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On the proportion of vanishing elements in finite groups *

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Abstract

We prove that the function $\mathrm{P_v}(G)$, measuring the proportion of the elements of a finite group G that are zeros of irreducible characters of G, takes very sparse values in a large segment of the $[0, 1]$ interval.

Keywords Character theory, zeros of irreducible characters.

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1 Introduction

We say that an element g of a finite group G is a *vanishing* element of G if there exists an irreducible character χ of G such that $\chi(g) = 0$, and we denote by $\mathrm{V}(G)$ the set of the vanishing elements of G.

A natural way to measure the relative abundance (or scarcity) of vanishing elements in a finite group G is to consider the proportion

$$\mathrm{P_v}(G) = \frac{|\mathrm{V}(G)|}{|G|}$$

which can be seen as the probability that a randomly chosen element of G turns out to be a vanishing element. In this setting, one can ask what are the possible values of the function $\mathrm{P_v}(G)$, where G varies in some relevant class of finite groups, and also what implications on the structure of the group G can be deduced from specific values of $\mathrm{P_v}(G)$. For example, as a consequence of a classical result by W. Burnside, G is abelian if and only if $\mathrm{P_v}(G) = 0$.

Recently, A. Moretó and P.H. Tiep proved in [12], confirming a conjecture proposed in [2], that if G is a finite group such that $\mathrm{P_v}(G) < \mathrm{P_v}(A_7) = \mathfrak{a} = \frac{1067}{1260} \simeq 0.846$, then G is solvable.

In [13], L. Morotti and H. Tong-Viet showed that a finite group G is abelian if and only if $\mathrm{P_v}(G) < \frac{1}{2}$, and that $\mathrm{P_v}(G) = \frac{1}{2}$ if and only if $G/\mathbf{Z}(G)$ is a Frobenius group with Frobenius complement of order 2. They also proved ([13, Theorem 1.6]) that if $\mathrm{P_v}(G) \leq \frac{2}{3}$, then $\mathrm{P_v}(G) \in \{0, \frac{1}{2}, \frac{2}{3}\}$ and observed that computer calculation seems to suggest that the only values of $\mathrm{P_v}(G)$ in the interval $[0, \mathfrak{a})$ are of the form $\frac{m-1}{m}$ for some integer $1 \leq m \leq 6$ (and these values are actually taken: see Remark 2.11). In this paper, we confirm this conjecture.

Theorem A. *Let G be a finite group such that $\mathrm{P_v}(G) < \mathfrak{a}$. Then*

$$\mathrm{P_v}(G) = \frac{m-1}{m}$$

for some integer $1 \leq m \leq 6$.

The sparseness of the values of the function $\mathrm{P_v}$ in the interval $[0, \mathfrak{a})$ looks rather striking, when compared to the following result by A. Miller (see [11]): defining the proportion $\mathrm{P_{ct}}(G)$ of zeros in the character table $CT(G)$ of a finite group G as the ratio of the number of zeros in $CT(G)$ by the total number of entries of $CT(G)$, then the set of values $\mathrm{P_{ct}}(G)$, when G varies in the class of the finite groups, is dense in $[0, 1]$.

The paper is organized as follows: in Section 2 and Section 3 we collect useful results and in Section 4 we prove that if $\mathrm{P_v}(G) < \mathfrak{a}$, then the set of the *non-vanishing* elements $G - \mathrm{V}(G)$ is, rather surprisingly, a *subgroup* of G (Theorem 4.3), a fact that easily yields Theorem A.

Our notation is standard and for character theory it follows [7]. All groups considered in the paper will be assumed, if not otherwise mentioned, to be finite groups.

2 Preliminaries

Let G be a finite group. We write $\mathrm{V}(G) = \{g \in G \mid \chi(g) = 0 \text{ for some } \chi \in \mathrm{Irr}(G)\}$ and we denote by $\mathrm{N}(G) = G - \mathrm{V}(G)$ its complement in G. As in [9], we call the elements of $\mathrm{N}(G)$ *non-vanishing* elements of G.

A basic observation: given a normal subgroup N of a group G and $g \in G$, if $gN \in \mathrm{V}(G/N)$ then, seeing by inflation the characters of G/N as characters of G whose kernel contains N, we have that $gN \subseteq \mathrm{V}(G)$. This implies that $\mathrm{P_v}(G/N) \leq \mathrm{P_v}(G)$, a fact that will be used freely in the rest of the paper.

We will use the following notation: for a positive integer m, $\mathsf{U}_m = \langle \zeta_m \rangle = \{\varepsilon \in \mathbb{C} \mid \varepsilon^m = 1\}$ is the group of the (complex) m-th roots of unity, and $\zeta_m = e^{\frac{2\pi}{m}\sqrt{-1}}$ is a primitive m-th root of unity. We denote by $\mathsf{U} = \bigcup_{m \geq 1} \mathsf{U}_m$ the infinite group of all roots of unity.

Lemma 2.1 ([10])**.** *Let m, n be a positive integers, $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathsf{U}_m$ and $p_1, p_2, \ldots, p_r$ the distinct prime divisors of m. Then $\alpha_1 + \cdots + \alpha_n = 0$ only if $n = n_1p_1 + \cdots + n_rp_r$ for suitable integers $n_i \in \mathbb{N}$.*

Lemma 2.2. *Let k be a positive integer and let $\varepsilon_i \in \mathsf{U}$, for $i = 1, 2, \ldots, k$, be roots of unity such that $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_k = 0$. Then, up to a suitable labelling of the elements ε_i:*

(a) *If $k \in \{2, 3\}$, then there exists a $\delta \in \mathsf{U}$ such that $\varepsilon_i = \delta\zeta_k^i$ for $i = 1, 2, \ldots, k$.*

If $k = 4$, then $\varepsilon_2 = -\varepsilon_1$ and $\varepsilon_4 = -\varepsilon_3$.

(b) *If $k = 6$, then one of the following holds:*

(b1) *there exist $\delta_0, \delta_1, \delta_2 \in \mathsf{U}$ such that, writing $i - 1 = 2q + r$, with q, r integers, $0 \leq r \leq 1$, $\varepsilon_i = \delta_q\zeta_2^r$ for $i = 1, 2, \ldots, 6$.*

(b2) *there exist $\delta_0, \delta_1 \in \mathsf{U}$ such that, writing $i - 1 = 3q + r$, with q, r integers, $0 \leq r \leq 2$, $\varepsilon_i = \delta_q\zeta_3^r$ for $i = 1, 2, \ldots, 6$.*

(b3) *there exists a $\delta \in \mathsf{U}$ such that $\varepsilon_i = \delta\zeta_5^i$ for $i = 1, 2, 3, 4$, $\varepsilon_5 = \delta\zeta_2\zeta_3$ and $\varepsilon_6 = \delta\zeta_2\zeta_3^2$.*

(c) *If $k = 6$ and all $\varepsilon_i \in \mathsf{U}_{2^n}$, for some $n \geq 1$, then $\varepsilon_2 = -\varepsilon_1$, $\varepsilon_4 = -\varepsilon_3$ and $\varepsilon_6 = -\varepsilon_5$.*

Proof. (a) and (b) follow from [14, Theorem 3.1], and (c) is a consequence of (b). $\square$

Lemma 2.3. *Let n be a positive integer and let $\varepsilon_i, \eta_i \in \mathsf{U}_{2^n}$, for $1 \leq i \leq 3$, be roots of unity such that $(*)$: $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$ and $\eta_1\eta_2\eta_3 = 1$.*

Then, writing $\Sigma = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \eta_1 + \eta_2 + \eta_3$ and $\Delta = \{\delta_i = \overline{\varepsilon_i}\eta_i \mid 1 \leq i \leq 3\}$.

(1) *If $\Delta \subseteq \mathsf{U}_2$, then $\Sigma \neq 0$.*

(2) *Assume $n \leq 2$ and fix the notation so that $o(\varepsilon_3) \leq o(\varepsilon_2) \leq o(\varepsilon_1)$ and $o(\eta_3) \leq o(\eta_2) \leq o(\eta_1) \leq o(\varepsilon_1)$. Then $\Sigma = 0$ if and only if $\{\varepsilon_1, \varepsilon_2\} = \{\zeta_4, -\zeta_4\}$ and $\eta_1 = \eta_2 = -1$. In particular, if $n \leq 2$ and $\eta_i = \overline{\varepsilon_i}$ for $i = 1, 2$, then $\Sigma \neq 0$.*

(3) *Assume $\Delta \subseteq \mathsf{U}_4$ and $o(\varepsilon_i) \geq 8$ for some $i \in \{1, 2, 3\}$. If $\Sigma = 0$, then either $\Delta = \{\zeta_4, -1\}$ or $\Delta = \{-\zeta_4, -1\}$.*

Proof. (1) Since $\Sigma = \varepsilon_1(1+\overline{\varepsilon_1}\eta_1)+\varepsilon_2(1+\overline{\varepsilon_2}\eta_2)+\varepsilon_3(1+\overline{\varepsilon_3}\eta_3)$, and $\delta_i = \overline{\varepsilon_i}\eta_i \in \{1, -1\}$ for $1 \leq i \leq 3$, the assumption $(*)$ implies that either $\Sigma = 2\varepsilon_i$ for some $i \in \{1, 2, 3\}$ or $\Sigma = 2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$. Hence, $\Sigma \neq 0$ by Lemma 2.1.

(2) If $\Sigma = 0$, then part (1) yields $n \neq 1$ and hence, by $(*)$ and our choice of notation, $o(\varepsilon_1) = o(\varepsilon_2) = 4$. Thus, $\varepsilon_1 + \varepsilon_2 + \overline{\varepsilon_1\varepsilon_2}$ is equal to either $2\zeta_4 - 1$, $-2\zeta_4 - 1$ or 1. If $o(\eta_1) = 4$, then similarly we have that $\eta_1 + \eta_2 + \overline{\eta_1\eta_2}$ is also equal to either $2\zeta_4 - 1$, $-2\zeta_4 - 1$ or 1, which contradicts $\Sigma = 0$. If $o(\eta_1) = 1$, then $\eta_1 + \eta_2 + \overline{\eta_1\eta_2} = 3$, and hence $\Sigma \neq 0$, a contradiction. Therefore, $o(\eta_1) = 2$ and, by $(*)$ and the choice of notation, $\Sigma = 0$ implies $\{\varepsilon_1, \varepsilon_2\} = \{\zeta_4, -\zeta_4\}$ and $\eta_1 = \eta_2 = -1$. The other implication is trivial, so (2) is proved.

(3) By part (1), at least one of the elements of Δ is a primitive 4-th root of unity and, as assumption $(*)$ implies that $\delta_1\delta_2\delta_3 = 1$, in fact exactly two of them are. So, by symmetry, we have just to show that $\delta_1 = \zeta_4$ and $\delta_2 = -\zeta_4$ leads to a contradiction.

In fact, $\Sigma = \varepsilon_1(1 + \zeta_4) + \varepsilon_2(1 - \zeta_4) + 2\overline{\varepsilon_1\varepsilon_2} = 0$ implies that $\varepsilon_1\frac{1+\zeta_4}{\sqrt{2}} + \varepsilon_2\frac{1-\zeta_4}{\sqrt{2}} = -\overline{\varepsilon_1\varepsilon_2}\sqrt{2}$. Since $\zeta_8 = \frac{1+\zeta_4}{\sqrt{2}}$, $\zeta_8^{-1} = \frac{1-\zeta_4}{\sqrt{2}}$ and $\sqrt{2} = \zeta_8 + \zeta_8^{-1}$, we get $\zeta_8\varepsilon_1 + \zeta_8^{-1}\varepsilon_2 + (\zeta_8 + \zeta_8^{-1})\overline{\varepsilon_1\varepsilon_2} = 0$ and, multiplying by ζ_8, we have $\zeta_4\varepsilon_1 + \varepsilon_2 + \zeta_4\overline{\varepsilon_1\varepsilon_2} + \overline{\varepsilon_1\varepsilon_2} = 0$. Now, part (a) of Lemma 2.2 implies that either (i) $\overline{\varepsilon_1\varepsilon_2} = -\varepsilon_2 = -\varepsilon_1$ or (ii) $\overline{\varepsilon_1\varepsilon_2} = -\zeta_4\varepsilon_1$ and $\zeta_4\overline{\varepsilon_1\varepsilon_2} = -\varepsilon_2$. One checks that (i) is satisfied only if $\varepsilon_1 = \varepsilon_2 = -1$, while (ii) implies $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathsf{U}_4$, against the assumption $o(\varepsilon_i) \geq 8$ for some $i \in \{1, 2, 3\}$. □

We now collect some useful results about vanishing elements.

Lemma 2.4. *Let G be a group and $z \in \mathbf{Z}(G)$. Then $x \in \mathrm{V}(G)$ if and only if $zx \in \mathrm{V}(G)$. In particular, $\mathbf{Z}(G) \subseteq \mathrm{N}(G)$.*

Proof. For $\chi \in \mathrm{Irr}(G)$, let $\mathfrak{X} : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation of G affording χ, and let λ be the irreducible constituent of $\chi_{\mathbf{Z}(G)}$. Hence, $\chi(xz) = \mathrm{Tr}(\mathfrak{X}(xz)) = \mathrm{Tr}(\lambda(z)\mathfrak{X}(x)) = \lambda(z)\chi(x)$ and the assertion follows. □

Lemma 2.5. *Let A and B be normal subgroups of the group G such that $A \cap B = 1$. Then, for $a \in A$, we have $a \in \mathrm{V}(G)$ if and only if $aB \in \mathrm{V}(G/B)$.*

Proof. Let $a \in A$ and assume that $a \in \mathrm{V}(G)$. Let $\chi \in \mathrm{Irr}(G)$ be such that $\chi(a) = 0$ and let α an irreducible constituent of χ_A. By Clifford's theorem

$$\chi(a) = e \sum_{i=1}^{t} \alpha^{x_i}(a)$$

where $e = [\chi_A, \alpha] \neq 0$ and $\{x_1, x_2, \ldots, x_t\}$ is a transversal of the inertia subgroup $I = I_G(\alpha)$ in G. As $\chi(a) = 0$ and $e \neq 0$, we see that $\sum_{i=1}^{t} \alpha^{x_i}(a) = 0$.

Consider now $\alpha \times 1_B \in \mathrm{Irr}(A \times B)$ and observe that also $I_G(\alpha \times 1_B) = I$. Choosing an irreducible character ψ of I lying over $\alpha \times 1_B$, Clifford correspondence yields that $\chi_0 = \psi^G$ is an irreducible character of G. Since χ_0 lies over $\alpha \times 1_B$ and B is normal in G, the kernel of χ_0 contains B and $\chi_0 \in \mathrm{Irr}(G/B)$. Setting $e_0 = [(\chi_0)_{A \times B}, \alpha \times 1_B]$, we have

$$\chi_0(a) = e_0 \sum_{i=1}^{t} (\alpha \times 1_B)^{x_i}(a) = e_0 \sum_{i=1}^{t} \alpha^{x_i}(a) = 0$$

and hence $aB \in \mathrm{V}(G/B)$.

Conversely, we have already observed that, by inflation of characters, $aB \in \mathrm{V}(G/B)$ implies $a \in \mathrm{V}(G)$. $\square$

The next lemma will be used repeatedly.

Lemma 2.6. *Let A be a normal subgroup of the group G. Then, for $a \in A$, $a \in \mathrm{V}(G)$ if and only if there exists a character $\alpha \in \mathrm{Irr}(A)$ such that $\alpha^G(a) = 0$.*

Proof. Consider $\alpha \in \mathrm{Irr}(A)$ and let $\chi \in \mathrm{Irr}(G)$ be an irreducible character of G lying over α; so $e = [\chi_A, \alpha] \neq 0$. Let T be a transversal for $I = I_G(\alpha)$ in G. By the formula of induction of characters we have

$$|A|(\alpha^G)_A = \sum_{g \in G} \alpha^g = |I| \sum_{t \in T} \alpha^t$$

and by Clifford's theorem $\chi_A = e \sum_{t \in T} \alpha^t$. Hence, $e|A|(\alpha^G)_A = |I|\chi_A$ and, as $e \neq 0$, we conclude that, for every $a \in A$, $\alpha^G(a) = 0$ if and only if $\chi(a) = 0$.

Now, if $a \in A \cap \mathrm{V}(G)$, we just choose a $\chi \in \mathrm{Irr}(G)$ such that $\chi(a) = 0$ and an irreducible constituent α of χ_A, getting $\alpha^G(a) = 0$. Conversely, if $a \in A$ and $\alpha \in \mathrm{Irr}(A)$ are such that $\alpha^G(a) = 0$, then by the previous paragraph $\chi(a) = 0$ for every irreducible character χ of G lying over α, and then $a \in \mathrm{V}(G)$. $\square$

Lemma 2.7. *Let N be a normal subgroup of the group G.*

(1) *If G is solvable and $N/\mathbf{F}(N)$ is abelian, then $N - \mathbf{F}(N) \subseteq \mathrm{V}(G)$.*

(2) *Let $M \trianglelefteq G$ be such that $N \leq M$. If $I_M(\nu) = N$ for some $\nu \in \mathrm{Irr}(N)$, then $M - N \subseteq \mathrm{V}(G)$.*

Proof. Part (1) is Lemma 2.6 of [3]. To prove (2), observe that $I_M(\nu) = N$ implies $\mu = \nu^M \in \mathrm{Irr}(M)$ and that $\mu(x) = 0$ for every $x \in M - N$. Thus, it follows that $\mu^G(x) = 0$ for every $x \in M - N$, and hence $M - N \subseteq \mathrm{V}(G)$ by Lemma 2.6. □

Lemma 2.8 ([4, Corollary 1.3]). *Let $g \in H \leq G$ be such that $G = \mathbf{C}_G(g)H$. Then $g \in \mathrm{V}(G)$ if and only if $g \in \mathrm{V}(H)$.*

Lemma 2.9. *If G is a nilpotent group, then $\mathrm{N}(G) = \mathbf{Z}(G)$ and hence $\mathrm{P_v}(G) = \frac{m-1}{m}$, where $m = [G : \mathbf{Z}(G)]$.*

Proof. It follows from Theorem A and Theorem C of [9]. □

Lemma 2.10 ([1]). *Let G be a finite group and let P be a Sylow p-subgroup of G, p a prime number. Then $\mathbf{Z}(P) \cap \mathbf{O}_p(G) \subseteq \mathrm{N}(G)$.*

Remark 2.11. Let G be a Frobenius group with Frobenius kernel K and complement H. If K is an abelian p-group and H is cyclic of order m, then we have (using for instance Lemma 2.10 and part (1) of Lemma 2.7) $\mathrm{N}(G) = K$ and hence $\mathrm{P_v}(G) = \frac{m-1}{m}$.

Next, we quote the theorem by A. Moretó and P.H. Tiep mentioned in the Introduction.

Theorem 2.12 ([12]). *If G is a finite group such that $\mathrm{P_v}(G) < \mathfrak{a} = \mathrm{P_v}(A_7)$, then G is solvable.*

Finally, we state two lemmas for later use.

Lemma 2.13. *Let p be a prime number and let G be a group such that $\mathrm{cd}(G) = \{1, p\}$. If $[G : \mathbf{Z}(G)] \neq p^3$ and $|G'| > p$, then $[G : \mathbf{Z}(G)] = p|G'|$ and there exists a characteristic abelian subgroup A of G such that $[G : A] = p$.*

Proof. By [7, Theorem 12.11], if $[G : \mathbf{Z}(G)] \neq p^3$ then there exists a normal abelian subgroup A of G such that $[G : A] = p$. As G is nonabelian, then $\mathbf{Z}(G) \leq A$ and by [7, Lemma 12.12] we get that $|A| = |\mathbf{Z}(G)||G'|$ and hence $[G : \mathbf{Z}(G)] = p|G'|$. Since $|G'| > p$, by [7, Lemma 12.13] A is characteristic in G. □

Lemma 2.14. *Let A and B be groups such that $\mathbf{Z}(B) \leq A \leq B$ and B/A is cyclic. If every $\alpha \in \mathrm{Irr}(A)$ is B-invariant, then $\mathbf{Z}(A) = \mathbf{Z}(B)$. Moreover, if $A \neq B$, then A is nonabelian.*

Proof. Under the current assumptions, Brauer's permutation lemma ([7, Theorem 6.32]) implies that every conjugacy class of A is B-invariant. Hence, $\mathbf{Z}(A) \leq \mathbf{Z}(B)$; the other inclusion is clear.

Using that $\mathbf{Z}(A) = \mathbf{Z}(B)$ we argue that if A is abelian, then A is central in B and, since B/A is cyclic, this implies that $B = \mathbf{Z}(B)$, so $B = A$. □

3 Auxiliary results

Let A be an abelian group and let $\hat{A} = \mathrm{Irr}(A)$ be the dual group of A. We mention that $A \cong \hat{A}$ and that A and $\hat{\hat{A}}$ (the dual of $\hat{A}$) can be identified in a canonical way. For $B \leq A$, let $B^\perp = \{\alpha \in \hat{A} \mid \alpha(b) = 1,\ \forall\ b \in B\} \leq \hat{A}$, and for $V \leq \hat{A}$, let $V^\perp = \{a \in A \mid \lambda(a) = 1,\ \forall\ \lambda \in V\} \leq A$. Then $B^\perp \cong \widehat{A/B}$, $\hat{B} \cong \hat{A}/B^\perp$ and $B^{\perp\perp} = B$ (see for instance [6, V.6.4]). Finally, we recall that for every $y \in \mathrm{Aut}(A)$, y acts on $\hat{A}$ via $\alpha^y(a) = \alpha(a^{y^{-1}})$, where $\alpha \in \hat{A}$ and $a \in A$. We also write, as usual, $[A, y] = \langle [a, y] \mid a \in A\rangle$, where $[a, y] = a^{-1}a^y$ (the commutator in the semidirect product $A \rtimes \langle y\rangle$).

Lemma 3.1. *Let A be an abelian group, $y \in \mathrm{Aut}(A)$ such that $o(y) = 2$, and let $Q = A \rtimes \langle y\rangle$. Then*

(1) *The map $\gamma : A \to A$, defined by $\gamma(a) = [a, y]$ for $a \in A$, is a group homomorphism with $\mathrm{Im}(\gamma) = [A, y] = Q'$ and $\ker(\gamma) = \mathbf{C}_A(y) = \mathbf{Z}(Q)$. Moreover, $A/\mathbf{C}_A(y)$ and $[A, y]$ are isomorphic $\langle y\rangle$-modules, on which y acts as the inversion.*

(2) *$[A, y]^\perp = \mathbf{C}_{\hat{A}}(y)$ and $[\hat{A}, y]^\perp = \mathbf{C}_A(y)$.*

(3) *$[\hat{A}, y] \cong [A, y]$.*

Proof. (1) For $a, b \in A$, $[ab, y] = [a, y]^b[b, y] = [a, y][b, y]$ and $[a, y]^y = [a^y, y]$; so, γ is a morphism of $\langle y\rangle$-modules. Hence, $\mathrm{Im}(\gamma) = \{[a, y] \mid a \in A\} = [A, y]$ and, by elementary commutator computation, $[A, y] = Q'$. Moreover, $\ker(\gamma) = \mathbf{C}_A(y) = \mathbf{Z}(Q)$, as $\mathbf{Z}(Q) \leq A$, and $A/\mathbf{C}_A(y)$ and $[A, y]$ are isomorphic $\langle y\rangle$-modules. Finally, for every $a \in A$, $[a, y]^y = (a^{-1})^y a^{y^2} = (a^y)^{-1}a = [a, y]^{-1}$, so y acts as the inversion on both $[A, y]$ and $A/\mathbf{C}_A(y)$.

(2) Since y fixes $\alpha \in \hat{A}$ if and only if $[A, y] \leq \ker(\alpha)$, we have $[A, y]^\perp = \mathbf{C}_{\hat{A}}(y)$. The second equality hence follows by the canonical isomorphism of A and the double dual $\hat{\hat{A}}$.

(3) By part (1), $[A, y] \cong A/\mathbf{C}_A(y)$ and $A/\mathbf{C}_A(y) \cong \widehat{A/\mathbf{C}_A(y)} \cong \mathbf{C}_A(y)^\perp = [\hat{A}, y]$ by (2). $\square$

Given an element g of a finite group G and a prime number p, there exists a unique p-element $g_p \in G$, the *p-part* of g, such that $g = g_p h$, where g_p and h commute and the order of h is coprime to p; see [7, Lemma 8.18]. The argument in the proof of Lemma 8.18 of [7] shows that if n is any multiple of the order of g, say $n = mp^a$ with m coprime to p, then $g_p = g^{mm'}$, where m' is an integer such that $mm' \equiv 1 \pmod{p^a}$. We also remark that, if y is a p-element of G and y commutes with g, then $(gy)_p = g_p y$.

Proposition 3.2. *Let A be an abelian normal subgroup of the group G such that $[G : A] \leq 6$ and $[G : A] \neq 5$. For $a \in A$:*

(1) *if $[G : A] \neq 3$ and G has abelian Sylow 3-subgroups, then $a \in \mathrm{V}(G)$ if and only if $a_2 \in \mathrm{V}(G)$;*

(2) *if* $[G : A] = 3$*, then* $a \in \mathrm{V}(G)$ *if and only if* $a_3 \in \mathrm{V}(G)$*.*

Proof. Let $k = [G : A]$ and let $T = \{t_1, t_2, \ldots, t_k\}$ be a transversal for A in G. We can assume $k \neq 1$, as otherwise G is abelian and there is nothing to prove. Let Q be a Sylow 3-subgroup of G.

Let us assume, first, that $a \in A$ is a vanishing element of G. Then by Lemma 2.6 there exists an irreducible character α of A such that

$$\alpha^G(a) = \sum_{i=1}^{k} \alpha^{t_i}(a) = 0 \ .$$

We will show, for $p = 2$ if $k \neq 3$, and for $p = 3$ if $k = 3$, that $\alpha^G(a_p) = 0$, concluding by another application of Lemma 2.6 that $a_p \in \mathrm{V}(G)$.

Since A is abelian, α is an element of the dual group $\hat{A}$, i.e. it is a homomorphism $\alpha : A \to \mathsf{U}_n$, where $n = |A|$. For any prime p, writing $n = mp^a$, where m is coprime to p, and setting $c = mm'$, where m' is an inverse of m modulo p^a, by the discussion preceding this lemma we know that $a_p = a^c$ in A, $\alpha_p = \alpha^c$ in $\hat{A}$ and $(\alpha(a))_p = (\alpha(a))^c$ in U_n. Hence,

$$(\alpha(a))_p = \alpha_p(a) = \alpha(a_p). \tag{3.1}$$

Now, if $k = 2$, then clearly $\alpha^{t_2}(a) = -\alpha^{t_1}(a)$. It follows that $\alpha^{t_2}(a_2) = (\alpha^{t_2}(a))_2 = (\zeta_2\alpha^{t_1}(a))_2 = -\alpha^{t_1}(a_2)$, and hence $\alpha^G(a_2) = 0$.

If $k = 3$, then by part (a) of Lemma 2.2 there exists a root of unity $\delta \in \mathsf{U}_n$ such that, up to renumbering the elements of T, $\alpha^{t_i}(a) = \delta\zeta_3^i$, for $i = 1, 2, 3$. Denoting by δ_3 the 3-part of δ in U_n and using (3.1) for $\alpha^{t_i} \in \mathrm{Irr}(A)$, we have $(\alpha^{t_i}(a))_3 = \delta_3\zeta_3^i = \alpha^{t_i}(a_3)$ for $i = 1, 2, 3$. Hence, $\alpha^G(a_3) = \sum_{i=1}^{3} \alpha^{t_i}(a_3) = \delta_3 \sum_{i=1}^{3} \zeta_3^i = 0$.

If $k = 4$, then again by part (a) of Lemma 2.2, up to renumbering, $\alpha^{t_2}(a) = -\alpha^{t_1}(a)$ and $\alpha^{t_4}(a) = -\alpha^{t_3}(a)$. So, as before, $\alpha^{t_2}(a_2) = -\alpha^{t_1}(a_2)$ and $\alpha^{t_4}(a_2) = -\alpha^{t_3}(a_2)$, yielding $\alpha^G(a_2) = 0$.

Finally, we assume $k = 6$. By Lemma 2.2 we have three possible situations (of which only one, as we will show, can actually occur).

(i) Up to renumbering, $\alpha^{t_2}(a) = -\alpha^{t_1}(a)$, $\alpha^{t_4}(a) = -\alpha^{t_3}(a)$ and $\alpha^{t_6}(a) = -\alpha^{t_5}(a)$. In this case, as before, we deduce that $\alpha^G(a_2) = 0$.

(ii) There exist $\gamma, \delta \in \mathsf{U}_n$ such that, up to renumbering, $\alpha^{t_i}(a) = \gamma\zeta_3^i$, for $i = 1, 2, 3$ and $\alpha^{t_i}(a) = \delta\zeta_3^i$, for $i = 4, 5, 6$. So, the set of 3-parts $X_3 = \{(\alpha^{t_i}(a))_3 \mid 1 \leq i \leq 6\} = \delta_3\mathsf{U}_3 \cup \gamma_3\mathsf{U}_3$, being union of cosets of U_3 in U, must contain either 3 or 6 elements. But this is impossible: in fact, for $t \in T$, we have $(\alpha^t(a))_3 = \alpha^t(a_3) = \alpha(a_3^{t^{-1}})$, and this implies $|X_3| \leq |a_3^G|$, where a_3^G is the conjugacy class of a_3 in G. We rule out this case by observing that $|a_3^G| \leq 2$, because a_3 belongs to the abelian group Q and hence a_3 is central in AQ.

(iii) There exists $\delta \in \mathsf{U}_n$ such that, up to renumbering, $\alpha^{t_i}(a) = \delta\zeta_5^i$, for $i = 1, 2, 3, 4$, $\alpha^{t_5}(a) = -\delta\zeta_3$ and $\alpha^{t_6}(a) = -\delta\zeta_3^2$. In this case, $X_3 = \{(\alpha^{t_i}(a))_3 \mid 1 \leq i \leq 6\}$ is a coset of U_3 in U. But, as argued above, $|X_3| \leq |a_3^G| \leq 2$, which rules out also this case.

Conversely, let $a \in A$ and let p be a prime number such that $a_p \in \mathrm{V}(G)$. Let B be the p-complement of A and A_p be the Sylow p-subgroup of A; so, $A = A_p \times B$ and $a_p \in A_p$. By Lemma 2.5, then $a_pB \in \mathrm{V}(G/B)$ and, by inflation, $a_pB \subseteq \mathrm{V}(G)$. Since $a \in a_pB$, we conclude that $a \in \mathrm{V}(G)$. □

The next lemma shows that it is necessary to exclude the index 5 case in the statement of Proposition 3.2.

Lemma 3.3. *Let U and V be noncentral minimal normal subgroups of the group G. Assume that U is a 2-group, V is a 3-group and that $|G/UV| = 5$. Then $\mathrm{N}(G) = U \cup V$ and $\mathrm{P_v}(G) = \frac{133}{135} > \mathfrak{a}$.*

Proof. We first observe that the assumptions imply that G is solvable and that U and V are elementary abelian groups. Let $H = \langle x \rangle$ be a Sylow 5-subgroup of G. Since U and V are nontrivial irreducible H-modules and $|H| = 5$, then by [6, II.3.10] $|U| = 2^4$, $|V| = 3^4$ and G is a Frobenius group with kernel $U \times V$.

For any nontrivial element $u \in U$, the irreducibility of U as $\langle x \rangle$-module yields $U = \langle u^{x^i} \mid i = 0, \ldots, 4\rangle$. As $uu^xu^{x^2}u^{x^3}u^{x^4} = 1$, being clearly an element of the trivial subgroup $\mathbf{C}_U(H)$, we see that $U = \langle u \rangle \times \langle u^x \rangle \times \langle u^{x^2} \rangle \times \langle u^{x^4} \rangle$. Similarly, for $1 \neq v \in V$, $V = \langle v \rangle \times \langle v^x \rangle \times \langle v^{x^2} \rangle \times \langle v^{x^4} \rangle$. So, there exist characters $\beta \in \mathrm{Irr}(U)$ and $\gamma \in \mathrm{Irr}(V)$ such that $\beta(u) = \beta^x(u) = \beta^{x^2}(u) = \beta^{x^4}(u) = -1$, $\beta^{x^3}(u) = 1$, and $\gamma(v) = \gamma^{x^3}(v) = \gamma^{x^4}(v) = 1$, $\gamma^x(v) = \zeta_3$, $\gamma^{x^2}(v) = \zeta_3^2$. Let $\alpha = \beta \times \gamma \in \mathrm{Irr}(U \times V)$. Then,

$$\alpha^G(uv) = \sum_{j=0}^{4} \alpha^{x^j}(uv) = \sum_{j=0}^{4} \beta^{x^j}(u)\gamma^{x^j}(v) = 0.$$

Hence, by Lemma 2.6 $uv \in \mathrm{V}(G)$ for all $1 \neq u \in U$ and $1 \neq v \in V$. By part (1) of Lemma 2.7 $\mathrm{N}(G) \subseteq \mathbf{F}(G) = U \times V$, so we deduce that $\mathrm{N}(G) \subseteq U \cup V$. On the other hand, by Lemma 2.10, $U, V \subseteq \mathrm{N}(G)$ and hence $\mathrm{N}(G) = U \cup V$ and $\mathrm{P_v}(G) = \frac{133}{135} > \mathfrak{a}$. □

In the rest of the paper, we will use the so-called *bar convention*: if N is a normal subgroup of G, setting $\overline{G} = G/N$, we denote by $\overline{X}$ the image of a subset, or a subgroup, X of G under the natural projection of G onto $\overline{G}$. We recall a couple of basic facts about coprime action of groups.

Lemma 3.4. *Let p be a prime number and let P be a p-group acting via automorphisms on a group G.*

(1) *Let N be a P-invariant normal subgroup of G and let $\overline{G} = G/N$. If $|N|$ is coprime to p, then $\mathbf{C}_{\overline{G}}(P) = \overline{\mathbf{C}_G(P)}$.*

(2) *If G is abelian and $|G|$ is coprime to p, then $G = [G, P] \times \mathbf{C}_G(P)$.*

Proof. (1) follows from [8, Theorem 3.27] and (2) is [8, Theorem 4.34]. □

In the following, we denote by $\mathrm{c}(Q)$ the nilpotency class of a nilpotent group Q, by $\exp(Q)$ the exponent of Q and by $\mathbf{Z}_2(Q)$ the second term of the upper central series of Q.

Lemma 3.5. *Let A be an abelian normal 2-subgroup of the group G such that $[G : A] = 6$, and let $Q \in \mathrm{Syl}_2(G)$ and $P \in \mathrm{Syl}_3(G)$. Then*

(1) *There exists an element $y \in \mathbf{N}_Q(P)$ such that $y \notin A$ and $y^2 \in \mathbf{C}_A(P)$.*

(2) *If $\mathbf{C}_A(P) = 1$ and either $\exp(Q') \leq 2$ or $\mathrm{c}(Q) \leq 2$, then $A \subseteq \mathrm{N}(G)$.*

Assume further that P acts nontrivially on A and that Q is nonabelian. Then

(3) *$G - A \subseteq \mathrm{V}(G)$.*

(4) *If $\mathrm{P_v}(G) \leq \mathfrak{a}$, then $\mathbf{C}_A(P) \leq \mathbf{Z}(G)$.*

Proof. (1) Since $PA \trianglelefteq G$, by Frattini's argument $G = \mathbf{N}_G(P)A$, and hence $Q = \mathbf{N}_Q(P)A$ by Dedekind's lemma. Thus there exists an element $y \in \mathbf{N}_Q(P) - A$ and, consequently, $y^2 \in \mathbf{N}_A(P) = \mathbf{C}_A(P)$.

(2) By part (1), there exists an element $y \in \mathbf{N}_Q(P) - A$ such that $y^2 \in \mathbf{C}_A(P) = 1$. Write $P = \langle x \rangle$ and observe that $|P| = 3$ and that $T = \{x^i, x^i y \mid 1 \leq i \leq 3\}$ is a transversal of A in G. If Q is abelian, then $A \subseteq \mathrm{N}(G)$ by Lemma 2.10. So, we can assume that Q is nonabelian and then, by part (1) of Lemma 3.1, y acts as the inversion on Q'. If $\mathrm{c}(Q) = 2$, then $Q' \leq \mathbf{C}_A(y)$ and hence Q' is elementary abelian. Thus, in any case, we can assume that Q' is elementary abelian.

Let $a \in A$ and $\alpha \in \mathrm{Irr}(A)$ and write $\varepsilon_1 = \alpha(a)$, $\varepsilon_2 = \alpha^x(a)$, $\varepsilon_3 = \alpha^{x^2}(a)$, $\eta_1 = \alpha(a^y)$, $\eta_2 = \alpha^x(a^y)$, $\eta_3 = \alpha^{x^2}(a^y)$. The assumption $\mathbf{C}_A(P) = 1$ yields, for every $b \in A$, that $bb^x b^{x^2} = 1$ and hence $b^x = (bb^{x^2})^{-1}$. We deduce that $\varepsilon_3 = \overline{\varepsilon_1 \varepsilon_2}$ and that $\eta_3 = \overline{\eta_1 \eta_2}$. Writing $w = [a, y] \in Q'$, we recall that $o(w) \leq 2$ and hence that both $(\overline{\varepsilon_1}\eta_1)^2 = \alpha(w^2) = 1$ and $(\overline{\varepsilon_2}\eta_2)^2 = \alpha^x(w^2) = 1$. Hence, by part (1) of Lemma 2.3 we have that $\alpha^G(a) = \sum_{i=1}^3 \varepsilon_i + \sum_{i=1}^3 \eta_i \neq 0$. Since this is true for every $a \in A$ and $\alpha \in \mathrm{Irr}(A)$, Lemma 2.6 yields $A \subseteq \mathrm{N}(G)$.

For the rest of the proof, we assume that P acts nontrivially on A and that Q is nonabelian.

(3) G/A is isomorphic either to the cyclic group C_6 or to the symmetric group S_3. If $G/A \cong C_6$, then $Q \trianglelefteq G$. As Q is nonabelian and $[Q : A] = 2$, we have $\mathbf{Z}(Q) \leq A$; so, an application of Lemma 2.14 and of part (2) of Lemma 2.7 yields $Q - A \subseteq \mathrm{V}(G)$. Moreover, since P acts nontrivially on Q, $Q = \mathbf{F}(G)$ and by part (1) of Lemma 2.7 $G - Q \subseteq \mathrm{V}(G)$. Thus, $G - A \subseteq \mathrm{V}(G)$.

If $G/A \cong S_3$, then by lifting from G/A we get $G - PA \subseteq \mathrm{V}(G)$. As P acts nontrivially on A, then $A = \mathbf{F}(PA)$ and $PA - A \subseteq \mathrm{V}(G)$ by part (1) of Lemma 2.7 . Thus $G - A \subseteq \mathrm{V}(G)$.

(4) Let $y \in \mathbf{N}_Q(P) - A$. Then y normalizes $\mathbf{C}_A(P)$ and hence $\mathbf{C}_A(P) \trianglelefteq G$. Similarly, $[A, P] \trianglelefteq G$. Now, we argue that $Q' \leq [A, P]$. Otherwise, writing $K = P[A, P]$, G/K is a nonabelian 2-group

and $\mathrm{N}(G) \subseteq Z$, where $Z/K = \mathbf{Z}(G/K)$, by Lemma 2.9. As $\mathrm{P_v}(G) \leq \mathfrak{a}$, then $[G : Z] = 4$. But by part (3) $\mathrm{N}(G) \subseteq Z \cap A$, so $\mathrm{P_v}(G) \geq \frac{11}{12} > \mathfrak{a}$, a contradiction.

Thus, $Q' \leq [A, P]$ and $[Q, \mathbf{C}_A(P)] \leq [A, P] \cap \mathbf{C}_A(P) = 1$, where the last equality follows from part (2) of Lemma 3.4. So, $\mathbf{C}_A(P) \leq \mathbf{Z}(G)$. $\square$

Given an abelian p-group A, we denote by $\mathrm{rank}(A)$ its *rank* (i.e. the number of factors in a decomposition of A as a direct product of cyclic groups) and, for $a \in A$, $k \in \mathbb{Z}$ and $x \in \mathrm{Aut}(A)$, we shortly write a^{kx} instead of $(a^k)^x$. For a positive integer i, we define the characteristic subgroups $\Omega_i(A) = \{a \in A \mid o(a) \text{ divides } p^i\}$ and $A^i = \{a^i \mid a \in A\}$ of A.

In the rest of the section, we will address one of the most complicated cases that we have to face in the process of proving Theorem A. To avoid repetitions, we introduce the following setting, that will be used in the next three lemmas.

Setting 3.6. *Let A be an abelian normal 2-subgroup of the group G such that $[G : A] = 6$ and assume that A has a complement H in G. Let P be the Sylow 3-subgroup of H, y an involution of H and $Q = A\langle y\rangle$ a Sylow 2-subgroup of G. Assume that Q is nonabelian and that $\mathbf{C}_A(P) = 1$.*

Remark. If a cyclic group $P = \langle x\rangle$ of order 3 acts on an abelian 2-group A and $\mathbf{C}_A(P) = 1$, then $aa^xa^{x^2} = 1$ for every $a \in A$, because $aa^xa^{x^2} \in \mathbf{C}_A(P)$, (a similar result is true for every Frobenius action: see [6, V.8.9(d)]). Considering A as a module under the action of P, if B is a nontrivial P-invariant subgroup of A and $\mathrm{rank}(B) \leq 2$, then B is an indecomposable P-module and hence B is homocyclic of rank 2. Moreover, B is uniserial P-module whose only submodules are the subgroups $\Omega_i(B)$, $i \geq 0$, and whose composition factors are all isomorphic to $C_2 \times C_2$ (see, for instance, [5]). As a consequence, for every $1 \neq a \in A$, the P-submodule A_0 generated by a is $A_0 = \langle a, a^x\rangle = \langle a\rangle \times \langle a^x\rangle$, a fact that will be used several times in the rest of the section. Finally, we observe that A, being a direct sum of indecomposable P-modules, has even rank.

In the next three lemmas, we collect some facts related to Setting 3.6.

Lemma 3.7. *Assume Setting 3.6 and that $\mathrm{P_v}(G) \leq \mathfrak{a}$. Let $Z = \mathbf{Z}(Q)$ and $P = \langle x\rangle$. Then*

(1) *If $H \cong C_6$, then $Z \trianglelefteq G$ and $\exp(A/Z) = 2^{c-1}$, where $c = \mathrm{c}(Q) \leq 3$.*

(2) *If $H \cong C_6$ and B is a P-invariant subgroup of A such that $\mathrm{rank}(B) = 2$ and $A_0 = B \times B^y$ has index at most 4 in A, then $\exp(B) = 2$.*

(3) *If $H \cong S_3$, $A = Z \times Z^x$ and $\mathrm{rank}(A) \leq 4$, then $\exp(A) \leq 4$ and Z is not isomorphic to $(C_4)^2$.*

Proof. (1) Assume that H is cyclic of order 6, and observe that $Q \trianglelefteq G$ implies $Z \trianglelefteq G$. Write $\overline{G} = G/Z$. Note that $\overline{A} \neq 1$, as Q is nonabelian. By part (1) of Lemma 3.1 y acts as the inversion on $\overline{A}$, and this implies that $\mathrm{c}(Q) = \log_2(\exp(\overline{A})) + 1$. We prove this claim working by induction on $\exp(\overline{A})$: in fact, if $\exp(\overline{A}) = 2$, then $\overline{Q}$ is abelian and $\mathrm{c}(Q) = 2$; if $\exp(\overline{A}) > 2$, then $\mathbf{Z}(\overline{Q}) = \mathbf{C}_{\overline{A}}(y) = \Omega_1(\overline{A})$, and the claim follows because y still acts as the inversion on $\overline{A}/\mathbf{Z}(\overline{Q})$.

We have to prove that $\mathrm{c}(Q) \leq 3$. Working by contradiction, we assume $\mathrm{c}(Q) \geq 4$, so $\exp(\overline{A}) \geq 8$. Observe that, by part (1) of Lemma 3.4, $\mathbf{C}_{\overline{A}}(\overline{P}) = 1$ and that $\overline{Q}$ is nonabelian. Let $\overline{a} \in \overline{A}$ be such that $o(\overline{a}) \geq 8$ and let $\alpha \in \mathrm{Irr}(\overline{A})$ be such that $\alpha(\overline{a}) = \zeta_8$ and $\alpha(\overline{a}^x) = \zeta_8^5$: consider, for instance, an extension to $\overline{A}$ of a suitable character of $\langle \overline{a}, \overline{a}^x \rangle = \langle \overline{a} \rangle \times \langle \overline{a}^x \rangle$. Since $\overline{a}\overline{a}^x\overline{a}^{x^2} = 1$, $\alpha(\overline{a}^{x^2}) = (\alpha(\overline{a})\alpha(\overline{a}^x))^{-1} = \zeta_8^2$.

As already mentioned, $\overline{a}^y = \overline{a}^{-1}$; so, we have

$$\begin{aligned}\alpha^G(\overline{a}) &= \sum_{0 \leq i \leq 2} (\alpha^{x^i}(\overline{a}) + \alpha^{x^i y}(\overline{a})) = \sum_{0 \leq i \leq 2} (\alpha^{x^i}(\overline{a}) + \alpha^{x^i}(\overline{a}^{-1})) \\ &= \zeta_8 + \zeta_8^7 + \zeta_8^2 + \zeta_8^6 + \zeta_8^5 + \zeta_8^3 = 0.\end{aligned}$$

Hence, by Lemma 2.6, $\overline{a} \in \mathrm{V}(\overline{G})$. We conclude that $\overline{A} - \Omega_2(\overline{A}) \subseteq \mathrm{V}(\overline{G})$ and, since $\overline{G} - \overline{A} \subseteq \mathrm{V}(\overline{G})$ by part (3) of Lemma 3.5, we get the contradiction

$$\mathrm{P_v}(\overline{G}) \geq \frac{5}{6} + \frac{1}{6}\frac{|\overline{A}| - |\Omega_2(\overline{A})|}{|\overline{A}|} \geq \frac{5}{6} + \frac{1}{6} \cdot \frac{3}{4} > \mathfrak{a}.$$

Therefore, $\mathrm{c}(Q) \leq 3$.

(2) We recall that, since B is a P-invariant and $\mathrm{rank}(B) = 2$, B is homocyclic. We also remark that $A_0 \trianglelefteq G$. Working by contradiction and possibly replacing G with G/A_0^4, we can assume that $\exp(A_0) = \exp(B) = 4$ (note that the assumption $\mathbf{C}_A(P) = 1$ is preserved by part (1) of Lemma 3.4). Let $G_0 = A_0 H$, $Q_0 = A_0\langle y \rangle$ and let $W = \{b_1 b_2^y \mid b_1, b_2 \in B, b_1^2 = b_2^2\}$ (incidentally, we note that $W \leq A_0$; actually, $W = \mathbf{Z}_2(Q_0)$, since Q_0 is isomorphic to the wreath product $B \wr C_2$). We claim that $A_0 - W \subseteq \mathrm{V}(G)$. Let $c \in A_0 - W$; so there exist $b_0, b \in B$ such that $c = b_0 b^y$ and $b_0^2 \neq b^2$. Observe that $o(c) = 4$ and let $C = \langle c, c^x \rangle$. If $C \cap C^y \neq 1$, then $C^2 = (C^y)^2$, because this is the only nontrivial P-submodule of both C and C^y, so $c^{2y} = c^{2x^i}$ for some $0 \leq i \leq 2$. As $xy = yx$, one computes that $b^2(b_0^{2x^i})^{-1} = (b_0^{-2} b^{2x^i})^y \in B \cap B^y = 1$ and hence, using that $\mathbf{C}_A(x) = 1$, we get $i = 0$ and $b^2 = b_0^2$, a contradiction. Therefore, $C \cap C^y = 1$ and, as $\mathrm{rank}(A_0) = 4$ and $o(c) = 4 = \exp(A_0)$, we have $A_0 = C \times C^y$. Consider $\lambda, \mu \in \mathrm{Irr}(C)$ such that $\lambda(c) = \lambda(c^x) = -1$ and $\mu(c) = \zeta_4$, $\mu(c^x) = -\zeta_4$, and define $\gamma = \lambda \times \mu^y \in \mathrm{Irr}(A_0)$. Then, observing that $cc^x c^{x^2} = 1$ gives $\lambda(c^{x^2}) = 1 = \mu(c^{x^2})$,

$$\gamma^G(c) = \sum_{0 \leq i \leq 2} \gamma^{x^i}(c) + \sum_{0 \leq i \leq 2} \gamma^{x^i}(c^y) = \sum_{0 \leq i \leq 2} \lambda^{x^i}(c) + \sum_{0 \leq i \leq 2} \mu^{x^i}(c) = 0.$$

So, by Lemma 2.6, we conclude that $A_0 - W \subseteq \mathrm{V}(G_0)$ and hence $A_0 - W \subseteq \mathrm{V}(G)$ by Lemma 2.8. One can readily check that $|W| = |B||\Omega_1(B)|$ and hence $|W| = |A_0|/4$. Therefore, we get the contradiction $\mathrm{P_v}(G) \geq \frac{5}{6} + \frac{1}{6}\frac{|A_0 - W|}{|A|} = \frac{83}{96} > \mathfrak{a}$.

(3) Let now $H \cong S_3$, $A = Z \times Z^x$ and $\mathrm{rank}(A) \leq 4$.

We will first show that $e = \exp(A) \leq 4$. As $\mathrm{rank}(A) \leq 4$, we can write $Z = \langle u \rangle \times \langle w \rangle$ where $o(u) = e$ and $o(w) \geq 1$ (possibly, $w = 1$). Then $A = U \times W$, where $U = \langle u \rangle \times \langle u^x \rangle$ and $W = \langle w \rangle \times \langle w^x \rangle$. As $u^{xy} = u^{yx^2} = u^{x^2} = (uu^x)^{-1}$, we have $U \trianglelefteq G$ and, similarly, $W \trianglelefteq G$. Assume, working by contradiction, that $e \geq 8$ and consider the factor group $\overline{G} = G/U^8W$. Then $\overline{G} = \overline{A} \rtimes \overline{H}$ and $\overline{A} = \langle z \rangle \times \langle z^x \rangle$, where $z = \overline{u}$ and $o(z) = 8$. Let $I = \{1, 2, 3, 5, 6, 7\}$ and define

$$X = \{z^k z^{lx} \mid k, l \in I \text{ and } k - l \not\equiv 0 \pmod 4\}.$$

One checks that $|X| = 24$ and that, for an element $z^k z^{lx} \in X$,

$$(z^k z^{lx})^H = \{(z^k z^{lx})^h \mid h \in H\} = \{z^k z^{lx}, z^{-l} z^{(k-l)x}, z^{l-k} z^{-kx}, z^{k-l} z^{-lx}, z^{-k} z^{(l-k)x}, z^l z^{kx}\}.$$

By choosing a suitable representative of the conjugacy class $(z^k z^{lx})^H$, we can assume that both k and l are odd.

If $k \not\equiv -l \pmod 8$, we consider $\lambda \in \mathrm{Irr}(\overline{A})$ such that $\lambda(z) = \zeta_8$ and $\lambda(z^x) = \zeta_8^{-1}$. Then

$$\lambda^G(z^k z^{lx}) = \sum_{g \in (z^k z^{lx})^H} \lambda(g) = \zeta_8^{k-l} + \zeta_8^{-k} + \zeta_8^{l} + \zeta_8^{k} + \zeta_8^{-l} + \zeta_8^{l-k} = \sum_{i \in I} \zeta_8^i = 0.$$

If $k \equiv -l \pmod 8$, we consider $\mu \in \mathrm{Irr}(\overline{A})$ such that $\mu(z^k) = \zeta_8$ and $\mu(z^{lx}) = \zeta_8^5$. Then

$$\mu^G(z^k z^{lx}) = \sum_{g \in (z^k z^{lx})^H} \mu(g) = \zeta_8^6 + \zeta_8^7 + \zeta_8^3 + \zeta_8^5 + \zeta_8 + \zeta_8^2 = 0.$$

Hence, $X \subseteq \mathrm{V}(\overline{G})$. Observing that $\mathbf{C}_{\overline{A}}(\overline{P}) = 1$ by part (1) of Lemma 3.4, and that $\overline{Q}$ is nonabelian as $(z^x)^y = z^{yx^{-1}} = z^{x^{-1}} \neq z^x$, part (3) of Lemma 3.5 yields $\overline{G} - \overline{A} \subseteq \mathrm{V}(\overline{G})$. So, we get

$$\mathrm{P_v}(\overline{G}) \geq \frac{5}{6} + \frac{1}{6}\frac{|X|}{|\overline{A}|} = \frac{5}{6} + \frac{1}{6} \cdot \frac{24}{64} = \frac{43}{48} > \mathfrak{a},$$

against $\mathrm{P_v}(\overline{G}) \leq \mathrm{P_v}(G) \leq \mathfrak{a}$. Hence, $\exp(A) \leq 4$.

We will now show that $Z \ncong (C_4)^2$. Assume, working by contradiction, that $Z \cong (C_4)^2$ and let $Y = \{zw^x \mid z, w \in Z \text{ such that } Z = \langle z \rangle \times \langle w \rangle\}$. One readily checks that $|Y| = (|Z| - |Z^2|)(|Z| - |Z^2| - 4) = 96$.

We claim that $Y \subseteq \mathrm{V}(G)$. Let $a \in Y$, and write $a = zw^x$ where $z, w \in Z$ are elements such that $Z = \langle z \rangle \times \langle w \rangle$. Let $V = \langle z, z^x \rangle$ and $W = \langle w, w^x \rangle$; so, $A = V \times W$. Consider $\lambda \in \mathrm{Irr}(V)$ and $\mu \in \mathrm{Irr}(W)$ such that $\lambda(z) = \lambda(z^x) = \zeta_4$ and $\mu(w) = \zeta_4$ and $\mu(w^x) = 1$; then $\lambda(z^{x^2}) = -1$ and $\mu(w^{x^2}) = -\zeta_4$, as $bb^x b^{x^2} = 1$ for all $b \in A$.

Let $\alpha = \lambda \times \mu \in \mathrm{Irr}(A)$. Since $z, w \in Z = \mathbf{Z}(Q)$ and $xy = yx^2$, we have $\{a^h \mid h \in H\} = \{zw^x, z^x w^{x^2}, z^{x^2} w, zw^{x^2}, z^{x^2} w^x, z^x w\}$ and then

$$\begin{aligned}\alpha^G(a) = {} & \alpha(zw^x) + \alpha(z^x w^{x^2}) + \alpha(z^{x^2} w) + \alpha(zw^{x^2}) + \alpha(z^{x^2} w^x) + \alpha(z^x w) = \\ & \alpha(w)(\alpha(z^x) + \alpha(z^{x^2})) + \alpha(w^x)(\alpha(z) + \alpha(z^{x^2})) + \alpha(w^{x^2})(\alpha(z) + \alpha(z^x)) = \\ & \zeta_4(\zeta_4 - 1) + (\zeta_4 - 1) + (-\zeta_4)(\zeta_4 + \zeta_4) = 0.\end{aligned}$$

Hence, $Y \subseteq \mathrm{V}(G)$ and, as also $G - A \subseteq \mathrm{V}(G)$ by part (3) of Lemma 3.5, we get $\mathrm{P_v}(G) \geq \frac{5}{6} + \frac{1}{6}\frac{|Y|}{|A|} = \frac{5}{6} + \frac{1}{6}\frac{96}{16^2} > \mathfrak{a}$, a contradiction. □

Lemma 3.8. *Assume Setting 3.6 and that H is isomorphic to C_6. Suppose that* $\mathrm{rank}(A) = 2$, $\exp(A) = 8$ *and* $\mathrm{c}(Q) > 2$. *Then one of the following is true.*

(1) $\mathrm{N}(G) = A^2$ *and* $\mathrm{P_v}(G) = \frac{23}{24} > \mathfrak{a}$.

(2) $\mathrm{N}(G) = A$ *and* $\mathrm{P_v}(G) = \frac{5}{6}$.

Proof. We observe that $Q \trianglelefteq G$ and then both Q' and $Z = \mathbf{Z}(Q)$ are P-submodules of the uniserial P-module $A \cong C_8 \times C_8$. Since $\mathrm{c}(Q) > 2$, Q' is not contained in Z and hence $Z = \Omega_1(A)$ and $Q' = A^2$. Since $\mathrm{c}(A^2\langle y\rangle) = 2$, part (2) of Lemma 3.5 applied to the subgroup A^2H yields $A^2 \subseteq \mathrm{N}(A^2H)$ and hence $A^2 \subseteq \mathrm{N}(G)$ by Lemma 2.8. We also recall that $\mathrm{N}(G) \subseteq A$ by part (3) of Lemma 3.5. Let $a \in A$ be such that $o(a) = 8$; by the Frobenius action of $P = \langle x\rangle$ on A, we have $A = \langle a\rangle \times \langle a^x\rangle$. Note that $aa^y \in \mathbf{C}_A(y) = Z = \{1, a^4, a^{4x}, a^{4x^2}\}$. We distinguish two cases.

(1) Assume first that $aa^y \in \{1, a^4\}$, so $a^y = a^k$ with $k \in \{-1, 3\}$. Let $\alpha \in \mathrm{Irr}(A)$ be such that $\alpha(a) = \zeta_8$ and $\alpha(a^x) = \zeta_8^5$; so, $\alpha(a^{x^2}) = (\alpha(a)\alpha(a^x))^{-1} = \zeta_8^2$. Hence,

$$\alpha^G(a) = \sum_{0\leq i\leq 2}(\alpha^{x^i}(a) + \alpha^{x^iy}(a)) = \sum_{0\leq i\leq 2}(\alpha^{x^i}(a) + \alpha^{x^i}(a^k)) = \sum_{j\in\{1,2,3,5,6,7\}}\zeta_8^j = 0,$$

so $a \in \mathrm{V}(G)$ by Lemma 2.6. Since $A = \langle a, a^x\rangle$ and y commutes with x, $a^y = a^k$ implies that $b^y = b^k$ for every element $b \in A$. Thus, the previous argument shows that $b \in \mathrm{V}(G)$ for all $b \in A$ such that $o(b) = 8$. So, $\mathrm{N}(G) = \mathrm{N}(G) \cap A \subseteq A^2$ and hence $\mathrm{N}(G) = A^2$ and $\mathrm{P_v}(G) = \frac{23}{24}$.

(2) Assume now that $aa^y \in \{a^{4x}, a^{4x^2}\}$; up to interchanging x and $x^2 = x^{-1}$, we can assume that $aa^y = a^{4x}$. We claim that $a \in \mathrm{N}(G)$. Arguing by contradiction, by Lemma 2.6 there exists a character $\alpha \in \mathrm{Irr}(A)$ such that $\alpha^G(a) = 0$. Let $K = \ker(\alpha)$ and observe that $Z \not\leq K$: otherwise, writing $\overline{G} = G/Z$, $\alpha \in \mathrm{Irr}(\overline{A})$ and $\alpha^{\overline{G}}(\overline{a}) = 0$, so $\overline{a} \in \mathrm{V}(\overline{G}) \cap \overline{A}$, against part (2) of Lemma 3.5 since $\mathrm{c}(\overline{Q}) = 2$. Hence, as $a^4a^{4x}a^{4x^2} = 1$ and $\{\alpha(a^{4x^i}) \mid 0 \leq i \leq 2\} = \{1, -1\}$, up to possibly exchanging α with one conjugate character α^{x^i} (observing that $\alpha^G = (\alpha^{x^i})^G$ for every i), we can assume $\alpha(a^{4x}) = 1$ and $\alpha(a^4) = \alpha(a^{4x^2}) = -1$. Recalling that $a^y = a^{4x}a^{-1}$, we have that $\alpha^{x^i}(a^y) = \alpha^{x^i}(a^{4x})\overline{\alpha^{x^i}(a)} = \alpha(a^{4x^{1-i}})\overline{\alpha^{x^i}(a)}$ for $0 \leq i \leq 2$ and then

$$\begin{aligned}\alpha^G(a) &= \alpha(a) + \alpha^x(a) + \alpha^{x^2}(a) + \alpha(a^y) + \alpha^x(a^y) + \alpha^{x^2}(a^y)\\ &= \alpha(a) + \alpha^x(a) + \alpha^{x^2}(a) + \overline{\alpha(a)} - \overline{\alpha^x(a)} - \overline{\alpha^{x^2}(a)}\end{aligned}$$

Hence, the assumption $\alpha^G(a) = 0$ gives

$$\alpha(a) + \overline{\alpha(a)} = \overline{\alpha^x(a)} - \alpha^x(a) + \overline{\alpha^{x^2}(a)} - \alpha^{x^2}(a).$$

Observing that the first member of the above equality is real, while the second member is purely imaginary, we deduce that $\overline{\alpha(a)} = -\alpha(a)$ and we get $\alpha(a)^4 = \alpha(a^4) = 1$, against the assumption $\alpha(a^4) = -1$. Hence, $a \in \mathrm{N}(G)$. Since $A = \langle a, a^x \rangle$ and y commutes with x, the map $\varphi : A \to A$ such that, for $b \in A$, $\varphi(b) = bb^y$, is a morphism of P-modules. Hence, we have $bb^y = (b^4)^x$ for every $b \in A$ and then by the previous argument we deduce that $A - A^2 \subseteq \mathrm{N}(G)$. Recalling that $A^2 \subseteq \mathrm{N}(G) \subseteq A$, we conclude that $A = \mathrm{N}(G)$ and $\mathrm{P_v}(G) = \frac{5}{6}$. ☐

Lemma 3.9. *Assume the setting 3.6 and that H is isomorphic to C_6. Let $A = CD$, where C is a P-invariant subgroup of rank 2, $D = [C, y]$ and $C \cap D = D^2 = \Omega_1(D)$. If $e = \exp(A) \geq 8$, then one of the following is true.*

(1) *$a^{\frac{e}{2}} = [a^2, y]$ for every $a \in A$, $C - C^2 \subseteq \mathrm{V}(G)$ and $\mathrm{P_v}(G) > \mathfrak{a}$.*

(2) *For a suitable generator x of P, $a^{\frac{e}{2}} = [a^2, y]^x$ for every $a \in A$, $A = \mathrm{N}(G)$ and $\mathrm{P_v}(G) = \frac{5}{6}$.*

Proof. As $D^2 = \Omega_1(D)$ and $D = [C, y] = [A, y] \neq 1$, we have that $\exp(D) = 4$ and, since $D = [C, y]$ has rank at most two and is P-invariant, we conclude that $D \cong (C_4)^2$. We also observe that $A = CC^y$ and that $e = \exp(C)$. Write $P = \langle x \rangle$. Let $a \in C$ be such that $o(a) = e$. Then $C = \langle a \rangle \times \langle a^x \rangle$, $D = \langle [a, y] \rangle \times \langle [a, y]^x \rangle$ and a is a generator of A as a H-module. As $1 \neq a^{\frac{e}{2}} \in \Omega_1(C) = D^2$, we have $a^{\frac{e}{2}} = ([a, y]^{x^k})^2 = [a^2, y]^{x^k}$ for some $k \in \{0, 1, -1\}$. We remark that then $g^{\frac{e}{2}} = [g^2, y]^{x^k}$ for all $g \in A$, because the two maps $g \mapsto [g^2, y]^{x^k}$ and $g \mapsto g^{\frac{e}{2}}$ are (by part (1) of Lemma 3.1 and $y \in \mathbf{Z}(H)$) endomorphisms of the H-module A and they coincide on the generator a. We distinguish two cases.

(1) Assume that $a^{\frac{e}{2}} = [a, y]^2$ and consider characters $\gamma \in \mathrm{Irr}(C)$ and $\delta \in \mathrm{Irr}(D)$ such that $\delta([a, y]) = \delta^x([a, y]) = \zeta_4$ and $\gamma(a) = \zeta_e$, $\gamma^x(a) = -\zeta_e$. Since $aa^x a^{x^2} = 1$, it follows that $\delta^{x^2}([a, y]) = -1$ and $\gamma^{x^2}(a) = -\zeta_e^{-2}$. As $e \geq 8$, δ and γ coincide on the subgroup $C \cap D$, and it follows (see, for instance, [7, Problem 4.4]) that there exists a character $\alpha \in \mathrm{Irr}(A)$ that extends both γ and δ. Thus, one computes that

$$\alpha^G(a) = \sum_{0 \leq i \leq 2} \alpha^{x^i}(a)(1 + \alpha^{x^i}([a, y])) = \sum_{0 \leq i \leq 2} \gamma^{x^i}(a)(1 + \delta^{x^i}([a, y])) = 0$$

and hence by Lemma 2.6 $a \in \mathrm{V}(G)$. We can apply this argument to every $a \in C$ such that $o(a) = e$, and hence $C - C^2 \subseteq \mathrm{V}(G)$. As $[A : C] = [C : C^2] = 4$, we conclude that

$$\mathrm{P_v}(G) \geq \frac{5}{6} + \frac{1}{6} \frac{|C - C^2|}{4|C|} = \frac{5}{6} + \frac{1}{6} \cdot \frac{3}{16} = \frac{83}{96} > \mathfrak{a}.$$

(2) For the remaining cases, up to interchanging the generators x and x^{-1} of P, we can assume that $g^{\frac{e}{2}} = [g^2, y]^x$ for every $g \in A$. We will show that $\mathrm{N}(G) = A$, so $\mathrm{P_v}(G) = \frac{5}{6}$. By part (3) of Lemma 3.5 we only have to show that $A \subseteq \mathrm{N}(G)$. Working by contradiction, we

assume that there exists an element $b \in A \cap \mathrm{V}(G)$ and we have, by Lemma 2.6, a character $\alpha \in \mathrm{Irr}(A)$ such that $\alpha^G(b) = \sum_{0 \leq i \leq 2} \alpha^{x^i}(b) + \sum_{0 \leq i \leq 2} \alpha^{x^i}(b^y) = 0$. Defining $\delta_i = \alpha^{x^i}(b)^{-1}\alpha^{x^i}(b^y) = \alpha^{x^i}([b, y])$, by part (1) of Lemma 2.3 there hence exists at least one $i_0 \in \{0, 1, 2\}$ such that δ_{i_0} is a primitive 4-th root of unity (recall that $[b, y] \in D$ and $\exp(D) = 4$). In particular, $1 \neq \delta_{i_0}^2 = \alpha^{x^{i_0}}([b^2, y]) = \alpha^{x^{i_0}}((b^{\frac{e}{2}})^{x^{-1}}) = (\alpha^{x^{i_0+1}}(b))^{\frac{e}{2}}$ implies that $o(\alpha^{x^j}(b)) = e \geq 8$ for some $j \in \{1, 2, 3\}$. So, part (3) of Lemma 2.3 gives that $\{\delta_i \mid 0 \leq i \leq 2\} = \{\omega, -1\}$, where either $\omega = \zeta_4$ or $\omega = -\zeta_4$. Up to replacing α with some conjugate character α^{x^i} (which we can do, since they induce the same character to G), we can hence assume that $\delta_0 = \omega = \delta_1$ and that $\delta_2 = -1$. So, $\alpha^G(b) = \alpha(b)+\alpha^x(b)+\omega\alpha(b)+\omega\alpha^x(b) = (\alpha(b)+\alpha^x(b))(1+\omega) = 0$ forces $\alpha^x(b) = -\alpha(b)$ and hence, since $\frac{e}{2}$ is even, we have $-1 = \delta_0^2 = \alpha([b^2, y]) = \alpha^x([b^2, y]^x) = \alpha^x(b^{\frac{e}{2}}) = \alpha(b^{\frac{e}{2}}) = \alpha([b^2, y]^x) = \alpha^{x^2}([b, y]^2) = \delta_2^2 = 1$, a contradiction. □

We are now ready to prove the main result of this section.

Proposition 3.10. *Let A be an abelian normal 2-subgroup of the group G such that $[G : A] = 6$. Let P be a Sylow 3-subgroup of G, let Q be a Sylow 2-subgroup of G, and assume that Q is nonabelian and that $\mathbf{C}_A(P) = 1$. If $\mathrm{P_v}(G) \leq \mathfrak{a}$, then $A = \mathrm{N}(G)$.*

Proof. By part (3) of Lemma 3.5, we know that $\mathrm{N}(G) \subseteq A$ and we have to prove that $A \subseteq \mathrm{N}(G)$. Let G be a counterexample of minimal order, and let $a_0 \in A$ such that $a_0 \in \mathrm{V}(G)$. Let $\hat{A} = \mathrm{Irr}(A)$ be the dual group of the abelian group A. By Lemma 2.6, there exists a character $\alpha \in \hat{A}$ such that $\alpha^G(a_0) = 0$. Write $K = \ker(\alpha)$. Then $K^\perp \cong \widehat{A/K} \cong A/K$ is a cyclic group of order $o(\alpha)$ and, as $\alpha \in K^\perp$, we see that $K^\perp = \langle \alpha \rangle$. It follows that, for $g \in G$, $(K^\perp)^g = (K^g)^\perp = \langle \alpha^g \rangle$.

Note that $\ker(\alpha^G) = \bigcap_{g \in G} K^g = K_G$ is the normal core of K in G; in particular, $K_G \leq K \leq A$. Let $\overline{G} = G/K_G$ and observe that all assumptions are inherited by $\overline{G}$. In fact, $\overline{A}$ is a normal abelian 2-subgroup of $\overline{G}$, $[\overline{G} : \overline{A}] = 6$, and $\mathbf{C}_{\overline{A}}(\overline{P}) = 1$ by part (1) of Lemma 3.4. Since $K_G \leq K$, we can see α as a character of $\mathrm{Irr}(\overline{A})$ and $\alpha^{\overline{G}}(\overline{a_0}) = \alpha^G(a_0) = 0$, so $\overline{a_0} \in \mathrm{V}(\overline{G})$ by Lemma 2.6. In particular, part (2) of Lemma 3.5 implies that $\overline{Q}$ is nonabelian. Hence, by the minimality of G, we conclude that $K_G = 1$.

Let $P = \langle x \rangle$. As $\mathbf{C}_A(P) = 1$, by Brauer's permutation lemma $\mathbf{C}_{\hat{A}}(P) = 1$ and hence $\lambda\lambda^x\lambda^{x^2} = 1_A$ for all $\lambda \in \hat{A}$. So, $K^{x^2} \geq K \cap K^x$ and then

$$K \cap K^x \cap K^y \cap K^{xy} = K_G = 1\,.$$

By part (1) of Lemma 3.5, there exists an involution $y \in \mathbf{N}_Q(P) - A$; hence, $Q = A\langle y \rangle$ and $H = \langle x, y \rangle$ is a complement of A in G. Set $B = K \cap K^x$; so, B is P-invariant and $B \cap B^y = 1$. By [6, V.6.4] we have that $B^\perp = K^\perp (K^x)^\perp = \langle \alpha, \alpha^x \rangle$ has rank at most 2, so $B^\perp = \langle \alpha \rangle \times \langle \alpha^x \rangle$, as $\mathbf{C}_{\hat{A}}(P) = 1$. Moreover,

$$\hat{A} = 1^\perp = (B \cap B^y)^\perp = B^\perp B^{\perp y} = \langle \alpha, \alpha^x \rangle \langle \alpha^y, \alpha^{xy} \rangle = \langle \alpha, \alpha^x \rangle \langle [\alpha, y], [\alpha^x, y] \rangle = B^\perp [\hat{A}, y].$$

In particular, $\mathrm{rank}(A) = \mathrm{rank}(\hat{A}) \le 4$ and $\exp(A) = \exp(\hat{A}) = o(\alpha)$. Write $e = \exp(A)$ and observe that $e \ge 4$ by part (2) of Lemma 3.5.

We split the rest of the proof into two cases: (1) $H \cong C_6$ and (2) $H \cong S_3$.

(1) Assume first $H \cong C_6$. In this case, $Q \trianglelefteq G$ and, setting $Z = \mathbf{Z}(Q)$, then $Z \trianglelefteq G$. By part (1) of Lemma 3.7 and part (2) of Lemma 3.5 we have $\exp(A/Z) = 4$ and $\mathrm{c}(Q) = 3$. Observe also that $B \cap Z = 1$, since $B \cap B^y = 1$.

As $\hat{A} = B^{\perp} B^{\perp y}$, we see that $[\hat{A}, y] = [B^{\perp}, y][B^{\perp y}, y] = [B^{\perp}, y] = \langle [\alpha, y], [\alpha, y]^x \rangle$ has rank two. By Lemma 3.1, $Q' = [A, y] \cong [\hat{A}, y]$ and $Q' \cong A/Z$, so Q' has rank two and $\exp(Q') = 4$. Since Q' is P-invariant, Q' is homocyclic and we conclude that $Q' \cong (C_4)^2$.

By Lemma 3.1 we also know that y acts as the inversion on both Q' and A/Z. It follows that $Q' \cap Z = \Omega_1(Q')$ and $A/\mathbf{Z}_2(Q) \cong (C_2)^2$.

As $B \cong BZ/Z$ and BZ/Z is a P-submodule of $A/Z \cong (C_4)^2$, we have three cases:

(1a): $B = 1$. In this case $A \cong \hat{A} = B^{\perp}$ is homocyclic of rank two. Since $Q' \not\le Z$, as $\mathrm{c}(Q) = 3$, and A is a uniserial P-module, we have $Z \le Q'$. So, $Z \cong (C_2)^2$ and $A \cong (C_8)^2$, a contradiction by Lemma 3.8.

(1b): $B \cong (C_2)^2$. Note that in this case $BZ/Z = \mathbf{C}_{A/Z}(y)$, so $BZ = \mathbf{Z}_2(Q)$. By part (2) of Lemma 3.5 applied to the subgroup BZH of G and Lemma 2.8, we hence see that $a_0 \notin BZ$. We also observe that $Z \not\le Q'$. In fact, if $Z \le Q'$, then $Z = \Omega_1(Q')$, so $B \cap Q' = 1$; this implies that Q'/Z is a complement of BZ/Z in $A/Z \cong (C_4)^2$, a contradiction.

Let $\overline{A} = A/B$. Since $\overline{A} \cong \widehat{A/B} \cong B^{\perp} \cong (C_e)^2$ and $\overline{a_0} \notin \overline{Z} = \overline{A}^2$, we have that $o(a_0) = e = \exp(A)$. Let $C = \langle a_0, a_0^x \rangle$ and observe that $[A : C] = 2^2$, so C is a maximal P-submodule of A. We also note that $e \ge 8$, as $e \le 4$ implies $|A| \le 8^2$, so $|Z| = 2^2$ and Z is a minimal P-submodule of A, which is impossible as $Z \not\le Q'$.

If $C^y \ne C$, then $A = CC^y$ by the maximality of C. It follows that $[C, y] = [A, y] = Q'$ and $A = C[C, y] = CQ'$. Also, $C \cap Q' = (Q')^2 = \Omega_1(Q')$, because $Q' \cong (C_4)^2$, and we get a contradiction by Lemma 3.9.

Hence, we can assume that $C^y = C$, so $C \trianglelefteq G$. As $\mathbf{C}_{A/C}(y)$ is a nontrivial P-submodule of the irreducible P-module A/C, we deduce that Q/C is abelian and hence that $Q' \le C$. Hence, since both $Z \not\le Q'$ and $Q' \not\le Z$, Z cannot be contained in the uniserial P-module C. By the maximality of C, it follows that $A = CZ$. So, $C/(Z \cap C) \cong A/Z \cong (C_4)^2$ and we conclude that $e = 8$ and $C \cong (C_8)^2$. We observe that C is H-invariant, that $\mathrm{c}(C\langle y \rangle) = 3$ since $Q' \le C$, and that $a_0 \in \mathrm{V}(CH)$ by Lemma 2.8. So, an application of Lemma 3.8 to the CH yields $C - C^2 \subseteq \mathrm{V}(CH)$ and then $C - C^2 \subseteq \mathrm{V}(G)$ by Lemma 2.8. Hence

$$\mathrm{P_v}(G) \ge \frac{5}{6} + \frac{1}{6}\frac{|C - C^2|}{2^2|C|} = \frac{83}{96} > \mathfrak{a},$$

a contradiction.

(1c): $B \cong (C_4)^2$. In this case, $A = B \times Z$ and, as $\operatorname{rank}(A) \leq 4$ and Z is nontrivial and P-invariant, we see that $Z \cong (C_f)^2$, where $f = \exp(Z)$. Since $B \cong (C_4)^2$ and as $|B \times B^y| = |B|^2 \leq |A|$, we have $f \geq 4$. If $f \in \{4, 8\}$, then $[A : B \times B^y] \leq 4$ and we get a contradiction by part (2) of Lemma 3.7. So, we have $f \geq 16$ and $f = e = \exp(A)$. Let $b \in B$ with $o(b) = 4$. Since $Q' = [BZ, y] = [B, y] = \langle [b, y], [b^x, y] \rangle$ and $[b, y]^{2x} = [b^x, y]^2$ is an element of order 2 of $\Omega_1(Q') = \Omega_1(Z)$, there exists an element (that depends on b) $z = z_b \in Z$ such that $o(z) = e$ and $z^{\frac{e}{2}} = [b, y]^{2x}$. Let $X_b = \{bz^i z^{jx} \mid i, j \text{ odd}\}$ and observe that, as $Z = \langle z \rangle \times \langle z^x \rangle$, we have $|X_b| = (\frac{e}{2})^2$. Consider an element $a = bz^i z^{jx} \in X_b$ and let $C = \langle a, a^x \rangle$ and $A_0 = CC^y \trianglelefteq G$. So, $A_0 = CD$, where $D = [C, y] = \langle [b, y], [b, y]^x \rangle = Q'$. As $e \geq 16$ and $o(b) = 4$, then $\Omega_2(C) = \langle a^{\frac{e}{4}}, a^{\frac{e}{4}x} \rangle \leq Z$ and $C \cap D = \Omega_2(C) \cap D = D^2 = \Omega_1(D)$. Also,

$$a^{\frac{e}{2}} = (bz^i z^{jx})^{\frac{e}{2}} = (z^i z^{jx})^{\frac{e}{2}} = [b, y]^{2x}[b, y]^{2x^2} = [b, y]^{-2} = [b, y]^2 = [bz^i z^{jx}, y]^2 = [a, y]^2,$$

where the fourth equality is a consequence of the Frobenius action of P on A. Since $\exp(A_0) = o(a) \geq 16$, an application of Lemma 3.9 to $G_0 = A_0 H$ yields $a \in \mathrm{V}(G_0)$ and, since $G = \mathbf{C}_G(a) G_0$, by Lemma 2.8 we conclude that $a \in \mathrm{V}(G)$. Hence, $X_b \subseteq \mathrm{V}(G)$ for every $b \in B - B^2$. As $|\bigcup_{b \in B - B^2} X_b| = |B - B^2| (\frac{e}{2})^2$, we conclude that $|A \cap \mathrm{V}(G)| \geq \frac{3}{4}|B|(\frac{e}{2})^2$. Hence

$$\mathrm{P_v}(G) = \frac{5}{6} + \frac{1}{6}\frac{|A \cap \mathrm{V}(G)|}{|A|} \geq \frac{5}{6} + \frac{1}{6}\frac{\frac{3}{4}|B|(\frac{e}{2})^2}{|B|e^2} = \frac{5}{6} + \frac{1}{6}\frac{3}{16} > \mathfrak{a},$$

a contradiction.

(2) Assume now $H \cong S_3$. As $H = \langle y, y^x \rangle$ and $Z = \mathbf{Z}(Q) = \mathbf{C}_A(y)$, we have

$$Z \cap Z^x = \mathbf{C}_A(y) \cap \mathbf{C}_A(y)^x = \mathbf{C}_A(\langle y, y^x \rangle) = \mathbf{C}_A(H) \leq \mathbf{C}_A(P) = 1.$$

An application of part (1) of Lemma 3.1 to the action of y on $\hat{A}$ yields $\mu^y = \mu^{-1}$ and hence $\mu^{xy} = \mu^{-x^2} \neq \mu^{-x}$ for every $\mu \in [\hat{A}, y]$, $\mu \neq 1_A$, so $[\hat{A}, y] \cap [\hat{A}, y]^x = 1_A$. Since by Lemma 3.1 $[\hat{A}, y]^\perp = Z$, it follows that

$$A = (1_A)^\perp = ([\hat{A}, y] \cap [\hat{A}, y]^x)^\perp = ZZ^x = Z \times Z^x.$$

Since $e = \exp(Z) \geq 4$ and $\operatorname{rank}(A) \leq 4$, by part (3) of Lemma 3.7 we deduce that $e = 4$ and that Z is not isomorphic to $(C_4)^2$. Hence, either $Z \cong C_4 \times C_2$ or $Z \cong C_4$. Write $Z = \langle z, w \rangle$, where $o(z) = 4$ and $o(w) \leq 2$. So, $A = V \times W$, where $V = \langle z, z^x \rangle$ and $W = \langle w, w^x \rangle$. As $z^{xy} = z^{yx^2} = z^{x^2} = (zz^x)^{-1}$, it follows that $V \trianglelefteq G$ and, similarly, $W \trianglelefteq G$. Let $a_0 \in A$ and $\alpha \in \operatorname{Irr}(A)$ be as defined in the first paragraph of the proof. Write $a_0 = z^k z^{lx} w_0$, where $w_0 \in W$ and recall that

$$\alpha^G(a_0) = \sum_{0 \leq i \leq 2} \alpha^{x^i}(a_0) + \sum_{0 \leq i \leq 2} \alpha^{x^i}(a_0^y) = 0.$$

Up to substituting α with a suitable conjugate α^h for some $h \in H$, we may assume that $o(\alpha(a_0)) = \max\{o(\alpha^h(a_0)) \mid h \in H\}$. Since $\alpha(a)\alpha^x(a)\alpha^{x^2}(a) = 1$ for all $a \in A$ and both $\alpha^{x^i}(a_0)$ and $\alpha^{x^i}(a_0^y)$

belong to U_4 for all $0 \le i \le 2$, an application of part (2) of Lemma 2.3 yields $o(\alpha(a_0)) = 4$ and $o(\alpha^{x^i}(a_0^y)) \le 2$ for all $0 \le i \le 2$. So, $o(\alpha([a_0, y])) = o(\alpha(a_0)^{-1}\alpha(a_0^y)) = 4$ and hence, as $[a_0, y] = [z^{lx}, y][w_0, y] = z^{-lx}z^{lx^2}[w_0, y]$ and $o([w_0, y]) \le 2$ (as $[w_0, y] \in W$), we deduce that l is odd. Up to interchanging z and z^{-1}, we can hence assume that $l = 1$. Let $u = z^2$, an involution, and observe that $(\alpha^{x^i}(a_0^y))^2 = 1$ for all $0 \le i \le 2$ implies that $(a_0^2)^y = (u^k u^x)^y = u^k u^{x^2} \in \ker(\alpha^{x^i})$ for all $0 \le i \le 2$, and hence $(u^k u^{x^2})^{x^j} \in K = \ker(\alpha)$ for all $0 \le j \le 2$. Now, both for $k \equiv 0 \pmod 2$ and $k \equiv 1 \pmod 2$, we conclude that $V^2 = \langle u, u^x \rangle \le K$. But $1 \ne V^2 \trianglelefteq G$ and this is a contradiction since $K_G = 1$. □

4 Proof of the main theorem

Proposition 4.1. *Let N be a proper nilpotent normal subgroup of the group G. Assume that N is nonabelian and that $\mathrm{N}(G) \subseteq N$. If $\mathrm{P_v}(G) \le \mathfrak{a}$, then $[G : N] = 3$ and there exists an abelian normal subgroup A of G such that $A \le N$, $[N : A] = 2$ and $\mathrm{N}(G) \subseteq A$.*

Proof. We prove that, whenever $N \ne G$ is a nonabelian nilpotent normal subgroup of G such that $\mathrm{N}(G) \subseteq N$, if $\mathrm{P_v}(G) \le \mathfrak{a}$ then $[G : N] = 3$ and there exists a subgroup $A \trianglelefteq G$ such that $A \le N$, $[N : A] = 2$ and $\mathrm{N}(G) \subseteq A$. It will hence follow that A is abelian, as A is nilpotent and $[G : A] \ne 3$.

Since N is a nilpotent, nonabelian and normal subgroup of G, there exists a G-invariant subgroup D of N such that, writing $\overline{G} = G/D$, $\overline{N}$ is a nonabelian p-group, for a prime p, and the commutator subgroup $\overline{N}'$ is a minimal normal subgroup of $\overline{G}$. We denote by K the preimage of $\overline{N}'$ in G. Setting $\overline{Z} = \mathbf{Z}(\overline{N})$, we have that $\overline{K} \le \overline{Z}$ and hence $\overline{K}$ is an irreducible G/N-module. The assumptions $G - N \subseteq \mathrm{V}(G)$ and $\mathrm{P_v}(G) \le \mathfrak{a}$ imply that $|G/N| \le 6$. If $G/N \not\cong S_3$, then G/N is abelian and, writing $|\overline{K}| = p^d$, by [6, II.3.10] d is the multiplicative order of p modulo n, where $n = |G/N/\mathbf{C}_{G/N}(\overline{K})| \le 6$, so $d \in \{1, 2, 4\}$. Recalling also that every field of prime order is a splitting field for S_3, we conclude that $|\overline{K}| \in \{p, p^2, p^4\}$

For a nontrivial $\lambda \in \mathrm{Irr}(\overline{K})$, let $\overline{Z_\lambda} \trianglelefteq \overline{N}$ be such that $\overline{Z_\lambda}/\ker(\lambda) = \mathbf{Z}(\overline{N}/\ker(\lambda))$. Then, for every $\varphi \in \mathrm{Irr}(\overline{N})$ that lies over λ, $\overline{Z_\lambda} = \mathbf{Z}(\varphi)$: in fact $\mathbf{Z}(\varphi) \le \overline{Z_\lambda}$, because $[\mathbf{Z}(\varphi), \overline{N}] \le \overline{K} \cap \ker(\varphi) = \ker(\lambda)$ as $\varphi_{\overline{K}} = \varphi(1)\lambda$, while the other inclusion is clear.

As $\overline{N}$ has nilpotency class 2, φ is fully ramified with respect to $\overline{N}/\overline{Z_\lambda}$ by Theorem 2.31 and Problem 6.3 of [7]. Hence, $[\overline{N} : \overline{Z_\lambda}] = \varphi(1)^2$ is a square $\ne 1$ (as φ lies over $\lambda \ne 1_{\overline{K}}$) and $\varphi(g) = 0$ for every $g \in \overline{N} - \overline{Z_\lambda}$.

Let

$$\overline{X_0} = \bigcup_{x \in G/N} \overline{Z_\lambda}^x$$

and let X_0 be the preimage of $\overline{X_0}$ in G. Then,

$$|\overline{X_0}| \le [G/N : I_{G/N}(\lambda)]|\overline{Z_\lambda}| \le |G/N||\overline{Z_\lambda}| \tag{4.1}$$

and in particular, denoting by Z_λ the preimage of $\overline{Z_\lambda}$ in G, we see that $|X_0|/|G| \leq |Z_\lambda|/|N|$.

By considering an irreducible character of $\overline{G}$ lying over φ, Clifford's theorem implies that $\overline{N} - \overline{X_0} \subseteq \mathrm{V}(\overline{G})$. So, by inflation $N - X_0 \subseteq \mathrm{V}(G)$ and, recalling that $G - N \subseteq \mathrm{V}(G)$, we deduce that $G - X_0 \subseteq \mathrm{V}(G)$. Hence,

$$\mathrm{P_v}(G) = \frac{|\mathrm{V}(G)|}{|G|} \geq \frac{|G - X_0|}{|G|} = 1 - \frac{|X_0|}{|G|} \geq 1 - \frac{1}{[N : Z_\lambda]} \ .$$

Now, $\mathrm{P_v}(G) \leq \mathfrak{a}$ implies that $[N : Z_\lambda] \leq 6$, and hence, since $1 \neq [N : Z_\lambda]$ is a square, we deduce that $p = 2$ and that $\varphi(1)^2 = [N : Z_\lambda] = 2^2$. As this is true for every nontrivial $\lambda \in \mathrm{Irr}(\overline{K})$ and every $\varphi \in \mathrm{Irr}(\overline{N})$ that lies over λ, we conclude that $\mathrm{cd}(\overline{N}) = \{1, 2\}$.

Next, we argue that $|\overline{K}| \neq 2$. Namely, if $|\overline{K}| = 2$ then λ is the only nontrivial character of $\mathrm{Irr}(\overline{K})$ and $\overline{Z_\lambda} = \mathbf{Z}(\overline{N}) \trianglelefteq \overline{G}$. This implies (by the definition of X_0) that $X_0 = Z_\lambda$ and hence, as $N < G$, $\mathrm{P_v}(G) = 1 - (1/[G : Z_\lambda]) \geq \frac{7}{8}$, against $\mathrm{P_v}(G) \leq \mathfrak{a}$. Note that, since $\overline{K}$ is an irreducible G/N-module, $|\overline{K}| \neq 2$ implies $|G/N| \neq 2, 4$.

If $|G/N| = 6$, then $|\overline{K}| = 2^2$ and, by the first inequality in (4.1), $|X_0| \leq 3|Z_\lambda| = \frac{3}{4}|N|$ (as $[N : Z_\lambda| = 4)$. So $\mathrm{P_v}(G) \geq 1 - \frac{3}{4[G:N]} > \mathfrak{a}$, a contradiction.

If $|G/N| = 5$, then every nontrivial irreducible G/N-module over the field of order two has order 2^4. So $|\overline{K}| = 2^4$ and $|\overline{N}/\overline{Z}| \neq 2^3$, as a trivial action of G/N on $\overline{N}/\overline{Z}$ would imply $\overline{Z_\lambda} \trianglelefteq \overline{G}$, thus $\overline{X_0} = \overline{Z_\lambda}$ and hence $\mathrm{P_v}(G) > \mathfrak{a}$. Then by Lemma 2.13 $|\overline{N}/\overline{Z}| = 2|\overline{K}|$ and there exists a characteristic abelian subgroup $\overline{A}$ of $\overline{N}$ such that $[\overline{N} : \overline{A}] = 2$. Since $\mathrm{P_v}(G) \leq \mathfrak{a}$, part (2) of Lemma 2.7 implies that all irreducible characters of $\overline{A}$ are $\overline{N}$-invariant and we get a contradiction by Lemma 2.14.

Therefore, we have proved that $|G/N| = 3$. Hence, $|\overline{K}| = 2^2$ and by Lemma 2.13 we see that $|\overline{N}/\overline{Z}| = 2^3$. Since G/N and $\overline{N}/\overline{Z}$ have coprime orders, $\overline{N}/\overline{Z}$ is a direct sum of indecomposable G/N-modules and, as nontrivial G/N-submodules of $\overline{N}/\overline{Z}$ are homocyclic of rank two, we deduce that there exists a normal subgroup A of G, with $D \leq A$, such that $\overline{Z} \leq \overline{A} \leq \overline{N}$ and $[\overline{N} : \overline{A}] = 2$. If $\overline{A}$ is nonabelian, then $\mathrm{cd}(\overline{A}) = \{1, 2\}$ and Lemma 2.13 applied to $\overline{A}$ yields $|\overline{A}'| = 2$, which is a contradiction because $\overline{A}' \trianglelefteq \overline{G}$ and $\overline{A}'$ is contained in the minimal normal subgroup $\overline{K}$ of $\overline{G}$. Hence $\overline{A}$ is abelian and by Lemma 2.14 there exists a character $\alpha \in \mathrm{Irr}(\overline{A})$ such that α is not $\overline{N}$-invariant. So, $I_{\overline{N}}(\alpha) = \overline{A}$ and by part (2) of Lemma 2.7 $\overline{N} - \overline{A} \subseteq \mathrm{V}(\overline{G})$. Hence, $\mathrm{N}(G) \subseteq A$, $A \trianglelefteq G$ and $[N : A] = 2$. As mentioned at the beginning of the proof, now it also follows that A is abelian. $\square$

The following result strengthens, under the special assumption $\mathrm{P_v}(G) < \mathfrak{a}$, the statement of Theorem D of [9].

Proposition 4.2. *Let G be a group such that $\mathrm{P_v}(G) < \mathfrak{a}$. Then $\mathrm{N}(G)$ is contained in an abelian normal subgroup of G.*

Proof. Clearly, we can assume $G \neq 1$. By Theorem 2.12, G is solvable and hence $F = \mathbf{F}(G) \neq 1$.

Write $\overline{G} = G/F$. Since $\mathrm{P_v}(\overline{G}) < \mathfrak{a}$, working by induction on $|G|$ we know that there exists a normal abelian subgroup $\overline{K}$ of $\overline{G}$ such that $\mathrm{N}(\overline{G}) \subseteq \overline{K}$. Since $\overline{\mathrm{N}(G)} \subseteq \mathrm{N}(\overline{G})$ by the usual lifting argument, then $\mathrm{N}(G) \subseteq K \trianglelefteq G$, where K is the preimage of $\overline{K}$ under the canonical homomorphism. Since $\mathbf{F}(K) = F$ and K/F is abelian, part (1) of Lemma 2.7 yields that $\mathrm{N}(G) \subseteq F$. We can hence assume that F is a proper subgroup of G, or we are done by Lemma 2.9. Now, if F is abelian there is nothing to prove, while if F is nonabelian we conclude by an application of Proposition 4.1. $\square$

In general, the set $\mathrm{N}(G)$ of the nonvanishing elements of G is not a subgroup of G (see, for instance, Lemma 3.3). However, as we are about to prove, this turns out to be the case if $\mathrm{P_v}(G) < \mathfrak{a}$.

Theorem 4.3. *Let G be a finite group such that $\mathrm{P_v}(G) < \mathfrak{a}$. Then $\mathrm{N}(G)$ is an abelian normal subgroup of G.*

Proof. By Proposition 4.2, $A = \langle \mathrm{N}(G) \rangle$ is an abelian normal subgroup of G. We will show that $\mathrm{N}(G)$ is a subgroup of G, so in fact $\mathrm{N}(G) = A$. In the following, we denote by A_p the Sylow p-subgroup of A, for a prime p.

Since $G - A \subseteq \mathrm{V}(G)$ and $\mathrm{P_v}(G) < \mathfrak{a} < \frac{6}{7}$, we have $|G/A| \leq 6$. If G is abelian, then there is nothing to prove as $\mathrm{N}(G) = G$. So we can assume G nonabelian and $[G : A] \geq 2$.

If $[G : A] \in \{2, 4\}$, then G has abelian Sylow 3-subgroups and by part (1) of Proposition 3.2 $\mathrm{N}(G) = \mathrm{N}(G) \cap A = (\mathrm{N}(G) \cap A_2) \times K$, where K is the normal 2-complement of G. Let $\overline{G} = G/K$, $\pi : G \to \overline{G}$ the natural projection and denote, as usual, by $\overline{X}$ the image $\pi(X)$ of any subset X of G. Since $\mathrm{N}(G) = (\mathrm{N}(G) \cap A_2) \times K$, by Lemma 2.5, $\overline{\mathrm{N}(G) \cap A_2} = \mathrm{N}(\overline{G}) \cap \overline{A}$ and hence $\mathrm{N}(G) = \pi^{-1}(\mathrm{N}(\overline{G}) \cap \overline{A})$. By Lemma 2.9 $\mathrm{N}(\overline{G}) = \mathbf{Z}(\overline{G})$, so we conclude that $\mathrm{N}(G)$ is a subgroup of G.

If $[G : A] = 3$, then we use the same argument of the previous paragraph, applied to the factor group over the normal 3-complement of G, and part (2) of Proposition 3.2.

Assume now that $[G : A] = 5$ and let R a Sylow 5-subgroup of G, L the normal 5-complement of G and $Z = \mathbf{Z}(G)$. Observe that $Z < A$, because G is nonabelian. We also remark that $R \cong G/L$ is abelian, since otherwise by Lemma 2.9 we have $\mathrm{P_v}(G/L) \geq 1 - \frac{1}{5^2}$, which is impossible as $\mathrm{P_v}(G/L) \leq \mathrm{P_v}(G) < \mathfrak{a}$. So, $R \cap A \leq Z$. By part (2) of Lemma 3.4, we have $L = [L, R] \times \mathbf{C}_L(R)$ and hence $A = [L, R] \times Z$. We claim that $[L, R] \subseteq \mathrm{N}(G)$. First, we argue that $|[L, R]|$ is coprime to either 2 or 3. In fact, since by part (1) of Lemma 3.4 (applied to the action of $R/R \cap A$ on L) G/Z is a Frobenius group with kernel $A/Z \cong [L, R]$, if 6 divides $|[L, R]|$, then G has a factor group satisfying the assumptions of Lemma 3.3, which is impossible because of the condition on the proportion of the vanishing elements. We also observe that $|[L, R]|$ is coprime to 5, since $R \cap A \leq Z$. Let now $a \in [L, R]$ and $\alpha \in \mathrm{Irr}(A)$. Then $\alpha^G(a) = \sum_{i=1}^{5} \alpha^{x_i}(a)$, where $\{x_1, \ldots, x_5\}$ is a transversal for A in G , and for every $i \in \{1, \ldots, 5\}$ $\alpha^{x_i}(a) \in \mathsf{U}_m$, where $m = |[L, R]|$. So, Lemma 2.1 implies that $\alpha^G(a) \neq 0$. By Lemma 2.6, we deduce that $[L, R] \subseteq \mathrm{N}(G)$ and by Lemma 2.4 we conclude that $A = \mathrm{N}(G)$.

Assume, finally, that $[G : A] = 6$. Let Q be a Sylow 2-subgroup of G and P a Sylow 3-subgroup of G. We argue that P is abelian. In fact, assuming that P is nonabelian, we consider the factor group $\overline{G} = G/M$, where M is the 3-complement of A, and we observe that $\overline{G} = \overline{P} \rtimes \overline{Q}$, where $\overline{P} \cong P$ and $|\overline{Q}| = 2$. If $[\overline{Q}, \overline{P}] = 1$, then G has a factor group isomorphic to P and by Lemma 2.9 we get the contradiction $\mathrm{P_v}(G) \geq \frac{8}{9}$. Thus, $\overline{P} = \mathbf{F}(\overline{G})$ and hence $\mathrm{N}(\overline{G}) \subseteq \overline{P}$ by Proposition 4.2. Since $\mathrm{P_v}(\overline{G}) < \mathfrak{a}$, we get a contradition by Proposition 4.1. Hence, P is abelian.

Denote by K the 2-complement of A and let $\overline{G} = G/K$. Since $\mathrm{N}(G) \subseteq A$, part (1) of Proposition 3.2 yields $\mathrm{N}(G) = \mathrm{N}(G) \cap A = (\mathrm{N}(G) \cap A_2) \times K$, and by Lemma 2.5 we deduce that $\overline{\mathrm{N}(G) \cap A} = \overline{\mathrm{N}(G) \cap A_2} = \mathrm{N}(\overline{G}) \cap \overline{A}$. So, $\mathrm{N}(G)$ is the preimage of $\mathrm{N}(\overline{G}) \cap \overline{A}$ via the canonical projection of G on $\overline{G}$, and we have to show that $\mathrm{N}(\overline{G}) \cap \overline{A}$ is a subgroup of $\overline{G}$. If the Sylow 2-subgroup $\overline{Q}$ of $\overline{G}$ is abelian, then $\mathrm{N}(\overline{G}) \cap \overline{A} = \overline{A}$ by Lemma 2.10. Hence, we can assume that $\overline{Q}$ is nonabelian. If $\overline{P}$ acts trivially on $\overline{A}$, then $\overline{P} \trianglelefteq \overline{G}$ and by Lemma 2.5 and Lemma 2.9 $(\mathrm{N}(\overline{G}) \cap \overline{A}) \times \overline{P}$ is the preimage, via the canonical projection of $\overline{G}$ onto $\overline{G}/\overline{P}$, of the group $\mathbf{Z}(\overline{G}/\overline{P}) \cap \overline{AP}/\overline{P}$, and again it follows that $\mathrm{N}(\overline{G}) \cap \overline{A}$ is a subgroup of $\overline{G}$. We can hence assume that $\overline{P}$ acts nontrivially on $\overline{A}$. Thus, by part (2) of Lemma 3.4 $\overline{A} = [\overline{A}, \overline{P}] \times \overline{Z}$, where $\overline{Z} = \mathbf{C}_{\overline{A}}(\overline{P}) \leq \mathbf{Z}(\overline{G})$ by part (4) of Lemma 3.5. By Lemma 2.4, $\mathrm{N}(\overline{G}) \cap \overline{A} = (\mathrm{N}(\overline{G}) \cap [\overline{A}, \overline{P}]) \times \overline{Z}$ and by Lemma 2.5 $(\mathrm{N}(\overline{G}) \cap [\overline{A}, \overline{P}])\overline{Z}/\overline{Z} = \mathrm{N}(\overline{G}/\overline{Z}) \cap \overline{A}/\overline{Z}$. Therefore, $\mathrm{N}(\overline{G}) \cap \overline{A}$ is the preimage of $\mathrm{N}(\overline{G}/\overline{Z}) \cap \overline{A}/\overline{Z}$ via the canonical projection of $\overline{G}$ on $\overline{G}/\overline{Z}$, and we are left to show that $\mathrm{N}(\overline{G}/\overline{Z}) \cap \overline{A}/\overline{Z}$ is a subgroup of $\overline{G}/\overline{Z}$. If $\overline{Q}/\overline{Z}$ is abelian, then $\mathrm{N}(\overline{G}/\overline{Z}) \cap \overline{A}/\overline{Z} = \overline{A}/\overline{Z}$ by Lemma 2.10. Hence, we can assume that $\overline{Q}/\overline{Z}$ is nonabelian. Since $\mathbf{C}_{\overline{A}/\overline{Z}}(\overline{PZ}/\overline{Z}) = 1$ by part (1) of Lemma 3.4 and $\mathrm{P_v}(\overline{G}/\overline{Z}) \leq \mathrm{P_v}(G) < \mathfrak{a}$, an application of Proposition 3.10 to the group $\overline{G}/\overline{Z}$ yields $\mathrm{N}(\overline{G}/\overline{Z}) = \overline{A}/\overline{Z}$, completing the proof. $\square$

Now Theorem A, which we state again, follows easily.

Theorem A. *Let G be a group such that $\mathrm{P_v}(G) < \mathfrak{a}$. Then $\mathrm{P_v}(G) = \frac{m-1}{m}$ for some integer $1 \leq m \leq 6$.*

Proof. By Theorem 4.3 we know that $A = \mathrm{N}(G)$ is a subgroup of G. Writing $m = [G : A]$, we have $\mathrm{P_v}(G) = 1 - \frac{1}{m} = \frac{m-1}{m}$ and, since $\mathrm{P_v}(G) < \mathfrak{a} < \frac{6}{7}$, clearly $1 \leq m \leq 6$. $\square$

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