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Polarization in relativistic fluids: a quantum field theoretical derivation

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We review the calculation of polarization in a relativistic fluid within the framework of statistical quantum field theory. We derive the expressions of the spin density matrix and the mean spin vector both for a single quantum relativistic particle and for a quantum free field. After introducing the formalism of the covariant Wigner function for the scalar and the Dirac field, the relation between spin density matrix and the covariant Wigner function is obtained. The formula is applied to the fluid produced in relativistic nuclear collisions by using the local thermodynamic equilibrium density operator and recovering previously known formulae. The dependence of these results on the spin tensor and pseudo-gauge transformations of the stress-energy tensor is addressed.

I. INTRODUCTION

The discovery of global polarization of Λ hyperons in relativistic nuclear collisions [1–5] has sparked a great interest in the theory of spin and polarization in relativistic fluids and relativistic matter in general. Several approaches have been proposed and several are currently pursued; amongst them, relativistic kinetic theory [6–11] and a phenomenological treatment of the spin tensor [12–14].

Yet, the most fundamental tool is quantum statistical field theory, which was used to derive the original formula [15] of polarization of quasi-free particles in a relativistic fluid at local thermodynamic equilibrium. All other approaches should, in the first place, reproduce the results obtained with this method, once the state of the system, that is its density operator, is chosen.

Carrying out the calculation of the polarization matrix, which for spin 1/2 particles boils down to the mean spin vector, in a quantum field theoretical framework is, however, not an easy task. In the derivation presented in ref. [15] some approximation were introduced and the final formula admittedly relied on the use of the canonical spin tensor, that is dependent on the specific set of quantum stress-energy and spin tensor (in other words of a the *pseudo-gauge* choice [16]). It is the purpose of this paper to review the derivation of the polarization of spin 1/2 particles step by step and fill some of the conceptual gaps. Particularly, the exact formula relating the spin density matrix and the mean spin vector to the covariant Wigner function will be found and it will be thereby conclusively demonstrated that the *expression* of the polarization is independent of the spin tensor. Furthermore, a general formula for particles with any spin S will be derived in the limit of distinguishable quantum particles, that is neglecting quantum statistics. For this purpose, many useful concepts in statistical quantum field theory will be thoroughly reviewed and discussed.

The paper is organized as follows: in Section II the general definitions of spin density matrix and mean spin vector will be given in a quantum relativistic framework. In Section III a formula for the spin density matrix and the mean spin vector will be derived for a single free quantum particle with spin S in a thermal bath with rotation and acceleration, by using only group theory techniques. In Section IV the covariant Wigner operator and function will be introduced for the free scalar and Dirac field. In Section V the formula for the mean spin vector of a free Dirac fermion will be obtained as a function of the covariant Wigner function while in Section VI the same formula will be obtained with a different method based on total angular momentum, already used in ref. [15]. In Section VII the density operator at local thermodynamic equilibrium will be discussed in detail with emphasis on its application in relativistic heavy ion collisions and the derivation of the formula of mean spin vector for a fermion in a relativistic fluid at local thermodynamic equilibrium will be outlined.

Notations and conventions

In this paper we use the natural units, with $\hbar = c = K = 1$.

We will use the relativistic notation with repeated indices assumed to be saturated, however, contractions of indices will be sometimes denoted with, e.g. $\beta_\mu p^\mu = \beta \cdot p$. The Minkowskian metric tensor is $\text{diag}(1, -1, -1, -1)$; for the Levi-Civita symbol we use the convention $\epsilon^{0123} = 1$.

Operators in Hilbert space will be denoted by a large upper hat (\hat{T}) while unit vectors with a small upper hat (\hat{v}). Noteworthy exception, the Dirac field which is expressed by Ψ without an upper hat.

The symbol Tr with a capital T stands for the trace over all states in the Hilbert space, whereas the symbol tr stands for a trace over polarization states or traces of finite dimensional matrices.

II. THE SPIN DENSITY MATRIX AND THE DEFINITION OF MEAN SPIN

In relativistic quantum mechanics, for a single massive particle, the spin angular momentum vector is defined as:

$$\hat{S}^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\hat{J}_{\nu\rho}\hat{P}_\sigma \quad (1)$$

where $\hat{J}_{\nu\rho}$ are the angular momentum-boost operators and \hat{P}_σ the energy-momentum operator. The operator in eq. (1) is also known as Pauli-Lubanski vector and it fulfills the following commutation relations:

$$\begin{aligned} [\hat{S}_\mu, \hat{P}_\nu] &= 0 \\ [\hat{S}_\mu, \hat{S}_\nu] &= -i\epsilon_{\mu\nu\rho\sigma}\hat{S}^\rho\hat{P}^\sigma \\ \hat{S} \cdot \hat{P} &= 0 \end{aligned} \quad (2)$$

Hence, if the ket $|p\rangle$ is an eigenvector of \hat{P} , so is $\hat{S}|p\rangle$. The restriction of \hat{S} to the eigenspace labelled by four-momentum p is defined as $\hat{S}(p)$. Since $\hat{S}(p) \cdot p = 0$, it can be decomposed onto three orthonormal spacelike four-vectors $n_1(p), n_2(p), n_3(p)$ orthogonal to p , forming a basis of the Minkowski space with the unit vector $\hat{p} = p/\sqrt{p^2}$:

$$\hat{S}(p) = \sum_{i=1}^3 \hat{S}_i(p) n_i(p) \quad (3)$$

It can be shown that the operators $\hat{S}_i(p)$ form a SU(2) algebra and are the generators of the so-called *little group* of massive particles. The third component $\hat{S}_3(p)$ can be diagonalized along with \hat{S}^2 which corresponding eigenvalues s and $S(S+1)$, S being *the* spin of the particle so that:

$$\hat{P}|p, s\rangle = p|p, s\rangle \quad \text{and} \quad \hat{S}_3(p)|p, s\rangle = s|p, s\rangle \quad (4)$$

The $\hat{n}_i(p)$, being orthogonal to p , can be written as:

$$n_i(p) = [p]\hat{e}_i \quad (5)$$

with \hat{e}_i the i -th unit space vector and $[p]$ the so-called *standard Lorentz transformation* bringing the timelike vector $p_0 = (m, 0, 0, 0)$ into the four-momentum p .

The choice of $[p]$ entails a specific physical meaning of the eigenvalue s . For instance, if θ, φ are the spherical coordinates of \mathbf{p} and ξ the rapidity of p ,

$$[p] \equiv \mathbf{R}_3(\varphi)\mathbf{R}_2(\theta)\mathbf{L}_3(\xi)$$

where $\mathbf{R}_k(\psi)$ are rotations around the axis k with angle ψ and $\mathbf{L}_k(\xi)$ a Lorentz boost along the direction k with hyperbolic angle ξ , is a typical choice of the standard Lorentz transformation which makes s the helicity of the particle. Finally, if the states are normalized according to ¹:

$$\langle p, r | q, s \rangle = 2\varepsilon \delta^3(\mathbf{p} - \mathbf{q}) \delta_{rs} . \quad (6)$$

we have, for the representation of a general Lorentz transformation Λ in the Hilbert space:

$$\hat{\Lambda}|p, r\rangle = \sum_s |\Lambda p, s\rangle D^S([\Lambda p]^{-1}\Lambda[p])_{sr} \quad (7)$$

where D^S stands for the $(2S+1)$ -dimensional irreducible representation of the proper orthochronous Lorentz group SO(1,3) (the so-called $(0, S)$ representation) or - in case - of its universal covering group SL(2,C). The transformation

$$W(\Lambda, p) = [\Lambda p]^{-1}\Lambda[p] \quad (8)$$

is the so-called *Wigner rotation*, as it leaves the unit time vector \hat{t} invariant.

¹ Throughout this paper the symbol ε stands for the on-shell energy, that is $\varepsilon = \sqrt{\mathbf{p}^2 + m^2}$

The mean value of S^μ of the operator (1) is the properly called *spin vector* (the polarization vector being the mean spin vector S^μ divided by S so that its maximal magnitude is always 1):

$$S^\mu = \text{Tr}(\hat{S}^\mu \hat{\rho})$$

where $\hat{\rho}$ is the density operator of the single quantum relativistic particle in the Hilbert space. Its restriction to the subspace of four-momentum p is the *spin density operator* $\hat{\Theta}(p)$, which is used to express the mean value of the spin vector for a particle with momentum p :

$$S^\mu(p) = \text{Tr}(\hat{S}^\mu \hat{\Theta}(p)) \quad (9)$$

The matrix:

$$\Theta(p)_{rs} \equiv \langle p, r | \hat{\rho} | p, s \rangle = \langle p, r | \hat{\Theta}(p) | p, s \rangle \quad (10)$$

is the spin density matrix, in the basis labelled by the eigenvalues of $\hat{S}_3(p)$. The matrix Θ , which is hermitian, positive definite and with normalized trace, contains the maximal information about the spin state of the particle. It is dependent on the chosen basis, yet the mean value (9) is independent thereof, that is of the standard Lorentz transformation associated with the eigenvalue s .

Plugging the (3) into the (9) we get:

$$\begin{aligned} S^\mu(p) &= \sum_r \sum_i \langle p, r | \hat{S}_i(p) \hat{\Theta}(p) | p, r \rangle n_i(p) = \sum_{r,s} \sum_i \langle p, r | \hat{S}_i(p) | p, s \rangle \langle p, s | \hat{\Theta}(p) | p, r \rangle n_i(p) \\ &= \sum_{r,s} \sum_i D^S(J^i)_{rs} \Theta(p)_{sr} n_i(p) = \sum_{i=1}^3 \text{tr}(D^S(J^i) \Theta(p)) [p] (\hat{e}_i)^\mu = \sum_{i=1}^3 [p]^\mu \text{tr}(D^S(J^i) \Theta(p)), \end{aligned} \quad (11)$$

where we have used the fact that $\hat{S}_i(p)$ are the generators of the little group $\text{SU}(2)$ algebra in the subspace spanned by $|p\rangle$; the $D^S(J^i)$ are the familiar matrices of the angular momentum generators for the representation with spin S . The above formula can be made covariant by introducing the definition of angular momenta:

$$D^S(J^i) = -\frac{1}{2} \epsilon^{i\lambda\nu\rho} D^S(J_{\lambda\nu}) \hat{t}_\rho$$

where \hat{t} is the unit time vector and the tensor $J_{\lambda\nu}$ now includes all Lorentz transformation generators, angular momentum and boosts. Since $\hat{t}_\rho = \delta_\rho^0$ we can extend the sum over i from 0 to 4, because of the Levi-Civita tensor and write:

$$S^\mu(p) = -\frac{1}{2} \epsilon^{\alpha\lambda\nu\rho} \hat{t}_\rho [p]^\mu \text{tr}(D^S(J_{\lambda\nu}) \Theta(p)) \quad (12)$$

Now:

$$\begin{aligned} [p]^\mu \epsilon^{\alpha\lambda\nu\rho} &= [p]^\mu_\alpha ([p]^\beta_\varphi [p]_\beta^{-1\lambda}) ([p]^\gamma_\chi [p]_\gamma^{-1\nu}) ([p]^\delta_\psi [p]_\delta^{-1\rho}) \epsilon^{\alpha\varphi\chi\psi} = ([p]^\mu_\alpha [p]^\beta_\varphi [p]^\gamma_\chi [p]^\delta_\psi \epsilon^{\alpha\varphi\chi\psi}) [p]_\beta^{-1\lambda} [p]_\gamma^{-1\nu} [p]_\delta^{-1\rho} \\ &= \det[p] \epsilon^{\mu\beta\gamma\delta} [p]_\beta^{-1\lambda} [p]_\gamma^{-1\nu} [p]_\delta^{-1\rho} = \epsilon^{\mu\beta\gamma\delta} [p]_\beta^{-1\lambda} [p]_\gamma^{-1\nu} [p]_\delta^{-1\rho} \end{aligned}$$

and, substituting in (12):

$$S^\mu(p) = -\frac{1}{2} \epsilon^{\mu\beta\gamma\delta} [p]_\delta^{-1\rho} \hat{t}_\rho [p]_\beta^{-1\lambda} [p]_\gamma^{-1\nu} \text{tr}(D^S(J_{\lambda\nu}) \Theta(p))$$

By definition of standard Lorentz transformation, and taking into account its orthogonality, we have:

$$[p]_\delta^{-1\rho} \hat{t}_\rho = \frac{p_\delta}{m}$$

so that the mean spin vector becomes:

$$S^\mu(p) = -\frac{1}{2m} \epsilon^{\mu\beta\gamma\delta} p_\delta [p]_\beta^{-1\lambda} [p]_\gamma^{-1\nu} \text{tr}(D^S(J_{\lambda\nu}) \Theta(p))$$

From group representation theory we know that:

$$[p]_\beta^{-1\lambda} [p]_\gamma^{-1\nu} D^S(J_{\lambda\nu}) = D^S([p])^{-1} D^S(J_{\beta\gamma}) D^S([p]) \quad (13)$$

hence the (12) can be finally cast as:

$$S^\mu(p) = -\frac{1}{2m}\epsilon^{\mu\beta\gamma\delta}p_\delta \text{tr} \left(D^S([p])^{-1} D^S(J_{\beta\gamma}) D^S([p]) \Theta(p) \right) \quad (14)$$

The “kinematic” part of the spin vector derivation for a single quantum relativistic particle is completed.

In a quantum field theory framework, the single-particle spin density matrix can be still defined with a formula which is a generalization of the (10):

$$\Theta(p)_{rs} = \frac{\text{Tr}(\hat{\rho} \hat{a}_s^\dagger(p) \hat{a}_r(p))}{\sum_t \text{Tr}(\hat{\rho} \hat{a}_t^\dagger(p) \hat{a}_t(p))}, \quad (15)$$

where $\hat{\rho}$ is the density operator for the Hilbert space of the field states. The $\hat{a}_r(p)$ are destruction operators of the particle with momentum p and spin state r . The introduction of creation and destruction operators makes it clear that one can define a polarization of particles only when particle is a sensible concept, that is for a non-interacting or a weakly interacting field theory. For instance, defining a spin density matrix of quarks and gluons makes sense only in the perturbative limit of QCD.

The calculation of $\Theta(p)$ is the crucial and hardest part of the procedure. Before tackling the full quantum field theory case, one can obtain a good approximation by using the single quantum relativistic particle formalism, which is the subject of the next section.

III. THE SINGLE PARTICLE LIMIT AND GLOBAL EQUILIBRIUM FACTORIZATION

In the fixed-number particle formalism, the Hilbert space of quantum states is the tensor product of single-particle Hilbert spaces. Neglecting symmetrization or anti-symmetrization of the states means disregarding quantum statistics effects and taking the limit of distinguishable particles. Moreover, if the particles are non-interacting, the full density operator can be written as the tensor product of single-particle density operators:

$$\hat{\rho} = \otimes_i \hat{\rho}_i$$

We can now set out to get the spin density matrix for the general *global* equilibrium density operator $\hat{\rho}$ [17, 18]:

$$\hat{\rho} = \frac{1}{Z} \exp \left[-b \cdot \hat{P} + \frac{1}{2} \varpi : \hat{J} \right], \quad (16)$$

where b is a constant time-like four-vector and ϖ a constant anti-symmetric tensor. In global equilibrium, the vector field:

$$\beta^\mu = b^\mu + \varpi^{\mu\nu} x_\nu \quad (17)$$

is the four-temperature vector [17] fulfilling Killing equation and:

$$\varpi_{\mu\nu} = -\frac{1}{2} (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu) \quad (18)$$

is called thermal vorticity; the relation (18) can be taken as a definition of thermal vorticity in non-equilibrium situation. The operators \hat{P} and \hat{J} in (16) are the conserved total four-momentum and total angular momentum-boosts, respectively. For a set of non-interacting distinguishable particles, we can write:

$$\hat{P} = \sum_i \hat{P}_i, \quad \hat{J} = \sum_i \hat{J}_i,$$

and consequently,

$$\hat{\rho}_i = \frac{1}{Z_i} \exp \left[-b \cdot \hat{P}_i + \frac{1}{2} \varpi : \hat{J}_i \right].$$

so that the single-particle spin density matrix reads:

$$\Theta(p)_{irs} = \frac{\langle p, r | \hat{\rho}_i | p, s \rangle}{\sum_r \langle p, r | \hat{\rho}_i | p, r \rangle}. \quad (19)$$

In order to calculate the right hand side of (19), one can take advantage of a noteworthy factorization:

$$\frac{1}{Z_i} \exp \left[-b \cdot \hat{P}_i + \frac{1}{2} \varpi : \hat{J}_i \right] = \frac{1}{Z_i} \exp \left[-\tilde{b} \cdot \hat{P}_i \right] \exp \left[\frac{1}{2} \varpi : \hat{J}_i \right], \quad (20)$$

where

$$\tilde{b}_\mu = \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \underbrace{(\varpi_{\mu\nu_1} \varpi^{\nu_1\nu_2} \dots \varpi_{\nu_{k-1}\nu_k})}_{k \text{ times}} b^{\nu_k}. \quad (21)$$

The (20) is a very useful formula, whose derivation is worth being shown in some detail.

Let us start with the following very simple observation concerning the composition of translations and Lorentz transformation in Minkowski space-time. Let x be a four-vector and apply the combination

$$\mathsf{T}(a) \Lambda \mathsf{T}(a)^{-1}$$

$\mathsf{T}(a)$ being a translation of some four-vector a and Λ a Lorentz transformation. The effect of the above combination on x reads:

$$x \mapsto x - a \mapsto \Lambda(x - a) \mapsto \Lambda(x - a) + a = \Lambda(x) + (\mathsf{I} - \Lambda)(a) = \mathsf{T}((\mathsf{I} - \Lambda)(a))(\Lambda(x))$$

Since x was arbitrary, we have:

$$\mathsf{T}(a) \Lambda \mathsf{T}(a)^{-1} = \mathsf{T}((\mathsf{I} - \Lambda)(a)) \Lambda$$

This relation has a representation of unitary operators in Hilbert space, which can be written in terms of the generators of the Poincaré group:

$$\exp[ia \cdot \hat{P}] \exp[-i\phi : \hat{J}/2] \exp[-ia \cdot \hat{P}] = \exp[i((\mathsf{I} - \Lambda)(a)) \cdot \hat{P}] \exp[-i\phi : \hat{J}/2] \quad (22)$$

where ϕ are the parameters of the Lorentz transformation². By taking ϕ infinitesimal, we can obtain a known relation about the effect of translations on angular momentum operators:

$$\exp[ia \cdot \hat{P}] \hat{J}_{\mu\nu} \exp[-ia \cdot \hat{P}] = \hat{\mathsf{T}}(a) \hat{J}_{\mu\nu} \hat{\mathsf{T}}(a)^{-1} = \hat{J}_{\mu\nu} - a_\mu \hat{P}_\nu + a_\nu \hat{P}_\mu$$

The left hand side of (22) can now be worked out by using the above relation:

$$\begin{aligned} \exp[ia \cdot \hat{P}] \exp[-i\phi : \hat{J}/2] \exp[-ia \cdot \hat{P}] &= \hat{\mathsf{T}}(a) \exp[-i\phi : \hat{J}/2] \hat{\mathsf{T}}(a)^{-1} \\ &= \exp[-i\phi : \hat{\mathsf{T}}(a) \hat{J} \hat{\mathsf{T}}(a)^{-1}/2] = \exp[-i\phi : (\hat{J} - a \wedge \hat{P})/2] = \exp[i\phi_{\mu\nu} a^\mu \hat{P}^\nu - i\phi_{\mu\nu} \hat{J}^{\mu\nu}/2] \end{aligned} \quad (23)$$

Hence, combining (23) with (22), we have obtained the factorization:

$$\exp[i\phi_{\mu\nu} a^\mu \hat{P}^\nu - i\phi_{\mu\nu} \hat{J}^{\mu\nu}/2] = \exp[i((\mathsf{I} - \Lambda)(a)) \cdot \hat{P}] \exp[-i\phi : \hat{J}/2] \quad (24)$$

Now:

$$i(\mathsf{I} - \Lambda)(a) = ia - i \sum_{k=0}^{\infty} \frac{(-i)^k}{2^k k!} (\phi : \mathsf{J})^k(a) = -i \sum_{k=1}^{\infty} \frac{(-i)^k}{2^k k!} (\phi : \mathsf{J})^k(a) \quad (25)$$

Setting:

$$V_\mu = i\phi_{\mu\nu} a^\nu$$

and taking into account that:

$$(\mathsf{J}_{\mu\nu})^\alpha_\beta = i (\delta_\mu^\alpha g_{\nu\beta} - \delta_\nu^\alpha g_{\mu\beta})$$

² Henceforth, by $:$ we will denote a double contraction of rank 2 tensors, e.g. $\phi : \hat{J} = \phi_{\mu\nu} \hat{J}^{\mu\nu}$

we have:

$$(\phi : J)(a)_\alpha = 2i\phi_{\alpha\beta}a^\beta = 2V_\alpha$$

Therefore, the right hand side of the eq. (25) becomes:

$$-i \sum_{k=1}^{\infty} \frac{(-i)^k}{2^k k!} (\phi : J)^k(a) = -i \sum_{k=1}^{\infty} \frac{(-i)^k}{2^{k-1} k!} (\phi : J)^{k-1}(V) = - \sum_{k=0}^{\infty} \frac{(-i)^k}{2^k (k+1)!} (\phi : J)^k(V)$$

Finally, the eq. (24) becomes:

$$\exp[-V \cdot \hat{P} - i\phi : \hat{J}/2] = \exp[-\tilde{V}(\phi) \cdot \hat{P}] \exp[-i\phi : \hat{J}/2] \quad (26)$$

where

$$\tilde{V}(\phi) \equiv \sum_{k=0}^{\infty} \frac{(-i)^k}{2^k (k+1)!} (\phi : J)^k(V) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \underbrace{(\phi_{\mu\nu_1} \phi^{\nu_1\nu_2} \dots \phi_{\nu_{k-1}\nu_k})}_{k \text{ times}} V^{\nu_k}$$

The eq. (26) can be read as the factorization of the exponential of a linear combinations of generators of the Poincaré group. For this reason, it must be derivable also by using the known formulae of the factorization of the exponential of the sum of matrices $\exp[A+B]$ in terms of exponentials of commutators of A and B . Indeed, it can be shown, by using the commutation relations of \hat{P} and \hat{J} , that one precisely gets the eq. (26) for *any* vector V and tensor ϕ , either real or complex. Hence, the formula (26) can be applied to factorize the density operator (16) by setting $\phi = i\varpi$:

$$\hat{\rho} = \frac{1}{Z} \exp[-b \cdot \hat{P} + \varpi : \hat{J}/2] = \frac{1}{Z} \exp[-\tilde{b}(\varpi) \cdot \hat{P}] \exp[\varpi : \hat{J}/2] \quad (27)$$

with:

$$\tilde{b}(\varpi) = \sum_{k=0}^{\infty} \frac{1}{2^k (k+1)!} (\varpi : J)^k b = \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \underbrace{(\varpi_{\mu\nu_1} \varpi^{\nu_1\nu_2} \dots \varpi_{\nu_{k-1}\nu_k})}_{k \text{ times}} b^{\nu_k}$$

We have thus proved the formula (20).

The factorization of the density operator in eq. (20) can now be applied to calculate the spin density matrix in eq. (19). The momentum-dependent factor $\exp(-\tilde{b} \cdot p)$ cancels out in the ratio, and one is left with:

$$\Theta(p)_{rs} = \frac{\langle p, r | \exp[\varpi : \hat{J}/2] | p, s \rangle}{\sum_t \langle p, t | \exp[\varpi : \hat{J}/2] | p, t \rangle}.$$

To derive its explicit form, we use an analytic continuation; namely, we first determine $\Theta(p)$ for imaginary ϖ and then continue the function to real values. If ϖ is imaginary, $\exp[\varpi : \hat{J}/2] \equiv \hat{\Lambda}$ is just a unitary representation of a Lorentz transformation, and then one can use known relations of Poincaré group representations [19] to obtain:

$$\Theta(p)_{rs} = \frac{\langle p, r | \hat{\Lambda} | p, s \rangle}{\sum_t \langle p, t | \hat{\Lambda} | p, t \rangle} = \frac{2\varepsilon\delta^3(\mathbf{p} - \mathbf{\Lambda}(p)) W(\Lambda, p)_{rs}}{2\varepsilon\delta^3(\mathbf{p} - \mathbf{\Lambda}(p)) \sum_t W(\Lambda, p)_{tt}} \quad (28)$$

where $\mathbf{\Lambda}(p)$ stands for the spacial part of the four-vector $\Lambda(p)$ and $W(\Lambda, p)$ is the Wigner rotation defined in the eq. (8). We thus have:

$$\Theta(p)_{rs} = \frac{D^S([p]^{-1}\Lambda[p])_{rs}}{\text{tr}(D^S(\Lambda))},$$

which seems to be an appropriate form to be analytically continued to real ϖ . However, the above form is not satisfactory yet as the continuation to real ϖ , that is:

$$D^S(\Lambda) = \exp\left[-\frac{i}{2}\varpi : \Sigma_S\right] \rightarrow \exp\left[\frac{1}{2}\varpi : \Sigma_S\right]$$

where $\Sigma_S = D^S(J)$ is the matrix representing the generators, does not give rise to a hermitian matrix for $\Theta(p)$ as it should. This problem can be fixed by taking into account that $W(p)$ is the representation of a rotation, hence unitary. We can thus replace $W(p)$ with $(W(p) + W(p)^{-1\dagger})/2$ in (28) and, by using the property of $\text{SL}(2, \mathbb{C})$ representations $D^S(A^\dagger) = D^S(A)^\dagger$ [20] we obtain:

$$\Theta(p) = \frac{D^S([p]^{-1}\Lambda[p]) + D^S([p]^\dagger\Lambda^{-1\dagger}[p]^{-1\dagger})}{\text{tr}(D^S(\Lambda) + D^S(\Lambda)^{-1\dagger})},$$

which will give a hermitian result because the analytic continuation of $\Lambda^{-1\dagger}$ reads ³:

$$D^S(\Lambda^{-1\dagger}) \rightarrow \exp\left[\frac{1}{2}\varpi : \Sigma_S^\dagger\right].$$

Altogether, the final expression of the spin density matrix reads:

$$\Theta(p) = \frac{D^S([p])^{-1} \exp[(1/2)\varpi : \Sigma_S] D^S([p]) + D^S([p])^\dagger \exp[(1/2)\varpi : \Sigma_S^\dagger] D^S([p])^{-1\dagger}}{\text{tr}(\exp[(1/2)\varpi : \Sigma_S] + \exp[(1/2)\varpi : \Sigma_S^\dagger])}, \quad (29)$$

which is manifestly hermitian.

The equation (29) can be further developed. By using the (13), we have:

$$D^S([p])^{-1} \exp\left[\frac{1}{2}\varpi : \Sigma_S\right] D^S([p]) = \exp\left[\frac{1}{2}\varpi^{\mu\nu} D^S([p])^{-1} \Sigma_{S\mu\nu} D^S([p])\right] = \exp\left[\frac{1}{2}\varpi^{\mu\nu} [p]_\mu^{-1\alpha} [p]_\nu^{-1\beta} \Sigma_{S\alpha\beta}\right].$$

which applies to the original $\text{SO}(1,3)$ matrices too. So, if we apply the Lorentz transformation $[p]$ to the tensor ϖ :

$$\varpi^{\mu\nu} [p]_\mu^{-1\alpha} [p]_\nu^{-1\beta} = \varpi_*^{\alpha\beta}(p) \quad (30)$$

we realize that $\varpi_*^{\alpha\beta}$ are the components of the thermal vorticity tensor in the rest-frame of the particle with four-momentum p . Note that these components are obtained by back-boosting with $[p]$ (which in fact is not a pure Lorentz boost in the helicity scheme as it includes a rotation). Finally, the equation (29) becomes:

$$\Theta(p) = \frac{D^S(\exp[(1/2)\varpi_*(p) : \Sigma_S]) + D^S(\exp[(1/2)\varpi_*(p) : \Sigma_S^\dagger])}{\text{tr}(\exp[(1/2)\varpi : \Sigma_S] + \exp[(1/2)\varpi : \Sigma_S^\dagger])}. \quad (31)$$

The thermal vorticity ϖ is usually $\ll 1$; in this case, the spin density matrix can be expanded in power series around $\varpi = 0$. Taking into account that $\text{tr}(\Sigma_S) = 0$, we have:

$$\Theta(p)_{rs} \simeq \frac{\delta_{rs}}{2S+1} + \frac{1}{4(2S+1)} \varpi_*(p)^{\alpha\beta} (\Sigma_{S\alpha\beta} + \Sigma_{S\alpha\beta}^\dagger)_{rs}$$

to first order in ϖ . Now the Σ_S matrices can be decomposed into representations of angular momentum and boosts:

$$\Sigma_{S\mu\nu} = D^S(J_{\mu\nu}) = \epsilon_{\mu\nu\rho\sigma} D^S(J^\rho) \hat{t}^\sigma - D^S(\mathbf{K}_\mu) \hat{t}_\nu + D^S(\mathbf{K}_\nu) \hat{t}_\mu$$

and taking into account that the $D^S(J^i)$ are hermitian while $D^S(\mathbf{K}^i)$ are anti-hermitian, we find:

$$\Theta(p)_{rs} \simeq \frac{\delta_{rs}}{2S+1} + \frac{1}{2(2S+1)} \varpi_*(p)^{\alpha\beta} \epsilon_{\alpha\beta\rho\nu} D^S(J^\rho)_{r'}^r \hat{t}^\nu. \quad (32)$$

By plugging (32) into (12), we get:

$$\begin{aligned} S^\mu(p) &= [p]_\kappa^\mu \frac{1}{2(2S+1)} \varpi_*(p)^{\alpha\beta} \epsilon_{\alpha\beta\rho\nu} \text{tr}(D^S(J^\rho) D^S(J^\kappa)) \hat{t}^\nu \\ &= -\frac{1}{2(2S+1)} \frac{S(S+1)(2S+1)}{3} [p]_\kappa^\mu \varpi_*(p)^{\alpha\beta} \epsilon_{\alpha\beta\rho\nu} g^{\rho\kappa} \hat{t}^\nu \\ &= -\frac{1}{2} \frac{S(S+1)}{3} [p]_\rho^\mu \varpi_*(p)_{\alpha\beta} \epsilon^{\alpha\beta\rho\nu} \hat{t}_\nu = -\frac{1}{2m} \frac{S(S+1)}{3} \varpi_{\alpha\beta} \epsilon^{\alpha\beta\mu\nu} p_\nu, \end{aligned} \quad (33)$$

³ Note that the Lorentz transformations in Minkowski space-time and their counterparts of the fundamental $(0, 1/2)$ representation of the $\text{SL}(2, \mathbb{C})$ group are henceforth identified. Particularly, the standard Lorentz transformation $[p]$ will indicate either a $\text{SO}(1,3)$ transformation or a $\text{SL}(2, \mathbb{C})$ transformation.

where, in the last equality, we have boosted the vector to the laboratory frame by using the Eq. (30).

For a fluid made of distinguishable particles at local thermodynamic equilibrium, the thermal vorticity ϖ is promoted to a function of space and time, so that the expression (33) gives rise to the integral average:

$$S^\mu(p) = -\frac{1}{2m} \frac{S(S+1)}{3} \epsilon^{\mu\alpha\beta\nu} p_\nu \frac{\int_\Sigma d\Sigma_\lambda p^\lambda f(x, p) \varpi_{\alpha\beta}(x)}{\int_\Sigma d\Sigma_\lambda p^\lambda f(x, p)} \quad (34)$$

with $f(x, p)$ the distribution function and Σ is a 3D hypersurface from where particles are emitted. The latter is basically the same formula obtained in refs. [15, 21] and should apply to particles with any spin.

IV. THE COVARIANT WIGNER FUNCTION

We now turn to the general quantum field formula of the spin density matrix, the equation (15). To develop this expression, we need to introduce an important quantity: the covariant Wigner operator. We will do this first for the scalar field, where the spin plays no role, and later for the Dirac field.

A. The scalar field

The covariant Wigner operator is defined as a Fourier transform of the two-point function of the quantum field:

$$\widehat{W}(x, k) = \frac{2}{(2\pi)^4} \int d^4y : \widehat{\psi}^\dagger(x + y/2) \widehat{\psi}(x - y/2) : e^{-iy \cdot k} \quad (35)$$

where $:$ stands for the normal ordering of creation and destruction operators; the appearance of a normal ordering implies that this definition is suitable for a free field or a field interacting with an external field. Even if (35) is not, strictly speaking, a local operator (it depends on the field in two points), its quasi-locality makes it a suitable tool to deal with local thermodynamic equilibrium in quantum statistical mechanics. Besides, its mean value, namely the covariant Wigner function:

$$W(x, k) = \text{Tr}(\widehat{\rho} \widehat{W}(x, k)) \quad (36)$$

where $\widehat{\rho}$ is the density operator, is an indispensable tool to reckon quantum corrections to classical kinetic theory [22]. From the equation (36) it turns out that the covariant Wigner function is real, but it does not need to be positive definite. Inserting in the eq. (35) the free scalar field expansion in plane waves:

$$\widehat{\psi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2\varepsilon} e^{-ip \cdot x} \widehat{a}(p) + e^{ip \cdot x} \widehat{b}^\dagger(p) \quad (37)$$

a_p, b_p being destruction operators of particles with four-momentum p normalized so as to:

$$[\widehat{a}(p), \widehat{a}^\dagger(p')] = 2\varepsilon \delta^3(\mathbf{p} - \mathbf{p}') \quad (38)$$

we get, for the covariant Wigner function:

$$\begin{aligned} W(x, k) &= \frac{2}{(2\pi)^7} \int \frac{d^3p}{2\varepsilon} \frac{d^3p'}{2\varepsilon'} \int d^4y e^{-iy \cdot (k - (p+p')/2)} e^{i(p-p') \cdot x} \langle \widehat{a}^\dagger(p) \widehat{a}(p') \rangle + e^{-iy \cdot (k + (p+p')/2)} e^{-i(p-p') \cdot x} \langle \widehat{b}^\dagger(p') \widehat{b}(p) \rangle \\ &\quad + e^{-iy \cdot (k - (p-p')/2)} e^{i(p+p') \cdot x} \langle \widehat{a}^\dagger(p) \widehat{b}^\dagger(p') \rangle + e^{-iy \cdot (k + (p-p')/2)} e^{-i(p+p') \cdot x} \langle \widehat{b}(p) \widehat{a}(p') \rangle \\ &= \frac{2}{(2\pi)^7} \int \frac{d^3p}{2\varepsilon} \frac{d^3p'}{2\varepsilon'} \int d^4y e^{-iy \cdot (k - (p+p')/2)} e^{i(p-p') \cdot x} \langle \widehat{a}^\dagger(p) \widehat{a}(p') \rangle + e^{-iy \cdot (k + (p+p')/2)} e^{i(p-p') \cdot x} \langle \widehat{b}^\dagger(p') \widehat{b}(p) \rangle \\ &\quad + e^{-iy \cdot (k - (p-p')/2)} e^{i(p+p') \cdot x} \langle \widehat{a}^\dagger(p) \widehat{b}^\dagger(p') \rangle + e^{-iy \cdot (k - (p-p')/2)} e^{-i(p+p') \cdot x} \langle \widehat{b}(p') \widehat{a}(p) \rangle \\ &= \frac{2}{(2\pi)^3} \int \frac{d^3p}{2\varepsilon} \frac{d^3p'}{2\varepsilon'} e^{i(p-p') \cdot x} \left[\delta^4(k - (p+p')/2) \langle \widehat{a}^\dagger(p) \widehat{a}(p') \rangle + \delta^4(k + (p+p')/2) \langle \widehat{b}^\dagger(p') \widehat{b}(p) \rangle \right] \\ &\quad + \delta^4(k - (p-p')/2) \left[e^{i(p+p') \cdot x} \langle \widehat{a}^\dagger(p) \widehat{b}^\dagger(p') \rangle + e^{-i(p+p') \cdot x} \langle \widehat{b}(p') \widehat{a}(p) \rangle \right] \end{aligned} \quad (39)$$

where $\langle \rangle$ stands for the mean $\text{Tr}(\hat{\rho})$. In the above equalities, we have taken advantage of the symmetric integration in the variables p, p' .

The expression (39) makes it apparent that the variable k of the Wigner function $W(x, k)$ is *not* on-shell, i.e. $k^2 \neq m^2$ even in the free case. This makes the definition of a particle distribution function $f(x, p)$ à la Boltzmann not straightforward in a quantum relativistic framework. In the book by De Groot [22] it is shown that a distribution function can be defined in the limit of slowly varying $W(x, k)$ on the microscopic scale of the Compton wavelength. In fact, we will show that in the case of the free scalar field a distribution function can be defined without introducing such an approximation. From equation (39), we can infer that W is made up of three terms which can be distinguished for the characteristic of k . For future time-like $k = (p + p')/2$, only the first term involving particles is retained; for past time-like $k = -(p + p')/2$, only the second term involving antiparticles; finally, for space-like $k = (p - p')/2$ the last term with the mean values of two creation/destruction operators is retained. In symbols:

$$W(x, k) = W(x, k)\theta(k^2)\theta(k^0) + W(x, k)\theta(k^2)\theta(-k^0) + W(x, k)\theta(-k^2) \equiv W_+(x, k) + W_-(x, k) + W_S(x, k) \quad (40)$$

Local operators quadratic in the field can be expressed as four-dimensional integrals over k of the covariant Wigner function (35). For instance, the mean value of the conserved current of the scalar field is defined as [22]:

$$j^\mu(x) = i\langle : \hat{\psi}^\dagger(x) \overleftrightarrow{\partial}^\mu \hat{\psi}(x) : \rangle = \int d^4k k^\mu W(x, k) \quad (41)$$

By using the decomposition (40) for $W(x, k)$, the current can be written as the sum of three terms. The particle term, by using the (39), reads:

$$\begin{aligned} j_+^\mu(x) &= \frac{1}{(2\pi)^3} \int \frac{d^3p}{\varepsilon} \frac{d^3p'}{2\varepsilon'} \frac{(p + p')^\mu}{2} e^{i(p-p') \cdot x} \langle \hat{a}^\dagger(p) \hat{a}(p') \rangle \\ &= \int \frac{d^3p}{\varepsilon} p^\mu \text{Re} \left(\frac{1}{(2\pi)^3} \int \frac{d^3p'}{2\varepsilon'} e^{i(p-p') \cdot x} \langle \hat{a}^\dagger(p) \hat{a}(p') \rangle \right) \end{aligned} \quad (42)$$

To obtain the last expression, we have taken advantage of the hermiticity of the density operator, implying:

$$\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle = \langle \hat{a}^\dagger(p') \hat{a}(p) \rangle^*$$

which makes it possible to swap the integration variables p and p' . The formula (42) brings out a function that we can properly identify as the particle distribution function or phase space density, as the real part of a *complex distribution function* $f_c(x, p)$:

$$f(x, p) = \text{Re} f_c(x, p) \quad (43)$$

where

$$f_c(x, p) = \frac{1}{(2\pi)^3} \int \frac{d^3p'}{2\varepsilon'} e^{i(p-p') \cdot x} \langle \hat{a}^\dagger(p) \hat{a}(p') \rangle \quad (44)$$

Similarly, the antiparticle term in the current leads to a distribution function $\bar{f}(x, p)$ which is obtained from (44) replacing $\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle$ with $\langle \hat{b}^\dagger(p) \hat{b}(p') \rangle$. Finally, the last term in the current, can be written, by using (41) and (39):

$$\begin{aligned} j_S^\mu(x) &= \frac{1}{(2\pi)^3} \int \frac{d^3p}{\varepsilon} \frac{d^3p'}{2\varepsilon'} (p - p')^\mu \text{Re} \left(e^{i(p+p') \cdot x} \langle \hat{a}^\dagger(p) \hat{b}^\dagger(p') \rangle \right) \\ &= \frac{1}{(2\pi)^3} \int \frac{d^3p}{\varepsilon} \frac{d^3p'}{2\varepsilon'} p^\mu \text{Re} \left[e^{i(p+p') \cdot x} \left(\langle \hat{a}^\dagger(p) \hat{b}^\dagger(p') \rangle - \langle \hat{a}^\dagger(p') \hat{b}^\dagger(p) \rangle \right) \right] \end{aligned}$$

where, again, we have used the hermiticity of the density operator. Thereby, we could define a mixed distribution function g_c :

$$g_c(x, p) = \frac{1}{(2\pi)^3} \int \frac{d^3p'}{2\varepsilon'} \left[e^{i(p+p') \cdot x} \left(\langle \hat{a}^\dagger(p) \hat{b}^\dagger(p') \rangle - \langle \hat{a}^\dagger(p') \hat{b}^\dagger(p) \rangle \right) \right]$$

and write the whole current as:

$$j^\mu(x) = \int d^4k k^\mu W(x, k) = \text{Re} \int \frac{d^3p}{2\varepsilon} p^\mu [f_c(x, p) - \bar{f}_c(x, p) + g_c(x, p)] = \int \frac{d^3p}{2\varepsilon} p^\mu [f(x, p) - \bar{f}(x, p) + g(x, p)]$$

If $g_c = 0$, which is the most common case, the current can be formally written as the familiar relativistic kinetic formula, that is an integral over on-shell four-momenta of the four-momentum vector multiplied by on-shell phase space densities.

In general, an algebraic relation between e.g. $W_+(x, k)$ and $f(x, p)$ does not exist. Nevertheless, interesting integral relations between them can be obtained. It is not hard to show that, integrating the (39) in k , one gets:

$$\int d^4k W(x, k) = \int \frac{d^3p}{\varepsilon} (f_c(x, p) + \bar{f}_c(x, p) + g_c(x, p)) = \int \frac{d^3p}{\varepsilon} (f(x, p) + \bar{f}(x, p) + g(x, p)) \quad (45)$$

where the last equality follows from the vanishing of the imaginary part of the integral. Furthermore, if we integrate the time component of the particle current (42) over the hypersurface $t = \text{const}$ one retrieves the total number of particles:

$$N = \int d^3x j_+^0(x) = \int d^3x \frac{d^3p}{\varepsilon} \varepsilon f(x, p) = \int d^3p \int d^3x f(x, p)$$

the last expression confirms that $f(x, p)$ is the actual density of particles in phase space. Furthermore, according to the eq. (44):

$$\frac{dN}{d^3p} = \int d^3x f(x, p) = \frac{1}{2\varepsilon} \langle \hat{a}^\dagger(p) \hat{a}(p) \rangle$$

which is the expected relation between the particle density in momentum space in view of the (38).

B. The Dirac field

A similar connection can be built up for the spin 1/2 particles and the Dirac field. In this case, the covariant Wigner operator is a 4×4 spinorial matrix ⁴:

$$\begin{aligned} \widehat{W}(x, k)_{AB} &= -\frac{1}{(2\pi)^4} \int d^4y e^{-ik \cdot y} : \Psi_A(x - y/2) \bar{\Psi}_B(x + y/2) : \\ &= \frac{1}{(2\pi)^4} \int d^4y e^{-ik \cdot y} : \bar{\Psi}_B(x + y/2) \Psi_A(x - y/2) : \end{aligned} \quad (46)$$

Ψ being the Dirac field:

$$\Psi_A(x) = \sum_r \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2\varepsilon} \hat{a}_r(p) u_r(p)_A e^{-ip \cdot x} + \hat{b}_r^\dagger(p) v_r(p)_A e^{ip \cdot x} \quad (47)$$

In the equation (47), the creation and destruction operators are normalized according to the (38) with the anticommutator replacing the commutator, and $u_r(p), v_r(p)$ are the spinors of free particles and antiparticles in their polarization state r (usually a helicity or third spin component) normalized so as to $\bar{u}_r u_s = 2m \delta_{rs}$, $\bar{v}_r v_s = -2m \delta_{rs}$. The covariant Wigner function is, again, the mean value of the Wigner operator in (46). The definition (46) has to be modified in full spinor electrodynamics [23] to preserve gauge invariance, but this can be neglected for the scope of this work. Because of the Dirac equation, the Wigner operator solves the equation:

$$\left(m - \not{k} - \frac{i}{2} \not{\partial} \right) \widehat{W}(x, k) = 0$$

By plugging the (47) into the (46), we obtain:

$$\begin{aligned} \widehat{W}(x, k)_{AB} &= \sum_{r,s} \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\varepsilon} \frac{d^3p'}{2\varepsilon'} e^{-i(p-p') \cdot x} \left[\delta^4(k - (p+p')/2) \hat{a}_s^\dagger(p') \hat{a}_r(p) u_r(p)_A \bar{u}_s(p')_B \right. \\ &\quad \left. - \delta^4(k + (p+p')/2) \hat{b}_r^\dagger(p) \hat{b}_s(p') v_r(p)_A \bar{v}_s(p')_B \right] \\ &\quad - \delta^4(k - (p-p')/2) \left[e^{-i(p+p') \cdot x} \hat{a}_r(p) \hat{b}_s(p') u_r(p)_A \bar{v}_s(p')_B + e^{i(p+p') \cdot x} v_r(p')_A \bar{u}_s(p)_B \hat{b}_r^\dagger(p') \hat{a}_s^\dagger(p) \right] \end{aligned} \quad (48)$$

⁴ It should be reminded that the normal ordering for fermion fields involves a minus sign for each permutation, e.g. $aa^\dagger := -a^\dagger a$. Therefore, taking into account anticommutation relations, for fields $:\Psi_A(x) \bar{\Psi}_B(y): := -:\bar{\Psi}_B(y) \Psi_A(x):$

It can be seen that in the fermion case, the covariant Wigner operator can be split into future time-like (particle), past time-like (antiparticle) and space-like parts corresponding to the three terms of the right hand side, just as for the scalar field:

$$\widehat{W}(x, k) = \widehat{W}(x, k)\theta(k^2)\theta(k^0) + \widehat{W}(x, k)_{AB}\theta(k^2)\theta(-k^0) + \widehat{W}(x, k)\theta(-k^2) \equiv \widehat{W}_+(x, k) + \widehat{W}_-(x, k) + \widehat{W}_S(x, k)$$

Yet, unlike for the scalar field, we cannot identify a distribution function in phase space by integrating the covariant Wigner function in k . We can make this clear by calculating the mean current of the Dirac field (without vacuum contribution) which is obtained from the Wigner function through the formula [22]:

$$j^\mu(x) = \langle : \bar{\Psi}(x)\gamma^\mu\Psi(x) : \rangle = \int d^4k \operatorname{tr}(\gamma^\mu W(x, k))$$

We confine ourselves to the particle term of the (48), which, once fed into the above formula yields:

$$j_+^\mu(x) = \int d^4k \operatorname{tr}(\gamma^\mu W(x, k)_+) = \sum_{r,s} \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\varepsilon} \frac{d^3p'}{2\varepsilon'} e^{-i(p-p')\cdot x} \langle \hat{a}_s^\dagger(p') \hat{a}_r(p) \rangle \bar{u}_s(p') \gamma^\mu u_r(p) \quad (49)$$

Unlike in the (42) we cannot factorize the momentum integration because the spinors u have different momenta as argument. It thus follows that a reasonable definition of a particle distribution function with spin indices $f(x, p)_{rs}$ is precluded, except in the limit of very slow variation of the Wigner function as derived in ref. [22].

Notwithstanding, it is possible to establish an exact relation between the density of particles in momentum space and the covariant Wigner function. First of all, it can be shown, from (49), that:

$$\partial_\mu j_+^\mu = 0$$

and likewise for j_-^μ , taking into account that the spinors $u(p)$ and $\bar{u}(p)$ fulfill the equations:

$$(\not{p} - m)u(p) = 0 \quad \bar{u}(p)(\not{p} - m) = 0$$

If the divergence of j_+ vanishes, we can integrate the particle current (49) over an arbitrary 3D space-like hypersurface to get a constant particle number, provided that boundary fluxes vanish. For instance, we can integrate j_+^0 over the hypersurface $t = \text{const}$ to get:

$$\begin{aligned} N &= \int d^3x j_+^0(x) = \int d^3x \int d^4k \operatorname{tr}(\gamma^0 W(x, k)_+) = \sum_{r,s} \int \frac{d^3p}{2\varepsilon} \frac{d^3p'}{2\varepsilon'} \delta^3(\mathbf{p} - \mathbf{p}') \langle \hat{a}_s^\dagger(p) \hat{a}_s(p') \rangle \bar{u}_s(p') \gamma^0 u_r(p) \\ &= \sum_{r,s} \int \frac{d^3p}{4\varepsilon^2} \langle \hat{a}_r^\dagger(p) \hat{a}_s(p) \rangle \bar{u}_s(p) \gamma^0 u_r(p) = \sum_{r,s} \int \frac{d^3p}{4\varepsilon^2} \langle \hat{a}_r^\dagger(p) \hat{a}_s(p) \rangle 2\varepsilon \delta_{r,s} = \int \frac{d^3p}{2\varepsilon} \sum_r \langle \hat{a}_r^\dagger(p) \hat{a}_r(p) \rangle \end{aligned}$$

whence we obtain the particle density in momentum space, as expected:

$$\frac{dN}{d^3p} = \frac{1}{2\varepsilon} \sum_r \langle \hat{a}_r^\dagger(p) \hat{a}_r(p) \rangle \quad (50)$$

An important feature of the Wigner operator of free fields is that integrating it over a 3D hypersurface, k becomes an on-shell vector. Indeed, from (48):

$$k^\mu \partial_\mu \widehat{W}_\pm(x, k) = k^\mu \partial_\mu \widehat{W}_S(x, k) = 0$$

because, in taking the derivative $k \cdot \partial$ the factor $(p - p') \cdot (p + p') = 0$ is generated in all of the terms; the same applies to the Wigner operator of the scalar field. Therefore, provided that suitable boundary conditions are fulfilled, the integral over a space-like 3D hypersurface:

$$\int_\Sigma d\Sigma_\mu k^\mu \widehat{W}(x, k)$$

is independent of the hypersurface Σ . Thus, we can choose Σ at the hyperplane $t = 0$ and obtain, from (48):

$$\begin{aligned}
\int_{t=0} d\Sigma_\mu k^\mu \widehat{W}(x, k) &= k^0 \int d^3x \widehat{W}(x, k) = \sum_{r,s} k^0 \int \frac{d^3p}{2\varepsilon} \frac{d^3p'}{2\varepsilon'} \delta^3(\mathbf{p} - \mathbf{p}') [\delta^4(k-p) \widehat{a}_s^\dagger(p) \widehat{a}_r(p) u_r(p) \bar{u}_s(p) \\
&\quad - \delta^4(k+p) \widehat{b}_r^\dagger(p) \widehat{b}_s(p) v_r(p) \bar{v}_s(p)_B] + \delta(k^0) \delta^3(\mathbf{k} - \mathbf{p}) [\delta^3(\mathbf{p} + \mathbf{p}') \widehat{a}_r(p) \widehat{b}_s(p') u_r(p) \bar{v}_s(p') + v_r(p') \bar{u}_s(p) \widehat{b}_r^\dagger(p') \widehat{a}_s^\dagger(p)] \\
&= \sum_{r,s} k^0 \int \frac{d^3p}{4\varepsilon^2} [\delta^4(k-p) \widehat{a}_s^\dagger(p) \widehat{a}_r(p) u_r(p) \bar{u}_s(p) - \delta^4(k+p) \widehat{b}_r^\dagger(p) \widehat{b}_s(p) v_r(p) \bar{v}_s(p)] \\
&= \sum_{r,s} \frac{1}{2} \delta(k^2 - m^2) [\theta(k^0) \widehat{a}_s^\dagger(k) \widehat{a}_r(k) u_r(k) \bar{u}_s(k) + \theta(-k^0) \widehat{b}_r^\dagger(-k) \widehat{b}_s(-k) v_r(-k) \bar{v}_s(-k)_B] ; \tag{51}
\end{aligned}$$

note that the mixed term vanished because of the factor $k^0 \delta(k^0)$. The last obtained result proves the above statement, that is the argument k of the integral over a 3D spacelike hypersurface Σ is an on-shell four vector and it nicely separates particle-antiparticle contribution. Indeed, the on-shell operators \widehat{w}_\pm can be defined such that:

$$\frac{1}{2\varepsilon_k} \delta(k^0 - \varepsilon_k) \widehat{w}_+(k) = \int d\Sigma_\mu k^\mu \widehat{W}_+(x, k) \quad \frac{1}{2\varepsilon_k} \delta(k^0 + \varepsilon_k) \widehat{w}_-(k) = \int d\Sigma_\mu k^\mu \widehat{W}_-(x, k) \tag{52}$$

so that, by comparing with the (51):

$$\widehat{w}_+(k) = \sum_{r,s} \frac{1}{2} \widehat{a}_s^\dagger(k) \widehat{a}_r(k) u_r(k) \bar{u}_s(k) \tag{53}$$

with k on-shell. A similar equation can be established for \widehat{W}_- and antiparticles. Note that, from (53):

$$(\not{k} - m) \widehat{w}_+(k) = \widehat{w}_+(k) (\not{k} - m) = 0 \tag{54}$$

Now, multiplying the (53) by $\bar{u}_r(k)$ to the left and $u_s(k)$ to the right, and keeping in mind the normalization of the spinors u , we get:

$$\bar{u}_r(k) \widehat{w}_+(k) u_s(k) = 2m^2 \widehat{a}_s^\dagger(k) \widehat{a}_r(k) \tag{55}$$

This formula will be used in the next section to express the spin density matrix. Now, setting $r = s$, summing over r and taking the mean value with the suitable density operator, we obtain:

$$\sum_r \bar{u}_r(k) w_+(k) u_r(k) = \text{tr} \left(w_+(k) \sum_r u_r(k) \bar{u}_r(k) \right) = 2m^2 \sum_r \langle \widehat{a}_r^\dagger(k) \widehat{a}_r(k) \rangle = 4m^2 \varepsilon \frac{dN}{d^3k}$$

where we have used the (50). Since:

$$\sum_r u_r(k) \bar{u}_r(k) = \not{k} + m$$

we have:

$$\varepsilon \frac{dN}{d^3k} = \frac{1}{4m^2} \text{tr} (w_+(k) (\not{k} + m))$$

and, by using the (54), we finally obtain the sought relation between the momentum spectrum and the covariant Wigner function:

$$\varepsilon \frac{dN}{d^3k} = \frac{1}{2m} \text{tr} w_+(k) = \frac{\varepsilon}{m} \int dk^0 \int d\Sigma_\mu k^\mu \text{tr} W_+(x, k) \tag{56}$$

V. FERMION POLARIZATION AND THE COVARIANT WIGNER FUNCTION

We are now in a position to derive an exact formula connecting the Wigner function to the spin density matrix and the spin vector for spin 1/2 fermions. The derivation of $\Theta(p)$ for particles is now a straightforward consequence of its definition (15) and of the (55):

$$\Theta(p)_{rs} = \frac{\bar{u}_r(p) w_+(p) u_s(p)}{\sum_t \bar{u}_t(p) w_+(p) u_t(p)}$$

This formula can be also written in an expanded form by using the actual covariant Wigner function by using the (52) and taking advantage of the cancellation of the Dirac deltas in the ratio:

$$\Theta(p)_{rs} = \frac{\int d\Sigma_\mu p^\mu \bar{u}_r(p) W_+(x, p) u_s(p)}{\sum_t \int d\Sigma_\mu p^\mu \bar{u}_t(p) W_+(x, p) u_t(p)} \quad (57)$$

keeping in mind that p is on-shell because of the integration. As we have emphasized in the previous section, as long as one deals with free fields, the integration hypersurface is arbitrary, and this will be important for the use of (57) in relativistic heavy ion collisions. The above matrix can be written in a more compact way by introducing 4×2 spinorial matrices U (and corresponding 2×4 \bar{U}) such that $U_{A,r}(p) = u_r(p)_A$:

$$\Theta(p) = \frac{\int d\Sigma_\mu p^\mu \bar{U}(p) W_+(x, p) U(p)}{\text{tr}_2 \int d\Sigma_\mu p^\mu \bar{U}(p) W_+(x, p) U(p)} \quad (58)$$

where, henceforth, we will make explicit the distinction between the trace over the polarization states tr_2 and the trace over the four spinorial indices tr_4 . In the Weyl's representation, which is deeply connected to the theory of Lorentz'group representations, these spinors can be written as [20, 24]:

$$U(p) = \sqrt{m} \begin{pmatrix} D^S([p]) \\ D^S([p]^{\dagger-1}) \end{pmatrix} \quad V(p) = \sqrt{m} \begin{pmatrix} D^S([p]C^{-1}) \\ D^S([p]^{\dagger-1}C) \end{pmatrix} \quad (59)$$

with $C = i\sigma_2$ (σ_i being Pauli matrices).

The mean spin vector can now be calculated from (14) by using the eq. (57) therein. We first observe that, since $S^\mu(p)$ is a real number, we can also express it as:

$$\begin{aligned} S^\mu(p) &= -\frac{1}{4m} \epsilon^{\mu\beta\gamma\delta} p_\delta [\text{tr}_2(D^S([p]^{-1})D^S(J_{\beta\gamma})D^S([p])\Theta(p)) + \text{tr}_2(D^S([p]^{-1})D^S(J_{\beta\gamma})D^S([p])\Theta(p))^*] \\ &= -\frac{1}{4m} \epsilon^{\mu\beta\gamma\delta} p_\delta [\text{tr}_2(D^S([p]^{-1})D^S(J_{\beta\gamma})D^S([p])\Theta(p)) + \text{tr}_2(D^S([p])^\dagger D^S(J_{\beta\gamma})^\dagger D^S([p]^{-1})^\dagger \Theta(p))] \end{aligned} \quad (60)$$

where we have taken advantage of the hermiticity of $\Theta(p)$ and the cyclicity of the trace. We can now use the (58) and work out the numerator first:

$$\text{tr}_2(D^S([p]^{-1})D^S(J_{\beta\gamma})D^S([p])\bar{U}(p)W_+(x, p)U(p) + \text{tr}_2(D^S([p])^\dagger D^S(J_{\beta\gamma})^\dagger D^S([p]^{-1})^\dagger \bar{U}(p)W_+(x, p)U(p)) \quad (61)$$

where U are the spinors defined in (59). This expression can be written in a more compact and familiar form in the Dirac spinorial formalism. We start by defining:

$$\Sigma_{\beta\gamma} = \begin{pmatrix} D^S(J_{\beta\gamma}) & 0 \\ 0 & D^S(J_{\beta\gamma})^\dagger \end{pmatrix} \quad (62)$$

which is just the generator of Lorentz transformations written for the full spinorial representation $(0, 1/2) \oplus (1/2, 0)$ of the Dirac field, equal to $(i/4)[\gamma_\beta, \gamma_\gamma]$. It can be readily seen that the (61) is equivalent to:

$$\frac{1}{m} \text{tr}_2(\bar{U}(p)\Sigma_{\beta\gamma}U(p)\bar{U}(p)W_+(x, p)U(p))$$

In general, if A is a 2×4 and B is a 4×2 matrix:

$$\text{tr}_2 AB = \text{tr}_4 BA$$

hence the above trace can be rewritten:

$$\frac{1}{m} \text{tr}_4(\Sigma_{\beta\gamma}U(p)\bar{U}(p)W_+(x, p)U(p)\bar{U}(p))$$

which can be worked out taking into account that:

$$U(p)\bar{U}(p) = \sum_r u_r(p)\bar{u}_r(p) = \not{p} + m$$

Likewise, the denominator of (57) can be rewritten as:

$$\text{tr}_2(\bar{U}(p)W_+(x, p)U(p)) = \text{tr}_4(W_+(x, p)U(p)\bar{U}(p)) = \text{tr}_4((\not{p} + m)W_+(x, p))$$

Putting all together, we can write the mean spin vector as:

$$S^\mu(p) = -\frac{1}{4m^2} \epsilon^{\mu\beta\gamma\delta} p_\delta \frac{\int d\Sigma_\lambda p^\lambda \text{tr}_4(\Sigma_{\beta\gamma}(\not{p} + m)W_+(x, p)(\not{p} + m))}{\int d\Sigma_\lambda p^\lambda \text{tr}_4((\not{p} + m)W_+(x, p))} \quad (63)$$

Likewise, one can also recast the equation (14) - the exact expression of the mean spin vector at global equilibrium in the Boltzmann limit - for spin 1/2 particles in the Dirac formalism. With the foregoing definitions and notations, the equation (31) can be rewritten as:

$$\Theta(p) = \frac{\bar{U}(p) \exp\left[\frac{1}{2}\varpi : \Sigma\right] U(p)}{\text{tr}_2(\bar{U}(p) \exp\left[\frac{1}{2}\varpi : \Sigma\right] U(p))}$$

which is a hermitian matrix. By using this equation, and the (60), (61) the mean spin vector of (14) can be rewritten by simply replacing the integrals of $W_+(x, p)$ with $\exp\left[\frac{1}{2}\varpi : \Sigma\right]$ in the equation (63). We thus obtain:

$$S^\mu(p) = -\frac{1}{4m^2} \epsilon^{\mu\beta\gamma\delta} p_\delta \frac{\text{tr}_4(\Sigma_{\beta\gamma}(\not{p} + m) \exp\left[\frac{1}{2}\varpi : \Sigma\right] (\not{p} + m))}{\text{tr}_4((\not{p} + m) \exp\left[\frac{1}{2}\varpi : \Sigma\right])} \quad (64)$$

Taking into account that the trace of an odd number of gamma matrices vanishes and the commutator:

$$[\Sigma_{\beta\gamma}, \gamma_\lambda] = -ig_{\beta\lambda}\gamma_\gamma + ig_{\gamma\lambda}\gamma_\beta \quad (65)$$

it can be readily shown that the (64) can be written in the simpler form:

$$S^\mu(p) = -\frac{1}{2m} \epsilon^{\mu\beta\gamma\delta} p_\delta \frac{\text{tr}_4(\Sigma_{\beta\gamma} \exp\left[\frac{1}{2}\varpi : \Sigma\right])}{\text{tr}_4(\exp\left[\frac{1}{2}\varpi : \Sigma\right])} \quad (66)$$

which looks certainly more suggestive and compact with respect to the general group-theoretical formula.

Also the more general equation (63) can be further developed and simplified. According to the (54):

$$\not{p} \int d\Sigma_\lambda p^\lambda W_+(x, p) = m \int d\Sigma_\lambda p^\lambda W_+(x, p)$$

the (63) becomes:

$$S^\mu(p) = -\frac{1}{2m} \epsilon^{\mu\beta\gamma\delta} p_\delta \frac{\int d\Sigma_\lambda p^\lambda \text{tr}_4(\Sigma_{\beta\gamma} W_+(x, p))}{\int d\Sigma_\lambda p^\lambda \text{tr}_4 W_+(x, p)} \quad (67)$$

which is already quite a suggestive formula. Furthermore, because of the (51) we can write:

$$\begin{aligned} \int d\Sigma_\lambda p^\lambda \text{tr}_4(\Sigma_{\beta\gamma} W_+(x, p)) &= \sum_{r,s} \frac{1}{4\varepsilon_p} \delta(p^0 - \varepsilon_p) \langle \hat{a}_s^\dagger(p) \hat{a}_r(p) \rangle \text{tr}_4(u_r(p) \bar{u}_s(p) \Sigma_{\beta\gamma}) \\ &= \sum_{r,s} \frac{1}{4\varepsilon_p} \delta(p^0 - \varepsilon_p) \langle \hat{a}_s^\dagger(p) \hat{a}_r(p) \rangle \bar{u}_s(p) \Sigma_{\beta\gamma} u_r(p) \end{aligned}$$

Now, by using the spinorial relation:

$$m \bar{u}_r(p) \Sigma^{\mu\nu} \gamma_\lambda u_s(p) = \bar{u}_r(p) \Sigma^{\mu\nu} u_s(p) p_\lambda - 2i \bar{u}_r(p) (\gamma^\mu p^\nu - \gamma^\nu p^\mu) \gamma_\lambda u_s(p)$$

the previous equation turns into:

$$\begin{aligned} \int d\Sigma_\lambda p^\lambda \text{tr}_4(\Sigma_{\beta\gamma} W_+(x, p)) &= \sum_{r,s} \frac{1}{4\varepsilon_p^2} \delta(p^0 - \varepsilon_p) \langle \hat{a}_s^\dagger(p) \hat{a}_r(p) \rangle m \bar{u}_s(p) \Sigma_{\beta\gamma} \gamma_0 u_r(p) \\ &\quad + 2i \sum_{r,s} \frac{1}{4\varepsilon_p^2} \delta(p^0 - \varepsilon_p) \langle \hat{a}_s^\dagger(p) \hat{a}_r(p) \rangle \bar{u}_s(p) (\gamma_\beta p_\gamma - \gamma_\gamma p_\beta) \gamma_0 u_r(p) \end{aligned} \quad (68)$$

The second term in the right hand side will not contribute to the mean spin vector (67) because of two momenta multiplying the Levi-Civita tensor. By using the (65), the first term in the right hand side of (68) can be rewritten:

$$\begin{aligned} & \sum_{r,s} \frac{m}{4\varepsilon_p^2} \delta(p^0 - \varepsilon_p) \langle \hat{a}_s^\dagger(p) \hat{a}_r(p) \rangle \left[\bar{u}_s(p) \frac{1}{2} \{ \gamma_0, \Sigma_{\beta\gamma} \} u_r(p) + i g_{0\gamma} \bar{u}_s(p) \gamma_\beta u_r(p) - i g_{0\beta} \bar{u}_s(p) \gamma_\gamma u_r(p) \right] \\ &= \sum_{r,s} \frac{m}{4\varepsilon_p^2} \delta(p^0 - \varepsilon_p) \langle \hat{a}_s^\dagger(p) \hat{a}_r(p) \rangle \left[\bar{u}_s(p) \frac{1}{2} \{ \gamma_0, \Sigma_{\beta\gamma} \} u_r(p) + 2i \delta_{rs} (p_\beta g_{0\gamma} - p_\gamma g_{0\beta}) \right] \end{aligned} \quad (69)$$

where we have used the relation

$$\bar{u}_s(p) \gamma^\lambda u_r(p) = 2p^\lambda \delta_{rs}$$

Again, the second term in the right hand side of (69) does not contribute to the mean spin vector because of the Levi-Civita tensor in (67). Now, since:

$$\{ \gamma_\lambda, \Sigma_{\beta\gamma} \} = \epsilon_{\sigma\lambda\beta\gamma} \gamma^\sigma \gamma^5 \quad (70)$$

we can finally rewrite the numerator of the mean spin vector (67) as:

$$\begin{aligned} & -\frac{1}{4} \epsilon^{\mu\beta\gamma\delta} \epsilon_{\sigma 0 \beta \gamma} p_\delta \sum_{r,s} \frac{1}{4\varepsilon_p^2} \delta(p^0 - \varepsilon_p) \langle \hat{a}_s^\dagger(p) \hat{a}_r(p) \rangle \bar{u}_s(p) \gamma^\sigma \gamma^5 u_r(p) \\ &= p_0 \sum_{r,s} \frac{1}{8\varepsilon_p^2} \delta(p^0 - \varepsilon_p) \langle \hat{a}_s^\dagger(p) \hat{a}_r(p) \rangle \bar{u}_s(p) \gamma^\mu \gamma^5 u_r(p) - \delta_0^\mu p_\sigma \sum_{r,s} \frac{1}{8\varepsilon_p^2} \delta(p^0 - \varepsilon_p) \langle \hat{a}_s^\dagger(p) \hat{a}_r(p) \rangle \bar{u}_s(p) \gamma^\sigma \gamma^5 u_r(p) \end{aligned}$$

The second term vanishes because:

$$\bar{u}_s(p) \not{p} \gamma^5 u_r(p) = m \bar{u}_s(p) \gamma^5 u_r(p) = 0$$

so we have, for the numerator of the (67):

$$\sum_{r,s} \frac{1}{8\varepsilon_p} \delta(p^0 - \varepsilon_p) \langle \hat{a}_s^\dagger(p) \hat{a}_r(p) \rangle \bar{u}_s(p) \gamma^\mu \gamma^5 u_r(p)$$

This expression can be rewritten in form of an integral over an arbitrary hypersurface of a divergence-free integrand, and so we get, by using again (51):

$$S^\mu(p) = \frac{1}{2} \frac{\int d\Sigma \cdot p \operatorname{tr}_4(\gamma^\mu \gamma^5 W_+(x, p))}{\int d\Sigma \cdot p \operatorname{tr}_4 W_+(x, p)} \quad (71)$$

Hence, the mean spin vector is proportional to the integral of the axial vector component of the covariant Wigner function over some arbitrary 3D space-like hypersurface. Note that $S(p)$ is actually orthogonal to the four-momentum p because $\operatorname{tr}_4(\not{p} \gamma^5 W_+) = 0$ [23] (this can be shown also by using the expansion (48)). This expression is consistent and extends the relation between $W(x, p)$ and $S^\mu(p)$ at $\mathcal{O}(\hbar)$ used in refs. [9, 11, 25] to determine the mean spin vector.

Finally, by using the inverse of the (70), that is:

$$\gamma^\mu \gamma^5 \delta_\lambda^\delta = \gamma^\delta \gamma^5 \delta_\lambda^\mu - \frac{1}{2} \epsilon^{\mu\delta\alpha\beta} \{ \gamma_\lambda, \Sigma_{\alpha\beta} \}$$

and taking into account that $\operatorname{tr}_4(\not{p} \gamma^5 W_+) = 0$ we obtain another form of the (71):

$$S^\mu(p) = -\frac{1}{4} \epsilon^{\mu\beta\gamma\delta} p_\delta \frac{\int d\Sigma_\lambda \operatorname{tr}_4(\{ \gamma^\lambda, \Sigma_{\beta\gamma} \} W_+(x, p))}{\int d\Sigma_\lambda p^\lambda \operatorname{tr}_4 W_+(x, p)} \quad (72)$$

In the numerator, the educated reader shall recognize the matrix defining the canonical spin tensor of the Dirac field. However, it should be pointed out that the appearance of this combination does not imply the need of a particular spin tensor to find the expression (72), unlike originally stated in ref. [15]. The point is that the polarization expression was obtained without any reference whatsoever to the the tensors that are used to express the energy-momentum and angular momentum of the fields; this was already implied in the expressions of the mean spin vector quoted in refs. [13, 26] and will be discussed in more detail in section VI.

Indeed, one could have chosen to express the relation between mean spin vector and covariant Wigner function through the (67) or equally well with the eq. (71); they are completely equivalent forms of the (72).

VI. POLARIZATION FROM THE ANGULAR MOMENTUM OPERATOR

We have seen how to calculate the mean spin vector from the spin density matrix definition in quantum field theory, see eq. (15). It is possible to calculate the same quantity with a different method. Assume that the total angular momentum tensor $J^{\mu\nu}$ can be decomposed on-shell in momentum space, so that we know the total angular momentum tensor of particles with given four-momentum p , say $\tilde{J}^{\mu\nu}(p)$. We could then be able to obtain the mean spin vector $S^\mu(p)$ by simply dividing this quantity by the number of particles with momentum p and multiplying by the usual Levi-Civita like in the definition (1):

$$S^\mu(p) = -\frac{1}{2m} \epsilon^{\mu\nu\rho\sigma} p_\sigma \frac{\tilde{J}_{\nu\rho}(p)}{\frac{dN}{d^3p}} \quad (73)$$

This definition makes perfect sense as all particles with given momentum p have the same rest frame and was indeed used in ref. [15] to derive the expression of the mean spin vector at local thermodynamic equilibrium. We will show that the (73) leads to the eq. (71) as well.

The first step is to prove that $\tilde{J}_{\nu\rho}(p)$ exists and to find its form for the free Dirac field. Indeed, it is possible to show, under quite general assumptions, that conserved *charges* of free fields can be written as integrals over on-shell momentum four-vectors. In general, conserved charges can be written as integrals over a space-like 3D-hypersurface Σ of a divergenceless current, also at operator level:

$$\hat{Q}^{\mu_1 \dots \mu_N} = \int_{\Sigma} d\Sigma_{\lambda} \hat{J}^{\lambda \mu_1 \dots \mu_N}$$

With a suitable choice of the 3D boundaries, the above integral is independent of the hypersurface Σ and it can then be calculated using any space-like Σ , e.g. $t = 0$. Suppose now that the normal-ordered current $:\hat{J}: (normal ordering is necessary to have vanishing currents in vacuum)$ can be expressed as an integral over k of some tensor functional of the covariant Wigner function $\widehat{W}(x, k)$:

$$:\hat{J}^{\lambda \mu_1 \dots \mu_N}: = \int d^4k \mathcal{F}[\widehat{W}(x, k)]^{\lambda \mu_1 \dots \mu_N} \quad (74)$$

This is the case, for instance, for the charge current $:\hat{j}^\mu:$ of the scalar field, for which the functional would simply be $k^\mu \widehat{W}(x, k)$, or the current of the Dirac field for which it would be $\text{tr}_4(\gamma^\mu \widehat{W}(x, k))$, but it also applies to stress-energy tensor and spin tensor, as we will see. Using eq. (74), and taking advantage of the independence of the integration hypersurface, the charge can be written as:

$$\begin{aligned} :\hat{Q}^{\mu_1 \dots \mu_N}: &= \int_{\Sigma} d\Sigma_{\lambda} : \hat{J}^{\lambda \mu_1 \dots \mu_N}: = \int_{\Sigma} d\Sigma_{\lambda} \int d^4k \mathcal{F}[\widehat{W}(x, k)]^{\lambda \mu_1 \dots \mu_N} \\ &= \int_{t=0} d^3x \int d^4k \mathcal{F}[\widehat{W}(x, k)]^{0 \mu_1 \dots \mu_N} = \int d^4k \left(\int_{t=0} d^3x \mathcal{F}[\widehat{W}(x, k)]^{0 \mu_1 \dots \mu_N} \right) \end{aligned} \quad (75)$$

For *free fields*, the integration over x generally implies that the four-momentum k is on shell. In fact, this depends on the specific form of the functional \mathcal{F} , but it holds for all cases of interest, and the proof is the same which led to the equation (51); after the last integration in d^3x a factor $\delta(k^2 - m^2)$ comes in which makes it possible to separate particle and antiparticle contribution and to express the generally conserved normal-ordered charge as:

$$\begin{aligned} :\hat{Q}^{\mu_1 \dots \mu_N}: &= \int d^4k \delta(k^2 - m^2) \hat{Q}(k)^{\mu_1 \dots \mu_N} = \int d^3k \left(\int dk^0 \delta(k^2 - m^2) \hat{Q}(k)^{\mu_1 \dots \mu_N} \right) \\ &\equiv \int \frac{d^3k}{2\varepsilon_k} (\hat{q}_+(k)^{\mu_1 \dots \mu_N} + \hat{q}_-(k)^{\mu_1 \dots \mu_N}) \end{aligned} \quad (76)$$

Altogether, the charge can be written as a sum over three-momenta of on-shell particles and antiparticles and a spectral decomposition in momentum space is obtained:

$$\frac{d : \hat{Q}^{\mu_1 \dots \mu_N} :}{d^3k} = \frac{1}{2\varepsilon_k} \hat{q}_+(k)^{\mu_1 \dots \mu_N} = \int dk^0 \int d\Sigma_{\lambda} \mathcal{F}[\widehat{W}_+(x, k)]^{\lambda \mu_1 \dots \mu_N} \quad (77)$$

and likewise for antiparticles. The operators $\hat{q}_{\pm}(k)$ are invariant by the addition of a total divergence to the current. For instance, for the vector current:

$$\hat{j}^{\lambda} \rightarrow \hat{j}^{\lambda} + \partial_{\alpha} \hat{A}^{\lambda\alpha}$$

where $\hat{A}^{\lambda\alpha}$ is an anti-symmetric tensor, the corresponding \hat{q}_{\pm} get changed by:

$$\hat{q}_{\pm} \rightarrow \hat{q}_{\pm} + \int dk^0 \int d\Sigma_{\lambda} \partial_{\alpha} \mathcal{A}[\widehat{W}_{\pm}(x, k)]^{\lambda\alpha}$$

where \mathcal{A} is the suitable functional of the Wigner operator associated to $\hat{A}^{\lambda\alpha}$. The integral over the 3D hypersurface, for fixed k , can be turned into a boundary surface integral by means of the Stokes theorem and so, provided that the suitable boundary conditions are enforced, vanishes.

We now look for the spectral decomposition of the angular momentum-boost operators, hence the $\tilde{J}^{\mu\nu}(p)$ of the equation (73). The angular momentum-boost operator is the generator of the Lorentz transformations and can be written as:

$$\hat{J}^{\mu\nu} = \int_{\Sigma} d\Sigma_{\lambda} \hat{\mathcal{J}}^{\lambda, \mu\nu} = \int_{\Sigma} d\Sigma_{\lambda} \left(x^{\mu} \hat{T}^{\lambda\nu} - x^{\nu} \hat{T}^{\lambda\mu} + \hat{\mathcal{S}}^{\lambda, \mu\nu} \right) \quad (78)$$

with Σ space-like hypersurface. There are two contributing terms: the so-called *orbital* part, depending on the stress-energy tensor, and the *spin tensor* operator $\hat{\mathcal{S}}$. The generator in (78) is invariant under a so-called *pseudo-gauge transformation* of the stress-energy and spin tensor [27] which amounts to add a divergence to the angular momentum-boost current $\hat{\mathcal{J}}^{\lambda, \mu\nu}$. The choice of a stress-energy and a spin tensor is just a matter of convenience, and, for the Dirac field, a convenient choice is the so-called *canonical* stress-energy and spin tensor:

$$\begin{aligned} : \hat{T}^{\mu\nu}(x) : &= \frac{i}{2} : \bar{\Psi}(x) \gamma^{\mu} \overleftrightarrow{\partial}^{\nu} \Psi(x) : = \int d^4k \, k^{\nu} \text{tr}_4(\gamma^{\mu} \widehat{W}(x, k)) \\ : \hat{\mathcal{S}}^{\lambda, \mu\nu}(x) : &= \frac{1}{2} : \bar{\Psi}(x) \{ \gamma^{\lambda}, \Sigma^{\mu\nu} \} \Psi(x) : = \frac{1}{2} \int d^4k \, \text{tr}_4(\{ \gamma^{\lambda}, \Sigma^{\mu\nu} \} \widehat{W}(x, k)) \end{aligned} \quad (79)$$

where their relations with the covariant Wigner function have been written down. Let us start with the *orbital* part of the angular momentum-boost operator:

$$: \hat{L}^{\mu\nu} : \equiv \int d\Sigma_{\lambda} \left(x^{\mu} : \hat{T}^{\lambda\nu} : - x^{\nu} : \hat{T}^{\lambda\mu} : \right) = \int d\Sigma_{\lambda} \int d^4k \left(x^{\mu} k^{\nu} \text{tr}_4(\gamma^{\lambda} \widehat{W}(x, k)) - x^{\nu} k^{\mu} \text{tr}_4(\gamma^{\lambda} \widehat{W}(x, k)) \right) \quad (80)$$

The subtlety here is that the functional \mathcal{F} , that we can write as:

$$\mathcal{F} = x^{\mu} k^{\nu} \text{tr}_4(\gamma^{\lambda} W(x, k)) - (\mu \leftrightarrow \nu)$$

explicitly depends on x and so the proof of the on-shellness of k must be reviewed, what is done in detail in Appendix A. The result of this analysis is that the orbital part of the angular momentum operator, with the canonical stress-energy tensor in (79), can be written as:

$$: \hat{L}^{\mu\nu} : = \int \frac{d^3k}{2\varepsilon_k} \left(k^{\mu} \hat{G}_{+}^{\nu}(k) - k^{\nu} \hat{G}_{+}^{\mu}(k) \right) + \text{antiparticle term}$$

with k on-shell and with \hat{G} a vector operator (see Appendix A). Thus, the orbital part of the angular momentum does not contribute to the mean spin vector because of the Levi-Civita tensor which makes the orbital part vanishing.

On the other hand, the canonical spin tensor term in (78) has an algebraic dependence on the Wigner function, according to the (79) and the equation (77) can be applied with

$$\mathcal{F}[\widehat{W}_{+}(x, k)] = \frac{1}{2} \text{tr}_4 \left(\{ \gamma^{\lambda}, \Sigma^{\mu\nu} \} \widehat{W}_{+}(x, k) \right)$$

so the spin part of the total angular momentum-boost tensor is:

$$: \hat{\mathcal{S}}^{\mu\nu} : = \int d^4k \int d\Sigma_{\lambda} \text{tr}_4 \left(\frac{1}{2} \{ \gamma^{\lambda}, \Sigma^{\mu\nu} \} \widehat{W}_{+}(x, k) \right) + \text{antiparticle term} \quad (81)$$

Therefore, its contribution to the function $\tilde{J}^{\mu\nu}(p)$ for particles in (73) is (with k renamed p):

$$\tilde{S}_{+}^{\mu\nu}(p) = \int dp^0 \int d\Sigma_{\lambda} \text{tr}_4 \left(\frac{1}{2} \{ \gamma^{\lambda}, \Sigma^{\mu\nu} \} W_{+}(x, p) \right)$$

Now, in the equation (73) we can replace $\tilde{J}^{\mu\nu}$ with the above expression and use the formula (56) for the particle density in momentum space, obtaining:

$$S^\mu(p) = -\frac{1}{4m}\epsilon^{\mu\nu\rho\sigma}p_\sigma \frac{\int dp^0 \int d\Sigma_\lambda \text{tr}_4(\{\gamma^\lambda, \Sigma^{\nu\rho}\}W_+(x,p))}{\frac{1}{2m\varepsilon}\text{tr}_4 w_+(p)} = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}p_\sigma \frac{\int dp^0 \int d\Sigma_\lambda \text{tr}_4(\{\gamma^\lambda, \Sigma^{\nu\rho}\}W_+(x,p))}{\int dp^0 \int d\Sigma \cdot p \text{tr}_4 W_+(x,p)} \quad (82)$$

where we have used the eq. (52) integrated in p^0 . We can also recast the above formula by taking advantage of the cancellation of $\delta(k^0 - \varepsilon_k)$ in the ratio:

$$S^\mu(p) = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}p_\sigma \frac{\int d\Sigma_\lambda \text{tr}_4(\{\gamma^\lambda, \Sigma^{\mu\nu}\}W_+(x,p))}{\int d\Sigma_\lambda p^\lambda \text{tr}_4 W_+(x,p)}$$

which is precisely the (71). Hence, the method described in this section leads to the same result obtained in section V.

It is worth stressing the independence of the expression of $S^\mu(p)$ in eq. (82) of the particular couple of stress-energy and spin tensor chosen to calculate the total angular momentum spectral decomposition $\tilde{J}^{\mu\nu}(p)$. If we had used the Belinfante symmetrized tensor:

$$:\hat{T}_B^{\mu\nu}(x) := \frac{i}{4} : \bar{\Psi}(x)\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi(x) + \bar{\Psi}(x)\gamma^\nu \overleftrightarrow{\partial}^\mu \Psi(x) : = \frac{1}{2} \int d^4k k^\nu \text{tr}_4(\gamma^\mu \widehat{W}(x,k)) + k^\mu \text{tr}_4(\gamma^\nu \widehat{W}(x,k)) \quad (83)$$

with associated vanishing spin tensor, for the derivation of the mean spin vector, we would have obtained the same expression (82). This happens because the Belinfante associated “orbital” angular momentum (which is actually the only term as $\hat{S}_B = 0$) implies more terms in the decomposition with respect to the equation (80) (this is discussed at the end of Appendix A).

To conclude, as it was already discussed at the end of section V, the *expression* of the particle polarization as a function of momentum is independent of the pseudo-gauge transformation of stress-energy and spin tensor. For the Dirac field, the canonical stress-energy and spin tensor are actually the most convenient to obtain it by the method presented in this section, and yet, the same expression could be derived by using the Belinfante pseudo-gauge. The appearance of the canonical spin tensor in the eq. (72) does not give it a special physical meaning and, indeed, the equivalent forms (71),(67) do not feature the canonical spin tensor. However, the *value* of the mean spin vector, as well as any other quantity, may depend on the spin tensor because the density operator at local thermodynamic equilibrium is sensitive to the pseudo-gauge transformations [16, 28]. Particularly, it is the Wigner function itself which acquires a dependence on the pseudo-gauge transformations through the density operator.

VII. LOCAL THERMODYNAMIC EQUILIBRIUM

We have seen in the previous sections how the spin density matrix and the mean spin vector relate to the covariant Wigner function. In turn, the covariant Wigner function depends on the density operator $\hat{\rho}$, see e.g. eq. (36) and it is thus necessary to know the density operator to calculate it.

For a relativistic fluid which, at some time, is believed to have achieved local thermodynamic equilibrium, a powerful approach is the Zubarev’s method of the stationary Non-Equilibrium Density Operator (NEDO) [29]. We refer the reader to the recent paper [30] for a detailed description.

This approach is especially well suited for the physics of relativistic nuclear collisions, where the system supposedly achieves Local Thermodynamic Equilibrium (LTE) at some finite early “time” (in the most used model, a finite hyperbolic time $\tau = \sqrt{t^2 - z^2}$, see ref. [31]), to form a Quark Gluon Plasma (QGP) which lives in a finite space-time region before breaking up at some 3D hypersurface Σ_{FO} (see figure 1). The actual density operator, in the Heisenberg representation, must be a fixed, time and space independent, operator and for a fluid at local thermodynamic equilibrium it is obtained by maximizing the entropy $S = -\text{tr}(\hat{\rho} \log \hat{\rho})$ with the constraints of energy-momentum and charge densities [18]. The result is:

$$\hat{\rho} = \frac{1}{Z} \exp \left[- \int_{\Sigma_0} d\Sigma n_\mu \left(\hat{T}^{\mu\nu}(x) \beta_\nu(x) - \zeta(x) \hat{j}^\mu(x) \right) \right] \quad (84)$$

where β is the four-temperature vector, ζ the ratio between chemical potential and temperature and Σ_0 is the initial 3D hypersurface where LTE is achieved. For relativistic nuclear collisions, this is supposedly the 3D hyperbolic hypersurface $\tau = \tau_0$, the Σ_{eq} in figure 1. It should be pointed out that the form of the local equilibrium density operator is pseudo-gauge dependent [16]; the above form applies to the Belinfante stress-energy tensor only, so in the rest of the section it will be understood that \hat{T} is the Belinfante symmetrized stress-energy tensor.

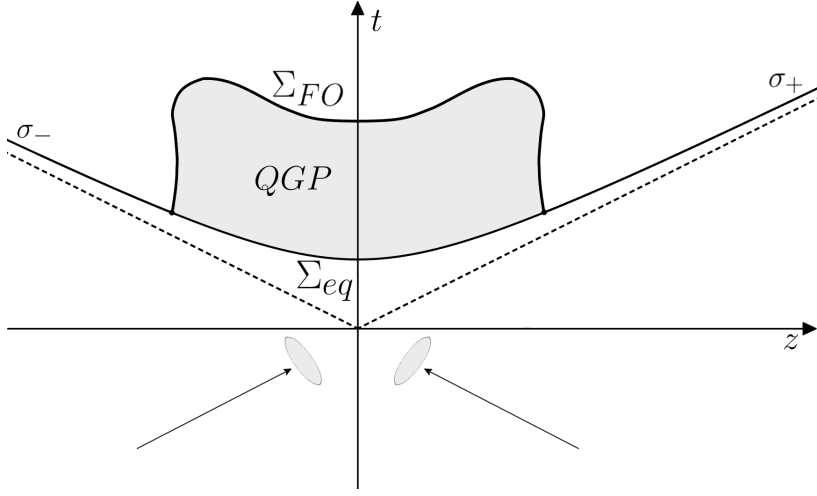


FIG. 1: Space-time diagram of a relativistic nuclear collision at very high energy. The hypersurface Σ_{eq} corresponds to the achievement of local thermodynamic equilibrium while Σ_{FO} is the hypersurface where the Quark Gluon Plasma decouples. The σ_{\pm} are described in the text.

However, the operator (84), as it stands, cannot be used to calculate the polarization of final state particles in practice. The reason is that the operators in the exponent of (84) are to be evaluated at the time τ_0 , when the system is in the QGP phase and the field operators are those of the fundamental QCD degrees of freedom, quarks and gluons, whereas the creation and destruction operators in a formula such as (15) or (48) are clearly those of the hadronic asymptotic states, which can be expressed in terms of the effective hadronic fields. Even if we were able to write the effective hadronic fields in terms of the fundamental quark and gluon fields, those should be evaluated at different times, that is the initial “time” τ_0 and the decoupling time, so that the full dynamical problem of interacting quantum field should be solved. It is indeed convenient to rewrite $\hat{\rho}_{LE}(\tau_0)$ in terms of the operators at some present “time” τ by means of the Gauss’ theorem, taking into account that \hat{T} and \hat{j} are conserved currents [30]. Being:

$$d\Sigma_{\mu} = d\Omega n_{\mu}$$

where \hat{n} is the unit vector perpendicular to the hypersurface and $d\Omega$ being the measure of a 4D region in spacetime, we have

$$-\int_{\Sigma(\tau_0)} d\Sigma_{\mu} \left(\hat{T}^{\mu\nu} \beta_{\nu} - \hat{j}^{\mu} \zeta \right) = -\int_{\Sigma(\tau)} d\Sigma_{\mu} \left(\hat{T}^{\mu\nu} \beta_{\nu} - \hat{j}^{\mu} \zeta \right) + \int_{\Omega} d\Omega \left(\hat{T}^{\mu\nu} \nabla_{\mu} \beta_{\nu} - \hat{j}^{\mu} \nabla_{\mu} \zeta \right), \quad (85)$$

where ∇ is the covariant derivative. The region Ω is the portion of spacetime enclosed by the two hypersurface $\Sigma(\tau_0)$ and $\Sigma(\tau)$ and the timelike hypersurface at their boundaries, where the flux of $(\hat{T}^{\mu\nu} \beta_{\nu}(x) - \hat{j}^{\mu} \zeta(x))$ is supposed to vanish [30]. Consequently, the stationary NEDO reads:

$$\hat{\rho} = \frac{1}{Z} \exp \left[-\int_{\Sigma(\tau_0)} d\Sigma_{\mu} \left(\hat{T}^{\mu\nu} \beta_{\nu} - \hat{j}^{\mu} \zeta \right) \right] = \frac{1}{Z} \exp \left[-\int_{\Sigma(\tau)} d\Sigma_{\mu} \left(\hat{T}^{\mu\nu} \beta_{\nu} - \hat{j}^{\mu} \zeta \right) + \int_{\Omega} d\Omega \left(\hat{T}^{\mu\nu} \nabla_{\mu} \beta_{\nu} - \hat{j}^{\mu} \nabla_{\mu} \zeta \right) \right] \quad (86)$$

In the case of heavy ion collisions the $\Sigma(\tau_0)$ - looking at figure 1 - is the 3D hypersurface Σ_{eq} , while the hypersurface $\Sigma(\tau)$ is usually the joining of the freeze-out hypersurface Σ_{FO} encompassing the QGP space-time region and the two side branches σ_{\pm} subsets of the $\Sigma(\tau_0)$. A peculiarity of the heavy ion collisions is that the hypersurface of “present” local equilibrium is partly time-like, that is $\hat{n} \cdot \hat{n} = -1$.

The density operator in (86) can be expanded perturbatively by identifying the two terms in its exponent:

$$\hat{A} = -\int_{\Sigma(\tau)} d\Sigma_{\mu} \left(\hat{T}^{\mu\nu} \beta_{\nu} - \hat{j}^{\mu} \zeta \right) \quad (87)$$

which is the supposedly, in hydrodynamics, the predominant term, and:

$$\hat{B} = \int_{\Omega} d\Omega \left(\hat{T}^{\mu\nu} \nabla_{\mu} \beta_{\nu} - \hat{j}^{\mu} \nabla_{\mu} \zeta \right) \quad (88)$$

which is supposedly the small term. The \hat{A} and \hat{B} terms correspond to the LTE at the current time and the dissipative correction respectively. Hence, the leading term of the expansion of the mean value of any operator is the local equilibrium one, that is:

$$O \simeq \text{tr}(\hat{\rho}_{\text{LE}} \hat{O}) = \frac{\text{tr}(\exp[\hat{A}] \hat{O})}{\text{tr}(\exp[\hat{A}])}$$

The convenient feature of this approach for the calculation of hydrodynamic constitutive equations, is the natural separation between non-dissipative terms - which are obtained by retaining the \hat{A} term - and the dissipative ones which are obtained by including \hat{B} .

The calculation of $W(x, k)_{\text{LE}}$, that is:

$$W(x, k)_{\text{LE}} = \frac{1}{Z_{\text{LE}}} \text{Tr} \left(\exp \left[- \int_{\Sigma} d\Sigma_{\mu} \left(\hat{T}^{\mu\nu}(y) \beta_{\nu}(y) - \hat{j}^{\mu}(y) \zeta(y) \right) \right] \hat{W}(x, k) \right) \quad (89)$$

can be tackled by taking advantage of the supposedly slow variation of the fields β and ζ in space-time compared with the variation of the Wigner operator over microscopic scales. Beforehand, it should be pointed out that the point x where the Wigner function is to be evaluated is, to a large extent, arbitrary. For, as we have seen in sections IV B and V, the 3D integration hypersurface of the Wigner function (see e.g. equation (57)) can be any hypersurface where the asymptotic hadronic fields are defined, one could choose a hyperplane at a sufficiently large value of the Minkowski time t so as to be completely outside the QGP spacetime region (see fig. 1), where hadronic fields cannot be used. However, this is not a convenient choice: at large times the fields β and ζ are no longer defined because the system is not a fluid anymore and it would then be difficult to estimate the Wigner function therein. A much better choice is an equivalent (from the viewpoint of the Gauss theorem) 3D hypersurface encompassing the QGP and much closer to where the hydrodynamic fields are still defined. This hypersurface Σ can be obtained by joining of the break-up hypersurface Σ_{FO} and the two branches σ_{\pm} , as discussed above. Now one can evaluate $W(x, k)_{\text{LE}}$ in space-time points where β and ζ exist, with the exception of the branches σ_{\pm} where the matter is not a fluid. Indeed, those branches involve the cold nuclear matter not participating the QGP formation and its contribution is usually neglected. Since the hydrodynamic-thermodynamic fields β and ζ are slowly varying, one can expand them in a Taylor series around x , the point where the Wigner operator is evaluated, and, retaining only the first order:

$$\beta_{\nu}(y) \simeq \beta_{\nu}(x) + \partial_{\lambda} \beta_{\nu}(x) (y - x)^{\lambda}$$

and similarly for ζ . Inserting in the eq. (89):

$$\begin{aligned} W(x, k)_{\text{LE}} & \quad (90) \\ & \simeq \frac{1}{Z_{\text{LE}}} \text{Tr} \left(\exp \left[- \int_{\Sigma} d\Sigma_{\mu} \left(\hat{T}^{\mu\nu}(y) [\beta_{\nu}(x) + \partial_{\lambda} \beta_{\nu}(x) (y - x)^{\lambda}] - \hat{j}^{\mu}(y) [\zeta(x) + \partial_{\lambda} \zeta(x) (y - x)^{\lambda}] \right) \right] \hat{W}(x, k) \right) \\ & = \frac{1}{Z_{\text{LE}}} \text{Tr} \left(\exp \left[- \beta_{\nu}(x) \int_{\Sigma} d\Sigma_{\mu} \hat{T}^{\mu\nu}(y) - \partial_{\lambda} \beta_{\nu}(x) \int_{\Sigma} d\Sigma_{\mu} (y - x)^{\lambda} \hat{T}^{\mu\nu}(y) - \zeta(x) \int_{\Sigma} d\Sigma_{\mu} \hat{j}^{\mu} \right. \right. \\ & \quad \left. \left. - \partial_{\lambda} \zeta(x) \int_{\Sigma} d\Sigma_{\mu} (y - x)^{\lambda} \hat{j}^{\mu} \right] \hat{W}(x, k) \right) \end{aligned}$$

This approximation corresponds to the hydrodynamic limit, where the mean value of local operators is determined by the local values of the thermodynamic fields. The gradient of β in the last equation can be split into the symmetric and the anti-symmetric part giving rise to:

$$-\frac{1}{2} \varpi_{\lambda\nu} \int_{\Sigma} d\Sigma_{\mu} (y - x)^{\lambda} \hat{T}^{\mu\nu}(y) - (y - x)^{\nu} \hat{T}^{\mu\lambda}(y) + \frac{1}{4} (\partial_{\lambda} \beta_{\nu} + \partial_{\nu} \beta_{\lambda}) \int_{\Sigma} d\Sigma_{\mu} (y - x)^{\lambda} \hat{T}^{\mu\nu}(y) + (y - x)^{\nu} \hat{T}^{\mu\lambda}(y) \quad (91)$$

where ϖ is the thermal vorticity (18). We can recognize in the first term of the above equation the total angular momentum operator, with a proviso: the above integration is over a 3D hypersurface $\Sigma \supset \Sigma_{FO}$ which is not fully space-like, in fact it has a time-like part. Notwithstanding, as the angular momentum-boost current is divergenceless and being $\Sigma = (\Sigma_{FO} \cup \sigma_{\pm})$ as discussed, we can again use the Gauss theorem and write:

$$\int_{\Sigma} d\Sigma_{\mu} (y - x)^{\lambda} \hat{T}^{\mu\nu}(y) - (y - x)^{\nu} \hat{T}^{\mu\lambda}(y) = \int_{\Sigma_{eq}} d\Sigma_{\mu} (y - x)^{\lambda} \hat{T}^{\mu\nu}(y) - (y - x)^{\nu} \hat{T}^{\mu\lambda}(y)$$

where Σ_{eq} is the initial, space-like local thermodynamic equilibrium. The latter is, by definition, the conserved total angular momentum-boost generator with center x , that is $\hat{J}_x^{\mu\nu}$.

The main contribution to the Wigner function supposedly arises from the terms surviving at the global equilibrium, occurring when $\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$ and $\partial_\mu \zeta = 0$. However, the symmetric term in (91) as well as the $\partial \zeta$ term in (90) may in principle contribute at LTE (they are non-dissipative, non-equilibrium terms) and it would be interesting to assess their quantitative effect. Assuming that they are negligible, we have:

$$W(x, k)_{\text{LE}} \simeq \frac{1}{Z} \text{Tr} \left(\exp \left[-\beta_\nu(x) \hat{P}^\nu + \frac{1}{2} \varpi_{\nu\lambda}(x) \hat{J}_x^{\nu\lambda} + \zeta(x) \hat{Q} \right] \hat{W}(x, k) \right) \quad (92)$$

This expression is the *global thermodynamic equilibrium* mean of the Wigner function $W(x, k)_{\text{GE}}$ with four-temperature and thermal vorticity values just equal to their values in the point x where the Wigner function is to be evaluated.

A. Polarization at local thermodynamic equilibrium

Working out (92) is, in principle, much easier than a complete local equilibrium calculation and yet, the exact form has not been determined so far. A possible approach is linear response theory, taking as the term $\varpi_{\lambda\nu}(x) \hat{J}_x^{\lambda\nu}$ in eq. (90) as the small term compared to the main term $-\beta_\nu(x) \hat{P}^\nu + \zeta(x) \hat{Q}$. This method, however, involves the calculation of complicated integral correlators between the angular momentum operator and the Wigner operator and, although viable, has never been attempted in literature.

In ref. [15] an educated *ansatz* was introduced based on the on-shell De Groot's approximation of the general form of the covariant Wigner function:

$$W(x, k) \simeq \frac{1}{2} \sum_{r,s} \int \frac{d^3 p}{\varepsilon} \delta^4(k - p) u_r(p) f(x, p)_{rs} \bar{u}_s(p) - \delta^4(k + p) v_r(p) \bar{f}(x, p)_{sr} \bar{v}_s(p) \quad (93)$$

where $f_{rs}(x, p)$ is a 2×2 distribution function and r, s label spin states. Now we know that the Boltzmann limit of (58) must yield the spin density matrix (29), i.e. by using the (59),

$$\frac{\int d\Sigma_\mu p^\mu \bar{U}(p) W_+(x, k)_{\text{GE}} U(p)}{\text{tr}_2 \int d\Sigma_\mu p^\mu \bar{U}(p) W_+(x, k)_{\text{GE}} U(p)} \longrightarrow \frac{\bar{U}(p) \exp \left[\frac{1}{2} \varpi : \Sigma \right] U(p)}{\text{tr}_2 (\bar{U}(p) \exp \left[\frac{1}{2} \varpi : \Sigma \right] U(p))} \quad (94)$$

Hence, a suitable form of f was assumed giving the correct Boltzmann (as well as the non-relativistic) limit:

$$f_{rs}(x, p) = \bar{u}_r(p) \exp \left[\beta \cdot p - \frac{1}{2} \varpi : \Sigma + I \right]^{-1} u_s(p) \quad (95)$$

The equations (93) and (95) together lead to the following form of the mean spin vector for spin 1/2 particles [15]:

$$S^\mu(p) = -\frac{1}{8m} \epsilon^{\mu\rho\sigma\tau} p_\tau \frac{\int_{\Sigma_{FO}} d\Sigma_\lambda p^\lambda n_F (1 - n_F) \varpi_{\rho\sigma}}{\int_{\Sigma_{FO}} d\Sigma_\lambda p^\lambda n_F} \quad (96)$$

where n_F is the Fermi-Dirac phase-space distribution function:

$$n_F = \frac{1}{\exp[\beta \cdot p - \mu q] + 1}$$

q being a charge of the particle and μ the corresponding chemical potential. The (96) is, in the Boltzmann limit, in full agreement with the equation (34) which is the first order formula obtained within a single particle framework. The problem to determine the exact form at global equilibrium including quantum statistics effects is - as mentioned - yet to be solved.

VIII. SUMMARY AND OUTLOOK

The calculation of polarization in a relativistic fluid stands out as a fascinating endeavour in quantum field theory. As we have seen, it requires the use of a broad range of concepts and theoretical tools and it involves intriguing fundamental physics problems such as the physical significance of the spin tensor. It should be emphasized that it is not just an academic problem: polarization in the QCD plasma has been observed in experiments and much of its

phenomenological potential as a probe of the hot QCD matter is still to be explored. In this regard, much theoretical and experimental work is ongoing. For a comprehensive review of the status of the subject, we refer the reader to the recent review [32].

The formula (96) is the benchmark for most estimates of polarization. While very successful in reproducing the global polarization of Λ hyperons in relativistic heavy ion collisions, a disagreement with the data was found as to the momentum dependence of polarization [32]. These discrepancies could be an effect of incorrect hydrodynamic initial conditions, resulting in a distorted thermal vorticity field at the freeze-out or they could possibly arise from missing theoretical ingredients and major corrections to the equation (34), which is a leading order formula in thermal vorticity. Even though thermal vorticity is apparently a small number in relativistic heavy ion collisions, a quantitative role of the yet unknown exact formula of the Wigner function (90) cannot be ruled out for the present. Similarly, dissipative corrections to the mean spin vector are quantitatively unknown thus far. Even the estimate of the first order correction to the formula (96) in the linear approximation with the operator (88) is a formidable task as it involves, in heavy ion collisions, the full non-perturbative QCD regime (there is an ongoing effort in this direction [33]). The theory of the polarization in relativistic fluid is still to be fully developed.

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Appendix A: Angular momentum decomposition

We shall prove that the orbital part of the angular momentum operator (80) can be written as an integral in momentum space of on-shell functions. We will confine ourselves to the proof for the particle term in (48), its extension to the anti-particle term and the proof of the vanishing of the mixed term being alike. By using the (48) and choosing the hyperplane $t = 0$ as integration hypersurface, for the particle term, we can write:

$$\int d^3x x^\mu k^\nu \text{tr}_4(\gamma^0 \widehat{W}(x, k)) \quad (\text{A1})$$

$$= \int d^3x x^\mu k^\nu \sum_{r,s} \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\varepsilon} \frac{d^3p'}{2\varepsilon'} e^{-i(p-p') \cdot x} \delta^4\left(k - \frac{p+p'}{2}\right) \hat{a}_s^\dagger(p') \hat{a}_r(p) \bar{u}_s(p') \gamma^0 u_r(p) \quad (\text{A2})$$

$$= 8 \int d^3x x^\mu k^\nu \sum_{r,s} \frac{1}{(2\pi)^3} \int \frac{d^3p}{4\varepsilon \varepsilon_{k,p}} e^{i(2k-2p) \cdot x} \delta\left(k^0 - \frac{\varepsilon + \varepsilon'_{k,p}}{2}\right) \hat{a}_s^\dagger(2k-p) \hat{a}_r(p) \bar{u}_s(2k-p) \gamma^0 u_r(p) \quad (\text{A3})$$

where:

$$\varepsilon_{k,p} = \sqrt{(2\mathbf{k} - \mathbf{p})^2 + m^2}$$

and it is understood that in the arguments of creation and destruction operators, as well as of spinors u , only the spatial part of the four-vector k , that is \mathbf{k} , enters.

For $\mu = 0$, the integration is straightforward as x^0 is constant on the hyperplane and we get, after integrating in d^3x :

$$\begin{aligned} & 8x^0 k^\nu \sum_{r,s} \int \frac{d^3p}{4\varepsilon \varepsilon_{k,p}} \delta^3(\mathbf{p} - \mathbf{k}) \delta\left(k^0 - \frac{\varepsilon + \varepsilon_{k,p}}{2}\right) \hat{a}_s^\dagger(2k-p) \hat{a}_r(p) \bar{u}_s(2k-p) \gamma^0 u_r(p) \\ &= x^0 k^\nu \sum_{r,s} \frac{1}{(2\pi)^3} \frac{1}{4\varepsilon_k^2} \delta(k^0 - \varepsilon) \hat{a}_s^\dagger(k) \hat{a}_r(k) \bar{u}_s(k) \gamma^0 u_r(k) = x^0 k^\nu \delta(k^0 - \varepsilon_k) \sum_r \frac{1}{2\varepsilon_k} \hat{a}_r^\dagger(k) \hat{a}_r(k) \end{aligned}$$

because $p' = 2k - p$ and $\mathbf{p} = \mathbf{k}$ implies in turn $k = p = p'$, hence k is on-shell; we have also used the known spinor relations.

For $\mu = i \neq 0$ we can replace x^μ with a derivative of the exponential and, integrating by parts:

$$\begin{aligned} & \int d^3x x^i k^\nu \text{tr}_4(\gamma^0 \widehat{W}(x, k)) \\ &= 4i \int d^3x k^\nu \sum_{r,s} \frac{1}{(2\pi)^3} \int \frac{d^3p}{4\varepsilon \varepsilon_{k,p}} \frac{\partial}{\partial p^i} e^{i(2k-2p) \cdot x} \delta\left(k^0 - \frac{\varepsilon + \varepsilon_{k,p}}{2}\right) \hat{a}_s^\dagger(2k-p) \hat{a}_r(p) \bar{u}_s(2k-p) \gamma^0 u_r(p) \\ &= 4i \int d^3x k^\nu \sum_{r,s} \frac{1}{(2\pi)^3} \int \frac{d^3p}{4\varepsilon \varepsilon_{k,p}} \frac{\partial}{\partial p^i} \left[e^{i(2k-2p) \cdot x} \delta\left(k^0 - \frac{\varepsilon + \varepsilon_{k,p}}{2}\right) \hat{a}_s^\dagger(2k-p) \hat{a}_r(p) \bar{u}_s(2k-p) \gamma^0 u_r(p) \right] \\ &\quad - 4i \int d^3x k^\nu \sum_{r,s} \frac{1}{(2\pi)^3} \int \frac{d^3p}{4\varepsilon \varepsilon_{k,p}} e^{i(2k-2p) \cdot x} \frac{\partial}{\partial p^i} \left[\delta\left(k^0 - \frac{\varepsilon + \varepsilon_{k,p}}{2}\right) \hat{a}_s^\dagger(2k-p) \hat{a}_r(p) \bar{u}_s(2k-p) \gamma^0 u_r(p) \right] \end{aligned}$$

The first term gives rise to a boundary integral which vanishes for fixed k and only the second term survives. We can now integrate in d^3x getting:

$$-\frac{i}{2} k^\nu \sum_{r,s} \int \frac{d^3p}{4\varepsilon \varepsilon_{k,p}} \delta^3(\mathbf{p} - \mathbf{k}) \frac{\partial}{\partial p^i} \left[\delta\left(k^0 - \frac{\varepsilon + \varepsilon_{k,p}}{2}\right) \hat{a}_s^\dagger(2k-p) \hat{a}_r(p) \bar{u}_s(2k-p) \gamma^0 u_r(p) \right] \quad (\text{A4})$$

There appear two derivative terms in the above expression: the derivative of the δ can be written as:

$$\frac{\partial}{\partial p^i} \delta\left(k^0 - \frac{\varepsilon + \varepsilon'_{k,p}}{2}\right) = -\frac{1}{2} \frac{\partial}{\partial k^0} \delta\left(k^0 - \frac{\varepsilon + \varepsilon_{k,p}}{2}\right) \frac{\partial}{\partial p^i} (\varepsilon + \varepsilon_{k,p}) = -\frac{1}{2} \frac{\partial}{\partial k^0} \delta\left(k^0 - \frac{\varepsilon + \varepsilon_{k,p}}{2}\right) \left(\frac{p^i}{\varepsilon} - \frac{2k^i - p^i}{\varepsilon_{k,p}}\right) \quad (\text{A5})$$

while the derivative of the factor including creation and destruction operators and spinors yields, taking into account

the $\delta^3(\mathbf{p} - \mathbf{k})$:

$$\begin{aligned} & \delta^3(\mathbf{p} - \mathbf{k}) \frac{\partial}{\partial p^i} \hat{a}_s^\dagger(2k - p) \hat{a}_r(p) \bar{u}_s(2k - p) \gamma^0 u_r(p) \\ &= \delta^3(\mathbf{p} - \mathbf{k}) \left[\left(\hat{a}_s^\dagger(p) \frac{\overleftrightarrow{\partial}}{\partial p^i} \hat{a}_r(p) \right) \bar{u}_s(p) \gamma^0 u_r(p) + \hat{a}_s^\dagger(p) \hat{a}_r(p) \left(\bar{u}_s(p) \frac{\overleftrightarrow{\partial}}{\partial p^i} \gamma^0 u_r(p) \right) \right] \end{aligned} \quad (\text{A6})$$

We can now plug the equations (A5) and (A6) into the (A4). The term (A5) vanishes because:

$$\begin{aligned} & \delta^3(\mathbf{p} - \mathbf{k}) \left(\frac{p^i}{\varepsilon} - \frac{2k^i - p^i}{\varepsilon_{k,p}} \right) k^\nu \frac{\partial}{\partial k^0} \delta \left(k^0 - \frac{\varepsilon + \varepsilon_{k,p}}{2} \right) = -\delta^3(\mathbf{p} - \mathbf{k}) \left(\frac{p^i}{\varepsilon} - \frac{2k^i - p^i}{\varepsilon_{k,p}} \right) \delta_0^\nu \delta \left(k^0 - \frac{\varepsilon + \varepsilon_{k,p}}{2} \right) \\ &= -\delta^3(\mathbf{p} - \mathbf{k}) \delta_0^\nu \delta(k^0 - \varepsilon) \left(\frac{k^i}{\varepsilon_k} - \frac{k^i}{\varepsilon_k} \right) = 0 \end{aligned}$$

and we are just left with the term from (A6).

We can now integrate in d^4k according to the equation (80). For $\mu = i$:

$$\int d^4k \int d^3x x^i k^\nu \text{tr}_4(\gamma^0 \widehat{W}(x, k)) \quad (\text{A7})$$

$$- \frac{i}{2} \int d^4k k^\nu \sum_{r,s} \frac{1}{4\varepsilon_k^2} \delta(k_0 - \varepsilon_k) \left[\left(\hat{a}_s^\dagger(k) \frac{\overleftrightarrow{\partial}}{\partial k^i} \hat{a}_r(k) \right) \bar{u}_s(k) \gamma^0 u_r(k) + \hat{a}_s^\dagger(k) \hat{a}_r(k) \left(\bar{u}_s(k) \frac{\overleftrightarrow{\partial}}{\partial k^i} \gamma^0 u_r(k) \right) \right] \quad (\text{A8})$$

$$= -\frac{i}{2} \int d^3k k^\nu \sum_{r,s} \frac{1}{4\varepsilon_k^2} \left[\left(\hat{a}_s^\dagger(k) \frac{\overleftrightarrow{\partial}}{\partial k^i} \hat{a}_r(k) \right) \bar{u}_s(k) \gamma^0 u_r(k) + \hat{a}_s^\dagger(k) \hat{a}_r(k) \left(\bar{u}_s(k) \frac{\overleftrightarrow{\partial}}{\partial k^i} \gamma^0 u_r(k) \right) \right] \quad (\text{A9})$$

$$(\text{A10})$$

with k^ν again on-shell. We can then conclude that:

$$\int d^4k \int_{t=0} d^3x x^\mu k^\nu \text{tr}_4(\gamma^0 \widehat{W}(x, k)) = \int \frac{d^3k}{2\varepsilon_k} \widehat{G}^\mu(k) k^\nu$$

with k on-shell and $\widehat{G}^0(k) = 0$ if $x^0 = 0$ is chosen.

Finally, we briefly address the calculation of the angular momentum tensor by using the Belinfante stress-energy tensor (83) where only the orbital part is involved. The calculation is very similar to the one just described, with the important difference that the second term of (83), obtained by swapping the indices of the first term in (83), leads to a term akin to the left hand side of the eq. (A1) with exchanged indices:

$$\int d^3x x^\mu k^0 \text{tr}_4(\gamma^\nu \widehat{W}(x, k))$$

However, the final result is not proportional to k^ν and a double derivative term appears just like in eq. (A7); therefore, this term is not cancelled by the Levi-Civita tensor in the calculation of the mean spin, unlike in the canonical case.