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Reproduction Capabilities of Penalized Hyperbolic-polynomial Splines

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Abstract

This paper investigates two important analytical properties of hyperbolic-polynomial penalized splines, HP-splines for short. HP-splines, obtained by combining a special type of difference penalty with hyperbolic-polynomial B-splines (HB-splines), were recently introduced by the authors as a generalization of P-splines. HB-splines are bell-shaped basis functions consisting of segments made of real exponentials $e^{\alpha x}$, $e^{-\alpha x}$ and linear functions multiplied by these exponentials, $x e^{+\alpha x}$ and $x e^{-\alpha x}$. Here, we show that these types of penalized splines reproduce functions in the space $\{e^{-\alpha x}, x e^{-\alpha x}\}$, that is they fit exponential data exactly. Moreover, we show that they conserve the first and second ‘exponential’ moments.

Keywords: Hyperbolic-polynomial splines, Penalized splines, Discrete penalty, P-splines, B-splines.

1. Introduction

This paper investigates the reproduction capabilities of hyperbolic-polynomial penalized splines. HP-splines, were recently introduced in [1] as a generalization of the better known P-splines (see [2, 3]), and combine a finite difference penalty with HB-splines, that piecewise consist of real exponentials and monomials multiplied by these exponentials. Numerical examples show that the exponential nature of HP-splines may turn out to be useful in applications when the data show an exponential trend [4].

The HP-splines we consider have segments in the four-dimensional space

$$\mathbb{E}_{4,\alpha} := \text{span}\{e^{\alpha x}, x e^{\alpha x}, e^{-\alpha x}, x e^{-\alpha x}\}, \alpha \in \mathbb{R}, \quad (1)$$

with the frequency α being an extra parameter to tune the smoother effects. Even though all details concerning their definition and construction can be already found in [1], the analysis of their reproduction capability is missing there. To fill the gap, here we show that these type of penalized splines reproduce functions in the space $\{e^{-\alpha x}, x e^{-\alpha x}\}$, that is fit exponential data of the latter type exactly. We also show that they conserve the first and second ‘exponential’ moments, showing that HP-splines are the natural generalization of P-splines even with respect to reproduction and moment preservation.

Given the data points (x_i, y_i) , $i = 1, \dots, m$, $x_1 < \dots < x_m$, the uniform knot partition $\Xi := \{x_1 := a = \xi_1 < \xi_2 < \dots < \xi_n = b := x_m\}$ with knots distance h , and denoting by $\{B_0^\alpha, \dots, B_{n+1}^\alpha\}$ a basis of the spline space- called HB-splines- with segments in $\mathbb{E}_{4,\alpha}$ (see [5]), the HP-spline approximating the given data is obtained by solving the minimization problem

$$\min_{a_0, \dots, a_{n+1}} \sum_{i=1}^m w_i \left(y_i - \sum_{j=0}^{n+1} a_j B_j^\alpha(x_i) \right)^2 + \lambda \sum_{j=2}^{n+1} \left((\Delta_2^{h,\alpha} \mathbf{a})_j \right)^2, \quad (2)$$

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where the minimum is with respect to the HP-spline coefficients $\mathbf{a} = (a_j)_{j=0}^{n+1}$. The values (w_1, \dots, w_m) are non-zero weights, $\Delta_2^{h,\alpha}$ is the difference operator acting on functions and on sequences respectively as (see [6] for this type of operators),

$$\Delta_2^{h,\alpha} u = u(x) - 2e^{-\alpha h} u(x-h) + e^{-2\alpha h} u(x-2h), \quad \text{with } h > 0 \text{ the knots distance,}$$

$$(\Delta_2^{h,\alpha} \mathbf{a})_j = a_j - 2e^{-\alpha h} a_{j-1} + e^{-2\alpha h} a_{j-2}, \quad j \in \mathbb{Z},$$

and λ is a regularization parameter that can be set in several different ways, e.g. with the discrepancy principle, the generalized cross-validation, or the L-curve method (see [7], for example). It is not difficult to see that the HP-spline can be written as

$$s_{hp}(x) = \sum_{j=0}^{n+1} \hat{a}_j B_j^\alpha(x), \quad (3)$$

where $\hat{\mathbf{a}} = (\hat{a}_0, \dots, \hat{a}_{n+1})^T \in \mathbb{R}^{(n+2)}$ is the solution of the linear system

$$\left(\mathbf{B}^{\alpha T} \mathbf{W} \mathbf{B}^\alpha + \lambda \mathbf{D}_2^{h,\alpha T} \mathbf{D}_2^{h,\alpha} \right) \mathbf{a} = \mathbf{B}^{\alpha T} \mathbf{W} \mathbf{y}, \quad (4)$$

where $\mathbf{y} = (y_1, \dots, y_m)^T \in \mathbb{R}^m$, $\mathbf{B}^\alpha \in \mathbb{R}^{m \times (n+2)}$ is the banded collocation matrix $\mathbf{B}^\alpha := (B_j^\alpha(x_i))_{i=1, \dots, m}^{j=0, \dots, n+1}$, \mathbf{W} is the diagonal matrix $\mathbf{W} := (\text{diag}(w_i))_{i=1, \dots, m} \in \mathbb{R}^{m \times m}$ and

$$\mathbf{D}_2^{h,\alpha} = \begin{bmatrix} 1 & -2e^{-\alpha h} & e^{-2\alpha h} & 0 & \dots & 0 \\ 0 & 1 & -2e^{-\alpha h} & e^{-2\alpha h} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & 1 & -2e^{-\alpha h} & e^{-2\alpha h} \end{bmatrix} \in \mathbb{R}^{n \times (n+2)}.$$

Note that for $\alpha \rightarrow 0$ the space $\mathbb{E}_{4,\alpha}$ reduces to $\{1, x, x^2, x^3\}$, HB-splines reduce to cubic B-splines and the difference operator $\Delta_2^{h,\alpha}$ reduces to the standard forward second order difference operator acting on a sequence \mathbf{a} as $(\Delta_2 \mathbf{a})_j = a_j - 2a_{j-1} + a_{j-2}$. Therefore, for $\alpha = 0$ HP-splines coincide with P-splines based on classical cubic B-splines proposed by Eilers and Max (see their recent monograph [8]). From now on, without loss of generality, we continue by assuming that \mathbf{W} is the identity matrix.

P-splines are known to have a number of useful properties, essentially inherited from B-splines and from the special type of penalty: they can fit polynomial data exactly, they can conserve the first two moments of the data and show no boundary effects. The aim of this paper is to investigate similar reproduction properties of HP-splines. As in the polynomial case, the HP-spline properties are essentially inherited from HB-splines and this is why in Section 2 we first prove that HB-splines reproduce $\mathbb{E}_{4,\alpha}$. Then, in Section 3, we show that, whatever the value of the smoothing parameter λ , HP-splines fit exponential data exactly as they reproduce $\mathbb{E}_{2,-\alpha} := \{e^{-\alpha x}, xe^{-\alpha x}\}$. Moreover, we show that HP-splines conserve the first and second ‘exponential’ moments. Section 4 draws conclusion and highlights future works.

2. HB-splines and their reproduction properties

As shown in [9], in the ‘cardinal’ situation –corresponding to integer spline knots–, HB-splines can be defined through convolution. For the spline space with segment in $\mathbb{E}_{4,\alpha}$, starting with the first order B-spline

$$B_\alpha^1(x) = e^{\alpha x} \chi_{[0,1]}(x), \quad (5)$$

where $\chi_{[0,1]}$ is the characteristic function with support $[0, 1]$, the four order cardinal HB-spline supported on $[0, 4]$ is obtained as

$$B_\alpha^1 = (B_\alpha^1 * B_\alpha^1 * B_{-\alpha}^1 * B_{-\alpha}^1) \quad \text{with} \quad \boldsymbol{\alpha} = (\alpha, \alpha, -\alpha, -\alpha),$$

hence, for any integer k , the HB-spline supported in $[k, k+4]$ is obtained by translation as $B_\alpha^1(\cdot - k)$. In case the knots are uniform but with a distance $h \neq 1$, the corresponding HB-splines are defined by dilation,

$$B_\alpha^h = B_\alpha^1\left(\frac{\cdot}{h}\right) = (B_\alpha^1 * B_\alpha^1 * B_{-\alpha}^1 * B_{-\alpha}^1)\left(\frac{\cdot}{h}\right), \quad (6)$$

and then translation. Alternatively, one can directly start with the scaled order-one HB-spline $B_\alpha^h(x) = e^{\frac{\alpha}{h}x} \chi_{[0,h]}(x)$ and use repeated convolution. In that case we see that when dealing with grid spacing h , the frequency α is scaled into $\frac{\alpha}{h}$, a fact that will also enter into the exponential reproduction discussion we are going to make.

Concerning the HB-spline reproduction, we prove the following result.

Proposition 2.1. *Let $\alpha = (\alpha, \alpha, -\alpha, -\alpha)$ be the vector of frequencies, and $B_{\alpha h}^h$ the HB-spline with uniform knots defined in (6) having support $[0, 4h]$. Then,*

$$f = \sum_{k \in \mathbb{Z}} c_k^f B_{\alpha h}^h(\cdot - hk), \quad \text{for all } f \in \mathbb{E}_{4,\alpha}.$$

Proof. The starting point is [9, Proposition 2] that yields the particularly simple reproduction formulas for the order-two HB-spline $B_{\alpha,\alpha}^1 = B_\alpha^1 * B_\alpha^1$

$$\sum_{k \in \mathbb{Z}} e^{\alpha k} B_{\alpha,\alpha}^1(x - k) = e^{\alpha x}, \quad \sum_{k \in \mathbb{Z}} (k+1) e^{\alpha k} B_{\alpha,\alpha}^1(x - k) = x e^{\alpha x},$$

providing, for $t_k = kh$, the reproduction formula for $e^{\alpha x}$. In fact, from $e^{\frac{\alpha}{h}x} = \sum_{k \in \mathbb{Z}} e^{\frac{\alpha}{h}t_k} B_{\alpha,\alpha}^h(x - t_k)$ we arrive at

$$e^{\alpha x} = \sum_{k \in \mathbb{Z}} e^{\alpha t_k} B_{\alpha h,\alpha h}^h(x - t_k). \quad (7)$$

Similarly, $\sum_{k \in \mathbb{Z}} (k+1) e^{\alpha k} B_{\alpha,\alpha}^1(x - k) = x e^{\alpha x}$ gives $\frac{x}{h} e^{\frac{\alpha}{h}x} = \sum_{k \in \mathbb{Z}} (k+1) e^{\frac{\alpha}{h}kh} B_{\alpha,\alpha}^h(x - hk)$ and hence the reproduction formula for $x e^{\alpha x}$, which is

$$x e^{\alpha x} = \sum_{k \in \mathbb{Z}} (t_k + h) e^{\alpha t_k} B_{\alpha h,\alpha h}^h(x - t_k). \quad (8)$$

Next, we use the fact that the convolution product of $B_{-\alpha h,-\alpha h}^h$ with the functions $f_1(x) = e^{\alpha x}$ or $f_2(x) = x e^{\alpha x}$ yields another exponential polynomial of the same type, that is

$$B_{-\alpha h,-\alpha h}^h * f_1 = a_0 f_1, \quad \text{and} \quad B_{-\alpha h,-\alpha h}^h * f_2 = b_0 f_1 + b_1 f_2, \quad \text{with } a_0, b_0, b_1 \in \mathbb{R}.$$

Therefore, since $B_\alpha^1 = B_{\alpha,\alpha}^1 * B_{-\alpha,-\alpha}^1$ and $B_{\alpha h}^h = \frac{1}{h} (B_{\alpha h,\alpha h}^h * B_{-\alpha h,-\alpha h}^h)$, if we convolve both side of (7) and (8) with $B_{-\alpha h,-\alpha h}^h$, we arrive at

$$a_0 f_1 = \sum_{k \in \mathbb{Z}} e^{\alpha t_k} h B_{\alpha h}^h(\cdot - t_k), \quad \text{and} \quad b_0 f_1 + b_1 f_2 = \sum_{k \in \mathbb{Z}} (t_k + h) e^{\alpha t_k} h B_{\alpha h}^h(\cdot - t_k),$$

that are the reproduction formulas for $B_{\alpha h}^h$ of a function in $\mathbb{E}_{2,\alpha} = \{e^{\alpha x}, x e^{\alpha x}\}$. Similarly we prove the reproduction of $\mathbb{E}_{2,-\alpha} = \{e^{-\alpha x}, x e^{-\alpha x}\}$ and therefore the reproduction of $\mathbb{E}_{4,\alpha}$. \square

3. HP-splines reproduction of $\mathbb{E}_{2,-\alpha}$ and moment preservation

Based on the HB-splines reproduction properties discussed in Proposition 2.1 in this section we show the reproduction capabilities of HP-splines, independently to the value of the smoothing parameter λ . Preliminarily, we provide the assumption we make on the HB-splines construction.

Assumptions 3.1. *Let the data points (x_i, y_i) , $i = 1, \dots, m$, $x_1 < \dots < x_m$, be given together with the uniform knots partition $\Xi := \{x_1 := a = \xi_1 < \xi_2 < \dots < \xi_n = b := x_m\}$ ($n < m$) extended with the uniform left and right extra knots $\xi_\ell = \xi_1 + (\ell - 1)h$, $\ell = -2, -1, 0$, $\xi_{n+\ell} = \xi_n + \ell h$, $\ell = 1, 2, 3$ where $h = (b - a)/(n - 1)$. Let $\{B_0^\alpha, \dots, B_{n+1}^\alpha\}$ be the spline basis with segments in $\mathbb{E}_{4,\alpha}$ consisting of the uniform HB-splines $B_0^\alpha := B_{\alpha h}^h(\cdot - \xi_{-2})$, with $B_{\alpha h}^h$ as in (6), and its translates $B_j^\alpha = B_0^\alpha(\cdot - jh)$, $j = 1, \dots, n + 1$.*

Proposition 3.2. *Under Assumptions 3.1, if the data are taken from a function $f \in \mathbb{E}_{2,-\alpha}$, i.e.,*

$$y_i = f(x_i), \quad i = 1, \dots, m, \quad \text{with } f \in \mathbb{E}_{2,-\alpha},$$

the HP-spline s_{hp} defined in (3) satisfies

$$s_{hp}(x_i) = y_i, \quad i = 1, \dots, m.$$

Proof. From Proposition 2.1 we know that HB-splines reproduce $\mathbb{E}_{2,-\alpha}$ meaning that there exists a sequence of coefficients c_j^f , $j = 0, \dots, n + 1$ satisfying

$$\sum_{j=0}^{n+1} c_j^f B_j^\alpha(x) = f(x), \quad x \in [a, b], \quad f \in \mathbb{E}_{2,-\alpha}. \quad (9)$$

Next, we observe that, due to the locality of HB-splines in $[a, b]$, for any selection of c_{-2}^f and c_{-1}^f , the expression (9) can be equivalently written as

$$\sum_{j=-2}^{n+1} c_j^f B_j^\alpha(x) = f(x) \quad x \in [a, b], \quad f \in \mathbb{E}_{2,-\alpha}. \quad (10)$$

With the notation $\hat{\mathbf{a}} = (c_0^f, \dots, c_{n+1}^f)$ assuming that the data are taken from a function $f \in \mathbb{E}_{2,-\alpha}$, it is not difficult to see that the solution of the linear system (4) is exactly $\hat{\mathbf{a}}$, since from (10) we have

$$y_i = \sum_{j=-2}^{n+1} \hat{a}_j B_j^\alpha(x_i), \quad i = 1, \dots, m. \quad (11)$$

Moreover, for $x \in [a, b]$ and $f \in \mathbb{E}_{2,-\alpha}$ it follows that $\Delta_2^{h,\alpha} f(x) = 0$. But, due to the uniformity of the knots we also have $B_{j+\ell}^\alpha(x) = B_j^\alpha(x - \ell h)$ from which it follows that $\Delta_2^{h,\alpha} B_j^\alpha(x) = B_j^\alpha(x) - 2e^{-\alpha h} B_{j+1}^\alpha(x) + e^{-2\alpha h} B_{j+2}^\alpha(x)$. In conclusion,

$$0 = \Delta_2^{h,\alpha} f(x) = \sum_{j=-2}^{n+1} \hat{a}_j \Delta_2^{h,\alpha} B_j^\alpha(x) = \sum_{j=0}^{n+1} (\Delta_2^{h,\alpha} \hat{\mathbf{a}})_j B_j^\alpha(x), \quad (12)$$

where the last equality is because B_{-2}^α , B_{-1}^α , B_{n+2}^α and B_{n+3}^α are zero in $[a, b]$. Hence, if we use the linear independence of HB-splines we arrive at $(\Delta_2^{h,\alpha} \hat{\mathbf{a}})_j = 0$ for $j = 0, \dots, n + 1$, that is to $\sum_{j=2}^{n+1} (\Delta_2^{h,\alpha} \hat{\mathbf{a}})_j^2 = 0$ which means that the model acts like non penalized regression and that the reproduction capabilities of the HB-splines transfer to the HP-splines. \square

Remark 3.3. It is important to remark that, even though HB-splines reproduce $\mathbb{E}_{4,\alpha}$, Proposition 3.2 shows that HP-splines reproduce $\mathbb{E}_{2,-\alpha}$ only. This limitation is due to the specific definition of the difference operator acting as shown in (12). A model based on $\Delta_2^{h,-\alpha}$ rather than $\Delta_2^{h,\alpha}$ would reproduce $\mathbb{E}_{2,\alpha}$.

Next, we show that HP-splines preserve the two ‘exponential’ moments

Proposition 3.4. Under Assumptions 3.1, denoting by $\hat{\mathbf{y}} = \mathbf{B}^\alpha \hat{\mathbf{a}}$ the vector of predicted values with elements $\hat{y}_i = s_{hp}(x_i)$, $i = 1, \dots, m$, we have

$$\sum_{i=1}^m e^{-\alpha x_i} \hat{y}_i = \sum_{i=1}^m e^{-\alpha x_i} y_i, \quad \text{and} \quad \sum_{i=1}^m x_i e^{-\alpha x_i} \hat{y}_i = \sum_{i=1}^m x_i e^{-\alpha x_i} y_i. \quad (13)$$

Proof. To see (13), we start from the two equations defining the reproduction of $\mathbb{E}_{2,-\alpha}$ in $[a, b]$

$$\sum_{j=0}^{n+1} c_j B_j^\alpha(x) = e^{-\alpha x}, \quad \text{and} \quad \sum_{j=0}^{n+1} d_j B_j^\alpha(x) = x e^{-\alpha x}, \quad \text{with} \quad c_j, d_j \in \mathbb{R}, \quad (14)$$

and evaluate them at x_i , $i = 1, \dots, m$. Hence, for $\mathbf{c} = (c_0, \dots, c_{n+1})$ and $\mathbf{e} = (e^{-\alpha x_1}, \dots, e^{-\alpha x_m})$ we have the equivalence

$$\sum_{j=0}^{n+1} c_j B_j^\alpha(x_i) = e^{-\alpha x_i}, \quad i = 1, \dots, m \quad \Leftrightarrow \quad \mathbf{B}^\alpha \mathbf{c} = \mathbf{e}, \quad (15)$$

and, for $\mathbf{d} = (d_0, \dots, d_{n+1})$ and $\mathbf{x}\mathbf{e} = (x_1 e^{-\alpha x_1}, \dots, x_m e^{-\alpha x_m})$, the equivalence

$$\sum_{j=0}^{n+1} d_j B_j^\alpha(x_i) = x_i e^{-\alpha x_i}, \quad i = 1, \dots, m \quad \Leftrightarrow \quad \mathbf{B}^\alpha \mathbf{d} = \mathbf{x}\mathbf{e}. \quad (16)$$

Now, in consideration of the linear independence of the HB-splines, with the same reasoning done in Proposition 3.2, from

$$0 = \Delta_2^{h,\alpha}(e^{-\alpha x}) = \Delta_2^{h,\alpha}\left(\sum_{j=-2}^{n+1} c_j B_j^\alpha(x)\right) = \sum_{j=0}^{n+1} (\Delta_2^{h,\alpha} \mathbf{c})_j B_j^\alpha(x), \quad x \in [a, b], \quad (17)$$

we can write

$$(\Delta_2^{h,\alpha} \mathbf{c})_j = 0, \quad j = 2, \dots, n+1,$$

from which we conclude $\mathbf{D}_2^{h,\alpha} \mathbf{c} = \mathbf{0}$. Similarly,

$$0 = \Delta_2^{h,\alpha}(x e^{-\alpha x}) = \Delta_2^{h,\alpha}\left(\sum_{j=-2}^{n+1} d_j B_j^\alpha(x)\right) = \sum_{j=0}^{n+1} (\Delta_2^{h,\alpha} \mathbf{d})_j B_j^\alpha(x), \quad x \in [a, b],$$

implies

$$(\Delta_2^{h,\alpha} \mathbf{d})_j = 0, \quad j = 2, \dots, n+1, \quad (18)$$

and therefore $\mathbf{D}_2^{h,\alpha} \mathbf{d} = \mathbf{0}$. Next, for the predicted values $\hat{\mathbf{y}} = \mathbf{B}^\alpha \hat{\mathbf{a}}$, from (4), we have

$$(\mathbf{B}^\alpha)^T (\mathbf{y} - \mathbf{B}^\alpha \hat{\mathbf{a}}) = \lambda \mathbf{D}_2^{h,\alpha T} \mathbf{D}_2^{h,\alpha} \hat{\mathbf{a}}. \quad (19)$$

Multiplication of both sides of (19) respectively by \mathbf{c} and \mathbf{d} yields

$$\mathbf{c}^T (\mathbf{B}^\alpha)^T (\mathbf{y} - \mathbf{B}^\alpha \hat{\mathbf{a}}) = \lambda \left(\mathbf{D}_2^{h,\alpha} \mathbf{c} \right)^T \mathbf{D}_2^{h,\alpha} \hat{\mathbf{a}},$$

and

$$\mathbf{d}^T (\mathbf{B}^\alpha)^T (\mathbf{y} - \mathbf{B}^\alpha \hat{\mathbf{a}}) = \lambda \left(\mathbf{D}_2^{h,\alpha} \mathbf{d} \right)^T \mathbf{D}_2^{h,\alpha} \hat{\mathbf{a}}.$$

Now, since $\mathbf{D}_2^{h,\alpha} \mathbf{c} = 0$ and $\mathbf{D}_2^{h,\alpha} \mathbf{d} = 0$ using (15) and (16) we arrive at

$$\mathbf{e}^T (\mathbf{y} - \hat{\mathbf{y}}) = \mathbf{0}, \quad \text{and} \quad \mathbf{x} \mathbf{e}^T (\mathbf{y} - \hat{\mathbf{y}}) = \mathbf{0},$$

which are the vector versions of (13). □

Remark 3.5. Note that the exponential moments (13), reduce to the classical moments preservation whenever $\alpha = 0$ that is to

$$\sum_{i=1}^m \hat{y}_i = \sum_{i=1}^m y_i, \quad \text{and} \quad \sum_{i=1}^m x_i \hat{y}_i = \sum_{i=1}^m x_i y_i.$$

We conclude the paper with some figures showing the exponential-reproduction capabilities of HP-splines. Figure 1 refers to data taken from the exponential functions e^{-x} while Figure 2 to data from the function $x e^{-x}$. They display the graph of the HP-spline (black ‘—’) approximating the data for different selections of α combined with different level of absolute Gaussian noise with zero mean and standard deviation σ , both specified in the figure captions. The data sites (red ‘*’) and the spline knots location (blue ‘◇’) are also given in the figures. The smoothing parameter is always $\lambda = 1$ since not relevant to our discussion. The exact exponential fit is evident in absence of noise (left) while it is almost attained in case of a moderate noise (middle) somehow shown also in case of a different selection of the frequency (right). For comparison, the graph of the P-splines approximating the data is also given (magenta ‘—.’) together with the graph of the function (blue ‘- -’).

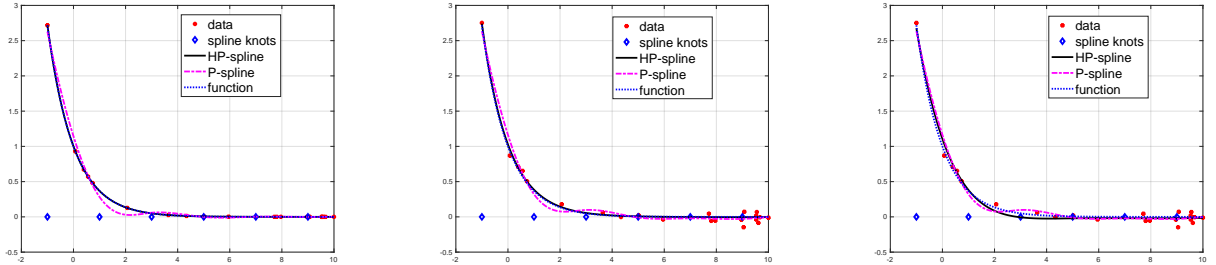


Figure 1: Function e^{-x} . From left to right values of (α, σ) : $(-1, 0)$, $(-1, 0.5 \cdot 10^{-2})$, $(-0.5, 0.5 \cdot 10^{-2})$.

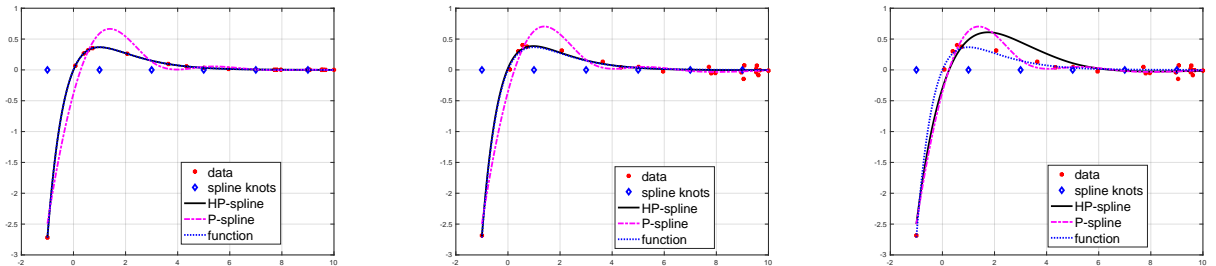


Figure 2: Function $x e^{-x}$. From left to right values of (α, σ) : $(-1, 0)$, $(-1, 0.5 \cdot 10^{-2})$, $(-0.5, 0.5 \cdot 10^{-2})$.

4. Conclusions

This paper enriches the study of HP-splines, penalized hyperbolic-splines with segments in the space $\mathbb{E}_{4,\alpha}$ consisting in the exponential polynomials $\{e^{\alpha x}, e^{-\alpha x}, xe^{\alpha x}, xe^{-\alpha x}\}$, where α is a real frequency. In particular, it investigates two important analytical properties of HP-splines: that they exactly fit functions in $\mathbb{E}_{2,-\alpha}$ and that they conserve the first and second ‘exponential’ moments, independently to the value of the smoothing parameter λ . A few numerical examples of reproduction are shown. A dynamic selection strategy of the parameter α , that certainly deserve more attention, is presently under investigation.

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