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Nonlocal Neumann boundary conditions: properties and problems

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Introduction

The aim of this Ph.D. thesis is to present new results regarding problems governed by the fractional p-Laplacian operator in presence of nonlocal boundary Neumann conditions. The whole work is based on the articles [26, 27, 51, 53, 54], and Chapter 7 presents some results that are not yet part of published articles.

More precisely, this thesis investigates problems of the form

$$\begin{cases} (-\Delta)^s_p u = f(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = g(x) & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with sufficiently smooth boundary,

$$(-\Delta)^s_p u(x) := C_{N,s,p} PV \int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+ps}}\, dy \tag{2}$$

is the fractional p-Laplacian and

$$\mathscr{N}_{s,p} u(x) := C_{N,s,p} \int_{\Omega} |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+ps}}\, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}, \tag{3}$$

is the nonlocal normal $p-$derivative, or $p-$Neumann boundary condition, which describes the natural Neumann boundary condition in presence of the fractional $p-$Laplacian. It extends the notion of nonlocal normal derivative introduced in [28] for the fractional Laplacian, i.e. for $p = 2$. In our situation, $p > 1$, $s \in (0, 1)$ and $C_{N,s,p}$ is a positive constant, which, for the sake of simplicity, from now on, we will choose to normalize setting $C_{N,s,p} = 1$.

The definition in (3) was introduced in [8], where basic integration by parts were given. Here, we start presenting some further properties of the associated operator, following [28], where a detailed description of the case $p = 2$ was given.

Nonlocal and fractional operators appear in many different applications, such as continuum mechanics and population dynamics, and have been studied by numerous authors. Far from giving an exhaustive list of references, here we only quote a few papers, where different situations have been faced. In [4] the authors consider problems analogous to the ones that we deal with in the case $p = 2$, and discuss the regularity up to the boundary. In [5], elliptic problems driven by the fractional Laplacian are studied in the whole of $\mathbb{R}^N$. In [6, 7] the authors consider critical problems involving the fractional Laplacian. In [11] the authors discuss existence, uniqueness and properties of solutions for general nonlinear equations. In [13] an extension problem for the fractional Laplacian is studied, through a by-now well-known Dirichlet-to-Neumann technique. In [40] the authors consider a nonlocal eigenvalue problem and discuss the behaviour of eigenvalues. In [42] the authors use an abstract linking theorem to find solutions for nonlocal Neumann problem, while in [43] they find solutions of Mountain Pass type for nonlocal equations. In [45], nonlocal equations in presence of a general integro-differential operator of fractional type are studied in order to find existence of solutions. In [46] nonlocal equations and their associated functional are studied in presence of critical Sobolev exponent. In [57] the authors consider resonant Neumann problems in presence of an indefinite and unbouded potential, while in [58] they study Robin problems with double resonance. In [59], nonlinear singular problems are taken into consideration in presence of an indefinite potential term. In [66, 67] the authors study nonlocal problems in presence of the fractional Laplacian. In [73], the existence of solutions is proved for fractional p-Laplacian equations with a perturbation. In [35] and [20] the authors consider different aspects of the obstacle problem in presence of the fractional Laplacian. In [3], a class of fractional double-phase problems is studied in presence of a nonlinearity with subcritical growth. A general overview on fractional operators and a relevant collection of results concerning fractional problems can be found in the monograph [44].

However, in the previous cases, if the problem is settled in a bounded domain, the associated boundary conditions is of *Dirichlet* type, namely

$$u = 0 \text{ in } \mathbb{R}^N \setminus \overline{\Omega}.$$

Concerning fractional problems with *Neumann* boundary conditions, we recall that Neumann boundary problems for the $p-$Laplacian were already introduced in [41], but the underlying operator was different from ours, since in their integral definition of fractional Laplacian only points in Ω were taken into account; more important, their Neumann boundary condition is a pointwise one, like that of [15], [16], [17], [47], [68] and [71]. Indeed, the Neumann boundary condition treated therein is of classical type, namely

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

which *is not* the condition we treat. Indeed, we will consider a more *natural Neumann* condition in the nonlocal case, mimicking the Dirichlet condition that is defined outside of Ω and setting

$$\mathscr{N}_{s,p}u = 0 \text{ in } \mathbb{R}^N \setminus \overline{\Omega},$$

where $\mathscr{N}_{s,p}$ is the normal nonlocal operator defined as

$$\mathscr{N}_{s,p}u(x) := \int_\Omega |u(x) - u(y)|^{p-2} \frac{u(x) - u(y)}{|x - y|^{N+ps}}\, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}.$$

The second aim of this thesis is to consider related Neumann problems with nonlinearities which may fail to satisfy the usual Ambrosetti-Rabinowitz condition and may verify a more recent one, introduced in [50], and already successfully exploited in other cases. For instance in [60] this condition is assumed for nonlinear Dirichelet problem driven by a nonhomogeneous differential operator, while in [56] for anisotropic double phase problems. Similar conditions are considered in order to obtain multiplicity results in [2] for a class of (p, q)-Schrödinger-Kirchhoff type equations and in [19] for a quasilinear elliptic local problem.

Let us now briefly describe the content of each chapter of this thesis.

In Chapter 1 we briefly recall the main motivations behind the definition of the nonlocal boundary Neumann conditions, as expressed in [8]. In addition, we give an interesting minimization result for functions satisfying the homogeneous Neumann conditions in the case $p = 2$, which is contained in [27]. In particular, we show that a function with fixed vulue inside of Ω minimizes the Gagliardo Seminorm if and only if it satisfies the homogeneous Neumann conditions outside of Ω.

In Chapter 2 we recall some notions of critical group theory and cohomological linking. The role of these known results will be crucial to prove existence of solutions in some subsequent chapters.

In Chapter 3 we give a general overview for the types of problems we will treat. First, we define the related functional space where solutions lay, and we show that it is a reflexive Banach space which is compactly embedded in suitable $L^q(\Omega)$ spaces. Then we give the definition of weak solutions and also discuss some properties. In particular, we prove that weak solutions satisfy the boundary conditions a.e. and a sort of maximum (or rigidity) principle. These results were published in [53].

Then we give a regularity result for solutions of general problems of the form

$$\begin{cases} (-\Delta)_p^s u = f(x, u) & \text{in } \Omega, \\ \mathscr{N}_{s,p}u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{4}$$

showing that, under natural growth conditions on f, all solutions are bounded. As far as we know, this is the first regularity result for solutions of (4) for general $p \in (1, \infty)$. In the case $p = 2$, as a consequence of related results for the Dirichlet case, we obtain that every weak solution of problem (4) is continuous on the whole of $\mathbb{R}^N$, a result related to those in [4, 26]. The proof of this result relies on a fractional version of De Giorgi's iteration method, which has been usefully employed in the case of fractional Dirichlet boundary conditions (for instance, see [33, 38]), and also for eigenvalues of the Neumann fractional problem (see [53]). Of course, due to the non local nature of the problem, the classical steps cannot be followed verbatim and several novelties are needed. These regularity results are contained in [51].

We also face the parabolic problem associated to this new class of operators, namely

$$\begin{cases} u_t(x,t) + (-\Delta)_p^s u(x,t) = 0 & \text{in } \Omega, \quad t > 0 \\ \mathscr{N}_{s,p} u(x,t) = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \quad t > 0 \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

In this case, we will prove that the mass is preserved and that the energy is monotone decreasing, as shown in [53], following the lines of [28]. Investigations on parabolic equations in presence of the fraction $p-$Laplacian have started in recent years, but only in presence of Dirichlet boundary conditions, and there are not many contributions, yet, see for instance [1], [37], [69], [70]. On the other hand, [28] is the first paper where linear parabolic problems with the associated Neumann boundary condition are considered, and, in this direction, we face the related nonlinear case written above.

In Chapter 4 we consider the eigenvalue problem associated to the Neumann condition introduced above, namely

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}. \end{cases}$$

In particular, we prove the existence of an unbounded sequence of eigenvalues and we show that some classical properties of the set of eigenvalues for the $p-$Laplacian still hold true in this case. In particular, we show that any eigenfunction is bounded in the whole of $\mathbb{R}^N$. These results can be found in [53].

In Chapter 5 we give existence results for different kinds of p-superlinear problems, focussing on wether or not the nonlinearity satisfies the Ambrosetti-Rabinowitz condition. The first problem that we consider is

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2} u + g(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}. \end{cases}$$

with $\lambda \geq 0$. As for the nonlinear source, we assume that g has $p-$superlinear growth and satisfies different sets of assumptions, including the usual Ambrosetti -Rabinowitz condition. In this case the natural geometric structure for the associated functional is the one of linking over cones, as introduced in [22], for which the suitable topological notions are introduced in Chapter 2. As usual when dealing with linking structures, it is natural to consider the eigenvalues of the underlying operator; in this case we will employ the sequence of eigenvalues found in [53] by using the Fadell-Rabinowitz index. We also recall that the use of linking theorems for fractional operators with Dirichlet boundary conditions has already appeared in related situations (see [63] and [64]). We also deal with the same problem under a different set of assumptions on g. However, in this case we have to deal with the notion of cohomological local splitting.

Then we consider problems which lack the Ambrosetti-Rabinowitz condition, and for this reason we will exploit a different general assumption, introduced in [50]. As a consequence, we have the additional complications related to the proof of the Cerami condition. We deal with two different problems, namely

$$\begin{cases} (-\Delta)^s_p u + |u|^{p-2}u = f(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p}u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases}$$

and

$$\begin{cases} (-\Delta)^s_p u = \lambda |u|^{p-2}u + f(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p}u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases}$$

with $\lambda \geq 0$. In the first case the natural geometric structure for the associated functional is of mountain pass type. In the second case it is again the one of linking over cones. In both these cases we prove the existence of two nontrivvial constant sign solutions, one positive and one negative. The results presented in Chapter 5 can be found in [51, 53, 54].

In Chapter 6 we study the problem

$$\begin{cases} (-\Delta)^s_p u = f(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p}u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{5}$$

in the asymptotically $p-$linear case, that is when

$$\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} = \lambda \in \mathbb{R}.$$

In particular, we prove that both in the resonant and the non-resonant case there exists a nontrivial solution of problem (5). The proofs of both these results are based on critical group theory and they originally appeared in [51].

In Chapter 7 we consider a mixed operator involving both local and nonlocal interactions, namely

$$-\alpha\Delta + \beta(-\Delta)^s,$$

with $s \in (0,1)$, $\alpha, \beta \in [0,\infty)$ and $\alpha + \beta > 0$. The Neumann boundary conditions that we consider depends on the different ranges of α and β according to the following setting. If $\alpha = 0$, we consider the nonlocal Neumann condition, thus prescribing that

$$\mathscr{N}_s u(x) := \int_\Omega \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy = 0 \qquad \text{for every } \ x \in \mathbb{R}^n \setminus \overline{\Omega}. \tag{6}$$

If instead $\beta = 0$, we prescribe the classical Neumann condition

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \ \partial\Omega. \tag{7}$$

Finally, if $\alpha \neq 0$ and $\beta \neq 0$, we prescribe *both* the classical and the nonlocal Neumann conditions, by requiring that

$$\begin{cases} \mathscr{N}_s u = 0 & \text{in } \ \mathbb{R}^n \setminus \overline{\Omega}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{8}$$

A remarkable fact is that the prescription in (8) is not an "overdetermined" condition, but it is consistent with the operator $-\alpha\Delta + \beta(-\Delta)^s$. The set of boundary/external Neumann conditions in (6), (7) and (8), in dependence of the different ranges of α and β, will be shortly denoted by "(α,β)-Neumann conditions".

We first deal with a generalized eigenvalue problem with weight, namely

$$\begin{cases} -\alpha\Delta u + \beta(-\Delta)^s u = \lambda m u & \text{in } \ \Omega, \\ \text{with } (\alpha,\beta)\text{-Neumann condition.} \end{cases}$$

with $m : \Omega \to \mathbb{R}$. Under suitable assumptions on the weight m we prove the existence of two unbounded sequences of eigenvalues, one being positive and the other negative. Moreover, we give conditions that ensures that the first eigenvalue is simple and the first eigenfunction can be taken to be nonnegative, in analogy with the classical properties of the first eigenpair for the Laplace operator. We also describe some properties for the space of eigenfunctions.

Then we consider the problem

$$\begin{cases} -\alpha\Delta u + \beta(-\Delta)^s u = (m - \mu u)u + \tau\, J \star u & \text{in } \Omega, \\ \text{with } (\alpha, \beta)\text{-Neumann condition.} \end{cases}$$

In this setting, the (α, β)-Neumann conditions provide an "ecological niche" for the population with density u, making Ω a natural environment in which a given species can live and compete for a resource m, according to a competition function μ. In this setting, the parameter τ, as modulated by the interaction kernel J, describes an additional birth rate due to further intercommunication than just with the closest neighbors, as it happens, for instance, in pollination.

As a matter of fact, the role of the (α, β)-Neumann conditions is precisely to make the boundary and the exterior of the niche Ω "reflective": namely when an individual exits the niche, it is forced to immediately come back into the niche itself, following the same diffusive process, see Section 2 in [28]. The first result for this problem is the existence of a nonnegative solution. Moreover, we give conditions to establish if the only possible solution is the trivial one, or if there exists a nonnegative solution different from the trivial one. The results presented for the eigenvalue problem and for the logistic problem can be found in [26, 27].

Then we consider the problem

$$\begin{cases} -\alpha\Delta u + \beta(-\Delta)^s u = g(x, u) & \text{in } \Omega, \\ \text{with } (\alpha, \beta) - \text{Neumann conditions,} \end{cases},$$

when $g(x, \cdot)$ is assumed to behave linearly at infinity. Under suitable assumptions on g we prove the existence of a weak solution. Depending on g, we have two different cases. In the first one we show that the associated functional is coercive and lower semicontinuous, so that a solution can be found applying the Weierstrass Theorem. For the second case, the geometry of the associated functional is of saddle type, so the existence of a solution is obtained using the Saddle Theorem.

Finally, we consider two problems which generalize previous ones already treated in this thesis, with the difference that here their are governed by the mixed operator instead of just the fractional Laplacian, here in the case $p = 2$. The first one is the parabolic problem

$$\begin{cases} u_t - \alpha\Delta u + \beta(-\Delta)^s u = 0 & \text{in } \Omega,\ t > 0, \\ \mathscr{N}_s u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x) & \text{in } \Omega. \end{cases}$$

As in Chapter 3, we show that classical solutions preserve their mass and that their energy decreases in time. In addition, we prove that classical solutions converge to a constant as time goes to infinity, as shown in [28] in the case $\alpha = 0$.

The second problem is the superlinear problem

$$\begin{cases} -\alpha\Delta u + \beta(-\Delta)^s u + u = f(x, u) & \text{in } \Omega, \\ \text{with } (\alpha, \beta)\text{-Neumann conditions.} \end{cases}$$

Similarly to Chapter 5, we consider this problem when f does not satisfy the Ambrosetti-Rabinowitz condition but satisfies the condition introduced in [50]. Also in this case, we prove the existence of two nontrivial constant sign solutions, one being positive and the other negative. In this case the geometry of the associated functional is of Mountain Pass type, so the strategy to find the two solutions is to apply the Mountain Pass Theorem to suitable truncated functionals.

Chapter 1

The nonlocal Neumann boundary condition

In this section we discuss some motivation behind the definition of the $p-$Neumann boundary conditions. Recalling that

$$\mathscr{N}_{s,p}u(x) := \int_{\Omega} |u(x)-u(y)|^{p-2}\frac{u(x)-u(y)}{|x-y|^{N+ps}}\,dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega},$$

the first advantage that we have is that the classic divergence theorem and of the integration by parts formula, namely

$$\int_{\Omega} \Delta u = \int_{\partial\Omega} \partial_\nu u$$

and

$$\int_{\Omega} \nabla u \nabla v = \int_{\Omega} v(-\Delta)u \int_{\partial\Omega} v\partial_\nu u,$$

have analogous formulations in the nonlocal case, and we give the results as stated in [8]:

Proposition 1.0.1. *Let u be any bounded C^2 function in $\mathbb{R}^N$. Then,*

$$\int_{\Omega} (-\Delta)^s_p u\,dx = -\int_{\mathbb{R}^N\setminus\Omega} \mathscr{N}_{s,p}u\,dx.$$

Proposition 1.0.2. *Let u and v be bounded C^2 functions in $\mathbb{R}^N$. Then,*

$$\frac{1}{2}\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} |u(x)-u(y)|^{p-2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dxdy$$
$$= \int_{\Omega} v(-\Delta)^s_p u\,dx + \int_{\mathbb{R}^N\setminus\Omega} v\mathscr{N}_{s,p}u\,dx.$$

These two results allow us to give the definition of weak solutions, which is fundamental since we want to find these solutions as critical points of suitable functionals.

Moreover, it is possible to give a probabilistic interpretation of the Neumann boundary conditions in the case of the homogeneous parabolic problem, as shown in [28]. If we consider $u(x,t)$ to be the probability distribution of the position of a particle moving randomly inside Ω, then when the particle exists Ω, it goes back immediately inside of Ω. More precisely, if the particle is in a point $x \in \mathbb{R}^N \setminus \Omega$, then it can go to any point $y \in \Omega$, and the probability to jump from x to y is proportional to $|x-y|^{-N-ps}$.

The next result is present in [27]. It covers the case $p = 2$ and shows that the functions satisfying the homogeneous boundary condition minimize the Gagliardo seminorm.

Theorem 1.0.3. *Let $u : \mathbb{R}^n \to \mathbb{R}$ with $u \in L^1(\Omega)$, and set, for all $x \in \mathbb{R}^n \setminus \overline{\Omega}$,*

$$E_u(x) := \int_\Omega \frac{u(z)}{|x-z|^{n+2s}}\, dz.$$

Then, if we define

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ \\ \dfrac{E_u(x)}{E_1(x)} & \text{if } x \in \mathbb{R}^n \setminus \overline{\Omega}, \end{cases} \tag{1.1}$$

we have

$$\iint_{\mathcal{Q}} \frac{|\tilde{u}(x)-\tilde{u}(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \leq \iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy. \tag{1.2}$$

Also, the equality in (1.2) *holds if and only if u satisfies $\mathscr{N}_s u = 0$ for every $x \in \mathbb{R}^n \setminus \overline{\Omega}$.*

Proof. We remark that the notation E_1 in (1.1) stands for E_u when $u \equiv 1$. Moreover, without loss of generality, we can suppose that

$$\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy < +\infty,$$

otherwise the claim in (1.2) is obviously true.

In addition,

$$\int_\Omega \int_\Omega \frac{|\tilde{u}(x)-\tilde{u}(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy, \tag{1.3}$$

so we only need to consider the integral on $(\mathbb{R}^n \setminus \Omega) \times \Omega$ (being the integral on $\Omega \times (\mathbb{R}^n \setminus \Omega)$ the same).

Setting $\varphi(x) := u(x) - \tilde{u}(x)$, for every $y \in \mathbb{R}^n \setminus \overline{\Omega}$ we have

$$\begin{aligned} \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx &= \int_\Omega \frac{|u(x)-\tilde{u}(y)-\varphi(y)|^2}{|x-y|^{n+2s}}\,dx \\ &= \int_\Omega \frac{|u(x)-\tilde{u}(y)|^2 - 2\varphi(y)(u(x)-\tilde{u}(y)) + |\varphi(y)|^2}{|x-y|^{n+2s}}\,dx. \end{aligned} \tag{1.4}$$

Now, we observe that, for every $y \in \mathbb{R}^n \setminus \overline{\Omega}$,

$$\int_\Omega \frac{u(x)-\tilde{u}(y)}{|x-y|^{n+2s}}\,dx = E_u(y) - \frac{E_u(y)}{E_1(y)}\,E_1(y) = 0.$$

Accordingly, (1.4) becomes

$$\begin{aligned} \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx &= \int_\Omega \frac{|\tilde{u}(x)-\tilde{u}(y)|^2 + |\varphi(y)|^2}{|x-y|^{n+2s}}\,dx \\ &\geq \int_\Omega \frac{|\tilde{u}(x)-\tilde{u}(y)|^2}{|x-y|^{n+2s}}\,dx, \end{aligned}$$

for every $y \in \mathbb{R}^n \setminus \overline{\Omega}$, and the equality holds if and only if $\varphi(y) = 0$. Integrating over $\mathbb{R}^n \setminus \Omega$ (or, equivalently, on $\mathbb{R}^n \setminus \overline{\Omega}$), we get

$$\int_{\mathbb{R}^n \setminus \Omega} \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \geq \int_{\mathbb{R}^n \setminus \Omega} \int_\Omega \frac{|\tilde{u}(x)-\tilde{u}(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

and the equality holds if and only if $\varphi \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. From this observation and (1.3) we obtain (1.2), as desired. □

Since we seek solutions as critical points of functionals that have at least a term involving the Gagliardo seminorm, Theorem 1.0.3 can be very useful when dealing with minimization arguments.

Chapter 2

Preliminaries

In this chapter we recall some known results that will be useful throughout this thesis.

2.1 Critical group theory

First, we start recalling some notions of critical group theory, see for example [48] and [62]. Let $\Phi : X \to \mathbb{R}$ be of class C^1. We denote by $K(\Phi)$ the set of critical points of Φ,

$$K(\Phi) = \{u \in X \,:\, \Phi'(u) = 0\}.$$

We also use the notation

$$\Phi^a = \{u \in X : \; \Phi(u) \le a\}, \quad a \in \mathbb{R}.$$

Assuming that $x \in X$ is an isolated critical point of Φ, the $k-$th (cohomological) critical groups of Φ at x is defined by

$$C^k(\Phi, x) = H^k(\Phi^c \cap U, \Phi^c \cap U \setminus \{x\})$$

for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $c = \Phi(x)$, U is a neighborhood of x such that $K(\Phi) \cap \Phi^c \cap U = \{x\}$, and $H^*(A, B)$ denotes the $*-$th Alexander-Spanier cohomology group with coefficients in $\mathbb{Z}_2$ of the topological pair (A, B). By the excision property of the cohomology groups, the definition of critical groups is independent of the particular choice of the neighborhood U.

We also recall some notions of critical group theory. For this, we recall that a real functional Φ defined in a Banach space X satisfies the (PS) condition if every sequence $(u_n)_n \subset X$ such that $(\Phi(u_n))_n$ is bounded and $\Phi'(u_n) \to 0$ in X' as $n \to \infty$ admits a strongly converging subsequence.

Moreover, we say that Φ satisfies the (C) condition if, for every sequence $(u_n)_n \subset X$ such that $(\Phi(u_n))_n$ is bounded and $(1 + \|u_n\|)\Phi'(u_n) \to 0$ in X' as $n \to \infty$, there exists a strongly converging subsequence of $(u_n)_n$.

Finally, we denote by $K(\Phi)$ the set of critical points of Φ.

We recall the following well-know facts (see [14] and [18].):

Proposition 2.1.1. *Let X be a Banach space, $u \in X$ and for all $\tau \in [0,1]$ let $\Phi_\tau \in C^1(X)$ be a functional such that $u \in K(\Phi_\tau)$. If there exists a closed neighborhood $U \subset X$ of u such that*
(i) Φ_τ satisfies (PS) in U for all $\tau \in [0,1]$,
(ii) $K(\Phi_\tau) \cap U = \{u\}$ for all $\tau \in [0,1]$,
(iii) the mapping $\tau \mapsto \Phi_\tau$ is continuous from $[0,1]$ to $C^1(U)$,
then $C^k(\Phi_1, u) = C^k(\Phi_0, u)$ for all $k \in \mathbb{N}_0$.

On the other hand, it is almost trivial to compute critical groups in extremal points:

Proposition 2.1.2. *Let X be a Banach space with $\dim(X) = \infty$, let $\Phi \in C^1(X)$ be a functional satisfying (C), and let $u \in K(\Phi)$ be an isolated critical point of Φ. Then:*
(i) if u is a local minimizer of Φ, then $C^k(\Phi, u) = \delta_{k,0}\mathbb{Z}_2$ for all $k \in \mathbb{N}_0$,
(ii) if u is a local maximizer of Φ, then $C^k(\Phi, u) = 0$ for all $k \in \mathbb{N}_0$.

Here, as usual, $\delta_{i,j}$ denotes the Kronecker symbol.

Definition 2.1.3. A functional Φ has a cohomological local splitting near 0 in dimention $k \in \mathbb{N}$ if there exist symmetric cones $X_\pm \subset X$ with $X_+ \cap X_- = \{0\}$ and $\rho > 0$ such that
(i) $i(X_- \setminus \{0\}) = i(X \setminus X_+) = k$,
(ii) $\Phi(u) \le \Phi(0)$ for all $u \in \overline{B}_\rho \cap X_-$, and $\Phi(u) \ge \Phi(0)$ for all $u \in \overline{B}_\rho \cap X_+$.
Here i denotes the $\mathbb{Z}_2$ cohomological index introduced in [32].

We shall use the following result (sse [23, Proposition 2.1]).

Proposition 2.1.4. *If X is a Banach space and $\Phi \in C^1(X)$ has a cohomological local splitting near 0 in dimension $k \in \mathbb{N}$, and 0 is an isolated critical point of Φ, then $C^k(\Phi, 0) \ne 0$.*

2.2 Cohomological linking

Now, we recall some notions on linking structures in connection with the Alexander-Spanier cohomology, referring to [22].

Definition 2.2.1. Let D, S, A, B be four subsets of a metric space X with $S \subseteq D$ and $B \subseteq A$. We say that (D, S) *links* (A, B), if $S \cap A = B \cap D = \emptyset$ and, for every deformation $\eta : D \times [0, 1] \to X \setminus B$ with $\eta(S \times [0, 1]) \cap A = \emptyset$, we have that $\eta(D \times \{1\}) \cap A \neq \emptyset$.

To prove the existence of critical points we will use a particular case of [34, Theorem 3.1]. A smooth version of such a result was already stated in [22, Theorem 2.2] under the validity of the Palais–Smale condition. However, the key point in the proof of [34, Theorem 3.1] is the possibility of defining deformations between sublevels, as it is possible under the validity of the Cerami condition. For this reason we recall that f satisfies the $(C)_c$ condition, $c \in \mathbb{R}$, if

for every $(u_n)_n$ such that $f(u_n) \to c$ and $(1 + \|u_n\|)f'(u_n) \to 0$ in X', then, up to a subsequence, $u_n \to u$ in X.

Hence, we will need the following version of [34, Theorem 3.1]:

Theorem 2.2.2. *Let X be a complete Finsler manifold of class C^1 and let $f : X \to \mathbb{R}$ be a function of class C^1. Let D, S, A, B be four subsets of X, with $S \subseteq D$ and $B \subseteq A$, such that (D, S) links (A, B) and such that*

$$\sup_S f < \inf_A f, \qquad \sup_D f < \inf_B f$$

(with $\sup \emptyset = -\infty$ *and* $\inf \emptyset = +\infty$*). Define*

$$c = \inf_{\eta \in \mathcal{N}} \sup f(\eta(D \times \{1\})),$$

where $\mathcal{N}$ is the set of deformations $\eta : D \times [0, 1] \to X \setminus B$ with $\eta(S \times [0, 1]) \cap A = \emptyset$. Then we have

$$\inf_A f \leq c \leq \sup_D f.$$

Moreover, if f satisfies $(C)_c$, then c is a critical value of f.

Definition 2.2.3. Let D, S, A, B be four subsets of X with $S \subseteq D$ and $B \subseteq A$; let m be a nonnegative integer and let $\mathbb{K}$ be a field. We say that (D, S) *links* (A, B) *cohomologically in dimension* m *over* $\mathbb{K}$ if $S \cap A = B \cap D = \emptyset$ and the restriction homomorphism $H^m(X \setminus B, X \setminus A; \mathbb{K}) \to H^m(D, S; \mathbb{K})$ is not identically zero.

The geometry we are interested in is described by the following

Theorem 2.2.4 ([22], Theorem 2.8)**.** *Let X be a real normed space and let $\mathcal{C}_-$, $\mathcal{C}_+$ be two cones such that $\mathcal{C}_+$ is closed in X, $\mathcal{C}_- \cap \mathcal{C}_+ = \{0\}$ and such that $(X, \mathcal{C}_- \setminus \{0\})$ links $\mathcal{C}_+$ cohomologically in dimension m over $\mathbb{K}$. Let $r_-, r_+ > 0$ and let*

$$D_- = \{u \in \mathcal{C}_- : \|u\| \le r_-\}, \qquad S_- = \{u \in \mathcal{C}_- : \|u\| = r_-\},$$

$$D_+ = \{u \in \mathcal{C}_+ : \|u\| \le r_+\}, \qquad S_+ = \{u \in \mathcal{C}_+ : \|u\| = r_+\}.$$

Then the following facts hold:

(a) (D_-, S_-) links $\mathcal{C}_+$ cohomologically in dimension m over $\mathbb{K}$;

(b) (D_-, S_-) links (D_+, S_+) cohomologically in dimension m over $\mathbb{K}$;

Moreover, let $e \in X$ with $-e \notin \mathcal{C}_-$, let

$$Q = \{u + te : u \in \mathcal{C}_-, \, t \ge 0, \, \|u + te\| \le r_-\},$$

$$H = \{u + te : u \in \mathcal{C}_-, \, t \ge 0, \, \|u + te\| = r_-\},$$

and assume that $r_- > r_+$. Then the following facts hold:

(c) $(Q, D_- \cup H)$ links S_+ cohomologically in dimension $m + 1$ over $\mathbb{K}$;

(d) $D_- \cup H$ links (D_+, S_+) cohomologically in dimension m over $\mathbb{K}$;

We will also take advantage of the following result

Corollary 2.2.5 ([22], Corollary 2.9)**.** *Let X be a real normed space and let $\mathcal{C}_-, \mathcal{C}_+$ be two symmetric cones in X such that $\mathcal{C}_+$ is closed in X, $\mathcal{C}_- \cap \mathcal{C}_+ = \{0\}$ and such that*

$$i(\mathcal{C}_- \setminus \{0\}) = i(X \setminus \mathcal{C}_+) < \infty.$$

Then the assertion (a)-(d) of Theorem 2.2.4 hold for $m = i(\mathcal{C}_- \setminus \{0\})$ and $\mathbb{K} = \mathbb{Z}_2$.

Going back to definitions (5.4) and (5.5), we have the following result, which is the transcription in our setting of [22, Theorem 3.2], and whose proof follows that one step-by-step.

Theorem 2.2.6. *Let $m \ge 1$ be such that $\lambda_m < \lambda_{m+1}$, then we have*

$$i(C_m^- \setminus \{0\}) = i(X \setminus C_m^+) = m$$

Finally, in order to use Theorem 2.2.2, the crucial tool is

Proposition 2.2.7 ([22], Proposition 2.4)**.** *If (D, S) links (A, B) cohomologically (in some dimension), then (D, S) links (A, B).*

Chapter 3

Functional space, regularity and parabolic problem

In this chapter we first discuss some general properties of the functional space where we seek solutions. In particular, we recall the definition of the space and of its norm, and we give a proof that is a Banach space (see Proposition 3.1.2). Moreover, we show that such a space is embedded in $L^p(B(0,R))$ for every $R > 0$ (see Remark 3.1.3) and that it is also compactly embedded in suitable $L^q(\Omega)$ spaces (see Remark 3.1.4). After giving the definition of weak solutions, we prove that these solutions satisfy the boundary conditition a.e. outside of Ω, that is $\mathscr{N}_{s,p}u = g$ a.e. in $\mathbb{R}^N \setminus \overline{\Omega}$ (see Theorem 3.1.6. We also show, as it is usual dealing with variational methods, that weak solutions of our problems can be seen as critical points of suitable functionals (see Proposition 3.1.7). Since the functionals that we consider always have a term involving the Gagliardo seminorm, we show that these type of functionals satisfies the so called (S) property (see Proposition 3.1.8), which will be very useful when proving the convergence of sequences. Lastly, we give a sort of maximum principle for our type of problems (see Proposition 3.1.9).

Then we give some regularity results for general problems; in particular, we prove that weak solutions are bounded in the general case (see Theorem 3.2.2 and Corollary 3.2.3 together with Remark 3.2.4). From the boundedness result and the boundary conditions, in the case $p = 2$, we prove that weak solutions are continuous outside of Ω (see Proposition 3.2.5). Moreover, using a result from [65], we are able to extend this continuity result in the whole R^N (see Theorem 3.2.6).

Finally, we consider a parabolic problem, showing that classical solutions have their mass preserved (see Proposition 3.3.1 and their energy that decreases in time (see Proposition 3.3.2).

The results in Section 3.1 are taken from [53, 54], in Section 3.2 from [51] and in Section 3.3 from [53].

3.1 Functional space

In this section we first introduce the functional space in which we work. To do that, fix a bounded domain with Lipschitz boundary $\Omega \subset \mathbb{R}^N$, $N \geq 1$, $g \in L^1(\mathbb{R}^N \setminus \Omega)$, and for $u : \mathbb{R}^N \to \mathbb{R}$ measurable, set

$$\|u\|_X := \left(\|u\|^p_{L^p(\Omega)} + \||g|^{\frac{1}{p}} u\|^p_{L^p(\mathbb{R}^N \setminus \Omega)} + \int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dxdy \right)^{\frac{1}{p}},$$

where $C\Omega = \mathbb{R}^N \setminus \Omega$, and

$$X := \{u : \mathbb{R}^N \to \mathbb{R} \quad \text{measurable such that } \|u\|_X < \infty\}.$$

Remark 3.1.1. It is clear that, Ω being "nice enough", in the previous setting we can equally write $\mathbb{R}^N \setminus \Omega$ in place of $\mathbb{R}^N \setminus \overline{\Omega}$. The abstract setting can be faced also for Ω less regular, replacing $\||g|^{\frac{1}{p}} u\|_{L^p(\mathbb{R}^N \setminus \Omega)}$ with $\||g|^{\frac{1}{p}} u\|_{L^p(\mathbb{R}^N \setminus \overline{\Omega})}$, which is the natural norm in the general framework.

We start recalling the following result, already stated in [8].

Proposition 3.1.2. *X is a reflexive Banach space with norm $\|\cdot\|_X$.*

Proof. First, we show that $\|\cdot\|_X$ is a norm. If $\|u\|_X = 0$, we have $\|u\|_{L^p(\Omega)} = 0$, so $u = 0$ a.e. in Ω. Moreover, we have

$$\int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dxdy = 0,$$

hence $|u(x) - u(y)| = 0$ in $\mathbb{R}^{2N} \setminus (C\Omega)^2$. In particular, we can take $x \in C\Omega$ and $y \in \Omega$ to obtain

$$u(x) = u(x) - u(y) = 0.$$

In this way, we have $u = 0$ a.e. in $\mathbb{R}^N$.

Now, we prove that X is complete, and to do this we take a Cauchy sequence $(u_k)_k$ in X. In particular, u_k is a Cauchy sequence in $L^p(\Omega)$ and so (up to a subsequence) there exists $u \in L^p(\Omega)$ such that u_k converges to u in $L^p(\Omega)$ and a.e. in Ω. This means that there exists $Z_1 \subset \Omega$ such that

$$|Z_1| = 0 \text{ and } u_k(x) \to u(x) \text{ for every } x \in \Omega \setminus Z_1. \tag{3.1}$$

We also define for every $U:\mathbb{R}^N\to\mathbb{R}$ and $(x,y)\in\mathbb{R}^{2N}$

$$T_U(x,y):=\frac{(U(x)-U(y))\chi_{\mathbb{R}^{2N}\setminus(C\Omega)^2(x,y)}}{|x-y|^{N/p+s}},$$

so

$$T_{u_k}(x,y)-T_{u_h}(x,y)=\frac{(u_k(x)-u_h(x)-u_k(y)+u_h(y))\chi_{\mathbb{R}^{2N}\setminus(C\Omega)^2(x,y)}}{|x-y|^{N/p+s}}.$$

Since u_k is a Cauchy sequence in X, for every $\varepsilon>0$ there exists $N_\varepsilon>0$ such that for $h,k\geq N_\varepsilon$ we have in particular

$$\varepsilon^p\geq\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2}\frac{|u_k(x)-u_h(x)-u_k(y)+u_h(y)|^p}{|x-y|^{N+ps}}\,dxdy=\|T_{u_k}-T_{u_h}\|^p_{L^p(\mathbb{R}^{2N})}.$$

So, T_{u_k} is a Cauchy sequence in $L^p(\mathbb{R}^{2N})$, and up to a subsequence we can assume that T_{u_k} converges to some T in $L^p(\mathbb{R}^{2N})$ and a.e. in $\mathbb{R}^{2N}$. This means that there exists $Z_2\subset\mathbb{R}^{2N}$ such that

$$|Z_2|=0 \text{ and } T_{u_k}(x,y)\to T_u(x,y) \text{ for every } (x,y)\in\mathbb{R}^{2N}\setminus Z_2. \tag{3.2}$$

For any $x\in\Omega$, we set

$$S_x:=\{y\in\mathbb{R}^N:\,(x,y)\in\mathbb{R}^{2N}\setminus Z_2\},$$
$$W:=\{(x,y)\in\mathbb{R}^{2N}:\,x\in\Omega \text{ and } y\in\mathbb{R}^N\setminus S_x\},$$
$$V:=\{x\in\Omega:\,|\mathbb{R}^N\setminus S_x|=0\}.$$

If we take $(x,y)\in W$, we have $y\in\mathbb{R}^N\setminus S_x$, so $(x,y)\notin\mathbb{R}^{2N}\setminus Z_2$ that is $(x,y)\in Z_2$. From this we get

$$W\subseteq Z_2. \tag{3.3}$$

From (3.3) and (3.2), we obtain $|W|=0$, so by the Fubini's Theorem we have

$$0=|W|=\int_\Omega|\mathbb{R}^N\setminus S_x|\,dx,$$

which implies that $|\mathbb{R}^N\setminus S_x|=0$ a.e. $x\in\Omega$. It follows that $|\Omega\setminus V|=0$. This together with (3.1) implies that

$$|\Omega\setminus(V\setminus Z_1)|=|(\Omega\setminus V)\cup Z_1|\leq|\Omega\setminus V|+|Z_1|=0.$$

In particular, $V\setminus Z_1\neq\emptyset$ (nay, $|V\setminus Z_1|=|\Omega|$), so we can take $x_0\in V\setminus Z_1$. From (3.1) we have

$$\lim_{k\to\infty}u_k(x_0)=u(x_0).$$

In addition, since $x_0 \in V$, we get $|\mathbb{R}^N \setminus S_{x_0}| = 0$. This means that for a.e. $y \in \mathbb{R}^N$, $(x_0, y) \in \mathbb{R}^{2N} \setminus Z_2$ and so

$$\lim_{k\to\infty} T_{u_k}(x_0, y) = T(x_0, y).$$

Moreover, since $\Omega \times (C\Omega) \subseteq \mathbb{R}^{2N} \setminus (C\Omega)^2$, we have

$$T_{u_k}(x_0, y) := \frac{u_k(x_0) - u_k(y)}{|x_0 - y|^{N/p+s}}$$

for a.e. $y \in C\Omega$. From this, we obtain

$$\begin{aligned}\lim_{k\to\infty} u_k(y) &= \lim_{k\to\infty} \big(u_k(x_0) - |x_0 - y|^{N/p+s} T_{u_k}(x_0, y)\big) \\ &= u(x_0) - |x_0 - y|^{N/p+s} T(x_0, y)\end{aligned}$$

for a.e. $y \in C\Omega$. This and (3.1) imply that u_k converges a.e. in $\mathbb{R}^N$, so we can say that u_k converges a.e. to some u in $\mathbb{R}^N$. Now, since u_k is a Cauchy sequence in X, for any $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that, for any $h \geq N_\varepsilon$,

$$\begin{aligned}\varepsilon^p &\geq \liminf_{k\to\infty} \|u_h - u_k\|_X^p \\ &\geq \liminf_{k\to\infty} \int_\Omega |u_h - u_k|^p \, dx + \liminf_{k\to\infty} \int_{\mathbb{R}^N\setminus\Omega} |g||u_h - u_k|^p \, dx \\ &\quad + \liminf_{k\to\infty} \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|(u_k - u_h)(x) - (u_k + u_h)(y)|^p}{|x - y|^{N+ps}} \, dxdy \\ &\geq \int_\Omega |u_h - u|^p \, dx + \int_{\mathbb{R}^N\setminus\Omega} |g||u_h - u|^p \, dx \\ &\quad + \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|(u_k - u)(x) - (u_k + u)(y)|^p}{|x - y|^{N+ps}} \, dxdy \\ &= \|u_h - u\|_X^p,\end{aligned}$$

where we used Fatou's Lemma. So u_h converges to u in X. Starting this procedure with a generic subsequence, we can conclude that X is complete.

As for the reflexivity, see [8]. □

Remark 3.1.3. From the definition of X, it follows that X is embedded in $L^p(B(0, R))$ for every $R > 0$. Indeed, by the convergence of the double integral, we get that for a.e. $x \in \Omega$

$$\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dy < \infty,$$

and so for every $R > 0$

$$\frac{1}{R^{N+ps}} \int_{B(x,R)} |u(x) - u(y)|^p \, dy < \infty.$$

In addition, we have

$$\int_{B(x,R)} |u(y)|^p \, dy \le 2^{p-1} \int_{B(x,R)} |u(x) - u(y)|^p \, dy + 2^{p-1} |u(x)|^p |B(x,R)| < \infty,$$

hence the claim follows.

Remark 3.1.4. Under the previous setting, X is embedded continuously in $W^{s,p}(\Omega)$. As a consequence, the standard compact embeddings in suitable $L^q(\Omega)$ spaces hold true, see [30].

The integration by parts formula in Proposition 1.0.2 leads to this natural definition:

Definition 3.1.5. Let $f \in L^{p'}(\Omega)$ and $g \in L^1(\mathbb{R}^N \setminus \overline{\Omega})$. We say that $u \in X$ is a weak solution of

$$\begin{cases} (-\Delta)_p^s u = f & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{3.4}$$

whenever

$$\frac{1}{2} \int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} \frac{J_p(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}} \, dxdy = \int_\Omega f v \, dx + \int_{\mathbb{R}^N \setminus \overline{\Omega}} g v \, dx \tag{3.5}$$

for every $v \in X$, where

$$J_p(u(x) - u(y)) := |u(x) - u(y)|^{p-2} (u(x) - u(y)).$$

As a consequence of this definition, we have the following result

Theorem 3.1.6. *Let u be a weak solution of* (3.4). *Then, $\mathscr{N}_{s,p} u = g$ a.e. in $\mathbb{R}^N \setminus \overline{\Omega}$.*

Proof. First, we take $v \in X$ such that $v \equiv 0$ in Ω as a test function in (3.5), obtaining

$$
\begin{aligned}
\int_{\mathbb{R}^N\setminus\overline{\Omega}} gv\,dx &= -\frac{1}{2}\int_{\Omega}\int_{\mathbb{R}^N\setminus\overline{\Omega}} \frac{J_p(u(x)-u(y))v(y)}{|x-y|^{N+ps}}\,dydx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N\setminus\overline{\Omega}}\int_{\Omega} \frac{J_p(u(x)-u(y))v(x)}{|x-y|^{N+ps}}\,dydx \\
&= -\int_{\Omega}\int_{\mathbb{R}^N\setminus\overline{\Omega}} \frac{J_p(u(x)-u(y))v(y)}{|x-y|^{N+ps}}\,dydx \\
&= -\int_{\mathbb{R}^N\setminus\overline{\Omega}} v(y)\int_{\Omega} \frac{J_p(u(x)-u(y))}{|x-y|^{N+ps}}\,dxdy \\
&= -\int_{\mathbb{R}^N\setminus\overline{\Omega}} v(y)\mathscr{N}_{s,p}u(y)\,dy.
\end{aligned}
$$

Therefore,

$$
\int_{\mathbb{R}^N\setminus\overline{\Omega}} (\mathscr{N}_{s,p}u(x)-g(x))v(x)\,dx = 0
$$

for every $v \in X$ which is 0 in Ω. In particular, this is true for every $v \in C_c^\infty(\mathbb{R}^N \setminus \overline{\Omega})$, and so $\mathscr{N}_{s,p}u(x) = g(x)$ a.e. in $\mathbb{R}^N \setminus \overline{\Omega}$. □

From the definition of weak solution, we have the following

Proposition 3.1.7. *Let $f \in L^{p'}(\Omega)$ and $g \in L^1(\mathbb{R}^N \setminus \Omega)$. Let $I_g : X \to \mathbb{R}$ be the functional defined as*

$$
I_g(u) := \frac{1}{2p}\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy - \int_{\Omega} fu\,dx - \int_{\mathbb{R}^N\setminus\Omega} gu\,dx
$$

for every $u \in X$. Then any critical point of I_g is a weak solution of problem (3.4).

Proof. We only show that I_g is well defined on X. Indeed, if $u \in X$ we have

$$
\left|\int_{\Omega} fu\,dx\right| \le \|f\|_{L^{p'}(\Omega)}\|u\|_{L^p(\Omega)} \le C\|u\|_X.
$$

In addition,

$$
\left|\int_{\mathbb{R}^N\setminus\Omega} gu\,dx\right| \le \int_{\mathbb{R}^N\setminus\Omega} |g|^{\frac{1}{p'}}|g|^{\frac{1}{p}}|u| \le \|g\|_{L^1(\mathbb{R}^N\setminus\Omega)}^{\frac{1}{p'}}\||g|^{\frac{1}{p}}u\|_{L^p(\mathbb{R}^N\setminus\Omega)} \le C\|u\|_X.
$$

Then, if $u \in X$, we have

$$
|I_g(u)| \le C\|u\|_X < \infty.
$$

The computation of the first variation of I_g is standard. □

Now we give the following result, which will be useful in any case and which makes precise the statement in [53] related to the (S) property.

Proposition 3.1.8. *Set $A(u) = [u]^p$. Then the functional $A' : X \to X'$ satisfies the $(S)_+$ property, that is for every sequence $(u_n)_n$ such that $u_n \rightharpoonup u$ in X as $n \to \infty$ and*

$$\limsup_{n\to\infty} \langle A'(u_n), u_n - u\rangle_{X',X} \le 0, \tag{3.6}$$

then $u_n \to u$ in X as $n \to \infty$.

Proof. Assume that $u_n \rightharpoonup u$ in X and $\limsup\langle A'(u_n), u_n - u\rangle_{X',X} \le 0$. First of all, A is convex, of class C^1 and weakly lower semicontinuous in X, so that $A(u) \le \liminf A(u_n)$.

Moreover, the linear functional $\langle A'(u), \cdot\rangle_{X',X}$ is in X'. So, since $u_n \rightharpoonup u$ in X,

$$\langle A'(u), u_n - u\rangle_{X',X} \to 0 \tag{3.7}$$

as $n \to \infty$. By the convexity of A, we get that A' is a monotone operator, so that

$$\langle A'(u_n) - A'(u), u_n - u\rangle_{X',X} \ge 0.$$

By (3.6) we get

$$0 \le \limsup_{n\to\infty} \langle A'(u_n) - A'(u), u_n - u\rangle_{X',X} \le 0,$$

and so

$$\lim_{n\to\infty} \langle A'(u_n) - A'(u), u_n - u\rangle_{X',X} = 0. \tag{3.8}$$

Hence, (3.7) and (3.8) imply that

$$\lim_{n\to\infty} \langle A'(u_n), u_n - u\rangle_{X',X} = 0. \tag{3.9}$$

Again by the convexity of A we have that

$$A(u) \ge \langle A'(u_n), u - u_n\rangle_{X',X} \ge A(u_n).$$

By (3.9), $A(u) \ge \limsup A(u_n)$, and so

$$A(u) = \lim_{n\to\infty} A(u_n).$$

By the compact embedding of X into $L^p(\Omega)$ we also have $u_n \to u$ in $L^p(\Omega)$. In the end, $\|u_n\| \to \|u\|$. Hence, by the uniform convexity of X (recall that $1 < p < \infty$) , we obtain that u_n converges strongly to u in X as $n \to \infty$. $\square$

The next result gives a sort of maximum principle.

Proposition 3.1.9. *Let $f \in L^{p'}(\Omega)$ and $g \in L^1(\mathbb{R}^N \setminus \Omega)$. Let $u \in X$ be a weak solution of* (3.4) *with $f \geq 0$ and $g \geq 0$. Then, u is constant.*

Proof. First, we notice that $v \equiv 1$ belongs to X. So, using it as a test function in (3.5) we obtain

$$0 \leq \int_\Omega f\,dx = -\int_{\mathbb{R}^N \setminus \Omega} g\,dx \leq 0.$$

Hence, $f = 0$ a.e. in Ω and $g = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. Now, taking $v = u$ as a test function again in (3.5), we get

$$\int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\,dxdy = 0,$$

so u must be constant. □

From now on, we concentrate on homogeneous boundary conditions, so that $g \equiv 0$.

Denoting by X' the dual of X, we can define the operator $A : X \to X'$ such that

$$\begin{aligned}\langle A(u), v\rangle = &\int_\Omega |u|^{p-2}uv\,dx \\ &+ \int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} \frac{J_p(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}}\,dxdy\end{aligned}$$

for all $u, v \in X$. In this way A is $(p-1)$-homogeneous and odd, and such that

$$\langle A(u), u\rangle = \|u\|_X^p, \qquad |\langle A(u), v\rangle| \leq \|u\|_X^{p-1}\|v\|_X.$$

By the uniform convexity of X, A satisfies the (S) property, that is, for all $(u_n)_n$ in X such that $u_n \rightharpoonup u$ in X and $\langle A(u_n), u_n - u\rangle \to 0$, then $u_n \to u$ in X, see [62, Proposition 1.3].

3.2 Regularity

In this section we prove some L^∞ a priori estimates and some regularity results for solutions of problems of the type

$$\begin{cases} (-\Delta)_p^s u + |u|^{p-2}u = f(x, u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}. \end{cases} \tag{3.10}$$

We just suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following condition:

(f_1) there exists $a > 0$ such that

$$|f(x,t)| \leq a(|t|^{q-1} + |t|^{r-1})$$

for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$, with $1 \leq q \leq p \leq r \leq p_s^*$.

As usual, p_s^* is the critical fractional Sobolev exponent, namely

$$p_s^* = \begin{cases} \frac{pN}{N-ps} & \text{if } N < ps, \\ \infty & \text{if } N \geq ps. \end{cases}$$

We tacitly assume that in (f_1) we have $r < \infty$ also when $N \geq ps$.

Remark 3.2.1. Clearly, if we prove an estimate for a solution of (3.10), then it will also be true for the problem

$$\begin{cases} (-\Delta)_p^s u = \tilde{f}(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases}$$

since $\tilde{f}(x,u) = f(x,u) - |u|^{p-2}u$ still satisfies condition (f_1).

We follow the lines of the analogous proofs in [38] for the Dirichlet case, but, while adapting the original proofs to our situation, we take the opportunity to give more details.

The first result is the following.

Theorem 3.2.2. *If hypotesis (f_1) holds with $1 \leq q \leq p \leq r \leq p_s^*$ satisfying*

$$1 + \frac{q}{p} > \frac{r}{p} + \frac{r}{p_s^*},$$

then there exist $K > 0$ and $\alpha > 1$, both depending on $p, q, r, s, a, |\Omega|$, such that every weak solution u of (3.10) belongs to $L^\infty(\Omega)$ and

$$\|u\|_\infty \leq K(1 + \|u\|_r^\alpha).$$

Proof. First, we fix a weak solution $u \in X$ of (3.10) with $u^+ \neq 0$. Take $\rho \geq \max\{1, \|u\|_r^{-1}\}$ and set $v := (\rho\|u\|_r)^{-1}u$. In this way, $v \in X$, and setting $\|v\|_r = \rho^{-1}$, we have that v is a weak solution of the problem

$$\begin{cases} (-\Delta)_p^s v + |v|^{p-2}v = (\rho\|u\|_r)^{1-p} f(x, \rho\|u\|_r v) & \text{in } \Omega, \\ \mathscr{N}_{s,p} v = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}. \end{cases} \tag{3.11}$$

For all $n \in \mathbb{N}$ we set $v_n := (v - 1 + 2^{-n})^+$, so that $v_n \in X$, $v_0 = v^+$, $0 \leq v_{n+1}(x) \leq v_n(x)$ for all $n \in \mathbb{N}$ and $v_n(x) \to (v(x) - 1)^+$ a.e. in Ω as $n \to \infty$. We also have the following inclusion:

$$\{v_{n+1} > 0\} \subseteq \{0 < v < (2^{n+1} - 1)v_n\} \cup \{v_n > 2^{-n-1}\}. \tag{3.12}$$

Now, we set $R_n := \|v_n\|_r^r$ for all $n \in \mathbb{N}$. So, $R_0 = \|v^+\|_r^r \leq \rho^{-r}$ and $(R_n)_n$ is nonincreasing in $[0, 1]$. We claim that $R_n \to 0$ as $n \to \infty$.

By Hölder's inequality, the fractional Sobolev inequality and (3.12) we have

$$R_{n+1} \leq |\{v_{n+1} > 0\}|^{1-\frac{r}{p_s^*}} \|v_{n+1}\|_{p_s^*}^r \leq C|\{v_n > 2^{-n-1}\}|^{1-\frac{r}{p_s^*}} \|v_{n+1}\|^r,$$

for all $n \in \mathbb{N}$. Using Chebichev's inequality we finally obtain

$$R_{n+1} \leq C\, 2^{\left(r - \frac{r^2}{p_s^*}\right)(n+1)} R_n^{1-\frac{r}{p_s^*}} \|v_{n+1}\|^r. \tag{3.13}$$

Now, we need an estimate on $\|v_{n+1}\|$. To do that, we introduce the inequality

$$|\xi^+ - \eta^+|^p \leq |\xi - \eta|^{p-2}(\xi - \eta)(\xi^+ - \eta^+), \tag{3.14}$$

for all $\xi, \eta \in \mathbb{R}$. Testing (3.10) with v_{n+1} and using (3.14), with $\xi^+ = v_{n+1}(x)$ and $\eta^+ = v_{n+1}(y)$, we get

$$\begin{aligned}
\|v_{n+1}\|^p &\leq \langle A(v), v_{n+1}\rangle \\
&= \int_\Omega (\rho\|u\|_r)^{1-p} f(x, \rho\|u\|_r v) v_{n+1}\, dx \\
&\leq a \int_{\{v_{n+1}>0\}} ((\rho\|u\|_r)^{q-p}|v|^{q-1} + (\rho\|u\|_r)^{r-p}|v|^{r-1}) v_{n+1}\, dx \\
&\leq a(\rho\|u\|_r)^{r-p} \int_{\{v_{n+1}>0\}} ((2^{n+1} - 1)^{q-1} v_n^q + (2^{n+1} - 1)^{r-1} v_n^r)\, dx \\
&\leq a\, 2^{(r-1)(n+1)} (\rho\|u\|_r)^{r-p} \int_{\{v_{n+1}>0\}} (v_n^q + v_n^r)\, dx.
\end{aligned}$$

Now,

$$\int_\Omega v_n^q \leq |\Omega|^{1-\frac{q}{r}} \left(\int_\Omega v_n^r\right)^{\frac{q}{r}}$$

and

$$\int_\Omega v_n^r \leq \left(\int_\Omega v_n^r\right)^{1-\frac{q}{r}} \left(\int_\Omega v_n^r\right)^{\frac{q}{r}} \leq \left(\int_\Omega (v^+)^r\right)^{1-\frac{q}{r}} \left(\int_\Omega v_n^r\right)^{\frac{q}{r}},$$

so we finally get

$$\|v_{n+1}\|^p \le C\,2^{(r-1)(n+1)}(\rho\|u\|_r)^{r-p} R_n^{\frac{q}{r}}.$$

Combining the last estimate with (3.13), we have

$$R_{n+1} \le C\,2^{\left(r-\frac{r}{p}+\frac{r^2}{p}-\frac{r^2}{p_s^*}\right)(n+1)}(\rho\|u\|_r)^{\frac{r^2}{p}-r} R_n^{1+\frac{q}{p}-\frac{r}{p_s^*}},$$

which can be written as

$$R_{n+1} \le C\,H^n(\rho\|u\|_r)^{\frac{r^2}{p}-r} R_n^{1+\beta}, \tag{3.15}$$

with

$$H := 2^{r-\frac{r}{p}+\frac{r^2}{p}-\frac{r^2}{p_s^*}} > 0$$

and

$$0 < \beta := \frac{q}{p} - \frac{r}{p_s^*} < 1.$$

Setting $\gamma := r\beta + r - r^2/p > 0$ and $\eta := H^{-\frac{1}{\beta}} \in (0,1)$, we can take

$$\rho = \max\{1, \|u\|_r^{-1}, \eta^{-\frac{1}{\gamma}}\|u\|_r^{\frac{1}{\gamma}\left(\frac{r^2}{p}-r\right)}\}.$$

Now, we prove by induction that

$$R_n \le C\frac{\eta^n}{\rho^r}. \tag{3.16}$$

Indeed, $R_0 = \|v^+\|_r^r \le \rho^{-r}$. Now we can assume that (3.16) holds for some $n \in \mathbb{N}$, and so by (3.15)

$$R_{n+1} \le C\,H^n(\rho\|u\|_r)^{\frac{r^2}{p}-r}\left(\frac{\eta^n}{\rho^r}\right)^{1+\beta} = C\frac{\eta^n}{\rho^r}\frac{\|u\|_r^{\frac{r^2}{p}-r}}{\rho^\gamma} \le C\frac{\eta^{n+1}}{\rho^r}.$$

Since $\eta \in (0,1)$, from (3.16) $R_n \to 0$ as $n \to \infty$. This implies that $v_n \to 0$ a.e. in Ω, and so $v(x) \le 1$ a.e. in Ω. A similar argument on $-v$ leads to $v \in L^\infty(\Omega)$ and $\|v\|_\infty \le 1$. So $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \le \rho\|u\|_r = \max\{\|u\|_r, 1, \eta^{-\frac{1}{\gamma}}\|u\|_r^{1+\frac{1}{\gamma}\left(\frac{r^2}{p}-r\right)}\} \le K(1+\|u\|_r^\alpha),$$

for some $K > 0$ and $\alpha > 1$. □

In the next result, we assume $q = p$ in (f_1). Then, if we take $\|u\|_r$ sufficiently small, the L^∞ estimate can be formulated in terms of the L^r norm of the solution.

Corollary 3.2.3. *If hypotesis* (f_1) *holds with* $q = p \leq r < p_s^*$*, then there exists a constant* $K \in (0, 1)$*, depending on* $s, p, r, N, |\Omega|$*, such that, for every weak solution* $u \in X$ *of* (3.10) *with* $\|u\|_r \leq K$*, we have* $u \in L^\infty(\Omega)$ *and*

$$\|u\|_\infty \leq K^{-1}\|u\|_r.$$

Proof. Consider $\varepsilon \in (0, 1)$ and let $u \in X$ be a weak solution of (3.10) with $u^+ \neq 0$ and $\|u\|_r \leq \varepsilon$. Now we set $v := \varepsilon^{-1}u$, so $v \in X$ and $\|v\|_r \leq 1$. As before, for all $n \in \mathbb{N}$ we can set $v_n = (v-1+2^{-n})^+$ and $R_n = \|v_n\|_r^r$. Arguing as in the proof of Theorem 3.2.2 we can obtain the inequality

$$R_{n+1} \leq C\, H^n R_n^{1+\beta}, \tag{3.17}$$

for some $H > 1$ and $0 < \beta < 1$. Setting $\eta = H^{-\frac{1}{\beta}}$, we have $K := \eta^{\frac{1}{\beta r}} < 1$. Now, if $\|u\|_r < K$, we can take ε such that $\|u\|_r = \delta = K\varepsilon \in (0, \varepsilon)$. Now, we want to prove that

$$R_n \leq C\frac{\delta^r}{\varepsilon^r}\eta^n. \tag{3.18}$$

Indeed, by definition $R_0 \leq \delta^r/\varepsilon^r$. Then, if (3.17) holds for some $n \in \mathbb{N}$, by (3.18)

$$R_{n+1} \leq C\, H^n\left(\frac{\delta^r}{\varepsilon^r}\eta^n\right)^{1+\beta} = C\frac{\delta^r}{\varepsilon^r}\frac{\delta^{r\beta}}{\varepsilon^{r\beta}}\eta^n = C\frac{\delta^r}{\varepsilon^r}\eta^{n+1}.$$

From (3.17), $R_n \to 0$ as $n \to \infty$ and so $v(x) \leq 1$ a.e. in Ω. Arguing in a similar way on $-v$, we have $\|v\|_\infty \leq 1$, and so

$$\|u\|_\infty \leq \varepsilon = K^{-1}\|u\|_r.$$

Since $\varepsilon \in (0, 1)$, for every solution u with $\|u\|_r < K$, we can write $\|u\|_r = K\varepsilon$ and obtain the desired estimate.

□

Remark 3.2.4. We recall that from the Neumann boundary condition we have

$$\|u\|_{L^\infty(\mathbb{R}^N)} = \|u\|_{L^\infty(\Omega)}.$$

Indeed, if we take $x \in \mathbb{R}^N \setminus \overline{\Omega}$, then from $\mathscr{N}_{s,p}u = 0$ we have

$$u(x)\int_\Omega \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{N+ps}}\,dy = \int_\Omega \frac{|u(x)-u(y)|^{p-2}u(y)}{|x-y|^{N+ps}}\,dy.$$

Now we can assume that u is not constant, otherwise it would be bounded, and obtain

$$|u(x)| = \left|\frac{\int_\Omega \frac{|u(x)-u(y)|^{p-2}u(y)}{|x-y|^{N+ps}}\,dy}{\int_\Omega \frac{|u(x)-u(y)|^{p-2}}{|x-y|^{N+ps}}\,dy}\right| \leq \|u\|_{L^\infty(\Omega)},$$

which proves our claim (see also [53, Proposition 3.4]).

Now we want to prove that every weak solution of problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathscr{N}_s u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{3.19}$$

is continuous. The goal is to use [65, Theorem 1.2]. To do this, we need to prove that every weak solution is continuous outside of Ω.

Proposition 3.2.5. *Let $u \in X$ be a weak solution of problem* (3.19)*. Then, $u \in C(\mathbb{R}^N \setminus \Omega)$.*

Proof. From [53, Theorem 2.8] we know that $\mathscr{N}_s u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, that is

$$u(x) = \frac{\int_\Omega \frac{u(y)}{|x-y|^{N+2s}}\, dy}{\int_\Omega \frac{1}{|x-y|^{N+2s}}\, dy},$$

for a.e. $x \in \mathbb{R}^N \setminus \Omega$. Clearly, this can be written as a quotient of convolutions:

$$\frac{u\chi_\Omega * \frac{1}{|x|^{N+2s}}}{\chi_\Omega * \frac{1}{|x|^{N+2s}}},$$

which is continuous in $\mathbb{R}^N \setminus \Omega$. So, u is equal a.e. to a continuous function in $\mathbb{R}^N \setminus \Omega$, hence it is continuous in this set. □

Theorem 3.2.6. *Let $p = 2$ and let $u \in X$ be a weak solution of problem* (3.19)*. Then, $u \in C(\mathbb{R}^N)$.*

Proof. In light of Proposition 3.2.5, it is enough to apply [65, Theorem 1.2] to obtain the continuity for weak solutions of problem (3.19) on the whole of $\mathbb{R}^N$. □

3.3 The parabolic equation

In this section, we consider the problem

$$\begin{cases} u_t(x,t) + (-\Delta)_p^s u(x,t) = 0 & \text{in } \Omega, \quad t > 0 \\ \mathscr{N}_{s,p} u(x,t) = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \quad t > 0 \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{3.20}$$

We show that the solutions of (3.20) preserve their mass and have energy that decreases in time, as proved in [28] for $p = 2$. To do so, we assume that u is a classical solution of (3.20), so that (3.20) holds pointwise. In particular, we can differentiate with respect to time.

Proposition 3.3.1. *Let u be a classical solution of* (3.20) *such that u is bounded and $|u_t(x,t)| + |(-\Delta)_p^s u(x,t)| \le K$ for all $t > 0$ and all $x \in \Omega$. Then, for all $t > 0$*

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx,$$

which means that the total mass is preserved.

Proof. By the dominated convergence theorem and Proposition 1.0.1, we have

$$\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega u_t\,dx = -\int_\Omega (-\Delta)_p^s u\,dx = \int_{\mathbb{R}^N\setminus\Omega} \mathscr{N}_{s,p} u\,dx = 0.$$

So, $\int_\Omega u\,dx$ does not depend on t, as desired. □

Proposition 3.3.2. *Under the assumptions of Proposition* 3.3.1*, the energy*

$$E(t) = \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|u(x,t)-u(y,t)|^p}{|x-y|^{N+ps}}\,dxdy$$

is decreasing in time $t > 0$.

Proof. From Proposition 1.0.2, we have

$$\begin{aligned} E'(t) &= \frac{d}{dt}\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|u(x,t)-u(y,t)|^p}{|x-y|^{N+ps}}\,dxdy \\ &= p\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} J_p(u(x,t)-u(y,t))\frac{u_t(x,t)-u_t(y,t)}{|x-y|^{N+ps}}\,dxdy \\ &= 2p\int_\Omega u_t(-\Delta)_p^s u\,dx = -2p\int_\Omega \left|(-\Delta)_p^s u\right|^2\,dx \le 0, \end{aligned}$$

since u is a solution of (3.20), and so the energy is decreasing. □

Chapter 4

The eigenvalue problem

In this chapter we consider the nonlinear eigenvalue problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \end{cases} \tag{4.1}$$

If (4.1) admits a weak solution $u \in X$ (notice that now $g \equiv 0$), that is

$$\frac{1}{2} \int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} \frac{J_p(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}} \, dxdy = \lambda \int_\Omega |u|^{p-2} uv \, dx$$

for all $v \in X$, then we say that λ is an eigenvalue of $(-\Delta)_p^s$ with $p-$Neumann boundary conditions and associated λ-eigenfunction u. As in the classical case, we call the set of all the eigenvalues the point spectrum of $(-\Delta)_p^s$ in X and, for further references, we denote it by $\sigma(s, p)$.

In this chapter, we recall some results proved in [53] about eigenvalues and eigenfunctions. In particular we prove that there exists a diverging sequence of eigenvalues (see Proposition 4.1.2), that the eigenfunctions corresponding to the first eigenvalue are just constant functions, and that every other eigenfunction changes sign (see Proposition 4.1.3). Moreover, we show that all eigenfunctions are bounded in the whole $\mathbb{R}^N$ (see Proposition 4.2.1).

4.1 Existence of eigenvalues

First of all we observe that for $\lambda = 0$ constant functions are all 0-eigenfunctions. Since all the eigenvalues are obviously non negative, we have that $\lambda_1 = 0$ is the first eigenvalue. Moreover,

$$\int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} |u(x) - u(y)|^{p-2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}} \, dxdy = 0$$

for all $v \in X$ implies u constant, so all the λ_1-eigenfunctions are just constant functions.

As usual, we can construct a sequence $(\lambda_k)_k$ of eigenvalues for problem (4.1), analogously to the Dirichlet case treated in [39], setting

$$\lambda_k = \inf_{A\in\mathcal{F}_k} \sup_{u\in A} \frac{[u]_{s,p}^p}{2},$$

with

$$[u]_{s,p}^p = \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy.$$

Here, if $\mathcal{F}$ is the family of all nonempty, closed, symmetric subsets of $S = \{u \in X :\ \int_\Omega |u|^p = 1\}$, for all $k \in \mathbb{N}$ we have set

$$\mathcal{F}_k = \{A \in \mathcal{F} :\ i(A) \geq k\},$$

while $i(A)$ is the cohomological index of Fadell and Rabinowitz [32].

In order to prove that λ_k is an eigenvalue for every $k \in \mathbb{N}$, we proceed in the standard way: set $\varphi(u) = \frac{[u]_{s,p}^p}{2}$, $I(u) = \|u\|_{L^p(\Omega)}^p$ and let $\bar{\varphi}$ be the restriction of φ to S.

Proposition 4.1.1. *The functional $\bar{\varphi}$ satisfies the Palais-Smale condition at any level $c \in \mathbb{R}$.*

Proof. Let $(u_n)_n \subset S$ and $(\mu_n)_n \subset \mathbb{R}$ be such that $\varphi(u_n) \to c$ as $n \to \infty$ and $\varphi'(u_n) - \mu_n I'(u_n) \to 0$ in X'. We have

$$\|u_n\|_X^p = 1 + \varphi(u_n) \to 1 + c,$$

so $(u_n)_n$ is bounded in X. Up to a subsequence, we have $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^p(\Omega)$ for some $u \in X$ as $n \to \infty$, see Remark 3.1.4. In particular, $u \in S$. We also get that $\varphi(u_n) - \mu_n \to 0$, and so $\mu_n \to c$. Now, we have

$$\begin{aligned} |p\langle A(u_n), u_n - u\rangle| &= |\langle I'(u), u_n - u\rangle + \langle \varphi'(u), u_n - u\rangle| \\ &= |\langle I'(u), u_n - u\rangle + \mu_n\langle I'(u), u_n - u\rangle + o(1)| \\ &\leq |1 + \mu_n| \|u_n - u\|_{L^p(\Omega)}^p + o(1) \to 0. \end{aligned}$$

So, by the (S) property of A, we get that $u_n \to u$ in X. □

Now we can give the desired result for the sequence $(\lambda_k)_k$.

Proposition 4.1.2. *For all $k \in \mathbb{N}$, λ_k is an eigenvalue of* (4.1). *In addition, $\lambda_k \to \infty$.*

The proof is standard, see for example the proof of [39, Proposition 2.2]. We also recall that in [24] a characterization of the second eigenvalue is given, together with the asymptotic for $p \to \infty$.

Now we show that every eigenfunction, except the ones corresponding to the first eigenvalue, changes sign.

Proposition 4.1.3. *Let $v \in X$ be a solution to* (4.1) *such that $v > 0$ in Ω. Then $\lambda = 0$, hence v is constant.*

Proof. We assume that $v \in X$ is strictly positive solution of (4.1) such that $I(u) = 1$, and take $u \in X$ a 0-eigenfunction with $I(u) = 1$. We set $v_\varepsilon(x) = v(x) + \varepsilon$, $u_\varepsilon(x) = u(x) + \varepsilon$ and

$$\sigma_t^\varepsilon(x) = (tu_\varepsilon(x)^p + (1-t)v_\varepsilon(x)^p)^{\frac{1}{p}}$$

for $x \in \mathbb{R}^N$, $t \in [0,1]$. It follows that $\sigma_t^\varepsilon \in X$ and

$$\varphi(\sigma_t^\varepsilon) \leq t\varphi(u) + (1-t)\varphi(v)$$

for all $t \in [0,1]$, see [33, Lemma 4.1]. From this, we have

$$\varphi(\sigma_t^\varepsilon) - \varphi(v) \leq t(\varphi(u) - \varphi(v)) = -t\lambda \tag{4.2}$$

for all $t \in [0,1]$ and ε small enough. Moreover, from the convexity of φ we get

$$\begin{aligned}&\varphi(\sigma_t^\varepsilon) - \varphi(v) \geq \\ &\frac{p}{2}\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} J_p(v(x)-v(y))\frac{\sigma_t^\varepsilon(x) - \sigma_t^\varepsilon(y) - (v(x)-v(y))}{|x-y|^{N+ps}}\,dxdy,\end{aligned} \tag{4.3}$$

for all $t \in [0,1]$ and ε small enough. Taking $\sigma_t^\varepsilon - v_\varepsilon$ as a test function in the weak formulation of (4.1) for the couple (v, λ), we obtain

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} J_p(v(x)-v(y))&\frac{\sigma_t^\varepsilon(x) - \sigma_t^\varepsilon(y) - (v_\varepsilon(x)-v_\varepsilon(y))}{|x-y|^{N+ps}}\,dxdy, \\ &= \lambda\int_\Omega v(x)^{p-1}(\sigma_t^\varepsilon(x) - v_\varepsilon(x))\,dx.\end{aligned} \tag{4.4}$$

Finally, from (4.2)–(4.4) we get

$$p\lambda\int_\Omega v(x)^{p-1}\frac{\sigma_t^\varepsilon(x) - v_\varepsilon(x)}{t}\,dx \leq -\lambda, \tag{4.5}$$

for all $t \in (0,1]$ and ε small enough. From the concavity of the p-th root follows that

$$\sigma_t^\varepsilon(x) - v_\varepsilon(x) \geq t(u_\varepsilon(x) - v_\varepsilon(x)) = t(u-v)(x)$$

in Ω. So, we can apply Fatou's Lemma in (4.5), obtaining

$$\lambda \int_\Omega \left(\frac{v(x)}{v_\varepsilon(x)}\right)^{p-1} (u_\varepsilon(x)^p - v_\varepsilon(x)^p)\, dx \leq -\lambda$$

for ε small enough. Since $v > 0$ in Ω, from the dominated convergence Theorem and $I(u) = I(v) = 1$, when $\varepsilon \to 0^+$ we get

$$0 \leq -\lambda.$$

Since all the eigenvalues are non negative, we have $\lambda = 0$ and so v belongs to the first eigenspace, as claimed. □

4.2 Boundedness

In this section we want to prove the boundedness of eigenfunctions in the whole of $\mathbb{R}^N$, starting as in [33] to get the bound in Ω, and exploiting the $p-$Neumann condition to get the bound in the complementary set of Ω. More precisely, we have that the $L^\infty-$norm in Ω estimates the $L^\infty-$norm in the $\mathbb{R}^N \setminus \Omega$.

Proposition 4.2.1. *Let $s \in (0,1)$, $p > 1$, and $u \in X$ be a solution of* (4.1) *for some $\lambda \geq 0$. Then $u \in L^\infty(\mathbb{R}^N)$ and*

$$\|u\|_{L^\infty(\mathbb{R}^N)} = \|u\|_{L^\infty(\Omega)}.$$

Proof. First, we prove that u is bounded in Ω, concentrating on the case $ps \leq N$, the case $ps > N$ being trivial by the fractional Morrey-Sobolev embedding. As in [33], we only have to prove that u_+ is bounded in Ω, since both $u_\pm$ are solutions, so we can get a bound for the negative part in the same way. To do that, it is enough to prove that

$$\|u\|_{L^\infty(\Omega)} \leq 1 \quad \text{when} \quad \|u\|_{L^p(\Omega)} \leq \delta, \tag{4.6}$$

where $\delta > 0$ is still to be determined. Indeed, we can scale the function verifying (4.6), so there is no restriction in this.

Now, for all $k \geq 0$, we define the function

$$w_k := (u - (1 - 2^{-k}))_+,$$

see [33], also for the following facts: $w_k \in X$ and

$$\begin{aligned} w_{k+1}(x) \le w_k(x) \quad \text{a.e. in} \quad \Omega, \\ u(x) < (2^{k+1}-1)w_k(x) \quad \text{for} \quad x \in \{w_{k+1} > 0\}, \end{aligned} \tag{4.7}$$

and the inclusions

$$\{w_{k+1} > 0\} \subseteq \{w_k > 2^{-(k+1)}\}$$

hold true for every $k \ge 0$. Moreover, for every function v

$$|v(x) - v(y)|^{p-2}(v_+(x) - v_+(y))(v(x) - v(y)) \ge |v_+(x) - v_+(y)|^p, \tag{4.8}$$

for all $x, y \in \mathbb{R}^N$.

Now, we want to prove (4.6) using a standard argument relying on estimating the decay of $U_k := \|w_k\|^p_{L^p(\Omega)}$. First of all, using (4.8) with $v = u - (1 - 2^{-k-1})$ we obtain

$$\|w_{k+1}\|^p_X \le \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{J_p(u(x) - u(y))(w_{k+1}(x) - w_{k+1}(y))}{|x-y|^{N+ps}}\, dxdy + U_{k+1}.$$

Taking w_{k+1} as a test function in (4.1) and then using (4.7), we get

$$\begin{aligned} \|w_{k+1}\|^p_X &\le \lambda \int_{\{w_{k+1}>0\}} |u(x)|^{p-2}u(x)w_{k+1}(x)\, dx + U_{k+1} \\ &\le (\lambda(2^{k+1}-1)^{p-1} + 1)U_k. \end{aligned}$$

Using the fractional Sobolev embeddings, as in [33], we get

$$U_{k+1} \le c\|w_{k+1}\|^p_X |\{w_{k+1} > 0\}|^{\frac{N}{pseigen}},$$

where $c > 0$ depends on N, p, s. Proceeding as in [33], we get that u is bounded in Ω.

Now, take $x \in \mathbb{R}^N \setminus \overline{\Omega}$. Since u is bounded in Ω, from (4.1) we get

$$u(x) \int_\Omega \frac{|u(x) - u(y)|^{p-2}}{|x-y|^{N+ps}}\, dy = \int_\Omega \frac{|u(x) - u(y)|^{p-2}u(y)}{|x-y|^{N+ps}}\, dy.$$

If u is constant, the result is trivial. On the other hand, if u is not constant, from Theorem 3.1.6 we have

$$|u(x)| = \left| \frac{\displaystyle\int_\Omega \frac{|u(x) - u(y)|^{p-2}u(y)}{|x-y|^{N+ps}}\, dy}{\displaystyle\int_\Omega \frac{|u(x) - u(y)|^{p-2}}{|x-y|^{N+ps}}\, dy} \right| \le \|u\|_{L^\infty(\Omega)},$$

and so $\|u\|_{L^\infty(\mathbb{R}^N\setminus\Omega)} \le \|u\|_{L^\infty(\Omega)}$, which concludes the proof. □

Chapter 5

p-superlinelar problems

In this chapter we deal with nonlinear problems involving a source term which is assumed to be p-superlinear. First, we focus our attention on problems with the nonlinar term that satisfies the so called Ambrosetti-Rabinowitz condition. Under suitable assumptions on the nonlinearity, we prove the existence of a nontrivial solution using an argument of linking over cones (see Theorem 5.1.2). Slightly changing the assumption on the nonlinear term, we prove the existence of a nontrivial solution for a similar problem, by using tools of critical group theory (see Theorem 5.1.4).

In the second part of the chapter we deal with problems with nonlinearities which do not satisfy the Ambrosetti-Rabinowitz condition, but more general condition, introduced in [50]. The first problem that we consider in this case has the associated functional with the geometric structure of mountain pass type, and so we prove the existence of two nontrivial constant sign solutions, one positive and one negative, applying the Mountain Pass Theorem to suitable truncated functionals (see Theorem 5.2.3). The second problem has the associated functional with the geometric structure of linking over cones type, and so we prove the existence of two nontrivial constant sign solutions considering suitable truncated functionals and using an argument similar to the one used in Theorem 5.1.2 (see Theorem 5.2.6).

The results in Section 5.1 are taken from [51, 54] and those in Section 5.2 from [51, 53].

5.1 Problems with the Ambrosetti-Rabinowitz condition

In this section we first consider the problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u + g(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}. \end{cases} \tag{5.1}$$

We recall that $p \in (1,\infty)$, Ω is a bounded domain with Lipschitz boundary, $\lambda \geq 0$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is the map $x \mapsto g(x,t)$ is measurable for every $t \in \mathbb{R}$ and the map $t \mapsto g(x,t)$ is continuous for a.e. $x \in \Omega$.

Of course, we shall assume growth conditions on g which will ensure that any critical point of the C^1 functional $I : X \to \mathbb{R}$ defined as

$$I(u) = \frac{1}{2p} \int\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy - \frac{\lambda}{p}\int_\Omega |u|^p\, dx - \int_\Omega G(x,u)\, dx \tag{5.2}$$

is a weak solution of (5.1).

Here we will further assume the following hypotheses on g:

(g_1) there exist constants $a_1, a_2 > 0$ and $q > p$ such that for every $t \in \mathbb{R}$ and for a.e. $x \in \Omega$
$$|g(x,t)| \leq a_1 + a_2|t|^{q-1},$$
where $q < \frac{pN}{N-ps}$ if $N > ps$;

(g_2) $g(x,t) = o(|t|^{p-1})$ as $t \to 0$ uniformly a.e. in Ω;

(g_3) denoting $G(x,t) = \int_0^t g(x,\tau)\, d\tau$, there exist $\mu > p$ and $R \geq 0$ such that for every t with $|t| > R$ and for a.e. $x \in \Omega$
$$0 < \mu G(x,t) \leq g(x,t)t,$$
and there exist $\tilde{\mu} > p$, $a_3 > 0$ and $a_4 \in L^1(\Omega)$ such that for every $t \in \mathbb{R}$ and a.e. $x \in \Omega$,
$$G(x,t) \geq a_3|t|^{\tilde{\mu}} - a_4(x); \tag{5.3}$$

(g_4) if $R > 0$, then $G(x,t) \geq 0$ for every $t \in \mathbb{R}$ and a.e. $x \in \Omega$.

Remark 5.1.1. Condition (5.3) was introduced in [49] to complete the Ambrosetti-Rabinowitz condition in presence of a Carathéodory functions.

For each λ_m, we can define the cones

$$C_m^- := \left\{ u \in X : \int\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy \leq \lambda_m \int_\Omega |u|^p\, dx \right\} \tag{5.4}$$

$$C_m^+ := \left\{ u \in X : \int\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy \geq \lambda_{m+1} \int_\Omega |u|^p\, dx \right\}. \tag{5.5}$$

Our first existence result is

Theorem 5.1.2. *If hypotheses* $(g_1)-(g_4)$ *hold, then problem* (5.1) *admits a nontrivial weak solution.*

In order to prove Theorem 5.1.2 it will be enough to apply Theorem 2.2.2 to the functional I defined in (5.2) under the validity of the Palais-Smale condition (of course, if the Cerami condition holds, the Palais-Smale condition holds, as well); hence, we will apply Theorem 2.2.2 in the version of [22, Theorem 2.2], where the Palais-Smale condition is assumed.

Thus, now we prove that I satisfies the Palais-smale condition at any level $c \in \mathbb{R}$ - $(PS)_c$ for short -, that is

for every sequence $(u_n)_n$ in X such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in X',
there exists a strongly converging subsequence of $(u_n)_n$.

Proposition 5.1.3. *Under the assumptions of Theorem* 5.1.2*, I satisfies* $(PS)_c$ *for every* $c \in \mathbb{R}$.

Proof. Let $(u_n)_n$ in X be such that $I(u_n) \to c$ and $I'(u_n) \to 0$ and fix $k \in (p, \mu)$. We re-write the functional in the following way:

$$\begin{aligned} I(u) &= \frac{1}{2p} \int\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy + \frac{1}{2p}\int_\Omega |u|^p dx \\ &\quad - \left(\frac{\lambda}{p} + \frac{1}{2p}\right) \int_\Omega |u|^p\, dx - \int_\Omega G(x,u)\, dx \\ &= \frac{1}{2p}\|u\|^p - \left(\frac{\lambda}{p} + \frac{1}{2p}\right) \int_\Omega |u|^p\, dx - \int_\Omega G(x,u)\, dx. \end{aligned}$$

We observe that

$$kI(u_n) - \langle I'(u_n), u_n\rangle \leq M + N\|u_n\| \tag{5.6}$$

for some $M, N > 0$ and all $n \in \mathbb{N}$. On the other hand, by (g_3) and (g_1) we have

$$\begin{aligned}
&kI(u_n) - \langle I'(u_n), u_n\rangle \\
&= \left(\frac{k}{2p} - \frac{1}{2}\right)\|u_n\|^p - \left(\frac{k}{p} - 1\right)\left(\lambda + \frac{1}{2}\right)\int_\Omega |u_n|^p\,dx \\
&+ \int_\Omega \big(g(x,u_n)u_n - kG(x,u_n)\big)\,dx \\
&\geq \left(\frac{k}{2p} - \frac{1}{2}\right)\|u_n\|^p - \left(\frac{k}{p} - 1\right)\left(\lambda + \frac{1}{2}\right)\int_\Omega |u_n|^p\,dx \\
&+ (\mu - k)\int_\Omega G(x,u_n)\,dx - C_R
\end{aligned}$$

for some constant $C_R \geq 0$. By (5.3), we get

$$\begin{aligned}
&kI(u_n) - \langle I'(u_n), u_n\rangle \\
&\geq \left(\frac{k}{2p} - \frac{1}{2}\right)\|u_n\|^p - \left(\frac{k}{p} - 1\right)\left(\lambda + \frac{1}{2}\right)\int_\Omega |u_n|^p\,dx \\
&+ (\mu - k)a_3\int_\Omega |u_n|^{\tilde{\mu}}\,dx - C
\end{aligned}$$

for some constant $C \geq 0$. By the Hölder and the Young inequalities, we get that for any $\varepsilon > 0$ we have that for every $u \in X$

$$\|u\|_p^p \leq \varepsilon\|u\|_{\tilde{\mu}}^{\tilde{\mu}} + C_\varepsilon.$$

Thus, we obtain

$$\begin{aligned}
&kI(u_n) - \langle I'(u_n), u_n\rangle \\
&\geq \left(\frac{k}{2p} - \frac{1}{2}\right)\|u_n\|^p + \left[(\mu - k)a_3 - \varepsilon\left(\frac{k}{p} - 1\right)\left(\lambda + \frac{1}{2}\right)\right]\int_\Omega |u_n|^{\tilde{\mu}}\,dx - \tilde{C}_\varepsilon
\end{aligned}$$

for some $\tilde{C}_\varepsilon > 0$. Taking ε small enough, we get

$$kI(u_n) - \langle I'(u_n), u_n\rangle \geq \left(\frac{k}{2p} - \frac{1}{2}\right)\|u_n\|^p - \tilde{C}_\varepsilon.$$

This together with (5.6) implies that $(u_n)_n$ is bounded in X. Up to a subsequence, we can assume that $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^p(\Omega)$ as $n \to \infty$. By assumption, we have

$$\langle I'(u_n), u_n - u\rangle \to 0.$$

On the other hand

$$\begin{aligned}
&\langle A'(u_n), u_n - u\rangle \\
&\quad = \langle I'(u_n), u_n - u\rangle + \lambda \int_\Omega |u_n|^{p-2}u_n(u_n - u)\,dx + \int_\Omega g(x,u_n)(u_n-u)\,dx.
\end{aligned}$$

Since $u_n \to u$ in $L^p(\Omega)$, from (g_1) we obtain that

$$\int_\Omega |u_n|^{p-2}u_n(u_n-u)\,dx \to 0$$

and

$$\int_\Omega g(x,u_n)(u_n-u)\,dx \to 0;$$

so $\langle A'(u_n), u_n - u\rangle_{X',X} \to 0$ as $n \to \infty$. By Proposition 3.1.8 we get that $u_n \to u$ in X, as desired. □

Now we are ready to prove Theorem 5.1.2.

Proof. Let $(\lambda_m)_m$ be the sequence of eigenvalues defined in Proposition 4.1.2. Since this sequence is divergent, there exists $m \geq 1$ such that $\lambda_m \leq 2\lambda + 1 < \lambda_{m+1}$. Defining C_m^- and C_m^+ as in (5.4) and (5.5), we have that C_m^-,C_m^+ are two symmetric closed cones in X with $C_m^- \cap C_m^+ = \{0\}$. We recall that by Theorem 2.2.6 we have

$$i(C_m^- \setminus \{0\}) = i(X \setminus C_m^+) = m.$$

Now, by (g_1) and (g_2) it is standard to see that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|G(x,t)| \leq \frac{\varepsilon}{2p}|t|^p + C_\varepsilon|t|^q$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$. As a consequence, taking $u \in C_m^+$, by the inequality in (5.5) and the Sobolev inequality, we have that

$$\begin{aligned}
I(u) &\geq \frac{1}{2p}\|u\|^p - \frac{2\lambda+1}{2p}\int_\Omega |u|^p dx - \frac{\varepsilon}{2p}\int_\Omega |u|^p dx - C_\varepsilon \int_\Omega |u|^q dx \\
&\geq \frac{1}{2p}\|u\|^p - \frac{1}{2p\lambda_{m+1}}\left(2\lambda+1+\varepsilon\right)[u]^p - C_\varepsilon\int_\Omega |u|^q dx \\
&\geq \frac{1}{2p}\left(1 - \frac{2\lambda+1+\varepsilon}{\lambda_{m+1}}\right)\|u\|^p - C\|u\|^q
\end{aligned}$$

for some $C > 0$.

Hence, choosing ε small enough, there exists $r_+ > 0$ and $\alpha > 0$ such that, if $\|u\| = r_+$, then $I(u) \geq \alpha$.

On the other hand, taking $u \in C_m^-$, $e \in X \setminus C_m^-$ and $t > 0$, by (5.3) we get that

$$I(u+te) \leq \frac{2^{p-2}}{p}\left(\int\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy + t^p \int\int_\Omega \frac{|e(x)-e(y)|^p}{|x-y|^{N+ps}}\,dxdy\right) - \frac{\lambda}{p}\int_\Omega |u+te|^p\,dx - a_3 t^{\tilde{\mu}} \int_\Omega \left|\frac{u}{t}+e\right|^{\tilde{\mu}} dx + \|a_4\|_1 \to -\infty$$

as $t \to +\infty$. In conclusion, there exists $r_- > r_+$ such that $I(v) \leq 0$ when $v \in C_m^- + (\mathbb{R}^+ e)$ and $\|v\| \geq r_-$.

Defining D_-, S_+, Q and H as in Theorem 2.2.4, by Corollary 2.2.5 we have that $(Q, D_- \cup H)$ links S_+ cohomologically in dimension $m+1$ over $\mathbb{Z}_2$. In particular, $(Q, D_- \cup H)$ links S_+ by Proposition 2.2.7. In addition, I is bounded on Q, $I(u) \leq 0$ for every $u \in D_- \cup H$ and $I(u) \geq \alpha > 0$ for every $u \in S_+$. By Proposition 5.1.3 $(PS)_c$ holds. Finally, by applying Theorem 2.2.2 with $S = D_- \cup H$, $D = Q$, $A = S_+$ and $B = \emptyset$, I admits a critical value $c \geq \alpha$, hence there exists a critical point u with $I(u) = c > 0$. It follows that u is a nontrivial weak solution of (5.1). □

Actually, the result in Theorem 5.1.2 can be improved in a certain sense. To show this, we consider a similar problem, namely

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u + g(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}. \end{cases} \tag{5.7}$$

Here, g is a Carathéodory function satisfying the following hypotheses:

(g_1') there exist $a > 0$ and $r \in (p, p_s^*)$ such that

$$|g(x,t)| \leq a(1+|t|^{r-1})$$

a.e. in Ω and for all $t \in \mathbb{R}$.

(g_2') there exist $\mu > p$ and $R > 0$ such that

$$0 < \mu G(x,t) \leq g(x,t)t \tag{5.8}$$

a.e. in Ω and for all $|t| \geq R$, where $G(x,t) = \int_0^t g(x,\tau)\,d\tau$, and

$$G(x,t) \geq C_0|t|^\mu - C_1 \tag{5.9}$$

a.e. in Ω and for all $t \in \mathbb{R}$, for some $C_0, C_1 > 0$.

(g_3')

$$\lim_{t\to 0}\frac{g(x,t)}{|t|^{p-1}}=0$$

uniformly a.e. in Ω.

We remark that in this case we have dropped condition (g_4), but in return we have to make assumpions depending on the relation between λ and the sequence of eigenvalues λ_k from Proposition 4.1.2.

The main result is the following one, which corresponds to [38, Theorem 4.1]:

Theorem 5.1.4. *If hypotheses (g_1')-(g_3') and one of the following hold:*

(i) $2\lambda \neq \lambda_k$ *for all* $k \in \mathbb{N}$,

(ii) $2\lambda = \lambda_k$, $k \in \mathbb{N}$, *and* $G(x,t) \geq 0$ *a.e. in* Ω *and for all* $|t| \leq \delta$ *for some* $\delta > 0$,

(iii) $2\lambda = \lambda_k$, $k \in \mathbb{N}$, *and* $G(x,t) \leq 0$ *a.e. in* Ω *and for all* $|t| \leq \delta$ *for some* $\delta > 0$,

then problem (5.7) *admits a nontrivial solution.*

We define the functional associated to (5.7) as

$$E(u) := \frac{1}{2p}[u]^p - \frac{\lambda}{p}\int_\Omega |u|^{p-2}u\,dx - \int_\Omega G(x,u)\,dx,$$

so that critical points of E are solution of (5.7).

Lemma 5.1.5. *The functional E is of class $C^1(X)$ and satisfies (PS). Moreover, for every $\eta < 0$ the set E^η is contractible.*

Proof. Let $(u_n)_n$ be a sequence in X such that $(E(u_n))_n$ is bounded and $E'(u_n) \to 0$ in X'. By (5.9) we have

$$\begin{aligned}\left(\frac{\mu}{p}-1\right)\frac{\|u_n\|^p}{4} &= \frac{\mu+p}{2}E(u_n) - \langle E'(u_n),u_n\rangle + \left(\frac{2\lambda+1}{2p}\right)\|u_n\|_p^p \\ &\quad + \int_\Omega\left(\frac{\mu+p}{2}G(x,u_n) - g(x,u_n)u_n\right)dx \\ &\leq C + \|E'(u_n)\|_{X'}\|u_n\| + \left(\frac{2\lambda+1}{2p}\right)\|u_n\|_p^p - \frac{\mu-p}{2}\|u_n\|_\mu^\mu,\end{aligned}$$

so $(u_n)_n$ is bounded in X. By [54, Proposition 3.2], E satisfies the (PS) condition.

Now, fix $u \in X \setminus \{0\}$. By (5.9) we have

$$E(\tau u) \leq \frac{\tau^p [u]^p}{p} - \frac{\lambda \tau^p \|u\|_p^p}{p} - C(\tau^\mu \|u\|_\mu^\mu - 1)$$

for all $\tau > 0$. So

$$\lim_{\tau \to \infty} E(\tau u) = -\infty \tag{5.10}$$

and E is unbounded below. Moreover, by (5.8)

$$\langle E'(u), u \rangle = pE(u) - \int_\Omega (pG(x,u) - g(x,u)u)\, dx \leq pE(u).$$

Taking $\eta < 0$ we have

$$\langle E'(u), u \rangle < 0 \text{ for every } u \in E^\eta. \tag{5.11}$$

Let $\partial B_1 := \{u \in X : \|u\| = 1\}$. In light of (5.10) and (5.11), for every $u \in \partial B_1$ we can find a maximal $\tau(u) > 0$ such that $E(\tau(u)u) = \eta$. The Implicit Function Theorem implies that $\tau \in C(\partial B_1)$. We can extend τ to all of $X \setminus \{0\}$ by setting

$$\tau^*(u) := \frac{1}{\|u\|} \tau \left(\frac{1}{\|u\|} \right) \text{ for all } u \in X \setminus \{0\}.$$

So $\tau^* \in C(X \setminus \{0\})$ and $E(\tau^*(u)u) = \eta$ for all $u \in X \setminus \{0\}$. In addition, $E(u) = \eta$ implies that $\tau^*(u) = 1$. Now we set

$$\tilde{\tau} := \begin{cases} 1, & \text{if } E(u) \leq \eta, \\ \tau^*(u), & \text{if } E(u) > \eta. \end{cases}$$

Then $\tilde{\tau} \in C(X \setminus \{0\})$.

Now we show that E^η is a strong deformation retract of $X \setminus \{0\}$. We consider the homotopy $h : [0,1] \times (X \setminus \{0\}) \to X \setminus \{0\}$ defines as

$$h(t,u) = (1-t)u + t\tilde{\tau}(u)u$$

for all $(t,u) \in [0,1] \times (X \setminus \{0\})$. Clearly, we have

$$h(0,u) = u \text{ and } h(1,u) = \tilde{\tau}(u)u \in E^\eta$$

for all $u \in X \setminus \{0\}$. Moreover, for all $(t,u) \in [0,1] \times E^\eta$

$$h(t,u) = u,$$

and so E^η is a strong deformation retract of $X \setminus \{0\}$.

By the radial retraction $r_0(u) = \frac{u}{\|u\|}$ we know that ∂B_1 is a retract of $X \setminus \{0\}$. So we can use the deformation

$$\tilde{h}(t,u) = (1-t)u + tr_0(u),$$

for all $(t,u) \in [0,1] \times (X \setminus \{0\})$, to see that $X \setminus \{0\}$ is deformable onto ∂B_1. Then ∂B_1 is a deformation retract of $X \setminus \{0\}$. So E^η and ∂B_1 are homotopy equivalent. Since X is infinite dimentional, ∂B_1 is contractible, hence E^η is contractible as well. □

Now we want to compute the critical groups of E at 0. So, for all $\tau \in [0,1]$, we define the functional

$$E_\tau(u) := \frac{1}{2p}[u]^p - \frac{\lambda}{p}\|u\|_p^p - \int_\Omega G(x,(1-\tau)u + \tau\theta(u))\,dx,$$

with $\theta \in C(\mathbb{R},[-\delta,\delta]$, where $\delta > 0$, and θ is such that

$$\theta(t) = \begin{cases} -\delta & \text{if } t \le -\delta, \\ t & \text{if } |t| \le \frac{\delta}{2}, \\ \delta & \text{if } t \ge \delta. \end{cases} \tag{5.12}$$

With this definition, it clearly follows that $E_\tau \in C^1(X)$ and $E = E_0$. We have the following result:

Lemma 5.1.6. *The point 0 is an isolated critical point of E_τ uniformly with respect to $\tau \in [0,1]$. Moreover*

$$C^k(E,0) = C^k(E_1,0)$$

for all $k \in \mathbb{N}_0$.

Proof. Since 0 is an isolated critical point of E, for $\varepsilon > 0$ small enough we have $K(E) \cap \overline{B}_\varepsilon(0) = \{0\}$. We want to prove the same for E_τ, that is for $\varepsilon > 0$ small enough

$$K(E_\tau) \cap \overline{B}_\varepsilon(0) = \{0\} \text{ for every } \tau \in [0,1], \tag{5.13}$$

and we argue by contradiction. So we assume that there exist two sequences, $(\tau_n)_n$ in $[0,1]$ and $(u_n)_n$ in $X \setminus \{0\}$, such that $E'_{\tau_n}(u_n) = 0$ for all $n \in \mathbb{N}$, and $u_n \to 0$ in X. For all $n \in \mathbb{N}$ and $(x,t) \in \Omega \times \mathbb{R}$ we set

$$g_n(x,t) := (1 - \tau_n + \tau_n\theta'(t)) + g(x,(1-\tau_n)u + \tau_n\theta(u)),$$

with $\theta \in C(\mathbb{R}, [-\delta, \delta]$ defined as in (5.12). By definition $g_n : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Moreover, Hypoteshes (g_1') and (g_3') imply that, for all $n \in \mathbb{N}$ and some $a' > 0$,

$$|\lambda|t|^{p-2}t + g_n(x,t)| \leq a'(|t|^{p-1} + |t|^{r-1}) \tag{5.14}$$

a.e. in Ω and for all $t \in \mathbb{R}$. Since u_n is a critical point of E'_{τ_n} for all $n \in \mathbb{N}$, it is also a weak solution of the auxiliary problem

$$\begin{cases} (-\Delta)_p^s u = \lambda|u|^{p-2}u + g_n(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{5.15}$$

In light of (5.14) there exists a constant $K > 0$, independent of n, such that for every weak solution $u \in X$ of (5.15) with $\|u\|_r < K$, then $u \in L^\infty(\Omega)$ and $\|u\|_\infty \leq K^{-1}\|u\|_r$ (see Corollary 3.2.3). From the continuous embedding $X \hookrightarrow L^r(\Omega)$ we know that $u_n \to 0$ in $L^r(\Omega)$, so $u_n \to 0$ in $L^\infty(\Omega)$ as well. Hence, if $n \in \mathbb{N}$ is big enough, we can take $u_n \in \overline{B}_\varepsilon(0)$ and $\|u_n\|_\infty \leq \delta/2$. Then, from the definition of E_τ we can see that

$$E'(u_n) = E'_{\tau_n}(u_n) = 0,$$

that is $u_n \in K(E) \cap \overline{B}_\varepsilon(0) \setminus \{0\}$, which is a contradiction, and so (5.13) holds true.

For all $\tau \in [0,1]$, the functional $E_\tau \in C^1(X)$ satisfies hypoteses analogous to (g_1')-(g_3') so, similarly as in the proof of Lemma 5.1.5, E_τ satisfies (PS) in $\overline{B}_\varepsilon(0)$. Moreover, the mapping $\tau \mapsto E_\tau$ is continuous in $[0,1]$. Now we can apply Propostion 2.1.1 to obtain $C^k(E,0) = C^k(E_1,0)$ for all $k \in \mathbb{N}_0$, as desired. □

Now we want to prove that, for every $\lambda > 0$, E has a non-trivial critical group at zero (recall that $(\lambda_k)_k$ is a sequence of eigenvalues of $(-\Delta)_s^p$ and $\lambda_1 = 0$).

Lemma 5.1.7. *If one of the following holds true for some $k \in \mathbb{N}$.*

(i) $\lambda_k < 2\lambda < \lambda_{k+1}$,

(ii) $\lambda_k = 2\lambda < \lambda_{k+1}$, *and $G(x,t) \geq 0$ a.e. in Ω and for all $|t| \leq \delta$ with $\delta > 0$,*

(iii) $\lambda_k < 2\lambda = \lambda_{k+1}$, *and $G(x,t) \leq 0$ a.e. in Ω and for all $|t| \leq \delta$ with $\delta > 0$,*

then $C^k(E,0) \neq 0$.

Proof. We want to prove that E has a cohomological local splitting near 0 in dimention $k \in \mathbb{N}$. As in [54], we set

$$C_k^- := \{u \in X : [u]^p \leq \lambda_k \|u\|_p^p\}, \quad C_k^+ := \{u \in X : [u]^p \geq \lambda_{k+1} \|u\|_p^p\},$$

which are symmetric closed cones with $C_k^- \cap C_k^+ = \{0\}$. From this definition, we have

$$\|u\|^p \leq (\lambda_k + 1)\|u\|_p^p \text{ for every } u \in C_k^-, \tag{5.16}$$

and

$$\|u\|^p \geq (\lambda_{k+1} + 1)\|u\|_p^p \text{ for every} u \in C_k^+. \tag{5.17}$$

Moreover, defining the manifold M as

$$M := \{u \in X : \|u\|_p^p = 1\},$$

by [54, Theorem 2.6] we have

$$i(C_k^- \setminus \{0\}) = i(X \setminus C_k^+) = k,$$

so the first condition for the local splitting is satisfied.

We remark that by (g_3'), for every $\varepsilon > 0$ there exists $\rho > 0$ such that, a.e. in Ω and for every $|t| < \rho$,

$$|g(x,t)| \leq \varepsilon |t|^{p-1}. \tag{5.18}$$

Integrating the inequalities in (5.18) and in (g_1') we get, for all $u \in X$,

$$\begin{aligned}\left|\int_\Omega G(x,u)\,dx\right| &\leq \int_{\Omega\cap\{|u|\leq\rho\}} \frac{\varepsilon|u|^p}{p}\,dx + \int_{\Omega\cap\{|u|>\rho\}} a\left(|u| + \frac{|u|^r}{r}\right)dx \\ &\leq \frac{\varepsilon\|u\|_p^p}{p} + C\|u\|_r^r.\end{aligned}$$

Now, the continuous embeddings of X in $L^p(\Omega)$ and in $L^r(\Omega)$, together with the arbitrarity of $\varepsilon > 0$, imply that

$$\int_\Omega G(x,u)\,dx = o(\|u\|^p) \text{ as } \|u\| \to 0. \tag{5.19}$$

Now we prove the second contion for the local splitting, that is, for ρ small enough,

$$\begin{aligned} &E(u) \leq 0 \text{ for all } u \in \overline{B}_\rho(0) \cap C_k^-, \\ &E(u) \geq 0 \text{ for all } u \in \overline{B}_\rho(0) \cap C_k^+, \end{aligned} \tag{5.20}$$

and we have to consider three different cases.

Assume (i). Then, for every $u \in C_k^- \setminus \{0\}$, by (5.19) and (5.16) we have

$$E(u) \leq \left(\frac{\lambda_k - 2\lambda}{\lambda_k + 1}\right) \frac{\|u\|^p}{2p} + o(\|u\|^p),$$

where the latter is negative if $\|u\|$ is small enough. On the other hand, for every $u \in C_k^+ \setminus \{0\}$, by (5.19) and (5.17)

$$E(u) \geq \left(\frac{\lambda_{k+1} - 2\lambda}{\lambda_{k+1} + 1}\right) \frac{\|u\|^p}{2p} + o(\|u\|^p),$$

and now the latter is positive if $\|u\|$ is small enough. So (5.20) holds for (i).

Now assume (ii). From Lemma 5.1.6 we know that $C^k(E, 0) = C^k(E_1, 0)$, so we consider E_1. Since $2\lambda = \lambda_k$, for every $u \in C_k^- \setminus \{0\}$, from (5.12) and (5.16) we have

$$E_1(u) \leq -\int_\Omega G(x, \theta(u))\, dx \leq 0.$$

If $u \in C_k^+ \setminus \{0\}$, from (5.17) we have

$$E_1(u) \geq \left(\frac{\lambda_{k+1} - 2\lambda}{\lambda_{k+1} + 1}\right) \frac{\|u\|^p}{2p} - \int_\Omega G(x, \theta(u))\, dx.$$

From (5.12) we have $\theta(t) = t$ if t is small enough, so taking $u \in \overline{B}_\rho(0)$ we get

$$E_1(u) \geq \left(\frac{\lambda_{k+1} - 2\lambda}{\lambda_{k+1} + 1}\right) \frac{\|u\|^p}{2p} + o(\|u\|^p),$$

and the latter is positive for small values of $\|u\|$.

In order to prove (5.20) in the case (iii) we argue similarly as in the case (ii). Then we get the inequalities

$$E_1(u) \leq \left(\frac{\lambda_k - 2\lambda}{\lambda_k + 1}\right) \frac{\|u\|^p}{2p} + o(\|u\|^p) \leq 0$$

for every $u \in \overline{B}_\rho(0) \cap C_k^-$, and

$$E_1(u) \geq -\int_\Omega G(x, \theta(u))\, dx \geq 0$$

for every $u \in \overline{B}_\rho(0) \cap C_k^+$.

So (5.20) holds true in every case. Recalling Lemma 5.1.6 for the cases (ii) and (iii), we can apply Proposition 2.1.4 to obtain $C^k(E, 0) \neq 0$, which concludes the proof. $\square$

Now we are ready to give the proof of Theorem 5.1.4:

Proof of Theorem 5.1.4. We argue by contradiction, and we assume

$$K(E) = \{0\}. \tag{5.21}$$

Let $\eta < 0$, so by Lemma 5.1.5 E^η is contractible. From (5.21) we know that there is no critical value for E in $[\eta, 0)$, and from Lemma 5.1.5 we know that E satisfies (PS) in X. So, by the Second Deformation Lemma, the set E^η is a deformation retract of $E^0 \setminus \{0\}$. In a similar way, since there is no critical value in $(0, +\infty)$, the set E^0 is a deformation retract of X. By the properties of critical groups and since E^η is contractible, for all $k \in \mathbb{N}_0$ we have

$$C^k(E, 0) = H^k(E^0, E^0 \setminus \{0\}) = H^k(X, E^\eta) = 0.$$

On the other hand, in all cases (i)-(iii), if we fix $k \in \mathbb{N}_0$, one of the assumptions of Lemma 5.1.7 has to hold, which implies $C^k(E, 0) \neq 0$, a contradiction. So (5.21) is false and there exists some $u \in K(E) \setminus \{0\}$, which is a nontrivial solution of (5.7). □

5.2 Problems without the Ambrosetti-Rabinowitz condition

In this section, we first consider the problem

$$\begin{cases} (-\Delta)_p^s u + |u|^{p-2}u = f(x, u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{5.22}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function such that $f(x, 0) = 0$ for almost every $x \in \Omega$. In addition, we assume the following hypotheses:

(f_1) there exists $a \in L^q(\Omega)$, $a \geq 0$, with $q \in ((p_s^*)', p)$, $c > 0$ and $r \in (p, p_s^*)$ such that

$$|f(x, t)| \leq a(x) + c|t|^{r-1}$$

for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$;

(f_2) denoting $F(x, t) = \int_0^t f(x, \tau) d\tau$, we have

$$\lim_{t \to \pm\infty} \frac{F(x, t)}{|t|^p} = +\infty$$

uniformly for a.e. $x \in \Omega$;

(f_3) if $\sigma(x,t) = f(x,t)t - pF(x,t)$, then there exist $\vartheta \geq 1$ and $\beta^* \in L^1(\Omega)$, $\beta^* \geq 0$, such that

$$\sigma(x,t_1) \leq \vartheta\sigma(x,t_2) + \beta^*(x)$$

for a.e. $x \in \Omega$ and all $0 \leq t_1 \leq t_2$ or $t_2 \leq t_1 \leq 0$;

(f_4)

$$\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0$$

uniformly for a.e. $x \in \Omega$.

As usual, in (f_1) we have denoted by p_s^* the fractional Sobolev exponent of order s, that is

$$p_s^* = \begin{cases} \dfrac{pN}{N-ps} & \text{if } ps < N, \\ \infty & \text{if } ps \geq N, \end{cases}$$

so that the embedding in $L^q(\Omega)$ of $W^{s,p}(\Omega)$ (and thus of X) is compact for every $q < p_s^*$.

Remark 5.2.1. A few comments on (f_3) are mandatory. Such a condition was introduced in [50] with $\vartheta = 1$. However, it is clear that assuming $\vartheta \geq 1$ enlarges the set of admissible *positive* (or definitely positive) functions σ's considered in [50] (as it happens for the model case $f(x,t) = |t|^{r-2}r$). On the other hand, if σ were negative, admitting $\vartheta < 1$ would make the situation more general. However, if $(f_1) - (f_4)$ hold for some $\vartheta > 0$, then $\sigma(x,t) > 0$ for a.e. $x \in \Omega$ and all t, at least for $|t|$ large, that is there exists $\bar{t} \geq 0$ such that $\sigma(x,t) > 0$ for a.e. $x \in \Omega$ and all $|t| > \bar{t}$. Indeed, reasoning with t positive, if for every $t > 0$ there exists $\tau > t$ such that $\sigma(x,\tau) \leq 0$, we get $\sigma(x,t) \leq \vartheta\sigma(x,\tau) + \beta^*(x) \leq \beta^*(x)$, that is $f(x,t)t - pF(x,t) \leq \beta^*(x)$ for a.e. $x \in \Omega$ and all t. As a consequence, $(F(t)t^{-p})' \leq \beta^*(x)t^{-p-1}$, and so

$$\frac{F(s)}{s^p} - \frac{F(t)}{t^p} \leq \frac{\beta^*(x)}{-p}\left(\frac{1}{s^p} - \frac{1}{t^p}\right)$$

for every $t < s$. Letting $s \to +\infty$, we get a contradiction with (f_2).

As a consequence, in (f_3) the requirement $\vartheta \geq 1$ is the most general one.

Now we are ready to give the definition of a weak solution of our problem.

Definition 5.2.2. Let $u \in X$. With the same assumption on f as above, we say that u is a weak solution of (5.22) if

$$\frac{1}{2}\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{J_p(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dxdy + \int_\Omega |u|^{p-2}uv\,dx = \int_\Omega f(x,u)v\,dx$$

for every $v \in X$.

With this definition, we have that any critical point of the functional $\mathscr{E} : X \to \mathbb{R}$ given by

$$\mathscr{E}(u) = \frac{1}{p}\|u\|^p - \int_\Omega F(x,u)\,dx$$

is a weak solution of (5.22).

We have the following result

Theorem 5.2.3. *If hypotheses* (f_1)-(f_4) *hold, then problem* (5.22) *admits two non-trivial constant sign solutions. More precisely, one solution is strictly positive in* $\mathbb{R}^N \setminus \overline{\Omega}$ *and the other one is strictly negative in* $\mathbb{R}^N \setminus \overline{\Omega}$. *In addition, if the equation holds pointwise, each solution has strict sign in the whole of* $\mathbb{R}^N$.

First, we introduce the functionals

$$\mathscr{E}_\pm(u) = \frac{1}{p}\|u\|^p - \int_\Omega F(x,u^\pm)\,dx,$$

where u^+ and u^- are the classical positive part and negative part of u. We want to prove that both $\mathscr{E}_\pm$ satisfies the Cerami condition, (C) for short, which states that any sequence $(u_n)_n$ in X such that $(\mathscr{E}_\pm(u_n))_n$ is bounded and $(1+\|u_n\|)\mathscr{E}'_\pm(u_n) \to 0$ as $n \to \infty$ admits a convergent subsequence.

We will also use the following inequality:

$$|x^- - y^-|^p \le |x-y|^{p-2}(x-y)(y^- - x^-), \tag{5.23}$$

for any $x, y \in \mathbb{R}$.

Proposition 5.2.4. *Under the assumptions of Theorem* 5.2.3, $\mathscr{E}_\pm$ *satisfies the* (C) *condition.*

Proof. We do the proof for $\mathscr{E}_+$, the proof for $\mathscr{E}_-$ being analogous.

Let $(u_n)_n$ in X be such that

$$|\mathscr{E}_+(u_n)| \leq M_1 \tag{5.24}$$

for some $M_1 > 0$ and all $n \geq 1$, and

$$(1 + \|u_n\|)\mathscr{E}'(u_n) \to 0 \tag{5.25}$$

in X' as $n \to \infty$. From (5.25) we have

$$|\mathscr{E}'_+(u_n)(h)| \leq \frac{\varepsilon_n h}{1 + \|u_n\|}$$

for every $h \in X$ and with $\varepsilon_n \to 0$ as $n \to \infty$, that is

$$\begin{aligned} \Bigg| \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} & \frac{J_p(u_n(x) - u_n(y))(h(x) - h(y))}{|x - y|^{N+ps}}\, dxdy + \int_\Omega |u_n|^{p-2} u_n h\, dx \\ & - \int_\Omega f(x, u_n^+) h\, dx \Bigg| \leq \frac{\varepsilon_n h}{1 + \|u_n\|}. \end{aligned} \tag{5.26}$$

Taking $h = u_n^-$ in (5.26), we obtain

$$\left| \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{J_p(u_n(x) - u_n(y))(u_n^-(x) - u_n^-(y))}{|x - y|^{N+ps}}\, dxdy + \int_\Omega |u_n^-|^p dx \right| \leq \varepsilon_n.$$

By (5.23), we have

$$\begin{aligned} & \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|u_n^-(x) - u_n^-(y)|^p}{|x - y|^{N+ps}}\, dxdy \\ & \leq \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{J_p(u_n(x) - u_n(y))(u_n^-(x) - u_n^-(y))}{|x - y|^{N+ps}}\, dxdy, \end{aligned}$$

which leads to

$$\|u_n^-\|^p \leq \varepsilon_n.$$

So, we have that

$$u_n^- \to 0 \quad \text{in } X \quad \text{as } n \to \infty. \tag{5.27}$$

Now, if we take $h = u_n^+$ in (5.26), we obtain

$$\begin{aligned} & - \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{J_p(u_n(x) - u_n(y))(u_n^+(x) - u_n^+(y))}{|x - y|^{N+ps}}\, dxdy \\ & - \int_\Omega |u_n^+|^p\, dx + \int_\Omega f(x, u_n^+) u_n^+\, dx \leq \varepsilon_n. \end{aligned} \tag{5.28}$$

From (5.24) we have

$$\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dxdy + \int_\Omega |u_n|^p dx - p\int_\Omega F(x,u_n^+)\,dx \le pM_1$$

for $M_1>0$ and all $n\ge 1$, and since

$$\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{J_p(u_n(x)-u_n(y))(u_n^-(x)-u_n^-(y))}{|x-y|^{N+ps}}\,dxdy + \int_\Omega |u_n^-|^p dx \to 0$$

as $n\to\infty$, we get

$$\begin{aligned}&\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{J_p(u_n(x)-u_n(y))(u_n^+(x)-u_n^+(y))}{|x-y|^{N+ps}}\,dxdy\\ &+\int_\Omega |u_n^+|^p dx - p\int_\Omega F(x,u_n^+)\,dx \le M_2 \end{aligned} \tag{5.29}$$

for some $M_2>0$ and all $n\ge 1$. Adding (5.29) to (5.28) we obtain

$$\int_\Omega f(x,u_n^+)u_n^+\,dx - p\int_\Omega F(x,u_n^+)\,dx \le M_3$$

for some $M_3>0$ and all $n\ge1$, that is

$$\int_\Omega \sigma(x,u_n^+)\,dx \le M_3. \tag{5.30}$$

Now we want to prove that $(u_n^+)_n$ is bounded in X, and to do this we argue by contradiction. Passing to a subsequence if necessary, we assume that $\|u_n^+\|\to\infty$ as $n\to\infty$. Defining $y_n = u_n^+/\|u_n^+\|$, we can assume

$$y_n \rightharpoonup y \quad \text{in } X \quad \text{and } y_n \to y \quad \text{in } L^q(\Omega) \tag{5.31}$$

for every $q\in(p,p_s^*)$ and $y\ge 0$.

First we consider the case $y\ne 0$. We define $Z(y)=\{x\in\Omega : y(x)=0\}$, and so we have $|\Omega\setminus Z(y)|>0$ and $u_n^+\to\infty$ for almost every $x\in\Omega\setminus Z(y)$ as $n\to\infty$. By hypothesis (f_2), we have

$$\frac{F(x,u_n^+(x))}{\|u_n^+\|^p} = \frac{F(x,u_n^+(x))}{u_n^+(x)^p} y_n(x)^p \to \infty$$

for almost every $x\in\Omega\setminus Z(y)$. By Fatou's Lemma, we have

$$\int_\Omega \liminf_{n\to\infty} \frac{F(x,u_n^+(x))}{\|u_n^+\|^p}\,dx \le \liminf_{n\to\infty}\int_\Omega \frac{F(x,u_n^+(x))}{\|u_n^+\|^p}\,dx,$$

and so

$$\int_{\Omega} \frac{F(x, u_n^+(x))}{\|u_n^+\|^p}\, dx \to \infty \tag{5.32}$$

as $n \to \infty$.

As before, from (5.24) and (5.27) we have

$$-\frac{1}{p}\|u_n\|^p + \int_{\Omega} F(x, u_n^+)\, dx \leq M_4$$

for some $M_4 > 0$ and $n \geq 1$. Since $\|u_n\|^p \leq 2^{p-1}(\|u_n^+\|^p + \|u_n^-\|^p)$, we obtain

$$-\frac{2^{p-1}}{p}\|u_n^+\|^p + \int_{\Omega} F(x, u_n^+)\, dx \leq M_5$$

for some $M_5 > 0$, and so

$$-\frac{2^{p-1}}{p} + \int_{\Omega} \frac{F(x, u_n^+(x))}{\|u_n^+\|^p}\, dx \leq \frac{M_5}{\|u_n^+\|^p}.$$

Passing to the limit, we have

$$\limsup_{n\to\infty} \int_{\Omega} \frac{F(x, u_n^+(x))}{\|u_n^+\|^p}\, dx \leq M_6$$

for some M_6, which is in contradiction with (5.32), and this concludes the case $y \neq 0$.

Now,we deal with the case $y \equiv 0$. We consider the continuous functions $\gamma_n : [0, 1] \to \mathbb{R}$, defined as $\gamma_n(t) := \mathscr{E}_+(tu_n^+)$ with $t \in [0, 1]$ and $n \geq 1$. So, we can define t_n such that

$$\gamma_n(t_n) = \max_{t\in[0,1]} \gamma_n(t). \tag{5.33}$$

Now we define $v_n := (p\lambda)^{\frac{1}{p}} y_n \in X$ for $\lambda > 0$. From (5.31), it follows that $v_n \to 0$ in $L^q(\Omega)$ for all $q \in (p, p_s^*)$. Starting from (f_1) and performing some integration, we have

$$\int_{\Omega} F(x, v_n(x))\, dx \leq \int_{\Omega} a(x)|v_n(x)|\, dx + C \int_{\Omega} |v_n(x)|^r\, dx,$$

and so

$$\int_{\Omega} F(x, v_n(x))\, dx \to 0 \tag{5.34}$$

as $n \to \infty$. Since $\|u_n^+\| \to \infty$, there exists $n_0 \geq 1$ such that $(p\lambda)^{\frac{1}{p}}/\|u_n^+\| \in (0, 1)$ for all $n \geq n_0$. Then, from (5.33), we have

$$\gamma_n(t_n) \geq \gamma_n\left(\frac{(p\lambda)^{\frac{1}{p}}}{\|u_n^+\|}\right)$$

for all $n \geq n_0$. It follows that

$$\begin{aligned}\mathscr{E}_+(t_n u_n^+) &\geq \mathscr{E}_+((p\lambda)^{\frac{1}{p}} y_n) = \mathscr{E}_+(v_n) \\ &= \lambda \|y_n\|^p - \int_\Omega F(x, v_n(x))\, dx.\end{aligned}$$

From (5.34), we have

$$\mathscr{E}_+(t_n u_n^+) \geq \lambda + o(1),$$

and since λ is arbitrary we have

$$\mathscr{E}_+(t_n u_n^+) \to \infty \tag{5.35}$$

as $n \to \infty$. Now, $0 \leq t_n u_n^+ \leq u_n^+$ for all $n \leq 1$, so from (f_3) we get

$$\int_\Omega \sigma(x, t_n u_n^+)\, dx \leq \vartheta \int_\Omega \sigma(x, u_n^+)\, dx + \|\beta^*\|_1 \tag{5.36}$$

for all $n \geq 1$. In addition, we have $\mathscr{E}_+(0) = 0$, and from (5.24), (5.27) and (5.23), we have $\mathscr{E}_+(u_n^+) \leq M_7$ for some $M_7 > 0$. Together with (5.35), this implies that $t_n \in (0, 1)$ for all $n \geq n_1 \geq n_0$. Since t_n is a maximum point, we also have

$$\begin{aligned}0 &= t_n \gamma_n'(t_n) \\ &= \int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} \frac{J_p(t_n u_n(x) - t_n u_n(y))(t_n u_n^+(x) - t_n u_n^+(y))}{|x-y|^{N+ps}}\, dxdy \\ &+ \int_\Omega |t_n u_n^+|^p\, dx - \int_\Omega f(x, t_n u_n^+(x)) t_n u_n^+(x)\, dx,\end{aligned}$$

and so, from (5.23),

$$\|t_n u_n^+\|^p - \int_\Omega f(x, t_n u_n^+(x)) t_n u_n^+(x)\, dx \leq 0. \tag{5.37}$$

Adding (5.37) to (5.36), we get

$$\|t_n u_n^+\|^p - p \int_\Omega F(x, t_n u_n^+(x))\, dx \leq \vartheta \int_\Omega \sigma(x, u_n^+)\, dx + \|\beta^*\|_1,$$

which is

$$p\mathscr{E}_+(t_n u_n^+) \leq \vartheta \int_\Omega \sigma(x, u_n^+)\, dx + \|\beta^*\|_1.$$

So, from (5.35), we get

$$\int_\Omega \sigma(x, u_n^+)\, dx \to \infty \tag{5.38}$$

as $n \to \infty$. Combining (5.30) and (5.38) we obtain a contradiction, and so the claim follows.

We have proved that $(u_n^+)_n$ is bounded in X, so from (5.27) we have that $(u_n)_n$ is bounded in X. Hence, we can assume

$$u_n \rightharpoonup u \quad \text{in } X \quad \text{and } u_n \to u \quad \text{in } L^q(\Omega) \tag{5.39}$$

with $q \in (p, p_s^*)$. Taking $h = u_n - u$ in (5.26), we have

$$\begin{aligned} \|u_n\|^p &- \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{J_p(u_n(x) - u_n(y))(u(x) - u(y))}{|x-y|^{N+ps}}\, dxdy \\ &- \int_\Omega |u_n|^{p-2} u_n u \, dx - \int_\Omega f(x, u_n^+)(u_n - u)\, dx \le \varepsilon_n. \end{aligned} \tag{5.40}$$

From (f_1) and (5.39), we have

$$\int_\Omega |f(x, u_n^+(x))(u_n(x) - u(x))|\, dx \to 0$$

as $n \to \infty$. So, passing to the limit in (5.40), we get

$$\begin{aligned} \|u_n\|^p &- \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{J_p(u_n(x) - u_n(y))(u(x) - u(y))}{|x-y|^{N+ps}}\, dxdy \\ &- \int_\Omega |u_n|^{p-2} u_n u \, dx \to 0 \end{aligned}$$

as $n \to \infty$. This implies that $\|u_n\|^p \to \|u\|^p$, and so from the (S) property it follows that $u_n \to u$ in X. This concludes the proof that $\mathscr{E}_+$ satisfies the (C) condition. □

We can now give the proof of Theorem 5.2.3.

Proof of Theorem 5.2.3. We want to apply the Mountain Pass Theorem to $\mathscr{E}_+$. Since $\mathscr{E}_+$ satisfies the (C) condition from Proposition 5.2.4, we only have to verify the geometric conditions.

From (f_1) and (f_4), for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$F(x, u) \le \frac{\varepsilon}{p}|u|^p + C_\varepsilon |u|^r \tag{5.41}$$

for almost every $x \in \mathbb{R}^N$ and all $u \in \mathbb{R}$. Then, we have

$$\begin{aligned} \mathscr{E}_+(u) &= \frac{1}{p}\|u\|^p - \int_\Omega F(x, u^+)\, dx \\ &\ge \frac{1}{p}\|u\|^p - \frac{\varepsilon}{p}\|u\|_p^p - C_\varepsilon \|u\|_r^r \\ &\ge \frac{1 - \varepsilon C_1}{p}\|u\|^p - C_2\|u\|^r. \end{aligned}$$

From this, if $\|u\| = \rho$ small enough, we have $\inf_{\|u\|=\rho} \mathscr{E}_+(u) > 0$.

Now, we take $u \in X$ with $u > 0$ and $t > 0$, then

$$\begin{aligned} \mathscr{E}_+(u) &= \frac{t^p}{p}\|u\|^p - \int_\Omega F(x, tu)\, dx \\ &= \frac{t^p}{p}\|u\|^p - t^p \int_\Omega \frac{F(x, tu)}{(tu)^p} u^p\, dx. \end{aligned}$$

By Fatou's Lemma we have

$$\int_\Omega \liminf_{t\to\infty} \frac{F(x, tu)}{(tu)^p} u^p\, dx \leq \liminf_{t\to\infty} \int_\Omega \frac{F(x, tu)}{(tu)^p} u^p\, dx,$$

so from (f_2) we have

$$\int_\Omega \frac{F(x, tu)}{(tu)^p} u^p\, dx \to \infty$$

as $n \to \infty$. It follows that

$$\mathscr{E}_+(tu) \to -\infty$$

as $t \to \infty$, and so there exists $e \in X$ such that $\|e\| \geq \rho$ and $\mathscr{E}_+(e) > 0$.

Now, we can apply the Mountain Pass Theorem to $\mathscr{E}_+$ and obtain a non-trivial critical point u. In particular, we have

$$\begin{aligned} 0 = &\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{J_p(u_n(x) - u_n(y))(u^-(x) - u^-(y))}{|x-y|^{N+ps}}\, dxdy \\ &+ \int_\Omega |u|^p\, dx - \int_\Omega f(x, u^+)u^-\, dx \\ = &\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{J_p(u_n(x) - u_n(y))(u^-(x) - u^-(y))}{|x-y|^{N+ps}}\, dxdy + \int_\Omega |u|^p\, dx. \end{aligned}$$

From (5.23), we get

$$0 \geq \|u^-\|^p,$$

and so $u^- \equiv 0$. As a consequence, we have $\mathscr{E}_+(u) = \mathscr{E}(u)$, and so $u \geq 0$ is a solution of (5.22).

Suppose that there exists $x_0 \in \mathbb{R}^N \setminus \overline{\Omega}$ such that $u(x_0) = 0$. Then, from Theorem 3.1.6 we would get

$$\int_\Omega \frac{u^{p-1}(y)}{|x-y|^{N+sp}} dy = 0,$$

so that $u = 0$ in Ω and thus, using u as test function in the equation, $u = 0$ in $\mathbb{R}^N$, while u is non-trivial.

Now, assume that the equation holds pointwise and suppose by contradiction that there exists $x \in \Omega$ such that $u(x) = 0$. From (5.22) we would get

$$\int_{\mathbb{R}^N} \frac{u(y)^{p-1}}{|x-y|^{N+ps}}\, dy = 0.$$

This would imply that $u = 0$ a.e. in $\mathbb{R}^N$, which is a contradiction since the solution is non-trivial. It follows that $u > 0$ in $\mathbb{R}^N$.

Arguing in the same way for $\mathscr{E}_-$, we can find a non-trivial negative solution for (5.22). □

Now we deal with the problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u + f(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases}, \tag{5.42}$$

where $\lambda \geq 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x,0) = 0$ for almost every $x \in \Omega$. This time, we assume the following hypotheses on f, first introduced in [50]:

(f_1') there exists $a \in L^q(\Omega)$, $a \geq 0$, with $q \in ((p_s^*)', p)$, $c > 0$ and $r \in (p, p_s^*)$ such that

$$|f(x,t)| \leq a(x) + c|t|^{r-1}$$

for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$;

(f_2') denoting $F(x,t) = \int_0^t f(x,\tau) d\tau$, we have

$$\lim_{t \to \pm\infty} \frac{F(x,t)}{|t|^p} = +\infty$$

uniformly for a.e. $x \in \Omega$;

(f_3') if $\sigma(x,t) := f(x,t)t - pF(x,t)$, then there exist $\vartheta \geq 1$ and $\beta^* \in L^1(\Omega)$, $\beta^* \geq 0$, such that

$$\sigma(x,t_1) \leq \vartheta \sigma(x,t_2) + \beta^*(x)$$

for a.e. $x \in \Omega$ and all $0 \leq t_1 \leq t_2$ or $t_2 \leq t_1 \leq 0$;

(f_4')

$$\lim_{t \to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0$$

uniformly for a.e. $x \in \Omega$.

As before, we give the definition of a weak solution.

Definition 5.2.5. Let $u \in X$. We say that u is a weak solution of problem (5.1) if

$$\frac{1}{2}\iint_{\mathbb{Q}} \frac{J_p(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dxdy = \lambda \int_\Omega |u|^{p-2}uv\,dx + \int_\Omega f(x,u)v\,dx$$

for every $v \in X$.

Again, any critical point of the C^1 functional $\mathscr{E} : X \to \mathbb{R}$ defined as

$$\mathscr{E}(u) = \frac{1}{2p}\int\int_{\mathbb{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy - \frac{\lambda}{p}\int_\Omega |u|^p\,dx - \int_\Omega F(x,u)\,dx$$

is a weak solution of (5.1).

The main result of this section is the following.

Theorem 5.2.6. *If hypotheses* (f_1')-(f_4') *hold, then problem* (5.42) *admits two nontrivial constant sign solutions. More precisely, one solution is strictly positive and the other one is strictly negative in* $\mathbb{R}^N$.

First of all, we introduce the functionals

$$\begin{aligned}\mathscr{E}_\pm(u) = {} & \frac{1}{2p}\int\int_{\mathbb{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy + \frac{1}{p}\int_\Omega |u|^p dx \\ & - \frac{\lambda+1}{p}\int_\Omega |u^\pm|^p dx - \int_\Omega F(x,u^\pm)\,dx,\end{aligned}$$

where $u^+ := \max\{u,0\}$ and $u^- := \max\{-u,0\}$ are the classical positive part and negative part of u, respectively. Notice that $\mathscr{E}_+(u) = \mathscr{E}(u)$ for every $u \geq 0$ and $\mathscr{E}_-(u) = \mathscr{E}(u)$ for every $u \leq 0$.

The following algebraic inequalities will be very useful in the following:

$$|x^- - y^-|^p \leq |x-y|^{p-2}(x-y)(y^- - x^-), \tag{5.43}$$

$$|x^+ - y^+|^p \leq |x-y|^{p-2}(x-y)(x^+ - y^+), \tag{5.44}$$

$$|x-y|^p \leq 2^{p-1}(|x^+ - y^+|^p + |x^- - y^-|^p) \tag{5.45}$$

and

$$|x^\pm - y^\pm| \leq |x-y| \tag{5.46}$$

for any $x, y \in \mathbb{R}$. The proofs are obvious.

Proposition 5.2.7. *Under the assumptions of Theorem* 5.2.6, $\mathscr{E}_\pm$ *satisfies* $(C)_c$ *for every* $c \in \mathbb{R}$.

Proof. We do the proof for $\mathscr{E}_+$, the proof for $\mathscr{E}_-$ being analogous.

Let $(u_n)_n$ in X be such that

$$|\mathscr{E}_+(u_n)| \leq M_1 \tag{5.47}$$

for some $M_1 > 0$ and all $n \geq 1$, and

$$(1 + \|u_n\|)\mathscr{E}'_+(u_n) \to 0 \tag{5.48}$$

in X' as $n \to \infty$. From (5.48) we have

$$|\mathscr{E}'_+(u_n)(h)| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|}$$

for every $h \in X$ and with $\varepsilon_n \to 0$ as $n \to \infty$, that is

$$\begin{aligned} \Bigg| \frac{1}{2} \int\int_{\mathcal{Q}} \frac{J_p(u_n(x) - u_n(y))(h(x) - h(y))}{|x - y|^{N+ps}} \, dxdy + \int_\Omega |u_n|^{p-2} u_n h \, dx \\ - (\lambda + 1) \int_\Omega |u_n^+|^{p-2} u_n^+ h \, dx - \int_\Omega f(x, u_n^+) h \, dx \Bigg| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|}. \end{aligned} \tag{5.49}$$

Taking $h = -u_n^-$ in (5.49), we obtain

$$\frac{1}{2} \int\int_{\mathcal{Q}} \frac{J_p(u_n(x) - u_n(y))(u_n^-(y) - u_n^-(x))}{|x - y|^{N+ps}} \, dxdy + \lambda \int_\Omega |u_n^-|^p dx \leq \varepsilon_n, \tag{5.50}$$

and by (5.43) we get

$$\int\int_{\mathcal{Q}} \frac{|u_n^-(x) - u_n^-(y)|^p}{|x - y|^{N+ps}} \, dxdy + 2\lambda \int_\Omega |u_n^-|^p dx \leq 2\varepsilon_n.$$

As a consequence, we get that

$$u_n^- \to 0 \text{ in } X \text{ as } n \to \infty. \tag{5.51}$$

In particular, $(u_n^-)_n$ is bounded in X.

On the other hand, taking $h = -u_n^+$ in (5.49), we get

$$\begin{aligned} -\frac{1}{2} \int\int_{\mathcal{Q}} \frac{J_p(u_n(x) - u_n(y))(u_n^+(x) - u_n^+(y))}{|x - y|^{N+ps}} \, dxdy \\ + \lambda \int_\Omega |u_n^+|^p \, dx + \int_\Omega f(x, u_n^+) u_n^+ \, dx \leq \varepsilon_n. \end{aligned} \tag{5.52}$$

From (5.47) we know that

$$\frac{1}{2}[u_n]^p + \int_\Omega |u_n|^p \, dx - (\lambda + 1) \int_\Omega |u_n^+|^p \, dx - p \int_\Omega F(x, u_n^+) \, dx \leq pM_1 \tag{5.53}$$

for all $n \geq 1$. Now, by (5.50) and (5.51), we have that

$$\int\int_{\mathcal{Q}} \frac{J_p(u_n(x) - u_n(y))(u_n^-(x) - u_n^-(y))}{|x-y|^{N+ps}}\, dxdy \to 0,$$

and so from (5.53) we get

$$\begin{aligned} &\frac{1}{2}\int\int_{\mathcal{Q}} \frac{J_p(u_n(x) - u_n(y))(u_n^+(x) - u_n^+(y))}{|x-y|^{N+ps}}\, dxdy \\ &\qquad + \int_\Omega |u_n|^p\, dx - (\lambda+1)\int_\Omega |u_n^+|^p\, dx - p\int_\Omega F(x,u_n^+)\, dx \leq M_2 \end{aligned} \tag{5.54}$$

for some $M_2 > 0$ and all $n \geq 1$. Adding (5.54) to (5.52) we obtain

$$\int_\Omega |u_n|^p\, dx - \int_\Omega |u_n^+|^p\, dx + \int_\Omega f(x,u_n^+)u_n^+\, dx - p\int_\Omega F(x,u_n^+)\, dx \leq M_3$$

for some $M_3 > 0$ and all $n \geq 1$, which clearly implies

$$\int_\Omega \sigma(x,u_n^+)\, dx \leq M_3. \tag{5.55}$$

Now we claim that $(u_n^+)_n$ is bounded in X, as well. We argue by contradiction. Up to a subsequence, we assume that $\|u_n^+\| \to \infty$ as $n \to \infty$. Defining $y_n = u_n^+/\|u_n^+\|$, we can assume that

$$y_n \rightharpoonup y \quad \text{in } X \quad \text{and } y_n \to y \quad \text{in } L^q(\Omega) \tag{5.56}$$

for every $q \in (p, p_s^*)$ with $y \geq 0$ in Ω.

First we deal with the case $y \not\equiv 0$. We define $Z(y) = \{x \in \Omega : y(x) = 0\}$, and so we have $|\Omega \setminus Z(y)| > 0$ and $u_n^+ \to \infty$ for almost every $x \in \Omega \setminus Z(y)$ as $n \to \infty$. By (f_2'), we have

$$\frac{F(x,u_n^+(x))}{\|u_n^+\|^p} = \frac{F(x,u_n^+(x))}{u_n^+(x)^p} y_n(x)^p \to \infty$$

for almost every $x \in \Omega \setminus Z(y)$. From Fatou's Lemma we get that

$$\int_\Omega \liminf_{n\to\infty} \frac{F(x,u_n^+(x))}{\|u_n^+\|^p}\, dx \leq \liminf_{n\to\infty} \int_\Omega \frac{F(x,u_n^+(x))}{\|u_n^+\|^p}\, dx,$$

and so

$$\int_\Omega \frac{F(x,u_n^+(x))}{\|u_n^+\|^p}\, dx \to \infty \tag{5.57}$$

as $n \to \infty$.

Again from (5.47) we have

$$-\frac{1}{2p}[u_n]^p - \frac{1}{p}\int_\Omega |u_n|^p\,dx + \frac{\lambda+1}{p}\int_\Omega |u_n^+|^p\,dx + \int_\Omega F(x,u_n^+)\,dx \le M_4$$

for some $M_4 > 0$ and $n \ge 1$. From (5.45) we get

$$-\frac{2^{p-2}}{p}([u_n^+]^p+[u_n^-]^p)-\frac{1}{p}\int_\Omega |u_n|^p\,dx+\frac{\lambda+1}{p}\int_\Omega |u_n^+|^p\,dx+\int_\Omega F(x,u_n^+)\,dx \le M_4,$$

and from (5.51)

$$-\frac{2^{p-2}}{p}[u_n^+]^p + \frac{\lambda}{p}\int_\Omega |u_n^+|^p\,dx + \int_\Omega F(x,u_n^+)\,dx \le M_5,$$

for some $M_5 > 0$ and all $n \ge 1$, so that

$$\int_\Omega F(x,u_n^+)\,dx \le M_5 + c\|u_n^+\|^p$$

for some $c > 0$ and all $n \ge 1$. Dividing by $\|u_n^+\|^p$ and passing to the limit we obtain

$$\limsup_{n\to\infty}\int_\Omega \frac{F(x,u_n^+(x))}{\|u_n^+\|^p}\,dx \le M_6$$

for some M_6, which is in contradiction with (5.57), and this concludes the case $y \ne 0$.

Now, we deal with the case $y \equiv 0$. We consider the continuous functions $\gamma_n : [0,1] \to \mathbb{R}$, defined as

$$\gamma_n(t) := \mathscr{E}_+(tu_n^+)$$

for any $n \ge 1$. So, there exists $t_n \in [0,1]$ such that

$$\gamma_n(t_n) = \max_{t\in[0,1]} \gamma_n(t). \tag{5.58}$$

Now, fixed $\mu > 0$, we define $v_n := (p\mu)^{\frac{1}{p}} y_n \in X$. From (5.56) we get that $v_n \to 0$ in $L^q(\Omega)$ for all $q \in (p, p_s^*)$. From (f_1') we know that

$$\int_\Omega F(x,v_n(x))\,dx \le \int_\Omega a(x)|v_n(x)|\,dx + C\int_\Omega |v_n(x)|^r\,dx,$$

and so

$$\int_\Omega F(x,v_n(x))\,dx \to 0 \tag{5.59}$$

as $n \to \infty$. Since $\|u_n^+\| \to \infty$, there exists $n_0 \geq 1$ such that $(p\mu)^{\frac{1}{p}}/\|u_n^+\| \in (0,1)$ for all $n \geq n_0$. Then, from (5.58), we have

$$\gamma_n(t_n) \geq \gamma_n\left(\frac{(p\mu)^{\frac{1}{p}}}{\|u_n^+\|}\right)$$

for all $n \geq n_0$. Thus, we get

$$\begin{aligned}
&\mathscr{E}_+(t_n u_n^+) \geq \mathscr{E}_+((p\mu)^{\frac{1}{p}} y_n) = \mathscr{E}_+(v_n)\\
&= \frac{1}{2}\mu \int\int_\Omega \frac{|y_n(x) - y_n(y)|^p}{|x-y|^{N+ps}}\, dxdy - \frac{\lambda}{p}\int_\Omega v_n^p\, dx - \int_\Omega F(x, v_n(x))\, dx\\
&= \frac{\mu}{2}\|y_n\|^p - \frac{\mu}{2}\int_\Omega y_n^p dx - \frac{2\lambda+1}{2p}\int_\Omega v_n^p\, dx - \int_\Omega F(x, v_n(x))\, dx\\
&= \frac{\mu}{2} - \frac{2\lambda+1}{2p}\int_\Omega v_n^p\, dx - \int_\Omega F(x, v_n(x))\, dx.
\end{aligned}$$

From (5.59) and the fact that $v_n \to 0$ in $L^p(\Omega)$, we get that

$$\mathscr{E}_+(t_n u_n^+) \geq \frac{\mu}{2} + o(1),$$

where $o(1) \to 0$ as $n \to \infty$. Since μ is arbitrary, we have

$$\lim_{n\to\infty} \mathscr{E}_+(t_n u_n^+) = +\infty. \tag{5.60}$$

On the other hand, since $0 \leq t_n u_n^+ \leq u_n^+$ for all $n \leq 1$, from (f_3') we get

$$\int_\Omega \sigma(x, t_n u_n^+)\, dx \leq \vartheta \int_\Omega \sigma(x, u_n^+)\, dx + \|\beta^*\|_1 \tag{5.61}$$

for all $n \geq 1$.

In addition, we have that $\mathscr{E}_+(0) = 0$; moreover, from (5.46) we get that

$$\mathscr{E}_+(u_n^+) \leq \mathscr{E}_+(u_n) \leq M_1$$

for all $n \geq 1$ by (5.47). Together with (5.60), these two facts imply the existence of $n_1 \geq n_0$ such that $t_n \in (0,1)$ for all $n \geq n_1$, namely $t_n \neq 0$ and $t_n \neq 1$. Since t_n is a maximum point for γ_n, we have

$$\begin{aligned}
0 &= t_n \gamma_n'(t_n)\\
&= \frac{1}{2}\int\int_\Omega \frac{|t_n u_n^+(x) - t_n u_n^+(y)|^p}{|x-y|^{N+ps}}\, dxdy\\
&- \lambda \int_\Omega |t_n u_n^+|^p\, dx - \int_\Omega f(x, t_n u_n^+(x)) t_n u_n^+(x)\, dx.
\end{aligned} \tag{5.62}$$

Adding (5.62) to (5.61), we get

$$\frac{1}{2}\int\int_{\mathbb{Q}}\frac{|t_nu_n^+(x)-t_nu_n^+(y)|^p}{|x-y|^{N+ps}}\,dxdy$$
$$-\lambda\int_\Omega |t_nu_n^+|^p\,dx - p\int_\Omega F(x,t_nu_n^+(x))\,dx$$
$$\leq \vartheta\int_\Omega \sigma(x,u_n^+)\,dx + \|\beta^*\|_1,$$

which is

$$p\mathscr{E}_+(t_nu_n^+)\leq \vartheta\int_\Omega \sigma(x,u_n^+)\,dx + \|\beta^*\|_1.$$

So, from (5.60), we get

$$\lim_{n\to\infty}\int_\Omega \sigma(x,u_n^+)\,dx = \infty. \tag{5.63}$$

Comparing (5.55) and (5.63) we obtain a contradiction, and so the claim follows.

In conclusion, we have proved that $(u_n^+)_n$ is bounded in X, so from (5.45) and (5.51) we have that $(u_n)_n$ is bounded in X. Hence, we can assume that

$$u_n \rightharpoonup u \quad \text{in } X \quad \text{and } u_n\to u \quad \text{in } L^q(\Omega) \tag{5.64}$$

for every $q\in(p,p_s^*)$ as $n\to\infty$. Taking $h=u_n-u$ in (5.49), we have

$$\Bigg|\frac{1}{2}\int\int_{\mathbb{Q}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dxdy$$
$$-\frac{1}{2}\int\int_{\mathbb{Q}}\frac{J_p(u_n(x)-u_n(y))(u(x)-u(y))}{|x-y|^{N+ps}}\,dxdy+\int_\Omega |u_n|^{p-2}u_n(u_n-u)$$
$$-(\lambda+1)\int_\Omega |u_n^+|^{p-2}u_n^+(u_n-u)\,dx-\int_\Omega f(x,u_n^+)(u_n-u)\,dx\Bigg|\leq\varepsilon_n. \tag{5.65}$$

From (f_1') and (5.64), we have

$$\int_\Omega f(x,u_n^+(x))(u_n(x)-u(x))\,dx\to 0,$$

$$\int_\Omega |u_n|^{p-2}u_n(u_n-u)\to 0$$

and

$$\int_\Omega |u_n^+|^{p-2}u_n^+(u_n-u)\to 0$$

as $n \to \infty$. Passing to the limit in (5.65), we get

$$\int\int_{\Omega} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \, dxdy$$
$$- \int\int_{\Omega} \frac{J_p(u_n(x) - u_n(y))(u(x) - u(y))}{|x-y|^{N+ps}} \, dxdy \to 0$$

as $n \to \infty$. From Proposition 3.1.8 we can conclude that $u_n \to u$ in X and this concludes the proof that $\mathscr{E}_+$ satisfies $(C)_c$ for every $c \in \mathbb{R}$.

Proceeding analogously, we have that $\mathscr{E}_-$ satisfies $(C)_c$ for every $c \in \mathbb{R}$, as well. □

Now we are ready to give the proof Theorem 5.2.6.

Proof of Theorem 5.2.6. First, we want to apply Theorem 2.2.2 to $\mathscr{E}_+$. So, as before, let $(\lambda_m)_m$ be the sequence of eigenvalues defined in Proposition 4.1.2. As in the proof of Theorem 5.1.2, there exists $m \geq 1$ such that $\lambda_m \leq 2\lambda + 1 < \lambda_{m+1}$, and we use the same two symmetric closed cones C_m^- and C_m^+ with $C_m^- \cap C_m^+ = \{0\}$. By Theorem 2.2.6 we also have

$$i(C_m^- \setminus \{0\}) = i(X \setminus C_m^+) = m.$$

In a similar way to the proof of Theorem 5.1.2, by (f_1'), (f_4') and taking $u \in C_m^+$ we have

$$\begin{aligned}
\mathscr{E}_+(u) &\geq \frac{1}{2p}\|u\|^p - \frac{2\lambda+1}{2p}\int_\Omega |u^+|^p dx - \frac{\varepsilon}{2p}\int_\Omega |u^+|^p dx - C_\varepsilon \int_\Omega |u^+|^q dx \\
&\geq \frac{1}{2p}\|u\|^p - \frac{2\lambda+1}{2p}\int_\Omega |u|^p dx - \frac{\varepsilon}{2p}\int_\Omega |u|^p dx - C_\varepsilon \int_\Omega |u|^q dx \\
&\geq \frac{1}{2p}\|u\|^p - \frac{1}{2p\lambda_{m+1}}(2\lambda+1+\varepsilon)\,[u]^p - C_\varepsilon \int_\Omega |u|^q dx \\
&\geq \frac{1}{2p}\left(1 - \frac{2\lambda+1+\varepsilon}{\lambda_{m+1}}\right)\|u\|^p - C\|u\|^q
\end{aligned}$$

for some $C > 0$. So there exists $r_+ > 0$ and $\alpha > 0$ such that, if $\|u\| = r_+$ then $\mathscr{E}_+(u) \geq \alpha$.

On the other hand, taking $u \in C_m^-$, $e \in X \setminus C_m^-$ with $e^+ \neq 0$ and $t > 0$, from (f_2') we get

$$\mathscr{E}_+(u+te) \leq \frac{1}{2p}\|u+te\|^p - \frac{2\lambda+1}{2p}\int_\Omega |(u+te)^+|^p dx - \int_\Omega F(x,(u+te)^+)\,dx$$
$$\leq \frac{1}{2p}\|u+te\|^p \left(1 - \int_\Omega \frac{F(x,(u+te)^+)}{((u+te)^+)^p}\frac{((u+te)^+)^p}{\|u+te\|^p}\,dx\right) \to -\infty$$

as $t \to +\infty$. So, there exists $r_- > r_+$ such that $\mathscr{E}_+(u) \leq 0$ when $u \in C_m^- + \mathbb{R}^+ e$ and $\|u\| \geq r_-$.

Again, we define D_-, S_+, Q and H as in Theorem 2.2.4. By Corollary 2.2.5 we have that $(Q, D_- \cup H)$ links S_+ cohomologically in dimension $m+1$ over $\mathbb{Z}_2$. In particular, $(Q, D_- \cup H)$ links S_+. In addition, $\mathscr{E}_+$ is bounded on Q, $\mathscr{E}_+(u) \leq 0$ for every $u \in D_- \cup H$ and $\mathscr{E}_+(u) \geq \alpha > 0$ for every $u \in S_+$. Moreover, by Proposition 5.2.7 $(C)_c$ holds as well.

By Theorem 2.2.2, $\mathscr{E}_+$ admits a critical value $c \geq \alpha$, hence a critical point u with $\mathscr{E}_+(u) > 0$. In particular, we have

$$\begin{aligned}
0 = &-\frac{1}{2}\int\int_{\mathcal{Q}} \frac{J_p(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+ps}}\,dxdy - \int_\Omega |u|^{p-2}uu^-\,dx \\
&+ (\lambda+1)\int_\Omega |u^+|^{p-2}u^+u^-\,dx + \int_\Omega f(x,u^+)u^-\,dx \\
= &-\frac{1}{2}\int\int_{\mathcal{Q}} \frac{J_p(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+ps}}\,dxdy + \int_\Omega (u^-)^p dx.
\end{aligned}$$

From (5.43) we get

$$0 \geq \int\int_{\mathcal{Q}} \frac{|u^-(x)-u^-(y)|^p}{|x-y|^{N+ps}}\,dxdy + \int_\Omega (u^-)^p dx$$

so that $u^- \equiv 0$ and $u \geq 0$. As a consequence, $\mathscr{E}_+(u) = \mathscr{E}(u)$, and so $u \geq 0$ is a nontrivial solution of (5.42).

Arguing in the same way for $\mathscr{E}_-$, we can find a nontrivial negative solution v for (5.42).

By the maximum principle (see, for instance, [25] and [52] for the Robin problem and also [55] for some linear cases), we can conclude that $u > 0$ and $v < 0$ a.e. in $\mathbb{R}^N$. □

Chapter 6

Asymptotically p-linear problems

In this chapter we consider problems of the form

$$\begin{cases} (-\Delta)_p^s u = h(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{6.1}$$

where $h(x,t) = \lambda |t|^{p-2}t + g(x,t)$ with $g(x,t) = o(|t|^{p-1})$ as $t \to \infty$. In light of this, it is clear that $h(x,\cdot)$ is asimptotically p-linear at infinity, that is

$$\lim_{|t|\to\infty} \frac{h(x,t)}{|t|^{p-2}t} = \lambda$$

uniformly a.e. in Ω, for some $\lambda \in (0,\infty)$. We recall that in general a problem of this kind is said to be of resonant type if $\lambda \in \sigma(s,p)$. Otherwise, it it said to be of non-resonant type. However, we shall see that problem (6.1) is of resonat type when $\lambda + 1 \in \sigma(s,p)$.

We first deal with the non resonant case, and under suitable assumptions on h we prove the existence of a nontrivial solution (see Theorem 6.1.1). In the resonant case, we need to make some additional assumptions to prove the existence of a nontrivial solution (see Theorem 6.2.1). All the results in this chapter can be found in [51].

We assume that $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, with

$$H(x,t) = \int_0^t h(x,\tau)\, d\tau \text{ for all } (x,t) \in \Omega \times \mathbb{R},$$

and satisfying the following hypotheses:

(h_1) $|h(x,t)| \leq a(1 + |t|^{r-1})$ a.e. in Ω and for all $t \in \mathbb{R}$, with $a > 0$ and $1 < r < p_s^*$,

(h_2) $\lim_{|t|\to\infty} \frac{h(x,t)}{|t|^{p-2}t} = \lambda$ uniformly a.e. in Ω, with $\lambda > 0$,

(h_3) there exist $\delta > 0$ and $\mu \in (0, p)$ such that

$$h(x,t)t > 0, \quad \text{for } x \in \Omega, \quad 0 < |t| < \delta, \tag{6.2}$$

$$\mu H(x,t) - h(x,t)t \geq 0 \quad \text{for } x \in \Omega, \quad |t| < \delta. \tag{6.3}$$

As usual, we want the existence of solutions, so we define the functional

$$J(u) = \frac{1}{2p}[u]^p - \int_\Omega H(x,t)\, dx,$$

and we look for critical points of J. Here, we have used the usual symbol $[\cdot]$ for the Gagliardo seminorm

$$[u] := \left(\iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dxdy \right)^{1/p}$$

for every $u \in X$.

Remark 6.0.1. We can write J as

$$J(u) = \frac{1}{2p}\|u\|^p - \frac{1}{2p}\|u\|_p^p - \int_\Omega H(x,t)\, dx,$$

which is the functional associated with the problem

$$\begin{cases} (-\Delta)_p^s u + |u|^{p-2}u = \tilde{h}(x,u) & \text{in } \Omega, \\ \mathcal{N}_{s,p}u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases}$$

with $\tilde{h}(x,u) = |u|^{p-2}u + h(x,u)$. So, this problem admits the same solutions as problem (6.1). In light of this, we can see that the assumptions of [62, Theorem 5.7] and [62, Theorem 5.9] are satisfied, hence we just need to verify if the hypotheses hold in order to use them. Moreover, thanks to the added term we have

$$\lim_{|t|\to\infty} \frac{\tilde{h}(x,t)}{|t|^{p-2}t} = \lambda + 1$$

uniformly a.e. in Ω, hence we have resonance when $\lambda + 1 \in \sigma(s,p)$.

Clearly (h_3) implies that $f(x, 0) = 0$ a.e. in Ω, so $u = 0$ is a trivial solution of (6.1). We seek the existence of nontrivial solutions, which are critical points of J. Indeed, if there exists a critical point of J which is not isolated, then we have infinite solutions. So, we can assume that all critical points of J are isolated.

6.1 Non resonant case

in this section we deal with the non-resonant case, and the main result is the following:

Theorem 6.1.1. *If Hypotheses* (h_1)*-*(h_3) *hold with* $\lambda \in (\lambda_k, \lambda_{k+1}) \setminus \sigma(s,p)$ *for some* $k \in \mathbb{N}$*, then problem* (6.1) *admits a nontrivial solution.*

Proof. If $\lambda > 0$, then there exists $k \in \mathbb{N}$ such that $\lambda_k < \lambda + 1 < \lambda_{k+1}$. From [62, Theorem 5.7] there exist some $u \in X$ such that u is a critical point of J and $C^k(J,u) \neq 0$.

We now want to prove that $C^k(J,0) = 0$ for every $k \in \mathbb{N}$. First we note that (6.2) and (6.3) imply that there exist a constant c_0 such that

$$H(x,t) \geq c_0|t|^\mu, \quad \text{for } x \in \Omega, \quad |t| < \delta. \tag{6.4}$$

From (h_1) and (6.4) we have that

$$H(x,t) \geq c_0|t|^\mu - c_1|t|^r, \quad \text{for } x \in \Omega, \quad t \in \mathbb{R} \tag{6.5}$$

for some $c_1 > 0$ and $r \in (p, p_s^*)$. So, taking $u \in X$ and $\tau > 0$ we have

$$\begin{aligned} J(\tau u) &= \frac{\tau^p}{2p}[u]^p - \int_\Omega H(x,\tau u)\,dx \\ &\leq \frac{\tau^p}{2p}[u]^p - \int_\Omega (c_0|\tau u|^\mu - c_1|\tau u|^r)\,dx \\ &\leq \frac{\tau^p}{2p}[u]^p - c_0\tau^\mu\|u\|^\mu_{L^\mu(\Omega)} + c_1\tau^r\|u\|^r_{L^r(\Omega)}. \end{aligned}$$

Since $\mu < p < r$, for given $u \in X$ such that $u \neq 0$, there exists $\tau_0 = \tau_0(u) > 0$ such that

$$J(\tau u) < 0, \quad \text{for every } \tau \in (0, \tau_0). \tag{6.6}$$

Let $u \in X$ be such that $J(u) = 0$ and u is not a constant function. So, there exists $c_2 > 0$ such that $[u]^p \geq c_2\|u\|^p$. Then, from (h_1) and the embedding $X \hookrightarrow L^q(\Omega)$

$$\begin{aligned} \frac{d}{d\tau}J(\tau u)|_{\tau=1} = \langle J'(\tau u), u\rangle &= \frac{1}{2}[u]^p - \int_\Omega h(x,u)u\,dx \\ &= \frac{1}{2}\left(1 - \frac{\mu}{p}\right)[u]^p + \int_\Omega (\mu H(x,u) - h(x,u)u)\,dx \\ &\geq \frac{c_2}{2}\left(1 - \frac{\mu}{p}\right)\|u\|^p - c_3\int_\Omega |u|^r\,dx \\ &\geq \frac{c_2}{2}\left(1 - \frac{\mu}{p}\right)\|u\|^p - c_4\|u\|^r, \end{aligned}$$

for some $r \in (p, p_s^*)$ and $c_3, c_4 > 0$. So we can conclude that there exists $\rho > 0$ such that

$$\frac{d}{d\tau} J(\tau u)|_{\tau=1} > 0, \ \forall u \in X \text{ with } J(u) = 0 \text{ and } 0 < \|u\| \leq \rho. \tag{6.7}$$

Now we fix $\rho > 0$, and we want to prove that (6.7) implies that

$$J(\tau u) < 0, \text{ for } \tau \in (0, 1), \text{ for } u \in X \text{ with } J(u) < 0 \text{ and } \|u\| \leq \rho. \tag{6.8}$$

Clearly, if $\|u\| \leq \rho$ and $J(u) < 0$, then there exists $\vartheta \in (0, 1)$ such that $J(su) < 0$ for all $\tau \in (1 - \vartheta, 1)$ from the continuity of J. Suppose that there is a $\tau_0 \in (0, 1 - \vartheta]$ such that $J(\tau_0 u) = 0$ and $J(\tau u) < 0$ as $\tau_0 < \tau < 1$. Denoting $u_0 := \tau_0 u$, by (6.7) we have

$$\frac{d}{d\tau} J(\tau u_0)|_{\tau=1} > 0.$$

On the other hand $J(\tau u) - J(\tau_0 u) < 0$ implies that

$$\frac{d}{d\tau} J(\tau u)|_{\tau=\tau_0} = \frac{d}{d\tau} J(\tau u_0)|_{\tau=1} \leq 0,$$

which is a contradiction, so (6.8) holds. We notice that if u is a constant function, (6.4) implies (6.8).

Now we define a mapping $T : B_\rho(0) \to [0, 1]$ as

$$T(u) = \begin{cases} 1, & \text{for } u \in B_\rho(0) \text{ with } J(u) \leq 0 \\ \tau, & \text{for } u \in B_\rho(0) \text{ with } J(u) > 0, J(\tau u) = 0, \tau < 1. \end{cases}$$

By (6.6), (6.7) and (6.8), the mapping T is well defined. Moreover, if $J(u) > 0$, then there exists an unique $T(u) \in (0, 1)$ such that

$$J(T(u)u) = 0, \ J(\tau u) < 0 \ \forall \tau \in (0, T(u)) \text{ and } J(\tau u) > 0 \ \forall \tau \in (T(u), 1). \tag{6.9}$$

The Implicit Function Theorem, (6.7) and (6.9) imply that T is continuous in u. Now, we define a mapping $\eta : [0, 1] \times B_\rho(0) \to B_\rho(0)$ as

$$\eta(\tau u) = (1 - \tau)u + \tau T(u)u, \ \tau \in [0, 1], \ u \in B_\rho(0)$$

From the definition of T we have that η is a continuous deformation $(B_\rho(0), B_\rho(0) \setminus \{0\})$ to $(B_\rho(0) \cap J^0, B_\rho(0) \cap J^0 \setminus \{0\})$. Since $B_\rho(0) \setminus \{0\}$ is contractible, by the homotopy invariance of cohomology group, we have

$$C^k(J, 0) = H^k(B_\rho(0) \cap J^0, B_\rho(0) \cap J^0 \setminus \{0\}) = H^k(B_\rho(0), B_\rho(0) \setminus \{0\}) = 0$$

for every $k \in \mathbb{N}$. So, the critical point that we found above cannot be $u = 0$. This concludes the proof. □

6.2 Resonant case

In order to deal with the resonant case, we need to assume additional conditions to have compactness of critical sequences. For all $(x,t) \in \Omega \times \mathbb{R}$ we define

$$\mathcal{H}(x,t) := pH(x,t) - h(x,t)t$$

We have the following result:

Theorem 6.2.1. *If Hypotheses* (h_1)*-*(h_3) *hold with* $\lambda+1 \in \sigma(s,p)$*, and there exist* $k \in \mathbb{N}$*,* $h_0 \in L^1(\Omega)$ *such that one of the following holds:*

(i) $\lambda_k < \lambda+1 \leq \lambda_{k+1}$*,* $\mathcal{H}(x,t) \leq -h_0(x)$ *a.e. in* Ω *and for all* $t \in \mathbb{R}$*, and*

$$\lim_{|t|\to\infty} \mathcal{H}(x,t) = -\infty$$

uniformly a.e. in Ω

(ii) $\lambda_k \leq \lambda+1 < \lambda_{k+1}$*,* $\mathcal{H}(x,t) \geq h_0(x)$ *a.e. in* Ω *and for all* $t \in \mathbb{R}$*, and*

$$\lim_{|t|\to\infty} \mathcal{H}(x,t) = \infty$$

uniformly a.e. in Ω*.*

Then problem (6.1) *admits a nontrivial solution.*

Proof. Since $\lambda+1 \in \sigma(s,p)$, there exists some $k \in \mathbb{N}$ such that $\lambda+1 \in [\lambda_k, \lambda_{k+1}]$, which is a non degenerate interval. We set

$$\Psi(u) = J(u) - \frac{1}{p}\langle J'(u), u\rangle = -\frac{1}{p}\int_\Omega \mathcal{H}(x,u)\,dx$$

for all $u \in X$. Assume we are in the case (i). In order to apply [62, Theorem 5.9] we need to verify if the (H_+) condition holds, that is, Ψ is bounded from below and every sequence $(u_n)_n \subset X$ such that $\|u_n\| \to \infty$ and $v_n = u_n/\|u_n\|$ converges weakly to some $v \neq 0$ as $n \to \infty$ admits a subsequence such that

$$\lim_{n\to\infty} \Psi(\tau u_n) = +\infty, \ \forall \tau \geq 1.$$

Respectively, to have condition (H_-) we ask that Ψ is bounded from above and $\Psi(\tau u_n) \to -\infty$ for every $\tau \geq 1$ (see [62, p. 82]). Clearly, for all $u \in X$ we have

$$\Psi(u) \geq \frac{1}{p}\int_\Omega \mathcal{H}(x,u)\,dx,$$

so Ψ is bounded from below in X. Now let $(u_n)_n \subset X$ be a sequence such that $\|u_n\| \to \infty$ and $v_n = u_n/\|u_n\| \rightharpoonup v \neq 0$ as $n \to \infty$. In particular, we have that $v_n(x) \to v(x)$ a.e. in Ω as $n \to \infty$. From the Fatou Lemma we have, for all $\tau \geq 1$,

$$\liminf_{n\to\infty} \Psi(\tau u_n) \geq -\frac{1}{p} \int_\Omega \liminf_{n\to\infty} \mathcal{H}(x, \|u_n\|\tau v_n)\, dx = \infty,$$

so we can conclude that the (H_+) condition holds. Applying [62, Theorem 5.9] we have that J satisfies the (C) condition and there exists a critical point u such that $C^k(J,u) \neq 0$. As in the proof of Theorem 6.1.1 we can see that $C^k(J,0) = 0$ for every $k \in \mathbb{N}$, so that $u \neq 0$ is a nontrivial solution of (6.1).

For the case (ii) the argument is similar, with the difference that we need to verify condition (H_-) instead of (H_+). □

Chapter 7

Mixed operators

In this chapter we consider several problems involving a mixed operator with both local and nonlocal interations. To do so, first we introduce the operator and explain the setting in which we work.

We let $s \in (0,1)$ and α, $\beta \in [0,+\infty)$ with $\alpha+\beta > 0$, and we consider the mixed operator

$$-\alpha\Delta + \beta(-\Delta)^s. \tag{7.1}$$

As customary, the operator $(-\Delta)^s$ is the fractional Laplacian

$$(-\Delta)^s u(x) := \frac{1}{2}\int_{\mathbb{R}^n} \frac{2u(x) - u(x+\zeta) - u(x-\zeta)}{|\zeta|^{n+2s}}\, d\zeta,$$

where other normalization constants have been removed to ease the notation (in any case, additional normalizing constants do not affect our arguments, and they can also be comprised into the parameter β in (7.1) if one wishes to do so).

The first problem that we consider with this operator is a weighted eigenvalue problem. In this case, we prove the existence of two unbounded sequences of eigenvalues, giving a characterization of the first eigenvalue of each sequence (see Proposition 7.1.1). We also show that the first positive eigenvalue is simple and its corresponding eigenfunction can be taken to be nonnegative (see Proposition 7.1.2).

Then we study a problem involving a logistic equation, and under suitable assumptions we prove that there exists a nonnegative solution, obtained with a minimization argument (see Theorem 7.2.1). Depending on the resource term, we give conditions that allow to prove that either the only possible solution is the one identically zero, or that there exists a nontrivial solution, see Theorem 7.2.2 for the case in which the resource is favorable to life and Theorem 7.2.4 for the case in which it is hostile to life.

We also study a problem with a source term that has a linear behaviour at infinity. Depending on linear term we face two cases, in the first one the functional associated to the problem is coercive, while in the second the functional has a natural geometry of saddle type. In both cases we prove the existence of a nontrivial solution (see Theorem 7.3.1).

Then we study a parabolic problem in presence of the mixed operator. Similarly to Section 3.3 we prove, for clasical solutions, that total mass is preserved (see Proposition 7.4.1) and that the energy is decreasing in time (see Proposition 7.4.2). In addition, we prove that classical solutions converge to a constant as time goes to infinity (see Proposition 7.4.3).

Lastly we deal with a superlinear problem involving the mixed operator. As in Section 5.2, we focus our attention on the case where the nonlinear term does not satisfy the Ambrosetti-Rabinowitz condition. Under suitable assumptions on the nonlinearity, we prove the existence of two nontrivial constant sign solutions (see Theorem 7.5.2).

The results in Sections 7.1 and 7.2 are taken from [26, 27], while the results in Sections 7.3, 7.4 and 7.5 are not taken from a published article since they are part of a work in progress.

As usual, the mathematical framework in (7.1) is endowed by a spatial domain on which the corresponding equation takes place. For this, we take a bounded open set $\Omega \subset \mathbb{R}^n$ of class C^1. When $\beta = 0$, we take the additional hypothesis that

$$\Omega \text{ is connected.} \tag{7.2}$$

From the biological point of view, Ω represents the natural environment inhabited by a given biological population, whose density is described by a function $u : \mathbb{R}^n \to \mathbb{R}$ (as customary in nonlocal problems, one has to prescribe functions in all of the space to make sense of the fractional diffusive operators).

We prescribe external conditions to u in order to make Ω an ecological niche. To this end, see [27], we set a variational formulation related to the operator in (7.1) which endows the equation in the set Ω with a suitable Neumann condition (see [29] for a thorough description of the biological motivation). The functional space that we consider is

$$X_{\alpha,\beta} = X_{\alpha,\beta}(\Omega) := \begin{cases} H^1(\Omega) & \text{if } \beta = 0, \\ H^s_\Omega & \text{if } \alpha = 0, \\ H^1(\Omega) \cap H^s_\Omega & \text{if } \alpha\beta \neq 0, \end{cases} \tag{7.3}$$

where

$$H^s_\Omega := \left\{ u : \mathbb{R}^n \to \mathbb{R} \ \text{ s.t. } u \in L^2(\Omega) \ \text{ and } \ \iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy < \infty \right\},$$

and $\mathcal{Q}$ is the cross-shaped set on Ω given by

$$\mathcal{Q} := \big(\Omega \times \Omega\big) \cup \big(\Omega \times (\mathbb{R}^n \setminus \Omega)\big) \cup \big((\mathbb{R}^n \setminus \Omega) \times \Omega\big).$$

We observe that $X_{\alpha,\beta}$ is a Hilbert space with respect to the scalar product

$$\begin{aligned}(u,v)_{X_{\alpha,\beta}} := &\int_\Omega u(x)v(x)\,dx + \alpha \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx \\ &+ \frac{\beta}{2}\iint_{\mathcal{Q}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy,\end{aligned} \tag{7.4}$$

for every $u, v \in X_{\alpha,\beta}$.

We also define the seminorm

$$[u]^2_{X_{\alpha,\beta}} := \frac{\alpha}{2}\int_\Omega |\nabla u(x)|^2\,dx + \frac{\beta}{4}\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy. \tag{7.5}$$

Given $f \in L^2(\Omega)$, we say that $u \in X_{\alpha,\beta}$ is a solution of

$$-\alpha\Delta u + \beta(-\Delta)^s u = f \qquad \text{in } \Omega \tag{7.6}$$

with (α,β)-Neumann condition if

$$\begin{aligned}\alpha\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx + \frac{\beta}{2}\iint_{\mathcal{Q}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy \\ = \int_\Omega f(x)\,v(x)\,dx,\end{aligned} \tag{7.7}$$

for every $v \in X_{\alpha,\beta}$.

We remark that, formally, the external condition in (7.7) can be detected by taking v with $v = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$ (which produces a normal derivative prescription along $\partial\Omega$) and then by taking $v = 0$ in $\overline{\Omega}$ (which produces a nonlocal prescription in $\mathbb{R}^n \setminus \overline{\Omega}$): that is, formally, the external condition in (7.7) can be written in the form

$$\begin{cases} \mathscr{N}_s u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.8}$$

where ν is the exterior normal to Ω, and with the usual notation

$$\mathscr{N}_s u(x) := \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy \qquad \text{for every } \ x \in \mathbb{R}^n \setminus \overline{\Omega}, \tag{7.9}$$

the first condition in (7.8) being dropped when $\alpha = 0$, the second condition in (7.8) being dropped when $\beta = 0$.

We recall that the nonlocal Neumann prescription in (7.9) is precisely the one introduced in [28] in light of probabilistic consideration (i.e., a particle following a $\frac{s}{2}$-stable process is sent back to the original domain by following the same process). Also, as shown in [28], the setting in (7.9) provides a coherent functional analysis setting.

Moreover, we stress that the setting in (7.7) provides a "zero-flux" condition, in the sense that if (7.6) has a solution, then necessarily

$$\int_\Omega f(x)\, dx = 0, \tag{7.10}$$

as it can be seen by taking $v := 1$ in (7.7).

7.1 The eigenvalue problem

In this section we consider a generalized eigenvalue problem associated to equation (7.6) with (α, β)-Neumann condition.

Namely, we let $m : \Omega \to \mathbb{R}$ and we consider the weighted eigenvalue equation

$$\begin{cases} -\alpha\Delta u + \beta(-\Delta)^s u = \lambda m u & \text{in } \ \Omega, \\ \text{with } (\alpha, \beta)\text{-Neumann condition.} \end{cases} \tag{7.11}$$

According to (7.7) the notion of solution in (7.11) is in the weak sense in the space $X_{\alpha,\beta}$: namely we say that $u \in X_{\alpha,\beta}$ is a solution of (7.11) if

$$\begin{aligned} \alpha \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx + \frac{\beta}{2} \iint_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\, dx\, dy \\ = \lambda \int_\Omega m(x) u(x) v(x)\, dx, \end{aligned} \tag{7.12}$$

for every $v \in X_{\alpha,\beta}$.

To deal with the integrability condition of the weight m, it is convenient to consider the following "critical" exponent:

$$\underline{q} := \begin{cases} \dfrac{2^*}{2^*-2} & \text{if } \beta = 0 \text{ and } n > 2, \\ \dfrac{2^*_s}{2^*_s-2} & \text{if } \beta \neq 0 \text{ and } n > 2s, \\ 1 & \text{if } \beta = 0 \text{ and } n \leq 2, \text{ or if } \beta \neq 0 \text{ and } n \leq 2s, \end{cases}$$
$$= \begin{cases} \dfrac{n}{2} & \text{if } \beta = 0 \text{ and } n > 2, \\ \dfrac{n}{2s} & \text{if } \beta \neq 0 \text{ and } n > 2s, \\ 1 & \text{if } \beta = 0 \text{ and } n \leq 2, \text{ or if } \beta \neq 0 \text{ and } n \leq 2s. \end{cases} \tag{7.13}$$

As customary, the exponent 2^*_s denotes the fractional Sobolev critical exponent for $n > 2s$ and it is equal to $\frac{2n}{n-2s}$. Similarly, the exponent 2^* denotes the classical Sobolev critical exponent for $n > 2$ and it is equal to $\frac{2n}{n-2}$.

Furthermore, we suppose that

$$m \in L^q(\Omega), \quad \text{for some } q \in \big(\underline{q}, +\infty\big], \tag{7.14}$$

where $\underline{q}$ is given in (7.13).

In this setting, problem (7.11) admits a spectral decomposition of classical flavor, according to the following result:

Proposition 7.1.1. *Suppose that m^+, $m^- \not\equiv 0$ and*[1] *that*

$$\int_\Omega m(x)\,dx \neq 0. \tag{7.15}$$

Then, problem (7.11) *admits two unbounded sequences of eigenvalues:*

$$\cdots \leq \lambda_{-2} \leq \lambda_{-1} < \lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \ .$$

In particular, if

$$\int_\Omega m(x)\,dx < 0,$$

then

$$\lambda_1 = \min_{u \in X_{\alpha,\beta}} \left\{ [u]^2_{X_{\alpha,\beta}} \ \ s.t. \ \int_\Omega m(x) u^2(x)\,dx = 1 \right\} \tag{7.16}$$

[1] As customary, we use the standard notation

$$m^+(x) := \max\{0, m(x)\} \qquad \text{and} \qquad m^-(x) := \max\{0, -m(x)\}.$$

where we use the notation in (7.5). *If instead*

$$\int_\Omega m(x)\,dx > 0,$$

then

$$\lambda_{-1} = -\min_{u\in X_{\alpha,\beta}} \left\{ [u]^2_{X_{\alpha,\beta}} \;\; s.t. \;\; \int_\Omega m(x)u^2(x)\,dx = -1 \right\}.$$

The first positive eigenvalue λ_1, as given by Proposition 7.1.1, has the following structural properties:

Proposition 7.1.2. *Suppose that* $m^+ \not\equiv 0$ *and*

$$\int_\Omega m(x)\,dx < 0.$$

Then, the first positive eigenvalue λ_1 *of* (7.11) *is simple, and the first eigenfunction* e *can be taken such that* $e \geq 0$.

A similar statement holds if $m^- \not\equiv 0$ *and*

$$\int_\Omega m(x)\,dx > 0.$$

To deal with the eigenvalue problem in (7.11), it is convenient to recall the notation in (7.3) and to introduce the space

$$V_m := \left\{ u \in X_{\alpha,\beta} \;\; \text{s.t.} \;\; \int_\Omega m(x)u(x)\,dx = 0 \right\}. \tag{7.17}$$

To ease the notation, we will simply write V instead of V_m in what follows. We observe that, in view of (7.10),

$$\text{all the eigenfunctions of problem (7.11) belong to } V. \tag{7.18}$$

The proofs of Propositions 7.1.1 and 7.1.2 rely on classical functional analysis, revisited in a mixed local-nonlocal framework. We start these arguments by pointing out that a Poincaré-type inequality holds in the space V introduced in (7.17):

Lemma 7.1.3. *Let* m *be such that*

$$\int_\Omega m(x)\,dx \neq 0. \tag{7.19}$$

Then, recalling the notation in (7.5), *we have that*

$$\int_\Omega u^2(x)\,dx \leq C[u]^2_{X_{\alpha,\beta}}, \tag{7.20}$$

for every $u \in V$, *where* $C > 0$ *depends only on* n, Ω, s *and* m.

Proof. We argue by contradiction and we suppose that there exists a sequence of functions $u_k \in V$ such that

$$\int_\Omega u_k^2(x)\,dx = 1 \tag{7.21}$$

and

$$[u_k]^2_{X_{\alpha,\beta}} < \frac{1}{k}. \tag{7.22}$$

In particular, the sequence $(u_k)_k$ is bounded in $X_{\alpha,\beta}$ uniformly in k. As a consequence, from the compact embedding of $X_{\alpha,\beta}$ in $L^2(\Omega)$ (see e.g. Corollary 7.2 in [30] if $\alpha = 0$), we have that, up to a subsequence, u_k converges to some function $u \in L^2(\Omega)$ as $k \to +\infty$. Moreover, u_k converges to u a.e. in Ω as $k \to +\infty$, and $|u_k| \le h$ for some $h \in L^2(\Omega)$ for every $k \in \mathbb{N}$ (see e.g. Theorem IV.9 in [9]).

As a result, since $u_k \in V$, we can apply the Dominated Convergence Theorem to conclude that

$$\int_\Omega m(x)u(x)\,dx = 0. \tag{7.23}$$

In addition, we deduce from (7.21) that

$$\int_\Omega u^2(x)\,dx = 1. \tag{7.24}$$

On the other hand, by the Fatou Lemma, the lower semicontinuity of the L^2-norm and (7.22) we have that

$$\begin{aligned} &\frac{\alpha}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\beta}{4}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad\le \liminf_{k\to+\infty}\left(\frac{\alpha}{2}\int_\Omega |\nabla u_k|^2\,dx + \frac{\beta}{2}\int_\Omega\int_\Omega \frac{|u_k(x)-u_k(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right) \\ &\quad\le \lim_{k\to+\infty}\frac{1}{k} = 0. \end{aligned} \tag{7.25}$$

Now, if $\beta = 0$, this says that

$$\int_\Omega |\nabla u|^2\,dx = 0,$$

which implies that u is constant in Ω, thanks to (7.2). If instead $\beta \ne 0$, we have from (7.25) that

$$\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = 0,$$

which gives that u is constant in Ω. Hence in both case, we have that u is constant in Ω.

Moreover, we observe that u cannot vanish identically in Ω, in light of (7.24). Using these observations into (7.23) we conclude that

$$\int_\Omega m(x)\,dx = 0,$$

which is in contradiction with (7.19). This completes the proof of formula (7.20). □

We notice that, thanks to (7.20), the seminorm in (7.5) is actually a norm on the space V and it is equivalent to the norm on $X_{\alpha,\beta}$ given by (7.4). Moreover, the scalar product defined as

$$\langle u, v\rangle_{X_{\alpha,\beta}} := \alpha \int_\Omega \nabla u \cdot \nabla v\,dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy \tag{7.26}$$

is equivalent to the one in $X_{\alpha,\beta}$ given by (7.4). In this setting, we also denote

$$\|u\|_V := \sqrt{\langle u, v\rangle_{X_{\alpha,\beta}}}.$$

To complete the functional setting for the eigenvalue problem in (7.11), we also remark that V is closed with respect to the weak convergence:

Lemma 7.1.4. *The space V introduced in* (7.17) *is closed with respect to the weak convergence in V.*

Proof. We take a sequence of functions $u_j \in V$ weakly converging to some u, and we claim that $u \in V$. Indeed, we have that u_j weakly converges to u in $X_{\alpha,\beta}$, and $u \in X_{\alpha,\beta}$. Furthermore, by the compact embeddings (see e.g. Corollary 7.2 in [30] if $\alpha = 0$), $u_j \to u$ in $L^p(\Omega)$ for any $p \in [1, 2^*_s)$ if $\alpha = 0$ and for any $p \in [1, 2^*)$ if $\alpha \neq 0$. Moreover, u_j converges to u a.e. in Ω, and $|u_j| \le h$ for some $h \in L^p(\Omega)$ (see e.g. Theorem IV.9 in [9]). As a result, since $u_j \in V$, recalling (7.14), we can apply the Dominated Convergence Theorem to conclude that

$$\int_\Omega m(x)u(x)\,dx = 0,$$

which proves that $u \in V$, thus completing the proof of Lemma 7.1.4. □

With this preliminary work, we can give the proofs of Propositions 7.1.1 and 7.1.2 by relying on functional analysis methods:

Proof of Proposition 7.1.1. We notice that

$$\text{the simple eigenvalue } \lambda_0 = 0 \text{ has only constant functions as eigenfunctions.} \tag{7.27}$$

Indeed, if u is an eigenfunction associated to $\lambda_0 = 0$, then, by (7.12),

$$\alpha \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx + \frac{\beta}{2} \iint_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}}\, dx\, dy = 0, \tag{7.28}$$

for all functions $v \in X_{\alpha,\beta}$. In particular, taking u as test function in (7.28), we obtain that

$$\alpha \int_\Omega |\nabla u(x)|^2\, dx + \frac{\beta}{2} \iint_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\, dx\, dy = 0. \tag{7.29}$$

Now, if $\beta = 0$, formula (7.29) implies that

$$\int_\Omega |\nabla u(x)|^2\, dx = 0.$$

This, together with (7.2), gives that u is constant in Ω, thus proving (7.27) in this case.

If instead $\beta \neq 0$, we deduce from (7.29) that

$$\iint_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\, dx\, dy = 0,$$

which implies (7.27).

Now, to obtain the other eigenvalues, we restrict to the space V introduced in (7.17). We point out that the assumption in (7.15) guarantees that the Poincarè inequality in (7.20) holds true on the space V.

Also, we define the linear operator $T : V \to V$ by

$$\langle Tv, w \rangle_{X_{\alpha,\beta}} = \int_\Omega m(x) v(x) w(x)\, dx, \tag{7.30}$$

for every v, $w \in V$.

It is easy to see that T is symmetric. Furthermore, we claim that

$$T \text{ is compact.} \tag{7.31}$$

To prove this, we let $(u_j)_j$ be a bounded sequence in V. Then, $(u_j)_j$ is a bounded sequence in $X_{\alpha,\beta}$, and therefore there exists $u \in X_{\alpha,\beta}$ such that u_j weakly converges to u in $X_{\alpha,\beta}$ as $j \to +\infty$. Moreover, from Lemma 7.1.4, we have that $u \in V$.

Now, by the compact embeddings,

$$u_j \to u \text{ in } L^p(\Omega) \text{ for any } p \in [1, 2_s^*) \text{ if } \alpha = 0 \text{ and for any } p \in [1, 2^*) \text{ if } \alpha \neq 0. \tag{7.32}$$

Using (7.30) with $v := u_j - u$ and $w := Tu_j - Tu$, we deduce that

$$\|Tu_j - Tu\|_V^2 = \langle T(u_j - u), Tu_j - T_u \rangle_{X_{\alpha,\beta}} = \int_\Omega m(u_j - u)\big(Tu_j - Tu\big)\, dx. \tag{7.33}$$

Now we apply Hölder's inequality with exponents q, as given in (7.14), p, as given by (7.32), and either 2_s^* if $\alpha = 0$ or 2^* if $\alpha \neq 0$. In this way, using also the continuous embedding of V either in $L^{2_s^*}(\Omega)$ if $\alpha = 0$ or $L^{2^*}(\Omega)$ if $\alpha \neq 0$, we obtain from (7.33) that

$$\|Tu_j - Tu\|_V^2 \le C\|m\|_{L^q(\Omega)}\|u_j - u\|_{L^p(\Omega)}\|Tu_j - Tu\|_V,$$

for some positive constant C independent of j. This implies that

$$\|Tu_j - Tu\|_V \le C\|m\|_{L^q(\Omega)}\|u_j - u\|_{L^p(\Omega)}.$$

Accordingly, recalling (7.32), we obtain that $Tu_j \to Tu$ in V as $j \to +\infty$. This completes the proof of (7.31).

Now we observe that, in light of (7.12), and recalling (7.26) and (7.30), we can write the weak formulation of problem (7.11) as

$$\langle u, v \rangle_{X_{\alpha,\beta}} = \lambda \langle Tu, v \rangle_{X_{\alpha,\beta}} \qquad \text{for all } v \in X_{\alpha,\beta}. \tag{7.34}$$

Therefore, we can apply standard results in spectral theory of self-adjoint and compact operators to obtain the existence and the variational characterization of eigenvalues (see e.g. [21, Proposition 1.10]; see also [10] and the references therein for related classical results). □

Proof of Proposition 7.1.2. We first observe that if $\beta \neq 0$ and w is an eigenfunction according to (7.11), then

$$w \equiv 0 \text{ in } \Omega \text{ entails that } w \equiv 0 \text{ in the whole of } \mathbb{R}^n. \tag{7.35}$$

To check this, suppose that $w \equiv 0$ in Ω and write (7.11) explicitly as in (7.12), namely

$$\begin{aligned} \alpha \int_\Omega \nabla w(x) \cdot \nabla v(x)\, dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(w(x) - w(y))(v(x) - v(y))}{|x - y|^{n+2s}}\, dx\, dy \\ = \lambda \int_\Omega m(x)\, w(x)\, v(x)\, dx \end{aligned} \tag{7.36}$$

for all functions $v \in X_{\alpha,\beta}$. In particular, choosing $v := w$ in (7.36),

$$0 = \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(w(x) - w(y))^2}{|x - y|^{n+2s}} \, dx \, dy = \beta \iint_{\Omega \times (\mathbb{R}^n \setminus \Omega)} \frac{w^2(y)}{|x - y|^{n+2s}} \, dx \, dy.$$

Whence, if $\beta \neq 0$, it follows that $w(y) = 0$ for each $y \in \Omega$, thus establishing (7.35).

Now, we prove that

$$\text{all the eigenfunctions corresponding to } \lambda_1 \text{ do not change sign.} \tag{7.37}$$

For this, we let u be an eigenfunction corresponding to the first positive eigenvalue λ_1. In particular, recalling (7.16), we have that $u \in X_{\alpha,\beta}$ and

$$\int_{\Omega} m(x) u^2(x) \, dx = 1. \tag{7.38}$$

If u is either nonnegative or nonpositive, then (7.37) is established. Hence, we are left with the case in which u changes sign in Ω. In this case, we have that both $u^+ \not\equiv 0$ and $u^- \not\equiv 0$, and we claim that

$$\text{both } u^+ \text{ and } u^- \text{ are eigenfunctions corresponding to } \lambda_1. \tag{7.39}$$

To this end, we notice that

$$\int_{\Omega} u^2(x) \, dx = \int_{\Omega} (u^+(x))^2 \, dx + \int_{\Omega} (u^-(x))^2 \, dx. \tag{7.40}$$

Moreover, recalling (7.5), by inspection one sees that

$$\begin{aligned}
&[u]^2_{X_{\alpha,\beta}} \\
&= \alpha \int_{\Omega} |\nabla u|^2 \, dx + \beta \iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \\
&= \alpha \int_{\Omega} \left(|\nabla u^+|^2 + |\nabla u^-|^2 \right) \, dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{|u^+(x) - u^+(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \\
&\qquad + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{|u^-(x) - u^-(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \\
&\qquad - \beta \iint_{\mathcal{Q}} \frac{(u^+(x) - u^+(y))(u^-(x) - u^-(y))}{|x - y|^{n+2s}} \, dx \, dy \\
&\geq [u^+]^2_{X_{\alpha,\beta}} + [u^-]^2_{X_{\alpha,\beta}}.
\end{aligned} \tag{7.41}$$

This and (7.40) imply that u^+, $u^- \in X_{\alpha,\beta}$.

Also, in light of (7.38), we have that

$$1 = \int_\Omega m(x)u^2(x)\,dx = \int_\Omega m(x)(u^+(x))^2\,dx + \int_\Omega m(x)(u^-(x))^2\,dx.$$

Hence, using this and (7.41), and recalling the characterization of λ_1 given in (7.16),

$$\begin{aligned}\frac{1}{\lambda_1} = \frac{1}{[u]^2_{X_{\alpha,\beta}}} &= \frac{\displaystyle\int_\Omega m(x)u^2(x)\,dx}{[u]^2_{X_{\alpha,\beta}}}\\ &\le \frac{\displaystyle\int_\Omega m(x)(u^+(x))^2\,dx + \int_\Omega m(x)(u^-(x))^2\,dx}{[u^+]^2_{X_{\alpha,\beta}} + [u^-]^2_{X_{\alpha,\beta}}}.\end{aligned} \tag{7.42}$$

Now we claim that, for any a_1, a_2, b_1, $b_2 > 0$, either

$$\frac{a_1+a_2}{b_1+b_2} = \frac{a_1}{b_1} = \frac{a_2}{b_2}, \tag{7.43}$$

or

$$\frac{a_1+a_2}{b_1+b_2} < \max\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}\right\}. \tag{7.44}$$

Indeed, if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, then

$$\frac{a_1+a_2}{b_1+b_2} = \frac{a_2}{b_2}\cdot\frac{\frac{a_1}{a_2}+1}{\frac{b_1}{b_2}+1} = \frac{a_2}{b_2}\cdot\frac{\frac{a_1}{a_2}+1}{\frac{a_1}{a_2}+1} = \frac{a_2}{b_2},$$

that is (7.43). If instead we suppose that $\frac{a_1}{b_1} > \frac{a_2}{b_2}$ (being the case in which $\frac{a_1}{b_1} < \frac{a_2}{b_2}$ similar), then

$$\frac{a_1+a_2}{b_1+b_2} = \frac{b_1(a_1+a_2)}{b_1(b_1+b_2)} < \frac{a_1b_1+a_1b_2}{b_1(b_1+b_2)} = \frac{a_1(b_1+b_2)}{b_1(b_1+b_2)} = \frac{a_1}{b_1},$$

which proves (7.44).

Now, if we suppose that

$$\frac{\displaystyle\int_\Omega m(x)(u^+(x))^2\,dx}{[u^+]^2_{X_{\alpha,\beta}}} > \frac{\displaystyle\int_\Omega m(x)(u^-(x))^2\,dx}{[u^-]^2_{X_{\alpha,\beta}}}$$

then we deduce from (7.42) and (7.44), applied here with

$$\begin{aligned}&a_1 := \int_\Omega m(x)(u^+(x))^2\,dx, \quad a_2 := \int_\Omega m(x)(u^-(x))^2\,dx,\\ &b_1 := [u^+]^2_{X_{\alpha,\beta}}, \quad \text{and} \quad b_2 := [u^-]^2_{X_{\alpha,\beta}},\end{aligned}$$

that

$$\frac{1}{\lambda_1} < \frac{\displaystyle\int_\Omega m(x)(u^+(x))^2\,dx}{[u^+]^2_{X_{\alpha,\beta}}},$$

which contradicts the minimality of λ_1. Similarly, if

$$\frac{\displaystyle\int_\Omega m(x)(u^+(x))^2\,dx}{[u^+]^2_{X_{\alpha,\beta}}} < \frac{\displaystyle\int_\Omega m(x)(u^-(x))^2\,dx}{[u^-]^2_{X_{\alpha,\beta}}},$$

then

$$\frac{1}{\lambda_1} < \frac{\displaystyle\int_\Omega m(x)(u^-(x))^2\,dx}{[u^-]^2_{X_{\alpha,\beta}}},$$

which is again a contradiction with the minimality of λ_1.

As a consequence, we have that

$$\frac{\displaystyle\int_\Omega m(x)(u^+(x))^2\,dx}{[u^+]^2_{X_{\alpha,\beta}}} = \frac{\displaystyle\int_\Omega m(x)(u^-(x))^2\,dx}{[u^-]^2_{X_{\alpha,\beta}}}.$$

In this case, we can apply (7.43) and we obtain from (7.42) that

$$\frac{1}{\lambda_1} \le \frac{\displaystyle\int_\Omega m(x)(u^+(x))^2\,dx}{[u^+]^2_{X_{\alpha,\beta}}} = \frac{\displaystyle\int_\Omega m(x)(u^-(x))^2\,dx}{[u^-]^2_{X_{\alpha,\beta}}},$$

that is

$$\lambda_1 \ge \frac{[u^+]^2_{X_{\alpha,\beta}}}{\displaystyle\int_\Omega m(x)(u^+(x))^2\,dx} = \frac{[u^-]^2_{X_{\alpha,\beta}}}{\displaystyle\int_\Omega m(x)(u^-(x))^2\,dx}. \tag{7.45}$$

Now, if the inequality in (7.45) is strict, we have a contradiction with the minimality of λ_1. Accordingly,

$$\lambda_1 = \frac{[u^+]^2_{X_{\alpha,\beta}}}{\displaystyle\int_\Omega m(x)(u^+(x))^2\,dx} = \frac{[u^-]^2_{X_{\alpha,\beta}}}{\displaystyle\int_\Omega m(x)(u^-(x))^2\,dx}.$$

This implies that u^+ and u^- are both eigenfunctions corresponding to λ_1 (unless they are trivial) thus establishing (7.39).

Our next claim is to prove that

$$\text{either } u \equiv u^+ \text{ or } u \equiv u^-. \tag{7.46}$$

We observe that, if $\beta = 0$, then (7.46) follows from the standard maximum principle for the Laplace operator (see e.g. [31]).

If instead $\beta \neq 0$, we use (7.39) and (7.42) to see that

$$\frac{1}{\lambda_1} \leq \frac{\displaystyle\int_\Omega m(x)(u^+(x))^2\,dx + \int_\Omega m(x)(u^-(x))^2\,dx}{[u^+]^2_{X_{\alpha,\beta}} + [u^-]^2_{X_{\alpha,\beta}}} = \frac{1}{\lambda_1}.$$

In particular, equality holds in the latter formula, and accordingly, recalling (7.41), we have that

$$0 = -\iint_{\mathcal{Q}} \frac{(u^+(x) - u^+(y))(u^-(x) - u^-(y))}{|x-y|^{n+2s}}\,dx\,dy = \iint_{\mathcal{Q}} \frac{2u^+(x)u^-(y)}{|x-y|^{n+2s}}\,dx\,dy.$$

This gives that

$$u^+(x)u^-(y) = 0 \qquad \text{for all } (x,y) \in \mathcal{Q}. \tag{7.47}$$

We can also suppose that $u^+ \not\equiv 0$ (in $\mathbb{R}^n$ if $\beta \neq 0$ and in Ω if $\beta = 0$), otherwise $u \equiv u^-$ and we are done. This and (7.35) give that $u^+ \not\equiv 0$ in Ω. Hence, we can take $\bar{x} \in \Omega$ such that $u^+(\bar{x}) \neq 0$. From this and (7.47), we obtain that

$$u^+(\bar{x})u^-(y) = 0 \qquad \text{for all } y \in \mathbb{R}^n.$$

As a consequence, we find that $u^- \equiv 0$ in $\mathbb{R}^n$, which establishes (7.46).

In turn, the claim in (7.46) implies the one in (7.37), as desired.

We now prove that λ_1 is simple. First we show that

$$\text{the geometric multiplicity of } \lambda_1 \text{ is } 1. \tag{7.48}$$

For this, let u_1 and u_2 be eigenfunctions corresponding to λ_1. From (7.37) we know that u_2 does not change sign, hence (up to exchanging u_2 with $-u_2$), we can suppose that $u_2 \geq 0$ (in $\mathbb{R}^n$, if $\beta \neq 0$, and in Ω, if $\beta = 0$).

From this and (7.35), it follows that

$$\int_\Omega u_2(x)\,dx > 0.$$

As a result, we can define

$$a := \frac{\displaystyle\int_\Omega u_1(x)\,dx}{\displaystyle\int_\Omega u_2(x)\,dx},$$

and we find that

$$\int_\Omega \big(u_1(x) - au_2(x)\big)\,dx = 0. \tag{7.49}$$

In addition, from (7.37), we know that the eigenfunction $u_1 - au_2$ does not change sign, and therefore (7.49) entails that $u_1 - au_2 \equiv 0$ in Ω. This and (7.35) show that $u_1 - au_2 \equiv 0$ also in $\mathbb{R}^n$ when $\beta \neq 0$, and this proves that u_1 and u_2 are linearly dependent, giving (7.48), as desired.

Finally, we prove that

$$\text{the algebraic multiplicity of } \lambda_1 \text{ is } 1. \tag{7.50}$$

To this end, we recall the notation in (7.17) and (7.30), and we claim that

$$\operatorname{Ker}\big((I - \lambda_1 T)^2\big) = \operatorname{Ker}(I - \lambda_1 T), \tag{7.51}$$

where I is the identity in V.

To prove (7.51), let $u \in \operatorname{Ker}\big((I - \lambda_1 T)^2\big)$. Then, setting $U := u - \lambda_1 Tu$, we have that $U - \lambda_1 TU = 0$, and accordingly, by (7.34), U is an eigenfunction corresponding to λ_1.

From this fact and (7.48), we conclude that $U = te_1$ for some $t \in \mathbb{R}$, where e_1 is a given eigenfunction corresponding to λ_1.

As a result,

$$\begin{aligned} t\langle e_1, e_1\rangle_{X_{\alpha,\beta}} &= \langle U, e_1\rangle_{X_{\alpha,\beta}} = \langle u - \lambda_1 Tu, e_1\rangle_{X_{\alpha,\beta}} \\ &= \langle u, e_1 - \lambda_1 Te_1\rangle_{X_{\alpha,\beta}} = \langle u, 0\rangle_{X_{\alpha,\beta}} = 0, \end{aligned}$$

which implies that $t = 0$. This yields that $U = 0$ and therefore $u \in \operatorname{Ker}(I - \lambda_1 T)$. This shows that $\operatorname{Ker}\big((I - \lambda_1 T)^2\big) \subseteq \operatorname{Ker}(I - \lambda_1 T)$, and the other inclusion is obvious.

The proof of (7.51) is therefore complete. From (7.51), we obtain that for all $k \in \mathbb{N}$ with $k \geq 1$,

$$\operatorname{Ker}\big((I - \lambda_1 T)^k\big) = \operatorname{Ker}(I - \lambda_1 T),$$

and thus

$$\bigcup_{k=1}^{+\infty} \operatorname{Ker}\big((I - \lambda_1 T)^k\big) = \operatorname{Ker}(I - \lambda_1 T).$$

The latter has dimension 1, thanks to (7.48), and therefore the claim in (7.50) is established. □

7.2 Logistic equation

In this section we want to study a problem involving a logistic equation. To do so, we first explain the mathematical framework in which we work. We consider a bounded open set $\Omega \subset \mathbb{R}^n$ with boundary of class C^1: that is, we suppose that there exist $R > 0$ and $p_1, \dots, p_K \in \partial\Omega$ such that $\partial\Omega \subset B_R(p_1) \cup \dots \cup B_R(p_K)$, and, for each $i \in \{1, \dots, K\}$,

$$\begin{gathered}\text{the set } \Omega \cap B_R(p_i) \text{ is } C^1\text{-diffeomorphic to} \\ B_1^+ := \{(x_1, \dots, x_n) \in B_1 \text{ s.t. } x_n > 0\}.\end{gathered} \tag{7.52}$$

Given $s \in (0,1)$, α, $\beta \in [0, +\infty)$, with $\alpha + \beta > 0$, $m : \Omega \to \mathbb{R}$, $\mu : \Omega \to \big[\underline{\mu}, +\infty\big)$, with $\underline{\mu} > 0$, $\tau \in [0, +\infty)$ and $J \in L^1(\mathbb{R}^n, [0, +\infty))$ with

$$J(x) = J(-x) \tag{7.53}$$

and

$$\int_{\mathbb{R}^n} J(x)\,dx = 1, \tag{7.54}$$

we consider the mixed order logistic equation

$$-\alpha\Delta u + \beta(-\Delta)^s u = (m - \mu u)u + \tau\ J \star u \qquad \text{in } \Omega, \tag{7.55}$$

where

$$J \star u(x) := \int_\Omega J(x - y)\, u(y) dy.$$

When $\beta = 0$, we take the additional hypothesis that

$$\Omega \text{ is connected.} \tag{7.56}$$

In light of this, the problem we want to study is

$$\begin{cases} -\alpha\Delta u + \beta(-\Delta)^s u = (m - \mu u)u + \tau\ J \star u & \text{in } \Omega, \\ \text{with } (\alpha, \beta)\text{-Neumann condition.} \end{cases} \tag{7.57}$$

We say that $u \in X_{\alpha,\beta}$ is a solution of (7.57) if

$$\begin{aligned} \alpha \int_\Omega \nabla u(x) \cdot \nabla v(x)\,dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}}\,dx\,dy \\ = \int_\Omega \Big(\big(m(x) - \mu(x)u(x)\big)u(x) + \tau(x)\ J \star u(x) \Big) v(x)\,dx \end{aligned} \tag{7.58}$$

for all functions $v \in X_{\alpha,\beta}$.

We can give the first existence result:

Theorem 7.2.1. *Assume that*

$$m \in L^q(\Omega), \text{ for some } q \in \big(\underline{q}, +\infty\big]$$
$$\text{and } (m+\tau)^3\mu^{-2} \in L^1(\Omega).$$

Then, there exists a nonnegative solution of (7.57) *which can be obtained as a minimum of an energy functional.*

The next results give us conditions to ensure that there exist a nonnegative solution, or that the only possible solution is the trivial one.

Theorem 7.2.2. *Assume that*

$$m \in L^q(\Omega), \text{ for some } q \in \big(\underline{q}, +\infty\big]$$
$$\text{and } (m+\tau)^3\mu^{-2} \in L^1(\Omega).$$

Then,

(i) *if* $m \equiv 0$ *and* $\tau = 0$, *then the only solution of* (7.57) *is the one identically zero;*

(ii) *if*

$$\int_\Omega \big(m(x) + \tau\, J \star 1(x)\big)\, dx > 0 \tag{7.59}$$

and

$$\mu \in L^1(\Omega), \tag{7.60}$$

then (7.57) *admits a nonnegative solution* $u \not\equiv 0$.

A particular case of Theorem 7.2.2 is when the resource m is nonnegative. In this situation, Theorem 7.2.2(i) gives that no survival is possible without resources and pollination, i.e. when both m and τ vanish identically (unless also μ vanishes identically, then reducing the problem to that of mixed operator harmonic functions), whereas Theorem 7.2.2(ii) guarantees survival if at least one between the environmental resource and the pollination is favorable to life. Precisely, one can immediately deduce from Theorem 7.2.2 the following result:

Corollary 7.2.3. *Assume that*

$$m \in L^q(\Omega), \text{ for some } q \in \big(\underline{q}, +\infty\big],$$
$$m \text{ is nonnegative},$$
$$\text{and } (m+\tau)^3\mu^{-2} \in L^1(\Omega).$$

Then,

(i) *if* $m \equiv 0$ *and* $\tau = 0$, *then the only solution of* (7.57) *is the one identically zero;*

(ii) *if either* $m > 0$ *or* $\tau(J \star 1) > 0$ *in a set of positive measure and* $\mu \in L^1(\Omega)$, *then* (7.57) *admits a nonnegative solution* $u \not\equiv 0$.

Problems related to Corollary 7.2.3 have been studied in [12] under Dirichelet (rather than Neumann) boundary conditions.

From the biological point of view, assumption (7.59) states that the environment is "in average" favorable for the survival of the species. It is therefore a natural question to investigate the situation in which the environment is "mostly hostile to life". To study this phenomenon, when $m \in L^q$ with $q > n/2$, with $m^+ \not\equiv 0$ and

$$\int_\Omega m(x)\,dx < 0,$$

we denote[2] by λ_1 the first positive eigenvalue associated with the diffusive operator in (7.57). More precisely, we consider the weighted eigenvalue problem (7.11).

As it will be discussed in detail in Proposition 7.1.1 here and in [26], problem (7.11) admits the existence of two unbounded sequences of eigenvalues, one positive and one negative. In this setting, the smallest strictly positive eigengevalue will be denoted by λ_1. When we want to emphasize the dependence of λ_1 on the resource m, we will write it as $\lambda_1(m)$.

We also denote by e an eigenfunction corresponding to λ_1 normalized such that

$$\int_\Omega m(x)\,e^2(x)\,dx = 1.$$

The first eigenvalue will be an important threshold for the survival of the species, quantifying the role of the necessary pollination parameter τ in order to overcome the presence of an hostile behavior in average. The precise result that we obtain is the following one:

Theorem 7.2.4. *Assume that* $m \in L^q(\Omega)$, *for some* $q \in \big(\underline{q}, +\infty\big]$, *and* $(m+\tau)^3\mu^{-2} \in L^1(\Omega)$.

Then,

[2] As customary, we freely use in this paper the standard notation

$$m^+(x) := \max\{0, m(x)\} \qquad \text{and} \qquad m^-(x) := \max\{0, -m(x)\}.$$

(i) *if $m \le -\tau$, then the only solution of* (7.57) *is the one identically zero;*

(ii) *if $m^+ \not\equiv 0$, $\mu \in L^1(\Omega)$,*

$$\int_\Omega m(x)\, dx < 0, \tag{7.61}$$

and

$$\lambda_1 - 1 < \tau \int_\Omega (J \star e(x)) e(x)\, dx, \tag{7.62}$$

then (7.57) *admits a nonnegative solution $u \not\equiv 0$.*

Once again, in Theorem 7.2.4, the case described in (i) is the one less favorable to life, since the combination of both the resources and the pollination is in average negative, while the case in (ii) gives a lower bound of the pollination parameter τ which is needed for the survival of the species, as quantified by (7.62).

The proof of Theorem 7.2.1 is based on a minimization argument. More precisely, in order to deal with problem (7.57), we consider the energy functional $\mathscr{E} : X_{\alpha,\beta} \to \mathbb{R}$ defined as

$$\begin{aligned} \mathscr{E}(u) :=& \frac{\alpha}{2} \int_\Omega |\nabla u|^2\, dx + \frac{\beta}{4} \iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\ &+ \int_\Omega \left(\frac{\mu |u|^3}{3} - \frac{m u^2}{2} - \frac{\tau u (J \star u)}{2} \right) dx. \end{aligned} \tag{7.63}$$

As a technical remark, we observe that our objective here is to distinguish between trivial and nontrivial solutions, to detect appropriate conditions for the survival of the solutions and we do not indulge in the distinction nonnegative and nontrivial versus strictly positive solutions. For the reader interested in this point, we mention however that, under appropriate conditions, one could develop a regularity theory (see e.g. Theorems 3.1.11 and 3.1.12 in [36]) that allows the use of a strong maximum principle for smooth solutions (see e.g. Theorem 3.1.4 in [36]).

Now, we prove that the functional in (7.63) is the one associated with (7.57):

Lemma 7.2.5. *The Euler-Lagrange equation associated to the energy functional $\mathscr{E}$ introduced in* (7.63) *at a non-negative function u is* (7.57)*.*

Proof. We compute the first variation of $\mathscr{E}$, and we focus on the convolution term in (7.63) (being the computation for the other terms standard, see in particular Proposition 3.7 in [28] to deal with the term involving the

Gagliardo seminorm, which is the one producing the nonlocal Neumann condition).

For this, we set

$$\mathcal{J}(u) := \frac{\tau}{2} \int_\Omega u(x)(J \star u(x))\, dx.$$

For any $\phi \in X_{\alpha,\beta}$ and $\varepsilon \in (-1,1)$, we have

$$\begin{aligned}
&\mathcal{J}(u+\varepsilon\phi)\\
&= \frac{\tau}{2} \int_\Omega (u+\varepsilon\phi)(x)(J \star (u+\varepsilon\phi))(x)\, dx\\
&= \frac{\tau}{2} \int_\Omega \big(u(x)(J\star u)(x) + \varepsilon\big[u(x)(J\star\phi)(x) + \phi(x)(J\star u)(x)\big]\\
&\qquad\qquad + \varepsilon^2 \phi(x)(J\star\phi)(x)\big)\, dx.
\end{aligned}$$

Accordingly,

$$\frac{d\mathcal{J}}{d\varepsilon}(u+\varepsilon\phi)\Big|_{\varepsilon=0} = \frac{\tau}{2} \int_\Omega u(x)(J\star\phi)(x) + \phi(x)(J\star u)(x)\, dx. \tag{7.64}$$

Now, since J is even (recall (7.53)), we see that

$$\begin{aligned}
\int_\Omega u(x)(J\star\phi)(x)\, dx &= \int_\Omega u(x) \left(\int_\Omega J(x-y)\phi(y)\, dy\right) dx\\
&= \int_\Omega \phi(y) \left(\int_\Omega J(y-x)u(x)\, dx\right) dy = \int_\Omega \phi(x)(J\star u)(x)\, dx.
\end{aligned}$$

Using this in (7.64) we obtain that

$$\frac{d\mathcal{J}}{d\varepsilon}(u+\varepsilon\phi)\Big|_{\varepsilon=0} = \tau \int_\Omega \phi(x)(J\star u)(x)\, dx,$$

which concludes the proof. □

As a consequence of Lemma 7.2.5, to find solutions of (7.57), we will consider the minimizing problem for the functional $\mathscr{E}$ in (7.63). First, we show the following useful inequality:

Lemma 7.2.6. *Let $v, w \in L^2(\Omega)$. Then*

$$\int_\Omega |v(x)|\, \big|(J\star w)(x)\big|\, dx \le \|v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}. \tag{7.65}$$

Proof. By the Cauchy-Schwarz Inequality, we have

$$\int_\Omega |v(x)|\,\big|(J\star w)(x)\big|\,dx \le \|v\|_{L^2(\Omega)}\|J\star w\|_{L^2(\Omega)}. \tag{7.66}$$

Now, using the Young Inequality for convolutions with exponents 1 and 2 (see e.g. Theorem 9.1 in [72]), we obtain

$$\|J\star w\|_{L^2(\Omega)} = \|J*(w\chi_\Omega)\|_{L^2(\mathbb{R}^n)} \le \|J\|_{L^1(\mathbb{R}^n)}\|w\chi_\Omega\|_{L^2(\mathbb{R}^n)} = \|w\|_{L^2(\Omega)},$$

where (7.54) has been also used. This and (7.66) give (7.65), as desired. □

We are now able to provide a minimization argument for the functional in (7.63):

Proposition 7.2.7. *Assume that* $m\in L^q(\Omega)$*, for some* $q\in\big(\underline{q},+\infty\big]$*, where* $\underline{q}$ *has been introduced in* (7.13)*, and that*

$$(m+\tau)^3\mu^{-2}\in L^1(\Omega). \tag{7.67}$$

Let also

$$p := \frac{2q}{q-1}.$$

Then, the functional $\mathscr{E}$ *in* (7.63) *attains its minimum in* $X_{\alpha,\beta}$*. The minimal value is the same as the one occurring among the functions* $u\in L^p(\Omega)$ *for which*

$$\iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy < +\infty$$

and such that $\mathscr{N}_s u = 0$ *a.e. outside* Ω*.*

Moreover, there exists a nonnegative minimizer u*, and it is a solution of* (7.57)*.*

Proof. First, we notice that $p\in\Big[2,\frac{2\underline{q}}{\underline{q}-1}\Big)$ and

$$\frac{2}{p}+\frac{1}{q}=1. \tag{7.68}$$

By (7.65) we have that

$$\int_\Omega \frac{\tau u(J\star u)}{2}\,dx \le \frac{\tau}{2}\|u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} = \frac{\tau}{2}\int_\Omega |u(x)|^2\,dx. \tag{7.69}$$

Moreover, we use the Young Inequality with exponents 3/2 and 3 to obtain that

$$\frac{(m+\tau)u^2}{2} = \frac{\mu^{\frac{2}{3}}u^2}{2^{\frac{4}{3}}}\cdot\frac{m+\tau}{2^{-\frac{1}{3}}\mu^{\frac{2}{3}}} \le \frac{\mu|u|^3}{6}+\frac{2}{3}\frac{|m+\tau|^3}{\mu^2}.$$

From this and (7.69) we have that

$$\begin{aligned}\int_\Omega \frac{\mu|u|^3}{6}-\frac{mu^2}{2}-\frac{\tau u(J\star u)}{2}\,dx &\geq \int_\Omega \frac{\mu|u|^3}{6}-\frac{mu^2}{2}-\frac{\tau u^2}{2}\,dx\\ &\geq -\frac{2}{3}\int_\Omega \frac{|m+\tau|^3}{\mu^2}\,dx =: -\kappa.\end{aligned}\tag{7.70}$$

We point out that the quantity κ is finite, thanks to (7.67), and it does not depend on u.

Recalling (7.63), formula (7.70) implies that

$$\mathscr{E}(u)\geq \frac{\alpha}{2}\int_\Omega |\nabla u|^2\,dx+\frac{\beta}{4}\iint_{\mathcal{Q}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy+\int_\Omega\frac{\mu|u|^3}{6}\,dx-\kappa.\tag{7.71}$$

Now, we take a minimizing sequence u_j, and we observe that, in light of Theorem 1.0.3, we can assume that

$$\mathscr{N}_s u_j = 0 \text{ in } \in \mathbb{R}^n\setminus\overline{\Omega}, \text{ for every } j\in\mathbb{N}.\tag{7.72}$$

We can also suppose that

$$\begin{aligned}0=\mathscr{E}(0)&\geq\mathscr{E}(u_j)\\ &\geq \frac{\alpha}{2}\int_\Omega |\nabla u_j|^2\,dx+\frac{\beta}{4}\iint_{\mathcal{Q}}\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy+\int_\Omega\frac{\mu|u_j|^3}{6}\,dx-\kappa,\end{aligned}$$

where (7.71) has been also exploited. This implies that

$$\frac{\alpha}{2}\int_\Omega |\nabla u_j|^2\,dx+\frac{\beta}{4}\iint_{\mathcal{Q}}\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy+\int_\Omega\frac{\mu|u_j|^3}{6}\,dx\leq\kappa.$$

As a consequence,

$$\frac{\alpha}{2}\int_\Omega |\nabla u_j|^2\,dx+\frac{\beta}{4}\iint_{\mathcal{Q}}\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\leq\kappa.\tag{7.73}$$

Moreover, by the Hölder Inequality with exponents $3/2$ and 3,

$$\begin{aligned}\|u_j\|^2_{L^2(\Omega)}&\leq\left(\int_\Omega |u_j|^3\,dx\right)^{2/3}|\Omega|^{1/3}\leq\left(\int_\Omega\frac{\underline{\mu}|u_j|^3}{6}\,dx\right)^{2/3}\frac{6^{2/3}|\Omega|^{1/3}}{\underline{\mu}^{2/3}}\\ &\leq\left(\int_\Omega\frac{\mu|u_j|^3}{6}\,dx\right)^{2/3}\frac{6^{2/3}|\Omega|^{1/3}}{\underline{\mu}^{2/3}}\leq\frac{6^{2/3}|\Omega|^{1/3}\kappa}{\underline{\mu}^{2/3}}.\end{aligned}$$

From this and (7.73), and using compactness arguments, we can assume, up to a subsequence, that u_j converges to some $u\in L^p(\Omega)$ (for every $p\in[1,2^*_s)$

if $\alpha = 0$, and for every $p \in [1, 2^*)$ if $\alpha \neq 0$, see e.g. Corollary 7.2 in [30]) and a.e. in Ω, and also $|u_j| \leq h$ for some $h \in L^p(\Omega)$ for every $j \in \mathbb{N}$ (see e.g. Theorem IV.9 in [9]).

Hence, if $x \in \mathbb{R}^n \setminus \overline{\Omega}$, by the Dominated Convergence Theorem,

$$\int_\Omega \frac{u_j(y)}{|x-y|^{n+2s}}\,dy \longrightarrow \int_\Omega \frac{u(y)}{|x-y|^{n+2s}}\,dy,$$

as $j \nearrow +\infty$. Accordingly, in light of (7.72), when $x \in \mathbb{R}^n \setminus \overline{\Omega}$, we have

$$u_j(x) = \frac{\displaystyle\int_\Omega \frac{u_j(y)}{|x-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|x-y|^{n+2s}}} \longrightarrow \frac{\displaystyle\int_\Omega \frac{u(y)}{|x-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|x-y|^{n+2s}}} =: u(x), \tag{7.74}$$

as $j \nearrow +\infty$ (we stress that till now u was only defined in Ω, hence the last step in (7.74) is instrumental to define u also outside Ω). As a consequence, we obtain that u_j converges a.e. in $\mathbb{R}^n$.

Now, recalling (7.68), we have that

$$\begin{aligned}\limsup_{j\nearrow+\infty}\left|\int_\Omega m(u_j^2-u^2)\,dx\right| &\leq \limsup_{j\nearrow+\infty}\int_\Omega |m(u_j^2-u^2)|\,dx\\ &= \limsup_{j\nearrow+\infty}\int_\Omega |m(u_j-u)(u_j+u)|\,dx\\ &\leq \limsup_{j\nearrow+\infty}\|m\|_{L^q(\Omega)}\|u_j-u\|_{L^p(\Omega)}\|u_j+u\|_{L^p(\Omega)} = 0,\end{aligned}$$

so that

$$\lim_{j\nearrow+\infty}\int_\Omega m(u_j^2-u^2)\,dx = 0.$$

Also,

$$\int_\Omega \big(u_j(J\star u_j)-u(J\star u)\big)\,dx = \int_\Omega (u_j-u)(J\star u_j)\,dx + \int_\Omega (J\star u_j - J\star u)u\,dx. \tag{7.75}$$

Using (7.65) with $v := u_j - u$ and $w := u$, we obtain

$$\limsup_{j\nearrow+\infty}\int_\Omega |u_j-u|\,\big|J\star u_j\big|\,dx \leq \limsup_{j\nearrow+\infty}\|u_j-u\|_{L^2(\Omega)}\|u_j\|_{L^2(\Omega)} = 0. \tag{7.76}$$

Similarly, exploiting (7.65) with $v := u$ and $w := u_j - u$, we have

$$\begin{aligned}\limsup_{j\nearrow+\infty}\int_\Omega \big|J\star u_j - J\star u\big|\,|u|\,dx &= \limsup_{j\nearrow+\infty}\int_\Omega \big|J\star(u_j-u)\big|\,|u|\,dx \qquad (7.77)\\ &\leq \limsup_{j\nearrow+\infty}\|u_j-u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} = 0.\end{aligned}$$

From (7.75), (7.76) and (7.77) we conclude that

$$\lim_{j\nearrow+\infty}\int_\Omega (u_j(J\star u_j)-u(J\star u))\,dx=0.$$

We also have, by the Fatou Lemma and the lower semicontinuity of the L^2-norm,

$$\liminf_{j\nearrow+\infty}\iint_Q \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \geq \iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

$$\liminf_{j\nearrow+\infty}\int_\Omega |\nabla u_j|^2\,dx \geq \int_\Omega |\nabla u|^2\,dx$$

and

$$\liminf_{j\nearrow+\infty}\int_\Omega \frac{\mu|u_j|^3}{3}\,dx \geq \int_\Omega \frac{\mu|u|^3}{3}\,dx.$$

Gathering together these observations, we conclude that

$$\liminf_{j\nearrow+\infty}\mathscr{E}(u_j)\geq\mathscr{E}(u),$$

and therefore u is the desired minimum.

Also, since $\mathscr{E}(|u|)\leq\mathscr{E}(u)$, we can suppose that u is nonnegative. Finally, u is a solution of (7.57) thanks to Lemma 7.2.5. □

The claim of Theorem 7.2.1 follows from Proposition 7.2.7.

Now, we provide the proof of Theorem 7.2.2, relying also on the existence result in Theorem 7.2.1:

Proof of Theorem 7.2.2. Thanks to Theorem 7.2.1, we know that there exists a nonnegative solution to (7.57).

We now prove the claim in (i). For this, we assume that $m\equiv 0$ and $\tau=0$, and we argue towards a contradiction, supposing that there exists a nontrivial solution u of (7.57).

We notice that, since $u\geq 0$ and $\mu\geq\underline{\mu}>0$ in Ω,

$$\int_\Omega \mu u^3\,dx>0.$$

As a consequence, taking $v:=u$ in (7.58) we obtain that

$$0\leq\alpha\int_\Omega|\nabla u|^2\,dx+\frac{\beta}{2}\iint_Q\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy=-\int_\Omega\mu u^3\,dx<0,$$

which is a contradiction, and therefore the claim in (i) is proved.

Now we deal with the claim in (ii). From Theorem 7.2.1 we know that there exists a nonnegative solution u to (7.57) which is obtained by the minimization of the functional $\mathscr{E}$ in (7.63) (recall Proposition 7.2.7). We claim that

$$u \text{ does not vanish identically.} \tag{7.78}$$

To prove this, we show that

$$0 \text{ is not a minimizer for } \mathscr{E}. \tag{7.79}$$

For this, we consider the constant function $v \equiv 1$ and a small parameter $\varepsilon > 0$. Then

$$\begin{aligned}\mathscr{E}(\varepsilon v) &= -\frac{\varepsilon^2}{2}\left[\int_\Omega m + \tau(J\star 1)\,dx\right] + \frac{\varepsilon^3}{3}\int_\Omega \mu\,dx\\ &\le -c_1\varepsilon^2 + c_2\varepsilon^3,\end{aligned}$$

where

$$c_1 := \frac{1}{2}\int_\Omega m + \tau\,(J\star 1)\,dx \qquad \text{and} \qquad c_2 := \frac{1}{3}\|\mu\|_{L^1(\Omega)}.$$

We remark that $c_1 > 0$, thanks to (7.59), and $c_2 \in (0, +\infty)$, in light of (7.60). Then, for small ε we have $\mathscr{E}(\varepsilon v) < 0 = \mathscr{E}(0)$. This implies (7.79), which in turn proves (7.78). □

With this, we are now ready to give the proof of Theorem 7.2.4:

Proof of Theorem 7.2.4. Thanks to Theorem 7.2.1, we know that there exists a nonnegative solution to (7.57).

We first prove the claim in (i). For this, we assume that $m \le -\tau$, and we suppose by contradiction that there exists a nontrivial solution u of (7.57).

We observe that, applying (7.65) with $v := u$ and $w := u$,

$$\tau\int_\Omega u\,\big(J\star u\big)\,dx \le \tau\|u\|^2_{L^2(\Omega)} = \tau\int_\Omega u^2\,dx. \tag{7.80}$$

Hence, taking u as a test function in (7.58), using (7.80) and recalling that $u \ge 0$ and $\mu \ge \underline{\mu}$, we get

$$\begin{aligned}0 &\le \alpha\int_\Omega |\nabla u|^2\,dx + \frac{\beta}{2}\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\ &= \int_\Omega (m-\mu u)u^2\,dx + \tau\int_\Omega (J\star u)u\,dx\\ &\le -\tau\int_\Omega u^2\,dx - \underline{\mu}\int_\Omega u^3\,dx + \tau\int_\Omega u^2\,dx\\ &< 0.\end{aligned}$$

This is a contradiction, whence the first claim is proved.

Now we show the claim in (ii). From Theorem 7.2.1 we know that there exists a nonnegative solution u to (7.57) which is obtained by the minimization of the functional $\mathscr{E}$ in (7.63) (recall Proposition 7.2.7). We claim that

$$u \text{ does not vanish identically.} \tag{7.81}$$

To prove this, we show that

$$0 \text{ is not a minimizer for } \mathscr{E}. \tag{7.82}$$

For this, we take an eigenfunction e associated to the first positive eigenvalue λ_1, as given by Proposition 7.1.2. Namely, we take $e \in X_{\alpha,\beta}$ such that

$$\alpha \int_\Omega \nabla e \cdot \nabla v \, dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(e(x) - e(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy = \lambda_1 \int_\Omega mev \, dx, \tag{7.83}$$

for every $v \in X_{\alpha,\beta}$.

By taking $v := e$ in (7.83), we obtain that

$$\alpha \int_\Omega |\nabla e|^2 \, dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{|e(x) - e(y)|^2}{|x-y|^{n+2s}} \, dx \, dy = \lambda_1 \int_\Omega me^2 \, dx. \tag{7.84}$$

We also remark that, thanks to (7.61), we can use the characterization of λ_1 given in formula (7.16) of Proposition 7.1.1, and hence we can normalize e in such a way that

$$\int_\Omega me^2 \, dx = 1. \tag{7.85}$$

By Corollary 1.4 in [26], we know that

$$e \text{ is bounded.} \tag{7.86}$$

We also take $\varepsilon > 0$. Then, by (7.84) and (7.85),

$$\begin{aligned}
\mathscr{E}(\varepsilon e) &= \frac{\varepsilon^2}{2} \Bigg[\alpha \int_\Omega |\nabla e|^2 \, dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{|e(x) - e(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \\
&\qquad - \int_\Omega me^2 \, dx - \int_\Omega \tau (J \star e) e \, dx \Bigg] + \frac{\varepsilon^3}{3} \int_\Omega \mu e^3 \, dx \\
&= \frac{\varepsilon^2}{2} \left[(\lambda_1 - 1) \int_\Omega me^2 \, dx - \int_\Omega \tau (J \star e) e \, dx \right] + \frac{\varepsilon^3}{3} \int_\Omega \mu e^3 \, dx \\
&= \frac{\varepsilon^2}{2} \left[(\lambda_1 - 1) - \int_\Omega \tau (J \star e) e \, dx \right] + \frac{\varepsilon^3}{3} \int_\Omega \mu e^3 \, dx \\
&= -\frac{c_1}{2} \varepsilon^2 + c_2 \frac{\varepsilon^3}{3},
\end{aligned}$$

where

$$c_1 := 1 - \lambda_1 + \tau \int_\Omega (J \star e) e \, dx \qquad \text{and} \qquad c_2 := \int_\Omega \mu e^3 \, dx.$$

We notice that $c_1 > 0$, thanks to (7.62), and $c_2 \in \mathbb{R}$, in light of (7.86). As a consequence, for small ε we have that $\mathscr{E}(\varepsilon e) < 0 = \mathscr{E}(0)$, which proves (7.82). In turn, this implies (7.81), thus completing the proof of (ii). □

7.3 Sublinear problems

In this section we consider problems of the form

$$\begin{cases} -\alpha\Delta u + \beta(-\Delta)^s u = g(x,u) & \text{in } \Omega, \\ \text{with } (\alpha,\beta) - \text{Neumann conditions}, \end{cases}, \tag{7.87}$$

when $g(x,\cdot)$ behaves not more then linearly at infinity. More precisely, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that there exists $a \in L^2(\Omega)$ and $b > 0$ for which

$$|g(x,t)| \le a(x) + b|t| \tag{7.88}$$

for every $t \in \mathbb{R}$ and for a.e. $x \in \Omega$.

We consider the functional : $X_{\alpha,\beta} \to \mathbb{R}$ defined as

$$I(u) := \frac{\alpha}{2} \int_\Omega |\nabla u(x)|^2 \, dx + \frac{\beta}{4} \iint_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, dxdy - \int_\Omega G(x,u) \, dx,$$

where $G(x,t) := \int_0^t g(x,\sigma) d\sigma$, so that every critical point of I is a weak solution of (7.87).

Before giving our first result, let us set

$$\underline{\alpha}(x) := \liminf_{|t|\to\infty} \frac{g(x,t)}{t} \tag{7.89}$$

and

$$\overline{\alpha}(x) := \limsup_{|t|\to\infty} \frac{g(x,t)}{t} \tag{7.90}$$

for a.e. $x \in \Omega$.

Then we have the following result.

Theorem 7.3.1. *Assume that there exists $\tau > 0$ such that*

- $\overline{\alpha}(x) < \lambda_0 - \tau = -\tau$, *or*

- *there exists* $m \in \mathbb{N}_0$ *such that*

$$\lambda_m + \tau < \underline{\alpha}(x) \leq \overline{\alpha}(x) < \lambda_{m+1} - \tau. \tag{7.91}$$

Then problem (7.87) *admits a weak solution.*

We treat the proofs of the two cases two cases separately.

Proof of Theorem 7.3.1 *when* $\boldsymbol{\overline{\alpha}(x) < -\tau}$*. .*

First, we claim that

$$\limsup_{|t|\to\infty} \frac{G(x,t)}{t^2} \leq \frac{\overline{\alpha}(x)}{2}. \tag{7.92}$$

Indeed, by (7.90), for every $\varepsilon >$ there exists $K > 0$ such that

$$\frac{g(x,t)}{t} - \overline{\alpha}(x) < \varepsilon$$

for $|t| \geq K$. Integrating we get

$$G(x,t) \leq \frac{\overline{\alpha}(x) + \varepsilon}{2}(t^2 - K^2)$$

for $|t| \geq K$. On the other hand, (7.88) implies

$$|G(x,t)| \leq a(x)K + \frac{b}{2}K^2$$

for $|t| < K$. So we have

$$\frac{G(x,t)}{t^2} < \frac{a(x)K + \frac{b}{2}K^2 + \frac{\overline{\alpha}(x)+\varepsilon}{2}(t^2 - K^2)}{t^2},$$

hence

$$\limsup_{|t|\to\infty} \frac{G(x,t)}{t^2} \leq \frac{\overline{\alpha}(x)}{2},$$

as claimed.

Now we want to prove that

$$\lim_{\|u\|\to\infty} \frac{I(u)}{\|u\|^2} > 0, \tag{7.93}$$

where we have chosen

$$\|u\| := \left([u]^2 + \tau \int_\Omega u^2 dx\right)^{1/2},$$

which is a norm equivalent to the standard one in $X_{\alpha,\beta}$.

Take a sequence $(u_n)_n$ in $X_{\alpha,\beta}$ such that $\|u_n\| \to \infty$. Up to a subsequence, we can assume that $v_n := \frac{u_n}{\|u_n\|}$ converges to some u weakly in $X_{\alpha,\beta}$ and strongly in $L^2(\Omega)$. Moreover, $\|u\| \leq 1$ and

$$\frac{G(x,u_n)}{\|u_n\|^2} \leq \frac{a(x)|u_n| + bu_n^2/2}{\|u_n\|^2} \to \frac{b}{2}u^2$$

in $L^1(\Omega)$. By the generalized Fatou Lemma we have

$$\limsup_{n\to\infty} \int_\Omega \frac{G(x,u_n)}{\|u_n\|^2}\,dx \leq \int_\Omega \limsup_{n\to\infty} \frac{G(x,u_n)}{\|u_n\|^2}\,dx. \tag{7.94}$$

If $(u_n(x))_n$ is bounded, by (7.88) we find

$$\frac{G(x,u_n(x))}{\|u_n\|^2} \to 0,$$

while if $|u_n(x)| \to \infty$,

$$\limsup_{n\to\infty} \frac{G(x,u_n(x))}{\|u_n\|^2} = \limsup_{n\to\infty} \frac{G(x,u_n(x))}{u_n^2(x)} \frac{u_n^2(x)}{\|u_n\|^2} \leq \frac{\overline{\alpha}(x)}{2}u^2(x).$$

Hence, by (7.92),

$$\limsup_{n\to\infty} \frac{G(x,u_n)}{\|u_n\|^2}\,dx \begin{cases} < -\dfrac{\tau}{2}u^2 & \text{if } u \neq 0, \\ \leq 0 & \text{if } u = 0, \end{cases}.$$

Therefore

$$\begin{aligned} &\liminf_{n\to\infty} \frac{I(u_n)}{\|u_n\|^2} \\ =&\liminf_{n\to\infty} \left(\frac{1}{2} - \frac{\int_\Omega [G(x,u_n) + \tau u_n^2]dx}{\|u_n\|^2}\right) \begin{cases} > \frac{1}{2} & \text{if } u \neq 0, \\ \geq \frac{1}{2} & \text{if } u = 0, \end{cases} \end{aligned}$$

and the claim follows.

Finally, it is easy to show that I is lower semicontinuous, while it is coercive from (7.93). So we can apply the Weierstrass Theorem to find a minimum for I, which is a solution of problem (7.87). □

Now we deal with the case when there exists $m \in \mathbb{N}_0$ such that $\lambda_m + \tau < \underline{\alpha}(x) \leq \overline{\alpha}(x) < \lambda_{m+1} - \tau$. In this case we want to use the Saddle Theorem to prove the existence of a solution. First, we set $V_m = \text{Span}(e_0, ..., e_m)$

and $V_m^\perp = \overline{\mathrm{Span}(e_{m+1}, ...)}$. Since $\{e_i\}_{i\geq 0}$ are an orthonormal basis in $L^2(\Omega)$ made of eigenfunctions of a selfadjoint operator (the inverse of the operator $-\alpha\Delta+\beta(-\Delta)^s$), classical theories (for instance, see [9, Theorem 6.11]) imply that

$$[u]^2 \leq \lambda_m \int_\Omega u^2\,dx, \quad \forall u \in V_m \tag{7.95}$$

and

$$[u]^2 \geq \lambda_{m+1} \int_\Omega u^2\,dx, \quad \forall u \in V_m^\perp. \tag{7.96}$$

First we prove that I satisfies the geometric properties of the Saddle Theorem.

Proposition 7.3.2. *If* (7.91) *holds, I satisfies*

$$\limsup_{u\in V_m, \|u\|\to\infty} \frac{I(u)}{\|u\|^2} < 0 \tag{7.97}$$

and

$$\liminf_{u\in V_m^\perp, \|u\|\to\infty} \frac{I(u)}{\|u\|^2} > 0. \tag{7.98}$$

Proof. First we prove (7.97). Reasoning as above, from (7.89) we can prove that

$$\liminf_{t\to\infty} \frac{G(x,t)}{t^2} \geq \frac{\underline{\alpha}(x)}{2}. \tag{7.99}$$

So, taking an unbounded sequence $(u_n)_n$ in V_m with $\|u_n\| \to \infty$ as $n \to \infty$, and such that $u_n/\|u_n\|$ converges strongly to some u in $X_{\alpha,\beta}$ and in $L^2(\Omega)$, we have

$$\liminf_{n\to\infty} \frac{G(x,u_n)}{\|u_n\|^2} = \liminf_{n\to\infty} \frac{G(x,u_n)}{u_n^2}\frac{u_n^2}{\|u_n\|^2} \geq \frac{\underline{\alpha}(x)}{2}u^2.$$

From Fatou's Lemma

$$\liminf_{n\to\infty} \int_\Omega \frac{G(x,u_n)}{\|u_n\|^2}\,dx \geq \int_\Omega \liminf_{n\to\infty} \frac{G(x,u_n)}{\|u_n\|^2}\,dx \geq \int_\Omega \frac{\underline{\alpha}(x)}{2}u^2\,dx$$

when $|u_n(x)| \to \infty$ as $n \to \infty$, while

$$\frac{G(x,u_n)}{\|u_n\|^2} \to 0$$

if $(u_n(x))_n$ is bounded. So we can write

$$\liminf_{n\to\infty} \frac{G(x,u_n)}{\|u_n\|^2}\,dx \geq \frac{\underline{\alpha}(x)}{2}u^2.$$

Since $u \neq 0$ (recall that $\|u\| = 1$), by (7.95) we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{I(u_n)}{\|u_n\|^2} &\leq \limsup_{n\to\infty} \left(\frac{\lambda_m}{2} \int_\Omega \frac{u_n^2}{\|u_n\|^2}\, dx - \int_\Omega \frac{G(x,u_n)}{\|u_n\|^2}\, dx \right) \\
&\leq \frac{\lambda_m}{2} \int_\Omega u^2\, dx - \int_\Omega \frac{\underline{\alpha}(x)}{2} u^2\, dx \\
&< -\frac{\tau}{2} \int_\Omega u^2\, dx < 0,
\end{aligned}$$

thus (7.97) holds.

To prove (7.98) we follow the proof of (7.93), so we take a sequence $(u_n)_n$ in $V_m^\perp$ with $\|u_n\| \to \infty$ as $n \to \infty$ and arrive to

$$\limsup_{n\to\infty} \frac{G(x,u_n)}{\|u_n\|^2}\, dx \begin{cases} \leq \dfrac{\overline{\alpha}(x)}{2} u^2\, dx & u \neq 0 \\ \leq 0 & u = 0 \end{cases}$$

Then, if $u \neq 0$

$$\begin{aligned}
\liminf_{n\to\infty} \frac{I(u_n)}{\|u_n\|^2} &\geq \liminf_{n\to\infty} \left(\frac{1}{2} - \int_\Omega \frac{G(x,u_n)}{\|u_n\|^2}\, dx - \frac{\tau}{2} \int_\Omega \frac{u_n^2}{\|u_n\|^2}\, dx \right) \\
&\geq \frac{1}{2} - \frac{\overline{\alpha}(x)}{2} u^2\, dx - \frac{\tau}{2} \int_\Omega u^2\, dx \\
&> \frac{1}{2} - \frac{\lambda_{m+1}}{2} \int_\Omega u^2 \geq \frac{1}{2} - \frac{1}{2}[u]^2 \geq 0
\end{aligned}$$

where we have used (7.96) and $1 \geq \|u\| \geq [u]$. Clearly, if $u = 0$ we have

$$\liminf_{n\to\infty} \frac{I(u_n)}{\|u_n\|^2} \geq \frac{1}{2},$$

which proves (7.98). □

Now we want to prove that I satisfies the (PS) condition, that is for every sequence $(u_n)_n$ in $X_{\alpha,\beta}$ such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ in H', then there exists a converging susequence.

Proposition 7.3.3. *If* (7.91) *holds, functional* I *satisfies the (PS) condition.*

Proof. Let $(u_n)_n$ be a sequence such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ in $X'_{\alpha,\beta}$. We assume by contradiction that $\|u_n\| \to \infty$. Up to a subsequence, there exists $u \in X_{\alpha,\beta}$ such that $v_n := u_n/\|u_n\|$ converges to u weakly in $X_{\alpha,\beta}$ and strongly in $L^2(\Omega)$. Moreover

$$\frac{|g(x,u_n)|}{\|u_n\|} \leq \frac{a(x)}{\|u_n\|} + b|v_n|,$$

where the right hand side is bounded in $L^2(\Omega)$. So there exists $w \in L^2(\Omega)$ such that $g(x, u_n)/\|u_n\|$ converges weakly to w in $L^2(\Omega)$.

Now we claim that there exists a measurable function m such that $\underline{\alpha}(x) \leq m(x) \leq \overline{\alpha}(x)$ a.e. in Ω and $w(x) = m(x)u(x)$. If $u(x) > 0$, then clearly $u_n(x) = v_n(x)\|u_n\| \to +\infty$, so that

$$\liminf_{n\to\infty} \frac{g(x, u_n)}{\|u_n\|} = \liminf_{n\to\infty} \frac{g(x, u_n)}{u_n} \frac{u_n}{\|u_n\|} \geq \underline{\alpha}(x)u(x),$$

and similarly

$$\limsup_{n\to\infty} \frac{g(x, u_n)}{\|u_n\|} \leq \overline{\alpha}(x)u(x).$$

On the other hand, if $u(x) < 0$ we have

$$\liminf_{n\to\infty} \frac{g(x, u_n)}{\|u_n\|} \leq \underline{\alpha}(x)u(x)$$

and

$$\limsup_{n\to\infty} \frac{g(x, u_n)}{\|u_n\|} \geq \overline{\alpha}(x)u(x).$$

Moreover, if $u(x) = 0$, then

$$\frac{|g(x, u_n)|}{\|u_n\|} \leq \frac{a(x)}{\|u_n\|} + b|v_n| \to 0.$$

From the weak convergence of $g(x, u_n)/\|u_n\|$ to w, we can write the inequalities above as

$$\underline{\alpha}(x)u(x) \leq w(x) \leq \overline{\alpha}(x)u(x), \quad u(x) > 0,$$

$$\underline{\alpha}(x)u(x) \geq w(x) \geq \overline{\alpha}(x)u(x), \quad u(x) < 0.$$

In light of these inequalities, we set

$$m(x) := \begin{cases} \dfrac{w(x)}{u(x)} & \text{if } u(x) \neq 0 \\ \frac{1}{2}(\underline{\alpha}(x) + \overline{\alpha}(x)) & \text{if } u(x) = 0 \end{cases}$$

In this way, m is measurable, and so the claim is proved.

By assumption, we have

$$\lim_{n\to\infty} I'(u_n) = 0,$$

which implies

$$\lim_{n\to\infty}\left\langle I'(u_n), \frac{v}{\|u_n\|}\right\rangle$$
$$= \alpha\int_\Omega \nabla u\cdot\nabla v\,dx + \frac{\beta}{2}\iint_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy \quad (7.100)$$
$$-\int_\Omega muv\,dx = 0$$

for every $v \in X_{\alpha,\beta}$. Now we want to prove that (7.100) impies $u = 0$. Since V_m and $V_m^\perp$ are orthogonal, in $L^2(\Omega)$, as well as in $X_{\alpha,\beta}$, we write $u = u_1 + u_2$ with $u_1 \in V_m$ and $u_2 \in V_m^\perp$ for any $u \in X_{\alpha,\beta}$. Choosing $v = u_1$ and $v = u_2$ in (7.100), we get respectively

$$[u_1]^2 - \int_\Omega mu_1^2\,dx = \int_\Omega mu_1u_2\,dx.$$

and

$$[u_2]^2 - \int_\Omega mu_2^2\,dx = \int_\Omega mu_1u_2\,dx.$$

As a consequence

$$[u_1]^2 - \int_\Omega mu_1^2\,dx = [u_2]^2 - \int_\Omega mu_2^2\,dx. \quad (7.101)$$

By (7.95) we know that

$$[u_1]^2 - \int_\Omega mu_1^2\,dx \le \int_\Omega (\lambda_m - m)u_1^2\,dx \le 0, \quad (7.102)$$

while from (7.96)

$$[u_2]^2 - \int_\Omega mu_1^2\,dx \ge \int_\Omega (\lambda_{m+1} - m)u_2^2\,dx \ge 0. \quad (7.103)$$

Finally, from (7.101), (7.102) and (7.103) we can deduce that

$$0 \ge \int_\Omega (\lambda_m - m)u_1^2\,dx = [u_1]^2 - \int_\Omega mu_1^2\,dx = [u_2]^2 - \int_\Omega mu_1^2\,dx$$
$$\ge \int_\Omega (\lambda_{m+1} - m)u_2^2\,dx \ge 0,$$

so that all integrals are equal to zero. From the definition of m we also know that $\lambda_m - m \le \lambda_m - \underline{\alpha} < -\tau$ and $\lambda_{m+1} - m \ge \lambda_{m+1} - \overline{\alpha} > \tau$, so the only possibility is $u_1 = u_2 = 0$ as desired. This is a contradiction, and so $(u_n)_n$ is bounded in $X_{\alpha,\beta}$.

Since both the Laplacian and the fractional Laplacian satisfy the (S+) property, the strong convergence of u_n follows. □

Proof of Theorem 7.3.1 *when* $\lambda_m + \tau < \underline{\alpha}(x) \le \overline{\alpha}(x) < \lambda_{m+1} - \tau$. From Proposition 7.3.3 we know that the functional I satisfies the (PS) condition, so we just need to verify that the geometric conditions of the Saddle Theorem are satisfied.

From Proposition 7.3.2 we know that

$$\lim_{u \in V_m, \|u\| \to \infty} I(u) = -\infty$$

and

$$\lim_{u \in V_m^\perp, \|u\| \to \infty} I(u) = \infty.$$

Considering the limit in $V_m^\perp$, for every $M > 0$ there exists some $K > 0$ such that, for every $u \in V_m^\perp$ with $\|u\| \ge K$,

$$I(u) \ge M.$$

On the other hand, if $\|u\| \le K$, from (7.96)

$$\begin{aligned} I(u) &\ge -\int_\Omega a(x)|u|\,dx - \frac{b}{2}\int_\Omega |u|^2\,dx \ge -\|a\|_2\|u\|_2 - \frac{b}{2}\|u\|_2^2 \\ &\ge -\frac{\|a\|_2}{\lambda_{m+1}+1}K - \frac{b}{2}\frac{K^2}{(\lambda_{m+1}+1)^2} = C_K. \end{aligned}$$

So for every $u \in V_m^\perp$ we can write

$$I(u) \ge \min\{M, C_K\}.$$

Considering the limit in V_m, that is (7.97), we can choose $R > 0$ such that for every $u \in V_m$ with $\|u\| = R$, then

$$\sup I(u) < \min\{M, C_K\} \le \inf I(V_m^\perp).$$

We are now able to apply the Saddle Theorem to obtain a critical point of I, and thus a solution of (7.87).

□

7.4 Mixed parabolic equations

In this section we consider problem

$$\begin{cases} u_t - \alpha\Delta u + \beta(-\Delta)^s u = 0 & \text{in } \Omega,\ t > 0, \\ \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{7.104}$$

.

We want to show that classical solutions of this problem preserve their mass and have energy that decreases in time. In order to prove this, we first recall that

$$\int_\Omega (-\Delta)^s u = -\int_{\mathbb{R}^N\setminus\Omega} \mathcal{N}_s u \tag{7.105}$$

and

$$\frac{1}{2}\iint_{\mathcal{Q}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy = \int_\Omega v(-\Delta)^s u + \int_{\mathbb{R}^N\setminus\Omega} v\mathcal{N}_s u, \tag{7.106}$$

see Lemma 3.2 and Lemma 3.3 in [28]. We remark that if u and v are classical solutions of problem (7.104), then they are regular enough to apply these results.

Proposition 7.4.1. *Let $u(x,t)$ be a classical solutions of* (7.104)*, in the sense that u is bounded and $|u_t| + |\alpha\Delta u| + |\beta(-\Delta)^s u| \leq K$ for all $t > 0$. Then, for all $t > 0$,*

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx,$$

namely the total mass is preserved.

Proof. By the dominated convergence theorem, we have

$$\frac{d}{dt}\int_\Omega u = \int_\Omega u_t = \alpha\int_\Omega \Delta u - \int_\Omega \beta(-\Delta)^s u.$$

Using (7.105) and the divergence theorem

$$\alpha\int_\Omega \Delta u - \int_\Omega \beta(-\Delta)^s u = \alpha\int_{\partial\Omega}\frac{\partial u}{\partial\nu} + \beta\int_{\mathbb{R}^N\setminus\Omega}\mathcal{N}_s u = 0.$$

So $\int_\Omega u$ does not depend on t, which concludes the proof. □

Proposition 7.4.2. *Let $u(x,t)$ be a classical solutions of* (7.104)*, in the sense that u is bounded and $|u_t| + |\alpha\Delta u| + |\beta(-\Delta)^s u| \leq K$ for all $t > 0$. Then the energy*

$$E(t) = \alpha\int_\Omega |\nabla u|^2 dx + \frac{\beta}{2}\iint_{\mathcal{Q}} \frac{|u(x,t)-u(y,t)|^2}{|x-y|^{N+2s}}\,dxdy$$

is decreasing in time $t > 0$.

Proof. We compute $E'(t)$ and show that it is negative. Clearly, from (7.106), the integration by parts formula and the homogeneous boundary conditions, we have

$$\begin{aligned}
E'(t) &= \frac{d}{dt}\alpha \int_\Omega |\nabla u|^2 dx + \frac{\beta}{2}\frac{d}{dt} \iint_\Omega \frac{|u(x,t)-u(y,t)|^2}{|x-y|^{N+2s}}\, dxdy \\
&= 2\alpha \int_\Omega \nabla u \nabla u_t + \beta \iint_\Omega \frac{(u(x,t)-u(y,t))(u_t(x,t)-u_t(y,t))}{|x-y|^{N+2s}}\, dxdy \\
&= -2\alpha \int_\Omega u_t \Delta u + 2\beta \int_\Omega u_t (-\Delta)^s u \\
&= -2 \int_\Omega u_t(\alpha \Delta u - \beta(-\Delta)^s u).
\end{aligned}$$

Since u is a solution, we know that $u_t - \alpha\Delta u + \beta(-\Delta)^s u = 0$ in Ω, so

$$E'(t) = -2 \int_\Omega |\alpha \Delta u - \beta(-\Delta)^s u|^2 \leq 0,$$

where the equality holds if and only if u is constant. □

In the next result we show that solutions of (7.104) converges to a constant as $t \to +\infty$.

Proposition 7.4.3. *Let $u(x,t)$ be a classical solutions of (7.104), in the sense that u is bounded and $|u_t| + |\alpha\Delta u| + |\beta(-\Delta)^s u| \leq K$ for all $t > 0$. Then,*

$$u(x,t) \to \frac{1}{|\Omega|}\int_\Omega u_0\, dx \ \text{ in } L^2(\Omega)$$

as $t \to +\infty$.

Proof. We denote m as the total mass of u, that is

$$m := \frac{1}{|\Omega|}\int_\Omega u_0\, dx,$$

and we also define the function

$$A(t) := \int_\Omega |u-m|^2\, dx.$$

First, by Proposition 7.4.1 we have

$$A(t) = \int_\Omega (u^2 - 2mu + m^2)\, dx = \int_\Omega u^2\, dx - |\Omega| m^2.$$

Then, by (7.106) and the integration by parts formula we obtain

$$\begin{aligned} A'(t) &= 2\int_\Omega u_t u\,dx = 2\alpha\int_\Omega u\Delta u\,dx - 2\beta\int_\Omega u(-\Delta)^s u\,dx \\ &= -2\alpha\int_\Omega |\nabla u|^2\,dx - \beta\iint_\Omega \frac{|u(x,t)-u(y,t)|^2}{|x-y|^{N+2s}}\,dxdy, \end{aligned}$$

and so A is decreasing. Now, keeping in mind Proposition 7.4.1, using [28, Lemma 3.10] we get

$$A'(t) \le -2\alpha\int_\Omega |\nabla u|^2\,dx - c\int_\Omega |u-m|^2\,dx \le -c\int_\Omega |u-m|^2\,dx = -cA(t),$$

for some $c > 0$ which does not depend on t. Then it follows that

$$A(t) \le e^{-ct}A(0),$$

and so, passing to the limit

$$\lim_{t\to+\infty}\int_\Omega |u(x,t)-m|\,dx = 0,$$

which concludes the proof. □

7.5 Mixed superlinear problems

In this section we study a superlinear problem in presence of a mixed operator, namely

$$\begin{cases} -\alpha\Delta u + \beta(-\Delta)^s u + u = f(x,u) & \text{in } \Omega, \\ \text{with } (\alpha,\beta)\text{-Neumann conditions,} \end{cases} \tag{7.107}$$

where $f : \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $f(x,0) = 0$ for almost every $x\in\Omega$. In addition, we assume the following hypotheses:

(f_1) there exist $a\in L^q(\Omega)$, $a\ge 0$, with $q\in((2_s^*)',2)$, $c>0$ and $r\in(2,2_s^*)$ such that
$$|f(x,t)| \le a(x) + c|t|^{r-1}$$
for a.e. $x\in\Omega$ and for all $t\in\mathbb{R}$;

(f_2) denoting $F(x,t) = \int_0^t f(x,\tau)\,d\tau$, we have
$$\lim_{t\to\pm\infty}\frac{F(x,t)}{|t|^2} = +\infty$$
uniformly for a.e. $x\in\Omega$;

(f_3) if $\sigma(x,t) = f(x,t)t - 2F(x,t)$, then there exist $\theta > 1$ and $\beta^* \in L^1(\Omega)$, $\beta^* \geq 0$, such that
$$\sigma(x,t_1) \leq \theta\sigma(x,t_2) + \beta^*$$
for a.e. $x \in \Omega$ and all $0 \leq t_1 \leq t_2$ or $t_2 \leq t_1 \leq 0$;

(f_4)
$$\lim_{t\to 0} \frac{f(x,t)}{|t|^2} = 0$$
uniformly for a.e. $x \in \Omega$.

In (f_1) we have denoted by 2^*_s the fractional Sobolev exponent of order s, that is
$$2^*_s = \begin{cases} \frac{2N}{N-2s} & \text{if } 2s < N, \\ +\infty & \text{if } 2s \geq N, \end{cases}$$
so that the embedding in $L^r(\Omega)$ of $H^s(\Omega)$ (and thus of $X_{\alpha,\beta}$) is compact for every $r < 2^*_s$.

Definition 7.5.1. Let $u \in X_{\alpha,\beta}$. With the same assumptions on f as above, we say that u is a weak solution of (7.107) if
$$\alpha\int_\Omega \nabla u\nabla v\,dx + \frac{\beta}{2}\iint_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy + \int_\Omega uv\,dx = \int_\Omega f(x,u)v\,dx$$
for every $x \in X_{\alpha,\beta}$.

Following this definition, we have that any critical point of the functional $I : X_{\alpha,\beta} \to \mathbb{R}$, defined as
$$I(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)\,dx$$
is a weak solution of (7.107).

We have the following result

Theorem 7.5.2. *If hypotheses (f_1)-(f_4) hold, then problem* (7.107) *two non-trivial constant sign solutions.*

We introduce the functionals
$$I_\pm(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u^\pm)\,dx,$$

where u^+ and u^- are the classical positive part and negative part of u.

Now we want to prove that both $I_\pm$ satisfy the Cerami condition, (C) for short, which stats that any sequence $(u_n)_n$ in $X_{\alpha,\beta}$ such that $(I_\pm(u_n))_n$ is bounded and $(1+\|u_n\|I'_\pm(u_n) \to 0$ as $n \to \infty$ admits a convergent subsequence.

Proposition 7.5.3. *Under the assumptions of Theorem 7.5.2, the functionals $I_\pm$ satisfy the (C) condition.*

Proof. We give the proof for I_+, the proof for I_- being analogous.

Let $(u_n)_n$ in $X_{\alpha,\beta}$ be such that

$$|I_+(u_n)| \leq M_1 \tag{7.108}$$

for some $M_1 > 0$ and all $n \geq 1$, and

$$(1+\|u_n\|)I'_+(u_n) \to 0 \text{ in } X'_{\alpha,\beta} \text{ as } n \to \infty. \tag{7.109}$$

From (7.109) we have

$$|I'_+(u_n)h| \leq \frac{\varepsilon_n h}{1+\|u_n\|}$$

for every $h \in X_{\alpha,\beta}$ and with $\varepsilon_n \to 0$ as $n \to \infty$, that is

$$\begin{aligned} \Bigg|\alpha \int_\Omega \nabla u_n \nabla h\, dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(u_n(x)-u_n(y))(h(x)-h(y))}{|x-y|^{N+2s}}\, dxdy \\ + \int_\Omega u_n h\, dx - \int_\Omega f(x,u_n^+)h\, dx\Bigg| \leq \frac{\varepsilon_n h}{1+\|u_n\|}. \end{aligned} \tag{7.110}$$

Taking $h = u_n^-$ in (7.110), we get

$$\begin{aligned} \Bigg|\alpha \int_\Omega |\nabla u_n^-|^2\, dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(u_n(x)-u_n(y))(u_n^-(x)-u_n^-(y))}{|x-y|^{N+2s}}\, dxdy \\ + \int_\Omega |u_n^-|^2\, dx\Bigg| \leq \varepsilon_n \end{aligned} \tag{7.111}$$

Observing that

$$|u_n^-(x)-u_n^-(y)|^2 \leq (u_n(x)-u_n(y))(u_n^-(x)-u_n^-(y)),$$

we get

$$\|u_n^-\| \leq \varepsilon_n,$$

and so

$$u_n^- \to 0 \text{ in } X'_{\alpha,\beta} \text{ as } n \to \infty. \tag{7.112}$$

Now, taking $h = u_n^+$ in (7.110), we obtain

$$-\alpha \int_\Omega |\nabla u_n^+|^2 \, dx - \frac{\beta}{2} \iint_\Omega \frac{(u_n(x) - u_n(y))(u_n^+(x) - u_n^+(y))}{|x-y|^{N+2s}} \, dxdy \\ - \int_\Omega |u_n^+|^2 \, dx + \int_\Omega f(x, u_n^+) u_n^+ \, dx \leq \varepsilon_n. \tag{7.113}$$

From (7.108) we have

$$\alpha \int_\Omega |\nabla u_n|^2 \, dx + \frac{\beta}{2} \iint_\Omega \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} \, dxdy \\ + \int_\Omega |u_n(x)|^2 \, dx - 2 \int_\Omega F(x, u_n^+) \, dx \leq 2M_1$$

for $M_1 > 0$ and $n \geq 1$, which together with (7.111) leads to

$$\alpha \int_\Omega |\nabla u_n^+|^2 \, dx + \frac{\beta}{2} \iint_\Omega \frac{(u_n(x) - u_n(y))(u_n^+(x) - u_n^+(y))}{|x-y|^{N+2s}} \, dxdy \\ + \int_\Omega |u_n^+|^2 \, dx - 2 \int_\Omega F(x, u_n^+) \, dx \leq M_2 \tag{7.114}$$

For some $M_2 > 0$ and all $n \geq 1$. Adding (7.113) to (7.114) we obtain

$$\int_\Omega f(x, u_n^+) u_n^+ \, dx - 2 \int_\Omega F(x, u_n^+) \, dx \leq M_3$$

for some $M_3 > 0$ and all $n \geq 1$, that is

$$\int_\Omega \sigma(x, u_n^+) \, dx \leq M_3. \tag{7.115}$$

Now we want to prove that $(u_n^+)_n$ is bounded in $X_{\alpha,\beta}$, and we argue by contradiction. Passing to a subsequence if necessary, we assume that $\|u_n^+\| \to \infty$ as $n \to \infty$. Defining $y_n = u_n^+ / \|u_n^+\|$, we can assume that

$$y_n \rightharpoonup y \text{ in } X_{\alpha,\beta} \text{ and } y_n \to y \text{ in } L^q(\Omega) \tag{7.116}$$

for every $q \in (2, 2_s^*)$ and for some $y \geq 0$.

First, we deal with the case $y \neq 0$. We define the set

$$Z(y) = \{x \in \Omega : \, y(x) = 0\},$$

so that $|\Omega \setminus Z(y)| > 0$ and $u_n^+ \to \infty$ for a.e. $x \in \Omega \setminus Z(y)$ as $n \to \infty$. By hypothesis (f_2) we have

$$\lim_{n\to\infty} \frac{F(x, u_n^+(x))}{\|u_n^+\|^2} = \lim_{n\to\infty} \frac{F(x, u_n^+(x))}{u_n^+(x)^2} y_n(x)^2 = +\infty$$

for almost every $x \in \Omega \setminus Z(y)$. On the other hand, by Fatou's Lemma

$$\int_\Omega \liminf_{n\to\infty} \frac{F(x, u_n^+(x))}{\|u_n^+\|^2}\, dx \leq \liminf_{n\to\infty} \int_\Omega \frac{F(x, u_n^+(x))}{\|u_n^+\|^2}\, dx,$$

which leads to

$$\lim_{n\to\infty} \int_\Omega \frac{F(x, u_n^+(x))}{\|u_n^+\|^2}\, dx = +\infty. \tag{7.117}$$

From (7.108) we have

$$-\frac{1}{2}\|u_n\|^2 + \int_\Omega F(x, u_n^+(x))\, dx \leq M_4$$

for some $M_4 > 0$ and $n \geq 1$. Recalling that $\|u_n\|^2 \leq 2(\|u_n^+\|^2 + \|u_n^-\|^2)$, from (7.112) we obtain

$$-\|u_n^+\|^2 + \int_\Omega F(x, u_n^+(x))\, dx \leq M_5$$

for some $M_5 > 0$. Dividing by $\|u_n^+\|^2$,

$$-1 + \int_\Omega \frac{F(x, u_n^+(x))}{\|u_n^+\|^2}\, dx \leq \frac{M_5}{\|u_n^+\|^2}.$$

Passing to the limit, we have

$$\limsup_{n\to\infty} \int_\Omega \frac{F(x, u_n^+(x))}{\|u_n^+\|^2}\, dx \leq M_6$$

for some $M_6 > 0$, which is in contradiction with (7.117), and this concludes the case $y \neq 0$.

Now we deal with the case $y \equiv 0$. We consider the continuous functions $\gamma_n : [0,1] \to \mathbb{R}$, defined as $\gamma_n(t) = I_+(tu_n^+)$ with $t \in [0,1]$ and $n \geq 1$. So, we can define t_n such that

$$\gamma_n(t_n) = \max_{t\in[0,1]} \gamma_n(t). \tag{7.118}$$

Now we define $v_n = (2\lambda)^{\frac{1}{2}} y_n \in X_{\alpha,\beta}$ for $\lambda > 0$. From (7.116), $v_n \to 0$ in $L^q(\Omega)$ for all $q \in (2, 2_s^*)$. Performing some integration from (f_1) we have

$$\int_\Omega F(x, v_n(x))\, dx \leq \int_\Omega a(x)|v_n(x)|\, dx + C \int_\Omega |v_n(x)|^r\, dx,$$

which implies

$$\lim_{n\to\infty} \int_\Omega F(x, v_n(x))\, dx = 0. \tag{7.119}$$

Since $\|u_n^+\| \to \infty$, there exists $n_0 \geq 1$ such that $(2\lambda)^{\frac{1}{2}}/\|u_n^+\| \in (0,1)$ for all $n \geq n_0$. Then, from (7.118),

$$\gamma_n(t_n) \geq \gamma_n\left(\frac{(2\lambda)^{\frac{1}{2}}}{\|u_n^+\|}\right)$$

for all $n \geq n_0$. So,

$$I_+(t_n u_n^+) \geq I_+((2\lambda)^{\frac{1}{2}} y_n) = \lambda\|y_n\|^2 - \int_\Omega F(x, v_n(x))\,dx.$$

Then (7.119) implies that

$$I_+(t_n u_n^+) \geq \lambda\|y_n\|^2 + o(1),$$

and since λ is arbitrary we have

$$\lim_{n\to\infty} I_+(t_n u_n^+) = +\infty. \tag{7.120}$$

We observe that $0 \leq t_n u_n^+ \leq u_n^+$ for all $n \geq 1$, so from (f_3) we know that

$$\int_\Omega \sigma(x, t_n u_n^+)\,dx \leq \theta\sigma(x, u_n^+)\,dx + \|\beta^*\|_1 \tag{7.121}$$

for all $n \geq 1$. Clearly, $I_+(0) = 0$. In addition, (7.108) and (7.111) imply that $I_+(u_n^+) \leq M_7$ for some $M_7 > 0$. Together with (7.120) this implies that $t_n \in (0,1)$ for all $n \geq n_1 \geq n_0$. Since t_n is a maximum point

$$\begin{aligned}0 &= t_n\gamma_n'(t_n)\\ &= +\frac{\beta}{2}\iint_\Omega \frac{(t_n u_n(x) - t_n u_n(y))(t_n u_n^+(x) - t_n u_n^+(y))}{|x-y|^{N+2s}}\,dxdy\\ &+ \alpha\int_\Omega |\nabla t_n u_n^+|^2\,dx + \int_\Omega |t_n u_n^+|^2\,dx - \int_\Omega f(x, t_n u_n^+) t_n u_n^+\,dx,\end{aligned}$$

and recalling that

$$|t_n u_n^+(x) - t_n u_n^+(y))|^2 \leq (t_n u_n(x) - t_n u_n(y))(t_n u_n^+(x) - t_n u_n^+(y))$$

we have

$$\|t_n u_n^+\|^2 - \int_\Omega f(x, t_n u_n^+) t_n u_n^+\,dx \leq 0. \tag{7.122}$$

Adding (7.122) to (7.121) we obtain

$$\|t_n u_n^+\|^2 - 2\int_\Omega F(x, t_n u_n^+)\,dx \leq \theta\sigma(x, u_n^+)\,dx + \|\beta^*\|_1,$$

that is

$$2I_+(t_n u_n^+) \le \theta\sigma(x, u_n^+)\, dx + \|\beta^*\|_1.$$

Hence, from (7.120) follows that

$$\lim_{n\to\infty} \sigma(x, u_n^+)\, dx = +\infty. \tag{7.123}$$

Combining (7.115) and (7.123) we obtain a contradiction, which conludes the case $y \equiv 0$.

So, we proved that $(u_n^+)_n$ is bounded in $X_{\alpha,\beta}$, and from (7.112) we know that $(u_n)_n$ is bounded in $X_{\alpha,\beta}$. Then we can assume

$$u_n \rightharpoonup u \text{ in } X_{\alpha,\beta} \text{ and } u_n \to u \text{ in } L^q(\Omega) \tag{7.124}$$

with $q \in (2, 2^s)$ and for some $u \in X_{\alpha,\beta}$. Taking $h = u_n - u$ in (7.110) we have

$$\begin{aligned} \|u_n\|^2 - \alpha \int_\Omega \nabla u_n \nabla u\, dx - \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(u_n(x) - u_n(y))(u(x) - u(y))}{|x-y|^{N+2s}}\, dxdy \\ - \int_\Omega u_n u\, dx - \int_\Omega f(x, u_n^+)(u_n - u)\, dx \le \varepsilon_n \end{aligned} \tag{7.125}$$

From (f_1) and (7.124) we know that

$$\lim_{n\to\infty} \int_\Omega |f(x, u_n^+)(u_n - u)|\, dx = 0.$$

Passing to the limit in (7.125) we obtain

$$\begin{aligned} \lim_{n\to\infty} \Bigg(\|u_n\|^2 - \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(u_n(x) - u_n(y))(u(x) - u(y))}{|x-y|^{N+2s}}\, dxdy \\ - \alpha \int_\Omega \nabla u_n \nabla u\, dx - \int_\Omega u_n u\, dx \Bigg) = 0. \end{aligned}$$

This implies that $\|u_n\| \to \|u\|$, so from the S property we know that $u_n \to u$ in $X_{\alpha,\beta}$. Then I_+ satisfies the (C) condition, which concludes the proof. $\square$

We can now give the proof of Theorem 7.5.2.

Proof of Theorem 7.5.2. We want to apply the Mountain Pass Theorem to I_+. From Proposition 7.5.3 we know that I_+ satisfies the (C) condition, so we only have to verify the geometric conditions.

From (f_1) and (f_4), for every $\varepsilon > 0$ there exists C_ε such that

$$F(x, t) \le \frac{\varepsilon}{2}|t|^2 + C_\varepsilon |t|^r$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$. Then

$$\begin{aligned} I_+(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u^+)\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\|u\|_2^2 - C_\varepsilon\|u\|_r^r \\ &\geq \frac{1-\varepsilon C_1}{2}\|u\|^2 - C_2\|u\|_r^r. \end{aligned}$$

From this we kwon that, if $\|u\| = \rho$ is small enough,

$$\inf_{\|u\|=\rho} I_+(u) > 0.$$

Now, we take $u \in X_{\alpha,\beta}$ with $u > 0$ and $t > 0$, then

$$\begin{aligned} I_+(u) &= \frac{t^2}{2}\|u\|^2 - \int_\Omega F(x,tu)\,dx \\ &= \frac{t^2}{2}\|u\|^2 - t^2\int_\Omega \frac{F(x,tu)}{|tu|^2}u^2\,dx. \end{aligned}$$

By Fatou's Lemma

$$\int_\Omega \liminf_{n\to\infty} \frac{F(x,tu)}{|tu|^2}u^2\,dx \leq \liminf_{n\to\infty}\int_\Omega \frac{F(x,tu)}{|tu|^2}u^2\,dx,$$

so from (f_2) we know that

$$\lim_{n\to\infty}\int_\Omega \frac{F(x,tu)}{|tu|^2}u^2\,dx = +\infty.$$

Then

$$\lim_{n\to\infty} I_+(tu) = -\infty,$$

therefore there exists $e \in X_{\alpha,\beta}$ such that $\|e\| > \rho$ and $I_+(e) < 0$.

Now we can apply the Mountain Pass Theorem to I_+ to obtain a nontrivial critical point u. In particular, we have

$$\begin{aligned} 0 &= \alpha\int_\Omega |\nabla u^-|^2\,dx + \frac{\beta}{2}\iint_\mathcal{Q} \frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+2s}}\,dxdy \\ &+ \int_\Omega |u^-|^2\,dx - \int_\Omega f(x,u^+)u^-\,dx \\ &= \alpha\int_\Omega |\nabla u^-|^2\,dx + \frac{\beta}{2}\iint_\mathcal{Q} \frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+2s}}\,dxdy \\ &+ \int_\Omega |u^-|^2\,dx. \end{aligned}$$

Recalling that

$$|u^-(x) - u^-(y)|^2 \le (u(x) - u(y))(u^-(x) - u^-(y))$$

we obtain

$$0 \ge \|u^-\|^2,$$

and so $u^- \equiv 0$. Then $I_+(u) = I(u)$, and so $u \ge 0$ is a solution of (7.107). Arguing in the same way for I_-, we can find a nontrivial negative solution of (7.107). □

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