The Mathematics of Painting: The Birth of Projective Geometry in the Italian Renaissance[**]

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Abstract. We show how the birth of perspective painting in the Italian Renaissance led to a new way of interpreting space that resulted in the creation of projective geometry. Unlike other works on this subject, we explicitly show how the craft of the painters implied the introduction of new points and lines (points and lines at infinity) and their projective coordinates to complete the Euclidean space to what is now called projective space. We demonstrate this idea by looking at original paintings from the Renaissance, and by carrying out the explicit analytic calculations that underpin those masterpieces.

Keywords and phrases: Renaissance, Piero della Francesca, painting, perspective, analytic projective geometry, points and lines at infinity.

«Porticus aequali quamvis est denique ductu
stansque in perpetuum paribus suffulta columnis,
longa tamen parte ab summa cum tota videtur,
paulatim trahit angusti fastigia coni,
tecta solo iungens atque omnia dextera laevis
donec in obscurum coni conduxit acumen.»

Titus Lucretius Carus, *De rerum natura*, IV 426–431

1. Introduction

The birth of projective geometry through the contribution of Italian Renaissance painters is a topic that has originated a large and very interesting bibliography, some of which is referred to in this article. Most of the existing literature dwells on the evolution of the understanding of the techniques that painters and artists such as Leon Battista Alberti and Piero della Francesca developed to assist them (and other painters) in creating realistic representations of scenes. These techniques, of course, are a concrete translation of ideas that slowly germinated and were only later completely developed into a new branch of geometry that goes under the name of projective geometry.

The point of view that we are taking in this article, however, is to strengthen the linkage between the pictorial ideas and the mathematical underpinnings. More to the point, the entire architecture of prospec-

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tive painting consists in realizing that the space of vision cannot be represented through the usual Euclidean space, but requires the inclusion of new geometrical objects that, properly speaking, do not exist in the Euclidean space. We are referring here to what mathematicians call improper points and improper lines or, with a more suggestive term, points and lines at infinity. Unlike most other studies, for example [4] and [12], we use here the approach and the terminology from analytic projective geometry, rather than the proportion theory from Euclidean geometry, by introducing the notion of projective coordinates. Just as the birth of projective geometry was stimulated by pictorial necessities, we show here how the language of this new geometry can be applied to those necessities.

There are two main reasons for this approach. On one hand, we believe the projective terminology allows a simpler way to treat the technical task at hand, namely the identification of the technical processes that a painter needs to represent a scene. But there is a deeper reason: perspective is not simply a technique; rather it is a radical change of perspective (pun intended) on what space is. In order to formally perfect the process of representation, the mathematicians had to introduce new objects, new points, new lines, new planes. It is by introducing these objects that mathematicians were able to create a logically consistent view of the pictorial space, that allowed them a formally unimpeachable process through which what we see can be translated into what we draw. The new line, plane, space (which are now the projective line, the projective plane, the projective space) resemble (and contain) the old Euclidean line, plane, space, but perfect the nature of their properties. So, for example, while in the Euclidean plane we say that any two distinct lines intersect in a point *unless* they are parallel, in the new projective plane we can say that any two distinct lines intersect in a point, without exception. Projective geometry is not just a new and useful technique, it is a radically different way of representing the space around us.

We should add a couple of notes for the reader. Projective geometry is born of the necessity to understand the phenomenon of apparent intersection between parallel lines, and most of our article is devoted to this aspect. However, once the mathematics is clear, projective geometry allows the study of much more complex situations. For example, the same techniques that we illustrate in our article, can be utilized to determine how to represent the halo of a saint, or the shadow of a lamp against the wall of a church. This topic goes beyond the purposes of this article, but we did not want the reader to think that projective geometry exhausts its role with the study of points and lines. We should add that, like it often happens in mathematics, the theory of projective geometry and its developments has taken a life of its own, and it is now one of the most fertile and successful fields in all of mathematics.

To begin our analysis of the evolution of the prospective point of view in painting, we will look at a few paintings from the early renaissance. Specifically, in section 2, we will look at two Tuscan painters: Giotto, whose worldwide fame rests on his fresco cycle in Padova (where he depicted the life of Jesus and the life of the Virgin Mary), and possibly (attribution is disputed) on his frescos in Assisi (where he depicted the life of San Francesco), and the equally important Duccio di Buoninsegna, whose *Maestà* is visible at the Duomo in Siena.

Giotto was considered, at the time, the greatest living painter, and he is usually credited with being the link between the Byzantine style and the Renaissance, and the first to adopt a more naturalistic style. Giotto was an attentive observer of reality, as we can see by looking at the faces and figures in his paintings, but because of the lack of appropriate technique, his approach to architecture appears a mixture of artificial and fantastic.[1] In this section, we consider some of his works, as well as Duccio's paintings, to highlight both their early understanding of the need for new ideas, as well as their insufficient clarity on what those ideas would need to be.

Section 3 is devoted to the mathematical description of the process that is necessary for a faithful representation of a three-dimensional scene on a canvas. This section is where we are able to introduce the basic ideas that will lead to the projective space. How Leon Battista Alberti and Piero della Francesca understood such ideas is the subject of Section 4, where we go back to the original texts, and paintings, to illustrate the way in which the theory of projective geometry was applied in these more advanced works from the Renaissance. To be precise, we will show that in fact Leon Battista Alberti did not fully justify his technique (*costruzione legittima*), and so we have an example of a process which seems to work, while its own developers are not yet fully aware of its theoretical justification. The last Section, before our final conclusions, inverts the process, so to speak. Instead of discussing how to use geometry to represent a scene on the canvas, we will take a painting as a starting point, to reconstruct what the scene that the painter had in mind must have been. This is an interesting exercise, not only for the mathematician, but for the art historian as well, since this reconstruction can help shed light on some interpretation issues, as we will discuss in more detail in the section.

[1] The reader is referred to [18], [26] for a careful reconstruction of the path from natural to artificial perspective in the Middle-Ages and Renaissance. See also the bibliographies [19], [21], [25].

Figure 1. Giotto, La cacciata dei diavoli da Arezzo, *scene from "Storie di San Francesco", (1295–1299), fresco, Basilica Superiore di Assisi.*

2. Early Steps: Giotto (1267–1337) and Duccio di Buoninsegna (1255/60–1318/19)

If one takes a look at any of Giotto's frescos, the first thing that jumps to the eye is a really distorted sense of distances, positions, and sizes of the elements of the pictorial composition. As we see above (figure 1) in a fresco that represents San Francesco who chases away the devils from the city of Arezzo, the buildings have odd angles, the figures are too big, and it looks like we are watching the scene both from the top and from the side (note how we see the side of the walls surrounding Arezzo, but also the buildings inside the walls themselves). What is going on?

The answer to this question lies in the fact that Giotto is one of those painters who found themselves in a moment of epochal transformation. A moment in which painters understood that the way we see objects, and the way objects are, do not coincide. More specifically, when we think of a table, when we touch a table, we deal with a rectangle. This is what most tables are, and if we close our eyes and simply touch the table, we perceive a rectangle. Opposite sides are parallel, and the angles between contiguous sides are right angles (ninety degrees). But when we look at a table, or when we try to draw a table, something completely different appears. Now the angles become acute or obtuse, depending on where we are looking, and the parallel sides may not appear parallel anymore.

Figure 2. Duccio di Buoninsegna, L'ultima Cena, *scene from the back of the "Maestà", (1308–1311), tempera on wood, Museo dell'Opera del Duomo, Siena.*

So, the painter has to recognize a complex shift: if the table has to look right, it has to be drawn wrong. Instead of a rectangle, something else has to be drawn, in order to trick the viewer's brain into recognizing a properly positioned table. If the painter were a mathematician, he would recognize that there are two geometries that conflict with each other: the geometry of touching (the geometry of sculpture), and the geometry of seeing (the geometry of painting). But this must have been incredibly difficult for Giotto and his contemporaries back in the fourteenth century. This difficulty explains why his frescos appear so odd, and why the angles in the buildings that he depicts are so un-lifelike. It is because Giotto understood that right angles do not always appear as right, but in fact they need to be depicted as acute or obtuse. But he did not grasp, for example, the fact that parallel lines don't always appear parallel. The unrealistic sizes of the figures in his frescos are a consequence of a similar cognitive dissonance. Giotto realized that objects that are closer to us appear larger than objects at a distance. But he lacked the mathematics to figure out the precise proportions that should be used. As we will see in Section 4, it will only be with Leon Battista Alberti (1404–1472) that a precise method to address this issue will be developed.

This conflict is quite apparent in another great contemporary of Giotto, namely Duccio di Buoninsegna. In his *Maest*à, there is a section where Duccio paints a *Last Supper* (figure 2).

The central object in any such painting is the table, and when we look at this representation, we have the impression that the plates on the table are on

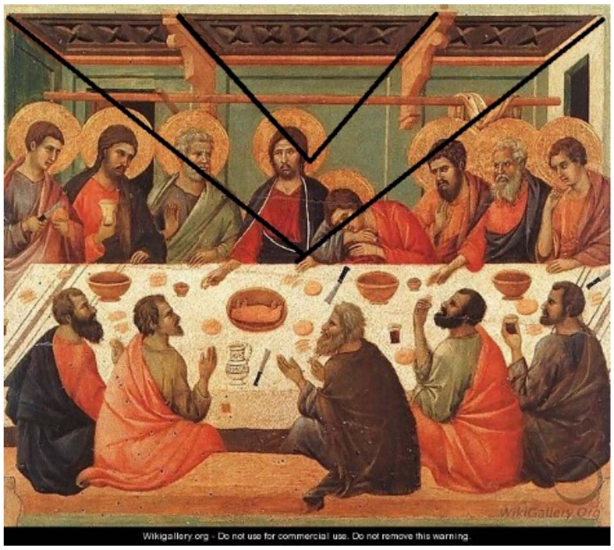

Figure 3.

the verge of falling on the floor. The reason for such an impression is that the table is represented not as a rectangle (Duccio like Giotto realized that this would not have worked), nor as a trapezoid (which is the correct representation). Rather, it is a parallelogram, in which the right angles are eliminated (as they should), but the parallelism among sides is preserved, thus offering a totally inadequate representation. One should also look at the ceiling and the beams in the ceiling itself. In the room, such beams are clearly parallel, and we know (we will get into more details later) that parallel lines must be represented as converging to a point. But, as we see in the modified picture above, while Duccio understands this, he seems not to know that all lines parallel to each other must converge in the same point. Instead, as we see (figure 3), the internal beams converge on the figure of Christ, while the external beams converge on the table. The outcome is a ceiling that is clearly wrong.

These two examples are not offered to demean these great painters, but rather to suggest how complex it must have been for those living in the XIII and XIV century, to realize how to go from Euclidean geometry to projective geometry. As we will see in the next sections, this process will lead to the understanding that a fundamental mathematical truth is hidden somewhere. And it was because of a few artists with great mathematical background, that this was finally understood. Before we get to that point, however, we will take a brief mathematical detour.

3. The Mathematics of Perspective

It is an interesting challenge to illustrate in modern terms how the effort to understand vision and painting leads to a new geometry, that mathematicians call projective geometry.

First of all, notice that the main approach of a painter to this problem does not discuss how the eye and the brain allow us to see the surrounding world, but only how the light and the colors reach the eye: in fact this is the environment in which a painter can mainly intervene. It is therefore reasonable to think the eye as a point of our 3-dimensional space, and set its position as the origin O of a system of Cartesian coordinates (x, y, z).[2]

We can then base our study on the experimental fact that light rays essentially propagate along straight lines in space, and hence assume that the vision relies upon what all the infinite rays entering O bring to the eye.[3] Every ray entering O brings a colored point, which comes from the object being viewed (possibly the sky). Therefore each ray entering the origin contributes to the vision with a colored point. Of course O brings no contribution to the vision.

Let us now imagine to be able to insert, between the observer, and the object which is observed, a canvas, possibly a transparent one. Then, it is clear that the ray that joins the object to the observer will intersect the canvas in one point and one alone. That point, with the color that the object has, becomes like a pixel on the canvas, and the entirety of the pixels that are generated in this way, is a faithful representation of the object itself. Note that if we were to take two different canvases, the eye would not be able to distinguish a difference in the images (we hope the reader will forgive our use of relatively modern terms such as pixel).

The beautifully simple, but not at all easy, mathematical idea that we have just described can be expressed by saying that we have represented each spatial ray r entering the origin O by means of one of its points only, $P(r)$, that lies on the chosen canvas and contains all the information that the ray brings to the eye. Of course, if we move the representing point $P(r)$ (with all the information that it carries) along the ray r, (in other words we change the canvas) the resulting view will not be affected at all. In principle, the representing point of a ray can be chosen to be *any* point of the ray. In practice, it is usually chosen to stay on a plane – the plane of the painting, its canvas – or, in different, more mathematical contexts, on a sphere (we will not discuss this more complex type of representation).

[2] This is clearly a simplification that does not model the anatomical aspects of vision.

[3] This interpretation of vision, a revolutionary one in the Renaissance, was due mainly to ibn Al-Haytham, also known as Alhazen, a well-known authority of the 11th century. Indeed, this visual theory was based on his *Book of Optics (Kitab al-Manazir)* [3]. On this topic see [5], [6], [7].

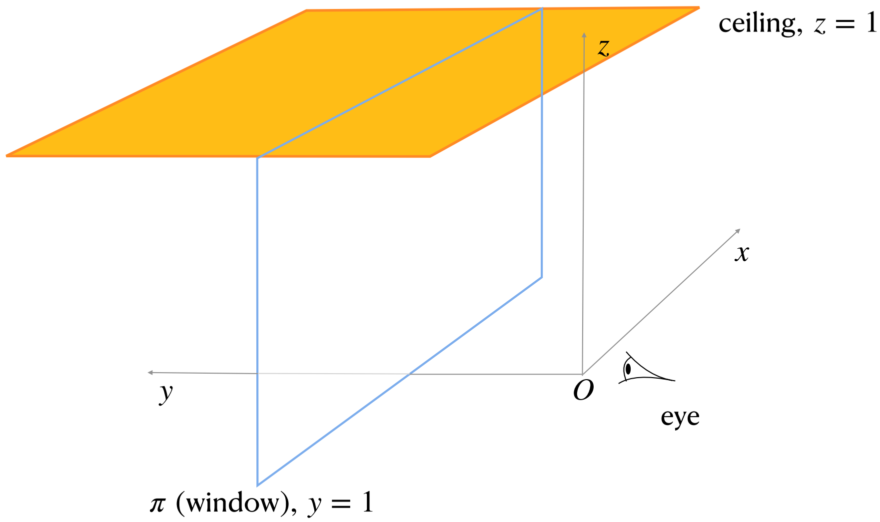

Figure 4.

To help the reader understand what follows, we suggest a simple experiment. As you go through the next few lines, we would ask that you sit in front of a window, looking at whatever lies in front of you.[4] And now, as you sit, imagine your eye to be the origin O of a system of coordinates (figure 4). The x-axis of the system will exit from your eye (the origin) and points to your right, the y-axis exits from O and points forward towards the window, and finally the z-axis is the vertical line from O up. Using the Cartesian coordinates so established, we will call π the plane of the window (we assume you are sitting upright, and therefore the window is perpendicular to the y-axis). If we assume the distance from the reader to the window to be one unit, we would mathematically express the equation of this plan as $y = 1$ (the mathematical notations will be useful in the sequel when we will write the equations of the transformations, but are not necessary for the understanding of the basic ideas). We will also identify the ceiling of the room with the plane of equation $z = 1$, i.e. with the horizontal plane located at a distance of 1 unit from the eye, above the head of the reader.

Now, two points of Cartesian coordinates (x, y, z) and (u, v, w) give the same contribution (pixel) $P(r)$ to the vision if they belong to the same ray r entering the origin (where the eye is located)[5]. Therefore we will denote the contribution (pixel) $P(r)$ to the vision, given by the ray r containing the point (x, y, z) and entering the origin, with the symbol $[x, y, z]$, and establish that $[x, y, z] = [tx, ty, tz]$ for all nonzero real numbers t. The idea is that $[x, y, z]$ and $[tx, ty, tz]$ will indicate the same pixel, positioned on different, parallel canvas.

Since we have taken the window to be represented by the equation $y = 1$, the pixel $[x, y, z]$ generated by (x, y, z) will correspond, on the window, to a point with $y = 1$; this can only be obtained by taking $t = 1/y$ and therefore the coordinate of the pixel on the window will be $(x/y, 1, z/y)$. (figure 5).

Rays that enter the eye at O will intersect the plane π (just like when you are watching the countryside from inside your home, the entering rays would all intersect the glass of the big window). If for all rays r we place the representative point $P(r)$ on the plane π than we have made the perfect theoretical painting that represents the landscape the eye is watching.

And here are a few surprises. A straight line s of the observed landscape will be seen by the eye through all the rays of the plane L that contains O and the line s. We can then represent s on the painting π as the intersection of L and π. This demonstrates (in an empirical way) one of the first results of projective geometry, namely the fact that a line is transformed (by projections) into another line (the reader is invited to reflect on what would happen, however, if the line were one of the rays).

With this in mind, if we are given the equations of a few beams of the ceiling of our room in the 3-space, we can for example compute the equations of the beams, and how they will appear in the painting π. As we have seen in Duccio's example in the previous section, the issue of representing ceiling beams was in fact one of the most difficult to understand.

In our Cartesian environment, let us consider 3

[4] The metaphor of the window was used by Leon Battista Alberti to explain his *costruzione legittima*. According to Gerard Simon, without the new ideas of ibn Al-Haytham on vision, Alberti's window would not have been thinkable: one of the many examples of historical encounters between Western and Arab cultures [20].

[5] Mathematically, this means that there is a nonzero real number t such that $(u, v, w) = t(x, y, z) = (tx, ty, tz)$.

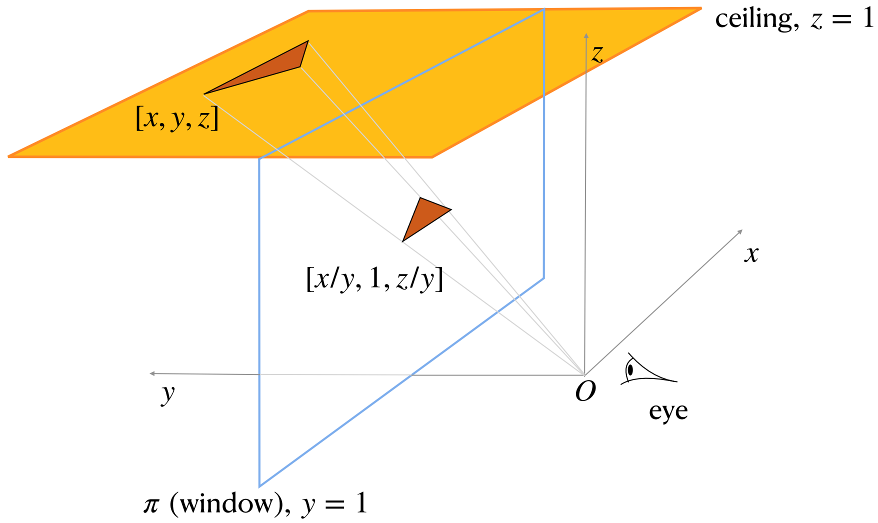

Figure 5.

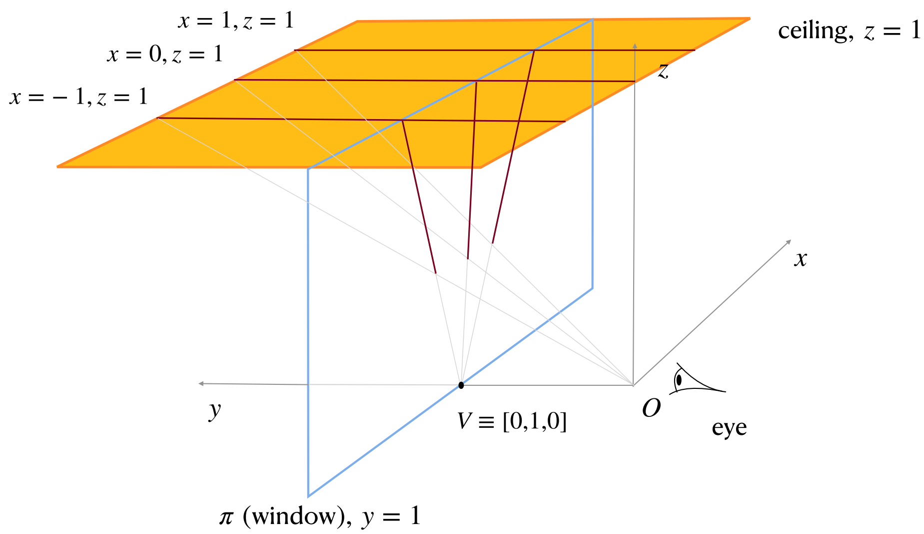

Figure 6.

parallel beams in the ceiling $z=1$, of equations

$$x=-1 \quad \text{and} \quad z=1$$
$$x=0 \quad \text{and} \quad z=1$$
$$x=1 \quad \text{and} \quad z=1,$$

respectively[6]. These three beams are all parallel to the y-axis as indicated in figure 6.

Note that a point on the first beam (the one with equations $x=-1, z=1$) will have coordinates $(-1,y,1)$, where the x and the z coordinates are fixed because of the planes and the y-coordinate is free to range in any way.

If we now search for the three sets of points that represent the contributions to the vision (pixels), coming from the three beams, we get (with arbitrary y)

$$[-1,y,1]$$
$$[0,y,1]$$
$$[1,y,1].$$

As we have already pointed out, to place them on the painting π of equation $y=1$, we just divide the (so called homogeneous) coordinates by (the arbitrary nonzero) y and get, with the established notations:

(1) $$[-1/y,1,1/y]$$
$$[0,1,1/y]$$

[6] The reader will note that we use two equations to represent a line. This is because one can think of a line as the intersection of two planes, each one with its own equation. In this particular case, we are looking at lines which are on the ceiling (and so all of their points have $z=1$), but also that are perpendicular to the x-axis and therefore have a fixed value for x (in the three cases, respectively, $x=1, x=0, x=-1$).

$$[1/y, 1, 1/y],$$

i.e., by putting $u = 1/y$,

$$(2) \qquad \begin{aligned} &[-u, 1, u] \\ &[0, 1, u] \\ &[u, 1, u]. \end{aligned}$$

These are the equations of three straight lines in π that are not only nonparallel, but that all meet at the point $V = (0, 1, 0)$.[7] (figure 6).

If the reader has followed the process, (s)he should be noticing that this process establishes a certain correspondence between the ceiling and the canvas. Every point of the ceiling has a corresponding point on the canvas, but not every point on the canvas comes from a point on the ceiling. Indeed, it is apparent that the points of the canvas $y = 1$ with a negative $z < 0$ coordinate cannot come from the ceiling: the reader will immediately see that these points come from the floor (ground) of the observed landscape (floor of equation, say, $z = -1$). Well, now all seems to be well understood…, but where is the point V coming from? If one tries to reconstruct the process we just described, one will realize that in fact the point V does not come either from any point on the ceiling, or from any point of the floor and this throws a monkey-wrench in our construction. What can be done to fix this apparent irregularity? What is the meaning of this surprising difficulty?

If we analyze carefully what we have done so far, we will notice that the point V is approached on the painting π by the pixels contributed by the rays coming from points of any of the three beams very far from the origin: when y becomes arbitrarily big in (1), $1/y = u$ approaches 0 (see (2)). If we now think of a few lines of the floor, parallel to the y-axis, and repeat for the floor $z = -1$ the procedure used for the ceiling, we will see that the point V is approached on the painting π by the pixels contributed by the rays coming from points of any of these lines of the floor, very far from the origin.

In some sense, the point V (which in the painter terminology is called *the vanishing point of the painting*[8]) is the image of the point on the ceiling that would belong to each of the beams, if they could continue to infinity.[9] In the same way, the point V is the image of the point on the floor that would belong to each of the chosen parallel lines, if they could continue to infinity. But of course the three beams, and the chosen lines of the floor, are parallel, and they have no point in common. What is happening? The answer (whose mathematical formalization we will describe shortly) is that in order for the correspondence between the canvas and the system ceiling-floor to be complete, we need to add a point (which doesn't belong either to the ceiling or to the floor), which is the intersection both of the parallel beams of the ceiling and of the chosen parallel lines of the floor. In fact, as we will discover shortly, even this addition will not be enough. Indeed, we will need to add to the system an entire line, in order to reconstruct a perfect correspondence.

To understand this last point, consider now a family of parallel beams on the ceiling that are not parallel to the y-axis. Consider for instance the three beams of equations (figure 7)

$$(3) \qquad \begin{aligned} &x = y \quad \text{and} \quad z = 1 \\ &x = y - 1 \quad \text{and} \quad z = 1 \\ &x = y - 2 \quad \text{and} \quad z = 1. \end{aligned}$$

These three beams contribute to the vision with the pixels denoted by, for arbitrary y

$$(4) \qquad \begin{aligned} &[y, y, 1] \\ &[y - 1, y, 1] \\ &[y - 2, y, 1], \end{aligned}$$

which placed on the painting π (of equation $y = 1$) become, for arbitrary nonzero y

$$(5) \qquad \begin{aligned} &[y/y, 1, 1/y] = [1, 1, 1/y] \\ &[(y-1)/y, 1, 1/y] = [1 - 1/y, 1, 1/y] \\ &[(y-2)/y, 1, 1/y] = [1 - 2/y, 1, 1/y], \end{aligned}$$

i.e., for an arbitrary nonzero u

$$(6) \qquad \begin{aligned} &[1, 1, u] \\ &[1 - u, 1, u] \\ &[1 - 2u, 1, u]. \end{aligned}$$

These are the equations of three straight lines in π that, again, are not only nonparallel, but that all meet at the point $W = (1, 1, 0)$ (figure 7). Again, the point W does not contribute with a pixel coming from the ceiling, and anyway belongs to the painting π. W is called *vanishing point for the given family of parallel beams.* The point W is approached on the painting

[7] Strictly speaking, the point V cannot be achieved because u is never zero, but we think it is clear what we are describing here.

[8] Lucretius, in his *De rerum natura* describes the vision of a colonnade which extends in front of us in the passage we used as incipit to this article: «Again, a colonnade may be of equal line from end to end and supported by columns of equal height throughout, yet, when its whole length is surveyed from one end, it gradually contracts into the point of a narrowing cone, completely joining roof to floor and right to left, until it has gathered all into the vanishing point of the cone.» Lucretius. *On the Nature of Things*, [17, IV, pp. 426–431]. This fascinating piece of poetry constitutes the first description of the vanishing point that has reached us.

[9] For an extensive explanation, see e.g. [11].

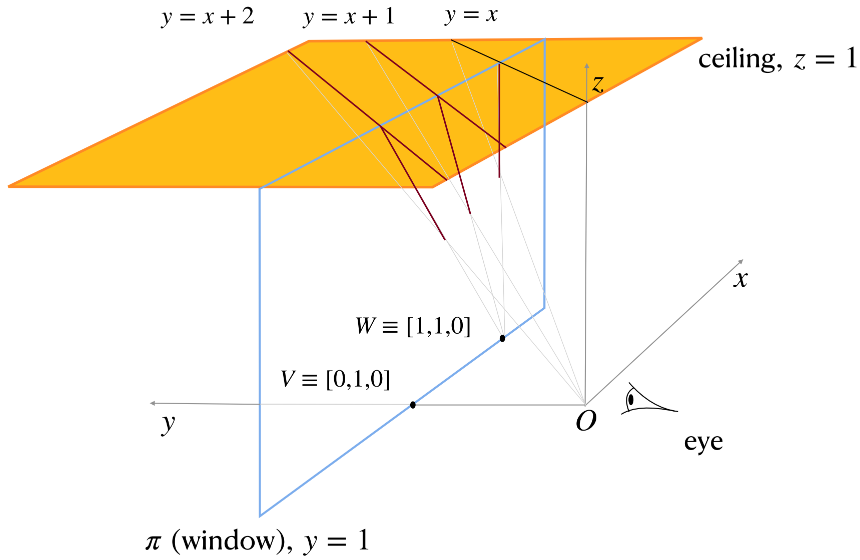

Figure 7.

π by the pixels contributed by the rays coming from points of any of the three beams listed in (3), very far from the origin: when y becomes arbitrarily big in (5), $1/y = u$ approaches 0 in (6). If we now think of a few lines of the floor, obtained by substituting $z = -1$ in place of $z = 1$ in formulas (3), and repeat the procedure used for the ceiling, we will see that the point W is approached on the painting π by the pixels contributed by the rays coming from points of any of these lines of the floor, very far from the origin.

The family of parallel lines that we considered in (3) are actually parallel to the bisecting line of the first and third quadrant of the xy plane of equation $z = 1$ (the ceiling): these lines could be diagonals of a square tessellation of the ceiling. In this situation, we see that the distance between the vanishing point of the painting $V = (0,1,0)$ and the vanishing point $W = (1,1,0)$ coincides with the distance of the eye of the observer from the plane of the painting π (!!). The distance of the eye of a painter from his painting can be encoded in the painting itself. This is the reason why Leon Battista Alberti e Piero della Francesca called W the *distance point*. Notice that the projective approach made the identification of the point W immediate.

The phenomenon we described above is not limited to a particular family of parallel lines. More generally, we will show that every family of parallel lines on the ceiling is represented, on the canvas, by a family of lines that converge to a point that doesn't come either from a point of the ceiling, or from a point of the floor, and that needs to be added in order to complete the correspondence of the canvas with the system ceiling-floor. And all these new points that we will add (and which we call *improper points* or *points at infinity*), will eventually be on a line (*improper line* or *line at infinity*), whose pictorial meaning we will describe in the next few pages.

Let us therefore consider an arbitrary family of parallel lines in the ceiling of equation $z = 1$. Three beams from this family have equations, for any nonzero m,

(7)
$$\begin{aligned} x &= (y-1)/m \quad \text{and} \quad z = 1 \\ x &= (y-2)/m \quad \text{and} \quad z = 1 \\ x &= (y-3)/m \quad \text{and} \quad z = 1. \end{aligned}$$

These three beams contribute to the vision with the pixels denoted by, for arbitrary y

(8)
$$\begin{gathered} [(y-1)/m, y, 1] \\ [(y-2)/m, y, 1] \\ [(y-3)/m, y, 1], \end{gathered}$$

which on the painting π (of equation $y = 1$) become, for arbitrary nonzero y

(9)
$$\begin{aligned} [(y-1)/my, 1, 1/y] &= [1/m - 1/my, 1, 1/y] \\ [(y-2)/my, 1, 1/y] &= [1/m - 2/my, 1, 1/y] \\ [(y-3)/my, 1, 1/y] &= [1/m - 3/my, 1, 1/y], \end{aligned}$$

i.e., for an arbitrary nonzero u

(10)
$$\begin{gathered} [1/m - u/m, 1, u] \\ [1/m - 2u/m, 1, u] \\ [1/m - 3u/m, 1, u]. \end{gathered}$$

These are the equations of three straight lines in π that, again, are not only nonparallel, but that all meet at the point $U = (1/m, 1, 0)$. Again, the point U does not contribute with a pixel coming from the ceiling,

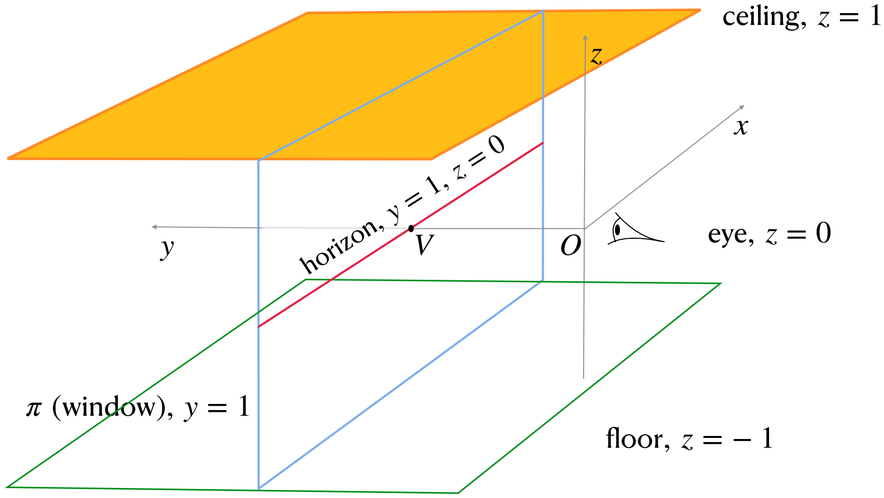

Figure 8.

and anyway belongs to the painting π. The point U is called *vanishing point for the given family of parallel lines*. If we now consider on the floor the family of lines analogous to the family described in (7), but with $z = -1$, we still find the point U, which does not contribute with a pixel coming from the floor, and anyway belongs to the painting π.

It is clear now that (being m arbitrary) the collection of all vanishing points of families of parallel lines of the ceiling or of the floor, contribute to the vision with all the pixels that on the painting π of equation $y = 1$ are of the form

$$[u, 1, 0],$$

i.e. with the line of the painting π of equation ($y = 1$ and) $z = 0$. For obvious and charming reasons, this line is called the *horizon*! (figure 8).

This construction is beautifully illustrated in the following painting of Andrea del Castagno (figure 9), in which we have highlighted (figure 10) three families of parallel lines, with the corresponding vanishing points and the resulting horizon (the reader is advised not to highlight such lines when visiting an art museum!).

In order for the correspondence that we have described to hold for every point, we need to add an improper point for every direction of lines. So, we now have an entire line of improper points on the system ceiling-floor, usually called the improper line. The image of the improper line under the correspondence we have described is thus the horizon on the painting.

In this way we have presented a formalized mathematical method to move the representative point of each ray of light (originating at the ceiling or at the floor) to the painting π.

The reader will appreciate the beautiful symmetry that is emerging. Just like the images of two parallel lines (whether on the ceiling or on the floor) intersect in a point, the vanishing point, that we imagine to be the image of the improper point shared by the parallel lines, so the images of two parallel planes (the ceiling and the floor) intersect in a line, the horizon, that is the image of the improper line that ceiling and floor share. A marvelous symmetry indeed!

The attentive reader will, however, note that while we have added a line (an improper one) to the system ceiling-floor, the full correspondence will require the addition of a line to the (infinite) canvas as well. Indeed, if we are now trying to describe (on the canvas) the line which is represented on the ceiling $z = 1$ by $y = 0$, we easily see that this is not possible. Pictorially, this is a consequence of the fact that the painter cannot represent, in the painting, the points that are vertically above his head. Mathematically, this is a consequence of the fact that the plane $y = 0$ does not intersect the plane π given by $y = 1$. Finally, if one looks at the coordinates $(x, y, 1)$ of a point on the ceiling, and allows y to become zero, one obtains the point $(x, 0, 1)$ which does not belong to the painting. Just like we did before, we would now need to add all these points (for all values of x) to the canvas, and thus complete the plane of the painting with an improper line.

In this new correspondence the following happens:

- Every point in the ceiling and on the floor (except those with $y = 0$) is represented by a point on the canvas.
- Every point on the canvas (except those on the horizon) are the representation of a point on the ceiling or on the floor.
- The points of the horizon can be thought of as images of the improper line that we have added to the system ceiling-floor.

Figure 9. Domenico Ghirlandaio, Ultima cena, *(∼1476), affresco, Cenacolo della Badia di Passignano, Abbazia di San Michele Arcangelo a Passignano, Tavarnelle Val di Pesa, Firenze.*

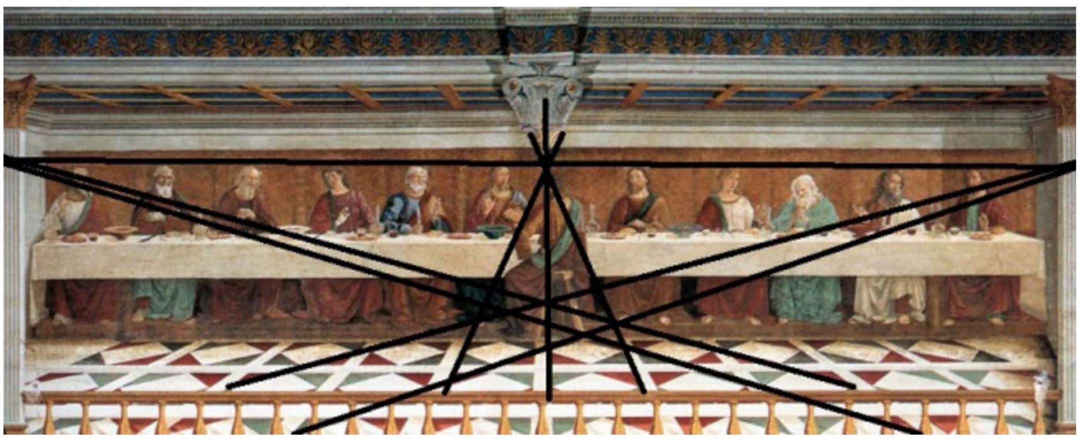

Figure 10.

- The points on the ceiling with $y = 0$ are represented on the improper line that we have added to the canvas.

4. Leon Battista Alberti (1404–1472) and Piero della Francesca (1416/17–1492)

Section 2 described some of the uncertainties that were plaguing the painters of the early Renaissance. Despite these uncertainties, these painters were feeling the strong need to change the nature and the subjects of their work. The interest was slowly shifting away from the ascetic body of the teachers of the medieval scholastics, and was turning to three-dimensional figures, the divine *Maestà* inside gothic churches, or the suggestive backgrounds of battles where the powerful soldiers and the vigor of the horses could find an effective representation. A philosophical development was forcing the painters towards a new understanding of their art as evidenced in the work of artists such as Paolo Uccello (figure 13), Mantegna (figure 11), Masaccio, and the Giambellino (Giovanni Bellini) (figure 12).

Among them Leon Battista Alberti (who wrote in 1435 the treatise *De pictura praestantissima* [1], where he offers a practical guide to perspective drawing) and the great painter and mathematician Piero della Francesca, who built on his knowledge of Euclid and Alberti, to write (towards the end of the XV century) *De prospectiva pingendi* [15], probably the ultimate text on prospective in painting.[10]

> For theory, when a separated from practice, is generally of very little use; but when the two chance to come together, there is nothing that is more helpful to our life, both because art becomes much richer and more perfect by the aid of science, and because the counsels and the writings of learned

[10] Field's essay [10] – an extensive comparison of the treatment of perspective in Alberti's *De pictura praestantissima* [1] and Piero della Francesca's *De prospectiva pingendi* [15] – contains a historically contextualized presentation of the main mathematical tools on which the theory and practice of perspective (and the very initial basis of projective geometry) relied upon. See also [8].

Figure 11. Andrea Mantegna, Cristo morto *(1475–1478), Tempera on canvas, Pinacoteca di Brera, Milano.*

> craftsmen have in themselves greater efficacy and greater credit than the words or works of those who know nothing but mere practice, whether they do it well or ill. And that all this is true is seen manifestly in Leon Batista Alberti, who, having studied the Latin tongue, and having given attention to architecture, to perspective, and to painting, left behind him books written in such a manner, that, since not one of our modern craftsmen has been able to expound these matters in writing, although very many of them in his own country have excelled him in working, it is generally believed; such is the influence of his writings over the pens and speech of the learned; that he was superior to all those who were actually superior to him in work.[11]

This is how Vasari, in his *Vite* [22], (essentially a collection of biographies of painters, sculptors, and architects) described the polyhedric character of Leon Battista Alberti, (1404–1472), architect, mathematician, humanist, musician, who was born in Genova but split his life between the papal court of Rome, and the courts of the Este in Ferrara, of the Malatesta in Rimini, of the Gonzaga in Modena, and belonged the circle of the Florentine humanists. It is important to note that his reflections on painting and sculpture were not simply the byproduct of his interest on painting techniques and on how to prospectively represent the human body, but rather were consequence of a deeper intellectual research.

Figure 12. Giovanni Bellini, The Blood of the Redeemer*, (1460–1465), The National Gallery, London.*

[11] *The Life of Leon Battista Alberti* in: Vasari's *Lives of the Artists*, [23]. See Vasari's *Le vite*: "…Non è cosa che più si convenga alla vita nostra, sì perché l'arte col mezzo della scienza diventa molto più perfetta e più ricca, sì perché gli scritti et i consigli de' dotti artefici hanno in sé molto maggiore efficacia et acquistansi maggior credito che le parole o l'opere di coloro che non sanno altro che il semplice esercizio, o bene o male che essi lo facciano: ché invero leggendo le istorie e le favole et intendendole, un capriccioso maestro megliora continovamente e fa le sue cose con più bontà e con maggiore intelligenza che non fanno gli illetterati. E che questo sia il vero si vede manifestamente in Leon Batista Alberti, il quale, per avere atteso alla lingua latina e dato opera alla architettura, alla prospettiva et alla pittura, lasciò i suoi libri scritti di maniera che, per non essere stato fra gli artefici moderni chi le abbia saputo distendere con la scrittura, ancora che infiniti ne abbiamo avuti più eccellenti di lui nella pratica", [22, 3 voll., vol. III, pp. 283–284]. See also [24, "Leon Battista Alberti", pp. 178–184].

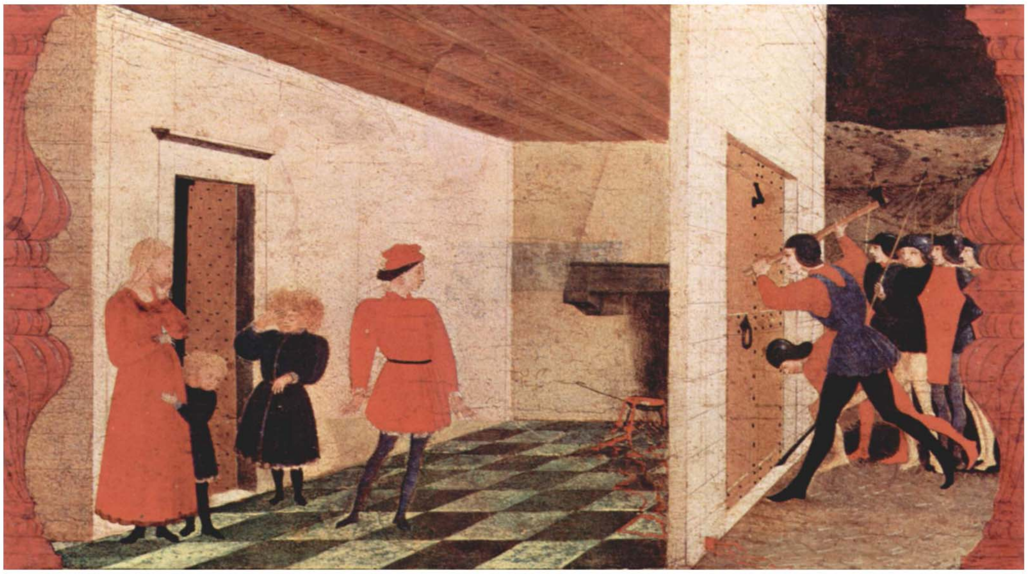

Figure 13. Paolo Uccello, Predella del Miracolo dell'ostia profanata*, (1467–1468), tempera on wood, Galleria Nazionale delle Marche, Urbino.*

In the opening of his treatise *De pictura praestantissima* [1], Leon Battista Alberti explicitly declares that he is not writing as mathematician, but as a painter. And in fact it is clear that the aim of his treatise is to provide painters a practical guide to the use of perspective, instead of investigating and discussing in detail the theoretical aspects and features of that subject, which – as he explicitly states – was certainly quite difficult and not yet well discussed by any author.[12]

> But throughout these whole Treatise I must beg my Reader to take Notice, that I speak of these Things, not as a Mathematician, but as a Painter; for the Mathematician considers the Nature and Forms of Things with the Mind only, absolutely distinct from all Kind of Matter: whereas it being my Intention to set Things in a Manner before the Eyes, it will be necessary for me to consider them in a Way less refined. And indeed I shall think I have done enough, if Painters, when they read me, can gain some Information in this difficult Subject, which has not, as I know of, been discussed hitherto by any Author.

In modern terms, we could say that what Leon Battista Alberti was doing was actually to scientifically present an algorithm that any painter could use to correctly set up the basics of his paintings, from the point of view of the perspective. More specifically, Alberti wanted to give a practical tool to correctly set up the floor, the vanishing point, and the horizon of a painting (see Section 3). Once done this, it was easier for a painter to fill in the painting with all the rest in a reasonably coherent form. This point of view explains also the reason why Alberti is in fact teaching how to represent in perspective a ground floor with square tiles: even in the case of an eventually uniform ground floor, the hidden presence of a fine square grid would help a lot the skilled painter to place objects and human figures properly and proportionally in the table.

It is of great interest to examine the three steps of the algorithm proposed by Leon Battista Alberti to paint a square-tile floor in perspective. In his treatise *De pictura praestantissima*, Alberti considers a square painting π whose side measures six *braccia fiorentine* (in modern terms approximately 348/354 cm). Since the established standard height of a human figure for a painter in those years was three *braccia fiorentine*, in practice Alberti chose to place:

- the horizontal basis of the painting on the floor;
- the point of view of the painter on the straight line orthogonal to the center of the painting π;
- the vanishing point at the center of the painting itself.

One other datum is that the square tiles of the floor to be painted have two sides parallel, and two orthogonal, to the painting π. Finally, here is Alberti's *costruzione legittima*.[13]

[12] Leon Battista Alberti, *The Architecture* [2, p. 241]. See also Leon Battista Alberti, *On Painting* [2, p. 37].

[13] The Renaissance texts on perspective are normally didactic manuals, whose authors take it for granted that perspective is a *vera scientia*. Alberti mentions but does not give a proof for his *costruzione legittima*. In his article, Elkins [9] presents an annotated (incomplete) proof of Alberti's *costruzione*, taken from two propositions of Piero's *De prospectiva pingendi*.

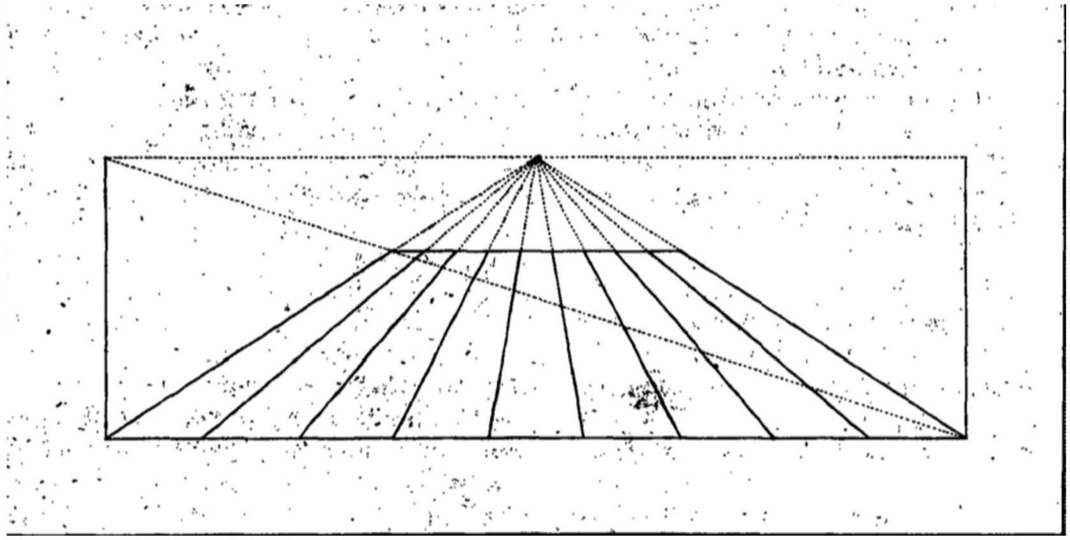

Figure 14. Leon Battista Alberti, Of Painting in three books, *"Book I", in [2].*

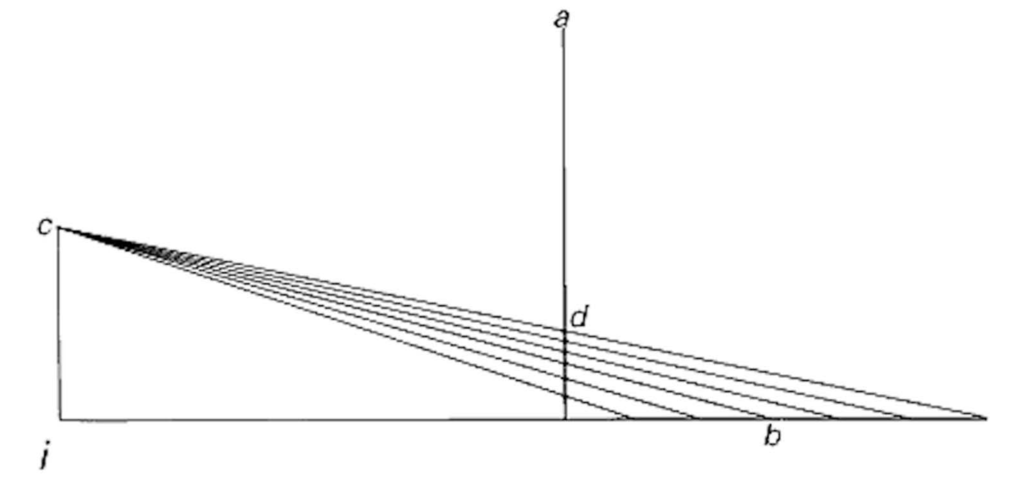

Figure 15.

Step 1. Design the projections of the "orthogonal" straight lines of the floor on the painting π. This step can be done formally as explained in Section 3. It can be practically performed as follows: it is enough to join each intersection of a straight line of the floor with the basis of the painting with the vanishing point (figure 14).

Step 2. Design the heights of the projections of the "parallel" straight lines of the floor on the painting π. The distance of the point of view of the painter from the vanishing point has to intervene in this step. Consider the set painter-painting-floor seen from someone on the right, staying on the plane of the painting π. Figure 15 shows how to construct these heights.

Step 3. Put together steps 1 and 2, and design the projection of the entire square-tile ground floor on the painting π. As shown in figure 16, it is enough to add to the painting obtained in Step 1 a horizontal line at each of the heights constructed in Step 2.

As the reader can see, the algorithmic construction illustrated by Leon Battista Alberti is very simple and does not require any knowledge of sophisticated mathematical theories: this aspect made it really innovative at that time.

Alberti, in his treatise *De pictura praestantissima* [1], gives several other interesting and useful techniques for the painters of his age, some of which are applications of the *costruzione legittima*. The process that we describe below was meant to help the painter to identify the appropriate size of figures at different places on the square-tile floor, exactly what Giotto would have needed in order to represent correctly the figure of San Francesco in the fresco we described in Section 2.

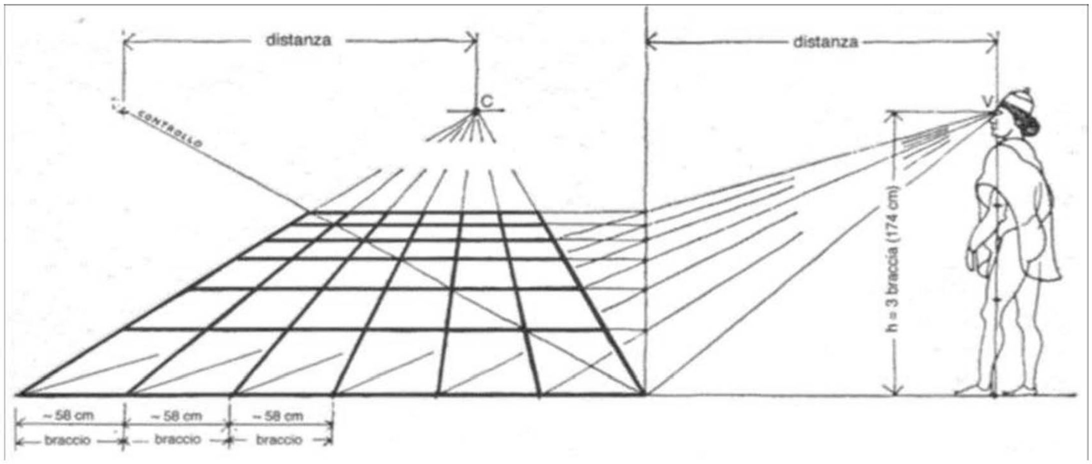

Figure 16.

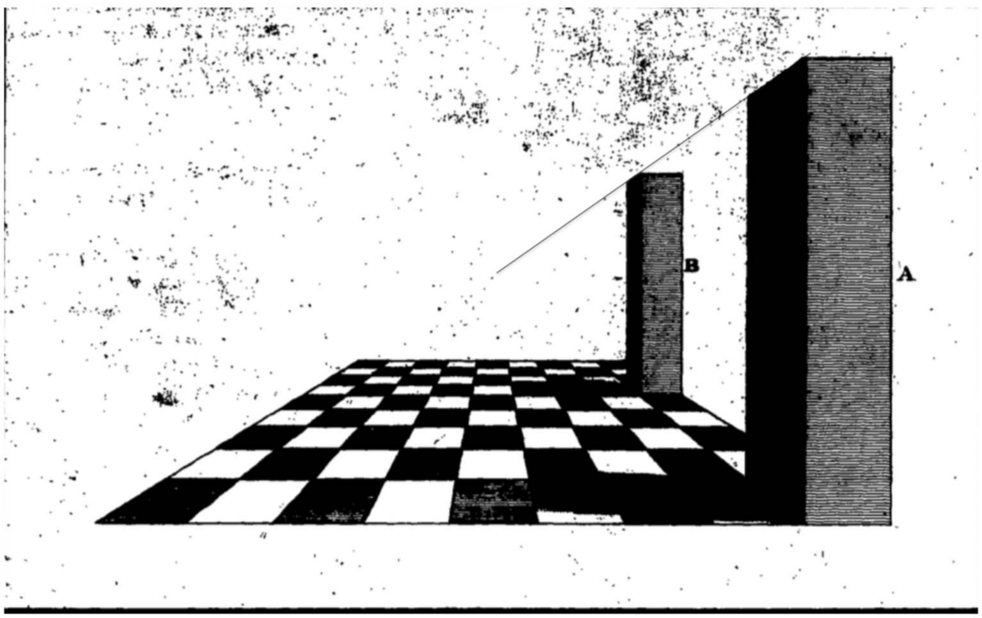

Figure 17. Leon Battista Alberti, Of Painting in three books*, "Book II", in [2].*

Note that the decision of placing the canvas on the floor, implies that only objects and figures placed on the floor along the basis of the painting are represented in 1-1 scale. And one can then use the projection of the side of a square tile parallel to the painting as a unit to give the measures of any object that is placed in the painting precisely on this side (figure 17).

We now show how to use this representation to calculate the distance of the eye of the painter from the painting (figure 18). Note that a horizontal straight line L exiting from the eye and making an angle of 45 degrees with the plane of the painting is parallel to one of the diagonals of the square tiles. Therefore, the line L, and all of the parallel diagonals, encounter the horizon of the painting at a same point A (see Section 3). Therefore, by extending a diagonal of a tile in the painting until it encounters the horizon, one can find the point A. And now, since the triangle with vertices the eye, the vanishing point, and the point A is isosceles (it has the angles equal to 45 degrees), then the distance between A and the vanishing point is equal to the distance of the eye from the painting. Therefore, by means of the side-of-tile-meter one can find the desired distance.

But Alberti, faithful to his promise to provide a practical manual and not a mathematical one, simply says to the painter: extend one of the diagonals of a tile of the floor until it encounters the horizon of the painting at A. Measure the distance between A and the vanishing point. That distance is equal to the distance of the eye of the painter from the painting itself.

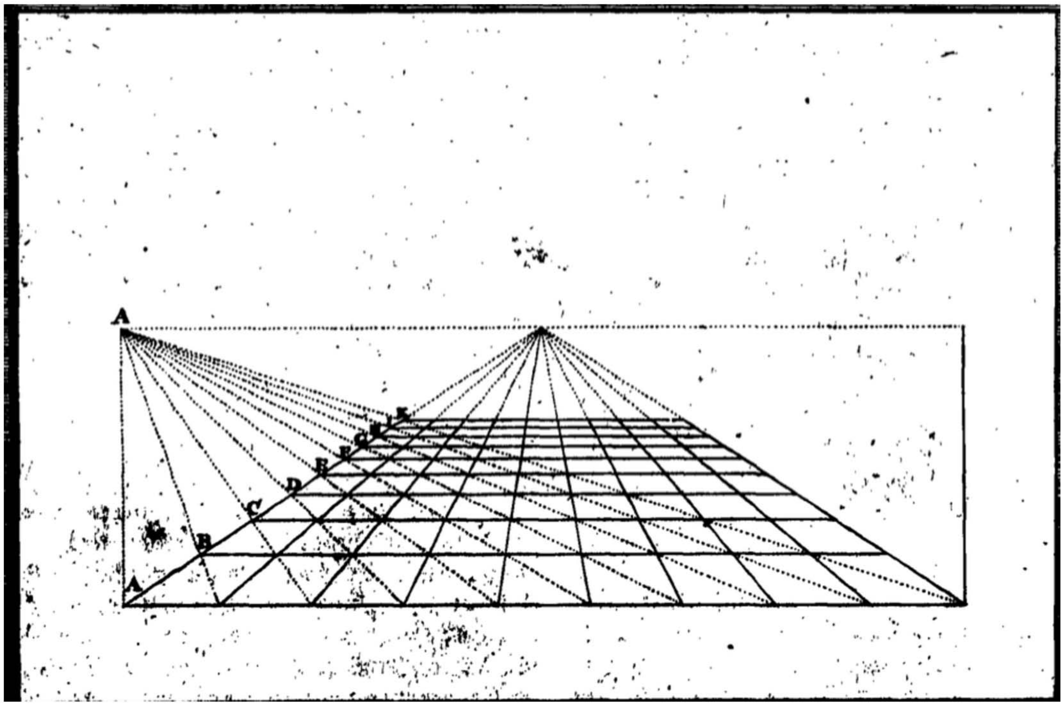

Figure 18. Leon Battista Alberti, Of Painting in three books*, "Book I", in [2].*

5. Reconstructing a Scene from a Painting

As we have shown in the previous sections, the entire purpose of perspective is to take a three-dimensional scene, and translate it into a two-dimensional scene (the painting) in a way that would fool the viewer into believing he is actually looking at the original scene.

But one could ask the inverse question. Can we reconstruct a real life scene, by just looking at its painting? Immediately, we should know that the answer is negative. In fact it is clear that when we go from three dimensions down to two dimensions we must lose some piece of information: the simplest way to convince ourselves of this consists in closing one eye and start walking around. It will become easily apparent that a single eye provides a lack of depth that may make some of the common chores difficult. This is essentially because the eye acts like a projection mechanism, and the image of a three dimensional object is there represented as a flat picture on the bottom of the retina. To remedy this difficulty, most animals have developed a system with two eyes.

We could say that a properly designed painting is like a 2D compression of the data of a 3D scene. And, at least in special situations, the originating scene can be appropriately reconstructed.

The first situation in which a reconstruction is possible is the one in which a painting has a floor. If this is the case, and if one knows both the distance D of the point of view O from the plane of the painting π, and the height H of the point of view from the floor, then all the vertical figures and objects that are standing on the floor can be well placed in 3D. This is made clear by the following Thales-style[14] drawings (figures 19, 20):

For those objects and figures that touch the floor, reconstruction is very easy: one has only to project from the point of view through the painting and reach the ground, on the other side of the painting. Hence the height of each vertical figure/object will become clear.

But how can one recover the measures of H and D, that are – as it is clear – fundamental for the reconstruction? These key measures belong to the world that was external to the painting at the time it was painted: after several centuries, the external world and the participating characters have all disappeared. The only hope is to find pieces of information concerning that world, encoded inside the painting. It is like if we need to enter the painting, in a new Mary Poppins-type walk.

Let us now see how this was done in a specific case [16], for Piero della Francesca's *Flagellazione* (figure 21), a first example – we should say the example –

[14] By this we mean a drawing that utilizes a theorem that is often referred to as Thales' Theorem, namely an important result in elementary geometry about the ratios of different line segments that arise if two intersecting lines are intercepted by two parallel lines.

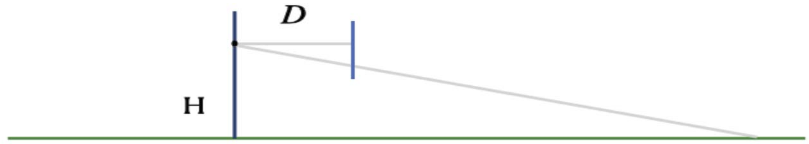

Figure 19.

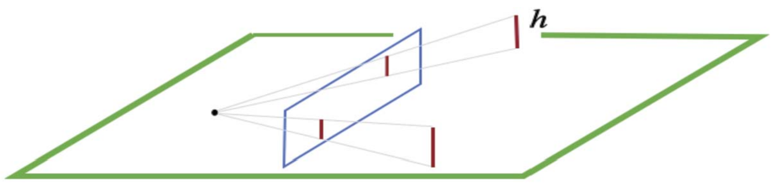

Figure 20.

Figure 21. Piero della Francesca, Flagellazione di Cristo, *(1444–1470), tempera on wood, Galleria Nazionale delle Marche, Urbino.*

of the mathematically well constructed theory of perspective contained in his *De prospectiva pingendi*[15].

First one notes that there are several human figures in the painting, whose knees all touch the line of the horizon (figure 22); since, in the early renaissance and as we have mentioned before when discussing Alberti, the height of the painted human figure was rigidly fixed to be three *braccia fiorentine*, one immediately deduces that the height of the knees, of the horizon, of the vanishing point V, and finally of point of view O of the painter turns out to be approximately 60 cm.

The determination of the distance D is even more challenging. Here is how it was performed in the case of the *Flagellazione*.

[15] In the extensive literature dedicated to the *Flagellazione* the article of Wittkower and Carter [27] offers an analysis made with particular attention to the technical aspects of the perspective and to the historical language used. In this essay the authors also trace the influence of the painter's perspective on real architecture.

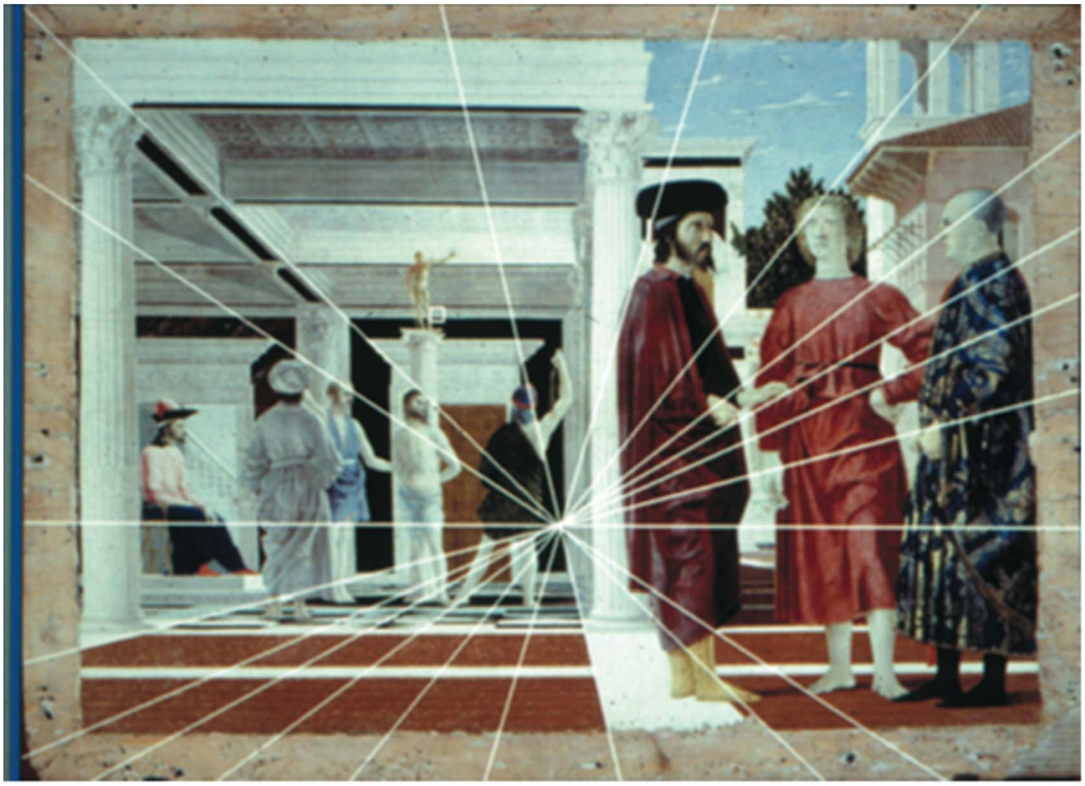

Figure 22.

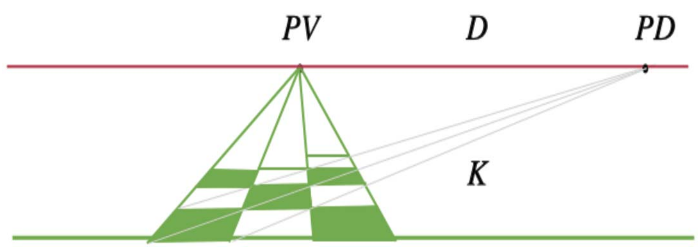

Figure 23.

As shown in figure 23, there is exactly one straight line L exiting from the point of view O (the eye of the painter), intersecting the horizon of the painting in a point PD (the distance point, see Section 3) on the right side of the vanishing point PV, and such that the triangle O, PV, PD, formed by the eye O, the vanishing point PV, and the point PD is isosceles and rectangle in PV.

As we have seen in Section 3, all straight lines of the space that are parallel to L, when represented in the painting, will have the same vanishing point PD. Therefore, if we find a line K in the painting π that is the projection of a straight line of the space parallel to L, then we can solve the problem: we intersect the extension of K with the horizon and find the point PD, and then we try to figure out the "real" distance between PV and PD, which will be the real distance between the eye O of the painter and the painting π.

Note that the floor of the *Flagellazione* has a rectangular tile, whose real sides are (horizontal and) parallel, respectively orthogonal, to the painting. If the tile of the floor were square, then one of the diagonals of a tile would be a possible line K we are searching for.

Further more, we see a decoration of the floor, inscribed in a rectangular tile near the column with the Christ, rendered as an ellipse in the painting. Of course this decoration could be in the reality either an ellipse or a circle.[16] If it were a circle, then we could deduce that the tile is a square, and that its diagonal

[16] This is due to the fact the circle and the ellipse are two possible sections of the visual cone that has vertex in the eye. In reality, and this goes beyond the purpose of this article, projective geometry gives us all the tools to describe the way in which circles are transformed when we project them from the ceiling-floor system to the painting.

is a possible line K. Then we could intersect its extension with the horizon and find PD, which will in turn suggest the distance between PV and PD, and hence the distance between the eye of the painter O and the painting π.

But now art history comes to our help. It appears that in the late 1400's no elliptical decorations were used in a floor of tiles, and hence it can be now demonstrated that the distance between the point of view O (the eye of the painter) and the painting is approximately cm 145, [16].

We conclude this section with a comment on the use of perspective not simply to represent reality, but to attribute additional meanings to it.

We believe that the mathematical analysis and reconstruction of the three dimensional scene of Piero's *Flagellazione* presented above could add a technical contribution to the historical-iconological one, in connection with the hermeneutical problem that alimented the main interpretations of this painting proposed in the last fifty years.[17]

The identification of the figures of the painting, and in particular of the three of them which appear in the foreground relies upon such a scant documentation, so that the iconological enigma hidden in the *Flagellazione* seems to remain still unsolved. In particular, the identification of the figure who appears on the right hand[18] side of the painting in the blue brocade tunic – likely an exponent of the noble Montefeltro family and possibly the patron of the painting – remains uncertain, [13, p. 62 and ff].

The representation of patrons is not a surprising fact (akin to the naming of buildings that we see on campuses around the world), but it is often somewhat unrelated to the painting itself. As an example, we can remind the reader of the Scrovegni Chapel in Padua, where Giotto depicts the patron (Enrico Scrovegni) in the act of donating the chapel to the Holy Virgin (figure 24).

Figure 24. Giotto, Last Judgment, *(∼1305), detail: Enrico Scrovegni gives to Madonna the model of Cappella Scrovegni, affresco, Cappella degli Scrovegni, Padova.*

In Piero's case, however, we believe we can read an attempt to place the patron exactly at the scene, through the use of perspective technique. Indeed, the fact that the level of the knees of the three gentlemen on the right is the same level for the knees of Jesus and his torturers, indicates very specifically that the two sides of the picture were rendered by the painter as if they were taking place at the same place and the same time. We see, therefore, perspective used not simply as a geometrical device, but as a narrative instrument: the painter has made, here, a very specific choice to insert the contemporary figures in a way that places them within the context of the historical event.

6. Conclusions

As we indicated in the introduction, this article is dedicated to the linear aspects of perspective. We have used the desire of renaissance painters to faithfully represent tables, ceiling beams, and square floor decorations, to create the new object that mathematician call the projective plane. This object (which will represent the floor and the ceiling) is nothing but the old plane, to which one must add new (improper) points to represent the vanishing points that the eye sees in a scene, as well as a new line, which is the line to which all improper points belong, and that is represented as the horizon. But the story of perspective and projective geometry does not end here. The next natural step, at least for a mathematician, is to study second degree equations, such as circles, ellipses, and other conic sections. From the point of view of the painter, this is also an urgent matter, as it relates to the representation of important everyday objects such as plates, carriage wheels, and windows, as well as not so everyday objects (yet very important in religious paintings) such as halos. Projective geometry, with ideas that go back to the ancient Greek mathematicians, provides a beautiful and very elegant solution to this problem, but this will be the object of a subsequent article.

[17] For a synthesis of the debate, see e.g., [13, p. 54 and ff].

[18] As a curiosity we point out that the painting that is reproduced in [14] is actually a specular image of the actual painting, a minor mistake that does not alter the interest of the article.

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