section 4). For  $X$  and  $Y$  regular schemes, endowed with a flat, proper and generically smooth map to the strictly henselian excellent trait  $S = \text{Spec } A$ , we may consider the dg-categories of singularities  $\text{Sing}(X_s) := \text{Coh}^b(X_s)/\text{Perf}(X_s)$  and  $\text{Sing}(Y_s) := \text{Coh}^b(Y_s)/\text{Perf}(Y_s)$  of the corresponding special fibers. Consider  $\mathcal{B} := \text{Sing}(s \times_S s)$  ( $s$  being the closed point in  $S$ , and  $s \times_S s$  being the *derived* fiber product); then  $\mathcal{B}$  is a monoidal dg-category for the convolution product coming from the *derived groupoid* structure of  $s \times_S s$ . Moreover,  $\mathcal{B}$  acts on both  $\text{Sing}(X_s)$  and  $\text{Sing}(Y_s)$ , in such a way that the tensor product  $\text{Sing}(X_s)^\circ \otimes_{\mathcal{B}} \text{Sing}(Y_s)$  makes sense as a dg-category over  $A$  (Proposition 4.1.7). Our Künneth formula for dg-categories of singularities is then the following result (Theorem 4.2.1 in the paper).

**Theorem C (Künneth formula for dg-categories of singularities).** *There is a canonical equivalence*

$$\text{Sing}(X_s)^\circ \otimes_{\mathcal{B}} \text{Sing}(Y_s) \simeq \text{Sing}(X \times_S Y)$$

*as dg-categories over  $A$ .*

Note that this result is peculiar to singularity categories: it is false if we replace  $\text{Sing}$  by  $\text{Coh}^b$ . Also notice that  $\mathcal{B}$  is defined as the convolution dg-category of a derived groupoid, and is an object in derived algebraic geometry that *cannot* be described within classical algebraic geometry. Finally, we prove that  $\text{Sing}(X_s)$  is smooth and proper (i.e. saturated) over  $\mathcal{B}$ , for any  $X$  regular scheme, endowed with a flat, proper and generically smooth map to  $S$ . This fact is well-known in characteristic zero (see, for instance, [Pr]) but is, in our opinion, a deep and surprising result in our general setting that includes positive equi-characteristic or mixed characteristic cases.

Smoothness and properness over  $\mathcal{B}$  is one half of the properties needed to apply the trace formula of Theorem A to  $\text{Sing}(X_s)$  over  $\mathcal{B}$ . Note that, without further hypothesis, it is however not true that  $\text{Sing}(X_s)$  is also admissible (with respect to  $r_\ell$ ).

Our fourth and final main result is a *categorical version of Bloch's Conductor formula for unipotent monodromy* (section 5) that we now describe.

In his seminal 1987 paper [Bl], S. Bloch introduced what is now called Bloch's intersection number  $[\Delta_X, \Delta_X]_S$ , for a flat, proper map of schemes  $X \rightarrow S$  where  $X$  is regular, and  $S$  is a henselian trait  $S$ . This number can be defined as the degree of the localized top Chern class of the coherent sheaf  $\Omega_{X/S}^1$  and measures the "relative" singularities of  $X$  over  $S$ . In the same paper, Bloch introduced his famous *conductor formula*, which can be seen as a conjectural computation of the Bloch's intersection number in terms of the arithmetic geometry of  $X/S$ . It reads as follows.

**Bloch's Conductor Conjecture (BCC)**

*We have an equality*

$$[\Delta_X, \Delta_X]_S = \chi(X_{\bar{k}}) - \chi(X_{\bar{K}}) - Sw(X_{\bar{K}}),$$

*where  $X_{\bar{k}}$  and  $X_{\bar{K}}$  denotes the special and generic geometric fibers of  $X$  over  $S$ ,  $\chi(-)$  denotes  $\mathbb{Q}_\ell$ -adic Euler characteristic, and  $Sw(-)$  is the Swan conductor.*

In [Bl] the above formula is proven in relative dimension 1. Further results implying special cases of BCC have been obtained since then by Kato-Saito [Ka-Sa] and others, the most recent one being a full proof in the geometric case by T. Saito [Sai] (see Section §5 for a more detailed discussion about the status of the art about BCC). In the mixed characteristic case, the conjecture is open in general outside the cases covered in [Ka-Sa]. In particular, for isolated singularities the above conjecture already appeared in Deligne's exposé [SGA7-I, Exp. XVI], and remains open.

In Section 5 we propose a first step towards a new understanding of Bloch's conductor formula using the results developed in the previous sections. We start by defining an analog of Bloch's intersection number, that we call the *categorical Bloch's intersection class* (Definition 5.2.1), and denote it by  $[\Delta_X, \Delta_X]_S^{\text{cat}}$ . It is an element in  $H^0(S_{\text{ét}}, r_\ell(\text{HH}(\mathcal{B}/A)))$ , where  $\mathcal{B} = \text{Sing}(s \times_S s)$  (as in Section 4, see above), and it is basically defined as an intersection class in the setting of non-commutative algebraic geometry. The precise comparison with the original Bloch's number is not covered in this work and will appear in a forthcoming paper.

It is easy to see that there are canonical inclusions  $\mathbb{Z} \hookrightarrow \mathbb{Q}_\ell \hookrightarrow H^0(S_{\text{ét}}, \text{HH}(r_\ell(\mathcal{B})/r_\ell(A)))$ , and for any  $\lambda \in \mathbb{Z}$ , we will write  $\lambda^\wedge$  its image under the canonical map  $\alpha : H^0(S_{\text{ét}}, \text{HH}(r_\ell(\mathcal{B})/r_\ell(A))) \rightarrow H^0(S_{\text{ét}}, r_\ell(\text{HH}(\mathcal{B}/A)))$ .

Our fourth main result is then the following theorem (Theorem 5.2.2 in the paper).

**Theorem D (Categorical BCC for unipotent monodromy)** *With the notations of BCC, if the monodromy action of the inertia on  $H^*(X_{\bar{K}}, \mathbb{Q}_\ell)$  is unipotent, then we have an equality*

$$[\Delta_X, \Delta_X]_S^{\text{cat}} = \chi(X_{\bar{k}})^\wedge - \chi(X_{\bar{K}})^\wedge.$$

Since unipotent monodromy action implies tame action, the Swan conductor vanishes for unipotent monodromy, so that Theorem D is completely analogous to BCC under this hypothesis.

We believe the main ideas in our proof of Theorem D are new in the subject, and might be also useful to answer other related questions in algebraic and arithmetic geometry. The key point in the proof of Theorem D is that, once Theorem B and C are established, it is a *direct consequence* of the trace formula for dg-categories (Theorem A). In fact, by the main result of [BRTV], the  $\ell$ -adic realization of the dg-category of singularities  $\text{Sing}(X_s)$  of the special fiber is the inertia invariant part of vanishing cohomology of  $X/S$  (suitably 2-periodized). Moreover, our Künneth formulas for inertia-invariant vanishing cycles (Theorem B above, and see Section 3 of the main text), and for dg-categories of singularities (Theorem C above, and see Section 4 of the main text), imply that  $\text{Sing}(X_s)$  is *smooth* and *proper* over  $\mathcal{B}$ . Finally, again our Künneth formulas *together with the hypothesis of unipotent monodromy* show that  $\text{Sing}(X_s)$  is also  *$r_\ell$ -admissible*, so that we are indeed in the position of applying our trace formula (Theorem A) to the identity endomorphism of  $\text{Sing}(X_s)$  over  $\mathcal{B}$ : Theorem D is then an immediate corollary of this trace formula.

We conclude this introduction by remarking that the hypothesis of unipotent inertia action in Theorem D is, of course, a bit restrictive: unipotent monodromy implies tame monodromy and our theorem does not deal with the interesting arithmetic aspects encoded by the Swan conductor. However we are convinced that we can go beyond the unipotent case, and prove some new cases of BCC conjecture (possibly the whole unrestricted conjecture), by considering  $\text{Sing}(X \times_S S')$  as a module over  $\mathcal{B}' := \text{Sing}(S' \times_S S')$ , where  $S' = \text{Spec } A' \rightarrow S$  is a totally ramified Galois extension of strictly henselian traits such that the inertia for  $S'$  acts unipotently on  $H^*((X \times_S S')_{\bar{K}'}, \mathbb{Q}_\ell)$ . Here  $K'$  is the fraction field

of  $A'$ , and the existence of such an extension  $S' \rightarrow S$  is guaranteed by Grothendieck local monodromy theorem ([SGA7-I, Exp. I, Théorème 1.2]). This strategy for BCC, as well as the comparison between the categorical and the classical Bloch numbers, will be investigated in a future work.

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## Notations.

- Throughout the text,  $A$  will denote a (discrete) commutative noetherian ring. When needed,  $A$  will be required to satisfy further properties (such as being excellent, local and henselian) that will be made precise in due course.
- $L(A)$  will denote the  $A$ -linear dg-category of (fibrant and) cofibrant  $A$ -dg-modules localized with respect to quasi-isomorphisms.
- $\mathbf{dgCat}_A$  will denote the Morita  $\infty$ -category of dg-categories over  $A$  (see §2.1).
- **Top** denotes the  $\infty$ -category of spaces (obtained, e.g. as the coherent nerve of the Dwyer-Kan localization of the category of simplicial sets along weak homotopy equivalences). **Sp** denotes the  $\infty$ -category of spectra.
- If  $R$  is a an associative and unital monoid in a symmetric monoidal  $\infty$ -category  $\mathcal{C}$ , we will write  $\mathbf{Mod}(R)$  or  $\mathbf{Mod}_{\mathcal{C}}(R)$  for the  $\infty$ -category of left  $R$ -modules in  $\mathcal{C}$  ([Lu-HA]).

## 2 A non-commutative trace formalism

### 2.1 $\infty$ -Categories of dg-categories

We denote by  $A$  a commutative ring. We recall here some basic facts about the  $\infty$ -category of dg-categories, its monoidal structure and its theory of monoids and modules.

We consider the category  $dgCat_A$  of small  $A$ -linear dg-categories and  $A$ -linear dg-functors. We recall that an  $A$ -linear dg-functor  $T \rightarrow T'$  is a Morita equivalence if the induced functor of the corresponding derived categories of dg-modules  $f^* : D(T') \rightarrow D(T)$  is an equivalence of categories (see [To1] for details). The  $\infty$ -category of dg-categories over  $S$  is defined to be the localisation of  $dgCat_A$  along these Morita equivalences, and will be denoted by  $\mathbf{dgCat}_S$  or  $\mathbf{dgCat}_A$ . Being the  $\infty$ -category associated to a combinatorial model category,  $\mathbf{dgCat}_A$  is a presentable  $\infty$ -category ([Lu-HTT, A.3.7.6]). As in [To1, § 4], the tensor product of  $A$ -linear dg-categories can be derived to a symmetric monoidal structure

on the  $\infty$ -category  $\mathbf{dgCat}_A$ . This symmetric monoidal structure moreover distributes over colimits making  $\mathbf{dgCat}_A$  into a presentable symmetric monoidal  $\infty$ -category. We have a notion of rigid, or, equivalently, dualizable, object in  $\mathbf{dgCat}_A$ . It is a well known fact that dualizable objects in  $\mathbf{dgCat}_A$  are precisely smooth and proper dg-categories over  $A$  (see [To2, Prop. 2.5]).

The compact objects in  $\mathbf{dgCat}_A$  are the dg-categories of finite type over  $A$  in the sense of [To-Va]. We denote their full sub- $\infty$ -category by  $\mathbf{dgCat}_A^{ft} \subset \mathbf{dgCat}_A$ . The full sub-category  $\mathbf{dgCat}_S^{ft}$  is preserved by the monoidal structure, and moreover any dg-category is a filtered colimit of dg-categories of finite type. We thus have a natural equivalence of symmetric monoidal  $\infty$ -categories

$$\mathbf{dgCat}_A \simeq \mathbf{Ind}(\mathbf{dgCat}_S^{ft}).$$

We will from time to time have to work in a bigger  $\infty$ -category, denoted by  $\mathbf{dgCAT}_A$ , that contains  $\mathbf{dgCat}_A$  as a non-full sub- $\infty$ -category. By [To2], we have a symmetric monoidal  $\infty$ -category  $\mathbf{dgCat}_A^{lp}$  of *presentable* dg-categories over  $A$  with morphisms given by colimit-preserving functors. We define  $\mathbf{dgCAT}_A$  the full sub- $\infty$ -category of  $\mathbf{dgCat}_A^{lp}$  consisting of all compactly generated dg-categories. The  $\infty$ -category  $\mathbf{dgCat}_A$  can be identified with the non-full sub- $\infty$ -category of  $\mathbf{dgCAT}_A$  which consists of compact objects preserving dg-functors. This provides a faithful embedding<sup>5</sup> of symmetric monoidal  $\infty$ -categories

$$\mathbf{dgCat}_A \hookrightarrow \mathbf{dgCAT}_A.$$

At the level of objects, this embedding sends a small dg-category  $T$  to the compactly generated dg-category  $\hat{T}$  of dg-modules over  $T^o$ . An equivalent description of  $\mathbf{dgCAT}_A$  is as the  $\infty$ -category of small dg-categories together with the mapping spaces given by the classifying space of *all* bi-dg-modules between small dg-categories. Objects and morphisms in  $\mathbf{dgCAT}_A$  (respectively, in  $\mathbf{dgCat}_A$ ) will be called *big* (respectively, *small*).

**Definition 2.1.1** *A monoidal  $A$ -dg-category is a unital and associative monoid in the symmetric monoidal  $\infty$ -category  $\mathbf{dgCat}_A$ . A module over a monoidal  $A$ -dg-category  $B$  will, by definition, mean a left  $B$ -module in  $\mathbf{dgCat}_A$  in the sense of [Lu-HA], and the  $\infty$ -category of (left)  $B$ -modules will be denoted by  $\mathbf{dgCat}_B$ .*

For a  $B$ -module  $T$ , we have a morphism  $\mu : B \otimes_A T \rightarrow T$  in  $\mathbf{dgCat}_A$ , that will be simply denoted by  $(b, x) \mapsto b \otimes x$ . For a monoidal  $A$ -dg-category  $B$ , we will denote by  $B^{\otimes\text{-op}}$  the monoidal  $A$ -dg-category where the monoid structure is the opposite to the one of  $B$ , i.e.  $b \otimes^{\text{op}} b' := b' \otimes b$ . Note that  $B^{\otimes\text{-op}}$  should not be confused with  $B^o$  (which is still a monoidal  $A$ -dg-category), where the “arrows” and not the monoid structure have been reversed, i.e.  $B^o(b, b') := B(b', b)$ . By definition, a *right*  $B$ -module is a (left)  $B^{\otimes\text{-op}}$ -module. The  $\infty$ -category of right  $B$ -modules will be denoted by  $\mathbf{dgCat}_{B^{\otimes\text{-op}}}$ , or simply by  $\mathbf{dgCat}^B$ . If  $B$  is a monoidal  $A$ -dg-category, then  $B^{\otimes\text{-op}} \otimes_A B$  is again a monoidal  $A$ -dg-category, and  $B$  can be considered either as a left  $B^{\otimes\text{-op}} \otimes_A B$  (denoted by  $B^L$ ), or as a right  $B^{\otimes\text{-op}} \otimes_A B$ -module (denoted by  $B^R$ ). For  $T$  a  $B$ -module, and  $T'$  a right  $B$ -module, then  $T' \otimes_A T$  is naturally a right  $B^{\otimes\text{-op}} \otimes_A B$ -module, and we define

$$T' \otimes_B T := (T' \otimes_A T) \otimes_{B^{\otimes\text{-op}} \otimes_A B} B^L$$

which is an object in  $\mathbf{dgCat}_A$ .

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<sup>5</sup>I.e. a functor which on mapping spaces is injective on  $\pi_0$  and an isomorphism on all the  $\pi_i$ 's.

Let  $B$  be a monoidal  $A$ -dg-category. We can consider the symmetric monoidal embedding  $\mathbf{dgCat}_A \hookrightarrow \mathbf{dgCAT}_A$ , so that the image  $\widehat{B}$  of  $B$  is a monoid in  $\mathbf{dgCAT}_A$ . The  $\infty$ -category of  $\widehat{B}$ -modules in  $\mathbf{dgCAT}_A$  is denoted by  $\mathbf{dgCAT}_B$ , and its objects are called *big  $B$ -modules*. The natural  $\infty$ -functor  $\mathbf{dgCat}_B \rightarrow \mathbf{dgCAT}_B$  is faithful and its image consists of all big  $B$ -modules  $\widehat{T}$  such that the morphism  $\widehat{B} \widehat{\otimes}_A \widehat{T} \rightarrow \widehat{T}$  is a small morphism.

It is known (see [To2]) that the symmetric monoidal  $\infty$ -category  $\mathbf{dgCAT}_A$  is rigid, that for any  $\widehat{T}$  its dual is given by  $\widehat{T}^\circ$ , and that the evaluation and coevaluation morphisms are defined by  $T$  considered as  $T^\circ \otimes_A T$ -module. This formally implies that if  $\widehat{T}$  is a big  $B$ -module, then its dual  $\widehat{T}^\circ$  is naturally a *right big  $B$ -module*. We thus have two big (i.e. morphisms in  $\mathbf{DGCAT}$ ) morphisms

$$\mu : \widehat{B} \widehat{\otimes}_A \widehat{T} \rightarrow \widehat{T}, \quad \mu^\circ : \widehat{T}^\circ \widehat{\otimes}_A \widehat{B} \rightarrow \widehat{T}^\circ.$$

$\mu$  has a right adjoint  $\mu^*$  (the restriction at the level of modules) which is presentable but not in general continuous. In fact,  $\mu^*$  is continuous iff  $\mu$  is a small morphism. When this is the case these morphisms also provide a third big morphism

$$h : \widehat{T}^\circ \widehat{\otimes}_A \widehat{T} \rightarrow \widehat{B}$$

obtained by duality from

$$\mu^* : \widehat{T} \rightarrow \widehat{B} \widehat{\otimes}_A \widehat{T}$$

We now make the following definitions.

**Definition 2.1.2** *Let  $B$  be a monoidal dg-category, and  $T$  a  $B$ -module. We say that:*

1.  $T$  is *cotensored (over  $B$ )* if the big morphism  $\mu^\circ$  defined above is a small morphism (i.e. a morphism in  $\mathbf{dgCat}_A$ );
2.  $T$  is *proper or enriched (over  $B$ )* if both the big morphisms  $\mu$  and  $h$  defined above are in fact small morphisms (i.e. morphisms in  $\mathbf{dgCat}_A$ ).

We can make the above definition more explicit as follows. Let  $B$  and  $T$  be as above; for two objects  $b \in B$  and  $x \in T$ , we can consider the dg-functor

$$x^b : T^\circ \rightarrow L(A)$$

sending  $y \in T$  to  $T(b \otimes y, x)$  where  $(b, x) \mapsto b \otimes x : B \otimes_A T \rightarrow T$  is the  $B$ -module structure on  $T$ . Then,  $T$  is cotensored over  $B$  if and only if for all  $b$  and  $x$  the above dg-module  $x^b : T^\circ \rightarrow L(A)$  is compact in the derived category  $D(T^\circ)$  of all  $T^\circ$ -dg-modules. When  $T$  is furthermore assumed to be triangulated, then this is equivalent to ask for the dg-module to be representable by an object  $x^b \in T$ . In a similar manner, we can phrase properness of  $T$  over  $B$  by saying that, for any  $x \in T$  and  $y \in T$ , the dg-module

$$B^\circ \rightarrow L(A)$$

sending  $b$  to  $T(b \otimes x, y)$  is compact (or representable, if  $B$  is furthermore assumed to be triangulated).

**Remark 2.1.3** It is important to notice that, by definition, when  $T$  is cotensored, then the big right  $B$ -module  $\widehat{T}^\circ$  is in fact a small right  $B$ -module (i.e. it is in the essential image of  $\mathbf{dgCat}_B \rightarrow \mathbf{dgCAT}_B$ ).

Equivalently, when  $T$  is cotensored, then  $T^\circ$  is naturally a right  $B$ -module inside  $\mathbf{dgCat}_A$ , with the right module structure given by the morphism in  $\mathbf{dgCat}_A$

$$\mu^\circ : T^\circ \otimes_A B \longrightarrow T^\circ$$

sending  $(x, b) \in T^\circ \otimes_A B$  to the cotensor  $x^b \in T^\circ$ .

We end this section by recalling some facts about existence of tensor products of modules over monoidal dg-categories. As a general fact, since  $\mathbf{dgCat}_A$  is a presentable symmetric monoidal  $\infty$ -category, for any monoidal dg-category  $B$  there exists a tensor product  $\infty$ -functor

$$\otimes_B : \mathbf{dgCat}_B \times \mathbf{dgCat}^B \longrightarrow \mathbf{dgCat}_A,$$

sending a left  $B$ -module  $T$  and a right  $B$ -module  $T'$  to  $T' \otimes_B T$  (see [Lu-HA]). Now, if  $T$  is a  $B$ -module which is also cotensored (over  $B$ ) in the sense of Definition 2.1.2, we have that  $T^\circ$  is a right  $B$ -module, and we can thus form

$$T^\circ \otimes_B T \in \mathbf{dgCat}_A.$$

When  $T$  is not cotensored, the object  $T \otimes_B T^\circ$  does not exist anymore. However, we can always consider the presentable dg-categories  $\widehat{T}$  and  $\widehat{T}^\circ$  as left and right modules over  $\widehat{B}$ , respectively, and their tensor product  $\widehat{T}^\circ \widehat{\otimes}_{\widehat{B}} \widehat{T}$  now only makes sense as a presentable dg-category which has no reason to be compactly generated, in general. Of course, when  $T$  is cotensored, this presentable dg-category is compactly generated and we have

$$\widehat{T^\circ \otimes_B T} \simeq \widehat{T}^\circ \widehat{\otimes}_{\widehat{B}} \widehat{T}.$$

**Remark 2.1.4** The following, easy observation, will be useful in the sequel. Let  $B$  be a monoidal dg-category, and assume that  $B$  is generated, as a triangulated dg-category, by its unit object  $1 \in B$ . Then, any big  $B$ -module is small (i.e. in the image of  $\mathbf{dgCat}_B \longrightarrow \mathbf{dgCat}_B$ ), and also cotensored.

## 2.2 The $\ell$ -adic realization of dg-categories

We denote by  $\mathcal{SH}_S$  the stable  $\mathbb{A}^1$ -homotopy  $\infty$ -category of schemes over  $S$  (see [Vo, Def. 5.7] and [Ro, § 2]). It is a presentable symmetric monoidal  $\infty$ -category whose monoidal structure will be denoted by  $\wedge_S$ . Homotopy invariant algebraic K-theory defines an  $\mathbb{E}_\infty$ -ring object in  $\mathcal{SH}_S$  that we denote by  $\mathbf{BU}_S$  (a more standard notation is  $KGL_S$ ). We denote by  $\mathbf{Mod}_{\mathcal{SH}_S}(\mathbf{BU}_S)$  the  $\infty$ -category of  $\mathbf{BU}_S$ -modules in  $\mathcal{SH}_S$ . It is a presentable symmetric monoidal  $\infty$ -category whose monoidal structure will be denoted by  $\wedge_{\mathbf{BU}_S}$ .

As proved in [BRTV], there exists a lax symmetric monoidal  $\infty$ -functor

$$\mathbf{M}^- : \mathbf{dgCat}_S \longrightarrow \mathbf{Mod}_{\mathcal{SH}_S}(\mathbf{BU}_S),$$

which will be denoted by  $T \mapsto \mathbf{M}^T$  (while it is denoted by  $T \mapsto \mathcal{M}_S^\vee(T)$  in [BRTV]). The precise construction of the  $\infty$ -functor  $\mathbf{M}^-$  is rather involved and uses in an essential manner the theory of non-commutative motives of [Ro] as well as the comparison with the stable homotopy theory of schemes. Intuitively, the  $\infty$ -functor  $\mathbf{M}^-$  sends a dg-category  $T$  to the homotopy invariant K-theory functor  $S' \mapsto \mathbf{HK}(S' \otimes_S T)$ . To be more precise, there is an obvious forgetful  $\infty$ -functor

$$\mathbf{U} : \mathbf{Mod}_{\mathcal{SH}_S}(\mathbf{BU}_S) \longrightarrow \mathbf{Fun}^\infty(Sm_S^{op}, \mathbf{Sp}),$$

to the  $\infty$ -category of presheaves of spectra on the category  $Sm_S$  of smooth  $S$ -schemes. For a given dg-category  $T$  over  $S$ , the presheaf  $\mathbf{U}(M^T)$  is defined by sending a smooth  $S$ -scheme  $S' = \mathbf{Spec} A' \rightarrow \mathbf{Spec} A = S$  to  $\mathbf{HK}(A' \otimes_A T)$ , the homotopy invariant non-connective K-theory spectrum of  $A' \otimes_A T$  (see [Ro, 4.2.3]).

The  $\infty$ -functor  $M^-$  satisfies some basic properties which we recall here.

1. The  $\infty$ -functor  $M^-$  is a localizing invariant, i.e. for any short exact sequence  $T_0 \hookrightarrow T \rightarrow T/T_0$  of dg-categories over  $A$ , the induced sequence

$$M^{T_0} \longrightarrow M^T \longrightarrow M^{T/T_0}$$

exhibits  $M^{T_0}$  as the fiber of the morphism  $M^T \rightarrow M^{T/T_0}$  in  $\mathbf{Mod}(\mathbf{BU}_S)$ .

2. The natural morphism  $\mathbf{BU}_S \rightarrow M^A$ , induced by the lax monoidal structure of  $M^-$ , is an equivalence of  $\mathbf{BU}_S$ -modules.
3. The  $\infty$ -functor  $T \mapsto M^T$  commutes with filtered colimits.
4. For any quasi-compact and quasi-separated scheme  $X$ , and any morphism  $p : X \rightarrow S$ , we have a natural equivalence of  $\mathbf{BU}_S$ -modules

$$M^{\mathrm{Perf}(X)} \simeq p_*(\mathbf{BU}_X),$$

where  $p_* : \mathbf{Mod}_{\mathcal{SH}_X}(\mathbf{BU}_X) \rightarrow \mathbf{Mod}_{\mathcal{SH}_S}(\mathbf{BU}_S)$  is the direct image of  $\mathbf{BU}$ -modules, and  $\mathrm{Perf}(X)$  is the dg-category of perfect complexes on  $X$ .

We now let  $\ell$  be a prime number invertible in  $A$ . We denote by  $Sh_{\mathbb{Q}_\ell}^{\mathrm{ct}}(S)$  the  $\infty$ -category of *constructible*  $\mathbb{Q}_\ell$ -adic complexes on the étale site  $S_{\mathrm{ét}}$  of  $S$ . It is a symmetric monoidal  $\infty$ -category, and we denote by

$$Sh_{\mathbb{Q}_\ell}(S) := \mathbf{Ind}(Sh_{\mathbb{Q}_\ell}^{\mathrm{ct}}(S))$$

its completion under filtered colimits (see [Ga-Lu, Def. 4.3.26]). According to [Ro, Cor. 2.3.9], there exists an  $\ell$ -adic realization  $\infty$ -functor

$$R_\ell : \mathcal{SH}_S \rightarrow Sh_{\mathbb{Q}_\ell}(S).$$

By construction,  $R_\ell$  is a symmetric monoidal  $\infty$ -functor sending a smooth scheme  $p : X \rightarrow S$  to  $p_! p^1(\mathbb{Q}_\ell)$ , or, in other words, to the relative  $\ell$ -adic homology of  $X$  over  $S$ .

We let  $\mathbb{T} := \mathbb{Q}_\ell[2](1)$ , and we consider the  $\mathbb{E}_\infty$ -ring object in  $Sh_{\mathbb{Q}_\ell}(S)$

$$\mathbb{Q}_\ell(\beta) := \bigoplus_{n \in \mathbb{Z}} \mathbb{T}^{\otimes n}.$$

In this notation,  $\beta$  stands for  $\mathbb{T}$ , and  $\mathbb{Q}_\ell(\beta)$  for the algebra of Laurent polynomials in  $\beta$ , so we might as well write

$$\mathbb{Q}_\ell(\beta) = \mathbb{Q}_\ell[\beta, \beta^{-1}].$$

As shown in [BRTV], there exists a canonical equivalence  $R_\ell(\mathbf{BU}_S) \simeq \mathbb{Q}_\ell(\beta)$  of  $\mathbb{E}_\infty$ -ring objects in  $Sh_{\mathbb{Q}_\ell}(S)$ , that is induced by the Chern character from algebraic K-theory to étale cohomology. We thus obtain a well-defined symmetric monoidal  $\infty$ -functor

$$R_\ell : \mathbf{Mod}_{\mathcal{SH}_S}(\mathbf{BU}_S) \rightarrow \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(S)}(\mathbb{Q}_\ell(\beta)),$$

from  $\mathbf{BU}_S$ -modules in  $\mathcal{SH}_S$  to  $\mathbb{Q}_\ell(\beta)$ -modules in  $Sh_{\mathbb{Q}_\ell}(S)$ . By pre-composing with the functor  $T \mapsto \mathbf{M}^T$ , we obtain a lax monoidal  $\infty$ -functor

$$r_\ell := \mathbf{R}_\ell \circ \mathbf{M}^- : \mathbf{dgCat}_S \longrightarrow \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(S)}(\mathbb{Q}_\ell(\beta)).$$

**Definition 2.2.1** *The  $\infty$ -functor defined above*

$$r_\ell : \mathbf{dgCat}_S \longrightarrow \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(S)}(\mathbb{Q}_\ell(\beta))$$

is called the  $\ell$ -adic realization functor for dg-categories over  $S$ .

From the standard properties of the functor  $T \mapsto \mathbf{M}^T$ , recalled above, we obtain the following properties for the  $\ell$ -adic realization functor  $T \mapsto r_\ell(T)$ .

1. The  $\infty$ -functor  $r_\ell$  is a *localizing invariant*, i.e. for any short exact sequence  $T_0 \hookrightarrow T \twoheadrightarrow T/T_0$  of dg-categories over  $A$ , the induced sequence

$$r_\ell(T_0) \longrightarrow r_\ell(T) \longrightarrow r_\ell(T/T_0)$$

is a fibration sequence in  $\mathbf{Mod}(\mathbb{Q}_\ell(\beta))$ .

2. The natural morphism

$$\mathbb{Q}_\ell(\beta) \longrightarrow r_\ell(A),$$

induced by the lax monoidal structure, is an equivalence in  $\mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(S)}(\mathbb{Q}_\ell(\beta))$ .

3. The  $\infty$ -functor  $r_\ell$  commutes with filtered colimits.

4. For any separated morphism of finite type  $p : X \longrightarrow S$ , we have a natural morphism of  $\mathbb{Q}_\ell(\beta)$ -modules

$$r_\ell(\mathbf{Perf}(X)) \longrightarrow p_*(\mathbb{Q}_\ell(\beta)),$$

where  $p_* : \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(S)}(\mathbb{Q}_{\ell,X}(\beta)) \longrightarrow \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(S)}(\mathbb{Q}_\ell(\beta))$  is induced by the direct image  $Sh_{\mathbb{Q}_\ell}^{\text{ct}}(X) \longrightarrow Sh_{\mathbb{Q}_\ell}^{\text{ct}}(S)$  of constructible  $\mathbb{Q}_\ell$ -complexes. If either  $p$  is proper, or  $A$  is a field, this morphism is an equivalence.

**Remark 2.2.2** The  $\ell$ -adic realization functor of Definition 2.2.1 can be easily “sheafified”, as follows. For  $X$  a separated, excellent scheme, and  $T$  a sheaf of dg-categories on  $X_{\text{Zar}}$  (i.e.  $T \in \mathbf{dgCat}_X := \lim \mathbf{dgCat}_{\mathcal{U}^\bullet}$ , where  $\mathcal{U}^\bullet$  is the Čech-nerve of any Zariski open cover  $\mathcal{U}$  of  $X$ ), by taking the limit along the Čech-nerve of a Zariski open cover of  $X$ , we define  $\mathbf{M}_X^- : \mathbf{dgCat}_X \rightarrow \mathbf{Mod}_{\mathcal{SH}_X}(\mathbf{BU}_X)$  and  $\mathbf{R}_{\ell,X} : \mathbf{Mod}_{\mathcal{SH}_X}(\mathbf{BU}_X) \longrightarrow \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(X)}(\mathbb{Q}_{\ell,X}(\beta))$ . The composite

$$r_{\ell,X} := \mathbf{R}_{\ell,X} \circ \mathbf{M}_X^- : \mathbf{dgCat}_X \longrightarrow \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(X)}(\mathbb{Q}_{\ell,X}(\beta))$$

is called the  $\ell$ -adic realization functor for dg-categories over  $X$ . Obviously we have  $r_\ell = r_{\ell,S}$ , for  $r_\ell$  as in Definition 2.2.1. Moreover, for  $X$  as above, any separated morphism of finite type  $p : X \longrightarrow S$  induces a commutative diagram<sup>6</sup>

$$\begin{array}{ccc} \mathbf{dgCat}_X & \xrightarrow{r_{\ell,X}} & \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(X)}(\mathbb{Q}_{\ell,X}(\beta)) \\ p_* \downarrow & & \downarrow p_* \\ \mathbf{dgCat}_S & \xrightarrow{r_\ell} & \mathbf{Mod}_{Sh_{\mathbb{Q}_\ell}(S)}(\mathbb{Q}_\ell(\beta)) \end{array}$$

<sup>6</sup>Note that the functor  $p_* : \mathbf{dgCat}_X \rightarrow \mathbf{dgCat}_S$  consists simply in viewing a dg-category over  $X$  as an  $A$ -dg-category via the morphism  $p$ .

## 2.3 The $\ell$ -adic Chern character

As explained in [BRTV], there is a symmetric monoidal  $\infty$ -category  $\mathcal{SH}_A^{nc}$  of non-commutative motives over  $A$ . As an  $\infty$ -category it is the full sub- $\infty$ -category of  $\infty$ -functors of (co)presheaves of spectra

$$\mathbf{dgCat}_A^{ft} \longrightarrow \mathbf{Sp},$$

satisfying Nisnevich descent and  $\mathbb{A}^1$ -homotopy invariance.

We consider  $\Gamma : Sh_{\mathbb{Q}_\ell}(S) \longrightarrow \mathbf{dg}_{\mathbb{Q}_\ell}$ , the global section  $\infty$ -functor, taking an  $\ell$ -adic complex on  $S_{\acute{e}t}$  to its hyper-cohomology. Composing this with the Dold-Kan construction  $\mathbb{R}\mathbf{Map}_{\mathbf{dg}_{\mathbb{Q}_\ell}}(\mathbb{Q}_\ell, -) : \mathbf{dg}_{\mathbb{Q}_\ell} \longrightarrow \mathbf{Sp}$  we obtain an  $\infty$ -functor

$$|-| : Sh_{\mathbb{Q}_\ell}(S) \longrightarrow \mathbf{Sp},$$

which computes hyper-cohomology of  $S_{\acute{e}t}$  with  $\ell$ -adic coefficients, i.e. for any  $E \in Sh_{\mathbb{Q}_\ell}(S)$ , we have natural isomorphisms

$$H^i(S_{\acute{e}t}, E) \simeq \pi_{-i}(|E|), i \in \mathbb{Z}.$$

By what we have seen in our last paragraph, the composite functor  $T \mapsto |r_\ell(T)|$  provides a (co)presheaf of spectra

$$\mathbf{dgCat}_A^{ft} \longrightarrow \mathbf{Sp},$$

satisfying Nisnevich descent and  $\mathbb{A}^1$ -homotopy invariance. It thus defines an object  $|r_\ell| \in \mathcal{SH}_A^{nc}$ . The fact that  $r_\ell$  is lax symmetric monoidal implies moreover that  $|r_\ell|$  is endowed with a natural structure of a  $\mathbb{E}_\infty$ -ring object in  $\mathcal{SH}_A^{nc}$ .

Each  $T \in \mathbf{dgCat}_A^{ft}$  defines a corepresentable object  $h^T \in \mathcal{SH}_A^{nc}$ , characterized by the ( $\infty$ -)functorial equivalence

$$\mathbb{R}\mathbf{Map}_{\mathcal{SH}_A^{nc}}(h^T, F) \simeq F(T),$$

for any  $F \in \mathcal{SH}_A^{nc}$ . The existence of  $h^T$  is a formal statement, however the main theorem of [Ro] implies that we have a natural equivalence of spectra

$$\mathbb{R}\mathbf{Map}_{\mathcal{SH}_A^{nc}}(h^T, h^A) \simeq \mathbf{HK}(T),$$

where  $\mathbf{HK}(T)$  stands for non-connective homotopy invariant algebraic  $K$ -theory of the dg-category  $T$ . In other words,  $T \mapsto \mathbf{HK}(T)$  defines an object in  $\mathcal{SH}_A^{nc}$  which is isomorphic to  $h^B$ . By Yoneda lemma, we thus obtain an equivalence of spaces

$$\mathbb{R}\mathbf{Map}^{lax-\otimes}(\mathbf{HK}, |r_\ell|) \simeq \mathbb{R}\mathbf{Map}_{\mathbb{E}_\infty\text{-Sp}}(\mathbb{S}, |r_\ell(A)|) \simeq *.$$

In other words, there exists a unique (up to a contractible space of choices) lax symmetric monoidal natural transformation

$$\mathbf{HK} \longrightarrow |r_\ell|,$$

between lax monoidal  $\infty$ -functors from  $\mathbf{dgCat}_A^{ft}$  to  $\mathbf{Sp}$ . We extend this to all dg-categories over  $A$ , as usual, by passing to Ind-completion  $\mathbf{dgCat}_A \simeq \mathbf{Ind}(\mathbf{dgCat}_A^{ft})$ .

**Definition 2.3.1** *The natural transformation defined above is called the  $\ell$ -adic Chern character. It is denoted by*

$$Ch_\ell : \mathbf{HK}(-) \longrightarrow |r_\ell(-)|.$$

**Remark 2.3.2** Definition 2.3.1 contains a built-in, formal Grothendieck-Riemann-Roch formula. Indeed, for any  $B$ -linear dg-functor  $f : T \rightarrow T'$ , the square of spectra

$$\begin{array}{ccc} \mathrm{HK}(T) & \xrightarrow{f_!} & \mathrm{HK}(T') \\ \mathrm{Ch}_{\ell, T} \downarrow & & \downarrow \mathrm{Ch}_{\ell, T'} \\ |r_{\ell}(T)| & \xrightarrow{f_!} & |r_{\ell}(T')| \end{array}$$

commutes up to a natural equivalence.

## 2.4 Trace formula for dg-categories

Let  $\mathcal{C}^{\otimes}$  be a symmetric monoidal  $\infty$ -category ([To-Ve-1], [Lu-HA, Definition 2.0.0.7]).

**Hypothesis 2.4.1** *The underlying  $\infty$ -category  $\mathcal{C}$  has small sifted colimits, and the tensor product preserves small colimits in each variable.*

**Definition 2.4.2** *Let  $\mathcal{C}^{\otimes}$  be a symmetric monoidal  $\infty$ -category satisfying Hypothesis 2.4.1. We denote by  $\mathbf{Alg}(\mathcal{C})$  the  $(\infty, 2)$ -category of algebras in  $\mathcal{C}^{\otimes}$  denoted by  $\mathbf{Alg}_{(1)}(\mathcal{C}^{\otimes})$  in [Lu-COB, Definition 4.1.11] (see also [Ha, Def. 4.40] for a more rigorous definition).*

Informally, one can describe  $\mathbf{Alg}(\mathcal{C})$  as the  $(\infty, 2)$ -category with:

- objects: associative unital monoids ( $=: E_1$ -algebras) in  $\mathcal{C}^{\otimes}$ .
- $\mathrm{Map}_{\mathbf{Alg}(\mathcal{C})}(B, B') := \mathbf{Bimod}_{B', B}(\mathcal{C}^{\otimes})$ , the  $\infty$ -category of  $(B', B)$ -bimodules.
- The composition of 1-morphisms (i.e. of bimodules) is given by tensor product.
- The composition of 2-morphisms (i.e. of morphisms between bimodules) is the usual composition.

**Definition 2.4.3** *Let  $B$  be an algebra in  $\mathcal{C}$  and  $X$  a left  $B$ -module. Let us identify  $X$  with a 1-morphism  $\underline{X} : 1_{\mathcal{C}} \rightarrow B$  in  $\mathbf{Alg}(\mathcal{C})$ . A right  $B$ -dual of  $X$  is defined as a right adjoint  $\underline{Y} : B \rightarrow 1_{\mathcal{C}}$  to  $\underline{X}$ .*

Unraveling the definition, we get that a right dual of  $X$  is a left  $B^{\otimes\text{-op}}$ -module  $Y$ , where  $(-)^{\otimes\text{-op}}$  denotes the opposite  $E_1$ -algebra structure, the unit  $u$  of adjunction (or *coevaluation*) is a map  $coev : 1_{\mathcal{C}} \rightarrow Y \otimes_B X$  in  $\mathcal{C}$ , the counit  $v$  of adjunction (or *evaluation*) is a map  $ev : X \otimes Y \rightarrow B$  of  $(B, B)$ -bimodules;  $u$  and  $v$  satisfy usual compatibilities.

Note that, if a right  $B$ -dual of  $X$  exists then it is unique up to an equivalence.

If the right  $B$ -module  $Y$  is the right  $B$ -dual of the left  $B$ -module  $X$ , then we can define the trace of any map  $f : X \rightarrow X$  of left  $B$ -modules, as follows.

Recall that we have a coevaluation map in  $\mathcal{C}$   $coev : 1_{\mathcal{C}} \rightarrow Y \otimes_B X$  in  $\mathcal{C}$  and an evaluation map of  $(B, B)$ -bimodules  $ev : X \otimes Y \rightarrow B$ .

Consider the graph  $\Gamma_f$  defined as the composite

$$1_{\mathcal{C}} \xrightarrow{coev} Y \otimes_B X \xrightarrow{id \otimes f} Y \otimes_B X.$$

We now elaborate on the evaluation map. Observe that

- $B \in \mathcal{C}$  has a left  $B \otimes B^{\otimes\text{-op}}$ -module structure that we will denote by  $B^L$ .
- $B \in \mathcal{C}$  has a right  $B \otimes B^{\otimes\text{-op}}$ -module structure (i.e. a left  $B^{\otimes\text{-op}} \otimes B$ -module), that we will denote by  $B^R$ .
- $ev : X \otimes Y \rightarrow B^L$  is a map of left  $B \otimes B^{\otimes\text{-op}}$ -modules.
- the composite  $ev' : Y \otimes X \xrightarrow{\sigma} X \otimes Y \xrightarrow{ev} B^R$  is a map of left  $(B^{\otimes\text{-op}} \otimes B)$ -modules

Apply  $(-) \otimes_{B \otimes B^{\otimes\text{-op}}} B^L$  to the composite

$$ev' : Y \otimes X \xrightarrow{\sigma} X \otimes Y \xrightarrow{ev} B^R$$

to get

$$ev_{HH} : (Y \otimes X) \otimes_{B \otimes B^{\otimes\text{-op}}} B^L \longrightarrow B^R \otimes_{B \otimes B^{\otimes\text{-op}}} B^L =: \mathbf{HH}_{\mathcal{C}}(B).$$

Now observe that  $(Y \otimes X) \otimes_{B \otimes B^{\otimes\text{-op}}} B^L \simeq Y \otimes_B X$  in  $\mathcal{C}$ .

Note that, by definition,  $\mathbf{HH}_{\mathcal{C}}(B)$  is (only) an object in  $\mathcal{C}$ , called the *Hochschild homology object* of  $B$ .

**Definition 2.4.4** *The non-commutative trace of  $f : X \rightarrow X$  over  $B$  is defined as the composite*

$$Tr_B(f) : 1_{\mathcal{C}} \xrightarrow{\Gamma_f} Y \otimes_B X \simeq (Y \otimes X) \otimes_{B \otimes B^{\otimes\text{-op}}} B^L \xrightarrow{ev_{HH}} B^R \otimes_{B \otimes B^{\otimes\text{-op}}} B^L =: \mathbf{HH}_{\mathcal{C}}(B).$$

$Tr_B(f)$  is a morphism in  $\mathcal{C}$ .

**Remark 2.4.5** Let  $B \in \mathbf{CAlg}(\mathcal{C}^{\otimes})$ , and let us still denote by  $B$  its image via the canonical map  $\mathbf{CAlg}(\mathcal{C}^{\otimes}) \rightarrow \mathbf{Alg}_{E_1}(\mathcal{C}^{\otimes})$ . In this case,  $\mathbf{Mod}_B(\mathcal{C}^{\otimes})$  is a symmetric monoidal  $\infty$ -category, and if  $X \in \mathbf{Mod}_B(\mathcal{C}^{\otimes})$  is a dualizable object (in the usual sense), then its (left and right) dual in  $\mathbf{Mod}_B(\mathcal{C}^{\otimes})$  is also a right-dual of  $X$  according to Definition 2.4.3. In this case, any  $f : X \rightarrow X$  in  $\mathbf{Mod}_B(\mathcal{C}^{\otimes})$ , has therefore *two possible traces*, a non-commutative one (as in Definition 2.4.4)

$$Tr_B(f) : 1_{\mathcal{C}} \longrightarrow \mathbf{HH}_{\mathcal{C}}(B)$$

which is a morphism in  $\mathcal{C}$ , and a more standard, commutative one

$$Tr_B^c(f) : B \longrightarrow B$$

which is a morphism on  $\mathbf{Mod}_B(\mathcal{C}^{\otimes})$ . The two traces are related by the following commutative diagram

$$\begin{array}{ccc} 1_{\mathcal{C}} & \xrightarrow{u_B} & B \\ Tr_B(f) \downarrow & & \downarrow Tr_B^c(f) \\ \mathbf{HH}_{\mathcal{C}}(B) & \xrightarrow{a} & B \end{array}$$

where  $a : \mathbf{HH}_{\mathcal{C}}(B) \rightarrow B$  is the canonical augmentation (which exists since  $B$  is commutative), and  $u_B : 1_{\mathcal{C}} \rightarrow B$  is the unit map of the algebra  $B$  in  $\mathcal{C}$ .

**The case of dg-categories.** Let us specialize the previous discussion to the case  $\mathcal{C}^\otimes = \mathbf{dgCat}_A$ . Let  $B$  be a monoidal dg-category, i.e. an associative and unital monoid in the symmetric monoidal  $\infty$ -category  $\mathbf{dgCat}_A$ .

**Proposition 2.4.6** *For any  $B$ -module  $T$  which is cotensored in the sense of Definition 2.1.2, the big  $B$ -module  $\widehat{T} \in \mathbf{dgCat}_B$  has a right  $B$ -dual in the sense of Definition 2.4.3, whose underlying big dg-category is  $\widehat{T}^\circ$ .*

**Proof.** This is very similar to the argument used in [To2, Prop. 2.5 (1)]. We consider  $\widehat{T}^\circ$ , and we define evaluation and coevaluation maps as follows.

The big morphism  $h$  introduced right before Definition 2.1.2

$$h : \widehat{T}^\circ \widehat{\otimes}_A \widehat{T} \longrightarrow \widehat{B},$$

whose domain is naturally a  $\widehat{B}$ -bimodule, can be canonically lifted to a morphism of bimodules. We choose this as our evaluation morphism.

The coevaluation is then obtained by duality. We start by the diagonal bimodule

$$T : (T^\circ \otimes_A T)^\circ \longrightarrow L(A) = \widehat{A}.$$

sending  $(x, y)$  to  $T(y, x)$ . This morphism naturally descends to  $(T^\circ \otimes_B T)^\circ$ , providing a dg-functor

$$(T^\circ \otimes_B T)^\circ \longrightarrow \widehat{A}.$$

Note that  $T^\circ$  is naturally a right  $B$ -module since  $T$  is assumed to be cotensored (and note that, otherwise,  $T^\circ \otimes_B T$  would not make sense). This dg-functor is an object in  $\widehat{T^\circ \otimes_B T} \simeq \widehat{T^\circ \widehat{\otimes}_A \widehat{T}}$ , and thus defines a coevaluation morphism

$$\widehat{A} \longrightarrow \widehat{T^\circ \widehat{\otimes}_A \widehat{T}}.$$

These two evaluation and coevaluation morphisms satisfy the required triangular identities, and thus make  $\widehat{T}^\circ$  a right dual to  $\widehat{T}$ .  $\square$

According to the previous Proposition, any cotensored  $B$ -module  $T$  has a big right dual, so it comes equipped with big evaluation and coevaluation maps.

**Definition 2.4.7** *For a monoidal dg-category  $B$ , and a  $B$ -module  $T \in \mathbf{dgCat}_B$ , we say that  $T$  is saturated over  $B$  if*

1.  $T$  is cotensored (over  $B$ ), and
2. the evaluation and coevaluation morphisms are small (i.e. are maps in  $\mathbf{dgCat}_A$ ).

In particular, if  $T$  saturated over  $B$ , and  $f : T \rightarrow T$  is a morphism in  $\mathbf{dgCat}_B$ , then the trace

$$Tr_B(f : T \rightarrow T) : A \rightarrow \mathbf{HH}(B/A) = B^R \otimes_{B^{\otimes\text{-op}} \otimes_A B} B^L$$

is also small, i.e. it is a morphism *inside*  $\mathbf{dgCat}_A$ .

Since our  $\ell$ -adic realization functor  $r_\ell$  is only lax-monoidal, in order to establish our trace formula for an endomorphism  $f : T \rightarrow T$  of a saturated  $B$ -module, we need to restrict to those saturated  $B$ -modules on which  $r_\ell$  is in fact symmetric monoidal.

**Definition 2.4.8** A saturated  $T \in \mathbf{dgCat}_B$  is called  $\ell^\otimes$ -admissible if the canonical map

$$r_\ell(T^{op}) \otimes_{r_\ell(B)} r_\ell(T) \longrightarrow r_\ell(T^{op} \otimes_B T)$$

is an equivalence in  $Sh_{\mathbb{Q}_\ell}(S)$ .

The trace  $Tr_B(f)$  is a map  $A \rightarrow \mathbf{HH}(B/A)$ , hence it induces a map in  $\mathbf{Sp}$

$$K(Tr_B(f)) : K(A) \rightarrow K(\mathbf{HH}(B/A))$$

which is actually a map of  $K(A)$ -modules (in spectra), since  $K$  is lax-monoidal. Hence it corresponds to an element denoted as

$$[\mathbf{HH}(T/B, f)] \equiv tr_B(f) \in K_0(\mathbf{HH}(B/A)).$$

Therefore, its image by the  $\ell$ -adic Chern character  $Ch_{\ell,0} := \pi_0(Ch_\ell)$

$$Ch_{\ell,0} : K_0(\mathbf{HH}(B/A)) \rightarrow Hom_{D(r_\ell(A))}(r_\ell(A), r_\ell(\mathbf{HH}(B/A))) \simeq H^0(S_{\acute{e}t}, r_\ell(\mathbf{HH}(B/A))),$$

is an element  $Ch_{\ell,0}([\mathbf{HH}(T/B; f)]) \in H^0(S_{\acute{e}t}, r_\ell(\mathbf{HH}(B/A)))$ .

On the other hand, the trace  $Tr_{r_\ell(B)}(r_\ell(f))$  of  $r_\ell(f)$  over  $r_\ell(B)$  is, by definition, a morphism

$$r_\ell(A) \simeq \mathbb{Q}_\ell(\beta) \rightarrow \mathbf{HH}(r_\ell(B)/r_\ell(A))$$

in  $\mathbf{Mod}_{r_\ell(A)}(Sh_{\mathbb{Q}_\ell}(S))$ .

We may further compose this with the canonical map

$$\mathbf{HH}(r_\ell(B)/r_\ell(A)) \rightarrow r_\ell(\mathbf{HH}(B/A))$$

(given by lax-monoidality of  $r_\ell(-)$ ), to get a map

$$r_\ell(A) \rightarrow r_\ell(\mathbf{HH}(B/A))$$

in  $\mathbf{Mod}_{r_\ell(A)}(Sh_{\mathbb{Q}_\ell}(S))$ . This is the same thing as an element denoted as

$$tr_{r_\ell(B)}(r_\ell(f)) \in \pi_0(|r_\ell(\mathbf{HH}(B/A))|) \simeq H^0(S_{\acute{e}t}, r_\ell(\mathbf{HH}(B/A))).$$

**Theorem 2.4.9** Let  $B$  a monoidal dg-category over  $A$ ,  $T \in \mathbf{dgCat}_B$  a saturated and  $\ell^\otimes$ -admissible  $B$ -module, and  $f : T \rightarrow T$  map in  $\mathbf{dgCat}_B$ . Then, we have an equality

$$Ch_{\ell,0}([\mathbf{HH}(T/B, f)]) = tr_{r_\ell(B)}(r_\ell(f))$$

in  $H^0(S_{\acute{e}t}, r_\ell(\mathbf{HH}(B/A)))$ .

**Proof.** This is a formal consequence of uniqueness of right duals, and of the resulting fact that traces are preserved by symmetric or lax symmetric monoidal  $\infty$ -functors under our admissibility condition. The key statement is the following lemma, left as an exercise to the reader, and applied to the case where  $F$  is our  $\ell$ -adic realization functor.

**Lemma 2.4.10** *Let*

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

*be a lax symmetric monoidal  $\infty$ -functor between presentable symmetric monoidal  $\infty$ -categories. Let  $B$  be a monoid in  $\mathcal{C}$ ,  $M$  a left  $B$ -module, and  $f : M \rightarrow M$  a morphism of  $B$ -modules. We assume that  $M$  has a right dual  $M^\circ$ , and that the natural morphism*

$$F(M^\circ) \otimes_{F(B)} F(M) \longrightarrow F(M^\circ \otimes_B M)$$

*is an equivalence. Then,  $F(M)$  has a right dual, and we have*

$$F(\mathrm{Tr}(f)) = i(\mathrm{Tr}(F(f)))$$

*as elements in  $\pi_0(\mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, F(\mathrm{HH}(B))))$ , where  $i$  is induced by the natural morphism  $\mathrm{HH}(F(B)) \rightarrow F(\mathrm{HH}(B))$ .*

□

### 3 Invariant vanishing cycles

This section gathers general results about inertia-invariant vanishing cycles (*I-vanishing cycles*, for short), their relations with dg-categories of singularities, and their behaviour under products. These results are partially taken from [BRTV], and the only original result is Proposition 3.4.2 that can be seen as a version of Thom-Sebastiani formula in the mixed-characteristic setting.

All along this section,  $A$  will be a strictly henselian excellent dvr with fraction field  $K = \mathrm{Frac}(A)$ , and perfect (hence algebraically closed) residue field  $k$ . We let  $S = \mathrm{Spec} A$ . All schemes over  $S$  are assumed to be separated and of finite type over  $S$ . We denote by  $i : s := \mathrm{Spec} k \rightarrow S$  the closed point of  $S$ , and  $j : \eta := \mathrm{Spec} K \rightarrow S$  its generic point. For an  $S$ -scheme  $X$ , we denote by  $X_s := X \times_S s$  its (geometric) special fiber, and  $X_\eta = X \times_S \eta$  its generic fiber. Accordingly, we write  $X_{\bar{\eta}} := X \times_S \mathrm{Spec} K^{sp}$  for the *geometric* generic fiber.

#### 3.1 Trivializing the Tate twist

We let  $\ell$  be a prime invertible in  $k$ , and we denote by  $p$  the characteristic exponent of  $k$ . As  $k$  is algebraically closed, we may, and will, choose once for all a group isomorphism

$$\mu_\infty(k) \simeq \mu_\infty(K) \simeq (\mathbb{Q}/\mathbb{Z})[p^{-1}]$$

between the group of roots of unity in  $k$  and the prime-to- $p$  part of  $\mathbb{Q}/\mathbb{Z}$ . Equivalently, we have chosen a given group isomorphism

$$\lim_{(n,p)=1} \mu_n(k) \simeq \hat{\mathbb{Z}}',$$

where  $\hat{\mathbb{Z}}' := \lim_{(n,p)=1} \mathbb{Z}/n$ . In particular, we have selected a topological generator of  $\lim_{(n,p)=1} \mu_n(k)$ , corresponding to the image of  $1 \in \mathbb{Z}$  inside  $\hat{\mathbb{Z}}'$ . The choice of the isomorphism above also provides a

specific induced isomorphism  $\mathbb{Q}_\ell(1) \simeq \mathbb{Q}_\ell$  of  $\mathbb{Q}_\ell$ -sheaves on  $S$ , where (1) denotes, as usual, the Tate twist. By taking tensor powers of this isomorphism, we get various induced isomorphisms  $\mathbb{Q}_\ell(i) \simeq \mathbb{Q}_\ell$  for all  $i \in \mathbb{Z}$ .

We recall that the absolute Galois group  $I$  of  $K$  (which coincides with the *inertia group* in our case) sits in an extension of pro-finite groups<sup>7</sup>

$$1 \longrightarrow P \longrightarrow I \longrightarrow I_t \simeq \hat{\mathbb{Z}}' \longrightarrow 1,$$

where  $P$  is a pro- $p$ -group (the *wild inertia* subgroup). For any continuous finite dimensional  $\mathbb{Q}_\ell$ -representation  $V$  of  $I$ , the group  $P$  acts by a finite quotient  $G_V$  on  $V$ . Moreover, the Galois cohomology of  $V$  can then be explicitly identified with the two-terms complex

$$V^G \xrightarrow{1-T} V^G$$

where  $T$  is the action of the chosen topological generator of  $I_t$ . This easily implies that for any  $\mathbb{Q}_\ell$ -representation  $V$  of  $I$ , the natural pairing on Galois cohomology

$$H^i(I, V) \otimes H^{1-i}(I, V^\vee) \longrightarrow H^1(I, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$$

is non-degenerate. In other words, if we denote by  $V^I$  the complex of cohomology of  $I$  with coefficients in  $V$ , we have a natural quasi-isomorphism  $(V^I)^\vee \simeq (V^\vee)^I[1]$ .

### 3.2 Reminders on actions of the inertia group

Let  $X \longrightarrow S$  be an  $S$ -scheme (separated and of finite type, according to our conventions). We recall from [SGA7-II, Exp. XIII, 1.2] that we can associate to  $X$  a *vanishing topos*  $(X/S)_{et}^\nu$  which is defined as (a 2-)fiber product of toposes

$$(X/S)_{et}^\nu := (X_s)_{et}^\sim \times_{s_{et}^\sim} \eta_{et}^\sim.$$

Since  $S$  is strictly henselian,  $s_{et}^\sim$  is in fact the punctual topos, and the fiber product above is in fact a product of topos. The topos  $\eta_{et}^\sim$  is equivalent to the topos of sets with continuous action of  $I = \text{Gal}(K^{sp}/k)$ , where  $K^{sp}$  denotes a separable closure of  $K$ . Morally,  $(X/S)_{et}^\nu$  is the topos of étale sheaves on  $X_s$ , endowed with a continuous action of  $I$  (see [SGA7-II, Exp. XIII, 1.2.4]).

As explained in [BRTV] we have an  $\ell$ -adic  $\infty$ -category  $\mathcal{D}((X/S)_{et}^\nu, \mathbb{Z}_\ell)$ . Morally speaking, objects of this  $\infty$ -category consist of the data of an object  $E \in \mathcal{D}(\bar{X}_s, \mathbb{Z}_\ell) = \text{Sh}_{\mathbb{Z}_\ell}(\bar{X}_s)$  together with a continuous action of  $I$ . We say that such an object is *constructible* if  $E$  is a constructible object in  $\mathcal{D}(\bar{X}_s, \mathbb{Z}_\ell) = \text{Sh}_{\mathbb{Z}_\ell}(\bar{X}_s)$ , and we denote by  $\mathcal{D}_c((X/S)_{et}^\nu, \mathbb{Z}_\ell)$  the full sub- $\infty$ -category of constructible objects.

**Definition 3.2.1** *The  $\infty$ -category of ind-constructible  $I$ -equivariant  $\ell$ -adic complexes on  $X_s$  is defined by*

$$\mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell) := \text{Ind}(\mathcal{D}_c((X/S)_{et}^\nu, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell).$$

*The full sub- $\infty$ -category of constructible objects is  $\mathcal{D}_c^I(X_s, \mathbb{Q}_\ell) := \mathcal{D}_c((X/S)_{et}^\nu, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ .*

---

<sup>7</sup>Note that the tame inertia quotient  $I_t$  is canonically isomorphic to  $\hat{\mathbb{Z}}'(1)$ , and it becomes isomorphic to  $\hat{\mathbb{Z}}'$  through our choice.

Note that since we have chosen trivialisations of the Tate twists,  $\mathbb{Q}_\ell(\beta)$  is identified with  $\mathbb{Q}_\ell[\beta, \beta^{-1}]$  where  $\beta$  is a free variable in degree 2. This is a graded algebra object in  $\mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell)$ , and we define the  $\infty$ -category  $\mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell(\beta))$  as the  $\infty$ -category of  $\mathbb{Q}_\ell(\beta)$ -modules in  $\mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell)$ , or equivalently, the 2-periodic  $\infty$ -category of ind-constructible  $\mathbb{Q}_\ell$ -adic complexes on  $(X/S)_{et}^\nu$ .

**Definition 3.2.2** *An object  $E \in \mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell(\beta))$  is constructible if it belongs to the thick triangulated sub- $\infty$ -category generated by objects of the form  $E_0(\beta) = E_0 \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(\beta)$  for  $E_0$  a constructible object in  $\mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell)$ .*

*The full sub- $\infty$ -category of  $\mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell(\beta))$  consisting of constructible objects is denoted  $\mathcal{D}_c^I(X_s, \mathbb{Q}_\ell(\beta))$ . Similarly, for any  $S$ -scheme  $X$ , we define  $\mathcal{D}_c(X, \mathbb{Q}_\ell(\beta))$  as the full sub- $\infty$ -category of objects in  $\mathcal{D}_{ic}(X, \mathbb{Q}_\ell(\beta))$  generated by  $E_0(\beta)$  for  $E_0$  constructible.*

Note that strictly speaking an object of  $\mathcal{D}_c^I(X_s, \mathbb{Q}_\ell(\beta))$  is not constructible in the usual sense, as its underlying object in  $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$  is 2-periodic.

The topos  $(X/S)_{et}^\nu$  comes with a natural projection  $(X/S)_{et}^\nu \rightarrow (X_s)_{et}^\sim$  whose direct image is an  $\infty$ -functor denoted by

$$(-)^I : \mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell) \rightarrow \mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$$

called the *I-invariants* functor. This  $\infty$ -functor preserves constructibility. The  $\infty$ -categories  $\mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell)$  and  $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$  carries natural symmetric monoidal structures and the  $\infty$ -functor  $(-)^I$  comes equipped with a natural lax symmetric monoidal structure (being induced by the direct image of a morphism of toposes). Moreover,  $(-)^I$  is the right adjoint of the symmetric monoidal  $\infty$ -functor  $U : \mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell) \rightarrow \mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell)$  endowing objects in  $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$  with the trivial action of  $I$ . This gives  $\mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell)$  the structure of a  $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$ -module via  $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell) \times \mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell) \rightarrow \mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell) : (E, F) \mapsto U(E) \otimes F$ . This bi-functor distributes over colimits, thus by the adjoint functor theorem, we get an enrichment of  $\mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell)$  over  $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$ . Note that  $\mathcal{D}_{ic}^I(X_s, \mathbb{Q}_\ell)$  is also enriched over itself.

It is important to notice that the *I-invariants* functor commutes with base change in the following sense. Let  $f : \text{Spec } k \rightarrow X_s$  be a geometric point. The morphism  $f$  defines a geometric point of  $(X_s)_{et}^\sim$  and thus induces a geometric morphism of toposes

$$\eta_{et}^\sim \rightarrow (X/S)_{et}^\nu.$$

We thus have an inverse image functor

$$f^* : \mathcal{D}_c^I(X_s, \mathbb{Q}_\ell) \rightarrow \mathcal{D}_c(\eta, \mathbb{Q}_\ell) = \mathcal{D}_c(I, \mathbb{Q}_\ell).$$

where  $\mathcal{D}_c(I, \mathbb{Q}_\ell)$  is the  $\infty$ -category of finite dimensional complexes of  $\ell$ -adic representations of  $I$ . As usual, the square of  $\infty$ -functors

$$\begin{array}{ccc} \mathcal{D}_c^I(X_s, \mathbb{Q}_\ell) & \xrightarrow{(-)^I} & \mathcal{D}_c(X_s, \mathbb{Q}_\ell) = Sh_{\mathbb{Q}_\ell}^{\text{ct}}(X_s) \\ f^* \downarrow & & \downarrow f^* \\ \mathcal{D}_c(I, \mathbb{Q}_\ell) & \xrightarrow{(-)^I} & \mathcal{D}_c(\mathbb{Q}_\ell) \end{array}$$

comes equipped with a natural transformation

$$f^*((-)^I) \Rightarrow (f^*(-))^I.$$

It can be checked that this natural transformation is always an equivalence. In particular, for any geometric point  $x$  in  $X_s$ , we have a natural equivalence of  $\ell$ -adic complexes  $(E)_x^{\mathbb{I}} \simeq (E_x)^{\mathbb{I}}$ , for any  $E \in \mathcal{D}_c^{\mathbb{I}}(X_s, \mathbb{Q}_\ell)$ .

The dualizing complex  $\omega$  of the scheme  $X_s$  can be used in order to obtain a dualizing object in  $\mathcal{D}_c^{\mathbb{I}}(X_s, \mathbb{Q}_\ell)$  as follows. We consider  $\omega$  as an object in  $\mathcal{D}_c^{\mathbb{I}}(X_s, \mathbb{Q}_\ell)$  endowed with the trivial  $\mathbb{I}$ -action. We then have an equivalence of  $\infty$ -categories

$$\mathbb{D}_I : \mathcal{D}_c^{\mathbb{I}}(X_s, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}_c^{\mathbb{I}}(X_s, \mathbb{Q}_\ell)^{op}$$

sending  $E$  to  $\mathbb{R}\underline{Hom}(E, \omega)$ , where  $\mathbb{R}\underline{Hom}$  denotes the natural enrichment of  $\mathcal{D}_c^{\mathbb{I}}(X_s, \mathbb{Q}_\ell)$  over itself. We obviously have a canonical biduality equivalence  $\mathbb{D}_I^2 \simeq id$ . The duality functor  $\mathbb{D}_I$  is compatible with the usual Grothendieck duality functor  $\mathbb{D}$  for the scheme  $X_s$  up to a shift, as explained by the following lemma.

**Lemma 3.2.3** *For any object  $E \in \mathcal{D}_c^{\mathbb{I}}(X_s, \mathbb{Q}_\ell)$ , there is a functorial equivalence in  $\mathcal{D}_c(X_s, \mathbb{Q}_\ell)$*

$$d : \mathbb{D}(E^{\mathbb{I}})[-1] \simeq (\mathbb{D}_I(E))^{\mathbb{I}}.$$

**Proof.** Taking  $\mathbb{I}$ -invariants is a lax monoidal  $\infty$ -functor, so we have a natural map  $E^{\mathbb{I}} \otimes \mathbb{D}_I(E)^{\mathbb{I}} \longrightarrow (E \otimes \mathbb{D}_I(E))^{\mathbb{I}}$ , that can be composed with the evaluation morphism  $E \otimes \mathbb{D}_I(E) \longrightarrow \omega$  to obtain  $E^{\mathbb{I}} \otimes \mathbb{D}_I(E)^{\mathbb{I}} \longrightarrow \omega^{\mathbb{I}}$ . As the action of  $\mathbb{I}$  on  $\omega$  is trivial, we have a canonical equivalence  $\omega^{\mathbb{I}} \simeq \omega \otimes \mathbb{Q}_\ell^{\mathbb{I}} \simeq \omega \oplus \omega[-1]$ . By projection on the second factor we get a pairing  $E^{\mathbb{I}} \otimes \mathbb{D}_I(E)^{\mathbb{I}} \longrightarrow \omega[-1]$ , and thus a map

$$\mathbb{D}_I(E)^{\mathbb{I}} \longrightarrow \mathbb{D}(E^{\mathbb{I}})[-1].$$

We claim that the above morphism is an equivalence in  $\mathcal{D}_c(X_s, \mathbb{Q}_\ell)$ . For this it is enough to check that the above morphism is a stalkwise equivalence. Now, the stalk of the above morphism at a geometric point  $x$  of  $X_s$  can be written as

$$(E(x)^\vee)^{\mathbb{I}} \longrightarrow (E(x)^{\mathbb{I}})^\vee[-1]$$

where  $E(x) := H_x^*(X, E) \in D_c(\mathbb{Q}_\ell)$  is the local cohomology of  $E$  at  $x$ , and  $(-)^\vee$  is now the standard linear duality over  $\mathbb{Q}_\ell$ . The result now follows from the following well-known duality for  $\mathbb{Q}_\ell$ -representations of  $\mathbb{I}$ : for any finite dimensional  $\mathbb{Q}_\ell$ -representation  $V$  of  $\mathbb{I}$ , the fundamental class in  $H^1(\mathbb{I}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$ , induces a canonical isomorphism of Galois cohomologies

$$H^*(\mathbb{I}, V^\vee) \simeq H^{1-*}(\mathbb{I}, V)^\vee.$$

□

### 3.3 Invariant vanishing cycles and dg-categories

From [SGA7-II, Exp. XIII] and [BRTV, 4.1], the vanishing cycles construction provides an  $\infty$ -functor

$$\phi : \mathcal{D}_c(X, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}_c^{\mathbb{I}}(X_s, \mathbb{Q}_\ell).$$

Applied to the constant sheaf  $\mathbb{Q}_\ell$ , we get this way an object denoted by  $\nu_{X/S}$  (or simply  $\nu_X$  if  $S$  is clear) in  $\mathcal{D}_c^{\mathbb{I}}(X_s, \mathbb{Q}_\ell)$ .

**Definition 3.3.1** *The I-invariant vanishing cycles of  $X$  relative to  $S$  (or I-vanishing cycles, for short) is the object*

$$\nu_X^I := (\nu_X)^I \in \mathcal{D}_c(X_s, \mathbb{Q}_\ell).$$

There are several possible descriptions of invariant vanishing cycles. First of all, by its very definition,  $\nu_X^I$  is related to the I-invariant nearby cycles  $\psi_X^I := (\psi_X)^I$  by means of an exact triangle in  $\mathcal{D}_c(X_s, \mathbb{Q}_\ell)$

$$\mathbb{Q}_\ell^I \longrightarrow \psi_X^I \longrightarrow \nu_X^I. \quad (2)$$

Another description, in terms of *local cohomology*, is the following. We let  $U = X_K$  be the open complement of  $X_s$  inside  $X$ , and  $j_X : U \hookrightarrow X$  and  $i_X : X_s \hookrightarrow X$  the corresponding immersions. Then, the I-vanishing cycles enters in an exact triangle in  $\mathcal{D}_c(X_s, \mathbb{Q}_\ell)$

$$\nu_X^I \longrightarrow \mathbb{Q}_\ell \longrightarrow i_X^!(\mathbb{Q}_\ell)[2]. \quad (3)$$

Triangle (3) follows from the octahedral axiom applied to the triangles (2) and

$$\mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell^I \simeq \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell[-1] \longrightarrow \mathbb{Q}_\ell[-1],$$

taking also into account the triangle

$$i_X^! \mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell \longrightarrow i_X^*(j_X)_* j_X^* \mathbb{Q}_\ell \simeq \psi_X^I.$$

We get one more description of  $\nu_X^I$  (or rather, of  $\nu_X^I(\beta) := \nu_X^I \otimes \mathbb{Q}_\ell(\beta)$ ) using the  $\ell$ -adic realization of the dg-category of singularities studied in [BRTV], at least when  $X$  is a regular scheme with smooth generic fiber. Let  $\text{Sing}(X_s) = \text{Coh}^b(X_s)/\text{Perf}(X_s)$  be the dg-category of singularities of the scheme  $X_s$ . This dg-category is naturally linear over the dg-categories  $\text{Perf}(X_s)$  and  $\text{Perf}(X)$ , and thus we can take its  $\ell$ -adic realization  $r_{\ell,X}(\text{Sing}(X_s))$  over  $X$  (see Remark 2.2.2) which is a  $\mathbb{Q}_{\ell,X}(\beta)$ -module in  $\mathcal{D}_{ic}(X, \mathbb{Q}_\ell)$  supported on  $X_s$ , hence can be identified with a  $\mathbb{Q}_{\ell,X_s}(\beta)$ -module in  $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$ . When  $X$  is a regular scheme and  $X_K$  is smooth over  $K$ , we have from [BRTV] a canonical equivalence in  $\mathcal{D}_c(X_s, \mathbb{Q}_\ell(\beta))$ <sup>8</sup>

$$\nu_X^I(\beta)[1] \simeq r_{\ell,X}(\text{Sing}(X_s)) \quad (4)$$

where  $\nu_X^I(\beta)$  stands for  $\nu_X^I \otimes \mathbb{Q}_\ell(\beta)$ .

We conclude this section with another description of  $\nu_X^I(\beta)$  (see equivalence (6)), this time in terms of *sheaves of singularities*. We need a preliminary result.

**Lemma 3.3.2** *We assume that  $S$  is excellent. Let  $p : X \rightarrow S$  be a separated morphism of finite type. We write  $r_{\ell,X} : \mathcal{SH}_X \rightarrow \text{Sh}_{\mathbb{Q}_\ell}(X)$ , for the  $\ell$ -adic realization over  $X$ , as in Remark 2.2.2. Then, we have*

1.  $r_{\ell,X}(\text{Perf}(X)) \simeq \mathbb{Q}_{\ell,X}(\beta)$  in  $\mathcal{D}_{ic}(X, \mathbb{Q}_\ell)$ .
2.  $r_{\ell,X}(\text{Coh}^b(X)) \simeq \omega_X(\beta)$  in  $\mathcal{D}_{ic}(X, \mathbb{Q}_\ell)$ , where  $\omega_X \simeq p^!(\mathbb{Q}_\ell)$  is the  $\ell$ -adic dualizing complex of  $X$ <sup>9</sup>.

<sup>8</sup>Strictly speaking, in [BRTV] the equivalence (4) is proved only after push-forward to  $S$  but the very same proof shows also the equivalence (4).

<sup>9</sup>Note that since  $S$  is excellent,  $X$  is excellent so that  $\omega_X$  exists by a theorem of Gabber ([ILO, Exp. XVII, Th. 0.2]).

3. There exists a canonical map  $\eta_X : \mathbb{Q}_{\ell, X}(\beta) \longrightarrow \omega_X(\beta)$  in  $\mathcal{D}_c(X, \mathbb{Q}_{\ell}(\beta))$ , called the 2-periodic  $\ell$ -adic fundamental class of  $X$ .

**Proof.** First of all, separated finite type morphisms of noetherian schemes are compactifiable (by Nagata's theorem), thus we can assume that  $p : X \rightarrow S$  is proper.

(1) follows immediately from [BRTV, Prop. 3.9 and formula (3.7.13)].

In order to prove (2) we first produce a map  $\alpha : r_{\ell}(\mathbf{Coh}^b(X)) \rightarrow \omega_X(\beta)$ . In the notations of [BRTV, §3] (note that [BRTV]'s notation for  $\mathbf{M}_X^T$  is  $\mathcal{M}_X^{\vee}(T)$ ), we first construct a map  $\alpha^{\text{mot}} : \mathcal{M}_X^{\vee}(\mathbf{Coh}^b(X)) \rightarrow p^!(\mathbf{BU}_S) =: \omega_X^{\text{mot}}$  in  $\text{SH}(X)$ , whose étale  $\ell$ -adic realization will be  $\alpha$ . Since  $p$  is proper,  $\alpha^{\text{mot}}$  is the same thing, by adjunction, as a map  $p_*(\mathcal{M}_X^{\vee}(\mathbf{Coh}^b(X))) \rightarrow \mathbf{BU}_S$  in  $\text{SH}(S)$ . Now,  $p_*(\mathcal{M}_X^{\vee}(\mathbf{Coh}^b(X)))$  is just  $\mathcal{M}_S^{\vee}(\mathbf{Coh}^b(X))$ , where  $\mathbf{Coh}^b(X)$  is viewed as a dg-category over  $S$ , via  $p$ . If  $Y$  is smooth over  $S$ , we have by [Pr, Prop. B.4.1], an equivalence of  $S$ -dg-categories

$$\mathbf{Coh}^b(X) \otimes_S \mathbf{Coh}^b(Y) \simeq \mathbf{Coh}^b(X \times_S Y) \quad (5)$$

Through this identification,  $\mathcal{M}_S^{\vee}(\mathbf{Coh}^b(X)) \in \text{SH}(S)$  is the  $\infty$ -functor  $Y \mapsto \text{KH}(\mathbf{Coh}^b(X \times_S Y))$ , and  $\text{KH}(\mathbf{Coh}^b(X \times_S Y))$  is equivalent to the G-theory spectrum  $\mathbf{G}(X \times_S Y)$  of  $X \times_S Y$ , by  $\mathbb{A}^1$ -invariance of G-theory. Since  $S$  is regular,  $\mathbf{BU}_S \simeq \mathbf{G}_S := \mathbf{G}(-/S)$  canonically in  $\text{SH}(S)$ , and we can take the map  $\mathcal{M}_S^{\vee}(\mathbf{Coh}^b(X)) \rightarrow \mathbf{BU}_S \simeq \mathbf{G}_S$  to be the push forward  $p_*$  on G-theories  $\mathbf{G}(X \times_S -) \rightarrow \mathbf{G}(-/S)$ . This gives us a map  $\alpha^{\text{mot}} : \mathcal{M}_X^{\vee}(\mathbf{Coh}^b(X)) \rightarrow p^!(\mathbf{BU}_S) =: \omega_X^{\text{mot}}$ . Now observe that by [BRTV, formula (3.7.13)], the étale  $\ell$ -adic realization of  $p^!(\mathbf{BU}_S)$  is canonically equivalent to  $p^!(\mathbb{Q}_{\ell}(\beta)) \simeq \omega_X(\beta)$  (since étale  $\ell$ -adic realization commutes with six operations, [BRTV, Rmk. 3.23]). Therefore, we get our map  $\alpha : r_{\ell}(\mathbf{Coh}^b(X)) \rightarrow \omega_X(\beta)$ . Checking that  $\alpha$  is an equivalence is a local statement, i.e. it is enough to show that if  $j : V = \text{Spec } A \hookrightarrow X$  is an open affine subscheme, then  $j^*(\alpha)$  is an equivalence. Now,  $j^*r_{\ell}(\mathbf{Coh}^b(X)) \simeq r_{\ell, V}(j^*\mathbf{Coh}^b(X)) \simeq r_{\ell, V}(\mathbf{Coh}^b(V))$  (where  $r_{\ell, V}$  denotes the  $\ell$ -adic realization over  $V$ ), and  $j^*\omega_X \simeq j^!\omega_X \simeq \omega_V$ , so  $j^*\alpha$  identifies with a map  $r_{\ell, V}(\mathbf{Coh}^b(V)) \rightarrow \omega_V(\beta)$ . Since  $V$  is affine and of finite type over  $S$ , we can choose a closed immersion  $i : V \hookrightarrow V'$ , with  $V'$  affine and smooth (hence regular) over  $S$ . Let  $h : V' \setminus V \hookrightarrow V'$  be the complementary open immersion. Since  $V'$  and  $V' \setminus V$  are regular, by Quillen localization and the properties of the nc realization functor  $\mathcal{M}^{\vee}$  (see [BRTV]), we get a cofiber sequence

$$\mathcal{M}_{V'}^{\vee}(\mathbf{Coh}^b(V)/V') \rightarrow \mathbf{BU}_{V'} \rightarrow h_*\mathbf{BU}_{V' \setminus V},$$

where the notation  $\mathbf{Coh}^b(V)/V'$  means that  $\mathbf{Coh}^b(V)$  is viewed as a dg-category over  $V'$ , via  $i$ . In other words,  $\mathcal{M}_{V'}^{\vee}(\mathbf{Coh}^b(V)/V') \simeq i_*\mathcal{M}_V^{\vee}(\mathbf{Coh}^b(V))$ . If we apply  $i^*$  to this cofiber sequence, and compare what we obtain to the application of  $i^*$  to the standard localization sequence

$$i_*i^!\mathbf{BU}_{V'} \rightarrow \mathbf{BU}_{V'} \rightarrow h_*h^*\mathbf{BU}_{V'} = h_*\mathbf{BU}_{V' \setminus V},$$

we finally get, after étale  $\ell$ -adic realization, that  $\omega_V(\beta) \simeq i^!\mathbb{Q}_{\ell}(\beta) \simeq r_{\ell, V}(\mathbf{Coh}^b(V))$ . This implies that  $j^*\alpha$  is also an equivalence.

By (1) and (2), the map in (3) is finally obtained by applying  $r_{\ell, X}$  to the inclusion  $\text{Perf}(X) \rightarrow \mathbf{Coh}^b(X)$ .  $\square$

**Remark 3.3.3** Note that Lemma 3.3.2 applies, in particular, to give a 2-periodic  $\ell$ -adic fundamental class map  $\eta_U : \mathbb{Q}_{\ell}(\beta) \longrightarrow \omega_U(\beta)$  for any open subscheme  $U \hookrightarrow X$  over  $S$ , whenever  $X$  is proper over  $S$ .

**Definition 3.3.4** Let  $X/S$  be a separated and finite type  $S$ -scheme. The sheaf of singularities of  $X$  is defined to be the cofiber of the 2-periodic  $\ell$ -adic fundamental class morphism  $\eta_X$  (Lemma 3.3.2 (3))

$$\omega_X^{\circ} := \text{Cofib}(\eta_X : \mathbb{Q}_{\ell}(\beta) \longrightarrow \omega_X(\beta)).$$

**Remark 3.3.5** By construction of  $\eta_X$  in Lemma 3.3.2,  $\omega_X^\circ$  is then the  $\ell$ -adic realization  $r_{\ell,X}(\text{Sing}(X))$  of the dg-category  $\text{Sing}(X)$ , considered as a dg-category over  $X$ , and this is true without any regularity hypothesis on  $X$ .

**Remark 3.3.6** If  $p : \mathcal{X} \rightarrow S$  is a proper lci map from a *derived* scheme  $\mathcal{X}$ , we can still define a 2-periodic  $\ell$ -adic fundamental class map  $\eta_{\mathcal{X}}$ , as in Lemma 3.3.2. This can be done by observing that the pushforward on G-theories along the inclusion of the truncation  $t_0\mathcal{X} \rightarrow \mathcal{X}$  is an equivalence, and that  $p$  being lci we have a natural inclusion  $\text{Perf}(\mathcal{X}) \rightarrow \text{Coh}^b(\mathcal{X})$ . We further observe that in this case, while the  $\infty$ -category  $\mathcal{D}_c(\mathcal{X}, \mathbb{Q}_\ell(\beta))$  only depends on the reduced subscheme  $(t_0\mathcal{X})_{\text{red}}$ , and the same is true for the objects  $\mathbb{Q}_{\ell,\mathcal{X}}(\beta)$  and  $\omega_{\mathcal{X}}(\beta)$ , in contrast, the morphism  $\eta_{\mathcal{X}}$  *does* depend on the derived structure on  $\mathcal{X}$ , and thus it is not a purely topological invariant.

Let us come back to  $X$  a regular scheme, proper over  $S$ . We have a canonical equivalence in  $\mathcal{D}_c(X_s, \mathbb{Q}_\ell(\beta))$

$$\nu_X^I(\beta)[1] \simeq \omega_{X_s}^\circ. \quad (6)$$

This is a reformulation of the equivalence (4), in view of Lemma 3.3.2.

**Remark 3.3.7** When  $X$  is not regular anymore, but still proper and lci<sup>10</sup> over  $S$ , there is nonetheless a natural morphism  $\nu_X^I(\beta)[1] \rightarrow \omega_{X_s}^\circ$ , constructed as follows. Consider again the triangle (3)

$$\nu_X^I \longrightarrow \mathbb{Q}_\ell \longrightarrow i_X^!(\mathbb{Q}_\ell)[2].$$

On  $X$ , we do have the 2-periodic  $\ell$ -adic fundamental class  $\eta_X : \mathbb{Q}_\ell(\beta) \rightarrow \omega_X(\beta)$ , and by taking its  $!$ -pullback by  $i_X^!$ , we get a morphism  $i_X^!(\mathbb{Q}_\ell)(\beta) \rightarrow \omega_{X_s}(\beta)$ . This produces a sequence of morphisms

$$\nu_X^I(\beta) \longrightarrow \mathbb{Q}_\ell(\beta) \longrightarrow i_X^!(\mathbb{Q}_\ell)[2](\beta) = i_X^!(\mathbb{Q}_\ell)(\beta) \longrightarrow \omega_{X_s}(\beta).$$

The resulting composite morphism  $\mathbb{Q}_\ell(\beta) \rightarrow \omega_{X_s}(\beta)$  is the 2-periodic  $\ell$ -adic fundamental class of  $X_s$ . Moreover, by construction, the composition  $\nu_X^I(\beta) \rightarrow \mathbb{Q}_\ell(\beta) \rightarrow \omega_{X_s}(\beta)$  is canonically the zero map, and this induces the natural morphism

$$\alpha_X : \nu_X^I(\beta)[1] \rightarrow \omega_{X_s}^\circ$$

we were looking for. Summing up, the morphism  $\alpha_X$  always exists for any proper, lci scheme  $X$  over  $S$ , and is an equivalence whenever  $X$  is regular.

### 3.4 A Künneth theorem for invariant vanishing cycles

In this section, we consider two separated, finite type  $S$ -schemes  $X$  and  $Y$ , such that  $X_K$  and  $Y_K$  are smooth over  $K$ , and both  $X$  and  $Y$  are regular and connected. For simplicity<sup>11</sup> we also assume that  $X$  and  $Y$  are flat over  $S$ . We set  $Z := X \times_S Y$ , and consider this as a scheme over  $S$ . We have  $Z_s \simeq X_s \times_s Y_s$ , and the  $\infty$ -category  $D_c^I(Z_s, \mathbb{Q}_\ell)$  comes equipped with pull-back functors

$$p^* : D_c^I(X_s, \mathbb{Q}_\ell) \longrightarrow D_c^I(Z_s, \mathbb{Q}_\ell) \longleftarrow D_c^I(Y_s, \mathbb{Q}_\ell) : q^*.$$

<sup>10</sup>Note that a morphism of finite type between regular schemes is lci, since we can check that its relative cotangent complex has perfect amplitude in  $[-1, 0]$ .

<sup>11</sup>In the non-flat case, the fiber product of  $X$  and  $Y$  over  $S$ , to be considered below, should be replaced by the derived fiber product.

By taking their tensor product, we get an external product functor

$$\boxtimes := p^*(-) \otimes q^*(-) : D_c^I(X_s, \mathbb{Q}_\ell) \times D_c(Y_s, \mathbb{Q}_\ell) \longrightarrow D_c^I(Z_s, \mathbb{Q}_\ell).$$

For two objects  $E \in D_c^I(X_s, \mathbb{Q}_\ell)$  and  $F \in D_c^I(Y_s, \mathbb{Q}_\ell)$ , we can define the Künneth morphism in  $D_c(Z_s, \mathbb{Q}_\ell)$

$$k : (E \boxtimes F)^I[-1] \longrightarrow E^I \boxtimes F^I$$

as follows. Since Grothendieck duality on  $Z_s$  is compatible with external products, in order to define  $k$  it is enough to define its dual

$$\mathbb{D}(E^I) \boxtimes \mathbb{D}(F^I) \longrightarrow \mathbb{D}((E \boxtimes F)^I)[1].$$

By Lemma 3.2.3, the datum of such a morphism is equivalent to that of a morphism

$$\mathbb{D}_I(E)^I \boxtimes \mathbb{D}_I(F)^I \longrightarrow \mathbb{D}_I(E \boxtimes F)^I.$$

We now define  $k$  as the map induced by the composite

$$\mathbb{D}_I(E)^I \boxtimes \mathbb{D}_I(F)^I \xrightarrow{\mu_{(-)^I}} (\mathbb{D}_I(E) \boxtimes \mathbb{D}_I(F))^I \xrightarrow{(\mu_{D_I})^I} \mathbb{D}_I(E \boxtimes F)^I$$

where  $\mu_{(-)^I}$  is the lax monoidal structure on  $(-)^I$ , and  $\mu_{D_I}$  the one on  $D_I$ <sup>12</sup>.

**Definition 3.4.1** *With the above notations, the I-invariant convolution of the two objects  $E \in D_c^I(X_s, \mathbb{Q}_\ell)$  and  $F \in D_c^I(Y_s, \mathbb{Q}_\ell)$  is defined to be the cone of the Künneth morphism, and denoted by  $(E \circledast F)^I$ . By definition it sits in a triangle*

$$(E \boxtimes F)^I[-1] \xrightarrow{k} E^I \boxtimes F^I \longrightarrow (E \circledast F)^I.$$

The main result of this section is the following proposition, relating the I-invariant convolution of vanishing cycles on  $X$  and  $Y$  to the dualizing complex of  $Z$ . It can be also considered as a computation of the  $\ell$ -adic realization of the  $\infty$ -category  $\mathbf{Sing}(Z)$  of singularities of  $Z$ .

Note that, as  $X$  and  $Y$  are generically smooth over  $S$ , so is  $Z$ , and thus the 2-periodic  $\ell$ -adic fundamental class map  $\eta_Z : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_Z(\beta)$  of Lemma 3.3.2 is an equivalence over the generic fiber. Therefore,  $\omega_Z^o$  is supported on  $Z_s$ , so that it can (and will) be considered canonically as an object in  $\mathcal{D}_c(Z_s, \mathbb{Q}_\ell)$ .

**Theorem 3.4.2** *With the above notations and assumptions, there is a canonical equivalence*

$$\omega_Z^o \simeq (\nu_X \circledast \nu_Y)^I(\beta)$$

in  $D_c(Z_s, \mathbb{Q}_\ell(\beta))$ .

**Proof.** The proof of this theorem will combine various exact triangles together with an application of Gabber's Künneth formula for nearby cycles.

To start with, the vanishing cycles  $\nu_Z$  of  $Z$  sits in an exact triangle in  $\mathcal{D}_c^I(Z_s, \mathbb{Q}_\ell)$

$$\mathbb{Q}_\ell \longrightarrow \psi_Z \longrightarrow \nu_Z,$$

---

<sup>12</sup>Note that  $\mu_{D_I}$  is in fact an equivalence.

where  $\psi_Z = \psi_Z(\mathbb{Q}_\ell)$  is the complex of nearby cycles of  $Z$  over  $S$ . According to [Be-Be, Lemma 5.1.1] or [Il1, 4.7]), we have a natural equivalence in  $\mathcal{D}_c^I(Z_s, \mathbb{Q}_\ell)$ , induced by external product

$$\psi_Z \simeq \psi_X \boxtimes \psi_Y.$$

The object  $\nu_Z$  then becomes the cone of the tensor product of the two morphisms in  $\mathcal{D}_c^I(Z_s, \mathbb{Q}_\ell)$

$$\mathbb{Q}_\ell \longrightarrow p^*(\psi_X) \quad \mathbb{Q}_\ell \longrightarrow q^*(\psi_Y)$$

where  $p$  and  $q$  are the two projections from  $Z$  down to  $X$  and  $Y$ , respectively. Now, cones of tensor products are computed via the following well known lemma.

**Lemma 3.4.3** *Let  $\mathcal{C}$  be a stable symmetric monoidal  $\infty$ -category, and*

$$u : x \rightarrow y \quad v : x' \rightarrow y'$$

*two morphisms. Let  $C(u)$  be the cone of  $u$ ,  $C(v)$  be the cone of  $v$ , and  $C(u \otimes v)$  the cone of the tensor product  $u \otimes v : x \otimes x' \rightarrow y \otimes y'$ . Then, there exists a natural exact triangle*

$$C(u) \otimes x' \oplus x \otimes C(v) \longrightarrow C(u \otimes v) \longrightarrow C(u) \otimes C(v).$$

*Proof of lemma.* Factor  $u \otimes v$  as  $x \otimes x' \xrightarrow{u \otimes \text{id}} y \otimes x' \xrightarrow{\text{id} \otimes v} y \otimes y'$ , and apply the octahedral axiom to the triangles

$$\begin{aligned} x \otimes x' &\xrightarrow{u \otimes \text{id}} y \otimes x' \xrightarrow{f} C(u) \otimes x' \\ y \otimes x' &\xrightarrow{\text{id} \otimes v} y \otimes y' \longrightarrow y \otimes C(v) \xrightarrow{d'}_{[1]} y \otimes x'[1], \end{aligned}$$

to get a triangle

$$C(u) \otimes x' \longrightarrow C(u \otimes v) \longrightarrow y \otimes C(v) \xrightarrow{\theta}_{[1]} C(u) \otimes x'[1]$$

together with the compatibility  $\theta = f[1] \circ d'$ . Now observe that  $\theta \circ (u \otimes \text{id}_{C(v)}) = 0$ , and apply the octahedral axiom to the triangles

$$\begin{aligned} x \otimes C(v) &\xrightarrow{u \otimes \text{id}} y \otimes C(v) \longrightarrow C(u) \otimes C(v), \\ y \otimes C(v) &\xrightarrow{\theta} C(u) \otimes x'[1] \xrightarrow{f} C(u \otimes v)[1] \end{aligned}$$

to conclude. ◇

Lemma 3.4.3 implies the existence of a natural exact triangle in  $\mathcal{D}_c^I(Z_s, \mathbb{Q}_\ell)$

$$\nu_X \boxplus \nu_Y \longrightarrow \nu_Z \longrightarrow \nu_X \boxtimes \nu_Y,$$

which, by taking I-invariants, yields an exact triangle in  $\mathcal{D}_c(Z_s, \mathbb{Q}_\ell)$

$$\mathbf{(T1)} \quad \nu_X^I \boxplus \nu_Y^I \longrightarrow \nu_Z^I \longrightarrow (\nu_X \boxtimes \nu_Y)^I.$$

**Lemma 3.4.4** *Let  $k$  be an algebraically closed field,  $s := \text{Spec } k$ ,  $p_X : X \rightarrow s$ ,  $p_Y : Y \rightarrow s$  be proper morphisms of schemes, and  $p_1 : Z := X \times_s Y \rightarrow X$ ,  $p_2 : Z := X \times_s Y \rightarrow Y$  the natural projections. If  $\omega_Z, \omega_X, \omega_Y$  denote the  $\mathbb{Q}_\ell$ -adic dualizing complexes of  $Z, X$ , and  $Y$ , respectively, there is a canonical equivalence*

$$a : p_1^* \omega_X \otimes p_2^* \omega_Y \longrightarrow \omega_Z.$$

*Proof of lemma.* We first exhibit the map  $a$ . We denote simply by  $\underline{\text{Hom}}_T(-, -)$  the derived internal hom in  $\mathcal{D}_c(T, \mathbb{Q}_\ell)$  (so that  $D := \underline{\text{Hom}}_Z(-, \omega_Z)$  is the  $\mathbb{Q}_\ell$ -adic duality on  $Z$ ). By adjunction, giving a map  $a$  is the same thing as giving a map  $\omega_X \rightarrow (p_1)_* \underline{\text{Hom}}_Z(p_2^* \omega_Y, p_2^! \omega_Y)$ . Since  $p_X$  is proper, by [SGA4-III, Exp XVIII, 3.1.12], we have a canonical equivalence

$$(p_1)_* \underline{\text{Hom}}_Z(p_2^* \omega_Y, p_2^! \omega_Y) \longrightarrow p_X^! (p_Y)_* \underline{\text{Hom}}_Y(\omega_Y, \omega_Y).$$

Therefore, we are left to define a map

$$\omega_X \simeq p_X^! \mathbb{Q}_\ell \rightarrow p_X^! (p_Y)_* \underline{\text{Hom}}_Y(\omega_Y, \omega_Y),$$

and we take  $p_X^!(\alpha)$  for this map, where  $\alpha$  is the adjoint to the canonical map  $\mathbb{Q}_\ell \simeq p_Y^* \mathbb{Q}_\ell \rightarrow \underline{\text{Hom}}_Y(\omega_Y, \omega_Y)$ . One can then prove that  $a$  is an equivalence, by checking it stalkwise.  $\square$

By Lemma 3.4.4, the dualizing complex  $\omega_{Z_s} = (Z_s \rightarrow s)^! \mathbb{Q}_\ell$  of  $Z_s \simeq X_s \times_s Y_s$  is canonically equivalent to  $\omega_{X_s} \boxtimes \omega_{Y_s}$ , and, through this equivalence, the virtual fundamental class of  $Z_s$

$$\eta_{Z_s} : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_{Z_s}(\beta) \simeq \omega_{X_s} \boxtimes \omega_{Y_s}(\beta)$$

is simply given by the external tensor product of the virtual fundamental classes of  $X_s$  and  $Y_s$ . By Lemma 3.4.3 we thus get a second exact triangle in  $\mathcal{D}_c(Z_s, \mathbb{Q}_\ell(\beta))$

$$\text{(T2)} \quad \omega_{X_s}^o \boxplus \omega_{Y_s}^o \longrightarrow \omega_{Z_s}^o \longrightarrow \omega_{X_s}^o \boxtimes \omega_{Y_s}^o.$$

There is a morphism from the triangle (T1) to the triangle (T2) which is defined using the natural morphism

$$\alpha_Z : \nu_Z^I(\beta)[1] \longrightarrow \omega_{Z_s}^o$$

introduced in Remark 3.3.7. In fact,  $Z$  is proper and lci over  $S$  (since  $X/S$  is flat and lci<sup>13</sup>, and being lci is stable under flat base change and composition), and the map  $\alpha_Z$  is defined for any proper, lci scheme  $Z$  over  $S$ , being an equivalence when  $Z$  is regular with smooth generic fiber. Using the compatible maps  $\alpha_X, \alpha_Y$  and  $\alpha_Z$  we get a commutative square

$$\begin{array}{ccc} \nu_X^I(\beta)[1] \boxplus \nu_Y^I(\beta)[1] & \longrightarrow & \nu_Z^I(\beta)[1] \\ \alpha_Z \oplus \alpha_Y \downarrow & & \downarrow \alpha_Z \\ \omega_{X_s}^o \boxplus \omega_{Y_s}^o & \longrightarrow & \omega_{Z_s}^o. \end{array}$$

<sup>13</sup>Again, note that a morphism of finite type between regular schemes is lci, since we can check that its relative cotangent complex has perfect amplitude in  $[-1, 0]$ .

This produces a morphism from triangle (T1) (tensoried by  $\mathbb{Q}_\ell(\beta)[1]$ ) to triangle (T2)

$$\begin{array}{ccccc} \nu_X^I(\beta)[1] \boxplus \nu_Y^I(\beta)[1] & \longrightarrow & \nu_Z^I(\beta)[1] & \longrightarrow & (\nu_X \boxtimes \nu_Y)^I(\beta)[1] \\ \downarrow & & \downarrow \alpha_Z & & \downarrow \\ \omega_{X_s}^o \boxplus \omega_{Y_s}^o & \longrightarrow & \omega_{Z_s}^o & \longrightarrow & \omega_{X_s}^o \boxtimes \omega_{Y_s}^o. \end{array} \quad (7)$$

Since  $X$  and  $Y$  are regular with smooth generic fibers, the maps  $\alpha_X$  and  $\alpha_Y$  are equivalences, therefore the leftmost vertical morphism is also an equivalence. Thus the right hand square is a cartesian square.

Now, the rightmost vertical morphism can be written, again using the equivalences  $\alpha_X$  and  $\alpha_Y$ , as

$$(\nu_X \boxtimes \nu_Y)^I(\beta)[1] \longrightarrow (\nu_X^I(\beta)[1]) \boxtimes_{\mathbb{Q}_\ell(\beta)} (\nu_Y^I(\beta)[1]) \simeq (\nu_X^I \boxtimes \nu_Y^I)[2](\beta)$$

This morphism is the Künneth map  $k$  of Definition 3.4.1 tensoried by  $\mathbb{Q}_\ell[2](\beta) \simeq \mathbb{Q}_\ell(\beta)$ , and thus its cone is  $(\nu_X \otimes \nu_Y)^I(\beta)$ . In order to finish the proof of the proposition it then remains to show that the cone of the middle vertical morphism in (7)

$$\alpha_Z : \nu_Z^I(\beta)[1] \longrightarrow \omega_{Z_s}^o$$

can be canonically identified with  $\omega_Z^o$ .

For this, we recall the exact triangle (3) tensoried by  $\mathbb{Q}_\ell(\beta)$

$$\mathbb{Q}_\ell(\beta) \longrightarrow i_Z^!(\mathbb{Q}_\ell(\beta)) \longrightarrow \nu_Z^I(\beta)[1].$$

Using the fundamental class  $\eta_Z : \mathbb{Q}_\ell(\beta) \rightarrow \omega_Z(\beta)$ , we get a morphism of triangles

$$\begin{array}{ccccc} \mathbb{Q}_\ell(\beta) & \longrightarrow & i_Z^!(\mathbb{Q}_\ell(\beta)) & \longrightarrow & \nu_Z^I(\beta)[1] \\ id \downarrow & & \downarrow i_Z^!(\eta_Z) & & \downarrow \\ \mathbb{Q}_\ell(\beta) & \longrightarrow & i_Z^!(\omega_Z(\beta)) & \longrightarrow & \omega_{Z_s}^o. \end{array}$$

The right hand square is thus cartesian, so that the cone of the vertical morphism on the right is canonically identified with the cone of the vertical morphism in the middle. By definition, this cone is  $i_Z^!(\omega_Z^o)$ . Since  $Z_K$  is smooth over  $K$ , the  $\ell$ -adic complex  $\omega_Z^o$  is supported on  $Z_s$ , and thus  $i_Z^!(\omega_Z^o)$  is canonically equivalent to  $\omega_{Z_s}^o$ , and we conclude.  $\square$

**Corollary 3.4.5** *We keep the same notations and assumptions as in Proposition 3.4.2, and we further assume one of the following conditions:*

1. *the I-action on  $\nu_X$  and on  $\nu_Y$  is tame, or*
2. *the reduced scheme  $(X_s)_{red}$  is smooth over  $k$ .*

*Then, there is a canonical equivalence*

$$\omega_Z^o \simeq (\nu_X \boxtimes \nu_Y)^I(\beta)$$

*in  $\mathcal{D}_c(Z_s, \mathbb{Q}_\ell(\beta))$ .*

**Proof.** It is enough to prove that under any one of the two assumptions,  $(\nu_X \otimes \nu_Y)^I$  is canonically equivalent to  $(\nu_X \boxtimes \nu_Y)^I$ . If the scheme  $(X_s)_{red}$  is smooth over  $k$ , then we have  $\nu_X^I(\beta) = 0$ . Indeed, triangle (3) can be then re-written

$$\nu_X^I \longrightarrow \mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell[2n+2]$$

where  $n$  is the dimension of  $X_s$ . By tensoring by  $\mathbb{Q}_\ell(\beta)$ , we get a triangle

$$\nu_X^I(\beta) \longrightarrow \mathbb{Q}_\ell(\beta) \xrightarrow{b} \mathbb{Q}_\ell[2n+2](\beta) \simeq \mathbb{Q}_\ell(\beta)$$

where  $b$  is an equivalence. Therefore,  $\nu_X^I(\beta) = 0$ , and, by definition of I-invariant convolution, this implies that  $(\nu_X \otimes \nu_Y)^I \simeq (\nu_X \boxtimes \nu_Y)^I$ .

Assume now that the action of I on  $\nu_X$  and  $\nu_Y$  is tame. This means that the action of I factors through the natural quotient  $I \rightarrow I_t$ , where  $I_t$  is the tame inertia group, which is canonically isomorphic to  $\hat{\mathbb{Z}}'$ , the prime-to- $p$  part of the profinite completion of  $\mathbb{Z}$ . As we have chosen a topological generator  $T$  of  $I_t$  (see Section 3.1), the actions of I are then completely characterized by the automorphisms  $T$  on  $\nu_X$  and  $\nu_Y$ . Moreover,  $\nu_X^I$  is then naturally equivalent to the homotopy fiber of  $(1 - T) : \nu_X \rightarrow \nu_X$ , and similarly for  $\nu_Y^I$ . From this it is easy to see that the Künneth map

$$(\nu_X \boxtimes \nu_Y)^I[-1] \longrightarrow \nu_X^I \boxtimes \nu_Y^I$$

fits in an exact triangle

$$(\nu_X \boxtimes \nu_Y)^I[-1] \longrightarrow \nu_X^I \boxtimes \nu_Y^I \longrightarrow (\nu_X \boxtimes \nu_Y)^I$$

where the second morphism is induced by the lax monoidal structure on  $(-)^I$ . We conclude that there is a natural equivalence  $(\nu_X \otimes \nu_Y)^I \simeq (\nu_X \boxtimes \nu_Y)^I$ .  $\square$

## 4 Künneth formula for dg-categories of singularities

### 4.1 The monoidal dg-category $\mathcal{B}$ and its action

We keep our standing assumptions:  $A$  is an excellent strictly henselian dvr with perfect residue field  $k$  and fraction field  $K$ . We let  $S = \text{Spec } A$  and  $s = \text{Spec } k$ , as usual, and choose a uniformizer  $\pi$  of  $A$ .

We let  $G := s \times_S^h s$  (derived fiber product), considered as a derived scheme over  $S$ . The derived scheme  $G$  has a canonical structure of groupoid in derived schemes acting on  $s$ . The composition in the groupoid  $G$  induces a convolution monoidal structure on the dg-category of coherent complexes on  $G$

$$\odot : \text{Coh}^b(G) \otimes_A \text{Coh}^b(G) \longrightarrow \text{Coh}^b(G).$$

More explicitly, we have a map of derived schemes

$$G \times_s G \xrightarrow{q} G,$$

defined as the projection on the first and third components  $s \times_S s \times_S s \rightarrow s \times_S s$ . We then define  $\odot$  by the formula

$$E \odot F := q_*(E \boxtimes_s F)$$

for two coherent complexes  $E$  and  $F$  on  $G$ . More generally, if  $X \rightarrow S$  is any scheme, with special fiber  $X_s$  (possibly a derived scheme, by taking the derived fiber at  $s$ ), the groupoid  $G$  acts naturally on  $X_s$  via the natural projection

$$q_X : G \times_s X_s \simeq (s \times_S s) \times_s (s \times_S X) \simeq s \times_S X_s \rightarrow X_s.$$

This defines an external action

$$\odot : \mathbf{Coh}^b(G) \otimes_A \mathbf{Coh}^b(X_s) \rightarrow \mathbf{Coh}^b(X_s)$$

by  $E \odot M := (q_X)_*(E \boxtimes_s M)$ .

The homotopy coherences issues for the above  $\odot$ -structures can be handled using the fact that the construction  $Y \mapsto \mathbf{Coh}^b(Y)$  is in fact a symmetric lax monoidal  $\infty$ -functor from a certain  $\infty$ -category of correspondences between derived schemes to the  $\infty$ -category of dg-categories over  $A$ . As a result,  $\mathbf{Coh}^b(G)$  is endowed with a natural structure of a monoid in the symmetric monoidal  $\infty$ -category  $\mathbf{dgCat}_A$ , and that, for any  $X/S$ ,  $\mathbf{Coh}^b(X_s)$  is naturally a module over  $\mathbf{Coh}^b(G)$  in  $\mathbf{dgCat}_A$ . However, for our purposes it will be easier and more efficient to provide explicit models for both  $\mathbf{Coh}^b(G)$  and its action on  $\mathbf{Coh}^b(X_s)$ . This will be done locally in the Zariski topology in a similar spirit to [BRTV, Section 2]; the global construction will then be obtained by a rather straightforward gluing procedure.

#### 4.1.1 The monoidal dg-categories $\mathcal{B}^+$ and $\mathcal{B}$

Let  $K_A$  be the Koszul commutative  $A$ -dg-algebra of  $A$  with respect to  $\pi$

$$K_A : A \xrightarrow{\pi} A$$

sitting in degrees  $[-1, 0]$ . The canonical generator of  $K_A$  in degree  $-1$  will be denoted by  $h$ . In the same way, we define the commutative  $A$ -dg-algebra

$$K_A^2 := K_A \otimes_A K_A$$

which is the Koszul dg-algebra of  $A$  with respect to the sequence  $(\pi, \pi)$ . As a commutative graded  $A$ -algebra,  $K_A^2$  is  $\mathit{Sym}_A(A^2[1])$ , and it is endowed with the unique multiplicative differential sending the two generators  $h$  and  $h'$  in degree  $-1$  to  $\pi$  (and  $hh'$  to  $\pi \cdot h' - \pi \cdot h$ ).

Moreover,  $K_A^2$  has a canonical structure of *Hopf algebroid* over  $K_A$ , in which the source and target map are the two natural inclusions  $K_A \rightarrow K_A^2$ , whereas the unit is given by the multiplication  $K_A^2 \rightarrow K_A$ . The composition (or coproduct) in this Hopf algebroid structure is given by

$$\Delta := \text{id} \otimes 1 \otimes \text{id} : K_A \otimes_A K_A = K_A^2 \rightarrow K_A^2 \otimes_{K_A} K_A^2 = K_A \otimes_A K_A \otimes_A K_A.$$

Finally, the antipode is the automorphism of  $K_A^2$  exchanging the two factors  $K_A$ . This structure of Hopf algebroid endows the dg-category  $\mathbf{Mod}(K_A^2)$ , of  $K^2(A)$ -dg-modules, with a unital and associative monoidal structure  $\odot$ . It is explicitly given for two object  $E$  and  $F$ , by the formula

$$E \odot F := E \otimes_{K_A} F$$

where the  $K_A \otimes_A K_A \otimes_A K_A$ -module on the rhs is considered as a  $K_A^2$  module via the map  $\Delta$ . The unit of this monoidal structure is the object  $K_A$ , viewed as a  $K_A^2$ -module by the multiplication  $K_A^2 \rightarrow K_A$ . It is not hard to see that  $\odot$  preserves cofibrant  $K_A^2$ -dg-modules; more generally it makes  $\mathbf{Mod}(K_A^2)$  into a monoidal model category in the sense of [Ho, Ch. 4]. Note however that the unit  $K_A$  is *not* cofibrant in this model structure.

**Definition 4.1.1** *The monoidal dg-category  $\mathcal{B}_{str}^+$  is defined to be  $\mathbf{Mod}^c(K_A^2)$ , the dg-category of all cofibrant dg-modules over  $K_A^2$  which are perfect over  $A$ , together with the unit object  $K_A$ . It is endowed with the monoidal structure  $\odot$  described above.*

By Appendix A, the localization of the dg-category  $\mathcal{B}_{str}^+$  along all quasi-isomorphisms, defines a monoidal dg-category.

**Definition 4.1.2** *The monoidal dg-category  $\mathcal{B}^+$  is defined to be the localization*

$$W_{eq}^{-1}(\mathcal{B}_{str}^+),$$

where  $W_{eq}$  is the set of quasi-isomorphisms. It is naturally a unital and associative monoid in the symmetric monoidal  $\infty$ -category  $\mathbf{dgCat}_A$ .

**Remark 4.1.3** Note that  $\mathcal{B}^+$  defined above is a model for  $\mathbf{Coh}^b(G)$ , for our derived groupoid  $G = s \times_S s$  above. Indeed, the commutative dga  $K_A^2$  is quasi-isomorphic to the normalization of the simplicial algebra  $k \otimes_A^{\mathbb{L}} k$ . Now, since  $G \rightarrow S$  is a closed immersion, a quasi-coherent complex  $E$  on  $G$  is coherent iff its direct image on  $S$  is coherent, hence perfect,  $S$  being regular. In particular, we have that  $\mathbf{Coh}^b(G)$  is equivalent to the dg-category of all cofibrant  $K_A^2$ -dg-modules which are perfect over  $A$ . The latter dg-category is also naturally equivalent to the localization of  $\mathcal{B}_{str}^+$  along quasi-isomorphisms<sup>14</sup>

We now introduce the weak monoidal dg-category  $\mathcal{B}$ , defined as a further localization of  $\mathcal{B}^+$ . This will be our main “base monoid” for the module dg-categories we will be interested in.

**Definition 4.1.4** *The weak monoidal dg-category  $\mathcal{B}$  is defined to be the localization*

$$\mathcal{B} := L_W(\mathcal{B}^+),$$

where  $W$  is the set of morphisms in  $\mathcal{B}^+$  whose cone is perfect as a  $K_A^2$ -dg-module.

As  $\mathcal{B}^+$  is itself defined as a localization of  $\mathcal{B}_{str}^+$ , if  $W_{pe}$  denotes the morphisms in  $\mathcal{B}_{str}^+$  whose cones are perfect over  $K_A^2$ , we have  $W_{eq} \subseteq W_{pe}$ , then  $\mathcal{B}$  can also be realized as localization of  $\mathcal{B}_{str}^+$  directly, as

$$\mathcal{B} \simeq L_{W_{pe}} \mathcal{B}_{str}^+.$$

Since the monoidal structure  $\odot$  is compatible with  $W_{pe}$ , this presentation implies that  $\mathcal{B}$  comes equipped with a natural structure of an associative and unital monoid in  $\mathbf{dgCat}_A$ . Note, moreover, that  $\mathcal{B}$  comes equipped with a natural morphism of monoids

$$\mathcal{B}^+ \longrightarrow \mathcal{B}$$

given by the localization map.

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<sup>14</sup>We leave to the reader to check that adding the unit object  $K_A$  does not change the localization.

### 4.1.2 The local actions

Let now  $X = \text{Spec } R$  be a regular scheme flat over  $A$ . As done above for the monoid structure on  $\mathcal{B}^+$ , we will define a *strict model* for  $\text{Coh}^b(X_s)$ , together with a *strict model* for the  $\text{Coh}^b(G)$ -action on  $\text{Coh}^b(X_s)$ . In order to do this, let  $K_R$  be the Koszul dg-algebra of  $R$  with respect to  $\pi$ , which comes equipped with a natural map  $K_A \rightarrow K_R$  of cdga's over  $A$ . We consider  $\text{Mod}^c(K_R)$ , the dg-category of all cofibrant  $K_R$ -dg-modules which are perfect as  $R$ -modules (note that  $R$  is regular, and see Remark 4.1.3). The same argument as in Remark 4.1.3 then shows that this dg-category is naturally equivalent to  $\text{Coh}^b(X_s)$ . Moreover,  $\text{Mod}^c(K_R)$  has a structure of a  $\mathcal{B}_{str}^+$ -module dg-category defined as follows. For  $E \in \mathcal{B}_{str}^+$ , and  $M \in \text{Mod}^c(K_R)$ , we can define

$$E \odot M := E \otimes_{K_A} M,$$

where, in the rhs, we have used the ‘‘right’’  $K_A$ -dg-module structure on  $E$ , i.e. the one induced by the composition

$$K_A \xrightarrow{\sim} A \otimes_A K_A \xrightarrow{\text{id} \otimes u} K_A \otimes_A K_A,$$

$u : A \rightarrow K_A$  being the canonical map. As  $E$  is either the unit or it is cofibrant over  $K_A^2$  (and thus cofibrant over  $K_A$ ),  $E \otimes_{K_A} M$  is again a cofibrant  $K_R$ -module, and again perfect over  $R$ , i.e.  $E \odot M \in \text{Mod}^c(K_R)$ . By localization along quasi-isomorphisms, (see Proposition 4.1.5 for details) we obtain that  $\text{Coh}^b(X_s)$  carries a natural  $\mathcal{B}^+$ -module structure as an object in the symmetric monoidal  $\infty$ -category  $\mathbf{dgCat}_A$ .

We now apply a similar argument in order to define a  $\mathcal{B}$ -action on  $\text{Sing}(X_s)$ . Let again  $X = \text{Spec } R$  be a regular scheme over  $A$ , and consider  $\text{Mod}^c(K_R)$  as a  $\mathcal{B}_{str}^+$ -module dg-category as above. Let  $W_{R,pe}$  be the set of morphisms in  $\text{Mod}^c(K_R)$  whose cones are perfect dg-modules over  $K_R$ . By localization we then get a  $\mathcal{B}$ -module structure on  $L_{W_{R,pe}} \text{Mod}^c(K_R)$ . Note that the localization  $L_{W_{R,pe}} \text{Mod}^c(K_R)$  is a model for the dg-category  $\text{Sing}(X_s)$ , which therefore comes equipped with the structure of a  $\mathcal{B}$ -module in  $\mathbf{dgCat}_A$ .

We gather the details of above constructions in the following

**Proposition 4.1.5** *Let  $X = \text{Spec } R$  be a regular scheme, flat over  $S = \text{Spec } A$ , and  $X_s$  its special fiber. Then there is a canonical  $\mathcal{B}^+$ -module structure (resp.,  $\mathcal{B}$ -module structure) on  $\text{Coh}^b(X_s)$  (resp., on  $\text{Sing}(X_s)$ ), inside  $\mathbf{dgCat}_A$ .*

**Proof.** This is an easy application of the localization results presented in Appendix A.

We first treat the case of  $\mathcal{B}^+$  and  $T := \text{Coh}^b(X_s)$ . If  $T^{\text{str}} := \text{Mod}^c(X_r)$  and  $W_{T,eq}$  denotes the quasi-isomorphisms in  $T^{\text{str}}$ , we have  $W_{T,eq}^{-1} T^{\text{str}} \simeq T$  in  $\mathbf{dgCat}_A$ . Analogously,  $W_{eq}^{-1} \mathcal{B}_{str}^+ \simeq \mathcal{B}^+$  in  $\mathbf{dgCat}_A$ . In order to apply the localization result of Appendix A, we need to prove that the tensor product  $\odot : \mathcal{B}_{str}^+ \otimes_A T^{\text{str}} \rightarrow T^{\text{str}}$  (defined in 4.1.2) sends  $W_{eq} \otimes id \cup id \otimes W_{T,eq}$  to  $W_{T,eq}$ . If  $L, L' \in \mathcal{B}_{str}^+$ ,  $E, E' \in T^{\text{str}}$ , and  $w' : L \rightarrow L'$ ,  $w : E \rightarrow E'$  are quasi-isomorphisms, then  $w' \odot id_E$  is again a quasi-isomorphism (because  $L$ , and  $L'$  are cofibrant over  $K_A^2$ , hence over  $K_A$ , and thus  $w'$  is in fact a homotopy equivalence), and the same is true for  $id_L \odot w$  (since  $L$  is cofibrant over  $K_A$ ). Therefore,  $\odot$  does send  $W_{eq} \otimes id \cup id \otimes W_{T,eq}$  to  $W_{T,eq}$ , and there is an induced canonical map  $(W_{eq} \otimes id \cup id \otimes W_{T,eq})^{-1} \mathcal{B}_{str}^+ \otimes_A T^{\text{str}} \rightarrow W_{T,eq}^{-1} T^{\text{str}} \simeq \text{Coh}^b(X_s)$ . By composing this with the natural equivalence  $(W_{eq} \otimes id \cup id \otimes W_{T,eq})^{-1} \mathcal{B}_{str}^+ \otimes_A T^{\text{str}} \rightarrow W_{eq}^{-1} \mathcal{B}_{str}^+ \otimes_A W_{T,eq}^{-1} T^{\text{str}}$  (Appendix A), we finally get our  $\mathcal{B}^+$ -module structure on  $\text{Coh}^b(X_s)$  inside  $\mathbf{dgCat}_A$ .

We now treat the case of  $\mathcal{B}$  and  $T = \text{Sing}(X_s)$ . Here, we consider the pairs  $(T^{\text{str}} = \text{Mod}^c(X_r), W_T)$  where  $W_T$  are the maps in  $T^{\text{str}}$  whose cones are perfect over  $K_R$ , and  $(\mathcal{B}_{\text{str}}^*, W)$ , where  $W$  are the maps in  $\mathcal{B}_{\text{str}}^+$  whose cones are perfect over  $K_A^2$ . We have  $W^{-1}\mathcal{B}_{\text{str}}^* \simeq \mathcal{B}$ , and  $W_T^{-1}T^{\text{str}} \simeq \text{Sing}(X_s)$ , and we need to prove that both  $W \odot id$  and  $id \odot W_T$  are contained in  $W_T$ . Let  $u : L \rightarrow L' \in W$  and  $v : E \rightarrow E' \in W_T$ , and  $C(-)$  denote the cone construction. We have  $C(id_L \odot v) \simeq L \otimes_{K_A} C(v)$  and  $C(u \odot id_E) \simeq C(u) \otimes_{K_A} E$ . By hypothesis,  $L$  is perfect over  $A$  hence over  $K_A$  (since  $K_A$  is perfect over  $A$ ), and  $C(v)$  is perfect over  $K_A$ , since  $X \rightarrow S$  is lci (as a map of finite type between regular schemes), and thus  $K_A \rightarrow K_R$  is derived lci (recall that  $X/S$  is flat so that  $X_s$  is also the derived fiber), and pushforward along a lci map preserves perfect complexes. So,  $C(id_L \odot v) \in W_T$ . On the other hand, the ‘‘right-hand’’ map  $K_A \rightarrow K_A^2$  (with respect to which  $L$  and  $L'$  are viewed as  $K_A$ -dg-modules in the definition of  $\odot$ ) is derived lci, hence  $C(u)$  is perfect over  $K_A$ , being perfect over  $K_A^2$  by hypothesis. Moreover, since  $s \rightarrow S$  is a closed immersion,  $E$  is perfect over  $K_A$  iff  $(X \rightarrow S)_*E$  is perfect (= coherent,  $S$  being regular) over  $S$ ; but  $(X_s \rightarrow X)_*E$  is perfect by hypothesis, and pushforward along  $X \rightarrow S$  preserves perfect complexes, since  $X/S$  is lci. Therefore  $C(u \odot id_E) \in W_T$ , and we deduce that  $\odot$  does send  $W \otimes id \cup id \otimes W_T$  to  $W_T$ . This gives us an induced canonical map  $(W \otimes id \cup id \otimes W_T)^{-1}\mathcal{B}_{\text{str}}^+ \otimes_A T^{\text{str}} \rightarrow W_T^{-1}T^{\text{str}} \simeq \text{Sing}(X_s)$ . By composing this with the natural equivalence  $(W \otimes id \cup id \otimes W_T)^{-1}\mathcal{B}_{\text{str}}^+ \otimes_A T^{\text{str}} \rightarrow W^{-1}\mathcal{B}_{\text{str}}^+ \otimes_A W_T^{-1}T^{\text{str}}$  (Appendix A), we finally get our  $\mathcal{B}$ -module structure on  $\text{Sing}(X_s)$  inside  $\mathbf{dgCat}_A$ .  $\square$

**Lemma 4.1.6** *The natural morphism*

$$\text{Coh}^b(X_s)^o \otimes_{\mathcal{B}^+} \mathcal{B} \longrightarrow \text{Sing}(X_s)$$

*is an equivalence of  $\mathcal{B}$ -modules.*

**Proof.** This is a reformulation of [BRTV, Proposition 2.43].  $\square$

### 4.1.3 The global actions

We now let  $X$  be a regular scheme, not necessarily affine anymore. We have by Zariski descent

$$\text{Coh}^b(X_s) \simeq \lim_{\text{Spec } R \subset X} \text{Mod}^c(K_R),$$

where the limit is taken over all affine opens  $\text{Spec } R$  of  $X$ . The right hand side of the above equivalence is a limit of dg-categories underlying  $\mathcal{B}^+$ -module structures (Proposition 4.1.5). As the forgetful functor from  $\mathcal{B}^+$ -modules to dg-categories reflects limits, this endows  $\text{Coh}^b(X_s)$  with a unique structure of  $\mathcal{B}^+$ -module.

In the same way, we have Zariski descent for  $\text{Sing}(X_s)$  in the sense that

$$\text{Sing}(X_s) \simeq \lim_{\text{Spec } R \subset X} L_{W_{R,\text{pe}}}(\text{Mod}^c(K_R)).$$

The right hand side of the above equivalence is a limit of dg-categories underlying  $\mathcal{B}$ -module structures (Proposition 4.1.5). As the forgetful functor from  $\mathcal{B}$ -modules to dg-categories reflects limits, this makes the dg-category  $\text{Sing}(X_s)$  into a  $\mathcal{B}$ -module in a natural way.

An important property of these  $\mathcal{B}^+$  and  $\mathcal{B}$ -module structures is given in the following proposition.

**Proposition 4.1.7** *Let  $X$  be a flat regular scheme over  $S$ .*

1. *The  $\mathcal{B}^+$ -module structure on  $\mathrm{Coh}^b(X_s)$  is cotensored.*
2. *The  $\mathcal{B}$ -module structure on  $\mathrm{Sing}(X_s)$  is cotensored.*

**Proof.** This follows from Remark 2.1.4 since the monoidal triangulated dg-categories  $\mathcal{B}^+$  and  $\mathcal{B}$  are both generated by their unit objects.  $\square$

## 4.2 Künneth formula for dg-categories of singularities

In the previous section we have seen that, for any regular scheme  $X$  over  $S$ , the dg-category  $\mathrm{Sing}(X_s)$  are equipped with a natural  $\mathcal{B}$ -module structure. In this section we compute tensor products of dg-categories of singularities over  $\mathcal{B}$ .

From a general point of view, let  $T \in \mathbf{dgCat}_A$  be a  $\mathcal{B}$ -module, and assume that  $T$  is also *co-tensored* over  $\mathcal{B}$  (Def. 4.1.7). Then  $T^\circ$  has a natural structure of a  $\mathcal{B}^{\otimes -op}$ -module given by co-tensorisation. By Proposition 4.1.7, we may take  $T = \mathrm{Sing}(X_s)$ , so that  $T^\circ$  is a  $\mathcal{B}^{\otimes -op}$ -module. In particular, if we have another regular scheme  $Y$ , we are entitled to consider the tensor product

$$\mathrm{Sing}(X_s)^\circ \otimes_{\mathcal{B}} \mathrm{Sing}(Y_s),$$

which is a well defined object in  $\mathbf{dgCat}_A$ . We further assume, for simplicity, that  $X$  and  $Y$  are also flat over  $S$ . The main result of this section is the following proposition, which is a categorical counterpart of our Künneth formula for vanishing cycles (Proposition 3.4.2).

**Theorem 4.2.1** *Let  $X$  and  $Y$  be two regular schemes, flat over  $S$ , with smooth generic geometric fibers. There is a natural equivalence in  $\mathbf{dgCat}_A$*

$$\mathrm{Sing}(X_s)^\circ \otimes_{\mathcal{B}} \mathrm{Sing}(Y_s) \simeq \mathrm{Sing}(X \times_S Y).$$

**Proof.** As  $\mathcal{B}$  is a localization of  $\mathcal{B}^+$ , the natural  $\infty$ -functor on enriched dg-categories (which is lax symmetric monoidal)

$$\mathbf{dgCat}_{\mathcal{B}} \longrightarrow \mathbf{dgCat}_{\mathcal{B}^+}$$

is fully faithful. Moreover, it commutes with tensor products in the following sense: for two objects  $T$  and  $T'$  in  $\mathbf{dgCat}_{\mathcal{B}}$ , there is a natural equivalence  $T^\circ \otimes_{\mathcal{B}} T' \simeq T^\circ \otimes_{\mathcal{B}^+} T'$ . Therefore, in order to prove the theorem it enough to construct an equivalence of dg-categories over  $A$

$$\mathrm{Sing}(X_s)^\circ \otimes_{\mathcal{B}^+} \mathrm{Sing}(Y_s) \simeq \mathrm{Sing}(X \times_S Y).$$

Now, we claim that the result is local on  $Z := X \times_S Y$ . Indeed, we have two prestacks of dg-categories on the small Zariski site  $Z_{zar}$ :

$$U \times_S V \mapsto \mathrm{Sing}(U_s)^\circ \otimes_{\mathcal{B}} \mathrm{Sing}(V_s) \quad U \times_S V \mapsto \mathrm{Sing}(U \times_S V)$$

for any two affine opens  $U \subset X$  and  $V \subset Y$ . These two prestacks are stacks of dg-categories, and thus we have equivalences in  $\mathbf{dgCat}_A$

$$\mathrm{Sing}(X_s)^\circ \otimes_{\mathcal{B}} \mathrm{Sing}(Y_s) \simeq \lim_{U \subset X, V \subset Y} \mathrm{Sing}(U_s)^\circ \otimes_{\mathcal{B}} \mathrm{Sing}(V_s) \tag{8}$$

$$\mathrm{Sing}(X \times_S Y) \simeq \lim_{U \subset X, V \subset Y} \mathrm{Sing}(U \times_S V). \quad (9)$$

The stack property (9) is proved in [BRTV, 2.3] (where it is moreover shown that this is a stack for the h-topology). The stack property (8) is then a consequence of the same descent argument for dg-categories of singularities. Indeed, we have the following lemma.

**Lemma 4.2.2** *Let  $Z$  be an  $S$ -scheme, and  $F$  be a stack of  $\mathcal{O}_Z$ -linear dg-categories. Assume that  $F$  is a  $\mathcal{B}^{\otimes\text{-op}}$ -module stack<sup>15</sup>, and let  $T_0$  be a  $\mathcal{B}$ -module dg-category. Then, the prestack  $F \otimes_{\mathcal{B}} T_0$  of dg-categories of  $Z_{\mathrm{zar}}$ , sending  $W \subset Z$  to  $F(W) \otimes_{\mathcal{B}} T_0$  is a stack.*

**Proof of Lemma 4.2.2.** This is an application of the main result of [To2]. We denote, as usual, by  $\widehat{T} := \mathbb{R}\underline{\mathrm{Hom}}(T, \widehat{A})$  the (non-small) dg-category of all  $T^o$ -dg-modules. By the main result of [To2], the dg-category

$$\lim_{W \subset Z} (F(\widehat{W}) \otimes_{\mathcal{B}} T_0)$$

is compactly generated, and its dg-category of compact objects is equivalent in  $\mathbf{dgCat}_A$  to  $\lim_{W \subset Z} F(W) \otimes_{\mathcal{B}} T_0$ . Moreover, we have

$$(F(\widehat{W}) \otimes_{\mathcal{B}} T_0) \simeq \widehat{F(W)} \widehat{\otimes}_{\widehat{\mathcal{B}}} \widehat{T}_0,$$

where  $\widehat{\otimes}$  is the symmetric monoidal structure on presentable dg-categories. As  $\widehat{\otimes}$  is rigid when restricted to compactly generated dg-categories, we have that  $\widehat{\otimes}_{\widehat{\mathcal{B}}}$  distributes over limits on both factors, and we have

$$(F(\widehat{Z}) \otimes_{\mathcal{B}} T_0) \simeq \lim_{W \subset Z} \widehat{F(W)} \widehat{\otimes}_{\widehat{\mathcal{B}}} \widehat{T}_0.$$

Passing to the sub-dg-categories of compact objects, we find that

$$F(Z) \otimes_{\mathcal{B}} T_0 \simeq \lim_{W \subset Z} F(W) \otimes_{\mathcal{B}} T_0$$

which is the statement of the lemma. ◇

Lemma 4.2.2 immediately implies the stack property (8).

We are thus reduced to the case where  $X$  and  $Y$  are both affine, and we have to produce an equivalence

$$\mathrm{Sing}(X_s)^o \otimes_{\mathcal{B}} \mathrm{Sing}(Y_s) \simeq \mathrm{Sing}(X \times_S Y)$$

that is compatible with Zariski localization on both  $X$  and  $Y$ .

In this case, we start by the following.

**Lemma 4.2.3** *There is a natural equivalence of dg-categories over  $A$*

$$\mathrm{Coh}^b(X_s)^o \otimes_{\mathcal{B}^+} \mathrm{Coh}^b(Y_s) \simeq \mathrm{Coh}_{Z_s}^b(Z),$$

where  $Z := X \times_S Y$ , and the right hand side is the dg-category of coherent complexes on  $Z$  with cohomology supported on the special fiber  $Z_s$ . This equivalence is furthermore functorial in  $X$  and  $Y$ .

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<sup>15</sup>I.e.,  $F$  is a stack on  $Z_{\mathrm{zar}}$  with values in the  $\infty$ -category of  $\mathcal{B}^{\otimes\text{-op}}$ -modules in  $\mathbf{dgCat}_A$ .

**Proof of lemma 4.2.3.** We use the strict models introduced in our last section. Let  $X := \text{Spec } B$  and  $Y := \text{Spec } C$ , and  $K_B, K_C$  the Koszul dg-algebras of  $B$  and  $C$  with respect to the element  $\pi$ . As in our previous section, we have the Hopf dg-algebroid  $K_A^2$  and its monoidal dg-category of modules  $\text{Mod}^c(K_A^2)$ . Now,  $\text{Mod}^c(K_A^2)$  acts on both  $\text{Mod}^c(K_B)$  and  $\text{Mod}^c(K_C)$ , the dg-categories of cofibrant  $K_B$  (resp.  $K_C$ ) dg-modules which are perfect over  $B$  (resp. over  $C$ ).

We define a dg-functor

$$\theta : (\text{Mod}^c(K_B))^o \otimes_{\text{Mod}^c(K_A^2)} \text{Mod}^c(K_C) \longrightarrow \text{Mod}^c(B \otimes_A C),$$

where  $\text{Mod}^c(B \otimes_A C)$  is the category of perfect  $B \otimes_A C$ -dg-modules. This dg-functor sends a pair of objects  $(E, F)$  to the object  $\mathbb{D}(E) \otimes_{K_A} F$ , where  $\mathbb{D}(E) = \underline{\text{Hom}}_{K_B}(E, K_B)$  is the  $K_B$ -linear dual of  $E$ . After localization with respect to quasi-isomorphisms, we get a well defined morphism in  $\mathbf{dgCat}_A$

$$\theta : \text{Coh}^b(X_s)^o \otimes_{\mathcal{B}^+} \text{Coh}^b(Y_s) \longrightarrow \text{Coh}^b(Z).$$

In order to finish the proof, we have to check the following two conditions:

1. The image of  $\theta$  generates (by shifts, sums, cones and retracts) the full sub-dg-category  $\text{Coh}_{Z_s}^b(Z)$ ;
2. The dg-functor  $\theta$  above is fully faithful.

Now, on the level of objects the dg-functor  $\theta$  sends a pair  $(E, F)$ , of coherent complexes on  $X_s$  and  $Y_s$ , to the coherent complex on  $Z$

$$j_*(\mathbb{D}(E) \boxtimes_k F),$$

where  $j : Z_s \hookrightarrow Z$  is the closed embedding,  $\boxtimes_k$  denotes the external product on  $Z_s = X_s \times_s Y_s$ , and  $\mathbb{D}(E)$  denotes the Grothendieck dual of  $E$  on the Gorenstein scheme  $X_s$ . It is known that coherent complexes of the form  $E \boxtimes_k F$  generate  $\text{Coh}^b(Z_s)$ . As the coherent complexes of the form  $j_*(G)$ , for  $G \in \text{Coh}^b(Z_s)$ , clearly generate the dg-category  $\text{Coh}_{Z_s}^b(Z)$ , we see that condition (1) above is satisfied.

It now remains to show that  $\theta$  is fully faithful. Given two pairs of objects  $(E, F), (E', F') \in \text{Coh}^b(X_s) \times \text{Coh}^b(Y_s)$ , we have the morphism induced by  $\theta$

$$\mathbb{R}\underline{\text{Hom}}(E', E) \otimes_{\mathcal{B}^+(1)} \mathbb{R}\underline{\text{Hom}}(F, F') \longrightarrow \mathbb{R}\underline{\text{Hom}}(j_*(\mathbb{D}(E) \boxtimes F), j_*(\mathbb{D}(E') \boxtimes F')),$$

where  $\mathcal{B}^+(1)$  denotes the algebra of endomorphism of the unit in  $\mathcal{B}^+$ . As we have already seen,  $\mathcal{B}^+ \simeq k[u]$  as an  $\mathbb{E}_1$ -algebra. As  $X_s$  and  $Y_s$  are Gorenstein schemes, the structure sheaf  $\mathcal{O}$  is a dualizing complex, and  $E \mapsto \mathbb{D}(E)$  is an (anti)equivalence. We thus have  $\mathbb{R}\underline{\text{Hom}}(E', E) \simeq \mathbb{R}\underline{\text{Hom}}(\mathbb{D}(E), \mathbb{D}(E'))$ , and the above morphism can thus be written in the form

$$\mathbb{R}\underline{\text{Hom}}(E, E') \otimes_{k[u]} \mathbb{R}\underline{\text{Hom}}(F, F') \longrightarrow \mathbb{R}\underline{\text{Hom}}(j_*(E \boxtimes F), j_*(E' \boxtimes F')),$$

where it is simply induced by the direct image functor  $j_*$ . Both the source and the target of the above morphism enter in a distinguished triangle. On the left hand side, for any two  $k[u]$ -dg-modules  $M$  and  $N$ , we have a triangle of  $A$ -dg-modules

$$(M \otimes_k N)[-2] \longrightarrow M \otimes_k N \longrightarrow M \otimes_{k[u]} N,$$

where the morphism on the right is the natural projection. This exact triangle follows from the isomorphism  $M \otimes_{k[u]} N \simeq (M \otimes_k N) \otimes_{k[u] \otimes_k k[u]} k[u]$ , and from the exact triangle of  $k[u]$  bi-dg-modules

$$(k[u] \otimes_k k[u])[-2] \xrightarrow{a} k[u] \otimes_k k[u] \xrightarrow{m} k[u],$$

where  $a$  is given by multiplication by  $(u \otimes 1 - 1 \otimes u)$ , and  $m$  is the product map. On the right hand side, we have, by adjunction,

$$\mathbb{R}\underline{Hom}(j_*(E \boxtimes F), j_*(E' \boxtimes F')) \simeq \mathbb{R}\underline{Hom}(j^*j_*(E \boxtimes F), E' \boxtimes F').$$

The adjunction map  $j^*j_* \rightarrow id$ , provides an exact triangle of coherent sheaves on  $Z_s$

$$E \boxtimes_k F[1] \longrightarrow j^*j_*(E \boxtimes_k F) \longrightarrow E \boxtimes_k F.$$

The coboundary map of this triangle

$$E \boxtimes_k F \longrightarrow E \boxtimes_k F[2]$$

is precisely given by the action of  $k[u]$ . We thus obtain another exact triangle

$$\begin{aligned} (\mathbb{R}\underline{Hom}(E, F) \otimes_k \mathbb{R}\underline{Hom}(E', F'))[-2] &\longrightarrow \mathbb{R}\underline{Hom}(E, F) \otimes_k \mathbb{R}\underline{Hom}(E', F') \longrightarrow \\ &\longrightarrow \mathbb{R}\underline{Hom}(j_*(E \boxtimes F), j_*(E' \boxtimes F')). \end{aligned}$$

By inspection, the morphism  $\theta$  is compatible with these two triangles, and provides an equivalence

$$\mathbb{R}\underline{Hom}(E, E') \otimes_{k[u]} \mathbb{R}\underline{Hom}(F, F') \longrightarrow \mathbb{R}\underline{Hom}(j_*(E \boxtimes F), j_*(E' \boxtimes F')).$$

Therefore  $\theta$  is fully faithful. ◇

Since  $X/S$  and  $Y/S$  are generically smooth, also  $Z = X \times_S Y$  is generically smooth over  $S$ . Therefore  $\text{Sing}(Z) \simeq \text{Coh}_{Z_s}^b(Z)/\text{Perf}_{Z_s}(Z)$ . Hence, in order to finish the proof of Proposition 4.2.1, we are left to prove that the image of  $\text{Perf}_{Z_s}(Z)$  under a quasi-inverse of  $\theta$  is generated by  $\text{Coh}^b(X_s)^\circ \otimes_{\mathcal{B}^+} \text{Perf}(Y_s)$  and  $\text{Perf}(X_s)^\circ \otimes_{\mathcal{B}^+} \text{Coh}^b(Y_s)$ . Now, the dg-category  $\text{Coh}^b(X_s)^\circ \otimes_{\mathcal{B}^+} \text{Perf}(Y_s)$  is generated by the image of the natural dg-functor

$$\text{Coh}^b(X_s)^\circ \otimes_A \text{Perf}(Y_s) \longrightarrow \text{Coh}^b(X_s)^\circ \otimes_{\mathcal{B}^+} \text{Perf}(Y_s),$$

and the dg-category  $\text{Perf}(X_s)^\circ \otimes_{\mathcal{B}^+} \text{Coh}^b(Y_s)$  is generated by the image of the natural dg-functor

$$\text{Perf}(X_s)^\circ \otimes_A \text{Coh}^b(Y_s) \longrightarrow \text{Perf}(X_s)^\circ \otimes_{\mathcal{B}^+} \text{Coh}^b(Y_s).$$

Therefore, what we have to prove is that  $\text{Perf}_{Z_s}(Z)$  is generated by objects of the form  $E \boxtimes_A F$ , for  $E \in \text{Coh}^b(X_s)$ ,  $F \in \text{Coh}^b(Y_s)$ , one of them being perfect.

We first notice that, if  $E$  or  $F$  is perfect, indeed,  $E \boxtimes_A F$  is a perfect complex on  $Z$ . To see this, we may localize on  $Z$ , and write  $X = \text{Spec } B$  and  $Y = \text{Spec } C$ . The objects  $E$  and  $F$  then correspond to bounded coherent complexes over  $B_s = B \otimes_A k$  and  $C_s = C \otimes_A k$  respectively. Assume that  $E$  is perfect (the complementary case being totally analogous). By dévissage we can assume that  $E = B_s$ , so that  $E \boxtimes_A F \simeq (B \boxtimes_A F) \otimes_A k$ . In other words, if we write  $j : Y_s \hookrightarrow Y$  and  $i : Z_s \hookrightarrow Z$  for the closed embeddings, and  $p : Z \rightarrow Y$  for the second projection, we have

$$E \boxtimes_A F \simeq i_* i^* p^*(j_*(F)).$$

But  $j_*(F)$  is perfect on  $Y$  (because  $Y$  is regular), so  $p^*j_*(F)$  is perfect on  $Z$ . Moreover, since  $i$  is an lci morphism,  $i_*i^*$  preserves perfect complexes on  $Z$ , and we conclude that, indeed,  $E \boxtimes_A F$  is a perfect complex on  $Z$ .

Finally, the fact that the image of  $\mathrm{Coh}^b(X_s)^\circ \otimes_{\mathcal{B}^+} \mathrm{Perf}(Y_s)$  and  $\mathrm{Perf}(X_s)^\circ \otimes_{\mathcal{B}^+} \mathrm{Coh}^b(Y_s)$  inside  $\mathrm{Coh}^b_{Z_s}(Z)$  generates the whole dg-category  $\mathrm{Perf}_{Z_s}(Z)$  follows from the fact that the image of  $\mathrm{Perf}(X_s)^\circ \otimes_A \mathrm{Perf}(Y_s)$  inside  $\mathrm{Perf}(Z)$  already generates  $\mathrm{Perf}_{Z_s}(Z)$ . Indeed, for two perfect complexes  $E$  on  $X_s$  and  $F$  on  $Y_s$ , the image of  $E \boxtimes_A F$  in  $\mathrm{Perf}(Z)$  is of the form  $i_*(E \boxtimes_k F) \oplus i_*(E \boxtimes_k F)[1]$ . As objects of the form  $E \boxtimes_k F$  generate  $\mathrm{Perf}(Z_s)$  we are done, and Proposition 4.2.1 is proved.  $\square$

### 4.3 Saturatedness

As a consequence of the Künneth formula for dg-categories of singularities we prove the following result.

**Proposition 4.3.1** *Let  $X$  be a regular and flat  $S$ -scheme.*

1. *If  $X$  is proper over  $S$ , then the dg-category  $\mathrm{Coh}^b(X_s)$  is proper over  $\mathcal{B}^+$ .*
2. *If  $X$  is proper over  $S$ , then the dg-category  $\mathrm{Sing}(X_s)$  is saturated over  $\mathcal{B}$ .*

**Proof.** (1) We have to show that the big morphism

$$h : \widehat{\mathrm{Coh}^b(X_s)^\circ \otimes_A \mathrm{Coh}^b(X_s)} \longrightarrow \widehat{\mathcal{B}^+} \simeq \widehat{\mathrm{Coh}^b(G)}.$$

is small. Here  $G = s \times_S s$ , and the morphism  $h$  is obtained as follows. The derived scheme  $G$  is a derived groupoid acting on  $X_s$  by means of the natural projection on the last two factors

$$\mu : G \times_s X_s = s \times_S s \times_S X \longrightarrow X_s.$$

The projection on the first and third factors provides another morphism

$$p : G \times_s X_s \longrightarrow X_s.$$

Thus, the morphisms  $p$  and  $\mu$  together define a morphism of derived schemes

$$q : G \times_s X_s \longrightarrow X_s \times_S X_s.$$

Finally, we denote by  $r : G \times_s X_s \longrightarrow G$  the natural projection. Now, for two coherent complexes  $E$  and  $F$  on  $X_s$ , we first form the external Hom complex  $\mathcal{H}om_A(E, F)$  which is a coherent complex on  $X_s \times_S X_s$ , and we have

$$h(E, F) \simeq r_*(q^*\mathcal{H}om_A(E, F)).$$

This is a quasi-coherent complex on  $G$ . Both  $q$  and  $r$  are local complete intersection morphisms of derived schemes, and moreover  $r$  is proper. This implies that  $q^*$  and  $r_*$  preserve coherent complexes, and thus that  $h(E, F)$  is coherent on  $G$ .

(2) We have  $\mathrm{Sing}(X_s) \simeq \mathrm{Coh}^b(X_s) \otimes_{\mathcal{B}^+} \mathcal{B}$ , thus (1) implies that  $\mathrm{Sing}(X_s)$  is proper over  $\mathcal{B}$ . To prove it is smooth we need to prove that the coevaluation big morphism  $A \longrightarrow \mathrm{Sing}(X_s)^\circ \otimes_{\mathcal{B}} \mathrm{Sing}(X_s)$  is a small morphism. Using our Künneth formula for dg-category of singularities (Proposition 4.2.1) this morphism corresponds to the data of an ind-object in  $\mathrm{Sing}(X \times_S X)$ . This object is the structure sheaf of the diagonal  $\Delta_X$  inside  $X \times_S X$  which is an object in  $\mathrm{Sing}(X \times_S X)$ . This shows that the coevaluation morphism is a small morphism and thus that  $\mathrm{Sing}(X_s)$  is indeed saturated over  $\mathcal{B}$ .  $\square$

**Remark 4.3.2** Proposition 4.3.1 (2) remains true if we only suppose that the singular locus of  $X_s$  is proper over  $s = \text{Spec } k$ .

## 5 Application to Bloch's conductor formula with unipotent monodromy

In this Section we first recall Bloch's Conductor Conjecture and the current state of the art for it, then we prove a version of Bloch's conductor under the hypothesis that the monodromy action is unipotent, where Bloch's number is replaced by a categorical Bloch class (see Definition 5.2.1). We are convinced that Bloch's number always agree with the categorical Bloch class but we will not prove (nor use) this fact in this paper. We are aware of a proof of this comparison in the geometric case (i.e. when the inclusion  $s \rightarrow S$  has a retraction) but we will defer this to another paper, where we hope to give the comparison also when  $S$  has mixed characteristic.

### 5.1 Bloch's Conductor Conjecture

Our base scheme is a discrete valuation ring  $S = \mathbf{Spec } A$ , with perfect residue field  $k$ , and fraction field  $K$ . Let  $p : X \rightarrow S$  be proper and flat morphism of finite type, and of relative dimension  $n$ . We assume that the generic fiber  $X_K$  is smooth over  $K$ , and that  $X$  is a regular scheme. We write  $\bar{K}$  for the separable closure of  $K$  (inside a fixed algebraic closure).

In his 1985 paper [Bl], Spencer Bloch formulated the following conductor formula conjecture which is a kind of vast arithmetic generalization of Gauss-Bonnet formula, where an intersection theoretic (coherent) term, the *Bloch's number*, is conjectured to be equal to an arithmetic (étale) term, the *Artin conductor*. We address the reader to [Bl] and [Ka-Sa] for more detailed definitions of the various objects involved in the statement.

**Conjecture 5.1.1 [Bloch's conductor Conjecture]** *Under the above hypotheses on  $p : X \rightarrow S$ , we have an equality*

$$[\Delta_X, \Delta_X]_S = \chi(X_{\bar{k}}, \ell) - \chi(X_{\bar{K}}, \ell) - \mathbf{Sw}(X_{\bar{K}}),$$

where  $\chi(Y, \ell)$  denotes the  $\mathbb{Q}_\ell$ -adic Euler characteristic of a variety  $Y$ , for  $\ell$  prime to the characteristic of  $k$ ,  $\mathbf{Sw}(X_{\bar{K}})$  is the Swan conductor of the  $\text{Gal}(\bar{K}/K)$ -representation  $H^*(X_{\bar{K}}, \mathbb{Q}_\ell)$ , and  $[\Delta_X, \Delta_X]_S$  is Bloch's number of  $X/S$ , i.e. the degree in  $\text{CH}_0(k) \simeq \mathbb{Z}$  of Bloch's localised self-intersection  $(\Delta_X, \Delta_X)_S \in \text{CH}_0(X_k)$  of the diagonal in  $X$ . The (negative of the) rhs is called the Artin conductor of  $X/S$ , and denoted by  $\text{Art}(X/S)$ .

Conjecture 5.1.1 is known to hold in several special cases that we recall below:

1. When  $k$  is of characteristic zero, Conjecture 5.1.1 follows from the work of [Kap]. When furthermore  $X_s$  has only isolated singularities the conductor formula was known as the *Milnor formula* stating that the dimension of the space of vanishing cycles equals the dimension of the Jacobian ring.
2. When  $S$  is equicharacteristic, Conjecture 5.1.1 has been proved recently in [Sai], based on Beilinson's theory of singular support of  $\ell$ -adic sheaves. The special subcase of isolated singularities already appeared in [SGA7-I, Exp. XVI].

3. When  $X$  is semi-stable over  $S$ , i.e. the reduced special fiber  $(X_s)_{red} \subset X$  is a normal crossing divisor, Conjecture 5.1.1 was proved in [Ka-Sa].

In view of the above list of results, one of the major open cases is that of isolated singularities in *mixed characteristic*, which is the conjecture appearing in Deligne's exposé [SGA7-I, Exp. XVI].

It is classical and easy to see that Conjecture 5.1.1 can be reduced to the case where  $S$  is excellent and strictly henselian (so that  $k$  algebraically closed)<sup>16</sup>. We will thus assume from now on that  $k$  is algebraically closed.

## 5.2 An analog of Bloch's Conjecture for unipotent monodromy

We start by introducing a *categorical variant* of Bloch's number  $[\Delta_X, \Delta_X]_S$  defined in terms of dg-categories of singularities.

The dg-category  $\text{Sing}(X_s)$  of singularities of the special fiber comes equipped with its canonical  $\mathcal{B}$ -module structure described in Proposition 4.1.7. As  $X$  is proper over  $S$ , we know by Proposition 4.3.1 that  $\text{Sing}(X_s)$  is saturated over  $\mathcal{B}$ . We are thus entitled to take the trace (Definition 2.4.4) of the identity of  $\text{Sing}(X_s)$ , which is a morphism

$$A \longrightarrow \text{HH}(\mathcal{B}/A)$$

in  $\mathbf{dgCat}_A$ . This trace morphism is then, by definition, determined by a perfect  $\text{HH}(\mathcal{B}/A)^o$ -dg-module  $\text{HH}(\text{Sing}(X_s)/\mathcal{B})$ , and thus provides us with a class in  $K$ -theory

$$[\text{HH}(\text{Sing}(X_s)/\mathcal{B})] \in K_0(\text{HH}(\mathcal{B}/A)).$$

The Chern character natural transformation (applied to  $\text{HH}(\mathcal{B}/A)$ , see Definition 2.3.1) of this class is an element  $Ch_{\ell, \text{HH}(\mathcal{B}/A)}([\text{HH}(\text{Sing}(X_s)/\mathcal{B})])$  in  $\pi_0|r_{\ell}(\text{HH}(\mathcal{B}/A))| = H^0(S_{\text{ét}}, r_{\ell}(\text{HH}(\mathcal{B}/A)))$ .

**Definition 5.2.1** *With the above notations and hypotheses on  $X/S$ , the categorical Bloch class of  $X/S$  is defined as*

$$[\Delta_X, \Delta_X]_S^{\text{cat}} := Ch_{\ell}([\text{HH}(\text{Sing}(X_s)/\mathcal{B})]) \in H^0(S_{\text{ét}}, r_{\ell}(\text{HH}(\mathcal{B}/A))).$$

Notice that though  $\mathcal{B}$  is just an associative algebra in  $\mathbf{dgCat}_A$  (an  $E_2$ -algebra over  $A$ ), its  $\ell$ -adic realization  $r_{\ell}(\mathcal{B})$  is in fact a commutative monoid in  $Sh_{\mathbb{Q}_{\ell}}(S)$ , naturally equivalent to  $i_*(\mathbb{Q}_{\ell}(\beta) \oplus \mathbb{Q}_{\ell}(\beta)[1])$  where  $i : s \rightarrow S$  is the inclusion of the closed point. Therefore there is a canonical augmentation  $\mathfrak{a} : \text{HH}(r_{\ell}(\mathcal{B})/r_{\ell}(A)) \rightarrow r_{\ell}(\mathcal{B})$ , hence an induced augmentation

$$\mathfrak{a} : H^0(S_{\text{ét}}, \text{HH}(r_{\ell}(\mathcal{B})/r_{\ell}(A))) = \pi_0|\text{HH}(r_{\ell}(\mathcal{B})/r_{\ell}(A))| \longrightarrow \pi_0|r_{\ell}(\mathcal{B})| = H^0(S_{\text{ét}}, r_{\ell}(\mathcal{B})) \simeq \mathbb{Q}_{\ell}$$

that is a left inverse to

$$\sigma : \mathbb{Q}_{\ell} \simeq H^0(S_{\text{ét}}, r_{\ell}(\mathcal{B})) \rightarrow H^0(S_{\text{ét}}, \text{HH}(r_{\ell}(\mathcal{B})/r_{\ell}(A)))$$

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<sup>16</sup>We first reduce to the complete dvr (hence henselian and excellent) case by proper base-change, and then we further reduce to the excellent, *strictly* henselian case.

(induced by  $r_\ell(\mathcal{B}) \rightarrow \mathrm{HH}(r_\ell(\mathcal{B})/r_\ell(A))$ ).

For  $\lambda \in \mathbb{Q}_\ell$ , we will simply write  $\lambda^\wedge$  for the image of  $\lambda$  via the composition

$$\mathbb{Q}_\ell \simeq H^0(S_{\acute{e}t}, r_\ell(\mathcal{B})) \xrightarrow{\sigma} H^0(S_{\acute{e}t}, \mathrm{HH}(r_\ell(\mathcal{B})/r_\ell(A))) \xrightarrow{\alpha} H^0(S_{\acute{e}t}, r_\ell(\mathrm{HH}(\mathcal{B}/A))).$$

We are now ready to prove our version of Bloch's Conductor Conjecture for unipotent monodromy.

**Theorem 5.2.2** *Let  $X/S$  be as in Conjecture 5.1.1, and assume further that the inertia subgroup  $I := \mathrm{Gal}(\bar{K}/K^{unr}) \subseteq \mathrm{Gal}(\bar{K}/K)$  acts unipotently on  $H^*(X_{\bar{K}}, \mathbb{Q}_\ell)$ . Then we have an equality*

$$[\Delta_X, \Delta_X]_S^{cat} = \chi(X_{\bar{k}}, \ell)^\wedge - \chi(X_{\bar{K}}, \ell)^\wedge$$

in  $H^0(S_{\acute{e}t}, r_\ell(\mathrm{HH}(\mathcal{B}/A)))$ .

**Remark 5.2.3** Since we have not proved that the map

$$\alpha : H^0(S_{\acute{e}t}, \mathrm{HH}(r_\ell(\mathcal{B})/r_\ell(A))) \rightarrow H^0(S_{\acute{e}t}, r_\ell(\mathrm{HH}(\mathcal{B}/A)))$$

is injective, the equality in Theorem 5.2.2 is not a priori an equality of integers (and it might even be the trivial equality  $0 = 0$  of classes in  $H^0(S_{\acute{e}t}, r_\ell(\mathrm{HH}(\mathcal{B}/A)))$ ). However, we conjecture that the canonical map  $u : \mathcal{B} \rightarrow \mathrm{HH}(\mathcal{B}/A)$  is in fact an  $\mathbb{A}^1$ -homotopy equivalence; this would imply that  $r_\ell(u) : r_\ell(\mathcal{B}) \rightarrow r_\ell(\mathrm{HH}(\mathcal{B}/A))$  is an equivalence in  $Sh_{\mathbb{Q}_\ell}(S)$ , so that  $H^0(S_{\acute{e}t}, r_\ell(\mathrm{HH}(\mathcal{B}/A))) \simeq H^0(S_{\acute{e}t}, r_\ell(\mathcal{B})) \simeq \mathbb{Q}_\ell$ . In particular  $\alpha$  would be injective, and the equality in Theorem 5.2.2 would indeed be an equality of integers.

**Remark 5.2.4** Recall that the inertia group  $I$  sits in an extension of pro-finite groups

$$1 \longrightarrow P \longrightarrow I \longrightarrow I_t \longrightarrow 1,$$

where  $I_t$  is the *tame inertia* quotient, and the *wild inertia*  $P$  is a pro- $p$ -subgroup. For any continuous finite dimensional  $\mathbb{Q}_\ell$ -representation  $V$  of  $I$ , the group  $P$  acts through a finite quotient on  $V$  (see [Se, §19]). Therefore, if  $I$  is supposed to act unipotently on  $V$ , then  $P$  acts necessarily trivially, i.e. the  $I$ -action factors through a  $I_t$ -action, i.e., by definition, the  $I$  action is *tame*. By definition, the Swan conductor  $Sw_I(V)$  (see e.g. [Ka-Sa, §6.1]) vanishes for a tame  $I$ -representation  $V$ . As a consequence, granting Remark 5.2.3, under the hypothesis of unipotent action of  $I$  on  $H^*(X_{\bar{K}}, \mathbb{Q}_\ell)$ , Conjecture 5.1.1 becomes Theorem 5.2.2 if Bloch's number  $[\Delta_X, \Delta_X]_S$  is replaced by the categorical Bloch class  $[\Delta_X, \Delta_X]_S^{cat}$ . Though we currently know a proof only in the geometric case, we are actually convinced that the categorical Bloch class is *always*, i.e. regardless any hypothesis on the monodromy action and on the mixed or equal characteristic property of  $S$ , an integer equal to Bloch's intersection number  $[\Delta_X, \Delta_X]_S$  appearing in Conjecture 5.1.1. This fact will not be used in this paper and will be more closely investigated in a future one.

**Proof of Thm. 5.2.2.** We may suppose  $S$  strictly henselian, so that  $I = \mathrm{Gal}(\bar{K}/K)$ . We want to apply our trace formula (Theorem 2.4.9) to  $T = \mathrm{Sing}(X_s)$  and  $f = id$ . In order to do this, we need to check that the conditions of being *saturated* and  $\ell^\otimes$ -*admissible* over  $\mathcal{B}$  are met by such  $T$ .

By Proposition 4.3.1, we know that  $T$  is *saturated over  $\mathcal{B}$* .

Let us now show that  $T$  is also  $\ell^\otimes$ -*admissible over  $\mathcal{B}$* , i.e. that the canonical map

$$\varphi : r_\ell(T^o) \otimes_{r_\ell(\mathcal{B})} r_\ell(T) \longrightarrow r_\ell(T^o \otimes_{\mathcal{B}} T)$$

is an equivalence.

First of all we notice that, since both the source and the target of  $\varphi$  are  $\ell$ -adic complexes on  $S$ , supported at  $s$ , it is enough to show that  $i^*(\varphi)$  is an equivalence,  $i : s \rightarrow S$  being the canonical inclusion (note that  $i^*i_*$  is equivalent to the identity functor).

Let us first elaborate on the target of  $i^*(\varphi)$ . By Künneth for dg-categories of singularities, we have the canonical equivalence of Proposition 4.2.1

$$T^o \otimes_{\mathcal{B}} T \simeq \mathbf{Sing}(X \times_S X).$$

Moreover, since a unipotent action of  $I$  is also tame, by Cor. 3.4.5 we have

$$r_\ell(\mathbf{Sing}(X \times_S X)) \simeq (p_{Z_s})_*(\nu_X \boxtimes \nu_X)^I(\beta),$$

where  $p_{Z_s} : Z_s = X_s \times_s X_s \rightarrow S$  is the composite  $Z_s \xrightarrow{q_{Z_s}} s \xrightarrow{i} S$ , and, as usual,  $\nu_X$  denotes vanishing cycles for  $X/S$  on  $X_s$ . Therefore

$$i^*r_\ell(\mathbf{Sing}(X \times_S X)) \simeq (q_{Z_s})_*(\nu_X \boxtimes \nu_X)^I(\beta)$$

in  $Sh_{\mathbb{Q}_\ell}(s)$ . Since  $q_{Z_s} = q_{X_s} \times_s q_{X_s}$ , where  $q_{X_s} : X_s \rightarrow s$  is the canonical map, we have

$$i^*r_\ell(\mathbf{Sing}(X \times_S X)) \simeq (q_{Z_s})_*(\nu_X \boxtimes \nu_X)^I(\beta) \simeq ((q_{X_s})_*(\nu_X) \otimes_{\mathbb{Q}_\ell} (q_{X_s})_*(\nu_X))^I(\beta) \quad (10)$$

in  $Sh_{\mathbb{Q}_\ell}(s)$ , that we may rewrite as

$$i^*r_\ell(\mathbf{Sing}(X \times_S X)) \simeq (\mathbb{H}(X_s, \nu_X) \otimes_{\mathbb{Q}_\ell} \mathbb{H}(X_s, \nu_X))^I(\beta) \quad (11)$$

where we have introduced the hypercohomology  $\mathbb{H}(X_s, \mathcal{E}) := (q_{X_s})_*(\mathcal{E})$ , for  $\mathcal{E} \in Sh_{\mathbb{Q}_\ell}(X_s)$ .

Let us now look more carefully at the source of the map  $i^*(\varphi)$ . By the main theorem of [BRTV] (see also formula (4)), we have

$$r_\ell(T) \simeq (p_{X_s})_*(\nu_X[-1]^I)(\beta)$$

where  $p_{X_s} : X_s \rightarrow S$  is the composite  $X_s \xrightarrow{q_{X_s}} s \xrightarrow{i} S$ . Therefore,

$$i^*(r_\ell(T^o) \otimes_{r_\ell(\mathcal{B})} r_\ell(T)) \simeq i^*(r_\ell(T^o)) \otimes_{i^*r_\ell(\mathcal{B})} i^*r_\ell(T) \simeq ((q_{X_s})_*(\nu_X[-1]^I) \otimes_{\mathbb{Q}_\ell^I} (q_{X_s})_*(\nu_X[-1]^I))^I(\beta) \quad (12)$$

in  $Sh_{\mathbb{Q}_\ell}(s)$ , that we may rewrite as<sup>17</sup>

$$i^*(r_\ell(T^o) \otimes_{r_\ell(\mathcal{B})} r_\ell(T)) \simeq (\mathbb{H}(X_s, \nu_X[-1]^I) \otimes_{\mathbb{Q}_\ell^I} \mathbb{H}(X_s, \nu_X[-1]^I))^I(\beta). \quad (13)$$

By recalling that  $\mathcal{E}(\beta) \simeq \mathcal{E}[-2](\beta)$ , for any  $\mathcal{E} \in Sh_{\mathbb{Q}_\ell}(X_s)$ , and by combining (11) and (13), we see that

$$i^*(\varphi) : i^*(r_\ell(T^o) \otimes_{r_\ell(\mathcal{B})} r_\ell(T)) \longrightarrow i^*r_\ell(T^o \otimes_{\mathcal{B}} T)$$

is then equivalent to  $\psi[-2](\beta)$ , where  $\psi$  is the lax-monoidal structure morphism for the functor  $(-)^I$ , applied to the pair of  $\ell$ -adic complexes  $(\mathbb{H}(X_s, \nu_X), \mathbb{H}(X_s, \nu_X))$ ,

$$\psi : \mathbb{H}(X_s, \nu_X)^I \otimes_{\mathbb{Q}_\ell^I} \mathbb{H}(X_s, \nu_X)^I \longrightarrow (\mathbb{H}(X_s, \nu_X) \otimes_{\mathbb{Q}_\ell} \mathbb{H}(X_s, \nu_X))^I. \quad (14)$$

The fact that the morphism  $\psi$  is an equivalence is a consequence of the following general lemma.

<sup>17</sup>Note that  $r_\ell(T) \simeq r_\ell(T^o)$  because the  $K$ -theory of a dg-category is canonically isomorphic to the  $K$ -theory of the opposite dg-category.

**Lemma 5.2.5** *Let  $\mathcal{D}_{uni}(\mathbf{I}, \mathbb{Q}_\ell)$  be the full sub- $\infty$ -category of  $\mathcal{D}_c(\text{Spec } K, \mathbb{Q}_\ell) \simeq \mathcal{D}_c^{\mathbf{I}}(\text{Spec } \bar{K}, \mathbb{Q}_\ell)$  consisting of all objects  $E$  for which the action of  $\mathbf{I}$  on each  $H^i(E)$  is unipotent. Then the invariant  $\infty$ -functor induces an equivalence of symmetric monoidal  $\infty$ -categories*

$$(-)^{\mathbf{I}} : \mathcal{D}_{uni}(\mathbf{I}, \mathbb{Q}_\ell) \simeq \mathcal{D}_{perf}(\mathbb{Q}_\ell[\epsilon_1]),$$

where  $\mathbb{Q}_\ell[\epsilon_1]$  is the free commutative  $\mathbb{Q}_\ell$ -dg-algebra generated by  $\epsilon_1$  in degree 1, and  $\mathcal{D}_{perf}(\mathbb{Q}_\ell[\epsilon_1])$  is its  $\infty$ -category of perfect dg-modules.

*Proof of lemma 5.2.5.* This is a well known fact. The  $\infty$ -functor

$$(-)^{\mathbf{I}} : \mathcal{D}_c(\text{Spec } K, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}_c(\mathbb{Q}_\ell)$$

is lax symmetric monoidal, so it induces a lax monoidal  $\infty$ -functor

$$(-)^{\mathbf{I}} : \mathcal{D}_c(\text{Spec } K, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}(\mathbb{Q}_\ell^{\mathbf{I}}).$$

It is easy to see that  $\mathbb{Q}_\ell^{\mathbf{I}}$  is equivalent to  $\mathbb{Q}_\ell[\epsilon_1]$ , and the choice of such an equivalence only depends on the choice of a generator of  $H^1(\mathbf{I}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$ . We thus have an induced lax symmetric monoidal  $\infty$ -functor

$$(-)^{\mathbf{I}} : \mathcal{D}_c(\text{Spec } K, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}(\mathbb{Q}_\ell[\epsilon_1]).$$

The above  $\infty$ -functor is in fact symmetric monoidal, as it preserves unit objects, and moreover  $\mathcal{D}_c(\text{Spec } K, \mathbb{Q}_\ell)$  is generated by the unit object  $\mathbb{Q}_\ell$ .

The lemma is then a direct consequence of the following fact: let  $x \in \mathcal{C}$  be a compact object in a compactly generated stable  $\infty$ -category  $\mathcal{C}$ , then the  $\infty$ -functor

$$\mathcal{C}(x, -) : \mathcal{C} \longrightarrow \text{Mod}_{\mathbf{Sp}}(\text{End}(x))$$

induces an equivalence of stable  $\infty$ -categories

$$\langle x \rangle \simeq \text{Perf}_{\mathbf{Sp}}(\text{End}(x)),$$

where  $\langle x \rangle \subset \mathcal{C}$  denotes the full stable  $\infty$ -category generated by the object  $x$ , and  $\text{End}(x)$  denotes the ring spectrum of endomorphisms of  $x$ .  $\diamond$

The above lemma implies that the map

$$\psi[-2] : \mathbb{H}(X_s, \nu_X[-1])^{\mathbf{I}} \otimes_{\mathbb{Q}_\ell^{\mathbf{I}}} \mathbb{H}(X_s, \nu_X[-1])^{\mathbf{I}} \longrightarrow (\mathbb{H}(X_s, \nu_X[-1]) \otimes_{\mathbb{Q}_\ell} \mathbb{H}(X_s, \nu_X[-1]))^{\mathbf{I}}$$

is an equivalence, and thus, as observed above, the same is true for  $i^*(\varphi)$ , and hence for the admissibility map

$$\varphi : r_\ell(T^o) \otimes_{r_\ell(\mathcal{B})} r_\ell(T) \longrightarrow r_\ell(T^o \otimes_{\mathcal{B}} T),$$

so that  $T$  is indeed  $\ell^\otimes$ -admissible over  $\mathcal{B}$  as we wanted.

Now that we have checked all the conditions for the trace formula of Theorem 2.4.9 to hold in our case, we get an equality

$$\text{Ch}_\ell([\text{HH}(T/\mathcal{B}, \text{id})]) = \text{tr}_{r_\ell(\mathcal{B})}(\text{id} : r_\ell(T) \rightarrow r_\ell(T)) \quad (15)$$

in  $H^0(S_{\text{ét}}, r_\ell(\mathbf{HH}(\mathcal{B}/A)))$ . Recall that

$$\text{tr}_{r_\ell(\mathcal{B})}(id : r_\ell(T) \rightarrow r_\ell(T)) := \alpha(\text{Tr}_{r_\ell(\mathcal{B})}(id : r_\ell(T) \rightarrow r_\ell(T))),$$

where  $\alpha : H^0(S_{\text{ét}}, \mathbf{HH}(r_\ell(\mathcal{B})/r_\ell(A))) \rightarrow H^0(S_{\text{ét}}, r_\ell(\mathbf{HH}(\mathcal{B}/A)))$  is the canonical map.

We are left to identify the two sides of (15).

The left hand side is, by definition, our categorical Bloch number  $[\Delta_X, \Delta_X]_S^{\text{cat}}$ .

Let us now investigate the r.h.s. of (15): we need to show that  $\text{Tr}_{r_\ell(\mathcal{B})}(r_\ell(id_T)) = \chi(X_{\bar{k}}, \ell) - \chi(X_{\bar{K}}, \ell)$ . As already noticed,  $r_\ell(\mathcal{B})$  is a commutative monoid in  $Sh_{\mathbb{Q}_\ell}(S)$ , naturally equivalent to  $i_*(\mathbb{Q}_\ell(\beta) \oplus \mathbb{Q}_\ell(\beta)[1])$  where  $i : s \rightarrow S$  is the inclusion of the closed point. Therefore there is a canonical augmentation  $\mathbf{a} : \mathbf{HH}(r_\ell(\mathcal{B})/r_\ell(A)) \rightarrow r_\ell(\mathcal{B})$ , hence an induced augmentation

$$\mathbf{a} : H^0(S_{\text{ét}}, \mathbf{HH}(r_\ell(\mathcal{B})/r_\ell(A))) = \pi_0|\mathbf{HH}(r_\ell(\mathcal{B})/r_\ell(A))| \longrightarrow \pi_0|r_\ell(\mathcal{B})| = H^0(S_{\text{ét}}, r_\ell(\mathcal{B})).$$

Consider the diagram

$$\begin{array}{ccc} H^0(S_{\text{ét}}, \mathbf{HH}(r_\ell(\mathcal{B})/r_\ell(A))) & \xrightarrow{\alpha} & H^0(S_{\text{ét}}, r_\ell(\mathbf{HH}(\mathcal{B}/A))) \\ \mathbf{a} \downarrow \uparrow \sigma & \nearrow \text{can} & \\ \mathbb{Q}_\ell \simeq H^0(S_{\text{ét}}, r_\ell(\mathcal{B})) & & \end{array}$$

where  $\text{can}$  is the map induced by the canonical map  $u : \mathcal{B} \rightarrow \mathbf{HH}(\mathcal{B}/A)$ , and  $\sigma$  is the map induced by  $r_\ell(u)$ , so that  $\mathbf{a} \circ \sigma = \text{id}$ . By compatibility between the non-commutative trace and the commutative trace (Remark 2.4.5)

$$\mathbf{a}(\text{Tr}_{r_\ell(\mathcal{B})}(r_\ell(id_T))) = \text{Tr}_{r_\ell(\mathcal{B})}^c(r_\ell(id_T))$$

Since  $r_\ell(\mathcal{B}) \simeq i_*(\mathbb{Q}_\ell(\beta) \oplus \mathbb{Q}_\ell(\beta)[1])$ , we have  $K_0(r_\ell(\mathcal{B})) \simeq \mathbb{Z}$  and the following commutative diagram

$$\begin{array}{ccccc} \mathbb{Q}_\ell \simeq H^0(S_{\text{ét}}, r_\ell(\mathcal{B})) & \xleftarrow{\sigma} & H^0(S_{\text{ét}}, \mathbf{HH}(r_\ell(\mathcal{B})/r_\ell(A))) & \xrightarrow{\mathbf{a}} & H^0(S_{\text{ét}}, r_\ell(\mathcal{B})) \\ & \searrow \text{Tr}(id_-) & \nearrow \text{Tr}^c(id_-) & \nearrow & \nearrow \\ K_0(r_\ell(\mathcal{B})) & & & & \\ & \searrow & & \nearrow & \\ & & \mathbb{Z} & & \end{array}$$

iso

where the maps with source  $\mathbb{Z}$  are the unique maps of commutative rings, and  $\mathbf{a} \circ \text{Tr}(id_-) = \text{Tr}^c(id_-)$  again by the functorial compatibility of non-commutative and commutative traces. By considering the class  $[r_\ell(T)]$  in  $K_0(r_\ell(\mathcal{B}))$ , and using that  $\mathbf{a} \circ \sigma = \text{id}$ , we deduce from the previous commutative diagram the equality of  $\ell$ -adic numbers

$$\sigma(\text{Tr}_{r_\ell(\mathcal{B})}^c(r_\ell(id_T))) = \text{Tr}_{r_\ell(\mathcal{B})}(r_\ell(id_T)). \quad (16)$$

Now, unfolding the definition of  $\text{Tr}_{r_\ell(\mathcal{B})}$ , and using Lemma 5.2.5, the  $\ell$ -adic number (16) can be described as follows. The dg-algebra  $\mathbb{Q}_\ell^I$  is such that  $K_0(\mathbb{Q}_\ell^I) \simeq \mathbb{Z}$ . Viewing  $\mathbb{Z}$  inside  $\mathbb{Q}_\ell$ , this isomorphism is induced by sending the class of dg-module  $E$  to the trace of the identity inside  $\mathbf{HH}_0(\mathbb{Q}_\ell^I) \simeq \mathbb{Q}_\ell$ . Using

the functoriality of the trace for the morphism of commutative dg-algebras  $\mathbb{Q}_\ell^I \longrightarrow \mathbb{Q}_\ell^I(\beta)$ , we see that (16) is simply the trace of the identity on  $\mathbb{H}(X_s, \nu_X[-1])$ , as an object in  $\mathcal{D}_{uni}(I, \mathbb{Q}_\ell)$ . This trace is easy to compute, as it equals 1 on the unit object  $\mathbb{Q}_\ell$ . Since the unit object generates  $\mathcal{D}_{uni}(I, \mathbb{Q}_\ell)$ , we have that the trace of the identity on any object  $E \in \mathcal{D}_{uni}(I, \mathbb{Q}_\ell)$  equals the Euler characteristic of the underlying complex of  $\mathbb{Q}_\ell$ -vector spaces.

We thus have shown that (16) can be rewritten as

$$Tr_{r_\ell(B)}(r_\ell(id_T)) = \sum_i (-1)^i \dim_{\mathbb{Q}_\ell} H^{i-1}(X_s, \nu_X).$$

However, by proper base change, the complex  $\mathbb{H}(X_s, \nu_X)$  appears in an exact triangle

$$H(X_{\bar{k}}, \mathbb{Q}_\ell) \longrightarrow H(X_{\bar{K}}, \mathbb{Q}_\ell) \longrightarrow \mathbb{H}(X_s, \nu_X)$$

and thus we have the equality

$$Tr_{r_\ell(B)}(r_\ell(id_T)) = \chi(X_{\bar{k}}, \ell) - \chi(X_{\bar{K}}, \ell)$$

in  $\mathbb{Z} \hookrightarrow \mathbb{Q}_\ell \hookrightarrow H^0(S_{\acute{e}t}, \mathrm{HH}(r_\ell(\mathcal{B})/r_\ell(A)))$ .

Therefore

$$[\Delta_X, \Delta_X]_S^{\mathrm{cat}} = \chi(X_{\bar{k}}, \ell)^\wedge - \chi(X_{\bar{K}}, \ell)^\wedge,$$

as required. □

## A Localizations of monoidal dg-categories

In this Appendix we recall some basic facts about localizations of dg-categories as introduced in [To1]. The purpose of the section is to explain the multiplicative properties of the localization construction. In particular, we explain how localization of strict monoidal dg-categories gives rise to monoids in  $\mathbf{dgCat}_A$ , and thus to monoidal dg-categories in the sense of our Definition 2.1.1.

Let  $T$  be a dg-category over  $A$ , together with  $W$  a set of morphisms in  $Z^0(T)$ , the underlying category of  $T$  (this is the category of 0-cycles in  $T$ , i.e.  $Z^0(T)(x, y) := Z^0(T(x, y))$ ). For the sake of brevity, we will just say that  $W$  is a set of maps in  $T$ . In other words, we allow  $W$  not to be strictly speaking a subset of the set of morphisms in  $T$ , but just a set together with a map  $W \rightarrow \mathrm{Mor}(T)$  from  $W$  to the set of morphisms in  $T$ . Recall that a localization of  $T$  with respect to  $W$ , is a dg-category  $L_W T$  together with a morphism in  $\mathbf{dgCat}_A$

$$l : T \longrightarrow L_W T$$

such that, for any  $U \in \mathbf{dgCat}_A$ , map induced by  $l$  on mapping spaces

$$\mathrm{Map}(L_W T, U) \longrightarrow \mathrm{Map}(T, U)$$

is fully faithful and its image consists of all  $T \rightarrow U$  sending  $W$  to equivalences in  $U$  (i.e. the induced functor  $[T] \rightarrow [U]$  sends elements of  $W$  to isomorphisms in  $[U]$ ).

As explained in [To1], localizations always exist, and are unique up to a contractible space of choices (because it represents an obvious  $\infty$ -functor). We will describe here a model for  $(T, W) \mapsto L_W T$  which will have nice properties with respect to tensor products of dg-categories. For this, let  $dgcat_A^{W,c}$  be the category of pairs  $(T, W)$ , where  $T$  is a dg-category with cofibrant hom's over  $A$ , and  $W$  a set of maps in  $T$ . Morphisms  $(T, W) \rightarrow (T', W')$  in  $dgcat_A^{W,c}$  are dg-functors  $T \rightarrow T'$  sending  $W$  to  $W'$ .

We fix once for all a factorization

$$\Delta_A^1 \xrightarrow{j} \tilde{I} \xrightarrow{p} \overline{\Delta}_A^1,$$

with  $j$  a cofibration and  $p$  a trivial fibration. Here  $\Delta_A^1$  is the  $A$ -linearisation of the category  $\Delta^1$  that classifies morphisms, and  $\overline{\Delta}_A^1$  is the linearisation of the category that classifies isomorphisms. For an object  $(T, W) \in dgcat_A^{W,c}$  we define  $W^{-1}T$  by the following cocartesian diagram in dg-categories

$$\begin{array}{ccc} \coprod_W \Delta_A^1 & \longrightarrow & T \\ \downarrow & & \downarrow \\ \coprod_W \tilde{I} & \longrightarrow & W^{-1}T, \end{array}$$

where  $\coprod_W \Delta_A^1 \rightarrow T$  is the canonical dg-functor corresponding to the set  $W$  of morphisms in  $T$ .

**Lemma A.0.1** *The canonical morphism  $l : T \rightarrow W^{-1}T$  defined above is a localization of  $T$  along  $W$ .*

**Proof.** According to [To1], the localization of  $T$  can be construed as the homotopy push-out of dg-categories

$$\begin{array}{ccc} \coprod_W \Delta_A^1 & \longrightarrow & T \\ \downarrow & & \downarrow \\ \coprod_W A & \longrightarrow & L_W T. \end{array}$$

The lemma then follows from the observation that when  $T$  has cofibrant hom's, then the push-out diagram defining  $W^{-1}T$  is in fact a homotopy push-out diagram.  $\square$

The construction  $(T, W) \rightarrow W^{-1}T$  clearly defines a functor

$$dgcat_A^{W,c} \rightarrow dgcat_A^c$$

from  $dgcat_A^{W,c}$  to  $dgcat_A^c$ , the category of dg-categories with cofibrant hom's. Moreover, this functor comes equipped with a natural symmetric colax monoidal structure. Indeed,  $dgcat_A^{W,c}$  is a symmetric monoidal category, where the tensor product is given by

$$(T, W) \otimes (T', W') := (T \otimes_A T', W \otimes id \cup id \otimes W').$$

We have a natural map

$$T \otimes_A T' \rightarrow (W^{-1}T) \otimes_A ((W')^{-1}T'),$$

which by construction has a canonical extension

$$(W \otimes id \cup id \otimes W')^{-1}(T \otimes_A T') \rightarrow (W^{-1}T) \otimes_A ((W')^{-1}T').$$

The unit in  $dgcat_A^{W,c}$  is  $(A, \emptyset)$ , which provides an canonical isomorphism  $(\emptyset)^{-1}A \simeq A$ . These data endow the functor  $(T, W) \mapsto W^{-1}T$  with a symmetric colax monoidal structure. By composing with the canonical symmetric monoidal  $\infty$ -functor  $dgcat_A^c \rightarrow \mathbf{dgCat}_A$ , we get a symmetric colax monoidal  $\infty$ -functor

$$dgcat_A^{W,c} \rightarrow \mathbf{dgCat}_A,$$

which sends  $(T, W)$  to  $W^{-1}T$ . By [To3, Ex. 4.3.3], this colax symmetric monoidal  $\infty$ -functor is in fact monoidal. We thus have a symmetric monoidal localization  $\infty$ -functor

$$W^{-1}(-) : dgcat_A^{W,c} \rightarrow \mathbf{dgCat}_A.$$

As a result, if  $T$  is a (strict) monoid in  $dgcat_A^{W,c}$ , then  $W^{-1}T$  carries a canonical structure of a monoid in  $\mathbf{dgCat}_A$ . This applies particularly to strict monoidal dg-categories endowed with a compatible notion of equivalences. By the MacLane coherence theorem, any such a structure can be turned into a strict monoid in  $dgcat_A^{W,c}$ , and by localization into a monoid in  $\mathbf{dgCat}_A$ . In other words, the localization of a monoidal dg-category along a set of maps  $W$  that is compatible with the monoidal structure, is a monoid in  $\mathbf{dgCat}_A$ . The same is true for dg-categories which are modules over a given monoidal dg-category.

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