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Abstract

We consider a general class of nonlinear dynamical systems with memory. We prove under which conditions periodic perturbations allow solutions with the same period. In particular, we consider memory kernels of Γ-type, which give prominence to critical phenomena in the past.

Key words: Nonlinear dynamics; memory effects; topological degree

1 Introduction

The functional differential equation in $\mathbb{R}^n$

$$\ddot{x}(t) = g\left(x(t), \dot{x}(t), \int_{-\infty}^{t} \mathcal{K}(t-s)\varphi(x(s), \dot{x}(s))\,\mathrm{d}s\right) , \tag{1}$$

where $g\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ and $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ are continuous maps, and $\mathcal{K}$ is a scalar kernel, falls within the class of delay-differential equations, the analysis of which has a rather long history, although often pertinent works deal with the scalar first-order case (see, e.g., [31], [43], [10], [24], [34], [48], [47], [27], [46] and references therein).

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The functional delay-structure above sketched appears appropriate when we describe the cooperation of discrete elements in space, the dynamics of which is affected by memory or involves composite processes developing at different rates. Concrete classes of phenomena the description of which may fall within the functional structure (1) emerge when we look at biological and mechanical systems.

- The compound eyes of arthropods are composed by hexagonal structures, called ommatidia, each containing a cluster of photoreceptors. The lateral inhibition to vision depends on how the structure of ommatidia reacts to the external excitation. In one such arthropods, namely the Limulus polyphemus, the lateral inhibition in the retina depends strongly on the distance between ommatidia, while the delay occurring between the response of one ommatidium and the inhibition of another seems to be constant. The vision time evolution of a Limulus can be then represented by a first-order differential equation with constant delay, as discussed by B. D. Coleman and G. H. Renninger in the 1970s (see [9], [11], [10], [24]). It is a special case of the scheme above.
- A constant delay seems to be natural when we aim at including the effects of digestion in predator-prey models. The delay can be selected as the average time needed for biomass transfer after predation from prey to predator, as discussed in reference [56]. At variance, when we consider the behavior in space of populations divided into immature and mature two age classes with the time delay being the maturation period, the functional structure (1) become insufficient for we need to consider the spatial variations, which imply a partial differential structure [32] not under analysis here.
- Jet-lag effects on humans suggest the opportunity of developing strategies that may control or even contrast them. Proposed mathematical models to describe quantitatively the phenomenon involve a population of oscillators with circadian effects [1]; so they require the insertion of a delay.
- Looking at a spatial scale smaller than the one of an entire organism, we see that transcription, translation, diffusion, and transport processes in gene networks have different duration: diffuse delay occurs, depending on the structure of gene networks. Such a delayed dynamics governs the transduction of genetic signals into phenotypic variations, the integration of information and stimuli of various nature. Its description falls within the functional structure (1) when the kernel vanishes below zero and φ depends only on x (see [55]). However, the explicit mathematical expression of gene mechanisms and signal-transduction pathways in gene networks remains to be further discussed. We need, in fact, explicit constitutive structures for the kernel and φ.
- At the scale of neuronal networks, brain plasticity is a sort of adaptive rewiring driven by the spontaneous synchronization of network dynamics [26]. Distributed delay can have physiological or pathological origin. The

functional structure (1) is appropriate for describing local neural network dynamics. However, the introduction of a noise could be appropriate [16], [44], [53], although not treated here. Neuroscience offers indeed a variety of intriguing dynamical problems (see pertinent remarks in P. Holmes' paper [25] and references therein).

- Looking at a more prosaic mechanical system, consider a mass point with position $q(t)$, constant mass m, and momentum $p(t)$. Take it as a *target particle*. Then, imagine that $N-1$ mass points are connected by harmonic springs with it, each with mass μ_i, position $\bar{q}_i(t)$, and momentum $\bar{p}_i(t)$. Assume that the potential is the weighed sum of quadratic terms $(q-\bar{q}_i)^2$, ranging i from 1 to $N-1$. The kinetic energy is the sum $\frac{p^2}{2m}+\sum_{i=1}^{N-1}\frac{\bar{p}_i^2}{2\mu_i}$. If we derive the balance equations from the resulting Hamiltonian, solve with respect to $\bar{q}_i$, we obtain that the resulting dynamics of the target particle is described by a structure falling within the scheme (1). Here, the $N-1$ particles behave as a thermal bath (see [54] for pertinent proofs and further detailed analyses).
- Consider a simple one-dimensional oscillator: a point mass connected to a fixed point by a damper and spring made of a complex material, i.e., an element that is per se a system described by a time-varying variable ν. Such a circumstance falls within the general model-building framework for the mechanics of complex continua, which include common phase-field approaches [8], [37], [36]. Consider now that the evolution of ν is affected by hereditary effects, as it may occur, more in general, in complex materials with memory [38]. Then, the scheme (1) applies. It is worth to notice that the representation of nonlinear oscillators including shape-memory devices (see, e.g., [2], [52]), above all when progressive degradation occurs, can be included to an extent in the general scheme that we analyze here; they foresee the inclusion of variables – say ν once again – describing the phase transition that characterize the shape-memory behavior.

Further examples of systems including delay and pertinent detailed discussions can be found in the books [17], [28], [40].

Here, we limit our analyses to the case in which the kernel $\mathcal{K}$ in equation (1) is the gamma probability distribution

$$\gamma_a^b(s)=\frac{a^b s^{b-1}\mathrm{e}^{-as}}{(b-1)!}\text{ for } s\geq 0,\qquad \gamma_a^b(s)=0 \text{ for } s<0,$$

with mean b/a and variance b/a^2. With this structure, since $t-s\geq 0$, for any x in C^1 the function $t\mapsto\int_{-\infty}^t\gamma_a^b(t-s)\varphi(x(s),\dot{x}(s))\,\mathrm{d}s$ is in C^1 also when $b=1$, even though γ_a^1 is not continuous. Our choice allows one to describe circumstances in which critical phenomena in the past influence more than others the memory.

Also, we extend equation (1) by introducing the effect of an external action, a *periodic living load* with period T, namely we consider the structure

$$\ddot{x}(t) = g\left(x(t), \dot{x}(t), \int_{-\infty}^{t} \gamma_a^b(t-s)\varphi\big(x(s), \dot{x}(s)\big)\,\mathrm{d}s\right) + \lambda f\big(t, x(t), \dot{x}(t)\big)\ , \quad (2)$$

where $f\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ is periodic of period T in the first variable, and λ is a non-negative real parameter. Such a view may model circumstances in which cells are submitted to the cyclic action of a fluid investing them (see [30]).

We analyze here the set of pairs (λ, x), called *T-forced pairs*, with $\lambda \geq 0$ and $x\colon \mathbb{R} \to \mathbb{R}^n$ a T-periodic solution of (2) corresponding to λ. In our investigation, we always regard such pairs as elements of the product space $[0,\infty)\times C_T^1(\mathbb{R}^n)$, where $C_T^1(\mathbb{R}^n)$ is the space of C^1, $\mathbb{R}^n$-valued, T-periodic maps with its natural C^1 topology. We point out that for any T-forced pair (λ, x), since f, g, and φ are assumed to be continuous, the map x is actually C^2.

We borrow some ideas and techniques from references [51] and [7]. Specifically, the former ones concern first-order differential equations defined on differentiable manifolds, with structure that is similar (although somewhat less general) to what we consider here. The latter ones deal with equations of the form (2) but only in the scalar case, in which the structure of $\mathbb{R}$ allows simpler statements of the results not requiring, in particular, any knowledge of degree theory otherwise necessary in higher dimensions.

In a wider perspective, this paper is indebted to the techniques developed mainly by M. Furi and collaborators in a series of papers on both ordinary and functional equations. We only mention here, as particularly pertinent to the present analysis, references [20] and [22].

2 A two-scale dynamics further motivating equation (2)

Consider a nonlinear oscillator with inner degrees of freedom (the mechanics of complex materials furnishes clear examples, although in a field setting; see [8], [37], [36]). Be $t \longmapsto x(t)$ the macroscopic motion and $t \longmapsto y(t)$ a map describing the internal evolution. Imagine that the macroscopic motion is nonlinear while the microscopic (internal) one is linear but coupled with the macroscopic one so that the governing equations (no more than the standard balance of momentum at the two different scales) are

$$\ddot{x}(t) = g(t, x(t), \dot{x}(t), y(t)) \quad (3)$$

and

$$\ddot{y}(t) = -(a+b)\dot{y}(t) - ab y(t) - x(t)\ . \quad (4)$$

Consider bounded solutions of this system, with $a, b > 0$.

- When $a \neq b$, variation of constant formula yields the unique bounded solution to equation (4):

$$y(t) = \frac{1}{a-b}\left(\int_{-\infty}^{t} \mathrm{e}^{-a(t-s)}x(s)\mathrm{d}s - \int_{-\infty}^{t} \mathrm{e}^{-b(t-s)}x(s)\mathrm{d}s\right), \tag{5}$$

  so that equation (3) becomes

$$\begin{aligned}\ddot{x}(t) &= g\left(t, x(t), \dot{x}(t), \frac{1}{a-b}\left(\int_{-\infty}^{t} \mathrm{e}^{-a(t-s)}x(s)\mathrm{d}s - \int_{-\infty}^{t} \mathrm{e}^{-b(t-s)}x(s)\mathrm{d}s\right)\right)\\ &= g\left(t, x(t), \dot{x}(t), \left(\frac{1}{a}\int_{-\infty}^{t} \gamma_a^1(t-s)x(s)\mathrm{d}s - \frac{1}{b}\int_{-\infty}^{t} \gamma_b^1(t-s)x(s)\mathrm{d}s\right)\right),\end{aligned} \tag{6}$$

  which is a system with two time delays, so it is a bit different from (2).
- However, when $a = b$, variation of constant formula yields the unique bounded solution

$$y(t) = \int_{-\infty}^{t} (t-s)\mathrm{e}^{-a(t-s)}x(s)\mathrm{d}s\,, \tag{7}$$

  which implies

$$\ddot{x}(t) = g\left(t, x(t), \dot{x}(t), \frac{1}{a^2}\int_{-\infty}^{t} \gamma_a^2(t-s)x(s)\mathrm{d}s\right), \tag{8}$$

  that is a special case of equation (2).

See also the book [40] for further discussions on the occurrence of gamma-type kernels.

3 Strategy adopted for the analysis of equation (2)

Our strategy is essentially based on the so-called *linear chain trick* (see, e.g., [5], [18], [19], [48], [51]). Specifically, we show that there is a correspondence between the T-periodic solutions to equation (2) and those of a non-linear system of $2n + kb$ ordinary differential equations:

$$\dot{\xi} = G(\xi) + \lambda F(t, \xi), \tag{9}$$

where $\lambda \geq 0$ and $\xi = (u, v_0, v_1, \ldots, v_b) \in \mathbb{R}^{2n+kb}$. The maps $G\colon \mathbb{R}^{2n+kb} \to \mathbb{R}^{2n+kb}$ and $F\colon \mathbb{R}\times\mathbb{R}^{2n+kb} \to \mathbb{R}^{2n+kb}$ are defined, for $u, v_0 \in \mathbb{R}^n$ and $v_1, \ldots, v_b \in$

$\mathbb{R}^k$, respectively, as

$$G(u, v_0, v_1, \dots, v_b) = \\ \Big(v_0, g(u, v_0, v_b), a\Big(\varphi(u, v_0) - v_1\Big), a(v_1 - v_2) \dots, a(v_{b-1} - v_b)\Big), \tag{10}$$

and

$$F(t, u, v_0, v_1, \dots, v_b) = \Big(\mathbf{0}_n, f(t, u, v_0), \underbrace{\mathbf{0}_k, \dots, \mathbf{0}_k}_{b \text{ times}}\Big),$$

where $\mathbf{0}_n$ and $\mathbf{0}_k$ denote the zero vectors on $\mathbb{R}^n$ and $\mathbb{R}^k$, respectively, and a appearing in G is a constant. Clearly, G and F are continuous and F is T-periodic in t.

Recall that $f\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$, $g\colon \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^k \to \mathbb{R}^n$, and $\varphi\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^k$ are continuous, hence so are F and G. Also, since f is T-periodic in the first variable, so is F.

4 Reduction steps

Given a function $x\colon \mathbb{R} \to \mathbb{R}^d$, denote by $x^{(i)}$ the i-th derivative of x, when defined. We set $x^{(0)} = x$. By $C^s_T(\mathbb{R}^d)$, $s = 0, 1, 2$, we indicate the Banach space of T-periodic C^s maps $x\colon \mathbb{R} \to \mathbb{R}^d$ with the C^s-norm

$$\|x\|_{C^s} = \sum_{i=0}^{s} \max_{t\in\mathbb{R}} |x^{(i)}(t)|.$$

Clearly, any C^s, $\mathbb{R}^d$-valued, T-periodic function is C^s-bounded, hence it belongs to $C^s_T(\mathbb{R}^d)$. Similarly, given $\mathcal{S} \subseteq \mathbb{R}^d$, by $C_T(\mathcal{S})$ we indicate the topological subspace of $C_T(\mathbb{R}^d)$ of T-periodic $\mathcal{S}$-valued functions. $C_T(\mathcal{S})$ is not complete unless $\mathcal{S}$ is a closed subset of $\mathbb{R}^d$.

The *linear chain trick* for equation (2) rests on the following two theorems. Their proofs follow those of Theorems 2.4 and 2.3 in reference [7]. So, we just sketch them.

Theorem 4.1 *Suppose that $(x_0, y_0, y_1 \dots, y_b)$ is a T-periodic solution of equation* (9)*, then x_0 is a T-periodic solution of equation* (2)*.*

The proof is based on the following general lemma (see [51, Lemma 3.3], see also [48, Prop. 2.7]):

Lemma 4.2 *Given any continuous and bounded function $z_0\colon \mathbb{R} \to \mathbb{R}^k$ and any nonzero number a, there exists a unique C^1 solution $z = (z_1, \dots, z_b)$, with*

$z_i\colon \mathbb{R} \to \mathbb{R}^k$, $i = 1, \ldots, b$, *of the system in* $\mathbb{R}^{kb}$

$$\dot{z}_i(t) = a\Big(z_{i-1}(t) - z_i(t)\Big),$$

that is bounded in the C^1 *norm. This solution is given by*

$$z_i(t) = \int_{-\infty}^{t} \gamma_a^i(t-s) z_0(s)\,\mathrm{d}s, \quad i = 1, \ldots, b.$$

In particular, for $i = 1$, we get

$$z_1(t) = \int_{-\infty}^{t} \gamma_a^1(t-s) z_0(s)\,\mathrm{d}s = a\int_{-\infty}^{t} \mathrm{e}^{-a(t-s)} z_0(s)\,\mathrm{d}s,$$

so that z_1 is actually a C^1 function, and thus so are all the z_i's for $i = 2, \ldots, b$.

Proof. [of Theorem 4.1] Let $(x_0, y_0, y_1 \ldots, y_b)$ be a T-periodic solution of equation (9). Let z_0 be the $\mathbb{R}^n$-valued, obviously bounded and continuous, $z_0(t) = \varphi(x_0(t), y_0(t))$ for $t \in \mathbb{R}$. Lemma 4.2 implies that

$$y_i(t) = \int_{-\infty}^{t} \gamma_a^i(t-s)\varphi(x_0(s), y_0(s))\,\mathrm{d}s\ , \quad i = 1, \ldots, b\ ,$$

is the unique C^1 solution of

$$\begin{cases} \dot{y}_1(t) = a\Big(\varphi(x_0(t), y_0(t)) - y_1(t)\Big), \\ \dot{y}_i(t) = a\Big(y_{i-1}(t) - y_i(t)\Big), \quad i = 2, \ldots, b. \end{cases}$$

From equation (9) we have, for all $t \in \mathbb{R}$,

$$\ddot{x}_0(t) = \dot{y}_0(t) = g\Big(x_0(t), y_0(t), y_b(t)\Big) + \lambda f\Big(t, x_0(t), y_0(t)\Big)\ ,$$

whence the assertion. ■

Theorem 4.3 *Suppose* x_0 *is a* T*-periodic solution of equation* (2)*. Take*

$$\begin{cases} y_0(t) := \dot{x}_0(t), \\ y_i(t) := \int_{-\infty}^{t} \gamma_a^i(t-s)\varphi(x_0(s), y_0(s))\,\mathrm{d}s, \quad i = 1, \ldots, b \end{cases} \tag{11}$$

for $t \in \mathbb{R}$*. Then,* $(x_0, y_0, y_1 \ldots, y_b)$ *is a* T*-periodic solution of equation* (9)*.*

Since $x_0 \in C_T^2(\mathbb{R})$, y_0 is C^1 and, being $\varphi\colon \mathbb{R}^{2n} \to \mathbb{R}^k$ continuous,

$$y_1(t) = \int_{-\infty}^{t} \gamma_a^1(t-s)\varphi(x(s), \dot{x}(s))\,\mathrm{d}s = \int_{-\infty}^{t} a\mathrm{e}^{-a(t-s)}\varphi(x(s), \dot{x}(s))\,\mathrm{d}s \tag{12}$$

is C^1 as well. Then, by the fundamental theorem of calculus, all y_i's, for $i = 2, \ldots, b$, are C^1 too.

Proof. Functions y_i's are in C^1_T. We claim that, for any $t \in \mathbb{R}$, the following equations hold:

$$\begin{cases} \dot{x}_0(t) = y_0(t), \\ \dot{y}_0(t) = g\Big(x_0(t), y_0(t), y_b(t)\Big) + \lambda f\Big(t, x_0(t), y_0(t)\Big), \quad \lambda \geq 0, \\ \dot{y}_1(t) = a\Big(\varphi(x_0(t), y_0(t)) - y_1(t)\Big), \\ \dot{y}_i(t) = a\Big(y_{i-1}(t) - y_i(t)\Big), \quad i = 2, \ldots, b. \end{cases}$$

The first equality follows by definition. For the second one, since x_0 is a solution of equation (2), by system (11), we find

$$\dot{y}_0(t) = \ddot{x}_0(t) = g\Big(x_0(t), y_0(t), y_b(t)\Big) + \lambda f\Big(t, x_0(t), y_0(t)\Big), \quad \forall t \in \mathbb{R}.$$

By differentiating equation (12) with respect to t, we get

$$\begin{aligned} \dot{y}_1(t) &= \varphi(x_0(t), y_0(t)) - a \int_{-\infty}^{t} \mathrm{e}^{-a(t-s)}\, \varphi(x_0(s), y_0(s))\, \mathrm{d}s \\ &= a\Big(\varphi(x_0(t), y_0(t)) - y_1(t)\Big), \end{aligned}$$

whence the third equality. Finally, for $i = 2, \ldots, b$, we proceed similarly by differentiating the second identity in system (11) with respect to t. Thus, recalling that when $i > 1$, we get $\gamma_a^i(0) = 0$ and $(\gamma_a^i)' = \gamma_a^{i-1} - \gamma_a^i$, so that

$$\begin{aligned} \dot{y}_i(t) &= \int_{-\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}t} \gamma_a^i(t-s)\, \varphi(x_0(s), y_0(s))\, \mathrm{d}s \\ &= \int_{-\infty}^{t} a\left(\gamma_a^{i-1}(t-s) - \gamma_a^i(t-s)\right) \varphi(x_0(s), y_0(s))\, \mathrm{d}s = a\left(y_{i-1}(t) - y_i(t)\right). \end{aligned}$$

This last identity proves our claim.

To see that $(x_0, y_0, y_1 \ldots, y_b)$ is T-periodic, observe that y_0 is T-periodic because it is the derivative of a periodic function. For $i = 1, \ldots, b$, by the T-periodicity of x_0 and y_0 we get

$$\begin{aligned} y_i(t+T) = &\int_{-\infty}^{t+T} \gamma_a^i(T + t - s)\varphi(x_0(s), y_0(s))\, \mathrm{d}s = \\ &\int_{-\infty}^{t} \gamma_a^i(t-\sigma)\varphi(x_0(\sigma), y_0(\sigma))\, \mathrm{d}\sigma = y_i(t), \end{aligned}$$

for any $t \in \mathbb{R}$. ■

5 Sets of T-periodic solutions

For the present analysis, the relation between known results on the set structure of harmonic solutions to periodically perturbed ordinary differential equations (see, e.g., for them [20]) and the problem discussed above is expedient here.

Definition 5.1 $(\lambda, \xi) \in [0, \infty) \times C_T(\mathbb{R}^{2n+kb})$ *is a* T-pair *for equation* (9) *when* ξ *is a* T*-periodic solution corresponding to* λ*. The pair* (λ, ξ) *is said to be* trivial *when* $\lambda = 0$ *and* ξ *is constant.*

For each $p \in G^{-1}(\mathbf{0}_{2n+kb})$, $(0, \overline{p})$ is a trivial T-pair for equation (9). Conversely, when $(0, \xi)$ is a trivial T-pair, $\xi = \overline{p}$ for some $p \in G^{-1}(\mathbf{0}_{2n+kb})$. In other words, the trivial T-periodic pairs of equation (9) correspond to the zeros of G.

Definition 5.2 $(\lambda, x) \in [0, \infty) \times C^1_T(\mathbb{R}^n)$*, with* $x \colon \mathbb{R} \to \mathbb{R}^n$ *a* T*-periodic solution of equation* (2) *corresponding to* λ *will be called a* T-forced pair*.* T*-forced pairs of the form* $(0, x)$ *with* x *constant are called* trivial.

In what follows we will denote by X and Y the sets of T-forced pairs of equation (2) and T-pairs of equation (9), respectively. We can prove a natural correspondence between X and Y that has the additional property of respecting the notion of triviality.

Remark 5.3 *Let* $(0, \overline{q})$*, with* $q \in \mathbb{R}^n$*, a trivial* T*-forced pair of equation* (2)*. Set*

$$\xi_0 := \Big(q, \mathbf{0}_n, \varphi(q, \mathbf{0}_k) \ldots, \varphi(q, \mathbf{0}_k)\Big) \in \mathbb{R}^{2n+kb}.$$

Then, $(0, \overline{\xi_0})$ *is a* T*-pair for equation* (9) *so that* $\xi_0 \in G^{-1}(\mathbf{0}_{2n+kb})$ *(see the argument proving Proposition 6.3). Conversely, any trivial* T*-pair of equation* (9) *must have the form* $(0, \overline{\xi_0})$*, where* $\xi_0 \in \mathbb{R}^{2n+kb}$ *is such that* $G(\xi_0) = \mathbf{0}_{2n+kb}$*. Thus, we have* $\xi_0 = (q, \mathbf{0}_n, \varphi(q, \mathbf{0}_k) \ldots, \varphi(q, \mathbf{0}_k))$ *for some* $q \in \mathbb{R}^n$ *and, consequently,* $(0, \overline{q})$ *is a trivial* T*-forced pair for equation* (9) *(once again see the argument proving Proposition 6.3 below).*

Consider the map $\mathcal{J} \colon [0, \infty) \times C^1_T(\mathbb{R}^n) \to [0, \infty) \times C_T(\mathbb{R}^{2n+kb})$ defined by:

$$\mathcal{J} \colon (\lambda, x_0) \mapsto (\lambda, x_0, y_0, y_1 \ldots, y_b)\,, \tag{13}$$

where

$$\begin{cases} y_0(t) := \dot{x}_0(t), \\ y_i(t) := \int_{-\infty}^{t} \gamma_a^i(t-s)\varphi(x_0(s), y_0(s))\,\mathrm{d}s, \quad i = 1, \ldots, b. \end{cases}$$

Lemma 5.4 *Let* $\mathcal{J}|_X : X \to Y$ *be the restriction to* X *of* $\mathcal{J}$*. Then,* $\mathcal{J}|_X$ *is a homeomorphism of* X *onto* Y*, which establishes a bijective correspondence between the trivial* T*-forced pairs in* X *and the trivial* T*-pairs in* Y*.*

Proof. Since φ is locally Lipschitz continuous, the continuity of $\mathcal{J}$ follows by construction; whence the continuity of $\mathcal{J}|_X$. Injectivity and surjectivity of $\mathcal{J}|_X : X \to Y$ follow from theorems 4.3 and 4.1.

The inverse map $\mathcal{J}|_X^{-1}$ acts merely as a projection onto the first two components, so it is continuous.

The last part of the assertion follows by Remark 5.3. ■

6 Branches of nontrivial T-forced pairs

Topological degree theory is the appropriate tool to determine connected sets of nontrivial T-forced pairs that branch off the zeros of Φ. For such a theory the pertinent bibliography is wide (see [15] and also [41], [33], [14]).

A map $\psi\colon V \subseteq \mathbb{R}^k \to \mathbb{R}^k$ is *proper* when for each compact set $K \subseteq \mathbb{R}^k$ the preimage $\psi^{-1}(K)$ is compact. Take $p \in \mathbb{R}^k$ and let f be an $\mathbb{R}^k$-valued function defined in a neighborhood of the closure $\overline{U}$ of an open set $U \subseteq \mathbb{R}^k$, if $f^{-1}(p) \cap U$ is compact, we say that the triple (f, U, p) is *admissible* in U with respect to p; in short we say that f is *admissible*. In this case an integer $\deg(f, U, p)$, called the *degree of f in U with respect to p*, is defined. Essentially, $\deg(f, U, p)$ indicates the number of $f^{-1}(p) \cap U$ elements. In fact, when f is C^1 and p is a regular value of f (that is either $f^{-1}(p) = \emptyset$ or the Jacobian matrix $J_q f$ of f at q is nonsingular for each $q \in f^{-1}(p)$), $f^{-1}(p)$ is a finite set. Then, we get

$$\deg(f, U, p) = \sum_{q \in f^{-1}(p)} \operatorname{sign} \det J_q f. \tag{14}$$

If we replace f with an approximating C^1 function, we do not alter the topological degree. It doesn't also change when p is substituted by sufficiently near regular value. Also, the set of regular values – we know – is dense in $\mathbb{R}^k$ (this is a consequence of Sard's theorem see, e.g., [41]). Thus, the formula above can be used as a basis for an operative definition of degree by setting, for arbitrary admissible triple

$$\deg(\phi, U, s) := \deg(f, U, r), \tag{15}$$

where f is a C^1 function and r is a regular value sufficiently close to ϕ and s, respectively. Other useful computational approaches can be indicated. We only mention those based on surface (e.g., [4], [45]) and volume integrals (e.g., [29], [42]).

In what follows the target point p will always be $\mathbf{0}_k$ so, to simplify notation, we will write $\deg(f, U)$ in place of $\deg(f, U, \mathbf{0}_k)$. Indeed, $\deg(f, U)$ can additionally be regarded as the degree (also called characteristic or rotation) of the map f interpreted as a tangent vector field on $\mathbb{R}^k$ (see [21] for an axiomatic approach).

As a matter of notation, for any $p \in \mathbb{R}^k$, by $\overline{p}\colon \mathbb{R} \to \mathbb{R}^k$ we mean the constant map $\overline{p}(t) \equiv p$. Also, given an open subset $\mathcal{O}$ of $[0,\infty) \times C_T^s(\mathbb{R}^k)$, $s \geq 0$, we will denote by $\widetilde{\mathcal{O}} \subseteq \mathbb{R}^k$ the open set $\widetilde{\mathcal{O}} := \big\{p : (0,\overline{p}) \in \mathcal{O}\big\}$. In particular, given $\mathcal{S} \subseteq \mathbb{R}^d$, if $\mathcal{O} \subseteq\subseteq \mathbb{R}^k$, we have $\widetilde{\mathcal{O}} \subseteq \mathcal{S}$.

6.1 Existence of T-pairs

First we recall a result of reference [20] (see also [49, Th. 3.1] and [50, Th. 3] for generalizations in different directions involving ordinary differential equations, [22, Th. 4.1] and [23, Th. 5.1] when the perturbation is allowed to contain a delay term). For the sake of convenience, we state that result in a self-contained form.

Theorem 6.1 (Thm. 3.3 of [20]) *Let $\mathfrak{f}\colon \mathbb{R} \times M \to \mathbb{R}^d$ and $\mathfrak{g}\colon M \to \mathbb{R}^d$ be continuous tangent vector fields defined on a (boundaryless) differentiable manifold $M \subseteq \mathbb{R}^d$, f being T-periodic in the first variable. Consider the following parametrized differential equation on M:*

$$\dot{x} = \mathfrak{g}(x) + \lambda\mathfrak{f}(t,x), \quad \lambda \geq 0. \tag{16}$$

Let Ω be an open set in $[0,\infty) \times C_T(M)$. Assume that $\deg(g,\widetilde{\Omega})$ is well defined and nonzero. Then, there exists a connected set Γ of nontrivial T-pairs of equation (16) in Ω the closure of which in $[0,\infty) \times C_T(M)$ intersects $\big\{(0,\overline{p}) : p \in g^{-1}(0)\big\}$ and is not contained in any compact subset of Ω. In particular, if M is closed in $\mathbb{R}^d$ and $\Omega = [0,\infty) \times C_T(M)$, the set Γ is unbounded.

In accord to Definition 5.1, in Theorem 6.1, (λ, x) is a T-pair for equation (16) when x is a T-periodic solution corresponding to λ. Also, if $\lambda = 0$ and x is constant, (λ, x) is *trivial*. In particular, the set $\big\{(0,\overline{p}) : p \in g^{-1}(0)\big\}$ consists of all trivial T-pairs of equation (16) that are contained in $\widetilde{\Omega}$.

By setting $d = 2n + kb$, $M = \mathbb{R}^{2n+kb}$, $\mathfrak{g} = G$ and $\mathfrak{f} = F$, we get the following immediate consequence of Theorem 6.1 concerning the set of T-pairs of (9):

Theorem 6.2 *Let $\mathcal{O}$ be open in $[0,\infty) \times C_T(\mathbb{R}^{2n+kb})$. Assume that $\deg(G,\widetilde{\mathcal{O}})$ is well defined and different from zero. Then, there exists a connected set Υ of nontrivial T-pairs in $[0,\infty) \times C_T(\mathbb{R}^{2n+kb})$ with non-compact closure in $\mathcal{O}$, which meets the set of trivial T-pairs contained in $\mathcal{O}$, namely the set $\big\{(0,\overline{p}) \in \mathcal{O} : G(p) = \mathbf{0}_{2n+kb}\big\}$.*

It is convenient to introduce a function $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Phi(u) = g\Big(u, \mathbf{0}_n, \varphi(u, \mathbf{0}_n)\Big)\ . \tag{17}$$

Proposition 6.3 *There is a one-to-one correspondence between the zeros of Φ and the rest points of G.*

Proof. Assume $(\bar{u}, v_0, v_1, \ldots, v_b) \in G^{-1}(\mathbf{0}_{2n+kb})$. Then, $\Phi(\bar{u}) = 0$, $v_0 = \mathbf{0}_n$, and $v_1 = v_2 = \ldots = v_b = \varphi(\bar{u}, 0_n)$. Conversely, for any $\bar{u} \in \Phi^{-1}(\mathbf{0}_n)$, one has $G(\bar{u}, \mathbf{0}_n, \varphi(\bar{u}, \mathbf{0}_n) \ldots, \varphi(\bar{u}, \mathbf{0}_n)) = \mathbf{0}_{2n+kb}$. ■

Indeed, the degree of G in special open subsets of $\mathbb{R}^{2n+kb}$ is related to that of Φ in the corresponding projected subsets of $\mathbb{R}^n$.

Theorem 6.4 *Let $U \subseteq \mathbb{R}^n$ be open. Put $W := U \times \mathbb{R}^n \times \mathbb{R}^{kb}$. Then,* $\deg(\Phi, U)$ *is well-defined and nonzero if and only if so is* $\deg(G, W)$*.*

The proof of this theorem is based on approximation and homotopy arguments that are far from the main point of this paper. It follows closely the lines of the analogous result in reference [7] (see also [51, Thm. 2.2]). For these reasons we will only sketch its proof.

Proof.

(i) We introduce the vector field $\mathcal{G}\colon \mathbb{R}^{2n+kb} \to \mathbb{R}^{2n+kb}$ as follows:

$$\begin{aligned}&\mathcal{G}(u, v_0, v_1, \ldots, v_b) := \\ &\qquad (v_0, g(u, v_0, v_b), a(\varphi(u, v_0) - v_b), a(v_1 - v_2), \ldots, a(v_{b-1} - v_b))\,.\end{aligned}$$

Then, we prove that if either $\mathcal{G}$ or G (as defined in (10)) is admissible for the degree (with respect to $\mathbf{0}_{2n+kb}$) in some open set $V \subseteq \mathbb{R}^{2n+kb}$, so is the other and we have

$$\deg(\mathcal{G}, V) = \deg(G, V). \tag{18}$$

This is accomplished by the homotopy degree invariance because the map

$$\begin{aligned}&\mathcal{H}(\lambda, u, v_0, v_1, \ldots, v_b) = \\ &(v_0, g(u, v_0, v_b), a(\varphi(u, v_0) - [\lambda v_b + (1-\lambda) v_1]), a(v_1 - v_2) \ldots, a(v_{b-1} - v_b))\,,\end{aligned}$$

is an admissible homotopy since $\mathcal{H}(\lambda, u, v_0, v_1, \ldots, v_b) = 0$ if and only if $(u, v_0, v_1, \ldots, v_b) \in V \cap G^{-1}(0) = V \cap \mathcal{G}^{-1}(0)$.

(ii) Let $\psi\colon \mathbb{R}^d \to \mathbb{R}^d$ be a C^1 vector field on some Euclidean space $\mathbb{R}^d$. A zero $p \in \psi^{-1}(0)$ of ψ is a *nondegenerate zero* (of f) if the Jacobian matrix $\psi'(p)$ is nonsingular; this means, equivalently, that p is a regular point of ψ, in particular p is isolated. So, we proceed to prove that all zeros of Φ are nondegenerate if and only if all zeros of $\mathcal{G}$ are nondegenerate too.

Observe first that, similarly to Proposition 6.3, the zeros of Φ and $\mathcal{G}$ correspond. In fact, if $\mathcal{G}(u, v_0, v_1, \ldots, v_b) = 0$, we obtain $v_0 = 0$ and $v_1 = v_2 = \ldots = v_b = \varphi(u, 0)$ whence $\Phi(u) = 0$ and, conversely, for any $\bar{u} \in \Phi^{-1}(0)$, we find $\mathcal{G}(\bar{u}, 0, \bar{w} \ldots, \bar{w}) = 0$. As a consequence, by using admissibility, we find that if all zeros of Φ are isolated, so are those of $\mathcal{G}$. The *vice versa* holds true. Thus, to prove nondegeneracy, for any zero, say p of Φ, one can write the Jacobian matrix $J_P\mathcal{G}$ of $\mathcal{G}$ at the corresponding zero $P = \big(p, \mathbf{0}_n, \varphi(p, \mathbf{0}_n), \ldots, \varphi(p, \mathbf{0}_n)\big)$ in block matrix form. By computing its determinant, we get

$$\det J_P\mathcal{G} = (-1)^{k(b-1)} a^{kb} \det J_p\Phi \,, \tag{19}$$

where $J_p\Phi$ denotes the Jacobian matrix of Φ at p.

(iii) Assume that Φ is admissible on U and suppose all of its zeros are nondegenerate. Since Φ is admissible for the degree $\Phi^{-1}(0)$ is compact. Also, since the zeros of Φ correspond to those of $\mathcal{G}$, as above shown, we can prove that the set $\mathcal{G}^{-1}(\mathbf{0}_{2n+kb}) \cap W$ is compact as well. This implies that $\mathcal{G}$ is admissible on W. By Equation (19) we have

$$\operatorname{sign}\det J_{(u,\mathbf{0}_n,w)}\mathcal{G} = (-1)^{k(b-1)} \operatorname{sign}\det J_u\Phi \,,$$

where $w = \varphi(u, \mathbf{0}_n)$. By using the computational formula (14) we obtain

$$\deg(\mathcal{G}, W) = (-1)^{k(b-1)} \deg(\Phi, U). \tag{20}$$

(iv) On the basis of Sard's lemma, the homotopy property of the degree, and smooth approximation Weierstrass' theorem, we can assume, without modifying their degree, that G is C^1 with only nondegenerate zeros so that the same holds true for $\mathcal{G}$ and Φ. The assertion now follows from equations (18) and (20). ∎

7 Existence of T-pairs for equation (2)

Theorem 7.1 *Let $U \subseteq \mathbb{R}^n$ be open and such that $\deg(\Phi, U)$ is well-defined and different from zero. Let also $\Omega \subseteq [0, \infty) \times C^1_T(\mathbb{R}^n)$ be open and such that $\widetilde{\Omega} = U$. Then, in $X \cap \Omega$ there is a connected subset Γ of nontrivial T-forced pairs with relative closure to Ω non-compact; it intersects the set*

$$\Big\{(0, \overline{u}) \in \Omega : u \in U \cap \Phi^{-1}(0)\Big\}. \tag{21}$$

Proof. Let $\mathcal{O} = \Omega \times C_T(\mathbb{R}^n \times \mathbb{R}^{kb})$, so that, clearly, $\widetilde{\mathcal{O}} = \widetilde{\Omega} \times \mathbb{R}^n \times \mathbb{R}^{kb} = U \times \mathbb{R}^n \times \mathbb{R}^{kb}$. By Theorem 6.4 we have that $\deg(G, \widetilde{\mathcal{O}})$ is well-defined and nonzero. Thus, Theorem 6.2 yields a connected set Υ of nontrivial T-pairs for equation (9) the closure of which in $\mathcal{O}$ is non-compact and intersects the set $\big\{(0,\overline{p}) \in \mathcal{O} : G(p) = \mathbf{0}_{2n+kb}\big\}$. Finally, Lemma 5.4 implies that $\Gamma := \mathcal{J}^{-1}(\Upsilon)$ is a connected set as in the assertion. ■

8 Periodic solutions for small perturbations

There are situations in which the unbounded branch yielded by Theorem 7.1 is contained in the slice $\{0\} \times C^1_T(\mathbb{R}^n)$. A typical example is the resonant harmonic oscillator ($T = 2\pi$)

$$\ddot{x} = -x + \lambda \sin t,$$

By using J. A. Yorke's inequality ([57]; see also, e.g., [6]), we can show that this phenomenon cannot happen when the function g in equation (2) is Lipschitz continuous and the perturbation frequency is sufficiently high (i.e., $T > 0$ is small enough).

Yorke's inequality reads as follows:

Theorem 8.1 ([57]) *Let ξ be a nonconstant τ-periodic solution of $\dot{x} = F(x)$ where $F\colon W \subseteq \mathbb{R}^d \to \mathbb{R}^d$, is Lipschitz of constant L. Then the period τ of ξ satisfies*

$$\tau \geq 2\pi/L.$$

Suppose that g and φ are Lipschitz continuous, then so is the map G defined in the expression (10) with a possibly larger constant L_G. Assume also that the period T of the forcing term is such that $T < 2\pi/L_G$. Then, equation (9) cannot have non-constant T-periodic solutions for $\lambda = 0$. In other words, the nontrivial T-pairs of equation (9) cannot be contained in $\{0\} \times C_T(\mathbb{R}^{2n+kb})$. Consequently, if (λ, x) is a nontrivial T-forced pair of equation (2), it must be $\lambda > 0$. To prove this, assume that $(0, x)$ is a nontrivial T-forced pair. Then, $\mathcal{J}\big((0, x)\big)$ is a nontrivial T-pair of equation (9) and, as we have seen, this is impossible. Combining this argument with Theorem 7.1, we get the following result:

Theorem 8.2 *Let $U \subseteq \mathbb{R}^n$ be open and such that $\deg(\Phi, U)$ is well-defined and nonzero. Let also $\Omega \subseteq [0, \infty) \times C^1_T(\mathbb{R}^n)$ be open and such that $\widetilde{\Omega} = U$. Suppose that g and φ are Lipschitz continuous. If the period T of f is sufficiently short, equation (2) admits a T-periodic solution for all sufficiently small values of $\lambda > 0$.*

Proof. Let Γ be the connected set given by Theorem 7.1 with closure $\overline{\Gamma}$ in Ω. We show that the projection of $\overline{\Gamma}$ on the first component contains a nontrivial interval $[0, \lambda_*)$. Assume by contradiction that this is not the case. Since the projection is a continuous map and $\overline{\Gamma}$ is connected, the only possibility is $\overline{\Gamma} \subseteq \{0\} \times C^1_T(\mathbb{R}^n)$, but this is excluded by the remarks above. ■

The only regularity required to the perturbing term f is continuity. It is an obstacle to the use of arguments based on the standard implicit function theorem. However, an alternative, and not equivalent, approach at showing the existence of T-periodic solutions of equation (2) for small values of $\lambda > 0$ could be carried out by an analysis of the linearization (when possible) of equation (9) at the zeros of G, for $\lambda = 0$. A presentation of the latter technique, based on the notion of T-resonance, is discussed, e.g., in references [13, Ch. 7] and [12, Ch. 2]. A similar idea can be traced back to Poincaré (see, e.g., [39] ans also [3] for an application to multiplicity results). We do not explore further such possibilities.

9 Influence of the distribution variance

Consider the special case

$$\ddot{x}(t) = -x(t) - 3\int_{-\infty}^{t} \gamma_a^b(t-s)x'(s)\,\mathrm{d}s + \lambda(1+3x(t))\sin(2\pi t)\ , \qquad (22)$$

with $T = 1$.

We fix the average of γ_a^b to $1/2$ and allow the variance to take values $1/4$, $1/16$, and $1/28$.

Figure 1 below (where $\lambda = 1$) shows how the pertinent solutions, depicted in the phase space, vary one with respect to the other.

The previous example suggests to ask the behavior of the equation analyzed here when the variance of the gamma distribution tends to zero.

To analyze the circumstance we start mentioning a known lemma:

Lemma 9.1 *Let $y\colon \mathbb{R} \to \mathbb{R}^k$ be a bounded uniformly continuous function. Then,*

$$\lim_{n\to\infty} \int_0^{+\infty} \gamma_{na}^{nb}(s)y(t-s)\,\mathrm{d}s = y(t-b/a) \qquad (23)$$

for all $t \in \mathbb{R}$.

Recall, also, that the average of γ_{na}^{nb} is b/a, whereas its variance is given by $nb/(na)^2 = b/(na^2)$, so that as $n \to \infty$ the average remains constant while

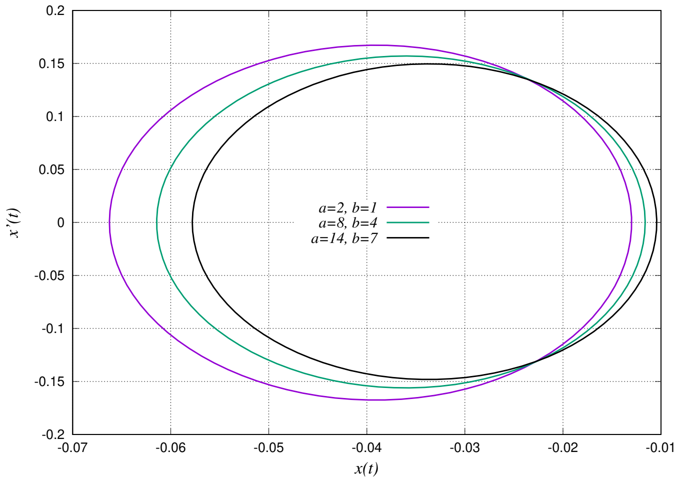

Figure 1. Representation in the phase space of periodic solutions to equation (22), by varying the kernel while maintaining fixed its average.

the variance tends to zero. For the sake of completeness we sketch the proof.

Proof. Since $\int_0^{+\infty} \gamma_{na}^{nb}(s)\,\mathrm{d}s = \int_{-\infty}^{+\infty} \gamma_{na}^{nb}(s)\,\mathrm{d}s = 1$, It is enough to prove that

$$\left\| \int_{-\infty}^{+\infty} \gamma_{na}^{nb}(s)[y(t-s) - y(t-b/a)]\,\mathrm{d}s \right\| \underset{n\to\infty}{\longrightarrow} 0$$

for all $t \in \mathbb{R}$.

For arbitrary $\varepsilon > 0$, since y is uniformly continuous, we can find $\ell > 0$ such that $|y(t-s) - y(t-b/a)| < \varepsilon$ for $s \in [t-\ell, t+\ell]$ and all $t \in \mathbb{R}$. Thus,

$$\left\| \int_{t-\ell}^{t+\ell} \gamma_{na}^{nb}(s)[y(t-s) - y(t-b/a)]\,\mathrm{d}s \right\| \le \int_{t-\ell}^{t+\ell} \varepsilon \gamma_{na}^{nb}(s)\,\mathrm{d}s \le \varepsilon.$$

Also, since $\gamma_{na}^{nb}(s) \to 0$ pointwise for all $s \neq b/a$, from Lebesgue's dominated convergence theorem we obtain that for sufficiently large n

$$\left\| \int_{t+\ell}^{+\infty} \gamma_{na}^{nb}(s)[y(t-s) - y(t-b/a)]\,\mathrm{d}s \right\| \le \int_{t+\ell}^{+\infty} 2 \sup_{\tau\in\mathbb{R}} \|y(\tau)\| \gamma_{na}^{nb}(s)\,\mathrm{d}s \le \varepsilon\,.$$

A similar inequality holds for

$$\left\| \int_{-\infty}^{t-\ell} \gamma_{na}^{nb}(s)[y(t-s) - y(t-b/a)]\,\mathrm{d}s \right\|,$$

so that

$$\left\| \int_{-\infty}^{+\infty} \gamma_{na}^{nb}(s)[y(t-s) - y(t-b/a)]\,\mathrm{d}s \right\| \le 3\varepsilon\,.$$

Then, the assertion follows because ε is chosen arbitrarily. ■

For y as in the lemma, with a change of variable we have

$$\lim_{n\to\infty}\int_{-\infty}^{t}\gamma_{na}^{nb}(t-s)y(s)\,\mathrm{d}s=\lim_{n\to\infty}\int_{0}^{+\infty}\gamma_{na}^{nb}(s)y(t-s)\,\mathrm{d}s=y(t-b/a)\ .$$

Therefore, for a T-periodic C^1 function x and a Lipschitz continuous φ, we may set $y(s)=\varphi\big(x(s),\dot{x}(s)\big)$, so that

$$\lim_{n\to\infty}\int_{-\infty}^{t}\gamma_{na}^{nb}(t-s)\varphi\Big(x(s),\dot{x}(s)\Big)\,\mathrm{d}s=\varphi\Big(x(t-b/a),\dot{x}(t-b/a)\Big).$$

In this sense, we may say that, by letting the gamma distribution variance to zero, equation (1) tends to a differential equation with delay equal to the distribution average.

10 Additional remarks

Equation (2) is a model structure for describing the dynamics of discrete structures with memory characterized by the prominence of special events localized in the past.

It can be profitably used, e.g., for models of population dynamics with social factors [35] in which we aim at considering the effects on memory of critical historical events.

However, this is only one possible choice. Another one emerges when we consider continuous schemes for the dynamics of materials with memory, even with complex material structure, i.e., with active microstructures (see, e.g., [38]). In that case, we have (generically non-linear) partial differential equations. However, if we discretize them in space and look at solutions in time for each node in the spatially discrete scheme, what remains is a system of ordinary differential equations with a structure falling in the scheme treated here in the appropriate cases.

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