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# Optimal control problems and their applications to insurance

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*To my family, especially Michele,*

*whose love has always supported me.*

# Abstract

In this thesis, we study a few optimization problems for an insurance company whose purpose is to maximize profits and/or minimize risks. Our results are collected in the next three chapters.

In the first chapter, we analyze the optimal investment and reinsurance problem of a company, endowed with forward dynamic utilities, in a stochastic factor model that allows for a double dependence between the financial and insurance markets. Precisely, we assume that the financial asset price and the insurance losses are both affected by a common stochastic factor which is described by a continuous time finite state Markov chain or a diffusion process. We construct a family of forward dynamic exponential utilities and we characterize the optimal portfolio strategy and the optimal proportional level of reinsurance. We perform some numerical experiments to further investigate our results. Moreover, we compare the forward approach with the classical one based on backward utilities, both analytically and numerically. We also discuss an extension of the conditional certainty equivalent.

In the second chapter, we study the dividend maximization problem and the ruin minimization problem, under the constraint that the terminal surplus of the insurance company follows a normal distribution with a given mean and a given variance, which may be set, e.g., to realize a Value at Risk or Expected Shortfall at some pre-specified confidence level. We suppose that the surplus is modeled by a Brownian motion with drift. When the company is allowed to distribuite dividends, we seek to maximize the expected discounted dividend payments or to minimize the ruin probability under the terminal distribution constraint. We find explicit expressions for the optimal strategies in both cases, when the dividend strategy is updated at discrete points in time and continuously in time. Instead, if the company buys reinsurance for part of its claim, we investigate the reinsurance retention level that minimizes the ruin probability and allows the net collective to achieve the target distribution. Due to

the fact that updating reinsurance contract is a complicated matter from the practical point of view, we study the case where the reinsurance retention level can be modified only once over a fixed time interval, typically of one or two years. In this setting, we find out that an admissible strategy is chosen at time zero, and we explicitly characterize the ruin minimizing strategy. We also discuss the implications of mantaining the initial retention level over the whole period and give the idea of how to deal with several strategy updates.

In the third chapter, we discuss the indifference pricing problem of a pure endowment (namely a contract that yields a fixed amount at maturity, provided the policyholder is alive at that time) for an insurance company, whose preferences are described by an exponential utility function. We propose a modeling framework where the mortality intensity of a reference population is stochastic and the risky asset price evolves according to a jump diffusion affected by regime changes. We determine the optimal investment strategies, with and without the insurance policy, and characterize the indifference price as a classical solution to a linear PDE with a suitable final condition and in terms of its probabilistic representation via an extension of the Feynman-Kac formula. Furthermore, we also investigate the indifference price for a portfolio of pure endowments and for a term life insurance. Finally, some numerical experiments are performed to address sensitivity analyses.

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# Introduction

Insurance is a preventive tool against risk, that is the possibility of the occurrence of future and uncertain events which, if they happen, could produce an unexpected damage, not only from a purely economic point of view. We can think of destabilizing events (such as road accidents, thefts or other malicious acts, environmental disasters) that harm people's assets and people themselves. Non-life insurance policies are used to cover potential losses resulting from the first occurrence, while life insurance treaties are used for the latter. Indeed, to hedge against the risk of a contingent or uncertain loss, people can buy an insurance contract, by which they receive financial protection or reimbursement from an insurance company. In case of non-life insurance, the objective of the company is to protect policyholders from the consequences that may arise from the occurrence of an accident by bearing a lower cost than they would otherwise have to face if they had to deal with it individually. Instead, a life insurance contract is a type of saving scheme: people usually purchase policies whose payoff depends on their remaining lifetime, investing their money in order to get a long-term profit. As a consequence, the insurance company is also subject to risks and it might happen that the available reserves are not sufficient to meet obligations. To handle risky situation to prevent ruin, a company may hedge its own risk in several ways, one of which is by taking reinsurance. Reinsurance brings various advantages: the company mitigates its risk exposure and can accept larger risks than its resources ordinarily would permit and this facilitates its growth and expansion. Another way to avoid bankruptcy is to invest capital, technical reserves and other available financial resources: trading financial assets (risky or not) coulde make a valuable contribution to operating results and enable the company to reduce premiums and increase dividends and bonuses, thereby improving its competitiveness.

Therefore, the wealth of the insurance company can be described by a dynamical system subject to random financial perturbations and other risky factors and which can be controlled in order to optimize some performance criterion, such as the maximization of the expteced utility of the terminal wealth, the maximization of the expected dividends value, the minimization of ruin probability and so on. In risk theory this leads to a dynamic stochastic optimization. In other words, the goal of the insurance company is to maximize the profit corresponding to its wealth (that is nothing but the reserve resulting from insurance premia received, reinsurance premia paid and claims covered, if financial investments are not involved). Specifically, the company aims to find the optimal strategy that increases the capital and/or minimizes the risk.

This thesis deals with the study of few optimization problems from the point of view of an insurance company, in different settings. Such problems are analyzed through the next three chapters.

In Chapter 1, we analyze the optimal investment and reinsurance problem of an insurance company, whose preferences are described via forward dynamic utilities of exponential type in a stochastic factor model which allows for a possible dependence between the financial and insurance markets. Endowing the company with forward utilities (as proposed by Musiela and Zariphopoulou [81]) permits modeling the variation of utility over time and with respect to some other stochastic factors that influence the market model. To the best of our knowledge, it is the first time that this type of insurance optimization problems are addressed via the forward approach. By stochastic control techniques, we construct a family of forward dynamic exponential utility and we characterize the optimal investment strategy and the optimal proportional level of reinsurance. We make a comparison with classical results on optimal investment and reinsurance problems in analogous settings under backward utility preferences (see e.g. Irgens and Paulsen [66], Liu and Ma [73]). Further, we discuss an extension of the conditional certainty equivalent, generalizing the classical notion of the certainty equivalent to the dynamic and stochastic environment involved, as in Frittelli and Maggis [53]. Finally, we perform a numerical analysis to highlight some features of the optimal strategy and the optimal value process under forward preferences. We highlight that in the forward approach, the company gives today's preferences and the utility is generated forward in time. Thereby, the main difference with the classical approach based on backward preferences is that it does not require to identify a trading horizon and set at the initial time a utility criterion to hold at some future date. As a consequence, forward utilities allocate the same value, in terms of utility of wealth, to the optimal investment over any investment horizon. This yields to a stochastic control problem where the value process is a semimartingale for any admissible strategy and a proper martingale for the optimal strategy. Further, since any market changes are absorbed by a utility function that updates forward in time according to the new conditions, in the forward case the optimal strategy consists of the myopic component only, whereas in the backward case there is an additional component which accounts for the part of risk linked to the stock price.

Chapter 2 investigates the problem of dividend maximization and the problem of ruin minimization for an insurance company whose purpose is to achieve a certain distribution of the surplus at a particular future date. Taking into account a target terminal distribution for the surplus of the company facilitates the calculation of risk measures as e.g. the Value at Risk (in short VaR) or Expected Shortfall (in short ES) and hence the Solvency Capital Requirement, which is among the most common. To the best of our knowledge, such a constraint on the terminal surplus distribution has not been considered in the literature before. In this framework we consider two problems. In the first one we allow the insurance company to pay dividends and seek the optimal strategy that maximizes the expected discounted dividend payments or minimizes the ruin probability, binding the terminal surplus to follow a normal distribution with a given mean and a given variance. We prove that the optimal strategy is completely decided at the beginning of the trading interval. Moreover we find that the ruin-minimizing strategy (which is also the strategy that leads to the minimal expected discounted dividend value) starts with small dividend rates and then increases when approaching the time horizon to achieve the target distribution. Second, we study optimal ruin minimizing reinsurance with a pre-determined final distribution of the wealth, assuming that ruin-checks are due at discrete deterministic points in time. Here, to mitigate the risk exposure, the insurance company buys a proportional reinsurance with an appropriate level of retention that can be updated at some apriori fixed dates. Under a terminal Gaussian surplus distribution, we show that the reinsurance strategy leading to the smallest ruin probability in a 2-period model, is deterministic, namely it is uniquely choosen initially. Moreover, the insurer acts in order to reduce the risk of ruin shortly before regulator's time check. To prove our results, we use purely probabilistic methods. Indeed, the discrete nature of the problem does not allow us to use the differential equation approach. We point out that the dividend related problems can be easily generalised to a continuous time framework whilst in the reinsurance setting we are able to fully analyze only the 2-period case. This is due to the fact that the retention level acts on both the drift and the volatility of the surplus process and, as a consequence, we can not compare different strategies 'path by path' as in the dividend setting.

In Chapter 3, we study the indifference pricing problem of pure endowment policies in a stochastic-factor model for an insurance company, which can also invest in a financial market. Precisely, we consider a market model where the hazard rate is stochastic (it is described by a general diffusion process) and the risky asset price is affected by long-term macroeconomic conditions, i.e. it evolves over time as a jump diffusion affected by a continuous time finite state Markov chain representing regimes of the economy. In this context, we evaluate a pure endowment, namely a life insurance contract which pays a fixed amount to the policyholder at maturity if and only if she/he is still alive at that date, through the principle of equivalent utility by comparing the maximal expected utility functions with and without issuing the contract. The indifference pricing method, initially proposed by Hodges and Neuberger [63], has been used extensively in finance, see Henderson and Hobson [59] for a survey, and then also in insurance, see e.g. Møller [79]. It is worth emphasizing that this is the first time that the utility indifference pricing method is used to price life insurance policies in such Markov-modulated financial-insurance market. Using the classical stochastic control approach based on the Hamilton-Jacobi-Bellman (in short HJB) equation, we solve two optimal investment problems. In particular, we provide the optimal investment strategies, with and without insurance liabilities, and we characterize the indifference price of a pure endowment via a classical solution to a linear partial differential equation (in short PDE), as the solution of a final value problem and in terms of its probabilistic representation by means of an extension of the Feynman-Kac formula. We also discuss the indifference price for a portfolio of pure endowment policies and for a term life insurance treaty. We conclude performing some numerical experiments in order to address sensitivity analyses.

**Chapter 1**

# Optimal investment and reinsurance under forward preferences

The aim of this chapter is to collect and discuss extensively the results of [32] and [33]. Precisely, we investigate an optimal investment and reinsurance problem for an insurance company whose preferences are described by forward dynamic utilities in a stochastic factor model allowing for a possible dependence between the financial and insurance markets.

Firstly, we provide some introductory considerations in order to motivate our choice to address a classical optimal portfolio problem using forward utilities, emphasizing the many advantages of this dynamic approach that allows agents to adjust their random preferences over time. Specifically, in Section 1.1, we place our study in the existing literature and point out the differences with the standard backward approach.

In Section 1.2, we describe the general setting of our model into details, explaining the mathematichal framework for the insurance and the financial markets. Our model is set to encompass a desirable characteristic of hybrid markets that is mutual dependence between the insurance/reinsurance business and the financial securities.

We consider forward preferences to describe the dynamic behavior of an insurance company whose purpose is to maximize the expected utility of its terminal wealth. In Section 1.3, we introduce the formal definition of forward utilities as solutions of dynamic optimization problems, and we formulate explicitly the problem faced by the insurance company.

Then, we construct a family of forward dynamic exponential utilities and characterize the optimal investment and proportional reinsurance strategy, by considering two different settings:

- in Section 1.4 we focus on zero-volatility forward preferences in an interdependent insurance insurance-financial market model affected by a stochastic factor described by a Markov chain;
- Section 1.5 is a follow-up: the problem is investigated under non-zero volatility forward utilities in a more general framework allowing for a double dependence between the insurance and financial markets.

In both cases, we manage to characterize a familiy of forward exponential utilites, penalizing the classical utility with a process that accounts for financial and actuarial frameworks, namely a process that depends on asset price, insurance and reinsurance premia. Moreover, we find the corresponding optimal investment and proportional reinsurance strategy for our class of forward preferences. We also perform some numerical experiments to further investigate our results, in particular we discuss the qualitative characteristics of the optimal investment portfolio and the optimal protection level implied by our model. We compare the forward performance approach with the standard backward one, pointing out similarities and differences, analitically and numerically. Finally, we analyze a dynamic version of the conditional certainty equivalent for forward preferences and then we make a comparison with backward utilities also in terms of conditional certainty equivalent. An Appendix, at the end of the chapter, collects some technical assumptions and proofs.

## 1.1 Motivation and literature review

Non-life insurance policies are intended to protect policyholder against unforeseeable events as, for example fire, water damage, earthquake, industrial catastrophes or car accidents, that may cause financial losses. For such protection an insured pays premiums, which consist of (periodic) payments and constitute the money income of the insurance company. The insurer may invest money to build up an asset position that allows to cover the policy risk. Insurance risk can be mitigated by buying a reinsurance. This is an agreement between a primary insurer and a secondary insurer: a primary insurer, for a definite premium, contracts with another insurer (or insurers) to carry a part or the whole of a risk assumed by the primary insurer. The most common reinsurance contracts can be divided in two types, called proportional and non-proportional reinsurance policies. Under proportional reinsurance, the reinsurer receives a premium and bears a portion of the losses based on a pre-negotiated percentage. Instead, with non-proportional reinsurance, the reinsurer is liable only if the insurer's losses exceed a specified amount, known as the priority or retention limit.

In this chapter we consider an insurance company that buys a proportional reinsurance policy and invests part of the capital and of the premia in a financial market. We study the optimal allocation of the wealth into reinsurance and investment to maximize the expected *forward dynamic utilities*. Indeed, to better describe the behavior of the insurance company, we consider dynamic preferences. Intuitively, a forward dynamic utility represents individual preferences of an agent, possibly changing over time, according to the available information. One of its advantages is that it allows for a significant flexibility in incorporating changing market opportunities and agents' attitudes in a dynamically consistent manner. This means to define the forward performance process as an adapted stochastic process parameterized by wealth, time and also by some stochastic factor (more or less correlated with the model), and constructed "forward in time", starting from an initial date. In this way agents specify their preferences when entering the market, without defining their risk profile at a future horizon time, unlike the classical backward approach. This idea to update preferences over time is not entirely new. The earliest attempt to model the variation of utility with respect to time dates back to the ninenties with the so called recursive utilities, see Epstein and Zin [50] and Duffie and Epstein [47]. Afterwards, Musiela & Zariphopoulou introduce the notion of forward dynamic preferences where utilities are determined today and they are generated for future times, via a self-generating criterion. In [81, 82, 83], they study optimal investment decision problems where agents track their risk preferences over time, for advancing the timing of future satisfaction or impatience. In other words, agents may dynamically adjust their preferences consistently with the information revealed over time and their impatience might be compensated for by the opportunities given to them, if they can be exploited in full according to their choices. This approach overcomes a few limitations of more traditional backward preferences. Classical literature on portfolio optimization under backward utilities is based on the assumption that a utility is exogenously chosen to hold at a future date (no earlier than the end of the investment horizon) and employed to make investment decisions for today; this means that, when entering the market, agents prescribe their risk profile at the horizon time and therefore cannot adapt it to changes in market conditions or update risk preferences. In addition, the investment horizon is fixed, and the portfolio is derived with respect to this reference date. For optimal reinsurance and investment problems employing classical backward preferences, we refer to Irgens and Paulsen [66], Liu and Ma [73], Gu et al. [56], Brachetta and Ceci [17], Brachetta and Schmidli [18], Cao et al. [22], Ceci et al. [28]. In the forward approach, instead of pre-specifying the utility function to be valid at some future time and identifying a trading horizon, the agent gives today's preferences and the utility is generated forward in time, that is, it naturally moves in the same direction of the market. The agent chooses the optimal strategy to maximize the expected forward utility of her wealth, *at any future time* $t \geq 0$. The main consequence of this approach is that a forward utility allocates the same value, in terms of utility of wealth, to the optimal investment over any investment horizon. This yields to a stochastic control problem where the solution can be obtained through the *Bellman optimality principle*: the value process is determined so that it enjoys the semimartingale property for any admissible strategy and it is a martingale for the optimal strategy. This reflects the natural idea that any sub-optimal strategy is under-performing, and that the expected performance of an optimal strategy at any future time is as good as today. In particular, the martingale property allows us to derive a PDE that characterizes the value function and the optimal investment and reinsurance strategy.

## 1.2 General setting

We fix a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ endowed with a filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \geq 0\}$, satisfying the usual conditions of completeness and right continuity. All processes introduced below are assumed to be $\mathbb{F}$-adapted.

Let $Y$ be an index that accounts for an environmental, social or even cultural factors and which may affect both the insurance and the financial markets. We assume that $Y$ is modeled by a Markov process and that it affects both the losses faced by the insurance company and the prices of the assets negotiated in the financial market, introducing a certain mutual dependence between the actuarial and the financial markets. Such type of dependence is empirically observed: see e.g. Tesselaar et al. [94], Baek et al. [7], Just and Echaust [69], Wang et al. [99], where the economic effects of climate changes and COVID-19 pandemic on both the insurance/reinsurance business and the financial market are analyzed.

Let $C = \{C_t,\ t \geq 0\}$ be the cumulative claim process that represents the total losses of the insurance company due to claims given by

$$C_t = \sum_{n=1}^{N_t} Z_n, \quad t \geq 0,$$

where $\{Z_n\}_{n\in\mathbb{N}}$ is a sequence of independent $\mathcal{F}_{T_N}$-random variables that indicate the claim amounts and where $T_n$ is the random time at which claims occur. Here, $N = \{N_t,\ t \geq 0\}$ is a doubly stochastic Poisson process that counts the number of claims and we assume that the stochastic intensity $\{\lambda(t, Y_{t-}),\ t \geq 0\}$ depends on the common index $Y$.

In the next paragraph, we show in detail a standard construction of the claim amount process.

**Mathematical construction of insurance losses.** In order to describe the losses of the insurance company, we retrace the classic construction of the claim amount process, see for instance [55, Chapter 2].

Define the process $\Lambda = \{\Lambda_t,\ t \geq 0\}$ as

$$\Lambda_t = \int_0^t \lambda(s, Y_{s-}), \mathrm{d}s, \quad t \geq 0, \tag{1.1}$$

where the function $\lambda : [0, +\infty) \times \mathbb{R} \to (0, +\infty)$ is measurable and satisfies

$$\mathbb{E}\left[\int_0^t \lambda(s, Y_{s-})\mathrm{d}s\right] < \infty, \tag{1.2}$$

for every $t \geq 0$. Therefore, the process $\Lambda$ is non-decreasing and it satisfies $\Lambda_0 = 0$ and $\Lambda_t < \infty$ $\mathbf{P}$-a.s. for every $t \geq 0$. Let $\eta = \{\eta_t,\ t \geq 0\}$ be a standard Poisson process (i.e. with intensity equal to 1) and let us consider the process $N = \{N_t,\ t \geq 0\}$ defined as $N_t = \eta_{\Lambda_t}$, for every $t \geq 0$. Then, $N$ is a doubly stochastic Poisson process (see Lemma 4 and the discussion on stochastic random measures in [55, Section 2.1]). The process $\{\lambda(t, Y_t),\ t \geq 0\}$ is called the *intensity* of $N$ and condition (1.2) implies that $N$ is non-explosive. Moreover, the compensated process $\tilde{N} = \{\tilde{N}_t,\ t \geq 0\}$, given by

$$\tilde{N}_t = N_t - \Lambda_t, \quad t \geq 0,$$

is an $(\mathbb{F}, \mathbf{P})$ martingale (see [19, Chapter II]). The jump times of the process $N$ describe the claim arrival times and are represented through the increasing sequence of (nonnegative) random variables $\{T_n\}_{n\in\mathbb{N}}$. Let $I \subset [0, +\infty)$ an arbitrary interval and let $\{Z_n\}_{n\in\mathbb{N}}$ be a sequence of independent and $I$-valued random variables independent of $N$ such that for each $n \in \mathbb{N}$, $Z_n$ is $\mathcal{F}_{T_n}$-measurable and $\mathbb{E}[e^{\xi Z_n}] < +\infty$, for every $n$ and for $\xi > 0$. For every $n \in \mathbb{N}$, $Z_n$ indicates the claim amount at time $T_n$. The distribution of claim amounts is described by the map $F : [0, +\infty) \times \mathbb{R} \times I \longrightarrow [0, 1]$ which is such that for each $(t, y) \in [0, +\infty) \times \mathbb{R}$, $F(\cdot, \cdot, z)$ is a distribution function, with $F(t, y, 0) = 0$.

Thus, for every $t \geq 0$, the cumulative claim process at time $t$ is given by

$$C_t = \sum_{n=1}^{N_t} Z_n.$$

In the sequel, it will be useful to describe $C$ in terms of its associated random measure defined as

$$m(\mathrm{d}t, \mathrm{d}z) = \sum_{n\in\mathbb{N}} \delta_{(T_n, Z_n)}(\mathrm{d}t, \mathrm{d}z),$$

where $\delta_{(t,z)}$ is the Dirac measure at point $(t, z) \in [0, +\infty) \times [0, +\infty)$; hence, the claim process $C$ reads as

$$C_t = \int_0^t \int_I z m(\mathrm{d}s, \mathrm{d}z), \quad t \geq 0.$$

We recall a set of properties of the random counting measure $m(\mathrm{d}t, \mathrm{d}z)$. For every $A \subset I$, the process $\{m((0, t] \times A)\}$ is a counting process that gives the number of claims with claim size in the set $A$. In particular, $m((0, t] \times I) = N_t$ is the total number of claims up to time $t$.

The dual predictable projection $\nu$ of the random measure $m(\mathrm{d}t, \mathrm{d}z)$ is given by

$$\nu(\mathrm{d}t, \mathrm{d}z) = F(t, Y_{t-}, \mathrm{d}z)\lambda(t, Y_{t-})\mathrm{d}t.$$

Moreover, for every non-negative, predictable random field $\Gamma = \{\Gamma(t, z),\ t \geq 0,\ z \in I\}$, such that

$$\mathbb{E}\left[\int_0^t \int_I \Gamma(s, z)\lambda(s, Y_{s-})F(s, Y_{s-}, \mathrm{d}z)\mathrm{d}s\right] < \infty,$$

for every $t \geq 0$, the process

$$\left\{\int_0^t \int_I \Gamma(s, z)\Big(m(\mathrm{d}s, \mathrm{d}z) - F(s, Y_{s-}, \mathrm{d}z)\lambda(s, Y_{s-})\mathrm{d}s\Big), \quad t \geq 0\right\}$$

is a martingale, see e.g. [19, Chapter $VIII$, Theorem T3] for further details. Consequently, it holds that

$$\mathbb{E}\left[\int_0^t \int_I \Gamma(s,z) m(\mathrm{d}s, \mathrm{d}z)\right] = \mathbb{E}\left[\int_0^t \int_I \Gamma(s,z) F(s, Y_{s-}, \mathrm{d}z) \lambda(s, Y_{s-}) \mathrm{d}s\right],$$

for every $t \geq 0$.

From now on, we consider the following set of assumptions.

**Assumption 1.1.** *It holds that*

$$\mathbb{E}\left[\int_0^t \int_I z \lambda(s, Y_{s-}) F(s, Y_{s-}, \mathrm{d}z) \mathrm{d}s\right] < \infty, \qquad \mathbb{E}\left[\int_0^t \int_I e^{\xi z} \lambda(s, Y_{s-}) F(s, Y_{s-}, \mathrm{d}z) \mathrm{d}s\right] < \infty,$$
$$\mathbb{E}\left[\int_0^t \int_I z e^{\xi z} \lambda(s, Y_{s-}) F(s, Y_{s-}, \mathrm{d}z) \mathrm{d}s\right] < \infty, \quad \mathbb{E}\left[\int_0^t \int_I z^2 e^{\xi z} \lambda(s, Y_{s-}) F(s, Y_{s-}, \mathrm{d}z) \mathrm{d}s\right] < \infty,$$

*for every* $t \geq 0$ *and* $\xi > 0$.

Such integrability conditions will be used in some technical steps of the solution of the optimization problem. One of the consequences is that the cumulative claim process is non-explosive:

$$\mathbb{E}\left[C_t\right] = \mathbb{E}\left[\int_0^t \int_I z m(\mathrm{d}s, \mathrm{d}z)\right] = \mathbb{E}\left[\int_0^t \int_I z \lambda(s, Y_{s-}) F(s, Y_{s-}, \mathrm{d}z) \mathrm{d}s\right] < \infty,$$

for every $t \geq 0$.

The insurance company receives premia and, in order to mitigate the risk exposure, reinsures part of its claims by continuously purchasing a proportional reinsurance contract. We assume that both insurance and reinsurance premia depend on the index $Y$, and hence they are, potentially, stochastic. This is inline with the recent literature, see, e.g. Delong and Gerrard [45], Cao et al. [22], Ceci et al. [28]. In fact, the reinsurance premium is often a random variable since it is a function of loss amounts and/or the reinsurance treaty is often assorted with clauses (such as paid reinstatements, sliding scale premium or profit commission): it might be an idea to handle with a premium which consists of an initial fixed amount plus a stochastic part, as described in Walhin [96], Albrecher and Haas [1], Campana and Ferretti [21]. It has also been observed in empirical studies that having a dynamic premium would strongly reduce the risk of insolvency; for example in Assa and Boonen [5] it is underlined the economic impact of COVID-19 in the UK and related policy implications.

Classical premium calculation principles can be extended to accommodate this assumption by *conditioning* on the value of $Y_t$. As remarked, for instance, in Delong and Gerrard [45], the conditional version of the expected value principle keeps the property that premium rate is proportional to the claim arrival intensity. This is also the case for the conditional variance principle. Another possibility is to exploit the so called intensity-adjusted variance premium principle, introduced by Brachetta and Ceci [17], that leads to a reinsurance strategy which explicitly depends on the claim intensity. It is worth noting that in real life, continuously updating reinsurance over time is complicated: unlike the financial products, sometime legal cost may involved, reinsurance policies can hardly be changed. Thus, our dynamic reinsurance seems to clash with reality. However, under some classical premium calculation rules, we obtain deterministic premia (see below), in line with the common practice. In addition, it is also worth mentioning that in the case of a reinsurance premium of deterministic type, the reinsurance contract may be too expensive for the insurance company that then might even decide not to buy it. A similar reasoning applies to the insurance premium. Hence, in order to prevent the agreements between insurance and reinsurance companies from vanishing, we consider stochastic premia. Precisely, we consider an insurance gross premium process of the form $\{a(t, Y_t),\ t \geq 0\}$, and a reinsurance contract of proportional type, with premium rate process $\{b(t, Y_t, \Theta_t),\ t \geq 0\}$, where $\Theta = \{\Theta_t,\ t \geq 0\}$, represents the protection level, for some functions $a : [0, +\infty) \times \mathbb{R} \to [0, +\infty)$ and $b : [0, +\infty) \times \mathbb{R} \times [0, 1] \to [0, +\infty)$. In particular, at any time $t \geq 0$, $\Theta_t$ represents the percentage of losses which are covered by the reinsurance. We assume that functions $a(t, y)$ and $b(t, y, \Theta)$ are jointly continuous with respect to the pair $(t, y)$ and the triplet $(t, y, \Theta)$, respectively. Throughout the section, we will also assume the following integrability conditions

$$\mathbb{E}\left[\int_0^t a(s, Y_s)\mathrm{d}s\right] < \infty, \qquad \mathbb{E}\left[\int_0^t b(s, Y_s, 1)\mathrm{d}s\right] < \infty, \tag{1.3}$$

for every $t \geq 0$.

**Remark 1.1.** *Conditional versions of some classical premium calculation principle read as follows. Under the expected value principle, for every $t \geq 0$ we get that*

$$a(t, Y_t) = (1 + \delta^I)\lambda(t, Y_t) \int_I zF(t, Y_t, \mathrm{d}z), \quad b(t, Y_t, \Theta_t) = (1 + \delta^R)\Theta_t\lambda(t, Y_t) \int_I zF(t, Y_t, \mathrm{d}z),$$

*where* $\delta^I > 0$, $\delta^R > 0$ *represent the insurance and reinsurance safety loading respectively, and for the variance principle it holds that*

$$a(t, Y_t) = \lambda(t, Y_t) \left( \int_I zF(t, Y_t, \mathrm{d}z) + \delta^I \int_I z^2 F(t, Y_t, \mathrm{d}z) \right),$$
$$b(t, Y_t, \Theta_t) = \Theta_t \lambda(t, Y_t) \left( \int_I zF(t, Y_t, \mathrm{d}z) + \Theta_t \delta^R \int_I z^2 F(t, Y_t, \mathrm{d}z) \right),$$

*for every* $t \geq 0$. *We point out that both the expected value principle and the variance premium principle lead to deterministic optimal reinsurance strategies, as in the classical case. Indeed, companies often do not change reinsurance policies in practice because they do not generally incur extra costs, unlike what happens when they deal with financial products (legal cost are always involved). However, it may make sense to consider that a pandemic, a political crisis or other destabilizing events could affect also a reinsurance contract. One of the characteristics of these simple premium calculation rules is that the optimal reinsurance strategy does not explicitly depend neither the stochastic factor nor the claim arrival intensity, even in the conditional case. This does not happen for more sophisticated premium evaluation principles such as the modified variance principle or the intensity-adjusted risk principle. Thus, to underline the effect that a factor model may have on the reisurance strategy, following Schmidli [89], in some of our numerical experiments we employ the modified variance principle, under which premia are given by*

$$a(t, Y_t) = \lambda(t, Y_t) \int_I zF(t, Y_t, \mathrm{d}z) + \delta^I \frac{\int_I z^2 F(t, Y_t, \mathrm{d}z)}{\int_I zF(t, Y_t, \mathrm{d}z)},$$
$$b(t, Y_t, \Theta_t) = \Theta_t \lambda(t, Y_t) \int_I zF(t, Y_t, \mathrm{d}z) + \delta^R \Theta_t \frac{\int_I z^2 F(t, Y_t, \mathrm{d}z)}{\int_I zF(t, Y_t, \mathrm{d}z)},$$

*for every* $t \geq 0$. *In some other experiments, instead, we consider insurance and reinsurance premia calculated under the intensity-adjusted variance principle, that is*

$$a(t, Y_t) = \lambda(t, Y_t) \int_I zF(t, Y_t, \mathrm{d}z) + \delta^I \lambda(t, Y_t)(1 + T\lambda(t, Y_t)) \int_I z^2 F(t, Y_t, \mathrm{d}z),$$
$$b(t, Y_t, \Theta_t) = \Theta_t \lambda(t, Y_t) \int_I zF(t, Y_t, \mathrm{d}z) + \delta^R \Theta_t^2 \lambda(t, Y_t)(1 + T\lambda(t, Y_t)) \int_I z^2 F(t, Y_t, \mathrm{d}z),$$

*for all* $T \geq t \geq 0$, *with* $T$ *denoting the maturity of the reinsurance contract. In both cases we obtain dynamic optimal reinsurance strategies that turn to be adapted to information via the dependence on the process* $Y$.

We make the following set of assumptions that extend the usual natural hypotheses on premia to the stochastic case.

**Assumption 1.2.** *The function* $b(t, y, \Theta)$ *has continuous partial derivatives* $\frac{\partial b(t,y,\Theta)}{\partial \Theta}$, $\frac{\partial^2 b(t,y,\Theta)}{\partial \Theta^2}$ *in* $\Theta \in [0, 1]$ *and it is such that*

(*i*) $b(t, y, 0) = 0$, *for all* $(t, y) \in [0, +\infty) \times \mathbb{R}$, *since the cedent does not need to pay for a null protection;*

(*ii*) $\frac{\partial b(t,y,\Theta)}{\partial \Theta} \geq 0$, *for all* $(t, y, \Theta) \in [0, +\infty) \times \mathbb{R} \times [0, 1]$, *because the premium is increasing with respect to the protection level;*

(*iii*) $b(t, y, 1) > a(t, y)$, *for all* $(t, y) \in [0, +\infty) \times \mathbb{R}$, *for preventing a profit without risk;*

*In the sequel,* $\frac{\partial b(t,y,0)}{\partial \Theta}$ *and* $\frac{\partial b(t,y,1)}{\partial \Theta}$ *are understood as right and left derivatives, respectively.*

The insurance company surplus (or reserve) process $R^\Theta = \{R^\Theta_t,\ t \geq 0\}$ satisfies the stochastic differential equation (in short SDE)

$$\mathrm{d}R^\Theta_t = a(t, Y_t)\mathrm{d}t - b(t, Y_t, \Theta_t)\mathrm{d}t - (1 - \Theta_{t-})\mathrm{d}C_t, \quad R^\Theta_0 = R_0 > 0. \tag{1.4}$$

Conditions (1.3) imply in particular that the surplus process $R^\Theta$ is well defined and $\mathbb{E}\left[R^\Theta_t\right] < \infty$, for all $t \geq 0$.

Beyond that, the insurance company is allowed to invest part of its premia in a financial market where investment possibilities are given by a riskless asset with value process $S^0 = \{S^0_t,\ t \geq 0\}$ and a stock with price process $S = \{S_t,\ t \geq 0\}$. We assume zero interest rate, that is, $S^0_t = 1$ for every $t \geq 0$, and that $S$ satisfies the SDE

$$\mathrm{d}S_t = \mu(t, Y_t, S_t)\mathrm{d}t + \sigma(t, Y_t, S_t)\mathrm{d}W^S_t, \quad S_0 = s > 0, \tag{1.5}$$

where $W^S = \{W^S_t,\ t \geq 0\}$ is a standard Brownian motion independent of the random measure $m(\mathrm{d}t, \mathrm{d}z)$. The functions $\mu : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ and $\sigma : [0, +\infty) \times \mathbb{R}^2 \to (0, +\infty)$, representing the appreciation rate and the volatility of the stock, respectively, are assumed to be measurable and such that the pair $(Y, S)$ is a Markov process. We also assume the Novikov condition

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^t \left(\frac{\mu(r,Y_r,S_r)}{\sigma(r,Y_r,S_r)}\right)^2 \mathrm{d}r}\right] < \infty, \tag{1.6}$$

for every $t \geq 0$, which implies the existence of a risk-neutral measure for $S$ and ensures that the financial market does not admit arbitrage opportunities.

We consider the problem of an insurance company with an initial wealth $x_0$, which invests its surplus in the financial market and buys a proportional reinsurance. For every $t \geq 0$, we denote by $\Pi_t$ the total amount of wealth invested in the risky asset at time $t$, and hence $X_t - \Pi_t$ is the capital invested in the riskless asset at time $t$. We assume that short-selling and borrowing from the bank account are allowed and accordingly we take $\Pi_t \in \mathbb{R}$ for every $t \geq 0$. Moreover, for every $t \geq 0$, let $\Theta_t \in [0,1]$ be the dynamic protection level at time $t$ corresponding to the reinsurance contract. We consider only self-financing strategies. Then, the wealth of the insurance company associated with the investment-reinsurance strategy $H = (\Pi, \Theta) = \{(\Pi_t, \Theta_t),\ t \geq 0\}$ satisfies the SDE

$$\mathrm{d}X_t^H = \mathrm{d}R_t^\Theta + \Pi_t \frac{\mathrm{d}S_t}{S_t} + (X_t^H - \Pi_t)\frac{\mathrm{d}S_t^0}{S_t^0} \tag{1.7}$$

with $X_0^H = x_0 \geq 0$.

**Remark 1.2.** *We observe that in our market model, the wealth process could attains negative values. Indeed, Schmidli [89] observes that "... The event of ruin almost never occurs in practice. If an insurance company observes that their surplus is decreasing they will immediately increase their premia. On the other hand an insurance company is built up on different portfolios. Ruin in one portfolio does not mean bankruptcy." In fact, in real life, a company may easily have access to large amount of liquidity, for example by borrowing from the bank. Moreover, the company usually updates its choices over time, increasing the price of the policies or looking for other financial investments in order to recover a possible loss trend. This means that if the wealth becomes negative, there are ways to make it positive again. In other words, technical ruin does not mean that the insurance company stops operating in the market. Thus, based on these considerations, we can allow the wealth process to be negative. Notice that, from the mathematical point of view, dealing with a negative wealth is not a problem due to the fact that we deal with forward utilities of exponential type.*

## 1.3 Forward utilities

We aim to study an optimal investment and reinsurance decision problem for the insurance company, following a forward approach. Therefore, we assume that the preferences of the insurance company are described by a dynamic utility function, which depends on wealth certainly but also on time and possibly on some other additional stochastic drivers. The insurance company starts with today' specification of its initial utility, without pre-committing an investment horizon and a terminal utility function at the beginning and then moves forward in time, modifying its preferences in relation to the available information, via a self-generating criterion, according to the following definition (see also Definition 2.1 in [82]).

**Definition 1.1.** *Fix a normalization point $t_0 \geq 0$. An adapted process $U = \{U_t(x, t_0),\ t \geq t_0\}$ is a dynamic performance process (normalized at $t_0$) if*

*(a) the function $x \to U_t(x, t_0)$ is increasing and concave for all $t \geq t_0$;*

*(b) for every self-financing strategy $H$, and for all $t, T$ such that $t_0 \leq t \leq T$ it holds that*

$$U_t(X_t^H, t_0) \geq \mathbb{E}\left[U_T(X_T^H, t_0)|\mathcal{F}_t\right];$$

*(c) there exists a self-financing strategy $H^*$ such that, for all $t, T$ such that $t_0 \leq t \leq T$, it holds that*

$$U_t(X_t^{H^*}, t_0) = \mathbb{E}\left[U_T(X_T^{H^*}, t_0)|\mathcal{F}_t\right];$$

*(d) at $t = t_0$,*

$$U_{t_0}(x, t_0) = u_0(x),$$

*where $u_0(x)$ is a concave and increasing function of wealth.*

From now on the time point $t_0$ is our starting point ($t_0$ is usally called *normalization point*) and all processes and filtrations will be considered for $t \geq t_0$. We work under exponential preferences, that is we choose the initial utility function of exponential type, i.e. $u_0(x) = -e^{-\gamma x}$, with the risk aversion coefficient $\gamma > 0$. Then, in this case Definition 1.1 describes a forward dynamic exponential utility and can be re-formulated as follows.

**Definition 1.2.** *Let* $t_0 \geq 0$. *An* $\mathbb{F}$*-adapted stochastic process* $U = \{U_t(x,t_0):\ t \geq t_0\}$ *is a* forward dynamic exponential utility (in short FDU)*, normalized at* $t_0$*, if for all* $t, T$ *such that* $t_0 \leq t \leq T$*, it satisfies the stochastic optimization criterion*

$$U_t(x,t_0) = \begin{cases} -e^{-\gamma x}, & t = t_0, \\ \max_{H\in\mathcal{A}} \mathbb{E}\left[U_T(X_T^H,t_0)|\mathcal{F}_t\right], & t_0 < t \leq T, \end{cases}$$

*with* $X^H$ *given by* (1.7)*,* $X_{t_0}^H = x \in [0,+\infty)$ *and* $\gamma > 0$*, for a suitable class* $\mathcal{A}$ *of admissible strategies which is characterized later.*

This definition reflects the fact that the insurance company tracks its risk preferences over time and its optimal strategy is associated with the martingale property along the optimal wealth trajectory. Indeed, the rationale behind this definition is that at a certain time $t_0$ (for instance, $t_0 = 0$), the insurance company specifies its utility which is based on the available information. As time goes by, market conditions may change and hence the insurance company might be willing to modify its preferences accordingly.

By construction, we get that forward dynamic exponential utilities have two important features: (i) there is no constraint on the length of the trading horizon and so there is no need to specify a priori a utility to be valid at the maturity, i.e. the investor does not fix today investment preferences that will hold at a future date; (ii) a forward dynamic exponential utility coincides with the dynamic value function of the optimization problem it generates, at all intermediate times. An important characteristic of the forward approach is that a forward dynamic utility might not be unique, as argued, for instance in Musiela and Zariphopoulou [82].

To represent a forward exponential utility, we penalize the classical exponential utility with a stochastic process that describes the insurance company dynamic preferences; this penalizing process depends on market coefficients, collected premia and paid premia but it may also be linked to other sources of risk which affect the combined financial-insurance market. Precisely, we define a penalizing process $P = \{P_t,\ t \geq t_0\}$ as

$$P_t = \int_{t_0}^t g(s, X_s^H, S_s, Y_s)\mathrm{d}s + \int_{t_0}^t h(s, X_s^H, S_s, Y_s)\mathrm{d}W_s^P, \quad t \geq 0, \tag{1.8}$$

where $g : [t_0,+\infty) \times \mathbb{R} \times (0,+\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ and $h : [t_0,+\infty) \times \mathbb{R} \times (0,+\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ are two measurable functions such that

$$\mathbb{E}\left[\int_{t_0}^{t} |g(s, X_s^H, S_s, Y_s)|\mathrm{d}s + \int_{t_0}^{t} h^2(s, X_s^H, S_s, Y_s)\mathrm{d}s\right] < \infty, \tag{1.9}$$

for every $t \geq t_0$ and every $H = (\Theta, \Pi) \in [0, 1] \times \mathbb{R}$. Here, $W^P = \{W_t^P,\ t \geq t_0\}$ is a standard Brownian motion which is $\rho^S$-correlated with $W^S$ and possibly depends on $Y$.

Now, we define the set of admissible reinsurance-investment strategies.

**Definition 1.3.** *An* admissible strategy *is a pair of predictable processes* $H = (\Theta, \Pi) = \{(\Theta_t, \Pi_t),\ t \geq t_0\}$*, representing the proportion of reinsured claims and the total amount invested in the risky asset, respectively, such that* $\Theta = \{\Theta_t,\ t \geq t_0\}$ *takes values in* $[0, 1]$ *and* $\Pi = \{\Pi_t,\ t \geq t_0\}$ *is* $\mathbb{R}$*-valued and such that*

$$\mathbb{E}\left[\int_{t_0}^{t} \left(|\Pi_s||\mu(s, Y_s, S_s)| + \Pi_s^2\sigma^2(s, Y_s, S_s)\right) \mathrm{d}s\right] < \infty, \tag{1.10}$$

*and* $\mathbb{E}\left[e^{-\gamma X_t^H - P_t}\right] < \infty$*, for every* $t \geq t_0$*. We denote by* $\mathcal{A}$ *the set of admissible strategies. Whenever controls are restricted to the time interval* $[t, +\infty)$*, we will use the notation* $\mathcal{A}_t$*.*

Our goal is to characterize the forward dynamic exponential utility (**Problem 1**), i.e. to prove that the process $\{U_t(x, t_0),\ t \geq t_0\}$ defined as

$$U_t(x, t_0) = -e^{-\gamma x - P_t}, \quad (t, x) \in [t_0, +\infty) \times \mathbb{R},$$

where $P$ is given by (1.8), is a forward dynamic exponential utility, normalized at $t_0$.

In the sequel we make the integrability assumption.

**Assumption 1.3.** *For every* $t \geq t_0$*, and every* $H = (\Theta, \Pi) \in [0, 1] \times \mathbb{R}$*,*

$$\mathbb{E}\left[e^{\frac{1}{2}\int_{t_0}^{t} h^2(s, X_s^H, S_s, Y_s)\mathrm{d}s}\right] < \infty.$$

**Mathematical construction of the penalizing process.** Next, we describe the functions $g$ and $h$ given in (1.8), that identify the penalizing process. We would like to account for the market risk but also for the risk related to the claims that will occur. However, since part of the losses will be covered by the reinsurance company, it would be advisable for the penalizing process to take into account only the claims that the company will actually have to pay. Therefore losses will affect the penalization according to a certain protection level.

Firstly, we assume that

$$-\frac{\partial^2 b}{\partial \Theta^2}(t, y, \Theta) < \gamma \lambda(t, y) \int_I e^{\gamma(1-\Theta)z} z^2 F(t, y, \mathrm{d}z), \tag{1.11}$$

for every $(t, y, \Theta) \in [t_0, +\infty) \times \mathbb{R} \times [0, 1]$, and let $\widehat{\Theta}$ be the unique solution of the equation

$$\frac{\partial b}{\partial \Theta}(t, y, \Theta) = \lambda(t, y) \int_I z e^{\gamma(1-\Theta)z} F(t, y, \mathrm{d}z), \tag{1.12}$$

which exists in view of condition (1.11). Set

$$\bar{\Theta}(t, y) = \begin{cases} 0, & (t, y) \in \mathcal{D}_0 \\ 1, & (t, y) \in \mathcal{D}_1 \\ \widehat{\Theta}(t, y), & (t, y) \in (\mathcal{D}_0 \cup \mathcal{D}_1)^c, \end{cases} \tag{1.13}$$

where

$$\mathcal{D}_0 \equiv \left\{(t, y) \in [t_0, +\infty) \times \mathbb{R} \mid \lambda(t, y) \int_I z e^{\gamma z} F(t, y, \mathrm{d}z) \leq \frac{\partial b}{\partial \Theta}(t, y, 0)\right\}, \tag{1.14}$$

$$\mathcal{D}_1 \equiv \left\{(t, y) \in [t_0, +\infty) \times \mathbb{R} \mid \frac{\partial b}{\partial \Theta}(t, y, 1) \leq \lambda(t, y) \int_I z F(t, y, \mathrm{d}z)\right\} \tag{1.15}$$

and $(\mathcal{D}_0 \cup \mathcal{D}_1)^c$ is the complementary set of $\mathcal{D}_0 \cup \mathcal{D}_1$. We introduce the function $\varphi : [t_0, +\infty) \times \mathbb{R} \to \mathbb{R}$ defined as

$$\varphi(t, y) = \gamma b(t, y, \bar{\Theta}) + \lambda(t, y) \int_I \left(e^{\gamma(1-\bar{\Theta})z} - 1\right) F(t, y, \mathrm{d}z), \tag{1.16}$$

for every $(t, y) \in [t_0, +\infty) \times \mathbb{R}$, with $\bar{\Theta} = \bar{\Theta}(t, y)$ given by (1.12). Then, we assume that $g$ and $h$ satisfy

$$g(t, x, s, y) = -\gamma a(t, y) + \frac{1}{2} h^2(t, x, s, y) - \frac{1}{2\sigma^2(t, y, s)} \Big(\mu(t, y, s) - \rho^S \sigma(t, y, s) h(t, x, s, y)\Big)^2 + \varphi(t, y), \tag{1.17}$$

for every $(t, x, s, y) \in [t_0, +\infty) \times \mathbb{R} \times (0, +\infty) \times \mathbb{R}$.

**Remark 1.3.** *We observe that in view of Assumption 1.1 and* (1.3), $\mathbb{E}\left[\int_{t_0}^t \varphi(s, Y_s) \mathrm{d}s\right] < \infty$, *for each* $t \geq t_0$. *Therefore, if* $\mathbb{E}\left[\int_{t_0}^t h^2(s, X_s^H, S_s, Y_s) \mathrm{d}s\right] < \infty$, *for each* $t \geq t_0$ *and* $H = (\Theta, \Pi) \in [0, 1] \times \mathbb{R}$, *then* $\mathbb{E}\left[\int_{t_0}^t |g(s, X_s^H, S_s, Y_s)| \mathrm{d}s\right] < \infty$, *for each* $t \geq t_0$, *so that* (1.9) *is* *satisfied. Indeed, for each constant control* $H = (\Theta, \Pi) \in [0,1] \times \mathbb{R}$ *and* $t \geq t_0$*, we have*

$$
\begin{aligned}
&\mathbb{E}\left[\int_{t_0}^{t} |g(s, X_s^H, S_s, Y_s)| \mathrm{d}s\right] \\
&= \mathbb{E}\left[\int_{t_0}^{t} | - \gamma a(s, Y_s) + \frac{1}{2} h^2(t, X_s^H, S_s, Y_s) + \varphi(s, Y_s) \right. \\
&\qquad \left. - \frac{1}{2\sigma^2(s, Y_s, S_s)} \Big(\mu(s, Y_s, S_s) - \rho^S \sigma(s, Y_s, S_s) h(s, X_s^H, S_s, Y_s)\Big)^2 | \mathrm{d}s \right] \\
&\leq \mathbb{E}\left[\int_{t_0}^{t} \Big\{ \gamma |a(s, Y_s)| + |\varphi(s, Y_s)| + \frac{1}{2}(1 + (\rho^S)^2) h^2(t, X_s^H, S_s, Y_s) + \frac{\mu^2(s, Y_s, S_s)}{2\sigma^2(s, Y_s, S_s)} \right. \\
&\qquad \left. + \rho^S \left| \frac{\mu(s, Y_s, S_s)}{\sigma(s, Y_s, S_s)} \right| |h(s, X_s^H, S_s, Y_s)| \Big\} \mathrm{d}s \right] \qquad (1.18) \\
&\leq \mathbb{E}\left[\int_{t_0}^{t} \Big\{ \gamma |a(s, Y_s)| + |\varphi(s, Y_s)| + \Big(\frac{(1 + (\rho^S)^2)}{2} + (\rho^S)^2\Big) h^2(t, X_s^H, S_s, Y_s) + \frac{3\mu^2(s, Y_s, S_s)}{2\sigma^2(s, Y_s, S_s)} \Big\} \mathrm{d}s \right] \\
&\qquad (1.19) \\
&< \infty,
\end{aligned}
$$

*where we have used the triangular inequality in* (1.18)*, the Cauchy-Schwarz inequality in* (1.19)*, and the integrability conditions* (1.3) *and* (1.6)*.*

Loosely speaking, the process $\{e^{-P_t},\ t \geq t_0\}$ can be interpreted as the density of a probability measure, Indeed, by the definition of $P$, this measure encompasses the main features of the combined market: it depends on the risk aversion parameter $\gamma$ of the initial utility and it is affected by the financial market through the Sharpe ratio $\frac{\mu(t, Y_t, S_t)}{\sigma(t, Y_t, S_t)}$ and by the insurance market via insurance and reinsurance premia $(a(t, Y_t) - b(t, Y_t, \bar{\Theta}(t, Y_t)))$, the claim arrival intensity and the claim size.

**Remark 1.4.** *Since* $P_t = \int_{t_0}^{t} g(s, X_s^H, S_s, Y_s)\mathrm{d}s + \int_{t_0}^{t} h(s, X_s^H, S_s, Y_s)\mathrm{d}W_s^P,\ t \geq t_0$*, different choices of the functions* $g$ *and* $h$ *will result in a different penalizing process* $P$*. For example, if* $h(t, x, s, y) = 0$ *, we are in the zero-volatility case and, by* (1.17)*, the function* $g$ *does not depend on* $x$ *and is given by*

$$
g(t, s, y) = -\frac{1}{2}\left(\frac{\mu(t, y, s)}{\sigma(t, y, s)}\right)^2 - \gamma a(t, y) + \varphi(t, y).
$$

*We observe that this choice for* $g$ *also allows us to consider a function* $h$ *of form*

$$
h(t, s, y) = -\frac{2\rho^S}{1 - (\rho^S)^2} \frac{\mu(t, y, s)}{\sigma(t, y, s)}.
$$

*Other special cases are, e.g.,* $g(t,x,s,y) = \frac{1}{2}h^2(t,x,s,y)$ *or* $g(t,x,s,y) = \frac{1}{2}h^2(t,x,s,y) - \frac{1}{2}\frac{\mu^2(t,y,s)}{\sigma^2(t,y,s)} - \gamma a(t,y) + \varphi(t,y)$*, for every* $(t,x,s,y) \in [t_0,+\infty) \times \mathbb{R} \times (0,+\infty) \times \mathbb{R}$*. We outline that the Brownian motion* $W^P$ *driving the dynamics of the penalizing process plays an important role, due to the correlation with the stock price dynamics; that is the penalizing process includes part of the randomness coming from the stock price. Furthermore, it may also directly depend on the stochastic factor: for example, if the latter is a diffusion process driven by a Brownian motion correlated with* $W^P$*.*

The definition of the function $\varphi$, and hence of the function $g$, depends on the specific choice of $\bar{\Theta}$. Instead of taking $\bar{\Theta}$ as in (1.13), one could have taken other values in $[0,1]$. For instance, the choice $\bar{\Theta}(t,Y_t) = 1$, for all $t \geq t_0$, implies that we have $g(t,s,y) = -\frac{1}{2}\left(\frac{\mu(t,y,s)}{\sigma(t,y,s)}\right)^2 - \gamma(a(t,y) - b(t,y,1)$, which in turn leads to a dynamic utility that does not adjust for claims. Instead, taking $\bar{\Theta}(t,Y_t) = 0$, for all $t \geq t_0$, corresponds to set $g(t,s,y) = -\frac{1}{2}\left(\frac{\mu(t,y,s)}{\sigma(t,y,s)}\right)^2 - \gamma(a(t,y) - b(t,y,0)) + \lambda(t,y)\int_I \left(e^{\gamma z} - 1\right)F(t,y,\mathrm{d}z)$, which instead, implies that the penalizing process accounts for the whole claims amount. Our decision on the function $\bar{\Theta}(t,y)$ lies in the middle: in a certain sense, as we will see later, we would like to incorporate in the utility preferences the amount of claims that the insurance will not able to cover via the optimal strategy.

In order to solve **Problem 1**, we need to add details about the stochastic factor that affects the loss process and the risky asset price, specifying its mathematical features, as we will see in the next sections. In particular, the common factor can be modeled as a Markov chain (see Section 1.4) or as a general stochastic process of diffusion type (see Section 1.5); the proof is very similar, except for some technicalities.

## 1.4 Optimal investment and reinsurance in a regime-switching market model under forward preferences

In this section, we propose an interdependent insurance-financial market model where a common stochastic factor, which affects the stock price and the claim arrival intensity, is modeled as a continuous time finite state Markov chain. In this framework, by stochastic control techniques, we analytically construct a forward dynamic exponential utility and characterize the optimal investment and reinsurance strategy. We also perform numerical experiments and provide sensitivity analyses with respect to some model parameters. Moreover, we point out the differences between forward performance criteria and standard backward performance criteria in the case of independent markets, both analytically and numerically.

### 1.4.1 Setting

In order to describe the stochastic factor, we introduce in our filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}; \mathbb{F})$ a continuous time Markov chain $Y = \{Y_t,\ t \geq 0\}$ with finite state space $\mathcal{E} = \{e_1, \dots, e_K\}$, where $e_j$, with $j = 1, \dots, K$, denote the standard vectors of $\mathbb{R}^K$. Let $Q = (q_{ij})_{i,j=1,\dots,K}$ be the $K \times K$ matrix representing the switching intensity. The entries of the matrix satisfy $q_{ij} \geq 0$ for all $i \neq j$ and $q_{ii} = -\sum_{i \neq j} q_{ij}$. We also recall that $Y$ admits the following semimartingale decomposition

$$Y_t = Y_0 + \int_0^t QY_s \mathrm{d}s + M_t^Y, \quad t \geq 0$$

where $QY_s$ is the matrix-vector product and $M^Y = \{M_t^Y,\ t \geq 0\}$ is a martingale with respect to the natural filtration of $Y$. Due to the finite state nature of the Markov chain $Y$ we also get that, for any function $f : \mathcal{E} \to \mathbb{R}$, $f(Y_t) = \sum_{j=1}^K f_j \mathbf{1}_{\{Y_t = e_j\}}$, where $f_j = f(e_j)$, for all $j = 1, \dots, K$. Thus, we can relax many integrability conditions, required in Section 1.2.

Retracing the construction of insurance losses, we introduce the process $\Lambda$ as (1.1). In a regime-switching setting we only assume that function $\lambda : [0, +\infty) \times \mathcal{E} \to (0, +\infty)$ is such that

$$\int_0^{+\infty} \lambda(t, e_j) \mathrm{d}t < \infty, \tag{1.20}$$

for every $j = 1, \dots, K$, and $\lambda(\cdot, e_j)$ is Borel-measurable. Notice that condition (1.20) implies condition (1.2), since

$$\mathbb{E}\left[\int_0^{+\infty} \lambda(t, Y_t) \mathrm{d}t\right] \leq \max_{j=1,\dots,K} \int_0^{+\infty} \lambda(t, e_j) \mathrm{d}t < \infty,$$

and guarantees that the counting process $N$, representing the number of claims, is non-explosive and the compensated process $\tilde{N}$ is a martingale.

Finally, the claim arrival intensity $\{\lambda(t, Y_{t-}),\ t \geq 0\}$ is an $\mathbb{F}$-predictable process. In this section, we consider a setting where claim amounts $\{Z_n\}_{n \in \mathbb{N}}$ are given by a sequence of independent and identically distributed $\mathcal{F}_{T_n}$-random variables with continuous cumulative distribution function $F(z)$. To avoid too technicalities, we also assume that the function $F$

has compact support $I \subset [0, +\infty)$ [1]. Moreover, we suppose that they are independent of $N$ and $Y$. Nevertheless, it is reasonable to assume independence between the claim sizes and the number of claims: indeed, there is not always a link between the amount of the claims and their frequency of arrival. For example, in car insurance, an earthquake, accounted by the common factor $Y$, leads to an increase in road accidents, regardless their damage size.

The dual predictable projection $\nu$ of the random counting measure $m$ is given by

$$\nu(\mathrm{d}t, \mathrm{d}z) = F(\mathrm{d}z)\lambda(t, Y_{t-})\mathrm{d}t.$$

The cumulative claim process $C$ is non-explosive, without requiring any further conditions since it holds that

$$\mathbb{E}\left[C_t\right] = \mathbb{E}\left[\int_0^t \int_I z m(\mathrm{d}s, \mathrm{d}z)\right] = \mathbb{E}\left[\int_0^t \int_I z\lambda(s, Y_{s-})F(\mathrm{d}z)\mathrm{d}s\right] < \infty, \tag{1.21}$$

for every $t \geq 0$. We refer to Brémaud [19] (Chapter VIII, Section 1) for further details.

We suppose that both the claim premium rate and the reinsurance premium are subject to different regimes. The insurance gross premium is of the form $a(t, Y_t)$, for every $t \geq 0$, where $a : [0, +\infty) \times \mathcal{E} \to [0, +\infty)$ is a continuous function in $t \geq 0$, for all $j = 1, \dots, K$. We consider reinsurance contracts of proportional type, with protection level $\Theta = \{\Theta_t,\ t \geq 0\}$ and premium rate $\{b(t, Y_t, \Theta_t),\ t \geq 0\}$, for some function $b : [0, +\infty) \times \mathcal{E} \times [0, 1] \to [0, +\infty)$, which is jointly continuous with respect to $(t, \Theta)$, for every $e_j \in \mathcal{E}$, with $j = 1, \dots, K$.

Insurance and reinsurance premia are assumed to satisfy the classical premium properties listed in Assumption (1.2).

Moreover, from the continuity of the functions $a(t, e_j)$ with respect to $t$ and of the function $b(t, e_j, \Theta)$ with respect to $(t, \Theta)$, for all $j = 1, \dots, K$, and the finite state nature of the Markov chain $Y$, we have that for every $t \geq 0$,

$$|a(t, Y_t) - b(t, Y_t, \Theta_t)| \leq k(t), \quad \mathbf{P}\text{-a.s.}, \tag{1.22}$$

for some continuous function $k : [0, +\infty) \to [0, +\infty)$, since $\Theta_t \in [0, 1]$. In particular

[1] Specifically, we need a compact interval $I$ in order to avoid some technical requirements concerning the expected value. It is clear that for certain distributions such as the exponential one, our results can be applied thanks to integrability conditions (1.1). Therefore, this assumption of compactness for the values of the claim amount can be relaxed. In our setting, however, assuming this kind of support is not restrictive as typically claim amounts can be arbitrarily large but do not explode.

$\int_0^t k(s)\mathrm{d}s < \infty$, for all $t \geq 0$. Furthermore, the following implications hold:

$$\mathbb{E}\left[\int_0^t b(s, Y_s, \Theta_s)\mathrm{d}s\right] \leq \max_{\substack{\Theta\in[0,1],\\ j=1,\dots,K}} \int_0^t b(s, e_j, \Theta)\mathrm{d}s < \infty, \tag{1.23}$$

for every $t \geq 0$, and

$$\mathbb{E}\left[\int_0^t a(s, Y_s)\mathrm{d}s\right] \leq \max_{j=1,\dots,K} \int_0^t a(s, e_j)\mathrm{d}s < \infty, \tag{1.24}$$

for every $t \geq 0$. We emphasize that conditions (1.24) and (1.23) naturally descend from the mathematical properties of the finite state Markov chain $Y$.

For any strategy $\Theta \in [0, 1]$, the insurance company surplus (or reserve) process $R^\Theta = \{R_t^\Theta,\ t \geq 0\}$, given by (1.4), is well defined and for every $t \geq 0$,

$$\begin{aligned} |R_t^\Theta| &= \left|\int_0^t (a(s, Y_s)\mathrm{d}s - b(s, Y_s, \Theta_s))\mathrm{d}s - \int_0^t (1 - \Theta_{s-})\mathrm{d}C_s\right| \\ &\leq \left|\int_0^t (a(s, Y_s)\mathrm{d}s - b(s, Y_s, \Theta_s))\mathrm{d}s\right| + \left|\int_0^t (1 - \Theta_{s-})\mathrm{d}C_s\right| \\ &\leq \int_0^t k(s)\mathrm{d}s + C_t, \quad \mathbf{P} - \text{a.s.}, \end{aligned}$$

and hence $\mathbb{E}\left[|R_t^\Theta|\right] < \infty$, for every $t \geq 0$, in view of (1.22) and (1.21).

The financial market is characterized by a risk-free asset $S^0$ with zero interest rate and a risky asset $S$ which follows a regime-switching constant elasticity of variance (CEV) model, i.e.

$$\mathrm{d}S_t = S_t\left(\mu(Y_t)\mathrm{d}t + \sigma(Y_t)S_t^\beta \mathrm{d}W_t^S\right), \quad S_0 = s > 0, \tag{1.25}$$

where $-1 < \beta \leq 0$ is the coefficient of elasticity and the Brownian motion $W^S = \{W_t^S,\ t \geq 0\}$ is independent of the random measure $m(\mathrm{d}t, \mathrm{d}z)$ and also of the Markov chain $Y$. The functions $\mu : \mathcal{E} \to \mathbb{R}$ and $\sigma : \mathcal{E} \to (0, +\infty)$ are measurable functions representing the appreciation rate and the volatility of the stock, respectively. We also assume that the diffusion term is not degenerate, that is, $\sigma(e_j) > 0$, for every $j = 1, \dots, K$. Notice that functions $\mu$ and $\sigma$ may only take a finite number of values and therefore, they are bounded from above and below; in particular, it holds that $\underline{\mu} \leq \mu(Y_t) \leq \overline{\mu}$ and $0 < \underline{\sigma} \leq \sigma(Y_t) \leq \overline{\sigma}$, for every $t \geq 0$, where $\underline{\mu} = \min_{j=1,\dots,K} \mu(e_j)$, $\overline{\mu} = \max_{j=1,\dots,K} \mu(e_j)$, $\underline{\sigma} = \min_{j=1,\dots,K} \sigma(e_j)$, $\overline{\sigma} = \max_{j=1,\dots,K} \sigma(e_j)$. Consequently, the ratio $\frac{\mu(Y_t)}{\sigma(Y_t)}$ is also bounded from above and below for every $t \geq 0$.

**Remark 1.5.** *Let us comment the choice of a CEV model for the stock price process. This model was originally introduced by Cox and Ross [35] under the assumption that the elasticity coefficient is strictly negative, i.e.* $\beta < 0$*. Later, Emanuel and MacBeth [49] extended this paper to the case* $\beta > 0$*. The CEV model, as the elasticity parameter* $\beta$ *varies, allows considering several situations that usually arise in financial market models. Firstly, we notice that for specific choices of* $\beta \in \mathbb{R}$ *the stock price dynamics reduces to well known processes. For example, if* $\beta = 0$ *and the coefficients* $\mu$ *and* $\sigma$ *are constant, we get the classical Black & Scholes model, for* $\beta = -\frac{1}{2}$ *we end up with a Cox-Ingersoll-Ross process and when* $\beta = -1$ *the process* $S$ *becomes an Ornstein-Uhlenbeck process. It is clear that, depending on the values of* $\beta < 0$*, the local volatility is a decreasing function of the stock price which then may touch zero with positive probability in finite time and even become negative. Even though the probability is generally quite small, this is an unpleasant characteristic for modeling risky asset prices. On the other hand, if* $\beta > 0$*, the increasing local volatility is able to generate upward-sloping volatility skews and thus the price process* $S$ *may explode. The latter implies the presence of a stock price bubble, as shown in Heston et al. [60], and which is not a desirable feature for financial applications since there might exist arbitrage opportunities. Therefore both choices for the range of* $\beta$*, either* $\beta < 0$ *or* $\beta > 0$*, have advantages and drawbacks. In the literature it is common to take* $-1 < \beta \leq 0$*. Taking into consideration this range for* $\beta$ *and constant coefficients* $\mu$ *and* $\sigma$*, Delbaen and Shirakawa [42] show the existence of an equivalent martingale measure and provide many considerations on absence of arbitrage. In this work we opt for a regime-switching extension of the CEV model, since the drift and the volatility depend on the Markov chain which represents an exogenous factor affecting the market model. For further details about the calibration of the elasticity parameter in CEV models we also refer to, e.g., Dias et al. [46], Heath and Schweizer [58].*

As in Section (1.2), the insurance company, with an initial wealth $x_0$, subscribes a proportional reinsurance and invests continuously the remaining part of its wealth in the financial market, following a self-financing strategy. Then, in this regime-switching model, the wealth of the insurance company associated with the investment-reinsurance strategy $H = (\Pi, \Theta) = \{(\Pi_t, \Theta_t),\ t \geq 0\}$ satisfies the following SDE

$$\mathrm{d}X_t^H = \big\{a(t, Y_t) - b(t, Y_t, \Theta_t) + \Pi_t \mu(Y_t)\big\}\mathrm{d}t + \Pi_t \sigma(Y_t) S_t^{\beta} \mathrm{d}W_t^S - (1 - \Theta_{t-})\mathrm{d}C_t, \tag{1.26}$$

with $X_0^H = x_0 \geq 0$. Thus, the solution of the SDE (1.26) is given by

$$\begin{aligned} X_t^H = & x_0 + \int_0^t \left(a(s,Y_s) - b(s,Y_s,\Theta_s) + \Pi_s \mu(Y_s)\right) \mathrm{d}s \\ & + \int_0^t \Pi_s \sigma(Y_s) S_s^\beta \mathrm{d}W_s^S - \int_0^t \int_I (1-\Theta_{s-}) z m(\mathrm{d}s, \mathrm{d}z), \end{aligned} \tag{1.27}$$

for every $t \geq 0$.

The preferences of the insurance company are described by a dynamic forward utility of exponential type (see Definition 1.2). Now, we focus on the zero-volatility case, i.e. we assume that the penalizing process satisfies

$$P(t) = \int_{t_0}^t g(s, S_s, Y_s) \mathrm{d}s, \tag{1.28}$$

for all $t \geq t_0$. In this subsection we use the notation $P(t)$ to underline that the penalizing process is absolutely continuous with respect to the Lebesgue measure. Here, the function $g : [t_0, +\infty) \times (0, +\infty) \times \mathcal{E} \to \mathbb{R}$ is given by

$$g(t, s, e_i) = -\frac{1}{2}\left(\frac{\mu(e_i)}{\sigma(e_i) s^\beta}\right)^2 - \gamma a(t, e_i) + \varphi(t, e_i), \tag{1.29}$$

where the function $\varphi : [t_0, +\infty) \times \mathcal{E} \to \mathbb{R}$ is given by

$$\varphi(t, e_i) = \gamma b(t, e_i, \overline{\Theta}_t) + \lambda(t, e_i) \int_I \left(e^{\gamma(1-\overline{\Theta}_{t-})z} - 1\right) F(\mathrm{d}z), \tag{1.30}$$

with $\overline{\Theta}_t = \overline{\Theta}(t, Y_t)$ that satisfies:

$$\overline{\Theta}(t, e_i) = \begin{cases} 0, & (t, e_i) \in \mathcal{D}_0, \\ \hat{\Theta}(t, e_i), & (t, e_i) \in (\mathcal{D}_0 \cup \mathcal{D}_1)^c, \\ 1, & (t, e_i) \in \mathcal{D}_1, \end{cases} \tag{1.31}$$

where $(\mathcal{D}_0 \cup \mathcal{D}_1)^c$ is the complementary set of $\mathcal{D}_0 \cup \mathcal{D}_1$ that are given by

$$\begin{aligned} \mathcal{D}_0 & \equiv \left\{(t, e_i) \in [t_0, +\infty) \times \mathcal{E} \mid \lambda(t, e_i) \mathbb{E}\left[Z_1 e^{\gamma Z_1}\right] \leq \frac{\partial b}{\partial \Theta}(t, e_i, 0)\right\}, \\ \mathcal{D}_1 & \equiv \left\{(t, e_i) \in [t_0, +\infty) \times \mathcal{E} \mid \frac{\partial b}{\partial \Theta}(t, e_i, 1) \leq \lambda(t, e_i) \mathbb{E}\left[Z_1\right]\right\} \end{aligned}$$

and $\hat{\Theta}$ is the unique solution of the equation:

$$\frac{\partial b}{\partial \Theta}(t, e_i, \Theta) = \lambda(t, e_i) \int_I z e^{\gamma(1-\Theta)z} F(\mathrm{d}z). \tag{1.32}$$

The concavity assumption (1.11) reduces to

$$-\frac{\partial^2 b}{\partial \Theta^2}(t, e_i, \Theta) < \gamma \lambda(t, e_i) \int_I e^{\gamma(1-\Theta)z} z^2 F(\mathrm{d}z), \tag{1.33}$$

for every $(t, e_i, \Theta) \in [0, +\infty) \times \mathcal{E} \times [0, 1]$, which guarantees the existence of a unique solution to Equation (1.32).

**Remark 1.6.** *We observe that, from the mathematical point of view, the condition* (1.33) *ensures that the value function is globally concave with respect to* $\Theta$ *and hence that it admits a unique maximizer* $\Theta^* \in [0, 1]$. *This condition, for instance, can come from the concavity of the reinsurance premium* $b(t, e_j, \Theta)$ *with respect to the protection level* $\Theta$*, which is satisfied under classical premium calculation principle and implies that extreme cases (full reinsurance as well as no reinsurance) are never optimal.*

We point out that this special penalizing process (1.28) turns out by setting the function $h(t, x, s, y)$ equal to zero. We notice that the function $g$ does not depend on wealth explicitly; that is, the standard exponential utility is penalized by a process $P$ which accounts only for market coefficients, collected premia and paid premia.

### 1.4.2 Optimal investment and reinsurance

Our objective is to show that the process $\{U_t(x, t_0),\ t \geq t_0\}$ defined as

$$U_t(x, t_0) = -e^{-\gamma x - P(t)}, \quad (t, x) \in [t_0, +\infty) \times \mathbb{R},$$

where $P$ is given in (1.28), is a forward dynamic exponential utility, normalized at $t_0$, and to charaterize the optimal reinsurance and investment strategy.

First of all, recalling Definition 1.3, we specify that in this framework, an admissible strategy is a pair $H = (\Pi, \Theta)$ of $\mathbb{F}$-progressively measurable processes with values in $\mathbb{R} \times [0, 1]$, such that, for every $T \geq t_0$, $\mathbb{E}\left[e^{-\gamma X_T^H - P(T)}\right] < \infty$ and

$$\mathbb{E}\left[\int_{t_0}^T \left(|\Pi_s| + \Pi_s^2 S_s^{2\beta}\right) \mathrm{d}s\right] < \infty.$$

Next, we prove that the triplet $(X^H, S, Y)$ is a Markovian process and we compute its infinitesimal generator that will be useful in the sequel. Denote by $C_b^{1,2,2}$ the set of all bounded functions $f(t, x, s, e_j)$, with bounded first-order derivatives with respect to $t, x, s$ and bounded second-order derivatives with respect to $x, s$, for every $j = 1, \dots, K$. Let $\hat{\mathcal{L}}^H$ denotes the Markov generator of $(X^H, S, Y)$ associated with a constant control $H = (\Theta, \Pi) \in [0,1] \times \mathbb{R}$.

**Lemma 1.1.** *Let $f(\cdot,\cdot,\cdot,e_i) \in C_b^{1,2,2}$, for each $e_i \in \mathcal{E}$. For any constant strategy $H = (\Pi, \Theta) \in \mathbb{R} \times [0,1]$, the triplet $(X^H, S, Y)$ is a Markov process with infinitesimal generator $\hat{\mathcal{L}}^H$ given by*

$$\begin{aligned}\hat{\mathcal{L}}^H f(t,x,s,e_i) &= \frac{\partial f}{\partial t}(t,x,s,e_i) + \big[a(t,e_i) - b(t,e_i,\Theta) + \Pi\mu(e_i)\big]\frac{\partial f}{\partial x}(t,x,s,e_i)\\ &+ \sum_{j=1}^{K} f(t,x,s,e_j)q_{ij} + s\mu(e_i)\frac{\partial f}{\partial s}(t,x,s,e_i) + \frac{1}{2}\Pi^2\sigma^2(e_i)s^{2\beta}\frac{\partial^2 f}{\partial x^2}(t,x,s,e_i)\\ &+ s^{2\beta+2}\sigma(e_i)\frac{\partial^2 f}{\partial s^2}(t,x,s,e_i) + \Pi\sigma^2(e_i)s^{2\beta+1}\frac{\partial^2 f}{\partial x \partial s}(t,x,s,e_i)\\ &+ \lambda(t,e_i)\int_I \Big\{f\big(t, x-(1-\Theta)z, s, e_i\big) - f(t,x,s,e_i)\Big\}F(\mathrm{d}z).\end{aligned} \tag{1.34}$$

*Proof.* To prove the result, we first characterize the martingale $M^Y$ in the semimartingale decomposition of the Markov chain $Y$. Let $\{\tau_n\}_{n\in\mathbb{N}}$ be the sequence of jump times of $Y$ and denote by $m^Y$ the jump measure of $Y$, which is given by

$$m^Y([0,t],\{e_j\}) := \sum_{n\geq 1} \mathbf{1}_{\{Y_{\tau_n}=e_j\}}\mathbf{1}_{\{\tau_n \geq t\}},$$

with compensator

$$\nu^Y([0,t],\{e_j\}) = \int_0^t \sum_{\substack{i,j=1,\\ i\neq j}}^{K} q_{ij}\mathbf{1}_{\{Y_{r-}=e_i\}}\mathrm{d}r,$$

for every $t \geq 0$. Hence, we get that

$$Y_t = Y_0 + \int_0^t \sum_{j=1}^{K}(e_j - Y_{r-})q_{Y_{r-}j}\mathrm{d}r + \int_0^t \sum_{j=1}^{K}(e_j - Y_{r-})(m^Y - \nu^Y)(\mathrm{d}r,\{e_j\}),$$

for every $t \geq 0$ (with a slight abuse of notation we identify $q_{Y_{r-}j}|_{Y_{r-}=i} = q_{ij}$.). Now, let $f : [0,+\infty) \times \mathbb{R} \times (0,+\infty) \times \mathcal{E} \to \mathbb{R}$ be a function in $C_b^{1,2,2}$ and $H = (\Pi,\Theta) \in \mathbb{R} \times [0,1]$ constant. Then, the result follows by applying Itô's formula to $f(X^H, S, Y)$. Indeed, we get that $\{f(t, X_t^H, S_t, Y_t), t \geq 0\}$ has the semimartingale decomposition

$$f(t, X_t^H, S_t, Y_t) = f(0, X_0^H, S_0, Y_0) + \int_0^t \hat{\mathcal{L}}^H f(r, X_r^H, S_r, Y_r)\mathrm{d}r + M_t^f, \quad t \geq 0,$$

where $M^f = \{M_t^f,\ t \geq 0\}$ is the $(\mathbb{F}, \mathbf{P})$-martingale null at $t = 0$ given by

$$
\begin{aligned}
\mathrm{d}M_t^f &= \left(\sigma(Y_t) S_t^{\beta+1} \frac{\partial f}{\partial s}(t, X_t^H, S_t, Y_t) + \Pi_t \sigma(Y_t) S_t^{\beta} \frac{\partial f}{\partial x}(t, X_t^H, S_t, Y_t)\right) \mathrm{d}W_t^S \\
&+ \int_I \left(f\big(t, X_{t-}^H - (1-\Theta)z, S_t, Y_t\big) - f(t, X_{t-}^H, S_t, Y_t)\right) \left(m(\mathrm{d}t, \mathrm{d}z) - \lambda(t, Y_{t-}) F(\mathrm{d}z)\mathrm{d}t\right) \\
&+ \sum_{j=1}^{K} \left(f\big(t, X_t^H, S_t, e_j\big) - f(t, X_t^H, S_t, Y_{t-})\right) (m^Y - \nu^Y)(\mathrm{d}t, \{e_j\}).
\end{aligned}
$$

□

Let us introduce the following optimization problem

$$
\max_{H \in \mathcal{A}} \hat{\mathcal{L}}^H f(t, x, s, e_i) - g(t, s, e_i) f(t, x, s, e_i) = 0, \tag{1.35}
$$

for all $(t, x, s, e_i) \in [t_0, T) \times \mathbb{R} \times (0, +\infty) \times \mathcal{E}$, with the final condition

$$
f(T, x, s, e_i) = -e^{-\gamma x}, \tag{1.36}
$$

for all $(x, s, e_i) \in \mathbb{R} \times (0, +\infty) \times \mathcal{E}$, where $f$ is a a function in $C_b^{1,2,2}$, $g$ is the specific function given by (1.29) and we recall that the operator $\hat{\mathcal{L}}^H$ denotes the infinitesimal generator of the Markov process $(X^H, S, Y)$ defined in (1.34) associated with a constant control $H \in [0, 1] \times \mathbb{R}$.

After that, we establish a general verification result for this final value problem that will allow us to achieve our main goal that is to construct analytically forward utilities of exponential type (see Theorem 1.2 below).

**Theorem 1.1** (Verification Theorem)**.** *Let $t_0 \geq 0$ be the normalization point and $T \geq t_0$. Let $\overline{u} : [t_0, T] \times \mathbb{R} \times (0, +\infty) \times \mathcal{E} \longrightarrow (-\infty, 0)$ be a smooth solution of the HJB Equations* (1.35) *and* (1.36) *(i.e., the function $\overline{u}(\cdot, \cdot, \cdot, e_j) \in \mathcal{C}^{1,2,2}$, for all $j = 1, ..., K$), which satisfies*

(i) $\mathbb{E}\left[\displaystyle\int_{t_0}^{T} e^{-\int_{t_0}^{r} g(l, S_l, Y_l)\mathrm{d}l} \left(\Pi_r \sigma(Y_r) S_v^{\beta} \frac{\partial \overline{u}}{\partial x}(r, X_r^H, S_r, Y_r)\right)^2 \mathrm{d}r\right] < \infty,$

(ii) $\mathbb{E}\left[\displaystyle\int_{t_0}^{T} e^{-\int_{t_0}^{r} g(l, S_l, Y_l)\mathrm{d}l} \left(\sigma(Y_r) S_r^{\beta+1} \frac{\partial \overline{u}}{\partial s}(r, X_r^H, S_r, Y_r)\right)^2 \mathrm{d}r\right] < \infty,$

(iii) $\mathbb{E}\left[\displaystyle\int_{t_0}^{T} e^{-\int_{t_0}^{r} g(l, S_l, Y_l)\mathrm{d}l} \sum_{j=1}^{K} \left\{\overline{u}\big(r, X_r^H, S_r, e_j\big) - \overline{u}(r, X_r^H, S_r, Y_{r-})\right\} \nu^Y(\mathrm{d}r, \{e_j\})\right] < \infty,$

(iv) $\mathbb{E}\Big[\displaystyle\int_{t_0}^{T} e^{-\int_{t_0}^{r} g(l, S_l, Y_l)\mathrm{d}l} \lambda(r, Y_r)$
$\quad \times \displaystyle\max_{z \in I} \left|\overline{u}(r, X_{r-}^H - (1 - \Theta_{r-})z, S_r, Y_r) - \overline{u}(r, X_{r-}^H, S_r, Y_r)\right| \mathrm{d}r\Big] < \infty.$

*Then,* $u(t,x,s,e_i) \leq \overline{u}(t,x,s,e_i)$*, for every admissible control* $H \in \mathcal{A}$ *and for every* $(t,x,s,e_i) \in [t_0,T] \times \mathbb{R} \times (0,+\infty) \times \mathcal{E}$*.*

*Moreover, if* $\overline{u}(T,x,s,e_i) = u(T,x,s,e_i)$*, for every* $(x,s,e_i) \in \mathbb{R} \times (0,+\infty) \times \mathcal{E}$ *and there exists* $H^* \in \mathcal{A}$ *such that* $\hat{\mathcal{L}}^{H^*}\overline{u}(t,x,s,e_i) + g(t,x,s,e_i)\overline{u}(t,x,s,e_i) = 0$*, for every* $(t,x,s,e_i) \in [t_0,T[\times \mathbb{R} \times (0,+\infty) \times \mathcal{E}$*, then* $u = \overline{u}$ *in* $[t_0,T] \times \mathbb{R} \times (0,+\infty) \times \mathcal{E}$*.*

*Proof.* Let $H \in \mathcal{A}$ be an admissible control. Using Equations (1.25) and (1.26) and applying Itô's formula to $e^{-\int_{t_0}^t g(r,X_r^H,Y_r)\mathrm{d}r}\overline{u}(t,X_t^H,S_t,Y_t)$, we have that

$$\begin{aligned}
&e^{-\int_{t_0}^T g(r,S_r,Y_r)\mathrm{d}r}\overline{u}(T,X_T^H,S_T,Y_T) = e^{-\int_{t_0}^t g(r,S_r,Y_r)\mathrm{d}r}\overline{u}(t,x,s,e_i)\\
&+\int_t^T e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l}\Big[\hat{\mathcal{L}}^H\overline{u}(v,X_v^H,S_v,Y_v) + g(r,S_r,Y_r)\overline{u}(r,X_r^H,S_r,Y_r)\Big]\mathrm{d}r\\
&+\int_t^T e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l}\Pi_r\sigma(Y_r)S_r^\beta\frac{\partial\overline{u}}{\partial x}(r,X_r^H,S_r,Y_r)\mathrm{d}W_r^S\\
&+\int_t^T e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l}\sigma(Y_r)S_r^{\beta+1}\frac{\partial\overline{u}}{\partial s}(r,X_r^H,S_r,Y_r)\mathrm{d}W_r^S\\
&+\int_t^T\int_I e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l}\sum_{j=1}^K\left\{\overline{u}\big(r,X_r^H,S_r,e_j\big) - \overline{u}(r,X_r^H,S_r,Y_{r-})\right\}(m^Y-\nu^Y)(\mathrm{d}r,\{e_j\})\\
&+\int_t^T\int_I e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l}\left(\overline{u}\big(r,X_{r-}^H-(1-\Theta_{r-})z,S_r,Y_r\big) - \overline{u}(r,X_{r-}^H,S_r,Y_r)\right)\\
&\qquad\times(m(\mathrm{d}r,\mathrm{d}z) - \lambda(r,Y_{r-})F(\mathrm{d}z)\mathrm{d}r),
\end{aligned}$$

where $\hat{\mathcal{L}}^H$ is introduced in (1.34). Let $M = \{M_t,\ t \in [t_0,T]\}$ be the process given by

$$\begin{aligned}
M_t &= \int_{t_0}^t e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l}\Pi_r\sigma(Y_r)S_r^\beta\frac{\partial\overline{u}}{\partial x}(r,X_r^H,S_r,Y_r)\mathrm{d}W_r^S\\
&+\int_{t_0}^t e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l}\sigma(Y_r)S_r^{\beta+1}\frac{\partial\overline{u}}{\partial s}(r,X_r^H,S_r,Y_r)\mathrm{d}W_r^S\\
&+\int_{t_0}^t\int_I e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l}\sum_{j=1}^K\left\{\overline{u}\big(r,X_r^H,S_r,e_j\big) - \overline{u}(r,X_r^H,S_r,Y_{r-})\right\}(m^Y-\nu^Y)(\mathrm{d}r,\{e_j\})\\
&+\int_{t_0}^t\int_I e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l}\left(\overline{u}\big(r,X_{r-}^H-(1-\Theta_{r-})z,S_r,Y_r\big) - \overline{u}(r,X_{r-}^H,S_r,Y_r)\right)\\
&\qquad\times(m(\mathrm{d}r,\mathrm{d}z) - \lambda(r,Y_{r-})F(\mathrm{d}z)\mathrm{d}r),
\end{aligned}$$

and observe that integrability conditions (i), (ii), (iii), (iv) ensure that $M$ is an $(\mathbb{F},\mathbf{P})$-

martingale. Now, since $\overline{u}$ solves the HJB-equation in (1.35) and (1.36), we have

$$
\begin{aligned}
& e^{-\int_{t_0}^{T} g(r,S_r,Y_r)\mathrm{d}r}\overline{u}(T, X_T^H, S_T, Y_T) \leq e^{-\int_{t_0}^{t} g(r,S_r,Y_r)\mathrm{d}r}\overline{u}(t,x,s,e_i) \\
& + \int_t^T e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l}\Pi_r \sigma(Y_r) S_r^{\beta} \frac{\partial \overline{u}}{\partial x}(r, X_r^H, S_r, Y_r)\mathrm{d}W_r^S \\
& + \int_t^T e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l}\sigma(Y_r) S_r^{\beta+1} \frac{\partial \overline{u}}{\partial s}(r, X_r^H, S_r, Y_r)\mathrm{d}W_r^S \qquad (1.37) \\
& + \int_t^T \int_I e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l} \sum_{j=1}^{K} \left\{ \overline{u}\big(r, X_r^H, S_r, e_j\big) - \overline{u}(r, X_r^H, S_r, Y_{r-}) \right\} (m^Y - \nu^Y)(\mathrm{d}r, \{e_j\}) \\
& + \int_t^T \int_I e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}r} \left( \overline{u}\big(r, X_{r-}^H - (1-\Theta_{r-})z, S_r, Y_r\big) - \overline{u}(r, X_{r-}^H, S_r, Y_r) \right) \\
& \quad \times (m(\mathrm{d}r, \mathrm{d}z) - \lambda(r, Y_{r-}) F(\mathrm{d}z)\mathrm{d}r),
\end{aligned}
$$

for every $H \in \mathcal{A}$.

Then, taking the conditional expectation with respect to $X_t^H = x$, $S_t = s$ and $Y_t = e_i$ on both sides of Equation (1.37) leads to

$$
\mathbb{E}_{t,x,s,e_i}\left[ e^{-\int_{t_0}^{T} g(r,S_r,Y_r)\mathrm{d}r}\overline{u}(T, X_T^H, S_T, Y_T) \right] \leq e^{-\int_{t_0}^{t} g(r,S_r,Y_r)\mathrm{d}r}\overline{u}(t,x,s,e_i).
$$

By the final condition in Equation (1.36), we obtain

$$
\mathbb{E}_{t,x,s,e_i}\left[ -e^{-\gamma X_T^H - \int_t^T g(r,S_r,Y_r)\mathrm{d}r} \right] \leq \overline{u}(t,x,s,e_i),
$$

for every $H \in \mathcal{A}$. Hence, $u(t,x,s,e_i) \leq \overline{u}(t,x,s,e_i)$, as we wanted. Finally, we observe that if $H \in \mathcal{A}$ is the maximizer in the HJB-Equation (1.35), then the inequality above becomes an equality, which proves the second part of the statement. □

Now, we are ready to address the optimal investment and proportional reinsurance problem of our insurance company.

In the following theorem, we provide the analytic construction of a class of forward dynamic exponential utilities in order to describe the preferences of the company.

**Theorem 1.2.** *Let $t_0 \geq 0$ be the forward normalization point. Then, the process $\{U_t(x,t_0),\ t \geq t_0\}$, given for $x \in \mathbb{R}$ and $t \geq t_0$, by*

$$
U_t(x,t_0) = -e^{-\gamma x - P(t)}, \qquad (1.38)
$$

*with the process $\{P(t),\ t \geq t_0\}$ defined in (1.28), is a forward dynamic exponential utility, normalized at $t_0$.*

*Proof.* The proof consists in showing that the process $\{U_t(x,t_0),\ t \geq t_0\}$, introduced in (1.38), satisfies Definition 1.2 (equivalently, Definition 1.1 with the initial condition $u_0(x) = -e^{-\gamma x}$). Firstly, we see that $U_t(x,t_0)$ is $\mathcal{F}_t$-measurable for each $t \geq t_0$ and normalized at $t_0$, as the condition at $t = t_0$ is satisfied (i.e., $U_{t_0}(x,t_0) = -e^{-\gamma x}$). Next, we need to prove that for arbitrary $t, T$ such that $t_0 \leq t \leq T$,

$$-e^{-\gamma x - P(t)} = \max_{H\in\mathcal{A}} \mathbb{E}\left[-e^{-\gamma X_T^H - P(T)}\Big|\mathcal{F}_t\right]. \tag{1.39}$$

This means that for any self-financing strategy $H$ we get

$$-e^{-\gamma x - P(t)} \geq \mathbb{E}\left[-e^{-\gamma X_T^H - P(T)}\Big|\mathcal{F}_t\right],$$

and we will also show that there is a self financing strategy $H^* \in \mathcal{A}$ such that equality holds.

We notice that Equation (1.39) is equivalent to say that

$$-e^{-\gamma x} = \max_{H\in\mathcal{A}} \mathbb{E}\left[-e^{-\gamma X_T^H - (P(T)-P(t))}\Big|\mathcal{F}_t\right]. \tag{1.40}$$

We define the right-hand side of Equation (1.40) as

$$u(t,x,s,e_i) = \max_{H\in\mathcal{A}} \mathbb{E}_{t,x,s,e_i}\Big[-e^{-\gamma X_T^H - (P(T)-P(t))}\Big] = \max_{H\in\mathcal{A}} \mathbb{E}_{t,x,s,e_i}\Big[-e^{-\gamma X_T^H - \int_t^T g(r,S_r,Y_r)\mathrm{d}r}\Big], \tag{1.41}$$

for a function $u : [0,+\infty) \times \mathbb{R} \times (0,+\infty) \times \mathcal{E} \to (-\infty, 0)$. We proceed as follows.

**Step 1.** We first notice that

$$u(t,x,s,e_i) = e^{-\int_{t_0}^t g(r,S_r,Y_r)\mathrm{d}r} \max_{H\in\mathcal{A}} \mathbb{E}_{t,x,s,e_i}\Big[-e^{-\gamma X_T^H - \int_{t_0}^T g(r,S_r,Y_r)\mathrm{d}r}\Big].$$

Using the martingale property of the conditional expectation, if $u$ is sufficiently smooth (i.e., $u \in \mathcal{C}_b^{1,2,2}$), by Itô's formula and the product rule we get that $u$ solves the final value problem

$$\max_{H\in\mathcal{A}} \hat{\mathcal{L}}^H u(t,x,s,e_i) - g(t,s,e_i)u(t,x,s,e_i) = 0, \tag{1.42}$$

for all $(t,x,s,e_i) \in [t_0,T) \times \mathbb{R} \times (0,+\infty) \times \mathcal{E}$, with the final condition

$$u(T,x,s,e_i) = -e^{-\gamma x}, \quad (x,s,e_i) \in \mathbb{R} \times (t_0,+\infty) \times \mathcal{E}, \tag{1.43}$$

where we recall that $\hat{\mathcal{L}}^H$ denotes the infinitesimal generator of the Markov process $(X^H, S, Y)$ defined in (1.34) associated with a constant control $H$.

**Step 2.** Next, we choose $H^* = (\Pi^*, \Theta^*)$ such that $\Pi^*_t = \dfrac{\mu(Y_t)}{\gamma \sigma^2(Y_t) S_t^{2\beta}}$ and $\Theta^*_t = \overline{\Theta}(t, Y_t)$ as in Equation (1.31). We show that the function $u$ of the form

$$u(t, x, s, e_i) = u(x) = -e^{-\gamma x}, \quad x \in \mathbb{R}, \tag{1.44}$$

is the unique solution of the problem (1.42)–(1.43). Indeed, the function $u(x) = -e^{-\gamma x}$, with $x \in \mathbb{R}$, solves (1.42)–(1.43). To get uniqueness, we apply the Verification Theorem (see Theorem 1.1). We notice that conditions (*ii*) and (*iii*) of Theorem 1.1 are trivially satisfied since the function $u(x)$ does not depend on $s$ and $y$, and hence we just need to show that for every $t, T$ such that $t_0 \leq t \leq T$, we have

$$\mathbb{E}\Big[\int_t^{T\wedge\tau_n} \Big(e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l} \sigma(Y_r) S_r^\beta \Pi_r \frac{\partial u}{\partial x}(r, X_r^H, S_r, Y_r)\Big)^2 \mathrm{d}r\Big] < \infty,$$

$$\mathbb{E}\Big[\int_I \int_t^{T\wedge\tau_n} e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l} \Big| u\big(r, X_{r-}^H - (1-\Theta_{r-})z, S_r, Y_r\big)\big) - u(r, X_{r-}^H, S_r, Y_r) \Big|$$
$$\times \lambda(r, Y_{r-}) F(\mathrm{d}z)\mathrm{d}r\Big] < \infty,$$

for a suitable, non-decreasing sequence of random times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \tau_n = +\infty$. We define the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ by setting

$$\tau_n := \inf\Big\{t \geq t_0 : \; e^{\int_{t_0}^t |g(r,S_r,Y_r)|\mathrm{d}r} > n \vee \; X_t^H < -n\Big\}, \quad n \in \mathbb{N}.$$

Observe that, over the stochastic interval $[\![t_0, T\wedge\tau_n]\!]$ there is a finite value $\bar{n} \leq n$ such that $e^{\int_{t_0}^t |g(r,S_r,Y_r)|\mathrm{d}r} \leq \bar{n}$ and $X_t^H \geq -\bar{n}$, for all $t \in [\![t_0, T\wedge\tau_n]\!]$ (the existence of $\bar{n}$ is guaranteed from the fact that the process $\Big\{e^{\int_{t_0}^t |g(r,S_r,Y_r)|\mathrm{d}r}, t \geq t_0\Big\}$ is continuous and the process $X^H$ is the unique solution of Equation (1.27) and hence it does not explode in $[t_0, T]$). Since $\lim_{n\to+\infty} \tau_n = +\infty$, it holds that for large $n$, $T \wedge \tau_n = T$ and therefore $\bar{n}$ does not depend on $n$. Then, we get that

$$\begin{aligned}
&\mathbb{E}\left[\int_t^{T\wedge\tau_n} \Big(e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l} \Pi_r \sigma(Y_r) S_r^\beta \frac{\partial u}{\partial x}(r, X_r^H, S_r, Y_r)\Big)^2 \mathrm{d}r\right] \\
&= \mathbb{E}\left[\int_t^{T\wedge\tau_n} e^{-2\int_t^r g(l,S_l,Y_l)\mathrm{d}l} \Pi_r^2 \sigma^2(Y_r) S_r^{2\beta} \Big(\gamma e^{-\gamma X_r^H}\Big)^2 \mathrm{d}r\right] \\
&\leq \big(\bar{n}\gamma e^{\bar{n}\gamma}\big)^2 \max_{j=1,\dots,K} \sigma^2(e_j) \mathbb{E}\left[\int_t^T \Pi_r^2 S_r^{2\beta} \mathrm{d}r\right] < \infty,
\end{aligned}$$

since $\Pi$ is an admissible strategy. Moreover, we have that

$$\begin{aligned}
&\mathbb{E}\Big[\int_t^{T\wedge\tau_n}\int_I e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l}\Big|u(r,X^H_{r-}-(1-\Theta_{r-})z,S_r,Y_r)-u(r,X^H_{r-},S_r,Y_r)\Big| \\
&\times\lambda(r,Y_{r-})F(\mathrm{d}z)\mathrm{d}r\Big] \\
&=\mathbb{E}\left[\int_t^{T\wedge\tau_n}\int_I e^{-\int_t^r g(l,S_l,Y_l)\mathrm{d}l}e^{-\gamma X^H_{r-}}\left|e^{\gamma(1-\Theta_{r-})z}-1\right|\lambda(r,Y_{r-})F(\mathrm{d}z)\mathrm{d}r\right] \\
&\leq \bar{n}e^{\bar{n}\gamma}\mathbb{E}\left[\int_t^T\int_I e^{\gamma z}\lambda(r,Y_{r-})F(\mathrm{d}z)\mathrm{d}r\right] \\
&\leq \bar{n}e^{\bar{n}\gamma}\mathbb{E}\left[\int_t^T\lambda(r,Y_{r-})\mathrm{d}r\right]\int_I e^{\gamma z}F(\mathrm{d}z)<\infty,
\end{aligned}$$

since $I\subset[0,+\infty)$ is compact and the integrability condition (1.20) holds. Therefore, thanks to Theorem 1.1, the function $u(x)=-e^{-\gamma x}$ is the unique solution of the final value problem (1.42)–(1.43).

**Step 3.** The steps above prove that the value function $u(t,x,s,e_i)$ is given by $u(x)=-e^{-\gamma x}$. Hence, using the equality (1.41), we get that Equations (1.40) and (1.38) hold. Consequently, according to Definition 1.2, $U=\{U_t(x,t_0)=-e^{-\gamma x-P(t)},\ t\geq t_0\}$ is a forward dynamic exponential utility. □

Now, we characterize the optimal investment portfolio and the optimal reinsurance level for this family of forward dynamic exponential utilities in (1.38). Since it will be necessary to demonstrate that the optimal investment-reinsurance strategy is also admissible, according to Definition 1.3, let us state a preliminary result.

**Lemma 1.2.** *Let $T\geq 0$. Define the process $L=\{L_t,\ t\in[0,T]\}$ as*

$$L_t=e^{-\frac{1}{2}\int_0^t\frac{\mu^2(Y_r)}{\sigma^2(Y_r)S_r^{2\beta}}\mathrm{d}r-\int_0^t\frac{\mu(Y_r)}{\sigma(Y_r)S_r^{\beta}}\mathrm{d}W_r};$$

*then, $L$ is an $(\mathbb{F},\mathbf{P})$-martingale. Moreover, $L_T$ is the density of a probability measure $\widetilde{\mathbf{P}}$, equivalent to $\mathbf{P}$ on $\mathcal{F}_T$.*

*Proof.* For the ease of notation we now take $t_0=0$. The proof extends that of Theorem 2.3 in [42] to the regime-switching version of the CEV model. We summarize the main steps. Consider the couple $(Y,S)$ where $Y$ is a finite state Markov chain, with infinitesimal generator $Q$, and $S$ is a process with continuous trajectories. Consider the following equations

$$\mathrm{d}S_t=S_t\mu(Y_t)\mathrm{d}t+S_t^{1+\beta}\sigma(Y_t)\mathrm{d}\overline{W}_t$$

and

$$\mathrm{d}S_t = S_t^{1+\beta}\sigma(Y_t)\mathrm{d}\overline{W}_t,$$

where $\overline{W}$ is a Wiener measure. Next we denote by $\mathbf{P}$ the law of the couple $(Y, S)$, where $S$ satisfies the first equation, on the interval $[0, T]$ and by $\widetilde{\mathbf{P}}$ the law of the couple $(Y, S)$, where $S$ satisfies the first equation, on the interval $[0, T]$. Notice that the generator of the Markov chain $Y$ is the same under $\mathbf{P}$ and under $\widetilde{\mathbf{P}}$. Then, we can find a $\mathbf{P}$-Brownian motion $W$ and a $\widetilde{\mathbf{P}}$-Brownian motion $\widetilde{W}$, both independent of $Y$, such that

$$\mathrm{d}S_t = S_t\mu(Y_t)\mathrm{d}t + S_t^{1+\beta}\sigma(Y_t)\mathrm{d}W_t$$

and

$$\mathrm{d}S_t = S_t^{1+\beta}\sigma(Y_t)\mathrm{d}\widetilde{W}_t.$$

We denote by $\mathbb{F}^{Y,S}$ the filtration generated by the pair $(Y, S)$. Notice that, for instance, this coincides with the filtration generated by the processes $(Y, W)$, and moreover, because of independence of the couple $(Y, S)$ with the jump measure $m(\mathrm{d}t, \mathrm{d}z)$, describing the jumps of the claim process, we can extend our analysis to the whole filtration $\mathbb{F}$. The laws $\mathbf{P}$ and $\widetilde{\mathbf{P}}$ are measures on the product space $\mathcal{M} \times \mathcal{C}$, where $\mathcal{M}$ is the space of piecewise continuous functions of $[0, T]$ and $\mathcal{C}$ is the space of continuous functions on $[0, T]$. In order to show that $\mathbf{P}$ and $\widetilde{\mathbf{P}}$ are equivalent we define the sequence of stopping times

$$\eta_n = \inf\left\{t > 0 : \int_0^t S_r^{-2\beta}\mathrm{d}r \geq n\right\}.$$

Clearly, $\eta_n \to +\infty$ (since $0 \leq -\beta < 1$) and the density of $\widetilde{\mathbf{P}}$ with respect to $\mathbf{P}$ on $\mathcal{F}_{\eta_n \wedge T}$ is given by

$$L_{\eta_n \wedge T} = e^{-\frac{1}{2}\int_0^{\eta_n \wedge T} \frac{\mu^2(Y_t)}{\sigma^2(Y_t)S_t^{2\beta}}\mathrm{d}t - \int_0^{\eta_n \wedge T} \frac{\mu(Y_t)}{\sigma(Y_t)S_t^{\beta}}\mathrm{d}W_t}.$$

Because $\int_0^T S_t^{-2\beta}\mathrm{d}t < +\infty$ $\mathbf{P}$-a.s. and $\frac{\mu(Y_t)}{\sigma(Y_t)}$ is bounded for every $t \geq 0$, we have that $\widetilde{\mathbf{P}}$ is absolutely continuous with respect to $\mathbf{P}$ on $\mathcal{F}_T$. Conversely, we can repeat the same reasoning and use that $\int_0^T S_t^{-2\beta}\mathrm{d}t < +\infty$ $\widetilde{\mathbf{P}}$-a.s., which implies equivalence. Hence $L$ is a strictly positive martingale with $\mathbb{E}[L_T] = 1$.

We also observe that, the change of measure above does not alter the law (i.e., the infinitesimal generator) of the Markov chain $Y$ nor the law (i.e., the compensator) of the jump process $C$. Hence, $Y$ and $C$ have the same law under $\mathbf{P}$ and $\widetilde{\mathbf{P}}$. □

Next, we prove that the optimal investment-reinsurance policy associated to forward exponential utilities belongs to the set of admissible strategies $\mathcal{A}$.

**Proposition 1.1.** *Let $t_0 \geq 0$ be the forward normalization point. The optimal investment portfolio $\Pi_t^* = \Pi^*(t, S_t, Y_t)$ is given by*

$$\Pi^*(t, s, e_i) = \frac{\mu(e_i)}{\gamma \sigma^2(e_i) s^{2\beta}}, \tag{1.45}$$

*for every $(t, s, e_i) \in [t_0, +\infty) \times (0, +\infty) \times \mathcal{E}$. Assume that condition (1.33) holds for every $(t, e_i, \Theta) \in [t_0, +\infty) \times \mathcal{E} \times [0, 1]$. Then, the process $\Theta^* = \{\Theta_t^*, t \geq t_0\}$, where $\Theta_t^* = \overline{\Theta}(t, Y_t)$ and $\overline{\Theta}(t, e_j)$ is given in Equation (1.31), is the optimal reinsurance level.*

*Proof.* We observe that, because of the relation between the value process $U$ and the function $u(t, x, s, e_i)$ in (1.41), we can define the functions $\Psi^\Pi$ and $\Psi^\Theta$ as

$$\begin{aligned}
\Psi^\Pi(t, x, s, e_i, \Pi) &= \Pi \mu(e_i) \frac{\partial u}{\partial x}(t, x, s, e_i) + \frac{1}{2} \Pi^2 \sigma^2(e_i) s^{2\beta} \frac{\partial^2 u}{\partial x^2}(t, x, s, e_i) \\
&\quad + \Pi \sigma^2(e_i) s^{2\beta+1} \frac{\partial^2 u}{\partial x \partial s}(t, x, s, e_i) \\
\Psi^\Theta(t, x, s, e_i, \Theta) &= -b(t, e_i, \Theta) \frac{\partial u}{\partial x}(t, x, s, e_i) \\
&\quad + \lambda(t, e_i) \int_I \big(u\big(t, x - (1 - \Theta) z, s, e_i\big) - u(t, x, s, e_i)\big) \, F(\mathrm{d}z).
\end{aligned}$$

Then, for every $T \geq t_0$, the problem (1.42)–(1.43) can be written as

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t, x, s, e_i) + a(t, e_i) \frac{\partial u}{\partial x}(t, x, s, e_i) + \mu(e_i) s \frac{\partial u}{\partial s}(t, x, s, e_i) + \frac{1}{2} \sigma^2(e_i) s^{2\beta+2} \frac{\partial^2 u}{\partial s^2}(t, x, s, e_i) \\
&+ \sum_{j=1}^K u(t, x, s, e_i) q_{ij} - g(t, s, e_i) u(t, x, s, e_i) \\
&+ \max_{\Pi \in \mathbb{R}} \Psi^\Pi(t, x, s, e_i, \Pi) + \max_{\Theta \in [0,1]} \Psi^\theta(t, x, s, e_i, \Theta) = 0,
\end{aligned}$$

for all $(t, x, s, e_i) \in [t_0, T) \times \mathbb{R} \times (0, +\infty) \times \mathcal{E}$ with the final condition $u(T, x, s, e_i) = -e^{-\gamma x}$, for all $(x, s, e_i) \in \mathbb{R} \times (0, +\infty) \times \mathcal{E}$.

We start with the computation of the optimal investment strategy. Since $\Psi^\Pi(t, x, s, e_i, \Pi)$ is a polynomial function in $\Pi$, from the first and the second order conditions and the form of the function $u(t, x, s, e_i)$ in Equation (1.44), we get (1.45).

For the optimal reinsurance strategy, we apply a classical argument (see, e.g., [17] [Proposition 4.1]). Because of the assumptions on the function $b(t, e_i, \Theta)$ and the smoothness of function $u(t,x,s,e_i)$ in (1.44) with respect to $x$, $\Psi^\Theta$ is continuous in $\Theta \in [0,1]$ and twice continuously differentiable in $\Theta \in (0,1)$, for every $(t,x,s,e_i) \in [t_0,T] \times \mathbb{R} \times (0,+\infty) \times \mathcal{E}$, for all $T \geq t_0$, and its first and second partial derivatives are given by

$$\frac{\partial \Psi^\Theta}{\partial \Theta}(t,x,s,e_i,\Theta) = -\gamma e^{-\gamma x}\left\{\frac{\partial b}{\partial \Theta}(t,e_i,\Theta) - \lambda(t,e_i)\int_I e^{\gamma(1-\Theta)z} z F(\mathrm{d}z)\right\},$$

$$\frac{\partial^2 \Psi^\Theta}{\partial \Theta^2}(t,x,s,e_i,\Theta) = -\gamma e^{-\gamma x}\left\{\frac{\partial^2 b}{\partial \Theta^2}(t,e_i,\Theta) + \gamma\lambda(t,e_i)\int_I e^{\gamma(1-\Theta)z} z^2 F(\mathrm{d}z)\right\}.$$

By condition (1.11), $\Psi^\theta(t,x,s,e_i,\Theta)$ is also strictly concave in $\Theta \in [0,1]$, and hence it admits a unique maximizer $\Theta^* \in [0,1]$. Next we observe that, by concavity of $\Psi^\Theta$ with respect to $\Theta$, the function $\frac{\partial \Psi^\Theta}{\partial \Theta}(t,x,s,e_i,\Theta)$ is decreasing in $\Theta$ and it holds that

$$\frac{\partial \Psi^\Theta}{\partial \Theta}(t,x,s,e_i,1) \leq \frac{\partial \Psi^\Theta}{\partial \Theta}(t,x,s,e_i,\Theta) \leq \frac{\partial \Psi^\Theta}{\partial \Theta}(t,x,s,e_i,0), \tag{1.46}$$

for all $\Theta \in (0,1)$. Then, the following cases arise:

a. If $\Psi^\Theta$ is increasing in $\Theta \in [0,1]$, then the maximizer is realized for $\Theta^* = 1$.

b. If $\Psi^\Theta$ is decreasing in $\Theta \in [0,1]$, then the maximizer is realized for $\Theta^* = 0$.

c. If $\frac{\partial \Psi^\Theta}{\partial \Theta}(t,x,s,e_i,\hat{\Theta}) = 0$ for some $\hat{\Theta} \in [0,1]$, then $\Theta^* = \hat{\Theta}$.

We observe that $\Psi^\Theta$ is increasing if and only if $(t,e_i) \in \mathcal{D}_1$. Indeed, because of concavity of $\Psi^\Theta$ with respect to $\Theta$ (see (1.46)), we get that $\frac{\partial \Psi^\Theta}{\partial \Theta}(t,x,s,e_i,\Theta) > 0$ is equivalent to say that $\frac{\partial \Psi^\Theta}{\partial \Theta}(t,x,s,e_i,1) > 0$. This implies that $\Psi^\Theta$ is increasing if and only if $\frac{\partial b}{\partial \Theta}(t,e_i,1) \leq \lambda(t,e_i)\mathbb{E}[Z_1]$. Equivalently $\Psi^\Theta$ is increasing if and only if $(t,e_i) \in \mathcal{D}_0$, and finally $\frac{\partial \Psi^\Theta}{\partial \Theta}(t,x,s,e_i,\hat{\Theta}) = 0$ corresponds to solve Equation (1.32).

It only remains to show that the process $H^* = (\Pi^*,\Theta^*)$ is an admissible strategy. It is clear that $\Theta^*_t \in [0,1]$, for every $t \geq t_0$, and that $\Theta^*$ is $\mathbb{F}$-adapted and càdlàg, hence $\mathbb{F}$-progressively measurable. The investment strategy $\Pi^*$ is also $\mathbb{F}$-adapted and càdlàg (hence $\mathbb{F}$-progressively measurable), and for every $T \geq t_0$ it satisfies:

$$\begin{aligned}
&\mathbb{E}\left[\int_{t_0}^T \left(|\Pi^*_r| + (\Pi^*_r)^2 S_r^{2\beta}\right)\mathrm{d}r\right] \\
&= \mathbb{E}\left[\int_{t_0}^T \left(\left|\frac{\mu(Y_r)}{\gamma\sigma^2(Y_r)S_r^{2\beta}}\right| + \frac{\mu^2(Y_r)}{\gamma^2\sigma^4(Y_r)S_r^{2\beta}}\right)\mathrm{d}r\right] \leq c\mathbb{E}\left[\int_{t_0}^T S_r^{-2\beta}\mathrm{d}r\right] < \infty,
\end{aligned}$$

for some constant $c > 0$, where the first inequality here is implied by the boundedness of $\mu(Y_t)$ and $\sigma(Y_t)$, for every $t \in [t_0, T]$. To show that $\mathbb{E}\left[e^{-\gamma X_T^{H^*} - P(T)}\right] < \infty$, we observe that in view of (1.27), and recalling that $\Theta_t^* = \overline{\Theta}_t$, for every $t$, we have

$$\begin{aligned}
&-\gamma X_T^{H^*} - P(T) = -\gamma x_{t_0} - \frac{1}{2}\int_{t_0}^T \frac{\mu^2(Y_t)}{\sigma^2(Y_t) S_t^{2\beta}} \mathrm{d}t - \int_{t_0}^T \frac{\mu(Y_t)}{\sigma(Y_t) S_t^{\beta}} \mathrm{d}W_t^S \\
&+ \gamma \int_{t_0}^T \int_I (1 - \overline{\Theta}_{t-}) z m(\mathrm{d}t, \mathrm{d}z) - \int_{t_0}^T \lambda(t, Y_{t-}) \int_I \left(e^{\gamma(1-\overline{\Theta}_{t-})z} - 1\right) F(\mathrm{d}z)\mathrm{d}t,
\end{aligned}$$

where $X_{t_0}^H = x_{t_0} \in \mathbb{R}$. Then,

$$\begin{aligned}
\mathbb{E}\left[e^{-\gamma X_T^{H^*} - P(T)}\right] &= e^{-\gamma x_{t_0}} \mathbb{E}\left[e^{-\frac{1}{2}\int_{t_0}^T \frac{\mu^2(Y_t)}{\sigma^2(Y_t) S_t^{2\beta}} \mathrm{d}t - \int_{t_0}^T \frac{\mu(Y_t)}{\sigma(Y_t) S_t^{\beta}} \mathrm{d}W_t^S}\right. \\
&\left. \times e^{\gamma \int_{t_0}^T \int_I (1-\overline{\Theta}_{t-}) z m(\mathrm{d}t, \mathrm{d}z)} e^{-\int_{t_0}^T \lambda(t, Y_{t-}) \int_I \left(e^{\gamma(1-\overline{\Theta}_{t-})z} - 1\right) F(\mathrm{d}z)\mathrm{d}t}\right].
\end{aligned} \tag{1.47}$$

For $T > t_0$, we define the process $L = \{L_t,\ t \in [t_0, T]\}$ as

$$L_t = e^{-\frac{1}{2}\int_{t_0}^t \frac{\mu^2(Y_t)}{\sigma^2(Y_t) S_t^{2\beta}} \mathrm{d}t - \int_{t_0}^t \frac{\mu(Y_t)}{\sigma(Y_t) S_t^{\beta}} \mathrm{d}W_t^S};$$

then, $L$ is an (integrable) $(\mathbb{F}, \mathbf{P})$-martingale. Precisely, $L$ is an exponential martingale with expected value equal to 1 (see Lemma 1.2) and defines an equivalent change of probability measure, i.e., $L_T = \frac{\mathrm{d}\widetilde{\mathbf{P}}}{\mathrm{d}\mathbf{P}}\Big|_{\mathcal{F}_T}$. Moreover, the change of measure from $\mathbf{P}$ to $\widetilde{\mathbf{P}}$ does not modify the law of the Markov chain $Y$ and the compensator of the claim process $C$, since it only affects the Brownian motion $W$. This means that $Y$ and $C$ have the same law under $\mathbf{P}$ and under $\widetilde{\mathbf{P}}$. Equation (1.47) becomes:

$$\begin{aligned}
&e^{-\gamma x_{t_0}} \mathbb{E}\left[L_T e^{\gamma \int_{t_0}^T \int_I (1-\overline{\Theta}_{t-}) z m(\mathrm{d}t, \mathrm{d}z)} e^{-\int_{t_0}^T \lambda(t, Y_{t-}) \int_I \left(e^{\gamma(1-\overline{\Theta}_{t-})z} - 1\right) F(\mathrm{d}z)\mathrm{d}t}\right] \\
&= e^{-\gamma x_{t_0}} \widetilde{\mathbb{E}}\left[e^{\gamma \int_{t_0}^T \int_I (1-\overline{\Theta}_{t-}) z m(\mathrm{d}t, \mathrm{d}z)} e^{-\int_{t_0}^T \lambda(t, Y_{t-}) \int_I \left(e^{\gamma(1-\overline{\Theta}_{t-})z} - 1\right) F(\mathrm{d}z)\mathrm{d}t}\right] \\
&= e^{-\gamma x_{t_0}} \mathbb{E}\left[e^{\gamma \int_{t_0}^T \int_I (1-\overline{\Theta}_{t-}) z m(\mathrm{d}t, \mathrm{d}z)} e^{-\int_{t_0}^T \lambda(t, Y_{t-}) \int_I \left(e^{\gamma(1-\overline{\Theta}_{t-})z} - 1\right) F(\mathrm{d}z)\mathrm{d}t}\right],
\end{aligned}$$

where $\widetilde{\mathbb{E}}[\cdot]$ denotes the expected value computed under the probability measure $\widetilde{\mathbf{P}}$, and in the last equality we have used the fact that $Y$ and $C$ have the same law under $\mathbf{P}$ and under $\widetilde{\mathbf{P}}$. In particular,

$$e^{-\int_{t_0}^T \lambda(t, Y_{t-}) \int_I \left(e^{\gamma(1-\overline{\Theta}_{t-})z} - 1\right) F(\mathrm{d}z)\mathrm{d}t} \leq e^{\int_{t_0}^T \lambda(t, Y_{t-})\mathrm{d}t} \leq e^{\max_{j=1,\dots,K} \int_{t_0}^T \lambda(t, e_j)\mathrm{d}t} := c_T < \infty,\ \mathbf{P}\text{-a.s.}.$$

Finally,

$$
\begin{aligned}
&\mathbb{E}\left[e^{\gamma\int_{t_0}^T\int_I(1-\overline{\Theta}_{t-})zm(\mathrm{d}t,\mathrm{d}z)}e^{-\int_{t_0}^T\lambda(t,Y_t)\int_I\left(e^{\gamma(1-\overline{\Theta}_{t-})z}-1\right)F(\mathrm{d}z)\mathrm{d}t}\right]\\
&\leq c_T\mathbb{E}\left[e^{\gamma\int_{t_0}^T\int_I zm(\mathrm{d}t,\mathrm{d}z)}\right]=c_T\mathbb{E}\left[e^{\gamma\sum_{i=1}^{N_T}Z_i}\right]\\
&=c_T\sum_{n\geq 0}\mathbb{E}\left[e^{\gamma\sum_{i=1}^{N_T}Z_i}\middle|N_T=n\right]\mathbf{P}(N_T=n)=c_T\sum_{n\geq 0}\mathbb{E}\left[e^{\gamma Z_1}\right]^n\mathbf{P}(N_T=n)<\infty,
\end{aligned}
$$

which implies the assertion. □

**Remark 1.7.** *We stress that the optimal protection level* $\Theta^*$ *coincides with* $\overline{\Theta}$*, provided by* (1.31)*. Thus, considering a very general reinsurance premium described by the function* $b(t, e_j, \Theta)$*, condition* (1.33) *implies that the set* $\mathcal{D}_1$ *may be non-empty, and consequently that full reinsurance may be optimal for certain time periods and certain market conditions. From an economic point of view, we could say that if the reinsurance is sufficiently cheap, namely when the price of an infinitesimal protection is below a certain dynamic threshold, then full reinsurance is optimal. If instead the reinsurance premium is above a certain dynamic threshold, meaning that it is too much expensive, the best strategy is to not reinsure anything. Otherwise, that is when the cost lies in the middle, it is best to reinsure part of the claims and precisely the optimal protection level is provided by* (1.32)*, i.e. by equating the marginal reinsurance cost and the marginal gain.*

We notice that the optimal reinsurance level and the optimal investment portfolio do not depend on the normalization point $t_0$, which is consistent with the classical theory on forward dynamic utilities (see e.g. [81]). Moreover, we observe that a penalizing process with a different choice of $\bar{\Theta}$ would not lead to the same optimal protection level. Our choice is motivated by the fact that taking $\bar{\Theta}$ as in (1.31) means that the forward utility accounts for the amount of claims that are covered by the insurance (i.e. not reinsured claims), and hence represents a risk for the insurance company.

### 1.4.3 Comparison with the backward utility approach

Now, we examine the case of independent markets. Clearly, this is a simplification of the general framework considered above: there is no more dependence on the Markov chain in the CEV model since the exogenous factor does not influence the risky stock price. In this example we retrace several key characteristics that allow us to discuss some important differences between the forward and the standard backward performance criteria.

We consider an insurance framework as in Subsection 1.4.1. The financial market, instead consists of a riskless asset with price process $S_t^0 = 1$, for all $t \geq 0$, and a risky asset with price process $S$ whose drift and volatility are not affected by the factor $Y$, and hence its dynamics follows

$$\mathrm{d}S_t = S_t(\mu \mathrm{d}t + \sigma S_t^{\beta} \mathrm{d}W_t^S), \quad S_0 = s > 0,$$

with $\mu \in \mathbb{R}$ and $\sigma > 0$. The results can be easily extended to the case where drift and volatility are functions of time.

The wealth associated to a strategy $H = (\Pi, \Theta) \in \mathcal{A}$ is given by $X^H = \{X_t^H,\ t \geq t_0\}$ such that

$$\mathrm{d}X_t^H = \big\{a(t, Y_t) - b(t, Y_t, \theta_t) + \Pi_t \mu)\big\}\mathrm{d}t + \Pi_t \sigma S_t^{\beta} \mathrm{d}W_t^S - (1 - \Theta_{t-})\mathrm{d}C_t,$$

with $X_{t_0}^H = x_{t_0} \geq 0$ being the wealth at time $t_0$.

We can derive the optimal investment and reinsurance strategy $H^* = (\Pi^*, \Theta^*)$ under the forward dynamic exponential utility, which is given by $\Pi_t^* = \Pi^*(S_t)$ where

$$\Pi^*(s) = \frac{\mu}{\gamma \sigma^2 s^{2\beta}},$$

and $\Theta^*$ is given by Equation (1.31). The optimal value satisfies $U_t(x, t_0) = -e^{-\gamma x - P(t)}$, for all $t \geq t_0$ and $x \in \mathbb{R}$, where now the process $P(t)$ is given by

$$P(t) = \int_{t_0}^{t} \left( -\frac{1}{2} \frac{\mu^2}{\sigma^2 S_r^{2\beta}} - \gamma a(r, Y_r) + \varphi(r, Y_r) \right) \mathrm{d}r,$$

and we recall that the function $\varphi(t, e_i)$ is given in (1.30).

Next we compare the optimal strategies and the value processes arising under the forward and the standard backward utilities. We fix a time horizon $T > t_0$ which coincides with the end of the investment period, and consider the optimization problem (**Problem 2**)

$$\max_{H \in \mathcal{A}} \mathbb{E}\left[ -e^{-\gamma X_T^H} \right].$$

**Proposition 1.2.** *The optimal investment and reinsurance strategy* $H^{B,*} = (\Pi^{B,*}, \Theta^{B,*})$ *is given by*

$$\Pi^{B,*}(t, s) = \frac{\mu}{\gamma \sigma^2 s^{2\beta}} - \frac{2\beta J_1(t)}{\gamma \sigma s^{2\beta}} \tag{1.48}$$

*and* $\Theta^{B,*} = \overline{\Theta}$, *with* $\overline{\Theta}$ *provided in Equation* (1.31). *The optimal value function satisfies*

$$V(t,x,s,e_i) = -e^{-\gamma x - h^B(t,s,e_i)},$$

*where* $h^B(t,s,e_i) = J_1(t)s^{-2\beta} - J_2(t,e_i)$, *for every* $(t,x,s,e_i) \in [t_0,T] \times \mathbb{R} \times (0,+\infty) \times \mathcal{E}$. *The function* $J_1(t)$ *is given by* $J_1(t) = \dfrac{\mu^2}{2\sigma^2}(T-t)$, *for every* $t \in [t_0,T]$ *and the function* $J_2(t,e_i)$ *solves the following system of ODEs*

$$\begin{aligned}\frac{\mathrm{d}J_2}{\mathrm{d}t}(t,e_i) =& \gamma[a(t,e_i) - b(t,e_i,\overline{\Theta})] - \sum_{j=1}^{K} e^{J_2(t,e_j)-J_2(t,e_i)} q_{ij} + \frac{\mu^2}{2}\beta(2\beta+1)(T-t) \\ & - \lambda(t,e_i)\int_I \left(e^{\gamma(1-\overline{\Theta})z} - 1\right) F(\mathrm{d}z), \ t \in [t_0,T) \end{aligned} \tag{1.49}$$

*with the final condition* $J_2(T,e_i) = 0$, *for all* $i = 1,\dots,K$.

*Proof.* We notice that the optimization is taken over the set of admissible functions $\mathcal{A}$, even though in the backward case one would require that $\mathbb{E}\left[e^{-\gamma X_T^H}\right] < \infty$ in place of $\mathbb{E}\left[e^{-\gamma X_T^H - P(T)}\right] < \infty$. However, because of the assumptions on the model coefficients these two conditions are equivalent.

Suppose that the value function $V(t,x,s,e_i)$ is $\mathcal{C}^{1,2,2}$, then it solves the equation

$$\max_{H\in\mathcal{A}} \hat{\mathcal{L}}^H V(t,x,s,e_i) = 0, \quad (t,x,s,e_i) \in [0,T) \times \mathbb{R} \times (0,+\infty) \times \mathcal{E}, \tag{1.50}$$

where $\hat{\mathcal{L}}^H$ is the infinitesimal generator given in (1.34), with the terminal condition $V(T,x,s,e_i) = -e^{-\gamma x}$. We guess that the value function has the form $V(t,x,s,e_i) = -e^{-\gamma x - J_1(t)s^{-2\beta} + J_2(t,e_i)}$. Plugging this expression into (1.50) and taking the first order condition on $\Pi$ yields (1.48). The second order conditions guarantee that $\Pi^{B,*}$ is the optimal investment strategy. For the optimal reinsurance strategy $\Theta^{B,*}(t,e_i)$ we argue as in the proof of Proposition 1.1 and hence we get that $\Theta^{B,*}(t,e_i) = \Theta^*(t,e_i)$ given in Equation (1.31).

Next, we establish a verification result. Let $v(t,x,s,e_i)$ be a solution of the Equation (1.50) with the final condition $v(T,x,s,e_i) = -e^{-\gamma x}$ (that is $v(T,x,s,e_i) = V(T,x,s,e_i)$). Then, by Itô's formula it holds that (for simplicity, we omit the dependence of $X$ on the strategy

$H$)

$$v(T,X_T,S_T,Y_T)=v(t,x,s,e_i)+\int_t^T\hat{\mathcal{L}}^H v(r,X_r,S_r,Y_r)\mathrm{d}r$$
$$+\int_t^T\Pi_r\sigma(Y_r)S_r^{\beta}\frac{\partial v}{\partial x}(r,X_r,S_r,Y_r)\mathrm{d}W_r^S+\int_t^T\sigma(Y_r)S_r^{\beta+1}\frac{\partial v}{\partial s}(r,X_r,S_r,Y_r)\mathrm{d}W_r^S$$
$$+\int_t^T\sum_{j=1}^K\Big\{v(r,X_r,S_r,e_j)-v(r,X_r,S_r,Y_{r-})\Big\}\left(m^Y-\nu^Y\right)(\mathrm{d}r,\{e_j\})$$
$$+\int_t^T\int_I\{v(r,X_{r-}-(1-\Theta_{r-})z,S_r,Y_r)-v(r,X_{r-},S_r,Y_r)\}(m(\mathrm{d}r,\mathrm{d}z)-\lambda(r,Y_{r-})F(\mathrm{d}z)\mathrm{d}r).$$

Since $v$ satisfies Equation (1.50), we get that

$$\begin{aligned}v(T,X_T,S_T,Y_T)&\le v(t,x,s,e_i)\\&+\int_t^T\Pi_r\sigma(Y_r)S_r^{\beta}\frac{\partial v}{\partial x}(r,X_r,S_r,Y_r)\mathrm{d}W_r^S+\int_t^T\sigma(Y_r)S_r^{\beta+1}\frac{\partial v}{\partial s}(r,X_r,S_r,Y_r)\mathrm{d}W_r^S\\&+\int_t^T\sum_{j=1}^K\{v(r,X_r,S_r,e_j)-v(r,X_r,S_r,Y_{r-})\}\left(m^Y-\nu^Y\right)\left(\mathrm{d}r,\{e_j\}\right)\\&+\int_t^T\int_I\{v(r,X_{r-}-(1-\Theta_{r-})z,S_r,Y_r)-v(r,X_{r-},S_r,Y_r)\}\\&\quad\times\left(m(\mathrm{d}r,\mathrm{d}z)-\lambda(r,Y_{r-})F(\mathrm{d}z)\mathrm{d}r\right).\end{aligned}\tag{1.51}$$

Let

$$M_t=\int_{t_0}^t\Pi_r\sigma(Y_r)S_r^{\beta}\frac{\partial V}{\partial x}(r,X_r,S_r,Y_r)\mathrm{d}W_r^S+\int_{t_0}^t\sigma(Y_r)S_r^{\beta+1}\frac{\partial V}{\partial s}(r,X_r,S_r,Y_r)\mathrm{d}W_r^S$$
$$+\int_{t_0}^t\sum_{j=1}^K\Big\{V(r,X_r,S_r,e_j)-V(r,X_r,S_r,Y_{r-})\Big\}\left(m^Y-\nu^Y\right)\left(\mathrm{d}r,\{e_j\}\right)$$
$$+\int_{t_0}^t\int_I V(r,X_{r-}-(1-\Theta_{r-})z,S_r,Y_r)-V(r,X_{r-},S_r,Y_r)\left(m(\mathrm{d}r,\mathrm{d}z)-\lambda(r,Y_{r-})F(\mathrm{d}z)\mathrm{d}r\right).$$

If $M$ is an $(\mathbb{F},\mathbf{P})$-martingale, then taking the conditional expectation given $X_t=x$, $S_t=s$, $Y_t=e_i$ on both sides of inequality (1.51) yields

$$V(t,x,s,e_i)\le v(t,x,s,e_i),$$

and the equality holds if $H$ is a maximizer of Equation (1.50). Then, it only remains to prove that the function $V(t,x,s,e_i)=-e^{-\gamma x-J_1(t)s^{-2\beta}+J_2(t,e_i)}$ is such that the process $M$ is an

$(\mathbb{F}, \mathbf{P})$-martingale. To this aim, observe that $J_1(t)$ and $J_2(t, e_i)$ are both bounded in $[t_0, T]$ and we consider the localizing sequence of random times $\{\widetilde{\tau}_n\}_{n\in\mathbb{N}}$ defined as

$$\widetilde{\tau}_n := \inf\Big\{t \geq t_0 : \ S_t^{-2\beta} > n, \ X_t < -n\Big\}, \quad n \in \mathbb{N}.$$

Then, $\{\widetilde{\tau}_n\}_{n\in\mathbb{N}}$ is an increasing sequence, $\lim_{n\to\infty} \widetilde{\tau}_n \wedge T = T$ and hence we get that

$$\begin{aligned}
&\mathbb{E}\bigg[\int_{t_0}^{T\wedge\widetilde{\tau}_n} \gamma^2\sigma^2\Pi_r^2 S_r^{2\beta} V^2(r, X_r, S_r, Y_r)\mathrm{d}r\bigg] \\
&+\mathbb{E}\bigg[\int_{t_0}^{T\wedge\widetilde{\tau}_n} 4\beta^2\sigma^2 J_1^2(r) S_r^{-2\beta} V^2(r, X_r, S_r, Y_r)\mathrm{d}r\bigg] \\
&+\mathbb{E}\bigg[\int_{t_0}^{T\wedge\widetilde{\tau}_n} \Big|V(r, X_r, S_r, Y_{r-})\Big| \sum_{j=1}^{K} \Big(e^{J_2(r,e_j)-J_2(r,Y_{r-})} - 1\Big)\, \nu^Y(\mathrm{d}r, \{e_j\})\bigg] \\
&+\mathbb{E}\bigg[\int_{t_0}^{T\wedge\widetilde{\tau}_n} \lambda(r, Y_{r-})\Big|V(r, X_{r-}, S_r, Y_r)\Big| \max_{z\in I} \Big(e^{\gamma(1-\Theta_r)z} - 1\Big)\, \mathrm{d}r\bigg] < \infty,
\end{aligned}$$

which concludes the proof. □

Notice that applying the transformation $\widetilde{J}(t, e_i) = e^{J_2(t,e_i)}$, Equation (1.49) can be reduced to a linear ODE with a final condition, which has a unique solution.

Now, we comment on some differences with the foward approach.

First of all, the standard backward approach requires that a utility function to be valid at some future time $T$, is specified today, as soon as the company enters the market. Instead, in the forward approach, the utility is set to hold at the initial time, in relation to the available information, and may be updated as time goes by since the company usually modifies its preferences due to changes in market conditions or in its personal attitudes. In this sense the word "forward" is used: under forward preferences, the company acts in the same direction of time, and therefore it may capture information about the market in a dynamic and consistent way.

Next, we see that the forward and the backward problems share the same optimal reinsurance strategy. Instead, the optimal investment strategies are different in the two approaches, with the backward portfolio being always smaller than the forward one. In the forward case, we observe that the optimal strategy consists of the myopic component only, whereas in the backward case there is an additional component (always negative) which reflects the fact that the instantaneous variance of the percentage asset price change is not constant.

As for the value processes under backward and forward utility preferences, we note that they do not coincide in general, as argued also in Musiela and Zariphopoulou [81]. In some sense, they are similar: in both cases the value processes are of exponential type and they are affine in the wealth. However, that they are generated in completely different ways, as it results from the different multiplicative component which involves the function $h^B$ or the process $P$. Specifically, in the backward context the value function accounts for market incompleteness by estimating the future changes, via the function $h^B$. Instead, in the forward case the value function coincides with the forward utility at all intermediate times and it is adjusted dynamically over time, according to the arrival of new information. This is encompassed in the function $h$.

### 1.4.4 Numerical experiments

In this part, we resort to a numerical approach in order to get qualitative characteristics of optimal investment and reinsurance strategies implied by our model. We also provide few illustrations on the case of independent markets, making a comparison analysis with classical results obtained via backward utility preferences.

We have seen that the behavior of an interdependent insurance-financial market can be modeled by a finite number of regimes, each with its specific parameters. To simplify the economic interpretation and analysis, let us suppose that the exogenous index is described by a two-state Markov chain $Y$, that is, $\mathcal{E} = \{e_1, e_2\}$; without loss of generality we may assume that $e_1$ represents a more favorable state of the combined market and $e_2$ is a less favorable state. For instance, regime $e_1$ might be a market with good conditions, that is a market in which few claims occur and asset prices are expected to rise. On the other hand, with regime $e_2$ we may consider a market under bad conditions, such as the arrival of so many claims and the fall of financial assets. In the following, we refer to $e_1$ (respectively, $e_2$) as the *good* (respectively, *bad*) state.

The infinitesimal generator matrix $Q$ has entries $\{q_{ij}\}_{i,j\in\{1,2\}}$ such that $q_{12} > q_{21}$: this choice suggests that it is more likely for the market to switch from the good state to the bad state than the opposite.

For the sake of simplicity, we consider a claims arrival intensity of exponential type, i.e.,

$$\lambda(t, e_j) = \lambda_0 e^{k_1 t + k_2(e_j)}, \tag{1.52}$$

where $\lambda_0 = e^{Y_0}$, $k_1 > 0$, for every $t \in [0, +\infty)$, and the function $k_2(e_j) = j \cdot k_2$ for all $j = 1, 2$ and some $k_2 > 0$. Moreover, claim size distribution is assumed to be truncated exponential. We assume that insurance and financial operations take place in one year, starting from today (i.e., $t_0 = 0$); this means that we analyze our theoretical results in a time interval $[0, T]$, with $T = 1$. Insurance and reinsurance premia are computed according to the intensity-adjusted variance principle (see [17]), and hence, they are specifically given by

$$b(t, e_j, \Theta) = \lambda(t, e_j)\mathbb{E}\left[Z_1\right]\Theta + 2\delta_R \lambda(t, e_j)\mathbb{E}\left[Z_1^2\right]\left(1 + T\lambda(t, e_j)\right)\Theta^2, \tag{1.53}$$

$$a(t, e_j) = \lambda(t, e_j)\mathbb{E}\left[Z_1\right] + 2\delta_I \lambda(t, e_j)\mathbb{E}\left[Z_1^2\right]\left(1 + T\lambda(t, e_j)\right), \tag{1.54}$$

for $j = 1, 2$, and $\delta_R > 0$ and $\delta_I > 0$ denote the reinsurance and insurance safety loading, respectively. In Equations (1.53) and (1.54) we reported $T$ to underline the dependence on contracts maturity, which will be omitted later, plugging $T = 1$. Finally, we set the following parameter values to $q_{12} = 2$, $q_{21} = 1$, $k_1 = 0.5$, $k_2 = 1$ and we fix the insurance and the reinsurance safety loading to $\delta_I = 0.05$ and $\delta_R = 0.1$, respectively.

**Dependent Markets**

We consider the general financial market which consists of a locally risk-free asset $S^0$ with zero interest rate and a risky asset $S$ which follows a CEV model with drift and volatility that depend on the Markov chain $Y$ as described by the SDE (1.25). According to our interpretation of regimes $e_1$ and $e_2$, we assume that $\mu_1 > \mu_2$ and $\sigma_1 < \sigma_2$, where $\mu_j$ and $\sigma_j$ represent the expected rate of return and the volatility of the stock, respectively, in the $j$-th regime, for $j = 1, 2$. In fact, it is reasonable to associate to a good state for the combined market a higher appreciation rate and eventually smaller fluctuations, and viceversa lower rate of return and larger volatility to the bad state. This mechanism is well known in economics (see, e.g., French et al. [52] and Hamilton and Gang [57] that find evidence of these relationships between the regime of the market and the financial coefficients, using empirical data). We report the parameters choice for the financial coefficients $\mu_j$ and $\sigma_j$, for $j = 1, 2$ in Table 1.1. Our framework, however, also involves the actuarial market and the interpretation of

| **Regime** | $\boldsymbol{\mu}$ | $\boldsymbol{\sigma}$ |
|---|---|---|
| $e_1$ (good) | 0.1 | 0.1 |
| $e_2$ (bad) | 0.05 | 0.2 |

Table 1.1: Parameter set for the rate of return and the volatility of the stock price in the two market regimes.

the Markov chain is not of a purely economic nature, but may also incorporate reactions to events, such as natural disasters, pandemics or even climate and environmental states, which have an impact on both insurance losses and the general trend of financial assets. Equation (1.52), for instance, shows that the common factor $Y$ affects the claim arrival intensity in a way that the average number of claims is smaller in the good state and larger in the bad state.

To illustrate the typical sample path of an optimal strategy, we provide in Figure 1.1 the plot of one trajectory of the optimal dynamic investment and reinsurance strategy given by Proposition 1.1. We observe that both the investment portfolio and the protection level depend on different states and exhibit jumps at switching times of the Markov chain. Moreover, if the good regime is in force, then the insurance company opts to invest more in the risky stock and to reinsure a greater percentage of losses. The model specification considered in this example (i.e., the form of the intensity function, the claim size distribution and the reinsurance premium) implies that the optimal reinsurance level is given by $\Theta^*_t = \hat{\Theta}_t$ for all $t \in [0,1]$, where $\hat{\Theta}_t = \hat{\Theta}(t, e_j)$ is the solution of the equation

$$\mathbb{E}\left[Z_1\right] + 4\delta_R \theta \mathbb{E}\left[Z_1^2\right]\left(1 + \lambda(t, e_j)\right) = \int_{\mathcal{Z}} z e^{\gamma(1-\Theta)z} F(\mathrm{d}z).$$

We observe that the optimal reinsurance protection level decreases linearly piecewise. Indeed, using the *Implicit function theorem*, we check that the derivative of $\Theta^*$ with respect to time is negative for every $t$. Let $G(t, e_j, \Theta) = \mathbb{E}\left[Z_1\right] + 4\delta_R \Theta \mathbb{E}\left[Z_1^2\right]\left(1 + T\lambda(t, e_j)\right) - \int_I z e^{\gamma(1-\Theta)z} F(\mathrm{d}z)$, then, for every fixed $j = 1, \dots, K$ we have that

$$\frac{\mathrm{d}\Theta(t, e_j)}{\mathrm{d}t} = -\frac{\frac{\mathrm{d}G(t, e_j, \Theta(t, e_j))}{\mathrm{d}t}}{\frac{\mathrm{d}G(t, e_j, \Theta(t, e_j))}{\mathrm{d}\Theta}} = -\frac{4\Theta(t, e_j)\delta_R \mathbb{E}\left[Z_1^2\right]\lambda_0 k_1 e^{k_1 t + k_2(e_j)}}{4\delta_R \mathbb{E}\left[Z_1^2\right]\left(1 + e^{k_1 t + k_2(e_j)}\right) + \gamma \int_I z^2 e^{\gamma(1-\Theta(t, e_j))z} F(\mathrm{d}z)} < 0.$$

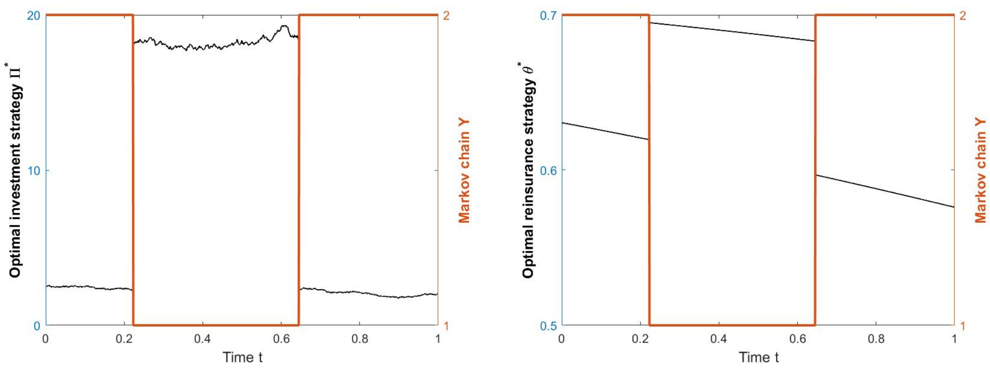



Figure 1.1: The optimal investment strategy (left panel) and the optimal reinsurance strategy (right panel), as functions of time, with parameters $S_0 = 1$, $\beta = -0.5$ and $\gamma = 0.5$.

In the sequel, we perform a sensitivity analysis of the optimal investment portfolio in order to study the effect of model coefficients on the insurance company decision, in both economic regimes. Specifically, our aim is to analyze how sensitive the optimal investment strategy $\Pi^*$ is to any change of the risk aversion $\gamma$ and the elasticity $\beta$, which are two parameters characterizing our market model.

First, in Figure 1.2, we analyze the effect of the elasticity coefficient $\beta$ on the optimal investment strategy at a certain date $t^* \in [0, 1]$. The left panel illustrates the situation when the price of the stock is smaller than 1. In this case, we notice that if $\beta$ grows up, then an increasing portion of the company's wealth is invested in the risky asset; this means that the optimal investment is positively correlated to the parameter of elasticity. Otherwise, the amount invested in the risky asset decreases as long as $\beta$ increases, as shown in the right panel. Further, it is worth noting that the strategy is more sensitive to variations of the elasticity parameter when the combined insurance-financial market is in the good regime (solid lines). Furthermore, we notice that the investment policy is always more aggressive if the market conditions are good.

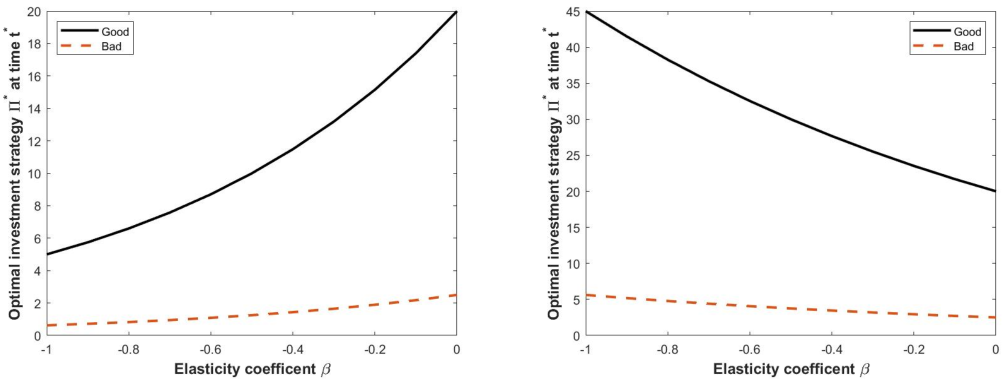



Figure 1.2: Optimal investment strategy at a fixed time $t^*$ as a function of elasticity coefficient $\beta$, for different values of stock price $S_{t^*}$, when the economic regime is $e_1$ (solid line) or $e_2$ (dashed line). Parameter values: $\gamma = 0.5$, $S_{t^*} = 0.5$ (left panel) and $S_{t^*} = 1.5$ (right panel).

Second, we are interested in knowing how much the company's risk aversion $\gamma$ affects the optimal portfolio. From Figure 1.3, we observe that the optimal investment is negatively correlated to the risk aversion parameter, under both regimes. As expected by (1.45), there is an inverse relationship on the values of the stochastic volatility; in other terms, if the risk aversion increases, then the insurance company finds it more convenient to invest in the risk-free asset. As before, we observe that if the bad regime is in force (dashed lines), the optimal investment policy is less aggressive and less affected by the coefficient changes.

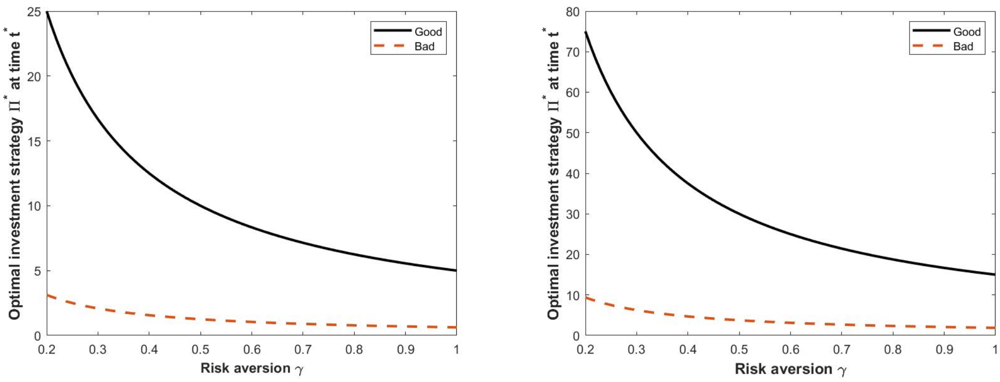



Figure 1.3: Optimal investment strategy at a fixed time $t^*$ as a function of risk aversion coefficient $\gamma$, for different values of stock price $S_{t^*}$ and constant elasticity coefficient $\beta$, when the market state is $e_1$ (solid line) or $e_2$ (dashed line). Parameter values: $\beta = -0.5$, $S_{t^*} = 0.5$ (**left** panel) and $S_{t^*} = 1.5$ (**right** panel).

**Independent Markets**

Now, we focus on the case of independent markets discussed in Section 1.4.3, in order to compare numerically the forward approach with the backward one.

For the numerical analysis below, we take $Y$ to be a two-state Markov chain, and let $(\bar{p}, 1-\bar{p})$ denote the stationary distribution of $Y$, i.e., $\bar{p} = \frac{q_{21}}{q_{12}+q_{21}}$. We calculate the appreciation rate and the volatility of the stock price, $\mu$ and $\sigma$, as the average of the values $\mu_1, \mu_2$ and $\sigma_1, \sigma_2$, according to the stationary distribution of $Y$, that is $\mu = \bar{p}\mu_1 + (1-\bar{p})\mu_2$ and $\sigma = \bar{p}\sigma_1 + (1-\bar{p})\sigma_2$.

We recall that insurance and reinsurance premia are evaluated via the intensity-adjusted variance principle and that the claim size distribution is exponential with expectation equal to 1.

In Figure 1.4 and Figure 1.5 we plot the difference between the optimal strategies under forward and backward utilities. Precisely, we plot that difference as a function of time in Figure 1.4, as a function of the elasticity parameter $\beta$ in the left panel of Figure 1.5 and as a function of the risk aversion coefficient $\gamma$ in the right panel of Figure 1.5. Thanks to Proposition 1.1 and Proposition 1.2, the optimal backward portfolio, in addition to the myopic component, has another term due to the fact that the corresponding value function is not updated dynamically over time. Indeed, by (1.48), it is clear that the optimal forward investment strategy $\Pi^*$ is more aggressive than the backward one $\Pi^{B,*}$. Figure 1.4 also shows that the difference between optimal strategies decreases over the time interval and it disappears at the end of trading horizon. Moreover, we get that the higher the initial price of the risky stock is, the higher the initial gap. A similar behavior is observed in the left and in the right panel of Figure 1.5, where we illustrate the difference in optimal initial portfolios with respect to the elasticity coefficient and the risk aversion parameter, respectively, for different initial values of the stock price.

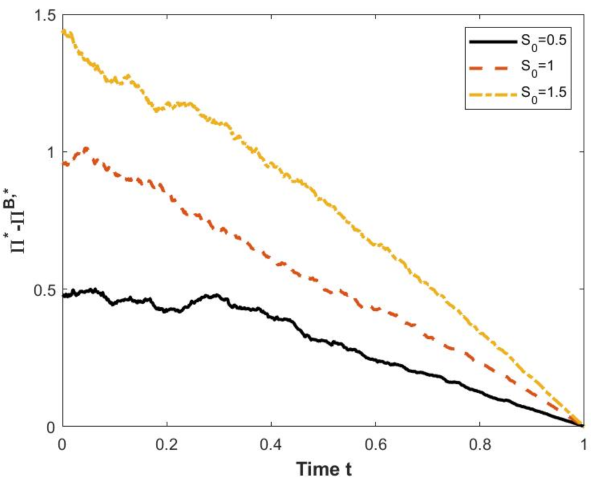
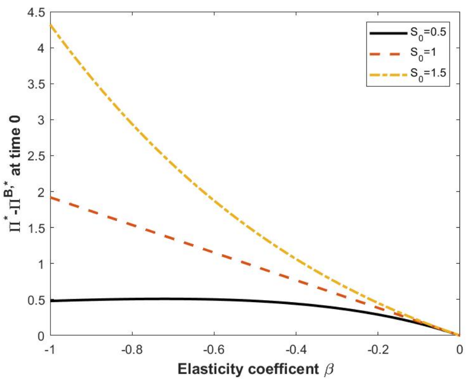



Figure 1.4: One trajectory of the optimal investment as a function of time for $\beta = -0.5$ and $\gamma = 0.5$.



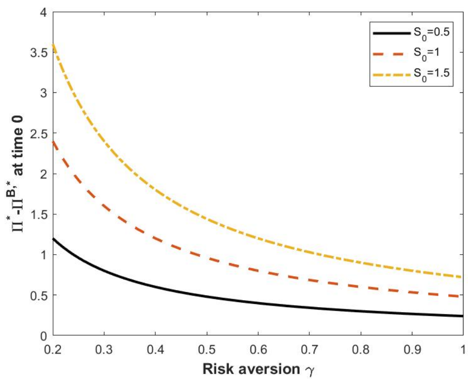




Figure 1.5: Left panel: Optimal investment as a function of elasticity coefficient at time 0 for $\gamma = 0.5$. Right panel: Optimal investment as a function of risk aversion parameter at time 0 for $\beta = -0.5$.

We conclude, comparing optimal value functions with respect to the stock price at the beginning of the trading interval, under both market regimes. Indeed, the optimal strategies under the forward and the backward criterion lead to different value functions. In particular, at the initial time, the optimal value corresponding to the backward utility is given by $V(0, x, s, e_i) = -e^{-\gamma x - J_1(0)s^\beta + J_2(0,e_i)}$, whereas the optimal value in the forward utility simply satisfies $U(x, 0) = -e^{-\gamma x}$. Figure 1.6 plots the difference between value functions at the initial time (in percentage), i.e., $\Delta_{VU}(s, e_i) := \frac{V(0,x,s,e_i)-U(x,0)}{U(x,0)}$ (notice that this quantity is independent of the initial wealth), as functions of the initial stock price, in states $e_1$ and $e_2$, assuming that the Markov chain $Y$ has only two states.

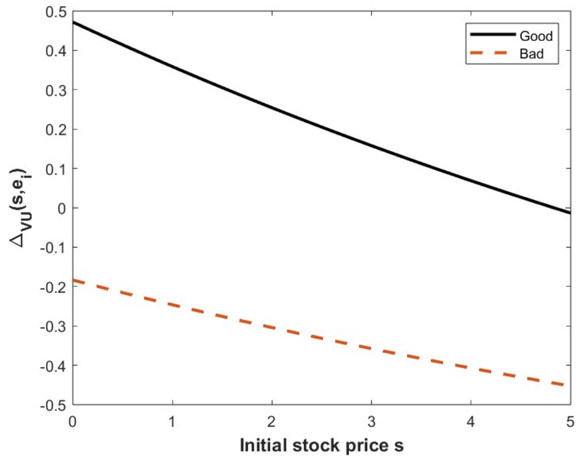



Figure 1.6: The effect of stock price on the difference between the backward optimal value functions and the forward one (in percentage) at time 0, when the market state is $e_1$ (solid line) or $e_2$ (dashed line). Parameter values: $\beta = -0.5$, $\gamma = 0.5$

We point out that the gap between the backward and the forward values at initial time is decreasing with respect to the stock price at $t = 0$, in both economic regimes. As a consequence, when market conditions are good, the difference between the two value functions decreases as the price of the risky asset increases.

## 1.5 Optimal investment and reinsurance under exponential forward preferences with non-zero volatility

In this section, we study the optimal investment and reinsurance problem of an insurance company, following a forward-looking approach, in a stochastic factor model allowing for a mutual dependence between the actuarial and the financial markets. Specifically, the common stochastic factor is modeled as a diffusion process. Moreover, the preferences of the company are described by a non-zero volatility forward dynamic utility of exponential type. Now, the penalizing process that reflects the insurance company dynamic preferences, depends on market coefficients, collected premia and paid premia but it is also linked to another source of risk which affects the interdependent market model. We provide an analytical construction of forward dynamic exponential utilities and we characterize the optimal reinsurance and investment strategy. Finally, we analyze a dynamic version of the conditional certainty equivalent (in short CCE) for forward utility preferences and we also compare it with the conditional certainty equivalent in the backward setting, both analytically and numerically.

### 1.5.1 Setting

In this section, the stochastic factor that affects the market model is described by a process $Y = \{Y_t,\ t \geq 0\}$ that solves the following SDE

$$\mathrm{d}Y_t = \alpha(t, Y_t)\mathrm{d}t + \beta(t, Y_t)\mathrm{d}W_t^Y, \quad Y_0 = y_0 \in \mathbb{R}, \tag{1.55}$$

where, $W^Y = \{W_t^Y,\ t \geq 0\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P}; \mathbb{F})$ and $\alpha, \beta : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are two measurable functions. We assume that there exists a unique strong solution to the SDE (1.55) such that

$$\mathbb{E}\left[\int_0^t |\alpha(s, Y_s)|\mathrm{d}s\right] < \infty, \qquad \mathbb{E}\left[\int_0^t \beta^2(s, Y_s)\mathrm{d}s\right] < \infty, \tag{1.56}$$

for every $t \geq 0$. For completeness, a set of classical sufficient conditions for uniqueness is given in Assumption 1.4, in Appendix 1.5.6.

The financial market consists of a risk-free asset with price process $S^0 = \{S_t^0,\ t \geq 0\}$ which is equal to 1 at any time and by a risky asset whose price process $S = \{S_t,\ t \geq 0\}$ is given by the following SDE

$$\mathrm{d}S_t = \mu(t, Y_t)S_t\mathrm{d}t + \sigma(t, Y_t)S_t\mathrm{d}W_t^S, \quad S_0 = s > 0, \tag{1.57}$$

where the process $W^S = \{W_t^S,\ t \geq 0\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P}; \mathbb{F})$, correlated with $W^Y$ with constant correlation coefficient $\rho \in [-1, 1]$, and independent of the random measure $m(\mathrm{d}t, \mathrm{d}z)$. Also here, the functions $\mu : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0, +\infty) \times \mathbb{R} \to (0, +\infty)$, representing the drift and the volatility of the stock price process, respectively, are assumed to be measurable and such that the system of equations (1.55)-(1.57) admits a unique strong solution. Hence, the pair $(Y, S)$ is a Markov process. Sufficient conditions for existence and uniqueness of the solution to the system (1.55)-(1.57) can be found, e.g. in [84, Theorem 5.2.1], and recalled in Appendix 1.5.6.

In this context, the Novikov condition required in (1.6) can be written as

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^t \left(\frac{\mu(s,Y_s)}{\sigma(s,Y_s)}\right)^2 \mathrm{d}s}\right] < \infty,$$

for every $t \geq 0$.

We notice that in our framework there is a possible double dependence between the financial and the insurance markets, since two different interactions between them are involved. The first one is realized by assuming that the claim arrival intensity, the claim amount distribution and the financial market coefficients (namely the appreciation rate and the volatility of the stock price) are functions of a common stochastic factor, the process $Y$. Indeed, exogenous events of different nature (such as social, cultural, geographical conditions, political decisions, natural events) may affect the claims that an insurance company experiences (their arrival frequency, their average number, their size and so on), as well as the performance of portfolios negotiated in the market. The second kind of interaction is due to the non-zero correlation between the Brownian motions $W^S$ and $W^Y$ driving the stock price and the factor process dynamics, respectively. This link between the two sources of noise can be viewed as an environmental contagion effect. We point out that, even if the correlation coefficient is zero, there remains an indirect dependence via $Y$.

We recall that for any given strategy $H = (\Theta, \Pi) \in [0,1] \times \mathbb{R}$, the wealth process $X^H = \{X_t^H,\ t \geq 0\}$ of the insurance company satisfies the SDE

$$\mathrm{d}X_t^H = \big\{a(t,Y_t) - b(t,Y_t,\Theta_t) + \Pi_t \mu(t,Y_t)\big\}\mathrm{d}t + \Pi_t \sigma(t,Y_t)\mathrm{d}W_t^S - (1-\Theta_{t-})\mathrm{d}C_t, \tag{1.58}$$

with $X_0^H = x_0 \geq 0$, being the initial wealth. Equivalently,

$$\begin{aligned} X_t^H = x_0 &+ \int_0^t \left(a(s,Y_s) - b(s,Y_s,\Theta_s) + \Pi_s \mu(s,Y_s)\right) \mathrm{d}s + \int_0^t \Pi_s \sigma(s,Y_s)\mathrm{d}W_s^S \\ &- \int_0^t \int_I (1-\Theta_{s-}) z m(\mathrm{d}s,\mathrm{d}z), \end{aligned} \tag{1.59}$$

for every $t \geq 0$.

The pair $(X^H, Y)$ is a Markov process with infinitesimal generator

$$\begin{aligned} &\tilde{\mathcal{L}}^H f(t,x,y) \\ &= \frac{\partial f}{\partial t}(t,x,y) + \big[a(t,y) - b(t,y,\Theta) + \Pi\mu(t,y)\big]\frac{\partial f}{\partial x}(t,x,y) \\ &+ \frac{1}{2}\Pi^2\sigma^2(t,y)\frac{\partial^2 f}{\partial x^2}(t,x,y) + \alpha(t,y)\frac{\partial f}{\partial y}(t,x,y) + \frac{1}{2}\beta^2(t,y)\frac{\partial^2 f}{\partial y^2}(t,x,y) \\ &+ \rho\Pi\sigma(t,y)\beta(t,y)\frac{\partial^2 f}{\partial x \partial y}(t,x,y) \\ &+ \lambda(t,y)\int_I \Big\{f\big(t, x-(1-\Theta)z, y\big) - f(t,x,y)\Big\}F(t,y,\mathrm{d}z), \end{aligned} \tag{1.60}$$

for every function $f:[0,+\infty)\times\mathbb{R}^2\to\mathbb{R}$ in $C^{1,2}$ which is sufficiently integrable and any given constant control $H=(\Theta,\Pi)\in[0,1]\times\mathbb{R}$, for every $(t,x,y)\in[0,+\infty)\times\mathbb{R}^2$.

We wish to maximize the profit of our insurance company that updates its risk profile forward in time. Since we set the normalization point $t_0=0$, in the sequel we can omit the dependence on the normalization point and we simply write $U_t(x)$ in place of $U_t(x,0)$, for notational convenience. In the literature, forward dynamics utilities with $t_0=0$ are often called *spot utilities*.

We consider a penalizing process $P$ given by (1.8) and we assume that Brownian motions $W^P$ and $W^Y$ are correlated, denoting by $\rho^Y\in[-1,1]$ the correlation coefficient. Moreover, the function $h$ is not necessarily equal to zero; in this way we manage to capture also the randomness coming from the stock price $S$ and the common factor $Y$.

In this framework, solving the **Problem 1**, introduced in Section 1.3, is equivalent to prove that the process $\{U_t(x),\ t\geq 0\}$ defined as

$$U_t(x)=-e^{-\gamma x-P_t},\quad (t,x)\in[0,+\infty)\times\mathbb{R},$$

where $P$ is given in (1.8), is a forward dynamic exponential utility, normalized at time 0.

It is easy to check that the process $(X^H,Y,P)$ is a Markovian triplet; its infinitesimal generator $\mathcal{L}^H$ is given by

$$\begin{aligned}
&\mathcal{L}^H f(t,x,y,p)\\
&\quad=\frac{\partial f}{\partial t}(t,x,y,p)+\big[a(t,y)-b(t,y,\Theta)+\Pi\mu(t,y)\big]\frac{\partial f}{\partial x}(t,x,y,p)\\
&\quad+\frac{1}{2}\Pi^2\sigma^2(t,y)\frac{\partial^2 f}{\partial x^2}(t,x,y,p)+\alpha(t,y)\frac{\partial f}{\partial y}(t,x,y,p)+\frac{1}{2}\beta^2(t,y)\frac{\partial^2 f}{\partial y^2}(t,x,y,p)\\
&\quad+\rho\Pi\sigma(t,y)\beta(t,y)\frac{\partial^2 f}{\partial x\partial y}(t,x,y,p)+g(t,x,y)\frac{\partial f}{\partial p}(t,x,y,p)+\frac{1}{2}h^2(t,x,y)\frac{\partial^2 f}{\partial p^2}(t,x,y,p)\\
&\quad+\rho^S\Pi\sigma(t,y)h(t,x,y)\frac{\partial^2 f}{\partial x\partial p}(t,x,y,p)+\rho^Y\beta(t,y)h(t,x,y)\frac{\partial^2 f}{\partial y\partial p}(t,x,y,p)\\
&\quad+\lambda(t,y)\int_I\Big\{f\big(t,x-(1-\Theta)z,y,p\big)-f(t,x,y,p)\Big\}F(t,y,\mathrm{d}z),
\end{aligned}\tag{1.61}$$

for every $(t,x,y,p)\in[0,+\infty)\times\mathbb{R}^3$ and for every function $f:[0,+\infty)\times\mathbb{R}^3\to\mathbb{R}$ in $C^{1,2}$ which is sufficiently integrable.

### 1.5.2 Optimal investment and reinsurance

In this subsection, we characterize a family of forward utilities (in order to describe the preferences of the insurance company) and the optimal investment portfolio and optimal protection level.

In the following theorem, we provide the analytic construction of a class of forward dynamic exponential utilities normalized at $t_0 = 0$.

**Theorem 1.3.** *The process* $\{U_t(x),\ t \geq 0\}$*, given for* $x \in \mathbb{R}$ *and* $t \geq 0$ *by*

$$U_t(x) = -e^{-\gamma x - P_t}, \tag{1.62}$$

*with the process* $P$ *defined in* (1.8)*, is a forward dynamic exponential utility, normalized at time* 0.

*Proof.* To show the result we prove that the process $\{U_t(x),\ t \geq 0\}$ defined in (1.62) verifies Definition 1.2 with the initial condition $u(x,p) = -e^{-\gamma x - p}$. By construction, for every $t \geq 0$ the random variable $U_t(x)$ is $\mathcal{F}_t$-measurable. Next, we need to show that for arbitrary $t, T$ such that $0 \leq t \leq T$, we have

$$-e^{-\gamma x - P_t} = \max_{H \in \mathcal{A}} \mathbb{E}\left[-e^{-\gamma X_T^H - P_T} \Big| \mathcal{F}_t\right]. \tag{1.63}$$

The equality (1.63) implies that $\{U_t(x),\ t \geq 0\}$ is a supermartingale for all admissible strategies $H \in \mathcal{A}$, and a martingale along some strategy $H^* \in \mathcal{A}$. In view of the Markov property of the process $(X^H, Y, P)$, we consider the function $u : [0, +\infty) \times \mathbb{R}^3 \to (-\infty, 0)$ given by

$$u(t,x,y,p) := \max_{H \in \mathcal{A}_t} \mathbb{E}_{t,x,y,p}\Big[-e^{-\gamma X_T^H - P_T}\Big], \tag{1.64}$$

where $\mathbb{E}_{t,x,y,p}$ denotes the conditional expectation given $X_t^H = x$, $Y_t = y$ and $P_t = p$, for every $(t,x,y,p) \in [0,T) \times \mathbb{R}^3$.

If $u(t,x,y,p)$ is $\mathcal{C}^1$ in $t$ and $\mathcal{C}^2$ in $(x,y,p)$, by applying Itô's formula, we get that the function $u(t,x,y,p)$ solves the final value problem

$$\max_{(\Theta,\Pi) \in [0,1] \times \mathbb{R}} \mathcal{L}^H u(t,x,y,p) = 0, \quad (t,x,y,p) \in [0,\infty) \times \mathbb{R}^3, \tag{1.65}$$

$$u(T,x,y,p) = -e^{-\gamma x - p}, \quad (x,y,p) \in \mathbb{R}^3, \tag{1.66}$$

with the operator $\mathcal{L}^H$ denoting the infinitesimal generator of the Markov process $(X^H, Y, P)$ associated with a constant control $H$, see (1.61).

Now, we choose $H^* = (\Theta^*, \Pi^*)$ given by

$$\Pi_t^* = \frac{\mu(t,Y_t)}{\gamma\sigma^2(t,Y_t)} - \rho^S \frac{h(t, X_t^{H^*}, Y_t)}{\gamma\sigma(t,Y_t)}$$

and $\Theta_t^* = \bar{\Theta}(t, Y_t)$, for each $t \in [0,T]$, with $\bar{\Theta}(t,y)$ as in (1.13); equality (1.63) holds and then the martingale property along $H^*$ is satisfied.

Next, we use a guess-and-verify approach and we show that the function $u(t,x,y,p)$ given by

$$u(t,x,y,p) = u(x,p) = -e^{-\gamma x - p}, \quad (x,p) \in \mathbb{R}^2, \tag{1.67}$$

provides the unique classical solution to the problem (1.65)-(1.66). Clearly, $u(x,p) = -e^{-\gamma x - p}$, with $(x,p) \in \mathbb{R}^2$, is $\mathcal{C}^\infty$ and it is easy to check that solves (1.65)-(1.66). Moreover, uniqueness follows from the Verification Theorem (see Theorem 1.5 in Appendix). Indeed, condition (ii) of Theorem 1.5 is trivially satisfied since the function $u(x,p)$ does not depend on $y$. Then, we just need to show the following conditions: for every $t, T$ such that $0 \le t \le T$,

$$\mathbb{E}\Big[\int_t^{T\wedge\tau_n} \Big(\sigma(s,Y_s)\Pi_s \frac{\partial u}{\partial x}(s, X_s^H, Y_s, P_s)\Big)^2 \mathrm{d}v\Big] < \infty,$$
$$\mathbb{E}\Big[\int_t^{T\wedge\tau_n} \Big(h(s, X_s^H, Y_s) \frac{\partial u}{\partial p}(s, X_s^H, Y_s, P_s)\Big)^2 \mathrm{d}v\Big] < \infty,$$
$$\mathbb{E}\left[\int_I \int_t^{T\wedge\tau_n} \Big| u\big(s, X_{s-}^H - (1-\Theta_{s-})z, Y_s, P_{s-})\big) - u(s, X_{s-}^H, Y_s, P_{s-}) \Big| \lambda(s,Y_s) F(s, Y_s, \mathrm{d}z) \mathrm{d}s\right] < \infty,$$

with $I \subset [0, +\infty)$ being an arbitrary interval, for a suitable, non-decreasing sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \tau_n = +\infty$. Therefore, for every $n \in \mathbb{N}$, we define

$$\tau_n = \inf\left\{s \in [t,T] : |P_s| > n \vee \ X_s^H < -n\right\}.$$

Over the stochastic interval $[\![t, T\wedge\tau_n]\!]$, since $X^H$ and $P$ do not explode, there is $\bar{n} \in \mathbb{N}$, with $\bar{n} \le n$, such that $\gamma X_t^H + P_t \ge -\bar{n}(\gamma+1)$. Hence, we get that

$$\begin{aligned}
&\mathbb{E}\left[\int_t^{T\wedge\tau_n} \Big(\sigma(s,Y_s)\Pi_s \frac{\partial u}{\partial x}(s, X_s^H, Y_s, P_s)\Big)^2 \mathrm{d}s\right] \\
&\quad = \mathbb{E}\left[\int_t^{T\wedge\tau_n} \sigma^2(s,Y_s)\Pi_s^2 \Big(\gamma e^{-\gamma X_s^H - P_s}\Big)^2 \mathrm{d}s\right] \le \Big(\gamma e^{(\gamma+1)\bar{n}}\Big)^2 \mathbb{E}\left[\int_t^T \sigma^2(s,Y_s)\Pi_s^2 \mathrm{d}s\right] < \infty,
\end{aligned}$$

since $\Pi$ is an admissible investment strategy. Moreover, we have that

$$\mathbb{E}\left[\int_t^{T\wedge\tau_n} \left(h(s, X_s^H, Y_s)\frac{\partial u}{\partial p}(s, X_s^H, Y_s, P_s)\right)^2 \mathrm{d}v\right]$$
$$= \mathbb{E}\left[\int_t^{T\wedge\tau_n} h^2(s, X_s^H, Y_s)\left(e^{-\gamma X_s^H - P_s}\right)^2 \mathrm{d}s\right] \leq \left(e^{(\gamma+1)\bar{n}}\right)^2 \mathbb{E}\left[\int_t^T h^2(s, X_s^H, Y_s)\mathrm{d}s\right] < \infty,$$

thanks to (1.9). Finally, we have that

$$\mathbb{E}\left[\int_t^{T\wedge\tau_n}\int_I \left|u(s, X_{s-}^H - (1-\Theta_{s-})z, Y_s, P_{s-})) - u(s, X_{s-}^H, Y_s, P_{s-})\right|\lambda(s, Y_s)F(s, Y_s, \mathrm{d}z)\mathrm{d}s\right]$$
$$= \mathbb{E}\left[\int_t^{T\wedge\tau_n}\int_I e^{-\gamma X_{s-}^H - P_{s-}}\left|e^{\gamma(1-\Theta_{s-})z} - 1\right|\lambda(s, Y_s)F(s, Y_s, \mathrm{d}z)\mathrm{d}s\right]$$
$$\leq e^{(\gamma+1)\bar{n}}\mathbb{E}\left[\int_t^T\int_I (e^{\gamma z} - 1)\lambda(s, Y_s)F(s, Y_s, \mathrm{d}z)\mathrm{d}s\right] < \infty,$$

by Assumption 1.1. Therefore, Theorem 1.5 applies and the function $u(x,p) = -e^{-\gamma x - p}$ is the unique solution of the problem (1.65)–(1.66).

We conclude that (1.63) holds, and then $\{U_t(x) = -e^{-\gamma x - P_t},\ t \geq 0\}$ is a forward dynamic exponential utility normalized at 0. □

Now, we characterize the optimal investment portfolio and the optimal reinsurance level for this family of forward dynamic exponential utilities in (1.62).

**Proposition 1.3.** *The optimal strategy $H^* = (\Theta^*, \Pi^*)$ is given by the optimal reinsurance protection level $\Theta^* = \{\Theta_t^*,\ t \geq 0\}$, where $\Theta_t^* = \bar{\Theta}_t = \bar{\Theta}(t, Y_t)$, with $\bar{\Theta}(t,y)$ defined by* (1.13), *and the optimal investment portfolio $\Pi^* = \{\Pi_t^*,\ t \geq 0\}$, where $\Pi_t^* = \Pi^*(t, X_t^{H^*}, Y_t)$, with*

$$\Pi^*(t,x,y) = \frac{\mu(t,y)}{\gamma\sigma^2(t,y)} - \rho^S\frac{h(t,x,y)}{\gamma\sigma(t,y)}, \tag{1.68}$$

*for every $(t,x,y) \in [0,+\infty) \times \mathbb{R}^2$.*

*Proof.* We consider the optimization problem defined by (1.65)–(1.66). Using the form of the function $u$, we observe that the problem can be written as

$$-\gamma a(t,y)u(x,p) - g(t,x,y)u(x,p) + \frac{1}{2}h^2(t,x,y)u(x,p)$$
$$+ \max_{\Theta\in[0,1]}\Psi_1(\Theta,t,x,y,p) + \max_{\Pi\in\mathbb{R}}\Psi_2(\Pi,t,x,y,p) = 0,$$

for all $(t,x,y,p) \in [0,+\infty) \times \mathbb{R}^3$ with the final condition $u(T,x,y,p) = -e^{-\gamma x - p}$, for all $(x,y,p) \in \mathbb{R}^3$, where the functions $\Psi_1$, $\Psi_2$ are defined as

$$\Psi_1(\Theta,t,x,y,p) = \gamma b(t,y,\Theta)u(x,p) + \lambda(t,y)\int_I u(x,p)(e^{\gamma(1-\Theta)z}-1)F(t,y,\mathrm{d}z),$$
$$\Psi_2(\Pi,t,x,y,p) = -\gamma\Pi\mu(t,y)u(x,p) + \frac{1}{2}\gamma^2\Pi^2\sigma^2(t,y)u(x,p) + \gamma\rho^S\Pi\sigma(t,y)h(t,x,y)u(x,p).$$

Now, we compute the optimal protection level $\Theta^*$. The function $\Psi_1$ is continuous in $\Theta$, due to the assumptions on the function $b(t,y,\Theta)$, and $\Theta \in [0,1]$, therefore a maximum exists. The first and second order derivatives of $\Psi_1$ are respectively given by

$$\frac{\partial\Psi_1}{\partial\Theta}(\Theta,t,x,y,p) = -\gamma u(x,p)\left\{\frac{\partial b}{\partial\Theta}(t,y,\Theta) - \lambda(t,y)\int_I e^{\gamma(1-\Theta)z}zF(t,y,\mathrm{d}z)\right\},$$
$$\frac{\partial^2\Psi_1}{\partial\Theta^2}(\Theta,t,x,y,p) = -\gamma u(x,p)\left\{\frac{\partial^2 b}{\partial\Theta^2}(t,y,\Theta) + \gamma\lambda(t,y)\int_I e^{\gamma(1-\Theta)z}z^2F(t,y,\mathrm{d}z)\right\},$$

and they are continuous in $\Theta$ and well defined thanks to Assumptions 1.1 and 1.2. In virtue of condition (1.11), $\Psi_1(\Theta,t,x,y,p)$ is strictly concave in $\Theta$ and hence it admits a unique maximizer $\Theta^* \in [0,1]$, whose measurability follows by classical selection theorems. Let $\widehat{\Theta}$ be the solution of the equation $\frac{\partial\Psi_1}{\partial\Theta}(\Theta,t,x,y,p) = 0$. If $\widehat{\Theta} \in (0,1)$, then $\widehat{\Theta}$ provides the optimal protection level; if $\widehat{\Theta} \geq 1$, then the optimal protection level is 1, which means full reinsurance is optimal; finally, if $\widehat{\Theta} \leq 0$, then the optimal protection level is 0, that is no reinsurance. Next, we describe the sets corresponding to these three cases. Recall the definition of sets $\mathcal{D}_0$ and $\mathcal{D}_1$ in definitions (1.14) and (1.15), respectively. From (1.11), we get that $\frac{\partial\Psi_1}{\partial\Theta}(\Theta,t,x,y)$ is decreasing in $\Theta \in [0,1]$, for every $(t,x,y,p) \in [0,+\infty) \times \mathbb{R}^3$, that is, $\frac{\partial\Psi_1}{\partial\Theta}(1,t,x,y,p) \leq \frac{\partial\Psi_1}{\partial\Theta}(\Theta,t,x,y,p) \leq \frac{\partial\Psi_1}{\partial\Theta}(0,t,x,y,p)$. We have:

(i) if $\frac{\partial\Psi_1}{\partial\Theta}(0,t,x,y,p) \leq 0$, then $\Theta^*(t,x,y,p) = 0$, i.e. no reinsurance is chosen. This is equivalent to say that $(t,y) \in \mathcal{D}_0$.

(ii) if $\frac{\partial\Psi_1}{\partial\Theta}(1,t,x,y,p) \geq 0$, then $\Theta^*(t,x,y,p) = 1$, i.e. full reinsurance is chosen. This corresponds to the case $(t,y) \in \mathcal{D}_1$.

(iii) the case $\frac{\partial\Psi_1}{\partial\Theta}(\widehat{\Theta},t,x,y,p) = 0$ for some $\widehat{\Theta} \in (0,1)$, corresponds to $(t,y) \in (\mathcal{D}_0 \cup \mathcal{D}_1)^c$.

To characterize the candidate for the optimal investment portfolio $\Pi^*$, we observe that the function $\Psi_2(\Pi,t,x,y,p)$ is continuous in $\Pi$. Then, taking the first order condition we get that $\Pi^*$ given in equation (1.68) is a stationary point of the function $\Psi_2(\Pi, t, x, y, p)$, which corresponds to a maximum since the second derivative with respect to $\Pi$ is negative.

Finally, we show that the pair $(\Pi^*, \Theta^*) \in \mathcal{A}$, since all required integrability conditions are satisfied. Both $\Theta^*$ and $\Pi^*$ are predictable; moreover, for every $t \geq 0$ it holds that

$$\mathbb{E}\left[\int_0^t \left(|\Pi_s^*||\mu(s,Y_s)| + (\Pi_s^*)^2\sigma^2(s,Y_s)\right)\mathrm{d}s\right]$$
$$\leq c_1\mathbb{E}\left[\int_0^t \frac{\mu^2(s,Y_s)}{\gamma\sigma^2(s,Y_s)}\mathrm{d}s\right] + c_2\mathbb{E}\left[\int_0^t h^2(s,X_s^H,Y_s)\mathrm{d}s\right] + c_3\mathbb{E}\left[\int_0^t \frac{\mu(s,Y_s)h(s,X_s^H,Y_s)}{\gamma\sigma(s,Y_s)}\mathrm{d}s\right] < \infty,$$

thanks to conditions (1.6) with $\beta = 0$, (1.9) with $t_0 = 0$ and the Cauchy-Schwarz inequality, for some positive constants $c_1, c_2, c_3$. Hence, condition (1.10) is satisfied. It remains to prove that $\mathbb{E}\left[e^{-\gamma X_t^{H^*} - P_t}\right] < \infty$ for each $t \geq 0$. In view of (1.59), (1.8) and (1.68), we have

$$\mathbb{E}\left[e^{-\gamma X_t^{H^*} - P_t}\right] = e^{-\gamma x_0}\mathbb{E}\Big[e^{-\frac{1}{2}\int_0^t \frac{\mu^2(s,Y_s)}{\sigma^2(s,Y_s)}\mathrm{d}s} e^{-\int_0^t \frac{\mu(s,Y_s)}{\sigma(s,Y_s)}\mathrm{d}W_s^S} e^{-\int_0^t \sqrt{1-(\rho^S)^2}h(s,X_s^H,Y_s)\mathrm{d}\widetilde{W}_s}$$
$$\times e^{-\frac{1}{2}\int_0^t (1-(\rho^S)^2)h^2(s,X_s^{H^*},Y_s)\mathrm{d}s} e^{\gamma\int_0^t\int_I (1-\bar{\Theta}_{s-})z m(\mathrm{d}s,\mathrm{d}z)} e^{-\int_0^t \lambda(s,Y_s)\int_I \left(e^{\gamma(1-\bar{\Theta}_{s-})z}-1\right)F(s,Y_s,\mathrm{d}z)\mathrm{d}s}\Big],$$

where $X_0^H = x_0 \geq 0$ is the initial wealth and $\widetilde{W} = \{\widetilde{W}_t,\ t \geq 0\}$ is an additional Brownian motion that is independent of $W^S$. We define the process $L = \{L_t,\ t \geq 0\}$ as

$$L_t = e^{-\frac{1}{2}\int_0^t \frac{\mu^2(r,Y_r)}{\sigma^2(r,Y_r)}\mathrm{d}r - \int_0^t \frac{\mu(r,Y_r)}{\sigma(r,Y_r)}\mathrm{d}W_r^S - \frac{1}{2}\int_0^t (1-(\rho^S)^2)h^2(r,X_r^{H^*},Y_r)\mathrm{d}r - \int_0^t \sqrt{1-(\rho^S)^2}h(r,X_r^H,Y_r)\mathrm{d}\widetilde{W}_r}.$$

Then, $L$ is a square integrable martingale thanks to condition (1.6) and Assumption 1.3. Therefore,

$$\mathbb{E}\left[e^{-\gamma X_t^{H^*} - P_t}\right] = e^{-\gamma x_0}\mathbb{E}\left[L_t e^{\gamma\int_0^t\int_I (1-\bar{\Theta}_{s-})z m(\mathrm{d}s,\mathrm{d}z)} e^{-\int_0^t \lambda(s,Y_s)\int_I \left(e^{\gamma(1-\bar{\Theta}_{s-})z}-1\right)F(s,Y_s,\mathrm{d}z)\mathrm{d}s}\right]$$
$$\leq e^{-\gamma x_0}\mathbb{E}\left[L_t^2\right]^{1/2}\mathbb{E}\left[e^{\gamma\int_0^t\int_I (1-\bar{\Theta}_{s-})z m(\mathrm{d}s,\mathrm{d}z)}\right]^{1/2},$$

because $\int_0^t \lambda(s,Y_s)\int_I \left(e^{\gamma(1-\bar{\Theta}_{s-})z} - 1\right)F(s,Y_s,\mathrm{d}z)\mathrm{d}s \geq 0$ $\mathbf{P}$-a.s. for all $t \geq 0$. Finally, we recall that $\mathbb{E}\left[L_t^2\right] < \infty$ and note that

$$\mathbb{E}\left[e^{2\gamma\int_0^t\int_I (1-\bar{\Theta}_{r-})z m(\mathrm{d}r,\mathrm{d}z)}\right] \leq \mathbb{E}\left[e^{2\gamma\sum_{i=1}^{N_t} Z_i}\right]$$
$$= \sum_{n\geq 0}\mathbb{E}\left[e^{2\gamma\sum_{i=1}^{N_t} Z_i}\middle| N_t = n\right]\mathbf{P}(N_t = n) = \sum_{n\geq 0}\prod_{i=1}^n \mathbb{E}\left[e^{2\gamma Z_i}\right]\mathbf{P}(N_t = n) < \infty,$$

thanks to the assumptions on the random variables $\{Z_n\}_{n\in\mathbb{N}}$ and the fact that the process $N$ does not explode in finite time. □

Similar observation as for the regime switching case can be done for this setting; see Remark 1.6 and subsequent comments.

### 1.5.3 Numerical experiments

In this subsection we conduct a numerical analysis in order to investigate some features of the optimal reinsurance-investment strategy and the optimal value process under forward utility preferences of exponential type.

In these experiments, we analyze our theoretical results in a time interval $[0, T]$, with $T = 1$, assuming that insurance and financial operations take place in one year, starting from today.

The proposed model specification is rich enough to incorporate several stochastic factor models well known in the literature. In our first example, the stochastic factor process is chosen to follow a Vasicek model, i.e.

$$\mathrm{d}Y_t = (\alpha_1 + \alpha_2 Y_t)\mathrm{d}t + \beta \mathrm{d}W_t^Y, \quad Y_0 = -0.2,$$

with constant coefficients $\alpha_1 = 0.2$, $\alpha_2 = -1$ and $\beta = 0.1$.

We also suppose that the function $\lambda$ is given by $\lambda(t, y) = \lambda_0 e^y$, for each $(t, y) \in [0, T] \times \mathbb{R}$, where $\lambda_0 = k e^{-Y_0}$ with $k > 0$, which guarantees that the intensity of the claims arrival process $N$ is positive. For the sake of semplicity, all random variables $\{Z_n\}_{n\in\mathbb{N}}$ have common distribution and the claim size distribution, specifically the claim size distribution is assumed as $\Gamma(\alpha_\Gamma, \beta_\Gamma)$, $\alpha_\Gamma, \beta_\Gamma > 0$. In particular, we set $k = 1$, $\alpha_\Gamma = 1$ and two different values of $\beta_\Gamma$, namely $\beta_\Gamma = 2$ which corresponds to larger losses and $\beta_\Gamma = 1/3$ to smaller claims.

We consider a risky asset price process with an affine appreciation rate and a uniformly elliptic Scott volatility, described by the following SDE

$$\mathrm{d}S_t = \mu(Y_t)S_t\mathrm{d}t + \sigma(Y_t)S_t\mathrm{d}W_t^S, \quad S_0 = 1,$$

where

$$\mu(y) := \mu_1 + \mu_2 y, \qquad \sigma(y) =: \bar{c}\sqrt{\epsilon_1 + e^{\epsilon_2 y}},$$

for every $y \in \mathbb{R}$. The Brownian motions $W^S$ and $W^Y$ are correlated with correlation coefficient $\rho$. We set the risk aversion coefficient to $\gamma = 0.5$.

First, we study the case where the reinsurance premium is calculated under the conditional modified variance principle, and given by

$$b(y, \Theta) = \alpha_\Gamma \beta_\Gamma \lambda(y)\Theta + \delta_R \beta_\Gamma \Theta, \tag{1.69}$$

for each $(y, \Theta) \in \mathbb{R} \times [0, 1]$. As pointed in Remark 1.1, this premium calculation principle allows keeping the dependence on $Y$ in the optimal reinsurance strategy, which in turn, means that the strategy adapts to the index value over time. Regarding the insurance safety loading $\delta_I$ and the reinsurance safety loading $\delta_R$, the condition $\delta_I < \delta_R < 2\delta_I$ usually holds true; thus, we set $\delta_I = 0.3$ and $\delta_R = 0.5$.

Under these parameters, we wonder how many claims should be reinsured in order to maximize the forward expected utility. In Figure 1.7, we plot the optimal reinsurance strategy, both when large claims (solid line) occur and when small claims (dashed line) occur. We observe that in the latter case, null reinsurance might be optimal for almost all negative values of the stochastic factor, whereas in case of large claims a big percentage of claim is always reinsured.

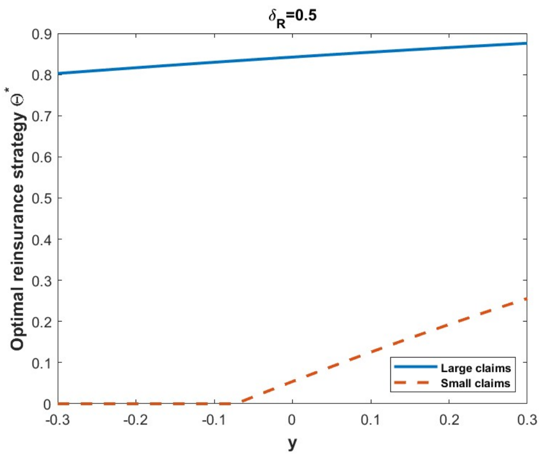



Figure 1.7: Optimal proportional reinsurance strategy with respect to the values of the index, for large claims (solid line) and small claims (dashed line)

Moreover, we note that the protection level in case of big claims is larger than in the case when claims are smaller. This means that, under the same claim arrival intensity, when expected claim amount is small, the insurance company purchases less reinsurance whilst for large losses it buys reinsurance with a higher protection level, to mitigate the risk.

Next, we analyze the behavior of the protection level over the time interval. Since $Y$ follows a Vasicek model with mean level 0.2, starting from $-0.2$, we can easily describe the evolution of the reinsurance strategy as time goes by. Indeed, simulating the Brownian motion $W^Y$, we notice from Figure 1.8 that trajectories of the optimal reinsurance strategy are accumulated around 0.85 in case of large claims and in the range $[0, 0.25]$ in case of smaller claims.

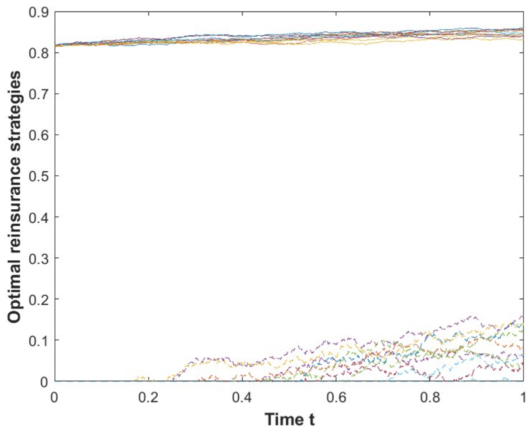



Figure 1.8: Some trajectory of the optimal proportional reinsurance strategy for large claims (solid line) and small claims (dashed line)

To better understand the form of the reinsurance strategy, in Figure 1.9 and Figure 1.10 we consider different safety loadings as the ones used in all the rest of this numerical section. Using a reinsurance premium of the form (1.69), we obtain that if reinsurance is expensive (which is the case, for example, of safety loadings at level $\delta_R = 0.9$, $\delta_I = 0.6$), the insurance company will reinsure fewer claims. In particular, as shown in Figure 1.9, in the case of small claims null reinsurance is optimal for every value of the index in the range $[-0.3, 0.3]$ [2].

On the other hand, if reinsurance is enough cheap, namely if the reinsurance safety loading is quite small (for instance, $\delta_R = 0.1$, $\delta_I = 0.07$), the insurance company prefers to protect itself as much as possible and thus will opt for a larger protection level $\Theta$.

As can be seen from Figure 1.10, in both cases (large claims and small claims), under our parameter setting, full reinsurance and null reinsurance are never optimal. The reinsurance level, however, is always large, as reinsurance is very cheap.

[2] The range of $y$ has been chosen according to the values in the simulations of the index $Y$.

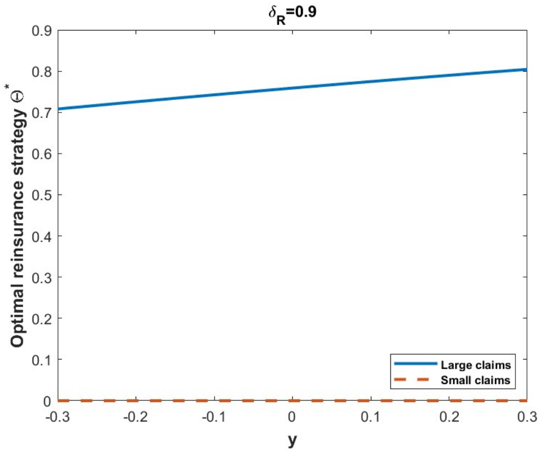



Figure 1.9: Optimal proportional reinsurance strategy with respect to the values of the index, for large claims (solid line) and small claims (dashed line), with safety loadings $\delta_R = 0.9$, $\delta_I = 0.6$.

Further, we point out that the optimal protection level is always quite high when large claims occur, always greater than what one get when the claim size is smaller: indeed Figure 1.7, 1.9 and 1.10 show that the solid line is always above the dashed one.

Now, we will highlight the characteristics of the optimal investment strategy corresponding to our family of forward exponential utility preferences. In order to consider a reinsurance policy neither too expensive nor too cheap, we restore the insurance safety loading at level $\delta_I = 0.3$ and the reinsurance safety loading at level $\delta_R = 0.5$. In this setting, we examine the optimal portfolio strategy. As pointed out by Remark 1.4, different choices of the functions $g$ and $h$ lead to a different forward utility and, as a consequence, to a different optimal investment strategy. In the sequel, we consider the following choices of the function $h$, for every $(t, x, y) \in [0, T] \times \mathbb{R}^2$:

(i) $h_1(t, x, y) = 0$. This corresponds to zero volatility utility function and the optimal investment portfolio is the myopic strategy $\Pi_1^*(y) = \frac{\mu(y)}{\gamma\sigma^2(y)}$. This means that when there is no additional stochastic part describing utility preferences, the optimal investment in the risky asset only depends on the drift and volatilty of the stock price.

(ii) $h_2(t, x, y) = h_2(y) = -2\dfrac{\rho^S}{1-(\rho^S)^2}\dfrac{\mu(y)}{\sigma(y)}$. In this case the insurance market only affects the drift of the penalizing process involved in the forward utility but not its volatility. The optimal investment strategy is given by $\Pi_2^*(y) = \dfrac{\mu(y)}{\gamma\sigma^2(y)} + \dfrac{\mu(y)}{\gamma\sigma^2(y)}\dfrac{\rho^S}{1-(\rho^S)^2}$ and

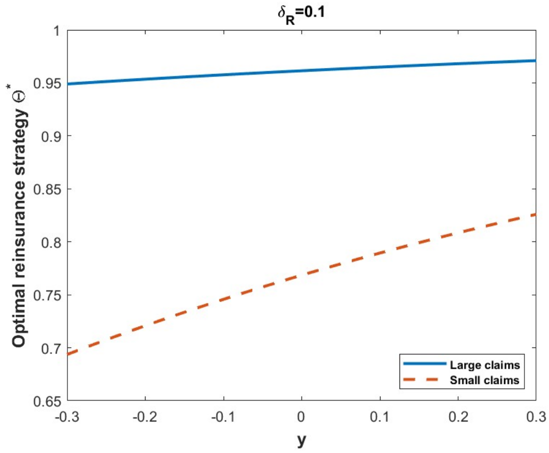



Figure 1.10: Optimal proportional reinsurance strategy with respect to the values of the index, for large claims (solid line) and small claims (dashed line), with safety loadings $\delta_R = 0.1$, $\delta_I = 0.07$.

consists of a myopic component and an additional term that accounts for the correlation between the stock price and the preferences of the insurer. The latter reflects the incremental changes in the optimal behavior due to the presence of two sources of risk which may also be linked to each other. Notice that the optimal investment reduces to the myopic component (namely the additional demand disappears) even when the penalizing process presents a stochastic noise which however is not correlated with that of the risky asset.

(iii) $h_3(t,x,y) = h_3(y) = \dfrac{\mu(y)}{\rho^S\sigma(y)} - \dfrac{1}{\rho^S}\sqrt{\varphi(y) - \gamma b(y,\Theta)}$. In this instance, the insurance company adjusts its preferences according to the state of the financial market, the collected premia and the paid premia. Interestingly, in this case the optimal portfolio depends on uncovered claims and the market volatility but it is not affected by the stock Sharpe ratio and the insurance and reinsurance premia. Indeed, the optimal investment strategy in this situation is $\Pi_3^*(y) = \dfrac{1}{\gamma\sigma(y)}\sqrt{\varphi(y) - \gamma b(y,\Theta)}$. This is due to the fact that since forward utilities are dynamically updated, the company can account into her preferences some randomness which therefore the portfolio must not take into account: for example, the risk arising from premia is processed in the utility rather than delegated to the insurance company actions.

(iv) $h_4(t,x,y) = -\bar{k}\frac{1}{\rho^S}\gamma\sigma(y)\bar{k}x + \frac{\mu(y)}{\rho^S\sigma(y)}$. Unlike the previous cases, here we have an explicit dependence on wealth. In particular, taking $h_4$ as an affine function of the wealth implies that the optimal strategy is of the form $\Pi_4^*(x) = \bar{k}x$, i.e. the insurance company would always invest in the risky asset the same percentage of the wealth. Notice that the optimal portfolio strategy is highly dependent on the scale parameter $\bar{k}$. In particular, if $\bar{k} > 1$ the insurance company would always borrow from the bank account and if $\bar{k} < 0$ it always short-sells the risky asset.

In Figure 1.11, we plot the optimal portfolio strategies corresponding to the first three choices of the function $h$, with respect to the value of the exogenous index which influences our combined market model. Taking, for example, $\beta_\Gamma = 2$, $\rho^S = 0.5$ and $\bar{k} = 0.5$, we consider two different parameter settings for the appreciation rate and the volatility of the stock price and we match them to obtain three different situations. In the left panel we consider the case where the stock price volatility is highly affected by the index and the drift is almost constant. We observe that all strategies decrease with respect to the common factor. However, the decline is not the same: $\Pi_2^*$ is the strategy most affected by the variation of the index whilst $\Pi_3^*$ is the less affected one. In the middle panel, instead, the effect of the index on the drift dominates the effect on volatility. Here, we see that strategies are increasing for all $y$ in the range $[-0.3, 0.3]$. Notice that $\Pi_2^*$ still assumes a wider range of values, hence also in this situation it is the most affected by the variation of the index. Finally, in the right panel we plot the optimal portfolio strategies when both the drift and the volatility highly depend on the values of the common index. We outline that strategies $\Pi_1^*$ and $\Pi_2^*$ are not monotone and $\Pi_2^*$ has the largest variation as for the other two cases.

### 1.5.4 Comparison with the backward utility approach

In this subsection, in order to more easily compare dynamic forward preferences with standard backward utilities, we focus on the zero-volatility case, namely we assume that the penalizing process satisfies

$$P(t) = \int_0^t g(s, Y_s)\mathrm{d}s,$$

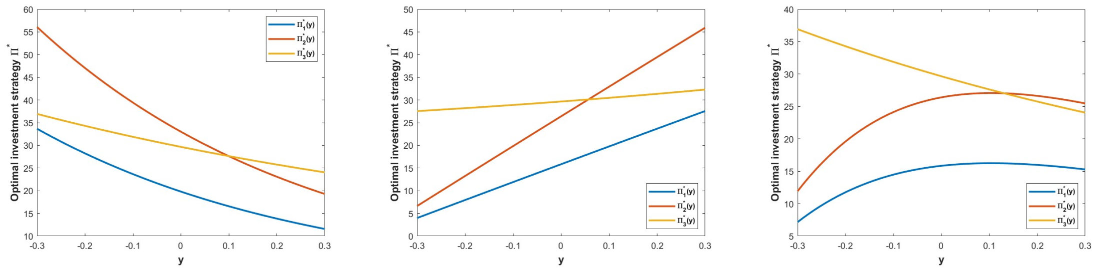


Figure 1.11: Optimal portfolio strategies with respect to the values of the index. under different forward preferences. Parameter settings: Left panel $\mu \in [0.094, 0.106]$, $\sigma \in [0.0748, 0.1354]$; Middle panel $\mu \in [0.02, 0.14]$, $\sigma \in [0.1002, 0.1008]$; Right panel $\mu \in [0.02, 0.14]$, $\sigma \in [0.0748, 0.1354]$

for all $t \geq 0$, where the function $g$ is given by

$$g(t,y) = -\frac{1}{2}\left(\frac{\mu(t,y)}{\sigma(t,y)}\right)^2 - \gamma a(t,y) + \varphi(t,y), \tag{1.70}$$

for all $(t,y) \in [0,+\infty) \times \mathbb{R}$, with $\varphi(t,y)$ introduced in (1.16).

The following result characterizes the forward utility process and the corresponding optimal strategy in the case where the additional stochastic part describing utility preferences is not involved.

**Corollary 1.3.1.** *The process $\{U_t(x),\ t \geq 0\}$, given for $x \in \mathbb{R}$ and $t \geq 0$ by*

$$U_t(x) = -e^{-\gamma x - \int_0^t g(s,Y_s)\mathrm{d}s}, \tag{1.71}$$

*with $g(t,y)$ defined in (1.70), is a forward dynamic exponential utility, normalized at 0. Moreover, the optimal strategy $H^* = (\Theta^*, \Pi^*)$ is given by the optimal reinsurance protection level $\Theta^* = \{\Theta_t^*,\ t \geq 0\}$, where $\Theta_t^* = \bar{\Theta}_t = \bar{\Theta}(t, Y_t)$, with $\bar{\Theta}(t,y)$ defined by (1.13), and the optimal investment portfolio $\Pi^* = \{\Pi_t^*,\ t \geq 0\}$, where $\Pi_t^* = \Pi^*(t, Y_t)$, with*

$$\Pi^*(t,y) = \frac{\mu(t,y)}{\gamma\sigma^2(t,y)}, \tag{1.72}$$

*for every $(t,y) \in [0,+\infty) \times \mathbb{R}$.*

*Proof.* It follows from Theorem 1.3 and Proposition 1.3 setting $h(t,x,y) = 0$, for every $(t,x,y) \in [0,+\infty) \times \mathbb{R}^2$. □

Next, we address the optimization problem under the backward approach, since our purpose is to compare the optimal strategy and the value of the optimal investment-reinsurance problem under dynamic forward zero-volatility utility to the classical backward utility, under the model setting outlined above. The first result provides the optimal strategy and the value function when the preferences of the company are described by the backward exponential utility function $u^B(x) = -e^{-\gamma x}$. Consider the backward reinsurance-investment problem

$$\max_{H \in \mathcal{A}} \mathbb{E}\left[-e^{-\gamma X_T^H}\right], \tag{1.73}$$

where $T \in (0, +\infty)$ is a fixed time horizon which coincides with the end of the investment period. We introduce the Cauchy problem

$$\begin{cases} \frac{\partial \phi}{\partial t}(t,y;T) + \frac{\partial \phi}{\partial y}(t,y;T)\left(\alpha(t,y) - \rho\frac{\mu(t,y)}{\sigma(t,y)}\beta(t,y)\right) + \frac{1}{2}\frac{\partial^2 \phi}{\partial y^2}(t,y;T)\beta^2(t,y), \\ \quad +\frac{1}{2}\left(\frac{\partial \phi}{\partial y}(t,y;T)\right)^2(1-\rho^2)\beta^2(t,y) - g(t,y) = 0, & (t,y) \in [0,T) \times \mathbb{R}, \\ \phi(T,y;T) = 0, & y \in \mathbb{R}. \end{cases} \tag{1.74}$$

**Theorem 1.4.** *Let $\phi(t,y;T)$ be the unique classical solution of the problem* (1.74)*. Then, the value function corresponding to the problem* (1.73) *is given by*

$$V(t,x,y;T) = -e^{-\gamma x - \phi(t,y;T)}.$$

*Moreover, the optimal Markovian reinsurance-investment strategy* $(\Theta^{*,B}, \Pi^{*,B}) = \{(\Theta_t^{*,B}, \Pi_t^{*,B}) = (\Theta^{*,B}(t,Y_t), \Pi^{*,B}(t,Y_t)),\ t \in [0,T]\}$ *is given by*

$$\Theta^{*,B}(t,y) = \bar{\Theta}(t,y),$$

$$\Pi^{*,B}(t,y) = \frac{\mu(t,y)}{\gamma\sigma^2(t,y)} - \rho\frac{\beta(t,y)\frac{\partial \phi(t,y;T)}{\partial y}}{\gamma\sigma(t,y)}, \tag{1.75}$$

*for every $(t,y) \in [0,T] \times \mathbb{R}$, where the function $\bar{\Theta}(t,y)$ is given in* (1.13)*.*

A sketch of the proof of Theorem 1.4 is provided in Appendix 1.5.6.

The results obtained so far deserve a few considerations. By Proposition 1.1 and Theorem 1.4, we immediately notice that the optimal reinsurance strategies under the backward and the forward utilities coincide. This can be explained at the mathematical level, using the same argument of the zero-correlation case. In fact the martingale driving the factor process $Y$, which affects the loss intensity, is orthogonal to the loss process. In this way, our model manages to capture an interesting effect that arises due to the nature of the forward utility, even when the financial and insurance frameworks are not independent.

As for the investment strategy, we note that forward and backward preferences induce different optimal investment portfolios. Moreover, by expressions (1.72) and (1.75), equality holds only if the factor process $Y$ and the price process $S$ are driven by uncorrelated Brownian motions. In particular, the optimal investment strategy under forward performances consists entirely of the myopic component. This means that it depends only on the risk aversion coefficient, the drift and the volatility of the stock, without taking into account other sources of risk, especially those related to the insurance business. This is a consequence of the fact that any changes in state of the market are absorbed by utility that updates forward in time according to the new conditions. The backward approach, instead, is based on the assumption that a future utility preference is set at the beginning of the investment period and does not change over time. Therefore, what needs to account for changes in market conditions must be the investment strategy which consists of a myopic part and an additional risk adjustment. The latter accounts for the part of risk correlated with the stock price. It is evidente that this additional demand vanishes if stochastic movements of the factor process $Y$ and the stock price process are orthogonal, i.e. the martingales driving the processes $Y$ and $S$ have zero predictable quadratic covariation.

One last comment should be made on the value functions: similarities and differences between the forward approach and the backward one. In both cases, they are exponential and they have a structure affine in the wealth. However, each of the two value functions is characterized also by another term that makes them very different. Indeed the multiplicative components $e^{-P(t)}$ and $e^{-\phi(t,y;T)}$ gather the effect due to market changes in very different ways. It is worth noting that $P$ also depends on $y$, which we omitted to keep the same notation as in the classical literature. If we compare the implicit expression of $P(t)$ given by $\frac{\partial P(t)}{\partial t} = g(t,y)$ with the initial condition $P(0) = 0$ and the PDE (1.74) satisfied by $\phi(t,y;T)$, we can conclude that both performance criteria process market changes in an aggregate way. But, observing carefully, the forward utility (which coincides with the forward value function that it generates) accounts for past observation, whereas in the backward case the value function estimates the future risk. It is clear that $P(t)$ and $\phi(t,y;T)$ are not easy to compare from the analytical point of view. As discussed in [81] these two processes are related to well known martingale measures, namely the minimal martingale measure and the minimal entropy measure.

It remains to establish existence (and uniqueness) of a classical solution to the PDE in (1.74), which is in general difficult to obtain because it is not linear. The next paragraph is devoted to this.

**Existence and uniqueness of a classical solution**

In the sequel, we provide sufficient conditions for existence and uniqueness of the solution to the PDE in (1.74), involved in the backward reinsurance-investment problem and, as a consequence, of the classical solution to the corresponding HJB equation associated with our optimization problem.

First of all, it is easy to see that, if Brownian motions $W^Y$ and $W^S$ have zero correlation, then the PDE becomes linear and a solution exists under suitable conditions on the model coefficients (see, for instance [86, Theorem 5.3] or [31, Theorem 1])

We also notice that the PDE is linear even when there is a perfect correlation (either positive or negative) between the two Brownian motions that drive the stochastic factor $Y$ and the stock price $S$. Consequently, also in this case it is easy to obtain existence of a classic solution of the problem (1.74).

Otherwise (i.e. for $\rho \neq 0, 1, -1$), similarly to [103], we introduce a transformation given by

$$\phi(t,y) = \kappa \ln(\xi(t,y)), \quad (t,y) \in [0,T] \times \mathbb{R},$$

for a suitable parameter $\kappa \in \mathbb{R} \setminus \{0\}$. Differentiating yields

$$\begin{aligned}
&\frac{\partial \phi}{\partial t}(t,y) = \frac{\kappa}{\xi(t,y)} \frac{\partial \xi}{\partial t}(t,y), \qquad \frac{\partial \phi}{\partial y}(t,y) = \frac{\kappa}{\xi(t,y)} \frac{\partial \xi}{\partial y}(t,y), \\
&\frac{\partial^2 \phi}{\partial y^2} = \frac{\kappa}{\xi(t,y)} \frac{\partial^2 \xi}{\partial y^2}(t,y) - \frac{\kappa}{\xi^2(t,y)} \left( \frac{\partial \xi}{\partial y}(t,y) \right)^2.
\end{aligned}$$

Substituting the above derivatives in (1.74) gives the following PDE

$$\begin{aligned}
&\frac{\partial \xi}{\partial t}(t,y) + \frac{\partial \xi}{\partial y}(t,y) \left( \alpha(t,y) - \rho \frac{\mu(t,y)}{\sigma(t,y)} \beta(t,y) \right) + \frac{1}{2} \frac{\partial^2 \xi}{\partial y^2}(t,y) \beta^2(t,y) \\
&\quad + \frac{1}{2\xi(t,y)} \frac{\partial \xi}{\partial y}^2 (t,y) \beta^2(t,y) \big( \kappa(1-\rho^2) - 1 \big) - g(t,y) \frac{\xi(t,y)}{\kappa} = 0,
\end{aligned} \tag{1.76}$$

for all $(t, y) \in [0, T) \times \mathbb{R}$, with the final condition $\xi(T, y) = 1$, for every $y \in \mathbb{R}$. The above expression suggests that if we take the parameter

$$\kappa = \frac{1}{1-\rho^2},$$

then the PDE (1.76) becomes linear. Indeed, the transformed solution $\xi(t, y)$ satisfies

$$\frac{\partial \xi}{\partial t}(t,y)+\frac{\partial \xi}{\partial y}(t,y)\left(\alpha(t,y)-\rho\frac{\mu(t,y)}{\sigma(t,y)}\beta(t,y)\right)+\frac{1}{2}\frac{\partial^2 \xi}{\partial y^2}(t,y)\beta^2(t,y)-g(t,y)\frac{\xi(t,y)}{\kappa}=0, \quad (1.77)$$

for all $(t, y) \in [0, T) \times \mathbb{R}$, with the final condition $\xi(T, y) = 1$, for every $y \in \mathbb{R}$.

Now, we provide some sufficient conditions that guarantee the existence of a classical solution of the equation (1.77), which is unique under the associated final condition, applying Theorem 1 of [58]. They prove that there exists only one classical solution and also provide a probabilistic representation by means of the Feynman–Kac formula. In order to apply this result, we introduce a new probability measure $\widetilde{\mathbf{P}}$ equivalent to $\mathbf{P}$, by setting

$$\left.\frac{\mathrm{d}\widetilde{\mathbf{P}}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_t} = \widetilde{L}_t = e^{-\left(\frac{1}{2}\int_0^t \rho^2\left(\frac{\mu(s,Y_s)}{\sigma(s,Y_s)}\right)^2 \mathrm{d}s + \int_0^t \rho\left|\frac{\mu(s,Y_s)}{\sigma(s,Y_s)}\right| \mathrm{d}W_s^Y\right)}, \qquad t \in [0, T].$$

Condition (1.6) ensures that the process $\widetilde{L} = \{\widetilde{L}_t,\ t \in [0, T]\}$ is a $\mathbf{P}$-martingale. By the Girsanov theorem, the process $\widetilde{W}^Y = \{\widetilde{W}_t^Y,\ t \in [0, T]\}$, defined as $\widetilde{W}_t^Y = W_t^Y + \rho\int_0^t \frac{\mu(s,Y_s)}{\sigma(s,Y_s)}\mathrm{d}s$, for every $t \in [0, T]$, is a $\widetilde{\mathbf{P}}$-Brownian motion. Thus, the dynamics of the stochastic factor $Y$ under $\widetilde{\mathbf{P}}$ are given by

$$\mathrm{d}Y_t = \left(\alpha(t, Y_t) - \rho\frac{\mu(t, Y_t)}{\sigma(t, Y_t)}\beta(t, Y_t)\right)\mathrm{d}t + \beta(t, Y_t)\mathrm{d}\widetilde{W}_t^Y, \quad Y_0 = y_0 \in \mathbb{R}.$$

**Proposition 1.4.** *Suppose that functions $\mu(t, y)$, $\sigma(t, y)$, $\alpha(t, y)$, $\beta(t, y)$ are locally Lipschitz continuous in $(t, y) \in [0, T] \times \mathbb{R}$. Suppose also that $a(t, y)$, $b(t, y, \Theta)$ and $\lambda(t, y)$ are bounded and Lipschitz-continuous in $(t, y) \in [0, T] \times \mathbb{R}$. In addition, let us assume that $\beta(t, y) \geq \eta$, for all $(t, y) \in [0, T] \times \mathbb{R}$. Then, there exists a unique positive and bounded classical solution of equation (1.77) which is given by*

$$\xi(t, y) = \mathbb{E}\left[e^{(1-\rho^2)\int_t^T g(s,Y_s)}\mathrm{d}s\right], \quad (1.78)$$

*for all $(t, y) \in [0, T) \times \mathbb{R}$, with terminal condition $\xi(T, y) = 1$, for every $y \in \mathbb{R}$.*

*Proof.* We apply Theorem 1 of [58] to prove existence and uniqueness of classical solution to equation (1.77) by verifying that their required conditions $(A1)$, $(A2)$, $(A3a')$-$(A3e')$ hold in our case. Consider a sequence of bounded sets $\{D_n = (-n,n),\ n \in \mathbb{N}\}$, such that $\bigcup_{n\in\mathbb{N}} D_n = \mathbb{R}$. Since functions $\mu(t,y)$, $\sigma(t,y)$, $\alpha(t,y)$, $\beta(t,y)$ are locally Lipschitz continuous in $(t,y) \in [0,T]\times\mathbb{R}$, conditions $(A1)$ and $(A2)$ of [58] for the coefficients $\alpha(t,y) - \rho\frac{\mu(t,y)}{\sigma(t,y)}\beta(t,y)$ and $\beta(t,y)$ are satisfied on $(t,y) \in [0,T]\times\mathbb{R}$. This also implies that $(A3a')$ holds. Moreover, $\beta^2$ is uniformly elliptic on $(t,y) \in [0,T]\times\bar{D}_n$, for every $n \in \mathbb{N}$; i.e. $(A3b')$ holds. The Hölder-continuity of functions $a$, $b$ and $\lambda$ imply that also the function $g$, defined by (1.70), is Hölder-continuous, as required by $(A3c')$. Further, in our problem $g \equiv 0$ and $h \equiv 1$; thus, condition $(A3d')$ is trivially satisfied. To complete the proof, we check for $(A3e')$. Thanks to [58, Lemma 2], it is sufficient to prove that the function $g$ is continuous and bounded from above, which is satisfied under our assumptions. Hence, the PDE (1.77) admits a unique classical solution $\xi(t,y)$ on $(t,y)\in[0,T]\times\mathbb{R}$ satisfying $\xi(T,y) = 1$, given by (1.78). □

### 1.5.5 The conditional certainty equivalent

In this subsection, we discuss the conditional certainty equivalent (in short CCE) introduced in [53], which extends the classical notion of the certainty equivalent.

**Definition 1.4.** *Let $X^H = \{X_t^H,\ t \geq 0\}$ be the wealth process corresponding to a constant strategy $H = (\Theta, \Pi) \in [0,1]\times\mathbb{R}$ and let $T > 0$ be a finite time horizon. For every $0 \leq t \leq T$, let $U_t(x)$ be the forward dynamic utility defined according to Definition* (1.2)*. Then, we define the* conditional certainty equivalent $C_t(X_T^H)$ *as the random variable given by*

$$C_t(X_T^H) = U_t^{-1}\Big(\mathbb{E}\left[U_T(X_T^H, 0)|\mathcal{F}_t\right], 0\Big),$$

*for every $t \in [0,T]$.*

The inverse exponential utility process $\{U_t^{-1}(x,0),\ t \geq 0\}$, and hence the CCE, is well defined (see [53, Lemma 1.1 and Definition 1.1]) and satisfies the properties below that can be directly derived from [53, Proposition 1.1]: for every $0 \leq t \leq s \leq T$ and every constant strategies $H, \widetilde{H} \in [0,1]\times\mathbb{R}$, we have

(i) $C_t(X_T^H) = C_t\big(C_s(X_T^H)\big)$.

(ii) $C_t(X_t^H) = X_t^H$.

(iii) If $C_s(X_T^H) \leq C_s(X_T^{\widetilde{H}})$ then $C_t(X_T^H) \leq C_t(X_T^{\widetilde{H}})$.
In particular, if $X_T^H \leq X_T^{\widetilde{H}}$ then $C_t(X_T^H) \leq C_t(X_T^{\widetilde{H}})$ and if $X_T^H = X_T^{\widetilde{H}}$ then $C_t(X_T^H) = C_t(X_T^{\widetilde{H}})$.

(iv) If $U(x,0) = \{U_t(x,0),\ t \in [0,T]\}$ is decreasing in time, then $C_t(X_T^H) \leq \mathbb{E}\left[C_s(X_T^H)|\mathcal{F}_t\right]$ and $\mathbb{E}\left[C_t(X_T^H)\right] \leq \mathbb{E}\left[C_s(X_T^H)\right]$. Moreover, $C_t(X_T^H) \leq \mathbb{E}\left[X_T^H|\mathcal{F}_t\right]$ and therefore $\mathbb{E}\left[C_t(X_T^H)\right] \leq \mathbb{E}\left[X_T^H\right]$.

The properties listed above have important financial implications. The first one, coming from property *(iii)*, is time consistency. That is, any two wealths with the same CCE at a given time $s$, have the same CCE at any time prior than $s$. Second, by property *(iv)*, we get that for dynamic utilities that are decreasing in time, the CCE is increasing. We point out that it is common practice to consider time-decreasing utilities since they reflect the impatience of the insurance company, namely the effective desire for accumulation mixed by the perspective undervaluation of the future. In this case, the guaranteed amount that a company would accept not to take the risk, becomes larger and larger as time goes, as a consequence of a the fact that the company has a better perception of the combined market conditions, which is represented by a smaller utility. Let us underline that the inequality $C_t(X_T^H) \leq \mathbb{E}\left[X_T^H|\mathcal{F}_t\right]$, also provided in *(iv)*, expresses the risk aversion of the company, similarly to the well-know inequality in the static environment.

In the sequel, we focus on the case of zero-volatility forward dynamic exponential utilities, discussed in Section 1.5.4.

In this case it is immediate to see that the monotonicity property with respect to time holds true. Indeed, if the function $g(t,y)$ defined in (1.70) is non-negative for every $(t,y) \in [0,+\infty) \times \mathbb{R}$, then given a constant strategy $H = (\Theta,\Pi) \in [0,1] \times \mathbb{R}$, the forward exponential utility process (1.71) is decreasing in time, and hence

$$C_t(X_T^H) = -\frac{1}{\gamma} \ln\left(\mathbb{E}\left[e^{-\gamma X_T^H - \int_0^T g(s,Y_s)\mathrm{d}s}\Big|\mathcal{F}_t\right]\right) - \frac{1}{\gamma}\int_0^t g(s,Y_s)\mathrm{d}s,$$

for every $t \in [0,T]$, i.e. the CCE is increasing in time.

Next, we will use the CCE to provide an additional comparison between forward dynamic and classical static backward exponential utilities (i.e. $U(x) = -e^{-\gamma x}$, $x \in \mathbb{R}$). We observe that CCE, for the static case is given by

$$\widetilde{C}_t(X_T^H) = -\frac{1}{\gamma}\ln\left(\mathbb{E}\left[e^{-\gamma X_T^H}\middle|\mathcal{F}_t\right]\right),$$

for every $t \in [0,T]$.

At time $t = 0$, CCEs under forward exponential dynamic utility and backward exponential static utility reduce, respectively, to

$$C_0(X_T^H) = -\frac{1}{\gamma}\ln\left(\mathbb{E}\left[e^{-\gamma X_T^H - \int_0^T g(s,Y_s)\mathrm{d}s}\right]\right), \qquad \widetilde{C}_0(X_T^H) = -\frac{1}{\gamma}\ln\left(\mathbb{E}\left[e^{-\gamma X_T^H}\right]\right).$$

Then, we have the following implication.

**Lemma 1.3.** *For every $t \in [0,T]$, the cumulative penalizing $P(T) - P(t) > 0$, $\mathbf{P}-a.s.$ if and only if $C_t(X_T^H) > \widetilde{C}_t(X_T^H)$, $\mathbf{P}-a.s.$, for any given constant strategy $H \in [0,1]\times\mathbb{R}$. In particular, $P(T) > 0$ if and only if $C_0(X_T^H) > \widetilde{C}_0(X_T^H)$.*

*Proof.* For every $t \in [0,T]$, we consider the difference

$$C_t(X_T^H) - \widetilde{C}_t(X_T^H) = -\frac{1}{\gamma}\ln\left(\mathbb{E}\left[e^{-\gamma X_T^H - \int_t^T g(s,Y_s)\mathrm{d}s}\middle|\mathcal{F}_t\right]\right) + \frac{1}{\gamma}\ln\left(\mathbb{E}\left[e^{-\gamma X_T^H}\middle|\mathcal{F}_t\right]\right)$$

This is larger than zero if and only if

$$\mathbb{E}\left[e^{-\gamma X_T^H}\middle|\mathcal{F}_t\right] > \mathbb{E}\left[e^{-\gamma X_T^H - \int_t^T g(s,Y_s)\mathrm{d}s}\middle|\mathcal{F}_t\right].$$

Since $e^{-\gamma X_T^H}$ and $e^{-\int_t^T g(s,Y_s)\mathrm{d}s}$ are both nonnegative random variables then we get that the inequality holds if and only if $0 < e^{-\int_t^T g(s,Y_s)\mathrm{d}s} < 1$. Recalling that $P(T) - P(t) = \int_t^T g(s,Y_s)\mathrm{d}s$ we get that $C_t(X_T^H) > \widetilde{C}_t(X_T^H)$ if and only if $P(T) - P(t) > 0$. Taking $t = 0$ we obtain the second inequality for the CCEs at the initial time. □

Lemma 1.3 provides necessary and sufficient conditions to establish a relationship between CCEs under forward exponential dynamic utility and backward exponential static utility, at any time $t$. First of all, we comment the result at time $t = 0$. At the beginning of time interval, the guaranteed amount that an insurance company endowed with a forward utility would accept not to take the risk of investing and reinsure its claims is larger than the guaranteed amount for an insurance company with static backward utility, i.e. $C_0(X_T^H) \geq \widetilde{C}_0(X_T^H)$, if and only if the total penalizing $P(T) = \int_0^T g(s,Y_s)\mathrm{d}s$ is nonnegative. Indeed, in this case we will have that the forward utility is smaller than the backward utility, and hence that, at time $t = 0$, the perception of the risk for the first company (the one with the forward utility) is lower than for the second company (the one with the backward). Such condition is satisfied, e.g., when the penalizing process is increasing, that is when $g(t, y) > 0$, for every $(t, y) \in [0, T] \times \mathbb{R}$.

When $t > 0$, the situation gets complicated. Due to the structure of the penalizing process, it is not easy to compare, in general, the values of $C_t(X_T^H)$ and $\widetilde{C}_t(X_T^H)$. However, under specific conditions some considerations can be made, as we see in the next Proposition.

**Proposition 1.5.** *If*

$$\frac{\mu^2(t,y)}{\sigma^2(t,y)} \geq -2\gamma a(t,y) + 2\min\left\{b(t,y,1), \lambda(t,y)\int_I \left(e^{\gamma z} - 1\right) F(t,y,\mathrm{d}z)\right\}, \tag{1.79}$$

*for every* $(t, y) \in [0, T] \times \mathbb{R}$, *then* $\widetilde{C}_t(X_T^H) \geq C_t(X_T^H)$, *for each* $t \in [0, T]$.

*Otherwise, let* $K := \inf_{(t,y)\in[0,T]\times\mathbb{R}} \gamma b(t, y, \bar{\Theta}) + \lambda(t, y) \int_I \left(e^{\gamma(1-\bar{\Theta})z} - 1\right) F(t, y, \mathrm{d}z) > 0$, *with* $\bar{\Theta}$ *given by* (1.12). *If* $a(t, y) < K/\gamma$ *for all* $(t, y) \in [0, T] \times \mathbb{R}$ *and*

$$\frac{\mu^2(t,y)}{\sigma^2(t,y)} \leq 2\left(-\gamma a(t,y) + K\right), \tag{1.80}$$

*for every* $(t, y) \in [0, T] \times \mathbb{R}$, *then* $C_t(X_T^H) \geq \widetilde{C}_t(X_T^H)$, *for each* $t \in [0, T]$.

*Proof.* Recall that, in the zero-volatility case, the function $g(t, y)$ is given by

$$g(t,y) = -\frac{1}{2}\frac{\mu^2(t,y)}{\sigma^2(t,y)} - \gamma a(t,y) + f(t,y,\bar{\Theta}),$$

where we have set $f(t, y, \Theta) = \gamma b(t, y, \Theta) + \lambda(t, y) \int_I \left(e^{\gamma(1-\Theta)z} - 1\right) F(t, y, \mathrm{d}z)$, for each $\Theta \in [0, 1]$. Notice that $f(t, y, \Theta)$ is convex in $\Theta$, then it admits minimum, which is given by $\bar{\Theta}$. Moreover, since $\Theta \in [0, 1]$, by convexity we get that

$$f(t,y,\bar{\Theta}) \leq \min\left\{\gamma b(t,y,1), \lambda(t,y)\int_I \left(e^{\gamma z} - 1\right) F(t,y,\mathrm{d}z)\right\}, \tag{1.81}$$

for every $(t, y) \in [0, T] \times \mathbb{R}$; Then, by (1.79) and (1.81), we get that $g(t, y) \leq 0$, for each $(t, y) \in [0, T] \times \mathbb{R}$, yielding $P(T) - P(t) \leq 0$, for every $t \in [0, T]$. Lemma 1.3 implies that $\widetilde{C}_t(X_T^H) \geq C_t(X_T^H)$, for each $t \in [0, T]$.

On the other hand, we note that $f(t, y, \bar{\Theta}) > 0$, for every $(t, y) \in [0, T] \times \mathbb{R}$. Indeed, if $\bar{\Theta} \neq \{0, 1\}$, then $b(t, y, \bar{\Theta}) > 0$ and $\int_I \left(e^{\gamma(1-\Theta)z} - 1\right) F(t, y, \mathrm{d}z) > 0$. For $\bar{\Theta} = 1$ $f(t, y, 1) =$

$\gamma b(t,y,1) > 0$ and for $\bar{\Theta} = 0$ it holds that $f(t,y,0) = \lambda(t,y)\int_I (e^{\gamma z} - 1)\, F(t,y,\mathrm{d}z) > 0$. We let $K > 0$ be the infimum over all $(t,y) \in [0,T] \times \mathbb{R}$ of the function $f(t,y,\bar{\Theta})$. If $a(t,y) < K/\gamma$ for all $(t,y) \in [0,T] \times \mathbb{R}$ and (1.80), we get that

$$g(t,y) = -\frac{1}{2}\frac{\mu^2(t,y)}{\sigma^2(t,y)} - \gamma a(t,y) + f(t,y,\bar{\Theta}) > 0,$$

which implies $P(T) - P(t) \geq 0$, for every $t \in [0,T]$. Finally, thanks to Lemma 1.3, we have that $\widetilde{C}_t(X_T^H) \leq C_t(X_T^H)$, for each $t \in [0,T]$. □

Proposition 1.5 is technical but meaningful. We could say that if market conditions are favorable, the insurance company with forward utility preferences has smaller certainty equivalent, meaning that they are more willing to risk. When we speak of favorable market, we mean the situation when the revenues from investment allow to cover for the payment of the claims, according to (1.79). In this case the amount of guaranteed money an insurance company would accept today instead of taking a risk of getting more money at a future date, is larger when preferences are described by a forward dynamic utility. Instead, if company's income (that comes from investing in a risky stock and collecting insurance premia) is not sufficient to cover reinsurance premia and payments of the remaining claims, then the relationship between CCEs reverts: in an unfavourable market, the forward approach leads to a lower willingness to risk.

In the numerical section we present two examples where each of the two conditions of Proposition 1.5 is satisfied.

### 1.5.6 Toy example: CCE and comparison with backward investment

We conclude this chapter with a comparison analysis between optimal strategies under forward and backward utilities. We give a toy example of our proposed model where the backward value function as well as the optimal strategy are characterized in closed form.

We consider a trading interval $[0,T]$, with $T = 1$, assuming that all market operations take place in one year, starting from today. We assume that the claim arrival intensity is a quadratic function of $y$, i.e. $\lambda(y) = \lambda_0(1 + y + \frac{1}{2}y^2)$, so that it stays positive. We specify that this choice of the function $\lambda$ corresponds to the second order approximation of $\lambda(y) = \lambda_0 e^y$, where $\lambda_0$ is a nonnegative constant, which is taken in the previous numerical experiments.

Now, we suppose that claims follow a Gamma distribution $\Gamma(\alpha_\Gamma, \beta_\Gamma)$, where $\alpha_\Gamma = \beta_\Gamma = 1$ (i.e. exponential distribution with mean 1).

Moreover, we employ the well-known expected value principle to calculate insurance and reinsurance premia, that is

$$a(t, Y_t) = (1 + \delta_I)\mathbb{E}[Z_1]\,\lambda(t, Y_t),$$
$$b(t, Y_t, \Theta) = (1 + \delta_R)\lambda(t, Y_t)\mathbb{E}[Z_1]\,\Theta_t.$$

For simplicity we denote $a = (1 + \delta_I)\mathbb{E}[Z_1]$ and $b^{(\Theta)} = (1 + \delta_R)\mathbb{E}[Z_1]\,\Theta_t$ and $\mathbb{E}[Z_1] = \int_I zF(\mathrm{d}z)$, where the insurance safety loading and the reinsurance safety loading now are set as $\delta_I = 0.4$ and $\delta_R = 0.7$, respectively. We notice that in this case the optimal reinsurance strategy is given by $\Theta^{*,B} = \min\{1, \bar{\Theta}\}$ where $\bar{\Theta}$ is the unique solution of the equation $(1 + \delta_R)\mathbb{E}[Z_1] = \int_I z e^{\gamma z \Theta} F(\mathrm{d}z)$. Clearly, $\Theta^{*,B}$ does not depend on the stochastic factor $Y$. We suppose that the dynamic of $S$ is given by

$$\mathrm{d}S_t = S_t(\mu_1 + \mu_2 Y_t)\mathrm{d}t + S_t \sigma \mathrm{d}W_t^S, \quad S_0 = 1,$$

where now $\sigma = \bar{c}\sqrt{\epsilon_1 + 1}$ is a non negative constant (it corresponds to the Scott volatility above taking $\epsilon_2 = 0$). As for the other parameters, we take $\mu_1 = 0.08$, $\mu_2 = 0.2$, $\bar{c} = 0.27$ and $\epsilon_1 = 0.01$. We also denote $c(\Theta) = \int_I \left(e^{\gamma(1-\Theta)z} - 1\right)F(\mathrm{d}z)$. In this case the function $g(t, y)$ is a quadratic function given by

$$g(t, y) = -\frac{\mu_1^2}{2\sigma^2} - \left(\gamma a - \gamma b(\Theta^{*,B}) - c(\Theta^{*,B})\right)\lambda_0(t)$$
$$-\left(\frac{\mu_1\mu_2}{\sigma^2} + \left(\gamma a - \gamma b(\Theta^{*,B}) - c(\Theta^{*,B})\right)\lambda_0(t)\right) y - \left(\frac{\mu_2^2}{2\sigma^2} + \left(\gamma a - \gamma b(\Theta^{*,B}) - c(\Theta^{*,B})\right)\frac{\lambda_0(t)}{2}\right) y^2.$$

To solve the PDE (1.74) we consider the following guess function:

$$\phi(t, y; T) = \phi^{(0)}(t) + \phi^{(1)}(t)y + \phi^{(2)}(t)y^2.$$

Then, the optimal investment strategy becomes $\Pi^{*,B}(t, y) = \frac{\mu_1 + \mu_2 y}{\gamma\sigma^2} + \rho\frac{\beta(t)(\phi^{(1)}(t) + 2\phi^{(2)}(t)y)}{\gamma\sigma}$. Plugging the guess function in (1.74) and collecting the coefficients of $y$, $y^2$ and the constant term leads to the following system of ODEs

$$\begin{aligned}\frac{\partial \phi^{(0)}}{\partial t}(t) =& -\phi^{(1)}(t)\big(\alpha_1(t) - \rho\frac{\mu_1\beta(t)}{\sigma}\big) - \frac{1}{2}\big(\phi^{(1)}(t)\big)^2(1-\rho^2)\beta^2(t) - \phi^{(2)}(t)\beta^2(t)\\ &+ \frac{\mu_1^2}{2\sigma^2} + \big(\gamma a - \gamma b(\Theta^{*,B}) - c(\Theta^{*,B})\big)\lambda_0(t)\\ \frac{\partial \phi^{(1)}}{\partial t}(t) =& -\phi^{(1)}(t)\big(\alpha_2(t) - \rho\frac{\mu_2\beta(t)}{\sigma}\big) - 2\phi^{(2)}(t)\big(\alpha_1(t) - \rho\frac{\mu_1\beta(t)}{\sigma^2}\big)\\ &- 2\phi^{(1)}(t)\phi^{(2)}(t)(1-\rho^2)\beta^2(t) + \frac{\mu_1\mu_2}{\sigma^2} + \big(\gamma a - \gamma b(\Theta^{*,B}) - c(\Theta^{*,B})\big)\lambda_0(t)\\ \frac{\partial \phi^{(2)}}{\partial t}(t) =& -2\phi^{(2)}(t)\big(\alpha_2(t) - \rho\frac{\mu_2\beta(t)}{\sigma}\big) - 2(\phi^{(2)})^2(1-\rho^2)\beta^2(t)\\ &+ \frac{\mu_2^2}{2\sigma^2} + \big(\gamma a - \gamma b(\Theta^{*,B}) - c(\Theta^{*,B})\big)\frac{\lambda_0(t)}{2},\end{aligned}$$

with the final conditions $\phi^{(0)}(T) = \phi^{(1)}(T) = \phi^{(2)}(T) = 0$. Regularity of the coefficients guarantees that a solution exists.

To highlight the effect of the common index on the optimal investment portfolio, we plot in Figure 1.12 the optimal investment strategies corresponding to forward and backward utilities as functions of the stochastic factor. Precisely, Figure 1.12 depicts the marginal impact on the optimal investment policy (both under forward and backward utilities) of the stochastic factor, for two different values of the correlation coefficient. Clearly, for the backward case, we have to fix an instant of time in order to represent the strategy as a function of the stochastic factor; for illustrative purposes, we check the strategy at the initial of the trading period and after 6 months, i.e. we choose the beginning and the middle of the time interval.

It is worth noting that the optimal portfolio varies less if the preferences of the company are described by forward utility. This means that the backward optimal portfolio is more sensitive to any variation of the stochastic factor with respect to the forward one, and this effect is amplified when the correlation coefficient is large (right panel). Moreover, we observe that this effect flattens out as maturity approaches: indeed, the big range of values at the beginning of the trading period shrinks a lot after 6 months. It is also clear that backward strategy gets closer and closer to the forward one, as the correlation coefficient approaches to zero, and they actually coincide when $\rho = 0$. The difference between the optimal strategy under the forward utility and the optimal strategy under the backward utility decreases with time: the additional risk adjustment characterizing the backward portfolio gets smaller and

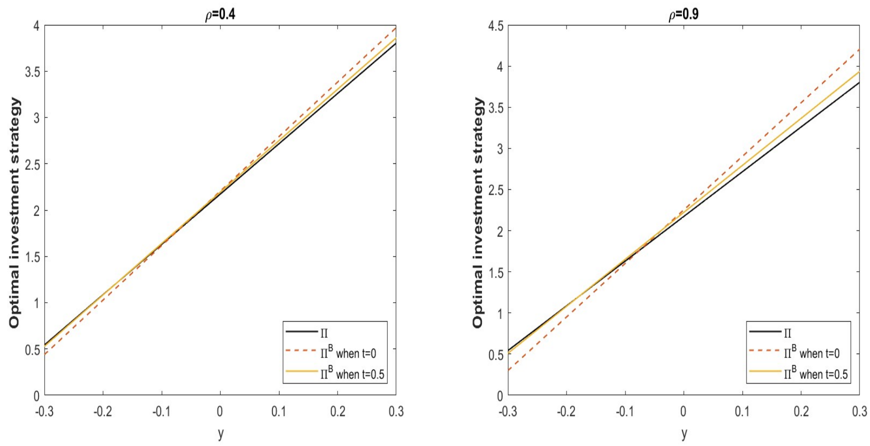



Figure 1.12: The optimal investment strategy when $\rho = 0.4$ (left panel) $\rho = 0.9$ (right panel), as functions of the stochastic factor. Solid (resp. dashed lines) line corresponds to the optimal portfolio under Forward (resp. Backward) utility.

smaller. Indeed, as time to maturity reduces, also the estimates of future risk in the backward case has a smaller effect on the value function and hence on the optimal strategy that gets closer and closer to the myopic component.

We conclude with a brief analysis of the CCE in order to outline some of its features. In Figure 1.13 we provide a trajectory of the process $R_t := \mathbb{E}\left[X_T^H|\mathcal{F}_t\right] - C_t(X_T^H)$, for $t \in [0,T]$, which expresses the risk aversion of the company during the time interval. This process is decreasing over time and disappears at maturity. Moreover, this process is mainly affected by big claims: when a large claim occurs, there is a downward peak. Next we compare the

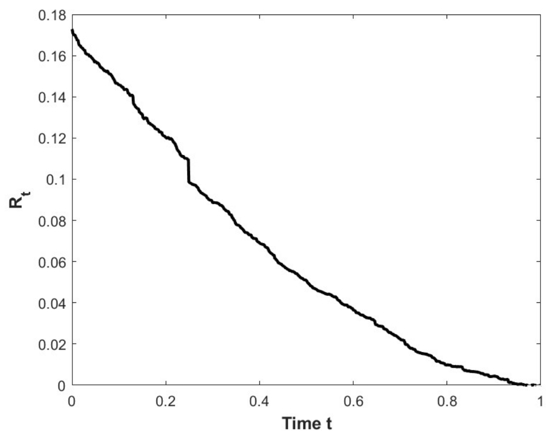



Figure 1.13: The risk aversion of the company with respect to time.

CCE for the forward and the backward utility preferences. To make the presentation more clear, we fix $\rho = 0$. Under the parameter setting that has been fixed in this subsection, (1.80) holds true. Therefore, by Proposition 1.4, we get that $P(T) - P(t) > 0$, $\mathbf{P}-$a.s. that in turns implies that $C_t(X_T^H) > \widetilde{C}_t(X_T^H)$, $\mathbf{P}-$a.s., thanks to Lemma 1.3. Figure 1.14 confirms this lower willingness to risk under unfavorable circumstances in the forward approach rather in the backward one.

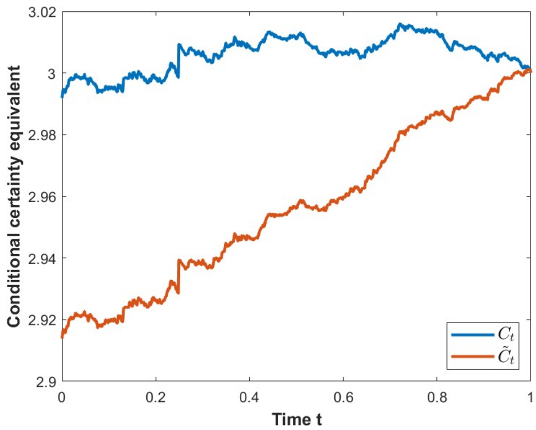



Figure 1.14: CCEs under forward exponential dynamic utility and backward exponential static utility, with $\beta_\Gamma = 1$ and $\bar{c} = 0.27$.

For comparison purposes we also analyze the case of a smaller claim size and a smaller stock volatility, taking $\beta_\Gamma = 1/3$ and $\bar{c} = 0.1$. As a result, condition (1.79) holds true and thus the relationship between the CCEs reverts, as shown in Figure 1.15.

This parameter setting represents a favorable market where the revenues from financial investments with the collected premia cover claim payments and hence the company does not buy reinsurance and instead invests a large part of its wealth in the risky asset. We obtain that the CCE under forward preferences is always smaller than the CCE under backward preferences, i.e. $C_t(X_T^H) < \widetilde{C}_t(X_T^H)$, $\mathbf{P}-$a.s., meaning that the company opts to risk more if its preferences are specified forward in time. We also notice that the CCE in the forward case fluctuates slowly around a specific value whereas in the backward case the CCE shows a remarkable decrease over time, due to the lower flexibility of the backward utility.

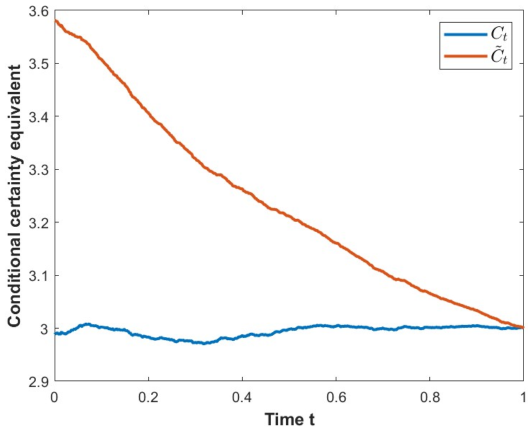



Figure 1.15: CCEs under forward exponential dynamic utility and backward exponential static utility, with $\beta_\Gamma = 1/3$ and $\bar{c} = 0.1$.

# Appendix

## A.1 Assumptions

This section contains a set of classical sufficient conditions on model coefficients under which existence and uniqueness of the solutions of the SDEs for the processes $Y$ and $S$ hold.

**Assumption 1.4.** *The functions $\alpha$ and $\beta$ satisfy:*

*i.* ***local Lipschitz-continuity**: for every $R > 0$ there is a constant $\kappa_R > 0$ such that for every $y_1, y_2 \in B(0,R)$[3] it holds that $|\alpha(t,y_1) - \alpha(t,y_2)| + |\beta(t,y_1) - \beta(t,y_2)| \leq \kappa_R |y_1 - y_2|$, for all $t \geq 0$;*

*ii.* ***sublinear grownth**: there is a constant $\kappa > 0$ such that for every $y \in \mathbb{R}$ it holds that $|\alpha(t,y)| + |\beta(t,y)| < \kappa(1 + |y|)$, for all $t \geq 0$.*

Under Assumption 1.4, classical results (see, for instance, Gihman and Skorohod [54]) yield that,for every initial condition $(t,y) \in [0,T] \times \mathbb{R}$, existence and uniqueness of a strong solution to equation (1.55) and that condition (1.56) is satisfied.

To ensure existence of equation (1.57), which describes the stock price process, we make the following assumption.

**Assumption 1.5.** *The functions $\mu$ and $\sigma$ satisfy:*

[3] $B(0,R)$ is the disc centered at the origin with radius $R$.

i. ***local Lipschitz-continuity***: *for every* $\bar{R} > 0$ *there is a constant* $\bar{\kappa}_{\bar{R}} > 0$ *such that for every* $y_1, y_2 \in B(0, \bar{R})$ *it holds that* $|\mu(t, y_1) - \mu(t, y_2)| + |\sigma(t, y_1) - \sigma(t, y_2)| \leq \bar{\kappa}_{\bar{R}} |y_1 - y_2|$, *for all* $t \geq 0$;

ii. ***sublinear grownth***: *there is a constant* $\bar{\kappa} > 0$ *such that for every* $y \in \mathbb{R}$ *it holds that* $|\mu(t, y)| + |\sigma(t, y)| < \bar{\kappa}(1 + |y|)$, *for all* $t \geq 0$.

Under Assumptions 1.4 and 1.5, the system of equations (1.55)-(1.57) admits a unique strong solution $(Y, S)$, with $S_t > 0$, $\mathbf{P}$-a.s., for every $t \geq 0$, see e.g. Øksendal [84, Theorem 5.2.1]. Moreover the pair $(Y, S)$ is an $(\mathbb{F}, \mathbf{P})$-Markov process and it holds that

$$\mathbb{E}\left[\int_0^t |\mu(s, Y_s)| \mathrm{d}s + \int_0^t \sigma(s, Y_s)^2 \mathrm{d}s\right] < \infty,$$

for every $t \geq 0$.

## A.2 Technical results

In this section, we collect some technical proofs.

### The verification theorem

Here, we prove a quite general verification result, which ensures that the function $u(t, x, y, p)$ defined in (1.67) is the unique solution of the optimization problem (1.64).

**Theorem 1.5** (Verification Theorem)**.** *Let* $T \geq 0$ *and let* $\bar{u} : [0, T] \times \mathbb{R}^3 \to (-\infty, 0)$ *be a classical solution of the final value problem* (1.65)-(1.66) *which satisfies*

(i) $\mathbb{E}\left[\int_0^T \left(\sigma(r, Y_r)\Pi_r \frac{\partial \bar{u}}{\partial x}(r, X_r^H, Y_r, P_r)\right)^2 \mathrm{d}r\right] < \infty$,

(ii) $\mathbb{E}\left[\int_0^T \left(\beta(r, Y_r) \frac{\partial \bar{u}}{\partial y}(r, X_r^H, Y_r, P_r)\right)^2 \mathrm{d}r\right] < \infty$,

(iii) $\mathbb{E}\left[\int_0^T \left(h(r, X_r^H, Y_r) \frac{\partial \bar{u}}{\partial p}(r, X_r^H, Y_r, P_r)\right)^2 \mathrm{d}r\right] < \infty$,

(iv) $\mathbb{E}\left[\int_0^T \lambda(r, Y_r) \int_I \left|\bar{u}\big(r, X_{r-}^H - (1 - \Theta_{r-})z, Y_r, P_{r-}\big) - \bar{u}(r, X_{r-}^H, Y_r, P_{r-})\right| F(r, Y_r, \mathrm{d}z) \mathrm{d}r\right] < \infty$.

(*a*) *Hence,* $u(t, x, y, p) \leq \bar{u}(t, x, y, p)$, *for every admissible control* $H \in \mathcal{A}$ *and for every* $(t, x, y, p) \in [0, T] \times \mathbb{R}^3$.

(*b*) *Moreover, if* $\bar{u}(T,x,y,p) = u(T,x,y,p)$*, for every* $(x,y,p) \in \mathbb{R}^3$ *and there exists* $H^* \in \mathcal{A}$ *such that* $\mathcal{L}^{H^*}\bar{u}(t,x,y,p) = 0$*, for every* $(t,x,y,p) \in [0,T) \times \mathbb{R}^3$*, then* $u = \bar{u}$ *in* $[0,T] \times \mathbb{R}^3$.

*Proof.* Let $H \in \mathcal{A}$ be an admissible control. Using equations (1.58) and (1.55) and applying Itô's formula to $\bar{u}(t, X_t^H, Y_t, P_t)$, we have that

$$\begin{aligned}\bar{u}(T, X_T^H, Y_T, P_T) &= \bar{u}(t,x,y,p) + \int_t^T \mathcal{L}^H \bar{u}(r, X_r^H, Y_r, P_r)\mathrm{d}r \\ &+ \int_t^T \Pi_r \sigma(r, Y_r)\frac{\partial \bar{u}}{\partial x}(r, X_r^H, Y_r, P_r)\mathrm{d}W_r^S + \int_t^T \beta(r, Y_r)\frac{\partial \bar{u}}{\partial y}(r, X_r^H, Y_r, P_r)\mathrm{d}W_r^Y \\ &+ \int_t^T h(r, X_r^H, Y_r)\frac{\partial \bar{u}}{\partial p}(r, X_r^H, Y_r, P_r)\mathrm{d}W_r^P \\ &+ \int_t^T \int_I \big(\bar{u}\big(r, X_{r-}^H - (1-\Theta_{r-})z, Y_r, P_{r-}\big) - \bar{u}(r, X_{r-}^H, Y_r, P_{r-})\big) \\ &\quad \times (m(\mathrm{d}r, \mathrm{d}z) - \lambda(r, Y_r)F(r, Y_r, \mathrm{d}z)\mathrm{d}r),\end{aligned}$$

where $\mathcal{L}^H$ is introduced in (1.61). Let $M = \{M_t,\ t \in [0,T]\}$ be the stochastic process given by

$$\begin{aligned}M_t &= \int_0^t \Pi_r \sigma(r, Y_r)\frac{\partial \bar{u}}{\partial x}(r, X_r^H, Y_r, P_r)\mathrm{d}W_r^S + \int_0^t \beta(r, Y_r)\frac{\partial \bar{u}}{\partial y}(r, X_r^H, Y_r, P_r)\mathrm{d}W_r^Y \\ &+ \int_t^T h(r, X_r^H, Y_r)\frac{\partial \bar{u}}{\partial p}(r, X_r^H, Y_r, P_r)\mathrm{d}W_r^P \\ &+ \int_0^t \int_I \big(\bar{u}\big(r, X_{r-}^H - (1-\Theta_{r-})z, Y_r, P_{r-}\big) - \bar{u}(r, X_{r-}^H, Y_r, P_{r-})\big) (m(\mathrm{d}r, \mathrm{d}z) - \lambda(r, Y_r)F(r, Y_r, \mathrm{d}z)\mathrm{d}r)\end{aligned}$$

and observe that integrability conditions (i), (ii), (iii), (iv) ensure that the process $M$ is a martingale. Now, since $\bar{u}$ solves equation (1.65) with final condition (1.66), we get

$$\begin{aligned}\bar{u}(T, X_T^H, Y_T, P_T) &\leq \bar{u}(t,x,y,p) + \int_t^T \Pi_r \sigma(r, Y_r)\frac{\partial \bar{u}}{\partial x}(r, X_r^H, Y_r, P_r)\mathrm{d}W_r^S \\ &+ \int_t^T \beta(r, Y_r)\frac{\partial \bar{u}}{\partial y}(r, X_r^H, Y_r, P_r)\mathrm{d}W_r^Y + \int_t^T h(r, X_r^H, Y_r)\frac{\partial \bar{u}}{\partial p}(r, X_r^H, Y_r, P_r)\mathrm{d}W_r^P \\ &+ \int_t^T \int_I \big(\bar{u}\big(r, X_{r-}^H - (1-\Theta_{r-})z, Y_r, P_{r-}\big) - \bar{u}(r, X_{r-}^H, Y_r, P_{r-})\big) (m(\mathrm{d}r, \mathrm{d}z) - \lambda(r, Y_r)F(r, Y_r, \mathrm{d}z)\mathrm{d}r),\end{aligned} \tag{1.82}$$

for every $H \in \mathcal{A}$. Thus, taking the conditional expectation with respect to $X_t^H = x$, $Y_t = y$ and $P_t = p$ on both sides of inequality (1.82), leads to

$$\mathbb{E}_{t,x,y,p}\Big[\bar{u}(T, X_T^H, Y_T, P_T)\Big] \leq \bar{u}(t,x,y,p).$$

By the final condition in equation (1.66), we obtain

$$\mathbb{E}_{t,x,y,p}\Big[ -e^{-\gamma X_T^H-(P_T-P_t)}\Big] \leq \bar{u}(t,x,y,p),$$

for every $H \in \mathcal{A}$. Hence, $u(t,x,y,p) \leq \bar{u}(t,x,y,p)$, as we wanted. Finally, we observe that if $H \in \mathcal{A}$ is the maximizer in equation (1.65) with final condition (1.66), then the inequality above becomes an equality, and we obtain statement $(b)$, which concludes the proof. □

**Proof of Theorem 1.4**

We notice that the optimization is taken over the set of admissible functions $\mathcal{A}$, even though in the backward case one would require that $\mathbb{E}\left[e^{-\gamma X_T^H}\right] < \infty$ in place of $\mathbb{E}\left[e^{-\gamma X_T^H - P_T}\right] < \infty$. However, because of the assumptions on model coefficients, these two conditions are equivalent. The proof of this result uses a guess-and-verify approach. Suppose that the value function $V(t,x,y)$ is $\mathcal{C}^1$ in $t$ and $\mathcal{C}^2$ in $x$ and $y$, then it solves the equation

$$\max_{(\Theta^B,\Pi^B)\in[0,1]\times\mathbb{R}} \tilde{\mathcal{L}}^H V(t,x,y) = 0, \quad (t,x,y) \in [0,T) \times \mathbb{R}^2, \tag{1.83}$$

where $\tilde{\mathcal{L}}^H$ is the infinitesimal generator given in (1.60), with the terminal condition $V(T,x,y) = -e^{-\gamma x}$, for every $x \in \mathbb{R}$. We guess that the value function has the form $V(t,x,y) = -e^{-\gamma x+\phi(t,y;T)}$, where $\phi(t,y;T)$ is the unique classical solution of the problem (1.74). Plugging this expression into (1.83) and taking the first order condition yields (1.74). The second order conditions imply that the optimal investment strategy $\Pi^{*,B}$ is given by (**??**) and the optimal reinsurance strategy is given by $\Theta^{*,B}(t,y)$ as in (1.13).

Next, we establish a verification result. Let $v(t,x,y)$ be a solution of the equation (1.83) with the final condition $v(T,x,y) = -e^{-\gamma x}$ (that is $v(T,x,y) = V(T,x,y)$). Then, by Itô's formula it holds that (we omit for simplicity the dependence of $X$ on the strategy $H$)

$$\begin{aligned} v(T,X_T,Y_T) &= v(t,x,y) + \int_t^T \tilde{\mathcal{L}}^H v(r,X_r,Y_r)\mathrm{d}r \\ &+ \int_t^T \Pi_r \sigma(r,Y_r)\frac{\partial v}{\partial x}(r,X_r,Y_r)\mathrm{d}W_r^S + \int_t^T \beta(r,Y_r)\frac{\partial v}{\partial y}(r,X_r,Y_r)\mathrm{d}W_r^Y \\ &+ \int_t^T \int_I v(r,X_{r-}-(1-\Theta_{r-})z,Y_r) - v(r,X_{r-},Y_r)\left(m(\mathrm{d}r,\mathrm{d}z) - \lambda(r,Y_r)F(r,Y_r,\mathrm{d}z)\mathrm{d}r\right). \end{aligned}$$

Since $v$ satisfies equation (1.83), we get that

$$v(T,X_T,Y_T) \le v(t,x,y) + \int_t^T \Pi_r \sigma(r,Y_r)\frac{\partial v}{\partial x}(r,X_r,Y_r)\mathrm{d}W_r^S + \int_t^T \beta(r,Y_r)\frac{\partial v}{\partial y}(r,X_r,Y_r)\mathrm{d}W_r^Y$$

$$+ \int_t^T \int_I v(r,X_{r-}-(1-\Theta_{r-})z,Y_r) - v(r,X_{r-},Y_r)\left(m(\mathrm{d}r,\mathrm{d}z) - \lambda(r,Y_r)F(r,Y_r,\mathrm{d}z)\mathrm{d}r\right) \quad (1.84)$$

If the process on the right side of (1.84) is a martingale, taking the expectation yields

$$V(t,x,y) \le v(t,x,y),$$

and the equality holds if $H$ is a maximizer of equation (1.83). Then, it only remains to prove that the function $V(t,x,y) = -e^{-\gamma x+\phi(t,y;T)}$ is such that

$$\begin{aligned}
M_t &= \int_0^t \Pi_r \sigma(r,Y_r)\frac{\partial V}{\partial x}(r,X_r,Y_r)\mathrm{d}W_r^S + \int_0^t \beta(r,Y_r)\frac{\partial V}{\partial y}(r,X_r,Y_r)\mathrm{d}W_r^Y \\
&\quad + \int_0^t \int_I V(r,X_{r-}-(1-\Theta_{r-})z,Y_r) - V(r,X_{r-},Y_r)\left(m(\mathrm{d}r,\mathrm{d}z) - \lambda(r,Y_r)F(r,Y_r,\mathrm{d}z)\mathrm{d}r\right) \\
&= \int_0^t \Pi_r \sigma(r,Y_r)\gamma e^{-\gamma X_r} e^{\phi(r,Y_r;T)}\mathrm{d}W_r^S + \int_0^t \beta(r,Y_r)\frac{\partial \phi}{\partial y}(r,Y_r;T)e^{-\gamma X_r}e^{\phi(r,Y_r;T)}\mathrm{d}W_r^Y \\
&\quad - \int_0^t \int_I e^{-\gamma X_r}e^{\phi(r,Y_r;T)}(e^{\gamma(1-\Theta_{r-})z}-1)\left(m(\mathrm{d}r,\mathrm{d}z) - \lambda(r,Y_r)F(r,Y_r,\mathrm{d}z)\mathrm{d}r\right)
\end{aligned}$$

is a martingale. To this aim, we consider the localizing sequence of random times

$$\widetilde{\tau}_n := \inf\left\{ s\in[t,T] : \ |\phi(t,Y_t;T)| > n, \ \left|\frac{\partial \phi}{\partial y}(t,Y_t;T)\right| > n, \ X_t < -n \right\}, \quad n \in \mathbb{N}.$$

Then, $(\widetilde{\tau}_n)_{n\in\mathbb{N}}$ is an increasing sequence, $\lim_{n\to\infty} \tau_n \wedge T = T$ and computations similar to those in the proof of Theorem 1.3 show that

$$\begin{aligned}
\mathbb{E}\Bigg[\int_0^{T\wedge\widetilde{\tau}_n} \Bigg\{ &\left(\Pi_r\sigma(r,Y_r)\frac{\partial V}{\partial x}(r,X_r,Y_r)\right)^2 + \left(\beta(r,Y_r)\frac{\partial V}{\partial y}(r,X_r,Y_r)\right)^2 \\
&+ \int_I |V(r,X_{r-}-(1-\Theta_{r-})z,Y_r) - V(r,X_{r-},Y_r)|\,\lambda(r,Y_r)F(r,Y_r,\mathrm{d}z)\Bigg\}\mathrm{d}r\Bigg] < \infty.
\end{aligned}$$

This concludes the proof.

Chapter 2

# Optimization problems in insurance with a terminal distribution constraint

The content of this chapter is based on the paper [34] which is a joint work with J. Eisenberg and K. Colaneri. We consider an insurance company whose objective is to choose a dividend payment or a reinsurance strategy leading to a certain surplus distribution. The question which strategy to prefer depends on the underlying objective functional: the value of expected discounted dividends or the ruin probability. Maximizing the first or minimizing the latter are classical optimization problems in insurance. The novelty of the problem studied in this chapter is to compute optimal strategies under the additional constraint that the terminal surplus follows a certain prescribed distribution. Such a problem is motivated by need to evaluate risk measures which are useful for the calculation of the capital required to ensure solvency of the system, according to Solvency II regulations. In other words, fixing a terminal wealth distribution would allow computing several risk measures at once, instead of choosing a specific constraint in the beginning of an optimization task.

The chapter is organized as follows. In Section 2.1 we introduce the topic, placing it with respect to related papers in the literature. Then, we describe the mathematical framework for the dividend setting and we maximize the expected discounted dividend payments in Section 2.2. After that, considering the same insurance setting, we address the ruin minimization problem in Section 2.3. We conclude discussing the reinsurance optimization problem in Section 2.4.

## 2.1 Motivation and literature review

In this chapter, we study two optimization settings for an insurance company, under the constraint that the terminal surplus at a deterministic and finite time follows a normal distribution with a given mean and a given variance. We are motivated by the need of computing risk measures, typically based on the distribution of a future loss at some fixed date. Indeed, fixing a terminal wealth distribution would allow computing several risk measures. This may be of a particular interest from a practical point of view since the calculation of the necessary capital reserves is one of the problems faced by practitioners on the almost daily basis. Measuring the solvency of a collective of risks remains a key task in insurance mathematics. The first attempt to describe mathematically this situation dates back to 1903 when Filip Lundberg proposes a model (well known as the classical risk model or Cramer-Lundberg model) for the surplus. He suggested to represent the total claim amount of an insurance company or of a collective of insured by a compound Poisson process and to assume that premia arrive at a constant positive rate. As a result, the surplus process, which is made of incomes (namely the initial capital and premia) and payments (namely the claims), is described by a jump process with drift that may eventually take negative values if for instance claims exceed premia. One way to assess the solvency is to look at the so called ruin probability, that is the probability that the surplus becomes negative in finite time (the aggregate claims exceed the collected premia) i.e. writes red numbers. This is a key topic in the classical risk model and has been studied in many settings since Harald Cramer republished the results of Lundberg in the 1930s. The Cramer-Lundberg model describes well the reality, with jumps that represent the claim sizes. However, from a mathematical point of view, it is not always easy to handle with jumps. As a consequence, explicit form representations of the ruin probability can be found just for a few claim size distributions, exponential distribution above all. For this reason, one often considers an approximation of a compound Poisson process by a diffusion, see, e.g., Schmidli [88, p. 226] for more details.

Taking into account this approximation, we address two classical optimization problems faced by an insurance company. Firstly, we concentrate on dividend payments: the objective of the insurance is to decide on the dividend rates in order to maximize the expected discounted dividends or to minimize the ruin probability under the terminal distribution constraint. Maximizing the expected dividends value is a well-known problem in insurance; therefore the literature about this optmization problems is quite rich; we refer to Asmussen and Taksar [4], Shreve et al. [91], Albrecher and Thonhauser [2], Avanzi [6], Hipp [62] and references therein for an overview of the existing results. We point out that the optimal dividend payout strategy in the most "unconstrained" settings turns out to be of a barrier or a band type, meaning that the strategy can change from "paying the maximal possible amount" to "paying nothing", in dependence on the current value of the surplus. These stategies, called bang-bang strategies, are very common in optimal control problems. However, switching abruptly betweenall and nothing would not be considered realistic for an insurance company. Moreover, solvency requirements imposed by regulators may not allow paying dividends according to the optimal, possibly bang-bang, strategy. Therefore, to make the models more realistic, one needs to put restrictions. For example, Paulsen [85] studies the optimal dividend problem taking into account a no-bankruptcy constraint, namely he requires that the company pay dividends only if its surplus has reached a certain amount. The latter is chosen as the smallest level such that the probability that the surplus will be negative is sufficiently low. In other words, a barrier is introduced into the model and the company does not distribute dividends if its surplus is below such barrier in order to avoid ruin in the future. Some years after, an extended setting with transaction costs and taxes is analyzed in Bai et al. [8]: also there, the purpose is to maximize expected discounted dividend payments, assuming that, whenever dividends are distributed, the probability of ruin should not exceed a predetermined level.

A similar approach to optimal dividend payment with different constraints can be found in Hipp [61]. Here, there is no constraint on the surplus directly but only on the probability that it becomes negative. Precisely, considering a stylized model for insurance business in discrete time, the optimal dividend problem is addressed assuming that the ruin probability remains under a given boundary.

It is worth mentioning also the paper of Thonhauser and Albrecher [95] where they maximize the total discounted utility of dividend payments including strictly positive transaction costs and imposing two constraints on the dividend strategy, namely a payment is allowed if it is greater than the tax amount and does not make the surplus negative.

The main novelty of our model is that the constraint is put on the probability distribution of the surplus at some terminal time. Indeed, we require that the distribution of the final surplus is of gaussian type with fixed exogenously given mean and variance. We point out that choosing a normal distribution as a target distribution is the natural choice since the surplus is modeled by a diusion process.

In our second problem, an insurance company may decide to reinsure part of its claims in order to reduce losses. We are looking at the surplus of an insurance company who buys proportional reinsurance. The purpose is to minimize the probability of ruin, under the constraint that the terminal surplus follows a gaussian distribution, with a given mean and a given variance.

To control the risk exposure and to be able to meet regulatory requirements, insurance companies need to pay attention to various constraints that reflect different practical considerations; prominent examples include insurers' budgetary, regulatory and reinsurers' participation constraints. For instance, Bernard and Tian [12], Lo [74, 75], Huang and Yin [65] search for the optimal reinsurance strategy under a constraint (that is a strictly positive surplus or a fixed risk measure under some prespecified boundary) on the loss at the terminal time.

Over the years, optimal investment and reinsurance problems have been considered with several constraints. For example, Bi et al. [14] address such type of optimization problems for an insurer under the criterion of mean-variance with the so-called bankruptcy prohibition, namely requiring that the wealth process of the insurer is not allowed to be below zero at any time. Instead, Choulli et al. [30], Bi and Cai [13], Wang and Siu [98] investigate optimal investment and reinsurance problems, under constraints strictly closed to VaR.

Finally, let us outline that the problem of choosing a reinsurance strategy to minimize the ruin probability (or equivalently to maximize the survival probability) has been investigated imposing a constraint on the risk measure (that is the VaR or the more general ES), see e.g. Zhang et al. [104] in a finite time interval and Chen et al. [29] under an infinite time horizon.

Here, the ruin minimization problem is faced for the first time with a constraint imposed on the distribution of the terminal wealth.

## 2.2 Maximizing dividends under a terminal distribution constraint

In this section, we consider an insurance company who is allowed to pay dividends. The dividend rate has to be chosen in such a way that the surplus at some future deterministic time $T$ achieves a given distribution. At the same time, the value of expected discounted dividends should be maximized.

We consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a finite time horizon $T > 0$ and a Brownian motion $W = \{W_t,\ t \in [0,T]\}$. We denote by $\mathbb{F} = \{\mathcal{F}_t,\ t \in [0,T]\}$ the natural complete and right continuous filtration of $W$, with $\mathcal{F}_T = \mathcal{F}$. We model the wealth of the insurance company in the interval $[0,T]$ by a Brownian motion with drift as

$$X_t^{\mathbf{0}} = x_0 + \bar{\mu}t + \bar{\sigma}W_t \ ,$$

for every $t \in [0,T]$, where $x_0 \geq 0$ represents the initial capital and $\bar{\mu}, \bar{\sigma} > 0$ denote the drift and the volatility, respectively.
The company is allowed to pay dividends in form of dividend rates $0 \leq c \leq \xi$ for some given $\xi > 0$. It means that the post-dividend wealth under a dividend strategy $\mathbf{c} = \{c_t,\ t \in [0,T]\}$ is given by

$$X_t^{\mathbf{c}} = x_0 + \bar{\mu}t - \int_0^t c_s \,\mathrm{d}s + \bar{\sigma}W_t \ , \tag{2.1}$$

for every $t \in [0,T]$. Our aim is to determine the strategies that maximize the expected discounted dividends and simultaneously lead to a normally distributed post-dividend terminal surplus $X_T^{\mathbf{c}}$. Specifically, we assume that the target distribution is Gaussian with the mean $x_0 + MT$, and the variance $\delta^2 T$, for some $M \in \mathbb{R}$ and $\delta > 0$.

At first, the company is only allowed to update a dividend strategy at $n \in \mathbb{N}$ equidistant time points $Tk/n$, $k \in \{0, .., n-1\}$ in the period $[0,T]$. Thus, an admissible strategy is a sequence $\mathbf{c} = (c_0, \ldots, c_{n-1})$ of dividend rates such that for all $k = 0, 1, \ldots, n-1$, $c_k \in [0, \xi]$ is an $\mathcal{F}_{\frac{kT}{n}}$-measurable random variable and the total surplus at time $T$ satisfies $X_T^{\mathbf{c}} \sim \mathcal{N}(x_0 + MT, \delta^2 T)$ [1]. We denote the set of admissible strategies by $\mathcal{A}_{(n)}$, where the subscript $(n)$ indicates the number of the allowed change points. Then accumulated dividends up to time

[1] $\mathcal{N}(x_0 + MT, \delta^2 T)$ indicates the Gaussian distribution with mean $x_0 + MT$ and variance $\delta^2 T$.

$t$ are given by

$$\sum_{k=0}^{n-1} c_k \Big( \frac{T(k+1)}{n} \wedge t - \frac{Tk}{n} \wedge t \Big).$$

We point out that dividends can be paid (up to time $T$) even if the surplus is negative, differently than in the classical dividend problems, see for instance [4]. When surplus becomes negative or touches zero (we can speak of technical ruin), the company usually continues to operate, since she has enough reserves to bridge a certain period of unfavourable business periods. Indeed, some insurance companies continue to pay dividends even during protracted crisis times: for instance, Munich Re did not reduce its dividends since at least 2006, see [87]. As a consequence, it seems reasonable to allow the company to proceed with dividend payments even if a technical ruin occurs. In this way our model manages to alleviates the drawback of stopping at the ruin time.

The following lemma indicates the range of achievable target expectations $x_0 + MT$ by a post-dividend Brownian surplus, see equation (2.1), at time $T$.

**Lemma 2.1.** *The parameter $M$ in the target distribution of the surplus at time $T$ has to fulfil $\bar{\mu} - \xi \leq M \leq \bar{\mu}$.*

*Proof.* For any admissible dividend strategy $\mathbf{c} = (c_0, \ldots, c_{n-1}) \in \mathcal{A}_{(n)}$, the distribution of the surplus in equation (2.1) at time $T$ is Gaussian with mean

$$x_0 + \left( \bar{\mu} - \sum_{k=0}^{n-1} \frac{\mathbb{E}[c_k]}{n} \right) T = x_0 + MT.$$

Using the fact that $0 \leq c_k \leq \xi$, for every $k \in \{0, \ldots n-1\}$, we get that

$$x_0 + (\bar{\mu} - \xi)\, T \leq x_0 + \left( \bar{\mu} - \frac{\sum_{k=0}^{n-1} \mathbb{E}[c_k]}{n} \right) T \leq x_0 + \bar{\mu} T \ ,$$

which proves the statement. □

Note that, for large values of $\xi$, the range of achievable means may include negative values. Although this is mathematically feasible, an insurance company would not pursue a strategy to achieve a negative expected surplus, but it would rather choose $M \in [0, \bar{\mu}]$ in order to obtain a expected net profit at time $T$, even if small. Next, we better identify the features of admissible strategies.

**Proposition 2.1.** *Any admissible strategy* $\mathbf{c} = (c_0, ..., c_{n-1}) \in \mathcal{A}_{(n)}$ *is such that* $\sum_{i=1}^{n-1} c_i$ *is* $\mathcal{F}_0$*-measurable, i.e. deterministic.*

*Proof.* Let $\mathbf{c} = (c_0, \ldots, c_{n-1})$ be an arbitrary admissible dividend strategy. The corresponding surplus at time $T$ is then given by

$$X_T^{\mathbf{c}} = x_0 + \bar{\mu}T - c_0\frac{T}{n} - \frac{T}{n}\sum_{k=1}^{n-1} c_k + \bar{\sigma}W_T \ . \tag{2.2}$$

We now identify the set of dividend strategies that lead to a normal distribution with mean $x_0 + MT$ and variance $\delta^2 T$. Let $Y$ be a generic random variable with $Y \sim \mathcal{N}(x_0 + MT, \delta^2 T)$. Then, for $\zeta \in \mathbb{R}$ it holds that $\mathbb{E}[e^{\zeta Y}] = e^{\zeta(x_0+MT)+\frac{\delta^2}{2}\zeta^2 T}$. Now, considering the surplus at time $T$ given by (2.2) and the fact that $\mathbf{c}$ is an admissible strategy, we get that $X_T^{\mathbf{c}} \sim \mathcal{N}(x_0 + MT, \delta^2 T)$ and it holds that

$$e^{\zeta(x_0+MT)+\frac{\delta^2}{2}\zeta^2 T} = \mathbb{E}[e^{\zeta X_T^C}] = e^{\zeta(x_0+\bar{\mu}T - c_0 T/n)}\mathbb{E}[e^{\zeta\bar{\sigma}W_T - \zeta\sum_{k=1}^{n-1} c_k T/n}] \ . \tag{2.3}$$

Let $\mathbf{Q}$ be a probability measure on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbf{P}$, with the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}}\Big|_{\mathcal{F}_T} = e^{-\zeta\bar{\sigma}W_T + \frac{\zeta^2\bar{\sigma}^2 T}{2}}$. Then, applying change of measure techniques in (2.3), we obtain

$$\mathbb{E}[e^{\zeta\bar{\sigma}W_T - \zeta\sum_{k=1}^{n-1} c_k T/n}] = e^{\frac{\zeta^2\bar{\sigma}^2 T}{2}}\mathbb{E}_{\mathbf{Q}}[e^{-\zeta\sum_{k=1}^{n-1} c_k T/n}] \ .$$

Together with (2.3), we get for all $\zeta \in \mathbb{R}$

$$e^{\zeta(x_0+MT)+\frac{\delta^2}{2}\zeta^2 T} = e^{\zeta(x_0+\bar{\mu}T - c_0 T/n)+\frac{\bar{\sigma}^2 T\zeta^2}{2}}\mathbb{E}_{\mathbf{Q}}[e^{-\zeta\sum_{k=1}^{n-1} c_k T/n}],$$

leading to

$$\mathbb{E}_{\mathbf{Q}}[e^{-\zeta\sum_{k=1}^{n-1} c_k T/n}] = e^{\zeta(M-\bar{\mu}+c_0/n)T + \frac{\delta^2-\bar{\sigma}^2}{2}\zeta^2 T} \ .$$

If $\delta^2 - \bar{\sigma}^2 > 0$, then by uniqueness of the moment generating functions, the variable $\sum_{k=1}^{n-1} c_k \frac{T}{n}$ follows a normal distribution with mean $(M - \bar{\mu} + c_0/n)T$ and variance $(\delta^2 - \bar{\sigma}^2)T$. Hence it has positive $\mathbf{Q}$-probability to attain negative values, which contradicts the equivalence of $\mathbf{Q}$ and $\mathbf{P}$, since $\sum_{k=1}^{n-1} c_k T/n \geq 0$ $P$-a.s.

If, instead, $\delta^2 - \bar{\sigma}^2 < 0$, there is no random variable with such a moment generating function.

Finally, if $\delta = \bar{\sigma}$, the variable $\sum_{k=1}^{n-1} c_k T/n$ must be a constant, i.e. deterministic. □

For the special case $n = 2$ we obtain the following corollary.

**Corollary 2.1.1.** *The set of admissible strategies $\mathcal{A}_{(2)}$ only consists of deterministic pairs $(c_0, c_1)$, i.e. $c_1$ is $\mathcal{F}_0$-measurable.*

Note that the dividend strategies act solely on the drift and do not affect the volatility of the wealth process. Due to this fact, we can compare different strategies by looking at the surplus "path by path". Moreover, in the case $n = 2$, both components are deterministic; that is the optimal dividend strategy is completely decided at the beginning of the trading period. This means that that once the dividend rate $c_0$, to be valid in $[0, T/2]$, is chosen, then $c_1$ is also uniquely determined at time $t = 0$ in order to achieve the final probability distribution. We will see in the reminder of the section that the optimal strategy is deterministic also for $n > 2$.

Let now $r > 0$ be the preference rate of the insurer. We call *return function* corresponding to a strategy $\mathbf{c} = (c_0, ..., c_{n-1}) \in \mathcal{A}_{(n)}$ the expected total present value of the dividends paid up to time $T$ , that is

$$V^{\mathbf{c}}(x) := \mathbb{E}_x \left[ \sum_{k=0}^{n-1} \frac{c_k}{r} e^{-r\frac{kT}{n}} \left(1 - e^{-r\frac{T}{n}}\right) \right] .$$

The function $V^{\mathbf{c}}(x)$ can be derived from the usual definition of return functions corresponding to continuous time dividend controls, as an integral over discounted dividend rates. Let $\zeta = \{\zeta_t, t \in [0, T]\}$ with $\zeta_t = c_k \mathbb{1}_{[t \in [Tk/n, T(k+1)/n)]}$ for $k \in \{0, .., n-1\}$. That is,

$$\begin{aligned} V^{\zeta}(x_0) &= \mathbb{E}_{x_0} \left[ \int_0^T e^{-rs} \zeta_s \mathrm{d}s \right] = \mathbb{E}_{x_0} \left[ \sum_{k=0}^{n-1} \int_{Tk/n}^{T(k+1)/n} e^{-rs} c_k \mathrm{d}s \right] \\ &= \mathbb{E}_x \left[ \sum_{k=0}^{n-1} \frac{c_k}{r} e^{-r\frac{kT}{n}} \left(1 - e^{-r\frac{T}{n}}\right) \right] \\ &= V^{\mathbf{c}}(x_0) . \end{aligned}$$

Note that the dependence on the initial capital $x_0$ is in this setting purely nominal. As stressed before, we do not stop our considerations at the time of ruin. The strategy will depend only on the parameters of the surplus process and the target distribution.

The target of the insurance company is to find a strategy $\mathbf{c}^* = (c_0^*, ..., c_{n-1}^*) \in \mathcal{A}_{(n)}$ leading to

$$V^{\mathbf{c}^*}(x_0) = \max_{\mathbf{c} \in \mathcal{A}_{(n)}} \mathbb{E}_x \left[ \sum_{k=0}^{n-1} \frac{c_k}{r} e^{-r\frac{kT}{n}} \left(1 - e^{-r\frac{T}{n}}\right) \right] . \tag{2.4}$$

To analyze Problem (2.4), we start with the case of two periods, i.e. $n = 2$.

### 2.2.1 A 2-period model

Suppose that the insurance company is allowed to update its dividend strategy only once, at time $T/2$. By Corollary 2.1.1, we get that the set of admissible dividend strategies $\mathcal{A}_{(2)}$ consists of all deterministic pairs $\mathbf{c} = (c_0, c_1)$ with $c_0, c_1 \in [0, \xi]$, and such that

$$\bar{\mu} - \frac{c_0 + c_1}{2} = M.$$

In this case we immediately get, from the fact that $c_0, c_1 \in \mathcal{F}_0$, that it must also hold that $\bar{\sigma}^2 = \delta^2$, otherwise the target distribution would not be reachable. In the next step, we investigate how to determine the optimal strategy.

**Proposition 2.2.** *The optimal strategy* $\mathbf{c}^* = (c_0^*, c_1^*)$ *is given by*

$$\begin{aligned} c_0^* &= \xi \wedge 2(\bar{\mu} - M), \\ c_1^* &= \begin{cases} 0, & \text{if } 2\bar{\mu} - 2M \leq \xi, \\ 2\bar{\mu} - 2M - \xi, & \text{if } 2\bar{\mu} - 2M > \xi. \end{cases} \end{aligned}$$

*Proof.* We consider the problem

$$\max_{(c_0,c_1)\in\mathcal{A}_{(2)}} \frac{c_0 T}{r}\left(1 - e^{-rT/2}\right) + \frac{c_1 T}{r} e^{-rT/2}\left(1 - e^{-rT/2}\right).$$

It is easy to see that, for $r > 0$, the discounting coefficient in the first period, $1 - e^{-rT/2}$, is larger than in the second period, $e^{-rT/2}(1 - e^{-rT/2})$. Therefore, to maximize the discounted dividends, $c_0$ must be chosen as big as possible. Taking into account that $0 \leq c_0 \leq \xi$ and that $c_0 + c_1 = 2(\bar{\mu} - M)$, we get that $c_0 = \min(\xi, 2(\bar{\mu} - M))$, and consequently, $c_1 = 2(\bar{\mu} - M) - \xi$ if $c_0 = \xi$ and $c_1 = 0$ if $c_0 = 2(\bar{\mu} - M)$. □

We notice that, in a two-period setting, the optimal dividend strategy is deterministic, i.e. it is optimal to pay dividends at the maximum rate in the first period and then adjust the strategy to achieve the target distribution in the second period. Such behavior is justified by the effect of discounting which has a larger impact in the time interval $[T/2, T]$.

### 2.2.2 An $n$-period model

We now extend our analysis to an $n$-period framework; this means that now the dividend strategy can be adjusted $n$ times in the interval $[0, T]$. Recall that, according to Proposition 2.1, the sum of dividend rates is necessarily deterministic.

We start considering the case $n = 3$, to better explain the mechanism for the computation of the optimal dividend strategy.

**Example 2.1.** *Let $n = 3$ and let $\mathbf{c} = (c_0, c_1, c_2)$ be an admissible strategy. The expected discounted total dividends are given by*

$$\frac{c_0}{r}(1 - e^{-rT/3}) + \frac{e^{-rT/3}(1 - e^{-rT/3})}{r}\mathbb{E}\Big[c_1 + c_2 e^{-rT/3}\Big] .$$

*We easily see that, that due to discounting ($r > 0$), the strategy $c_0$ to be applied in the first period has a larger weight than the others, hence, as in the two period model, it would be optimal to choos it the largest possible. Taking into account that $x_0 + MT = x_0 + \bar{\mu}T - \frac{(c_0+\mathbb{E}[c_1+c_2])T}{3}$ and that $c_k \in [0, \xi]$, for $k = 0, 1, 2$, we have that*

$$c_0 = \begin{cases} 3(\bar{\mu} - M), & \text{if } 3(\bar{\mu} - M) \leq \xi, \\ \xi, & \text{if } 3(\bar{\mu} - M) > \xi \end{cases} ,$$

*equivalently, $c_0 = \min(3\bar{\mu} - 3M, \xi)$. Now we move to the choice of $c_1, c_2$. After choosing $c_0$, we get that $\mathbb{E}[c_1 + c_2] = c_1 + c_2 = \max(0, 3(\bar{\mu} - M) - \xi)$, according to Proposition 2.1. If $c_0 = 3\bar{\mu} - 3M$, since $c_1$ and $c_2$ are non-negative, it holds that $c_1 = c_2 = 0$. If instead, $c_0 = \xi$, using the same argument like for $c_0$, we choose $c_1$ and $c_2$ so that $c_1$ is the largest possible value according to the constraints, i.e. $c_1 = \min(3(\bar{\mu} - M) - \xi, \xi)$, and $c_2 = \max(3(\bar{\mu} - M) - 2\xi, 0)$. Put in other words, if $2\xi \leq 3\bar{\mu} - 3M < 3\xi$, then $c_0 = c_1 = \xi$ and $c_2 = 3\bar{\mu} - 3M - 2\xi$. If $\xi < 3\bar{\mu} - 3M < 2\xi$, at time $T/3$ we determine both $c_1$ and $c_2$, depending on the current surplus so that*

$$\bar{\mu}T - (c_1 + c_2)T/3 = MT + \xi T/3.$$

*We stress that because $c_1 + c_2$ must be deterministic, we immediately get that $c_2$ is $\mathcal{F}_{T/3}$ measurable. That means, once $c_1$ is found, then $c_2$ is also determined, so that the constraint on the distribution is satisfied. Moreover, the value $3(\bar{\mu} - M) - \xi$ is the biggest possible choice* *for $c_1$. The deterministic strategy $\mathbf{c}^* = (c_0, c_1, c_2)$ where*

$$\begin{cases} c_0^* = \min(\xi, 3(\bar{\mu} - M)), \\ c_1^* = \max\big(\min(\xi, 3(\bar{\mu} - M) - \xi), 0\big), \\ c_2^* = \max(3(\bar{\mu} - M) - 2\xi, 0), \end{cases} \tag{2.5}$$

*fulfils all necessary conditions.*

*Next, we show that we cannot find a different, possibly stochastic, strategy with a higher expected discounted dividends value, meaning that the optimal strategy is indeed deterministic.*

*Let $\mathbf{c}^* = (c_0^*, c_1^*, c_2^*)$ be the strategy in (2.5) and let $\tilde{\mathbf{c}} = (\tilde{c}_0, \tilde{c}_1, \tilde{c}_2) \in \mathcal{A}_{(3)}$ be an arbitrary admissible strategy, i.e. such that $\tilde{c}_m \in [0, \xi]$ for $m = 0, 1, 2$, and $X_T^{\tilde{\mathbf{c}}} \sim \mathcal{N}(x_0 + MT, \delta^2 T)$.*

*Then, there exist two random variables $d_1, d_2$ such that $\mathbb{E}[d_1 + d_2] = c_0^* - \tilde{c}_0 \geq 0$, because $c_0^*$ is the largest possible dividend rate, and $\tilde{c}_1 = c_1^* + d_1$, $\tilde{c}_2 = c_2^* + d_2$. It holds that*

$$\begin{aligned} V^{\tilde{\mathbf{c}}}(x_0) &= \tilde{c}_0(1 - e^{-\frac{rT}{3}}) + (1 - e^{-\frac{rT}{3}})e^{-\frac{rT}{3}}\mathbb{E}[\tilde{c}_1 + \tilde{c}_2 e^{-\frac{rT}{3}}] \\ &= V^{\mathbf{c}^*}(x_0) - (c_0^* - \tilde{c}_0)(1 - e^{-\frac{rT}{3}}) + (1 - e^{-\frac{rT}{3}})e^{-\frac{rT}{3}}\mathbb{E}[d_1 + d_2 e^{-\frac{rT}{3}}] \\ &= V^{\mathbf{c}^*}(x_0) - (1 - e^{-\frac{rT}{3}})\left(\mathbb{E}[d_1 + d_2] - e^{-\frac{rT}{3}}\mathbb{E}[d_1 + d_2 e^{-\frac{rT}{3}}]\right), \end{aligned}$$

*where in the last equality we have used the fact that $(c_0^* - \tilde{c}_0) = \mathbb{E}[d_1 + d_2]$.*

*If $\mathbb{E}[d_2] < 0$, then, $\mathbb{E}[c_2^*] > \mathbb{E}[\tilde{c}_2] \geq 0$; hence, necessarily $c_2^* = 3(\bar{\mu} - M) - 2\xi > 0$ and $c_1^* = c_0^* = \xi$. Since $c_0^* + c_1^* + c_2^* = \tilde{c}_0 + \mathbb{E}[\tilde{c}_1] + \mathbb{E}[\tilde{c}_2]$, we get that $\tilde{c}_0 + \mathbb{E}[\tilde{c}_1] > c_0^* + \mathbb{E}[c_1^*] = 2\xi$ leading to a contradiction. Then, it must hold that $\mathbb{E}[d_2] \geq 0$. Now we have two cases:*

*i. if $\mathbb{E}[d_1] < -e^{-rT/3}\mathbb{E}[d_2]$, then it is immediate that $\mathbb{E}[d_1 + d_2] - e^{-\frac{rT}{3}}\mathbb{E}[d_1 + d_2 e^{-\frac{rT}{3}}] > 0$ which leads to $V^{\tilde{\mathbf{c}}} \leq V^{\mathbf{c}^*}$;*

*ii. if $\mathbb{E}[d_1] \geq -e^{-rT/3}\mathbb{E}[d_2]$, we get that*

$$c_0^* - \tilde{c}_0 = \mathbb{E}[d_1 + d_2] \geq \mathbb{E}[d_1 + d_2 e^{-\frac{rT}{3}}] \geq e^{-\frac{rT}{3}}\mathbb{E}[d_1 + d_2 e^{-\frac{rT}{3}}],$$

*which implies that $V^{\tilde{\mathbf{c}}} \leq V^{\mathbf{c}^*}$.*

*We conclude observing that if $\mathbb{E}[d_2] > 0$ then the inequality is strict and the strategy $(c_0^*, c_1^*, c_2^*)$, is optimal. If $\mathbb{E}[d_2] = 0$ we get that either $\mathbb{E}[d_1] > 0$, in which case the inequality is strict again, or $\mathbb{E}[d_1] = 0$, which corresponds to the case where $\tilde{\mathbf{c}} = \mathbf{c}$.*

The above example provides the argument for computing the optimal dividend strategy in an $n$-period framework.

**Proposition 2.3.** *Let $n$ be the number of sub-periods in the interval $[0,T]$ and let*

$$\kappa := \min\{m \geq 0 : \ n(\bar{\mu} - M) < (m+1)\xi\} \ .$$

*Then, an optimal strategy $\mathbf{c}^* = (c_0^*, c_1^*, \ldots, c_{n-1}^*)$ is given by*

$$\begin{cases} c_0^* = ... = c_{\kappa-1}^* = \xi \ , \\ c_\kappa^* = n(\bar{\mu} - M) - \kappa\xi \ , \\ c_{\kappa+1}^* = ... = c_{n-1}^* = 0 \ . \end{cases} \tag{2.6}$$

*Proof.* Firstly, we assume that $\kappa = n-1$; then, obviously, the optimal is to distribute dividends at maximum speed $\xi$ in all periods, except the last one where an update is needed for admissible reasons, i.e. the optimal strategy is $(\xi, ..., \xi, n(\bar{\mu} - M) - (n-1)\xi)$.

Let now $\kappa < n-1$ and let $\tilde{\mathbf{c}} = (\tilde{c}_0, ..., \tilde{c}_{n-1})$ be an admissible strategy. Like in Example 2.1, there exist $d_1, ..., d_{n-1}$ such that $\tilde{c}_m = c_m^* + d_m$ for $m \in \{1, ..., n-1\}$ and $\sum_{m=1}^{n-1} \mathbb{E}[d_m] = c_0^* - \tilde{c}_0 \geq 0$. Then we have that

$$V^{\tilde{\mathbf{c}}}(x_0) = V^{\mathbf{c}^*}(x_0) - (c_0^* - \tilde{c}_0)(1 - e^{-rT/n}) + (1 - e^{-rT/n}) \sum_{m=1}^{n-1} e^{-rTm/n} \mathbb{E}[d_m] \ .$$

We note that since $c_m^* = \xi$, for all $m \leq \kappa - 1$, and $c_m^* = 0$, for all $m > \kappa + 1$, it must hold $d_m \leq 0$, for $m \leq \kappa - 1$ and $d_m \geq 0$, for $m \geq \kappa + 1$.

Now we observe that, the function $t \to \sum_{m=1}^{n-1} e^{rt\frac{\kappa - m}{n}} \mathbb{E}[d_m]$ is decreasing, and hence it attains its maximum at $t = 0$, i.e. $\sum_{m=1}^{n-1} \mathbb{E}[d_m] \geq \sum_{m=1}^{n-1} e^{rT\frac{\kappa-m}{n}} \mathbb{E}[d_m]$ . Therefore, we conclude that

$$c_0^* - \tilde{c}_0 = \sum_{m=1}^{n-1} \mathbb{E}[d_m] \geq e^{-rT\frac{\kappa}{n}} \sum_{m=1}^{n-1} \mathbb{E}[d_m] \geq e^{-rT\frac{\kappa}{n}} \sum_{m=1}^{n-1} e^{rT\frac{\kappa-m}{n}} \mathbb{E}[d_m] \ .$$

The strict inequality holds true if there is at least one $m$ with $\mathbb{E}[d_m] \neq 0$. If instead $\mathbb{E}[d_m] = 0$, for all $m = 1, \ldots, n-1$, then strategies $\tilde{\mathbf{c}}$ and $\mathbf{c}^*$ coincide, i.e. in particular $d_m = 0$ almost surely for all $m = 0, \ldots, n-1$. This leads to $V^{\tilde{\mathbf{c}}} < V^{\mathbf{c}^*}$ if $\tilde{\mathbf{c}} \not\equiv \mathbf{c}^*$. □

**Remark 2.1** (Continuous time). *This procedure allows extending the setting to continuous time.*

*We denote by* $\mathcal{A}_{(\infty)}$ *the set of admissible strategies, consisting of the* $\mathbb{F}$*-adapted processes* $\mathbf{c} = \{c_t, t \in [0,T]\}$ *with* $0 \leq c_t \leq \xi$*, for every* $t \in [0,T]$ *and* $X_T^{\mathbf{c}}$ *from* (2.1) *normally distributed with mean* $x_0 + MT$ *and variance* $\delta^2 T$*. Letting* $n \to \infty$ *in the* $n$*-period models, the optimal strategies as given in* (2.6) *converge to a deterministic strategy in continuous time:*

$$c_s^* = \begin{cases} \xi & : t \leq T \wedge t^*, \\ 0 & : t > T \wedge t^*, \end{cases}$$

*where* $t^* = (\bar{\mu} - M)T/\xi$*. We assume* $t^* < T$*.*

*Let* $\tilde{\mathbf{c}} = \{\tilde{c}_t, t \in [0,T]\}$ *be an admissible strategy and define* $d_s := \tilde{c}_s - c_s^*$*. Since we would like to achieve the same final distribution with strategies* $\mathbf{c}^*$ *and* $\tilde{\mathbf{c}}$*, it must hold that* $\mathbb{E}[\int_0^T d_s] = 0$*.*

*Moreover, it is clear that* $d_s \leq 0$ *for* $s \leq t^*$*,* $d_s \geq 0$ *for* $s > t^*$*. As for the* $n$*-period models we get*

$$\begin{aligned} V^{\tilde{\mathbf{c}}}(x_0) &= \mathbb{E}_{x_0}\Big[\int_0^T e^{-rs}\tilde{c}_s \mathrm{d}s\Big] = V^{\mathbf{c}^*}(x_0) + \mathbb{E}_{x_0}\Big[\int_0^T e^{-rs} d_s \mathrm{d}s\Big] \\ &= V^{\mathbf{c}^*}(x_0) + e^{-rt^*}\mathbb{E}_{x_0}\Big[\int_0^T e^{-r(s-t^*)} d_s \mathrm{d}s\Big] \\ &\leq V^{\mathbf{c}^*}(x_0) + e^{-rt^*}\mathbb{E}_{x_0}\Big[\int_0^T d_s \mathrm{d}s\Big] = V^{\mathbf{c}^*}(x_0) \,. \end{aligned}$$

*A strict inequality holds true if* $\mathbb{E}[d_s] \neq 0$*, for all* $s \in \mathcal{T}$*, where* $\mathcal{T} \subseteq [0,T]$ *is a Lebesgue measurable non-zero set.*

*Therefore, in continuous time it is optimal to pay on the maximal rate as long as possible, and to pay nothing afterwards.*

Notice, that we have only considered the case of dividend rates. However, it is also possible to allow for lump sum payments. Then, because $r > 0$, it is clear that one should pay the amount $(\bar{\mu} - M)T$ directly at the beginning of the period, in both discrete and continuous time settings.

Taking into account the upper bound $\xi$ on the admissible dividend rates guarantees to avoid a lump sum payment at the initial time. Indeed, from the point of view of shareholders, it could be more attractive to get dividends over the whole period $[0,T]$. We point out that the value of $\xi$ is a management decision: in our framework it has to be small enough to distribute dividends over the whole period and, at the same time, large enough to achieve the target distribution.

## 2.3 Dividends minimizing the ruin probability

It is evident that paying dividends increases the probability of ruin and in many settings the optimal dividend strategy even leads to a certain ruin. Therefore, we now focus on the objective to minimize the probability of ruin, rather than maximize the cumulative dividends.

We consider the same framework like in Section 2.2 with a surplus, after dividends, described by equation (2.1). We recall that the set of achievable target means is given by $\bar{\mu} - \xi < M < \bar{\mu}$, thanks to Lemma 2.1. Moreover, by Proposition 2.1, the set of admissible strategies $\mathcal{A}_{(n)}$ is the set of all strategies $\mathbf{c} = (c_0, \ldots, c_{n-1})$, where $\sum_{m=1}^{n-1} c_m$ is $\mathcal{F}_0$-measurable, and $c_0 + \sum_{m=1}^{n-1} c_m = n(\bar{\mu} - M)$.

The goal of the insurance company is to minimize the ruin probability, which is given by

$$\min \mathbf{P}[\inf_{0 \le t \le T} X_t^{\mathbf{c}} < 0] \tag{2.7}$$

over all admissible dividend strategies $\mathbf{c} \in \mathcal{A}_{(n)}$.

We start by addressing Problem (2.7) in a two-period framework, like in Section 2.2.

### 2.3.1 A 2-period model

In the case $n = 2$, we denote by $\mathcal{A}_{(2)}$ the set of admissible strategies, as in Section 2.2.1. From Corollary 2.1.1, we know that all admissible strategies are of the form $\mathbf{c} = (c_0, c_1)$ with $c_0, c_1 \in [0, \xi]$ deterministic and $c_0 + c_1 = 2(\bar{\mu} - M)$. We target to minimize the ruin probability in the time interval $[0, T]$, i.e.

$$p(\mathbf{c}, x_0) := \mathbf{P}\left[\inf_{0 \le t \le T} X_t^{\mathbf{c}} < 0\right] ,$$

over all $\mathbf{c} \in \mathcal{A}_{(2)}$. We observe that the probability of ruin is closely linked to the initial capital $x_0$. Thus, differently than in Section 2.2, the dependence on the initial capital $x_0$ is crucial in this setting.

**Proposition 2.4.** *Let* $\mathbf{c} = (c_0, c_1)$ *and* $\tilde{\mathbf{c}} = (\tilde{c}_0, \tilde{c}_1)$ *be two admissible strategies, i.e.* $X^{\mathbf{c}}, X^{\tilde{\mathbf{c}}} \sim \mathcal{N}(x_0 + MT, \delta^2 T)$. *We assume that* $c_0 > \tilde{c}_0$. *Then,* $\tilde{\mathbf{c}}$ is better than $\mathbf{c}$, *in the sense that*

$$p(\tilde{\mathbf{c}}, x_0) < p(\mathbf{c}, x_0).$$

*Proof.* We first observe that at time $T$, both strategies $\mathbf{c}$ and $\tilde{\mathbf{c}}$ lead to the same distribution of the final surplus, i.e. $X_T^{\mathbf{c}}, X_T^{\tilde{\mathbf{c}}} \sim \mathcal{N}(x_0 + MT, \delta^2 T)$. Then, we have

$$\begin{aligned}
\inf_{0 \le t \le T} X_t^{\mathbf{c}} &= \inf_{0 \le s \le T} \begin{cases} x_0 + (\bar{\mu} - c_0)s + \bar{\sigma} W_s, & \text{if } s \le \frac{T}{2}, \\ x_0 + (\bar{\mu} - c_0)\frac{T}{2} + \bar{\sigma} W_s + (\bar{\mu} - c_1)(s - \frac{T}{2}), & \text{if } s \in (\frac{T}{2}, T], \end{cases} \\
&= \inf_{0 \le s \le T} \begin{cases} x_0 + (\bar{\mu} - c_0)s + \bar{\sigma} W_s, & \text{if } s \le \frac{T}{2}, \\ x_0 + \bar{\mu} s + c_0(s - T) + \bar{\sigma} W_s - 2(\bar{\mu} - M)(s - \frac{T}{2}), & \text{if } s \in (\frac{T}{2}, T], \end{cases} \\
&= \inf_{0 \le s \le T} \begin{cases} X_s^{\tilde{\mathbf{c}}} + (\tilde{c}_0 - c_0)s, & \text{if } s \le \frac{T}{2}, \\ X_s^{\tilde{\mathbf{c}}} + (c_0 - \tilde{c}_0)(s - T), & \text{if } s \in (\frac{T}{2}, T], \end{cases} \\
&< \inf_{0 \le t \le T} X_t^{\tilde{\mathbf{c}}} \,.
\end{aligned}$$

Therefore, for all $x_0 > 0$ we get that $p(\mathbf{c}, x_0) > p(\tilde{\mathbf{c}}, x_0)$. □

We note that, in a two-period setting, a strategy with a smaller dividend rate at the beginning leads to a smaller probability of ruin: this means that $c_0$ should be chosen as the smallest possible value in order to minimize the ruin probability. Taking into account the constraints related to a final gaussian distribution, we get the following result.

**Corollary 2.4.1.** *In a two-period framework, the ruin minimizing dividend strategy is* $\mathbf{c}^* = (c_0^*, c_1^*)$ *where*

$$\begin{cases} c_0^* = \max(2(\bar{\mu} - M) - \xi, 0) \,, \\ c_1^* = \min(\xi, 2(\bar{\mu} - M)) \,. \end{cases}$$

### 2.3.2 An $n$-period model

The extension to $n$-periods is obtained by replicating the reasoning of Proposition 2.4 and Corollary 2.4.1.

**Proposition 2.5.** *Let* $k := \min\{m \geq 0 : \ n(\bar{\mu} - M) < (m+1)\xi\}$. *Then, the ruin minimizing dividend strategy* $\mathbf{c} = (c_0, ..., c_{n-1})$ *fulfils*

$$\begin{cases} c_{n-1} = ... = c_{n-k} = \xi \ , \\ c_{n-k-1} = n(\bar{\mu} - M) - k\xi \ , \\ c_0 = ... = c_{n-k-2} = 0 \ . \end{cases}$$

**Remark 2.2** (Continuous time)**.** *Letting* $n \to \infty$ *will produce the following optimal strategy: we define* $t^*$ *as the time that realises* $(\bar{\mu} - M)T = \xi t^*$. *Then, the optimal dividend rate is* $c_t = 0$ *for all* $0 < t < t^*$ *and* $c_t = \xi$ *for* $t \geq t^*$.

To summarize, we point out that the strategy minimizing the ruin probability is deterministic, like in the dividend maximization problem discussed in Section 2.2. Moreover, by Proposition 2.5, we obtain that the dividend strategy leading to the minimal ruin probability starts with low payments in the very beginning and increases approaching the time horizon. This way of distributing dividends is exactly the opposite way that leads to a maximization of the expected discounted dividend payments. Indeed, the strategy minimizing the ruin probability is also the strategy that minimizes the value of expected discounted dividends, as it is reasonable.

## 2.4 Reinsurance with a target terminal distribution

In this section, we analyze the behavior of an insurance company who buys reinsurance for a certain branch of their business or a pool of insured claims, changing the setting of Sections 2.2 and 2.3.

We consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a fixed time horizon $T$. Let $Z$ be a random variable representing a claim size having positive finite first and second moments denoted by $\mathbb{E}[Z] = \mu_1$ and $\mathbb{E}[Z^2] = \mu_2$, respectively. We assume that the surplus of the insurance company is described by a Brownian motion with drift, approximating a Cramer-Lundberg model like, e.g., in Schmidli [88, p. 226]. Thus, for every $t \in [0, T]$, the surplus at time $t$ is given by

$$X_t = x_0 + \lambda\delta^I \mu_1 t + \sqrt{\lambda\mu_2} W_t \ ,$$

where $x_0$ is the initial capital whereas $\lambda, \delta^I > 0$ represent the claims arrival intensity and the insurance safety loading, respectively. Here, $W = \{W_t, \ t \in [0, T]\}$ is a Brownian motion. We also define by $\mathbb{F} = \{\mathcal{F}_t,\ t \in [0,T]\}$ the natural filtration of the Brownian motion $W$, under the usual hypotheses. In order to reduce future risks, the insurance company is allowed to buy proportional reinsurance with retention level $\Delta \in [0,1]$. This means that part of the losses will be covered by a reinsurance company and only the reminder is paid by the insurance company, specifically $\Delta$ is the percentage of claims which are not covered by the reinsurance. We note that the retention level is exactly equal to one minus the protection level, i.e. $\Delta = 1-\Theta$, where $\Theta$ represents the protection level. We assume that the reinsurance premium is calculated via the expected value principle, that is the reinsurance premium rate $b$ is given by

$$b(\Delta) = (1+\delta^R)\left(\lambda\mu_1 - \mathbb{E}[s(Z,\Delta)]\right),$$

where $\delta_R > 0$ denotes the reinsurance safety loading and $s(Z,\Delta) = (1-\Delta)Z$ is the so-called self-insurance function which is proportional to the claim size in our setting. Moreover, we use the same calculation principle for computing the price of the insurance policy: thus, the insurance premium rate $a$ given by

$$a = (1+\delta^I)\lambda\mu_1.$$

Then, the premium rate that remains to the insurer (i.e. the difference between the insurance premium and the reinsurance premium) is

$$c(\Delta) = \lambda(1+\delta^R)\mathbb{E}[s(Z,\Delta)] - \lambda\mu_1(\delta^R - \delta^I),$$

with $c(0) < 0$; see, e.g. [88, Ch. 2.2] for more details.

Under a reinsurance strategy $\boldsymbol{\Delta} = \{\Delta_t,\ t \in [0,T]\}$, the surplus is given by

$$X_t^{\boldsymbol{\Delta}} = x_0 + \lambda\mu_1 \int_0^t (\delta^R\Delta_s - (\delta^R - \delta^I))\mathrm{d}s + \sqrt{\lambda\mu_2}\int_0^t \Delta_s \mathrm{d}W_s\ ,$$

for every $t \in [0,T]$. Next, for all $t \in [0,T]$, we denote by

$$\bar{X}_t^{\boldsymbol{\Delta}} := X_t^{\boldsymbol{\Delta}} - x_0,$$

the net value of collective at time $t$, i.e. the part of the surplus that only accounts for insurance premia, reinsurance premia and claims.

The objective of the insurance company is collect enough premia to buy reinsurance and to pay the occurring claims. To achieve such level of *sustainability* the target of the insurance is to choose a reinsurance strategy such that at time $T$ the distribution of the net collective is normal with mean $MT$ and variance $\delta^2 T$, for some $M, \delta > 0$. To gain some intuition on the choice of mean and variance parameters, we may interpret $M$ as a (small) positive target gain while $\delta$ is fixed to fulfil $\mathbf{P}[-\bar{X}_T^{\Delta} > \ell] \leq 1 - \breve{\alpha}$ for some given $\ell > 0$ and $\breve{\alpha} \in (0, 1)$. The latter is a condition on the VaR at the confidence level $\breve{\alpha}$ (for instance $\breve{\alpha} = 99.5\%$, according to Solvency II requirements). [2] This can be interpreted as the required capital ensuring the system's solvency. Aiming at $\bar{X}_T^{\Delta} \sim \mathcal{N}(MT, \delta^2 T)$ as a target distribution is justified, for instance, by the existence of the closed form formulas for the VaR or ES for Gaussian random variables, which can be easily calculated. Indeed, denoting by $L_T$ the terminal loss at time $T$, we immediately get that

$$VaR_{\breve{\alpha}}(L_T) = -MT + \delta\sqrt{T}\Phi^{-1}(\breve{\alpha})$$

and

$$ES_{\breve{\alpha}}(L_T) = -MT + \delta\sqrt{T}\frac{\varphi(\Phi^{-1}(\breve{\alpha}))}{1 - \breve{\alpha}},$$

for $\breve{\alpha} \in (0, 1)$, where $\varphi$ and $\Phi$ indicate the density and the cumulative distribution function of the standard normal, respectively.

Our next step is to define the set of possible controls leading to the target distribution. We let $\mathcal{A}$ denote the set of strategies $\mathbf{\Delta} = \{\Delta_t,\ t \in [0, T]\}$ with $\Delta_t \in [0, 1]$, for all $t \in [0, T]$, that are $\mathbb{F}$-adapted and such that $\bar{X}_T^{\mathbf{\Delta}} \sim \mathcal{N}(MT, \delta^2 T)$.

In particular, we notice that deterministic controls make the terminal distribution of the net collective Gaussian, as we see in the following Example.

**Example 2.2** (Deterministic controls)**.** *In this example we deal with a deterministc retention level and we use the notation $\Delta(t)$ in place of $\Delta_t$ just to emphasize theis deterministic nature of the strategy. Let $\mathbf{\Delta} = \{\Delta(t),\ t \in [0, T]\}$ be a continuous deterministic reinsurance strategy, with $\Delta(t) \in [0, 1]$, for all $t \in [0, T]$. Then $\mathbf{\Delta}$ is an admissible control if the following two conditions hold:*

$$\begin{cases} \lambda\mu_1 \int_0^T (\delta^R \Delta(s) - (\delta^R - \delta^I))\mathrm{d}s = MT\ , \\ \lambda\mu_2 \int_0^T \Delta(s)^2 \mathrm{d}s = \delta^2 T\ . \end{cases}$$

[2] VaR is "the maximum loss that is not exceeded with a given high probability": it is simply a quantile that estimates how much a company might lose with a given probability level. In particular, $\ell$ represents the loss that the insurer can bear with at most probability $1 - \breve{\alpha}$.

*To make an example,* $\Delta(t) = \frac{A}{A+Ct}$ *is an admissible control for constants* $A, C$ *which satisfy*

$$\int_0^T \frac{A}{A+Cs}\mathrm{d}s = \frac{A}{C}\ln\Big(\frac{A+CT}{A}\Big) = \frac{MT + \lambda\mu_1(\delta^R - \delta^I)T}{\lambda\delta^R\mu_1}\ ,$$
$$\int_0^T \frac{A^2}{(A+Cs)^2}\mathrm{d}s = \frac{A}{C}\Big(1 - \frac{A}{A+CT}\Big) = \frac{\delta^2 T}{\lambda\mu_2}\ .$$

We wish to investigate the question which admissible reinsurance strategy minimizes ruin probability, if the ruin-checks are due at discrete deterministic points in time. In the sequel we restrict to the case where reinsurance strategies can be updated only at deterministic time points, which represent some apriori fixed checking dates, for instance all the four months or all the six months. In particular, we focus on the two period model, i.e. $n = 2$, due to a technical reason. Indeed, reinsurance controls affect both the drift and the volatility. Therefore, in this case, a pathwise comparison is not possible anymore, and the problem must be addressed with different techniques. In the case $n = 2$, we are still able to obtain an explicit solution with probabilistic methods. However, the problem becomes immediately more complicated when we increase the number of periods (see Section 2.4.4), even if we restrict to deterministic strategies.

### 2.4.1 Admissible strategies in a 2-period model

We denote the set of admissible strategies by $\mathcal{A}_{(2)}$, specifying, like before, the number of strategy updates up to time $T$. An admissible strategy is a pair $\boldsymbol{\Delta} = (\Delta_0, \Delta_1)$, where $\Delta_0$ is $\mathcal{F}_0$-measurable and $\Delta_1$ is $\mathcal{F}_{T/2}$-measurable. In this setting the retention level is updated only once, at time $T/2$. Hence, at time $T$ the net surplus satisfies

$$\begin{aligned}\bar{X}_T^{\boldsymbol{\Delta}} &= \frac{\lambda\mu_1\delta^R T}{2}(\Delta_0 + \Delta_1) - \lambda\mu_1(\delta^R - \delta^I)T + \sqrt{\lambda\mu_2}\Delta_0 W_{T/2} + \sqrt{\lambda\mu_2}\Delta_1(W_T - W_{T/2})\\ &= \bar{X}_{T/2}^{\Delta_0} + \Delta_1\frac{T}{2}\lambda\mu_1\delta^R - \lambda\mu_1(\delta^R - \delta^I)\frac{T}{2} + \sqrt{\lambda\mu_2}\Delta_1(W_T - W_{T/2})\ ,\end{aligned}$$

where $\bar{X}_{T/2}^{\Delta_0} = \lambda\mu_1(\delta^R\Delta_0 - \delta^R + \delta^I)\frac{T}{2} + \sqrt{\lambda\mu_2}\Delta_0 W_{T/2}$.

A precise characterization of admissible strategies is contained in the lemma below where we show that they are deterministic.

**Lemma 2.2.** *The set* $\mathcal{A}_{(2)}$ *consists of all strategies* $\boldsymbol{\Delta} = (\Delta_0, \Delta_1)$ *where* $\Delta_0, \Delta_1$ *are both*

*$\mathcal{F}_0$-measurable, taking values in* $[0,1]$*, and satisfying the following two conditions:*

$$\begin{aligned} \Delta_1 &= 2\frac{M+\lambda\mu_1(\delta^R-\delta^I)}{\lambda\delta^R\mu_1} - \Delta_0, \\ \Delta_1^2 &= \frac{2\delta^2}{\lambda\mu_2} - \Delta_0^2\,. \end{aligned} \tag{2.8}$$

*Proof.* Recall that for any normally distributed random variable $Y$ with mean $MT$ and variance $\delta^2 T$, the moment generating function is given by $\mathbb{E}\left[e^{\zeta Y}\right] = e^{\zeta MT+\frac{1}{2}\zeta^2\delta^2 T}$, for all $\zeta \in \mathbb{R}$. Let $\tilde{W}_{T/2} = W_T - W_{T/2}$; then $W_{T/2}$ and $\tilde{W}_{T/2}$ are independent. Since $\boldsymbol{\Delta}$ is chosen so that $X_T^{\boldsymbol{\Delta}} \sim \mathcal{N}(MT, \delta^2 T)$, it holds that

$$\mathbb{E}[e^{\zeta X_T^{\boldsymbol{\Delta}}}] = e^{\zeta MT+\frac{1}{2}\zeta^2\delta^2 T},$$

for all $\zeta \in \mathbb{R}$. Now, we let $\mathbf{Q}$ be a probability measure equivalent to $\mathbf{P}$, with the Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T} = e^{r\sqrt{\lambda\mu_2}\Delta_1\widetilde{W}_{T/2}-\frac{r^2}{2}\lambda\mu_2\Delta_1^2\frac{T}{2}}.$$

Using the independence of $\tilde{W}_{T/2}$ and $W_{T/2}$ and the change of measure we get that

$$\begin{aligned} e^{\zeta MT+\frac{1}{2}\zeta^2\delta^2 T} &= \mathbb{E}[e^{\zeta \bar{X}_T^{\Delta}}] = \mathbb{E}\left[e^{\zeta\bar{X}_{T/2}^{\Delta_0}+r\lambda\mu_1(\delta^R\Delta_1-\delta^R+\delta^I)\frac{T}{2}+\zeta\sqrt{\lambda\mu_2}\Delta_1\tilde{W}_{T/2}}\right] \\ &= e^{\zeta\lambda\mu_1(\Delta_0\delta^R-\delta^R+\delta^I)T/2+\frac{\zeta^2\lambda\mu_2\Delta_0^2 T}{4}}\,\mathbb{E}\left[e^{\zeta\lambda\mu_1(\delta^R\Delta_1-\delta^R+\delta^I)\frac{T}{2}+\zeta\sqrt{\lambda\mu_2}\Delta_1\tilde{W}_{T/2}}\right] \\ &= e^{\zeta\lambda\mu_1(\Delta_0\delta^R-\delta^R+\delta^I)T/2+\frac{\zeta^2\lambda\mu_2\Delta_0^2 T}{4}}\,\mathbb{E}_{\mathbf{Q}}\left[e^{\zeta\lambda\mu_1(\delta^R\Delta_1-delta^R+\delta^I)\frac{T}{2}+\frac{\zeta^2}{2}\lambda\mu_2\Delta_1^2\frac{T}{2}}\right] \\ &= e^{r\lambda\mu_1\delta^R\Delta_0\frac{T}{2}-r\lambda\mu_1(\delta^R-\delta^I)T}\,\mathbb{E}_{\mathbf{Q}}\left[e^{r\sqrt{\lambda\mu_2}\Delta_0 W_{\frac{T}{2}}+r\sqrt{\lambda\mu_2}\Delta_1\widetilde{W}_{\frac{T}{2}}+r\lambda\mu_1\delta^R\Delta_1\frac{T}{2}}\right] \\ &= e^{r\lambda\mu_1\delta^R\Delta_0\frac{T}{2}+r^2\lambda\mu_2\frac{\Delta_0^2}{2}\frac{T}{2}-r\lambda\mu_1(\delta^R-\delta^I)T}\,\mathbb{E}_{\mathbf{Q}}\left[e^{r\lambda\mu_1\delta^R\Delta_1\frac{T}{2}+\frac{1}{2}r^2\lambda\mu_2\Delta_1^2\frac{T}{2}}\right], \end{aligned}$$

for all $\zeta \in \mathbb{R}$. This can be simplified to

$$\mathbb{E}_{\mathbf{Q}}[e^{\zeta\lambda\mu_1(\delta^R\Delta_1-\delta^R+\delta^I)\frac{T}{2}+\frac{\zeta^2}{2}\lambda\mu_2\Delta_1^2\frac{T}{2}}] = e^{\zeta MT-\zeta\lambda\mu_1(\Delta_0\delta^R-\delta^R+\delta^I)\frac{T}{2}+\frac{1}{2}\zeta^2\delta^2 T-\frac{\zeta^2\lambda\mu_2\Delta_0^2 T}{4}}\,.$$

Deriving the above expression with respect to $\zeta$ and letting $\zeta = 0$, we observe that all moments of $\Delta_1$ correspond to the moments of a normal distribution, meaning that the moment generating function of $\Delta_1$ (written as a power series whose coefficients are the moments) corresponds to that of a normal distribution. Therefore, we conclude

$$\Delta_1 \sim \mathcal{N}\left(2\frac{M+\lambda\mu_1(\delta^R-\delta^I)}{\lambda\mu_1\delta^R} - \Delta_0, \frac{2\delta^2}{\lambda\mu_2} - \Delta_0^2\right)\,.$$

However, this is impossible because $\Delta_1$ can attain values only in $[0,1]$ $\mathbf{P}$-a.s. (hence also $\mathbf{Q}$-a.s.). As a result, $\Delta_1$ must be constant. $\square$

It is evident that not all arbitrary values of $M$ and $\delta$ are reachable, since $\Delta_0, \Delta_1$ can take values only in the interval $[0,1]$. In the next lemma we specify the ranges of $M$ and $\delta$.

**Lemma 2.3.** *If there exist $\Delta_0, \Delta_1 \in [0,1]$ such that condition (2.8) holds, then the target mean $M$ and the variance $\delta > 0$ satisfy:*

$$0 \le M \le \lambda\mu_1\delta^I,$$
$$\left(\frac{M + \lambda\mu_1(\delta^R - \delta^I)}{\lambda\mu_1\delta^R}\right)^2 \frac{M}{\lambda\mu_1\delta^R} \le \frac{\delta^2}{\lambda\mu_2} \le \min\left\{2\left(\frac{M + \lambda\mu_1(\delta^R - \delta^I)}{\lambda\mu_1\delta^R}\right)^2, 1\right\}. \tag{2.9}$$

*Proof.* From conditions (2.8) and the fact that $\Delta_0, \Delta_1$ take values in $[0,1]$, we get that $0 \le M \le \lambda\mu_1\delta^I$ and that $\frac{\delta^2}{\lambda\mu_2} \le 1$. Using again the conditions in (2.8) and substituting the value of $\Delta_1$ into the second equation, we get that $\Delta_0$ must solve

$$2\Delta_0^2 - 4\Delta_0 \frac{M + \lambda\mu_1(\delta^R - \delta^I)}{\lambda\mu_1\delta^R} + 4\left(\frac{M + \lambda\mu_1(\delta^R - \delta^I)}{\lambda\mu_1\delta^R}\right)^2 - \frac{2\delta^2}{\lambda\mu_1} = 0.$$

Imposing the existence of a real solution leads to

$$\frac{\delta^2}{\lambda\mu_2} \ge \left(\frac{M + \lambda\mu_1(\delta^R - \delta^I)}{\lambda\mu_1\delta^R}\right)^2.$$

Then, using the fact that $\Delta_0$ must take non-negative values, leads to the bound:

$$\frac{\delta^2}{\lambda\mu_2} \le \min\left\{2\left(\frac{M + \lambda\mu_1(\delta^R - \delta^I)}{\lambda\mu_1\delta^R}\right)^2, 1\right\}.$$

$\square$

Notice that the condition $\delta^I < \delta^R$ excludes arbitrage opportnities. As a consequence, in order to ensure the existence of a solution at least for the case $\delta^I = \delta^R$, we must have that $\frac{\delta^2}{\lambda\mu_2} \ge \left(\frac{M}{\lambda\mu_1\delta^R}\right)^2$, which is guaranteed by (2.9).

Lemma 2.3 prompts a clear trade-off between increasing profits and reducing risks. This is due to the fact that a reinsurance strategy controls both the mean and the volatility. We observe that increasing the retention level makes the mean larger and the same happens for the volatility. This means that in our reinsurance framework, the mean and the volatility of the surplus process move into the same direction, leading to interesting consequences for the ruin probability. Indeed, a greater retention level would make the drift of the net collective larger, meaning that it potentially can provide a good return pushing the surplus away from zero; however, at the same time, it increases the riskiness by making the volatility larger. For example, considering the parameters $\mu_1 = 0.22; \mu_2 = 0.05; \delta^I = 0.3; \delta^R = 0.35; \lambda = 2$, $M = 0.08 < 0.132$, we get that the admissible values of $\delta$ vary in the range $[0.2094, 0.2962]$. This means that, if an insurance company aims at getting an expected gain of 8% at the end of the observation period, it has to account for a relatively large risk of at least 21%.

We can write the range for $\delta$ as

$$\left(1 + \frac{M - \lambda\mu_1\delta^I}{\lambda\mu_1\delta^R}\right)^2 \leq \frac{\delta^2}{\lambda\mu_2} \leq \min\left\{2\left(1 + \frac{M - \lambda\mu_1\delta^I}{\lambda\mu_1\delta^R}\right)^2, 1\right\}.$$

From this expression it is more clear that if the target return is close to $\lambda\mu_1\delta^I$, the variance $\delta^2$ is approximately $\lambda\mu_2$, which corresponds to the case where no reinsurance is bought.

### 2.4.2 Ruin probabilities in a 2-period model

For the case $n = 2$, there are only two admissible retention levels, i.e. the pairs of strategies satisfying Conditions (2.8) are of the type $(\Delta_0, \Delta_1)$ and $(\Delta_1, \Delta_0)$.

We assume that $\frac{T}{2}$ and $T$ are the regulatory inspection dates and we investigate the survival probabilities at these dates. Specifically, we choose a reinsurance strategy in such a way the probability of having a positive surplus at both dates is maximized.

We now give a definition of ruin within this reinsurance setting. We say that *the ruin occurs if the insurance company showcases a negative surplus at any of the time points $T/2$ or $T$*. Then, an equivalent formulation of the problem is:

Find a reinsurance strategy that minimizes the ruin probability.

In mathematical terms, the problem is formulated as follows. Let $\boldsymbol{\Delta} = (\Delta_0, \Delta_1)$ and $\tilde{\boldsymbol{\Delta}} = (\Delta_1, \Delta_0)$ be the two admissible strategies. Without loss of generality, we assume that $\Delta_0 \leq \Delta_1$. For each strategy we define the corresponding survival probabilities:

$$p(\boldsymbol{\Delta}) = \mathbf{P}\left[\bar{X}^{\boldsymbol{\Delta}}_{T/2} > 0, \bar{X}^{\boldsymbol{\Delta}}_{T} > 0\right],$$
$$p(\tilde{\boldsymbol{\Delta}}) = \mathbf{P}\left[\bar{X}^{\tilde{\boldsymbol{\Delta}}}_{T/2} > 0, \bar{X}^{\tilde{\boldsymbol{\Delta}}}_{T} > 0\right].$$

Our aim is to decide which of these two probabilities, $p(\mathbf{\Delta})$ or $p(\tilde{\mathbf{\Delta}})$, is the largest. This means to decide which of the two admissible strategies is better according to this criterion.

We provide a numerical example where we get that it is convenient to reinsure less in the first period. In table 2.1 we illustrate survival probability for different values of $\delta^I < \delta^R$ so that $M$ and $\delta$ are achievable for $T = 1, \lambda = 2, \mu_1 = 0.22, \mu_2 = 0.05, \delta^R = 0.35, M = 0.05, \delta = 0.2$. The last two columns suggest that $p(\mathbf{\Delta}) > p(\tilde{\mathbf{\Delta}})$. Our empirical intuition is formally proved by next Proposition 2.6.

| $\delta^I$ | $\Delta_0$ | $\Delta_1$ | $\mathbf{p}(\Delta_0, \Delta_1)$ | $\mathbf{p}(\Delta_1, \Delta_0)$ |
|---|---|---|---|---|
| 0.25 | 0.4448 | 0.7760 | 0.4088 | 0.5117 |
| 0.26 | 0.3339 | 0.8298 | 0.3772 | 0.5372 |
| 0.27 | 0.2468 | 0.8597 | 0.3485 | 0.5561 |
| 0.28 | 0.1715 | 0.8778 | 0.3154 | 0.5720 |
| 0.29 | 0.1038 | 0.8884 | 0.2637 | 0.5857 |
| 0.3 | 0.0416 | 0.8935 | 0.1254 | 0.5967 |

Table 2.1: Admissible strategies and survival probabilities for different values of $\delta^I < \delta^R$.

**Proposition 2.6.** *Let $\Delta_0 < \Delta_1$. Then the strategy $(\Delta_1, \Delta_0)$ is better than the strategy $(\Delta_0, \Delta_1)$, i.e. $p(\tilde{\mathbf{\Delta}}) > p(\mathbf{\Delta})$.*

*Proof.* We first will derive an alternative representation for $p(\mathbf{\Delta})$ and $p(\tilde{\mathbf{\Delta}})$. Let $W = \{W_t,\ t \geq 0\}$ and $\hat{W} = \{\hat{W}_t,\ t \geq 0\}$ be two independent Brownian motions and denote

$$\bar{X}^{\Delta}_{T/2} = \lambda\mu_1 \left(\delta^R(1-\Delta) - \delta^I\right) T/2 + \sqrt{\lambda\mu_2}\Delta W_{T/2} \ ,$$
$$\hat{X}^{\Delta}_{T/2} = \lambda\mu_1 \left(\delta^R(1-\Delta) - \delta^I\right) T/2 + \sqrt{\lambda\mu_2}\Delta \hat{W}_{T/2} \ .$$

Then, the survival probabilities can be rewritten as

$$\begin{aligned} p(\mathbf{\Delta}) &= \mathbf{P}\left[\bar{X}^{\Delta_0}_{T/2} > 0, \bar{X}^{\Delta_0}_{T/2} + \hat{X}^{\Delta_1}_{T/2} > 0\right] \ , \\ p(\tilde{\mathbf{\Delta}}) &= \mathbf{P}\left[\bar{X}^{\Delta_1}_{T/2} > 0, \bar{X}^{\Delta_1}_{T/2} + \hat{X}^{\Delta_0}_{T/2} > 0\right] \ , \end{aligned} \tag{2.10}$$

having set $\hat{W}_{T/2} = W_T - W_{T/2}$. The advantage of this representation stands in the fact that for every $\Delta \in [0,1]$, $\bar{X}^{\Delta}_{T/2}$ and $\hat{X}^{\Delta}_{T/2}$ are independent. We observe that there exist standard

Brownian motions $W^0$ and $W^1$ such that

$$\begin{aligned}\sqrt{\lambda\mu_2}\Delta_0 W_t + \sqrt{\lambda\mu_2}\Delta_1 \hat{W}_t &= \sqrt{\lambda\mu_2(\Delta_0^2+\Delta_1^2)}W_t^0 \ ,\\ \sqrt{\lambda\mu_2}\Delta_1 W_t + \sqrt{\lambda\mu_2}\Delta_0 \hat{W}_t &= \sqrt{\lambda\mu_2(\Delta_0^2+\Delta_1^2)}W_t^1 \ ,\end{aligned} \tag{2.11}$$

for all $t \in [0,T]$. We now let

$$\begin{aligned}Y_t^0 &:= \lambda\mu_1\delta^R(\Delta_0+\Delta_1)t - 2\lambda\mu_1(\delta^R-\delta^I)t + \sqrt{\lambda\mu_2(\Delta_0^2+\Delta_1^2)}W_t^0,\\ Y_t^1 &:= \lambda\mu_1\delta^R(\Delta_0+\Delta_1) - 2\lambda\mu_1(\delta^R-\delta^I)t + \sqrt{\lambda\mu_2(\Delta_0^2+\Delta_1^2)}W_t^1,\end{aligned}$$

for all $t \in [0,T]$. Due to equations in (2.11) we get that for all $t \in [0,T]$, $Y_t^0 = Y_t^1 = 2Mt + \sqrt{2\delta^2}\hat{W}_t$; hence they are identically distributed.

Next, we write $\bar{X}_{T/2}^{\Delta_0}$ and $\bar{X}_{T/2}^{\Delta_1}$ in terms of $Y_{T/2}^0$ and $Y_{T/2}^1$. Since $\bar{X}_{T/2}^{\Delta_0}$ and $\bar{X}_{T/2}^{\Delta_1}$ are normally distributed, we have that

$$\begin{aligned}\bar{X}_{T/2}^{\Delta_0} &= \rho Y_{T/2}^0 + Z^0 \ , \quad \rho := \frac{Cov(\bar{X}_{T/2}^{\Delta_0}, Y_{T/2}^0)}{Var(Y_{T/2}^0)} = \frac{Var[\bar{X}_{T/2}^{\Delta_0}]}{Var[Y_{T/2}^0]} = \frac{\lambda\mu_2\Delta_0^2}{2\delta^2} \ ,\\ \bar{X}_{T/2}^{\Delta_1} &= \gamma Y_{T/2}^1 + Z^1 \ , \quad \gamma := \frac{Cov(\bar{X}_{T/2}^{\Delta_1}, Y_{T/2}^1)}{Var[Y_{T/2}^1]} = \frac{Var[\bar{X}_{T/2}^{\Delta_1}]}{Var[Y_{T/2}^1]} = \frac{\lambda\mu_2\Delta_1^2}{2\delta^2} = 1-\rho \ ,\end{aligned}$$

where $Y_{T/2}^0$ and $Z^0$, $Y_{T/2}^1$ and $Z^1$ are independent, since they are all normally distributed and $Cov(Y_{T/2}^0, Z^0) = Cov(Y_{T/2}^1, Z^1) = 0$. Expectations and variances of $Z^0$ and $Z^1$ are given by

$$\begin{cases}\mathbb{E}[Z^0] = \mathbb{E}[\bar{X}_{T/2}^{\Delta_0} - \rho Y_{T/2}^0] = \lambda\mu(\delta^R\Delta_0 - \delta^R + \delta^I)T/2 - 2\rho MT/2 \ ,\\ \mathbb{E}[Z^1] = \mathbb{E}[\bar{X}_{T/2}^{\Delta_1} - \gamma Y_{T/2}^1] = \lambda\mu(\delta^R\Delta_1 - \delta^R + \delta^I)T/2 - 2\gamma MT/2 = -\mathbb{E}[Z^0] \ ,\\ Var[Z^0] = \lambda\mu_2\Delta_0^2 T/2 - 2\rho^2\delta^2 T/2 = 2\delta^2\rho\gamma T/2 \ ,\\ Var[Z^1] = \lambda\mu_2\Delta_1^2 T/2 - 2\gamma^2\delta^2 T/2 = 2\delta^2\gamma\rho T/2 \ .\end{cases} \tag{2.12}$$

Using Fubini's theorem and the fact that $\mathbb{E}[Z^1] = -\mathbb{E}[Z^0]$, we get

$$\begin{aligned}p(\mathbf{\Delta}) &= \mathbf{P}\left[Y_{T/2}^0 + \frac{Z^0}{\rho} > 0,\ Y_{T/2}^0 > 0\right] = \mathbf{P}\left[\frac{Z^0}{\rho} > -Y_{T/2}^0,\ Y_{T/2}^0 > 0\right]\\ &= \int_0^\infty \left(1 - \Phi\left(\frac{-y - \frac{\mathbb{E}[Z^0]}{\rho}}{\sqrt{\delta^2\frac{\gamma}{\rho}T}}\right)\right) f_{Y_{T/2}^0}(y)\mathrm{d}y \ ,\\ p(\tilde{\mathbf{\Delta}}) &= \mathbf{P}\left[Y_{T/2}^1 + \frac{Z^1}{\gamma} > 0,\ Y_{T/2}^1 > 0\right] = \mathbf{P}\left[\frac{Z^1}{\gamma} > -Y_{T/2}^1,\ Y_{T/2}^1 > 0\right]\\ &= \int_0^\infty \left(1 - \Phi\left(\frac{-y + \frac{\mathbb{E}[Z^0]}{\gamma}}{\sqrt{\delta^2\frac{\rho}{\gamma}T}}\right)\right) f_{Y_{T/2}^1}(y)\mathrm{d}y \ ,\end{aligned}$$

where $\Phi$ is the standard normal distribution, $f_{Y^0_{T/2}}(y) = f_{Y^1_{T/2}}(y)$ are the densities of the random variables $Y^0_{T/2}$ and $Y^1_T/2$, respectively.

Since $\Phi$ is increasing, we consider the crucial quantities

$$z_0 := \frac{-y - \frac{\mathbb{E}[Z^0]}{\rho}}{\sqrt{\delta^2 \frac{\gamma}{\rho} \frac{T}{2}}} \quad \text{and} \quad z_1 := \frac{-y + \frac{\mathbb{E}[Z^0]}{\gamma}}{\sqrt{\delta^2 \frac{\rho}{\gamma} \frac{T}{2}}} \,.$$

We have the following two cases:

a. Firstly, we assume that $\mathbb{E}[Z^0] \leq 0$, with $\mathbb{E}[Z^0]$ from (2.12). Since $1-2\rho = \frac{\lambda\mu_2}{\delta^2}\left(\frac{\delta^2}{\lambda\mu_2} - \Delta_0^2\right) = \frac{\lambda\mu_2}{\delta^2}\frac{\Delta_1^2-\Delta_0^2}{2} > 0$, it holds that $z_0 > z_1$ for all $y > 0$. Then, we can immediately conclude, that $p(\mathbf{\Delta}) < p(\tilde{\mathbf{\Delta}})$ and hence the strategy $\tilde{\mathbf{\Delta}} = (\Delta_1, \Delta_0)$ is better than the strategy $\mathbf{\Delta} = (\Delta_0, \Delta_1)$.

b. Next, we assume that $\mathbb{E}[Z^0] > 0$, with $\mathbb{E}[Z^0]$ from (2.12). Note that since $\Delta_0 \leq \Delta_1$ and $\Delta_0, \Delta_1 \in [0,1]$, then either $\Delta_0 = \Delta_1 = 1$, in which case there is nothing to prove since $\mathbf{\Delta}$ and $\tilde{\mathbf{\Delta}}$ are equal, or it cannot hold that $\Delta_0 = 1$. The latter implies that there always exists an $y^* \in (0, +\infty)$ such that for $y < y^*$, it holds that $z_0 < z_1$, and the opposite holds true for $y > y^*$; see, e.g. the right panel in Figure 2.1.
Now, we consider the functions $\delta^I \to \Delta_0(\delta^I)$ and $\delta^I \to \Delta_1(\delta^I)$, for $\delta^I \in [0, \delta^R]$. We know that

$$\Delta_0(\delta^I) + \Delta_1(\delta^I) = 2\frac{M - \lambda\mu_1(\delta^R - \delta^I)}{\lambda\mu_1\delta^R} \quad \text{and} \quad \Delta_0(\delta^I)^2 + \Delta_1(\delta^I)^2 = 2\frac{\delta^2}{\lambda\mu_2},$$

meaning that $\Delta_0'(\delta^I) + \Delta_1'(\delta^I) = \frac{2}{\delta^R} > 0$ and $\Delta_0'(\delta^I)\Delta_0(\delta^I) + \Delta_1'(\delta^I)\Delta_1(\delta^I) = 0$. Consequently, since $\Delta_0 < \Delta_1$, we get that $\Delta_1'(\delta^I) > 0$ and $\Delta_0'(\delta^I) < 0$. This means that $\Delta_0$ is decreasing with respect to $\delta^I$ and $\Delta_1$ is increasing. Taking $\delta^I = \delta^R$, we note that, for any $a \in [0,1]$, it holds that

$$\bar{X}_t^{\Delta} = \Delta\{\lambda\mu_1\delta^R + \sqrt{\lambda\mu_2}W_t\} = \frac{\Delta}{a}\bar{X}_t^a \,,$$

for every $t \in [0,T]$. Using this fact, equations (2.10) and $\Delta_0 < \Delta_1$, we get

$$\begin{aligned} p(\boldsymbol{\Delta}) = \mathbf{P}\big[\bar{X}^{\Delta_0}_{T/2} > 0, \bar{X}^{\Delta_0}_{T/2} + \hat{X}^{\Delta_1}_{T/2}\big] &= \mathbf{P}\big[\bar{X}^{\Delta_1}_{T/2} > 0, \frac{\Delta_0}{\Delta_1}\bar{X}^{\Delta_0}_{T/2} + \hat{X}^{\Delta_0}_{T/2}\big] \\ &= \mathbf{P}\big[\bar{X}^{\Delta_1}_{T/2} > 0, \frac{\Delta_0^2}{\Delta_1^2}\bar{X}^{\Delta_1}_{T/2} + \hat{X}^{\Delta_0}_{T/2}\big] \\ &< \mathbf{P}\big[\bar{X}^{\Delta_1}_{T/2} > 0, \bar{X}^{\Delta_1}_{T/2} + \hat{X}^{\Delta_0}_{T/2}\big] \\ &= p(\tilde{\boldsymbol{\Delta}}) \;, \end{aligned}$$

which proves the statement in case $\delta^I = \delta^R$. Now, we let $\boldsymbol{\Delta}(\delta^I)$ and $\tilde{\boldsymbol{\Delta}}(\delta^I)$ be, respectively, the strategies $(\Delta_0, \Delta_1)$ and $(\Delta_1, \Delta_0)$ corresponding to $\delta^I \in (0, \delta^R)$. We assume that there is a $\bar{\delta}^I \in (0, \delta^R)$ such that $p(\boldsymbol{\Delta}(\bar{\delta}^I)) > p(\tilde{\boldsymbol{\Delta}}(\bar{\delta}^I))$. Then, by the intermediate value theorem, there exists a $\delta^{I^*} \in (\bar{\delta}^I, \delta^R)$ such that $p(\boldsymbol{\Delta}(\delta^{I^*})) = p(\tilde{\boldsymbol{\Delta}}(\delta^{I^*}))$. We suppose that $\boldsymbol{\Delta}(\delta^{I^*}) \neq \tilde{\boldsymbol{\Delta}}(\delta^{I^*})$. Let $\bar{X}$ be a random variable, independent of $Z^1$ and $Z^0$ with $\bar{X} \sim \mathcal{N}(MT, \delta^2 T)$.

$$\begin{aligned} 0 &= p(\boldsymbol{\Delta}(\delta^{I^*})) - p(\tilde{\boldsymbol{\Delta}}(\delta^{I^*})) \\ &= \mathbf{P}\left[\gamma\bar{X} + Z^1 > 0, \bar{X} > 0\right] - \mathbf{P}\left[\delta^I\bar{X} + Z^0 > 0, \bar{X} > 0\right] \\ &= \mathbf{P}\left[\max\left(-\frac{Z^1}{\gamma}, 0\right) < \bar{X} < \max\left(-\frac{Z^0}{\delta^I}, 0\right)\right] > 0 \;. \end{aligned}$$

The last inequality follows from the fact that $\bar{X}$, $Z^1$ and $Z^0$ are normally distributed. Hence, this contradiction yields that $\boldsymbol{\Delta}(\delta^{I^*}) = \tilde{\boldsymbol{\Delta}}(\delta^{I^*})$. However, since it holds that $\Delta_1' > 0$ and $\Delta_0' < 0$, for $\Delta_0 < \Delta_1$, that means $\Delta_0(\bar{\delta}^I) > \Delta_0(\delta^{I^*})$ and $\Delta_1(\bar{\delta}^I) < \Delta_1(\delta^{I^*})$, contradicting $\boldsymbol{\Delta}(\delta^{I^*}) = \tilde{\boldsymbol{\Delta}}(\delta^{I^*})$.

As a consequence, we can conclude that $\tilde{\boldsymbol{\Delta}} = (\Delta_0, \Delta_1)$ is always better than $\boldsymbol{\Delta} = (\Delta_1, \Delta_0)$. $\square$

In the following, we discuss the situations where $\mathbb{E}[Z^0] \leq 0$ (case *a* of the proof above), deriving sufficient conditions for this expected value to be negative.

**Lemma 2.4.** *If*

$$\frac{\mu_1 \delta^I}{\mu_2} \leq \frac{M}{\delta^2} \quad \textit{and} \quad \frac{\mu_1 \delta^R}{\mu_2} \geq \frac{2M}{\delta^2}, \tag{2.13}$$

*then* $\mathbb{E}[Z^0] \leq 0$, *with* $\mathbb{E}[Z^0]$ *given in* (2.12).

*Proof.* By expression in (2.12), if we substitute $\rho = \frac{\lambda\mu_2\Delta_0^2}{2\delta^2}$, we observe that $\mathbb{E}[Z_0] \leq 0$ reduces to

$$-\frac{\lambda\mu_2\Delta_0^2}{\delta^2}MT + \lambda\mu_1(\delta^R\Delta_0 - \delta^R + \delta^I)T \leq 0, \tag{2.14}$$

for all $\Delta_0 \in [0,1]$. To show that (2.14) holds for all $\Delta_0 \in [0,1]$, we consider the function

$$F(\Delta) = -\frac{\mu_2\Delta^2}{\delta^2}M + \mu_1(\delta^R\Delta - \delta^R + \delta^I).$$

This function is concave and has a maximum at $\Delta^* = \frac{\mu_1\delta^R\delta^2}{2\mu_2 M} > 0$. We note that $F(0) < 0$ and that $F(1) = -\frac{\mu_2}{\delta^2}M + \mu_1\delta^I$, which is negative if the first of condition in (2.13) holds true. Moreover, under the second condition in (2.13), we also get that $\Delta^* \geq 1$, which guarantees that $\mathbb{E}[Z] \leq 0$. □

Let us briefly comment on the conditions (2.13) from an economic point of view. Rewriting them as $\mu_1\delta^I \leq \frac{M\mu_2}{\delta^2}$ and $\mu_1\delta^R \geq \frac{2M\mu_2}{\delta^2}$, we immediately get that the reinsurance is costly and the income from the insurance premia is low. Therefore, by Proposition 2.6, we could say that if reinsurance is expensive and the insurance is cheap, it is optimal, in terms of survival probability, to reinsure more claims in the second part of the trading interval. In other words, under conditions (2.13), choosing a bigger retention level in the first period has the advantage that a larger drift can drive the surplus away from zero and hence it minimizes the ruin at times $T/2$ and $T$.

To better understand the different cases (i.e. $\mathbb{E}[Z_0] \leq 0$ and $\mathbb{E}[Z_0] > 0$), we let

$$G_0(y) = \left(1 - \phi\left(\frac{-y - \frac{\mathbb{E}[Z^0]}{\rho}}{\sqrt{\delta^2\frac{1-\rho}{\rho}T}}\right)\right) f_{Y^0_{T/2}}(y),$$

$$G_1(y) = \left(1 - \phi\left(\frac{-y + \frac{\mathbb{E}[Z^0]}{\gamma}}{\sqrt{\delta^2\frac{1-\gamma}{\gamma}T}}\right)\right) f_{Y^1_{T/2}}(y),$$

for all $y > 0$, so that

$$p(\boldsymbol{\Delta}) = \int_0^\infty G_0(y)\mathrm{d}y, \qquad p(\tilde{\boldsymbol{\Delta}}) = \int_0^\infty G_1(y)\mathrm{d}y.$$

Figure 2.1 represents the densities of the survival probability (i.e. $G_0(y)$ and $G_1(y)$) relative to the strategy $\boldsymbol{\Delta} = (\Delta_0, \Delta_1)$ (blue line) and the strategy $\tilde{\boldsymbol{\Delta}}(\Delta_1, \Delta_0)$ (red line), under given parameters.

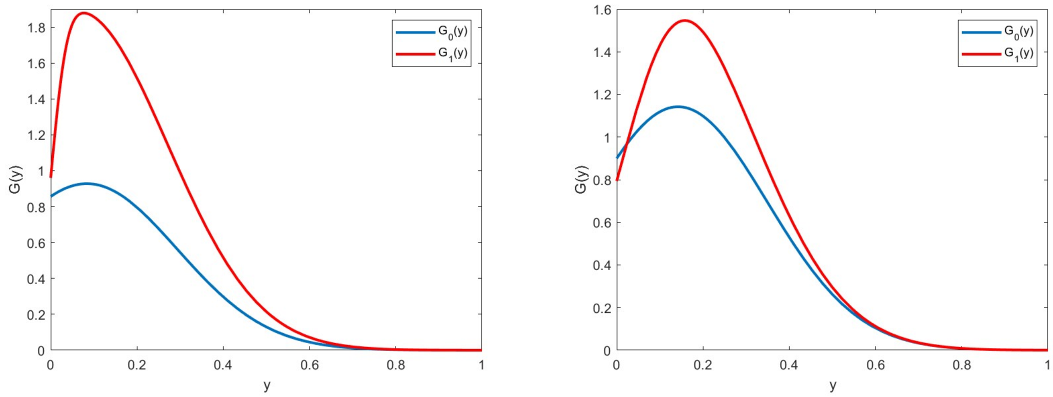



Figure 2.1: Survival density under the case $\mathbb{E}[Z^0] < 0$ (left panel), and under $\mathbb{E}[Z^0] > 0$ (right panel).

The left panel corresponds to the case where conditions (2.13) hold, i.e., insurance is cheap and reinsurance is expensive. Here, the survival probability of the strategy $\tilde{\mathbf{\Delta}}$ dominates that of the strategy $\mathbf{\Delta}$, for all values of $y$. Instead, in the right panel, there exist a level $y^* > 0$ (small) at which these two densities curves switch. However the area under the curve $G_1$ in the set $\{y > y^*\}$ largely compensates that in the set $\{y < y^*\}$. We point out that such point $y^*$ only exists in case $\mathbb{E}[Z^0] > 0$; in particular it corresponds to $y^* = \frac{2\mathbb{E}[Z^0]}{1-2\rho}$. We notice that such compensation of areas always applies, thanks to natural bounds on the value of $\delta^I$, such as $0 \leq \delta^I \leq \delta^R$. Hence, also when insurance premium is large and reinsurance is not too expensive, $p(\tilde{\mathbf{\Delta}}) > p(\mathbf{\Delta})$ holds true, see Proposition 2.6).

### 2.4.3 The penalization problem

Now, we face the following situation: the insurance company may decide to update or not the reinsurance contract at time $T/2$. If the contract is updated, the company will pay a penalty amounting to $PT$ at time $T/2$, with $P \geq 0$. In case of no changes, no penalty will be applied. The strategies corresponding to these two different scenarios are chosen to achieve a Gaussian distribution at time $T$ with the same target variance $\delta^2 T$. If the insurance company does not modify the retention level at time $T/2$, then the mean of the terminal net collective is $M'T$, uniquely determined by the condition on the target variance, with $M' > 0$. Otherwise (that is the case when the reinsurance strategy is modified in the middle of the trading period at $T/2$), the final expected wealth will be $M$ such that $0 < M < M'$.

Next, we show that considering such kind of penalization, it is more preferable to change the strategy at time $T/2$, even with a smaller expected mean, when the objective is to minimize ruin probability.

We assume that $M = M' - P$. Let $\hat{\mathbf{\Delta}} = (\hat{\Delta}, \hat{\Delta})$ be the strategy where the insurer decides to make no changes at time $T/2$ and let $\mathbf{\Delta} = (\Delta_0, \Delta_1)$ and $\tilde{\mathbf{\Delta}} = (\Delta_1, \Delta_0)$ be the admissible strategy where the insurer switches, with $\Delta_0 < \Delta_1$. We know, by Proposition 2.6, that strategy $\tilde{\mathbf{\Delta}}$ is better than $\mathbf{\Delta}$. The survival probability of strategy $\hat{\mathbf{\Delta}}$ is given by

$$p(\hat{\mathbf{\Delta}}) = \mathbf{P}\left[\bar{X}^{\hat{\Delta}}_{T/2} > 0,\ \bar{X}^{\hat{\Delta}}_{T} > 0\right].$$

We let $\hat{Y} = \bar{X}^{\hat{\Delta}}_{T}$. Then we get that $Y \sim \mathcal{N}(M'T, \delta^2 T)$ and we observe that

$$\bar{X}^{\hat{\Delta}}_{T/2} = \frac{1}{2}\hat{Y} + \hat{Z},$$

where

$$\hat{Z} \sim \mathcal{N}\left(\lambda\mu_1(\delta^R\hat{\Delta} - \delta^R + \delta^I)\frac{T}{2} - M'\frac{T}{2}, \frac{1}{4}\delta^2 T\right).$$

Since random variables $\hat{Y}$ and $\hat{Z}$ are independent, we get

$$p(\hat{\mathbf{\Delta}}) = \left[2\hat{Z} > -\hat{Y}, \hat{Y} > 0\right] = \int_0^\infty \left(1 - \phi\left(\frac{-y - 2\mathbb{E}[\hat{Z}]}{\sqrt{\delta^2 T}}\right)\right) f_{\hat{Y}}(y)\mathrm{d}y.$$

Next, for the strategy $\tilde{\mathbf{\Delta}}$ the survival probability is given by:

$$p(\tilde{\mathbf{\Delta}}) = \int_{PT}^\infty \left(1 - \phi\left(\frac{-y + PT - \frac{\mathbb{E}[Z^1]}{\gamma}}{\sqrt{\delta^2 \frac{\rho}{\gamma} T}}\right)\right) f_{\hat{Y}}(y)\mathrm{d}y,$$

where $Z^1 \sim \mathcal{N}\left(\lambda\mu_1(\delta^R\Delta_1 - \delta^R + \delta^I)T/2 - 2\gamma(M' - P)T/2, \gamma(1-\gamma)\delta^2 T\right)$, like in the proof of Proposition 2.6. If no penalization is involved (i.e. $P = 0$), there is a unique strategy, $\hat{\mathbf{\Delta}}$, that leads to the desired distribution for the net collective. Otherwise, considering a penalty for updating a reinsurance strategy (i.e. $P > 0$), the strategy $\hat{\mathbf{\Delta}}$ has a survival probability that is always smaller than the survival probability of the optimal strategy $\mathbf{\Delta}$ and larger than that of $\tilde{\mathbf{\Delta}}$, i.e. $p(\tilde{\mathbf{\Delta}}) \leq p(\hat{\mathbf{\Delta}}) \leq p(\mathbf{\Delta})$, as shown in Figure 2.2.

### 2.4.4 A 3-period model

As mentioned above, increasing the number of periods adds to the complexity of the problem. Here, the form of the survival probability does not allow deriving conditions that guarantee a

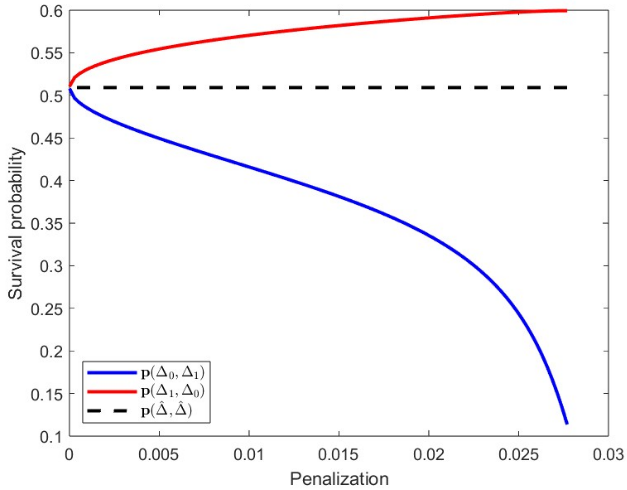



Figure 2.2: Survival probabilities under penalization. Red line corresponds to the survival probability of the strategy $\tilde{\mathbf{\Delta}}$, blue line to that of $\mathbf{\Delta}$ and the dashed line to that of the constant strategy $\hat{\mathbf{\Delta}}$.

clear dominance of one strategy over the others. In addition, the computational time increases with the number of periods. To better explain the difficulties that arise when $n > 2$, we briefly consider the case $n = 3$ and provide some intuition on how to deal with more than two updates.

Now, the admissible strategies are not necessarily deterministic and the optimal strategy may even not exist. However we still restrict to deterministic strategies $\mathbf{\Delta} = (\Delta_0, \Delta_1, \Delta_2)$. Then, in this example, it holds that

$$\begin{aligned}\Delta_0 + \Delta_1 + \Delta_2 &= 3\frac{M + \lambda\mu_1(\delta^R - \delta^I)}{\lambda\mu_1\delta^R} ,\\ \Delta_0^2 + \Delta_1^2 + \Delta_2^2 &= \frac{3\delta^2}{\lambda\mu_2} ,\\ \Delta_0, \Delta_1, \Delta_2 &\in [0, 1],\end{aligned}$$

which means that there are infinitely many combinations of $(\Delta_0, \Delta_1, \Delta_2)$ that lead to the target distribution. In particular, admissible triplets build (a part of) a circle as shown in Figure 2.3. In order to choose the ruin-minimizing strategy, we look at survival probability

$$p(\mathbf{\Delta}) = \mathbf{P}\left[\bar{X}_{\frac{T}{3}}^{\Delta_0} > 0, \bar{X}_{\frac{2T}{3}}^{\Delta_0,\Delta_1} > 0, \bar{X}_T^{\mathbf{\Delta}} > 0\right] .$$

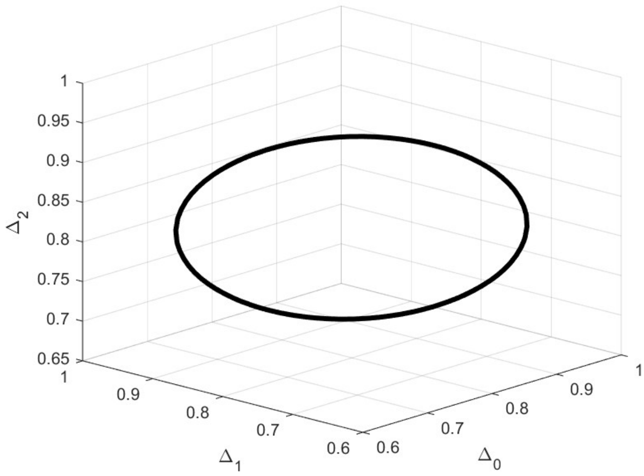



Figure 2.3: Admissible deterministic strategies for $n = 3$ and parameters $T = 1; \mu_1 = 0.15; \mu_2 = 0.06; \lambda = 1; \delta^R = 0.35; \delta^I = 0.2; M = 0.02; \delta = 0.2;$.

We define auxiliary random variables $\zeta^0, \zeta^1$ such that

$$\zeta^0 \sim \mathcal{N}\left(\lambda\mu_1(\delta^R\Delta_0 - \delta^R + \delta^I)\frac{T}{3} - M\rho_0 T,\ \rho_0(1-\rho_0)\delta^2 T\right),$$

$$\rho_2\zeta^0 + \zeta^1 \sim \mathcal{N}\left(-\lambda\mu_1\delta^R\Delta_2\frac{T}{3} + \lambda\mu_1(\delta^R - \delta^I)T + (1-\rho_1)MT,\ \rho_1(1-\rho_1)\delta^2 T\right),$$

which are correlated. Then, we have that

$$\begin{aligned} p(\boldsymbol{\Delta}) &= \mathbf{P}\left[\frac{\zeta^0}{\rho_0} > -\bar{X}_T^{\boldsymbol{\Delta}},\ \frac{\rho_2\zeta^0 + \zeta^1}{\rho_1} > -\bar{X}_T^{\boldsymbol{\Delta}},\ \bar{X}_T^{\boldsymbol{\Delta}} > 0\right] \\ &= \int_0^\infty \mathbf{P}\left[\frac{\zeta^0}{\rho_0} > y,\ \frac{\rho_2\zeta^0 + \zeta^1}{\rho_1} > y\right] f_Y(y)\mathrm{d}y\ , \end{aligned}$$

where $Y \sim \mathcal{N}(MT, \delta^2 T)$ and $f_Y(y)$ is the corresponding density.

We perform a numerical experiment in order to investigate the survival probability after choosing the retention level $\Delta_0$ for the first part of the time interval. In Figure 2.4, we plot the survival probability with respect to the first component $\Delta_0$ of the reinsurance strategy. It is clear that, once $\Delta_0$ is chosen, there are only two possible choices for $\Delta_1$ and $\Delta_2$. Suppose that, for instance $\Delta_1 > \Delta_2$. Thus, for a fixed $\Delta_0$, the possible strategies are $(\Delta_0, \Delta_1, \Delta_2)$ and $(\Delta_0, \Delta_1, \Delta_2)$. Figure 2.4 shows that the survival probability is maximized by taking the largest available value of $\Delta_0$ in the beginning and then choosing a retention level in the

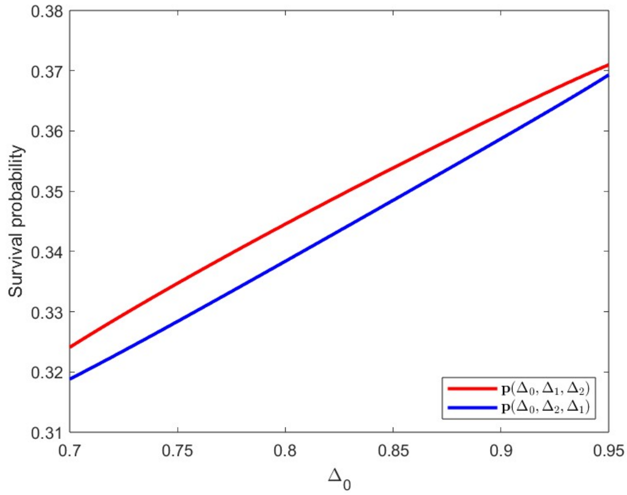



Figure 2.4: Survival probabilities as functions of the first component $\Delta_0$ of the reinsurance strategy $(\Delta_0, \Delta_1, \Delta_2)$.

second period which is greater than that one in the last time interval. Hence, the combination that leads to the higher value of survival probability is the sorted one, i.e. $(\Delta_0, \Delta_1, \Delta_2)$ with $\Delta_0 > \Delta_1 > \Delta_2$. This means that the insurance company increases the reinsurance coverage over time, especially shortly before the last checking date, hence acting in a risk averse way.

We conclude the section by showing that the sorted sequence of retention levels $\mathbf{\Delta} = (\Delta_0, \Delta_1, \Delta_2)$ leads to a bigger survival probability than the "unsorted" sequence $\tilde{\mathbf{\Delta}} = (\Delta_0, \Delta_2, \Delta_1)$, i.e. $p(\mathbf{\Delta}) > p(\tilde{\mathbf{\Delta}})$. In other words, any unsorted triplet will be overperformed by a sorted one, in terms of survival probability and thus also in terms of ruin probability. To prove this fact mathematically, we denote by $p^{x_0}(\cdot)$ the survival probability of a strategy, where $x_0$ is the initial capital. Then,

$$p^{x_0}(\mathbf{\Delta}) = \mathbf{P}\left[\bar{X}_{T/3}^{\Delta_0} > 0, \bar{X}_{2/3T}^{(\Delta_0,\Delta_1)} > 0, \bar{X}_{T}^{(\Delta_0,\Delta_1,\Delta_2)} > 0\right] = \mathbb{E}\left[\mathbb{1}_{\bar{X}_{T/3}^{\Delta_0} > 0}\, p^{\bar{X}_{T/3}^{\Delta_0}}((\Delta_1, \Delta_2))\right]$$
$$> \mathbb{E}\left[\mathbb{1}_{\bar{X}_{T/3}^{\Delta_0} > 0}\, p^{\bar{X}_{T/3}^{\Delta_0}}((\Delta_2, \Delta_1))\right] = p^{x_0}(\tilde{\mathbf{\Delta}}) \,,$$

where the inequality follows from the case $n = 2$.

**Chapter 3**

# Indifference pricing of pure endowments in a regime-switching market model

In this chapter we still consider an insurance company whose purpose is to maximize its profit but now our attention is focused on the problem of evaluating life insurance policies. Specifically, we study indifference pricing of mortality contingent claims in a stochastic-factor model for an insurance company endowed with exponential utility preferences. We propose a modeling framework where the hazard rate is described by an observable general diffusion process and the risky asset price evolves as a jump diffusion process affected by a continuous time finite state Markov chain representing regimes of the economy. Using the actuarial principle of equivalent utility, we characterize the indifference price for a pure endowment contract and provide its probabilistic representation. The indifference price has been determined by solving an equation involving two value functions, resulting from the stochastic control problems with and without insurance liabilities. As a consequence, we show that the price that makes the insurance company indifferent, in terms of expected utility, between not selling and selling the policy for that premium now and paying the benefits at maturity, is linked to a classical solution of a specific linear PDE with a proper terminal condition; it means that the indifference price solves a suitable final value problem.

The chapter is organized as follows. We introduce the indifference pricing approach in insurance in Section 3.1, by referring to the existing literature. Then, in Section 3.2 we describe the Markov-modulated financial-insurance market model. The pricing problem formulation via utility indifference pricing can be found in Section 3.3. In Section 3.4 we apply the classical approach based on the HJB equation to the resulting stochastic control problems and provide the Verification Theorems and describe the optimal investment strategies. The characterization of the indifference price of the pure endowment policy and a brief discussion on the indifference price for a portfolio of pure endowments and also for a term life insurance policy is given in Section 3.5. Last but not least, in Section 3.6, by performing some numerical experiments in case of a two-state Markov chain, we detect some interesting features of the indifference price, the optimal investment strategies and the value functions.

## 3.1 Indifference pricing in insurance

The evaluation of dynamic risks has been a fundamental issue in financial markets. One successful pricing technique consists of constructing a portfolio that accurately replicates the payoff of the product, whether it is a financial derivative, an insurance policy and so on. This traditional risk-neutral evaluation eliminates completely the risk but fails whenever the market is incomplete, namely whenever the market involves stochastic volatilities or random jumps as in our case. In case of incomplete market, various alternative pricing mechanisms have been developed, such as the superreplication (see e.g. Leland [70], Schweizer [90]), the local variance minimization via the instantaneous Sharpe ratio (see e.g. Bayraktar et al. [11], Delong [44]), the local risk-minimization in a partial information framework (see e.g. Ceci et al. [24, 25, 26]) and so called *utility indifference pricing method*. The latter relies heavily on risk preferences that are described by utility functions. Indeed, the indifference seller's price is defined at the level where the issuer of the contract is indifferent (in terms of expected utility) between entering the market on its own, or selling the claim and entering the market with the collected premium. In other words, the compensation at which the issuer is indifferent between the two alternative opportunities yields her/his indifference price which therefore can be determined by solving an equation involving two value functions, resulting from the stochastic control problems with and without incorporating the claim. Thus, this approach seems the most natural one, since it focuses more on the company preferences than on the market equilibrium. In the literature, the utility indifference pricing method was initially proposed by Hodges and Neuberger [64] for the valuation of European calls in the presence of transaction costs. After being refined and extended by Davis et al. [41], the methodology has gained much attention in the literature on pricing and hedging contingent claims, see e.g. Henderson and Hobson [59] for a survey. The indifference pricing approach has become a popular method for evaluating financial derivatives in incomplete markets and has been successfully applied to price insurance contracts in e.g. Young and Zariphopoulou [102], Moore and Young [80], Ludkovski and Young [76], Delong [43], Eichler et al. [48], Liang and Lu [71], Ceci et al. [27]. Precisely, [102] obtain explicit results for an exponential utility function by solving the HJB equation in a market driven by a geometric Brownian motion when the insurance risk is independent of the financial risk. This independence vashishes in Moore and Young [80] where a more general framework is studied considering a equity-linked pure endowment, namely an insurance policy whose payment amount is a function of the underlying risky asset. Ludkovski and Young [76] investigate pricing of pure endowments and life annuities in a fully stochastic model since they assume both stochastic interest rates and stochastic hazard rates governing the population mortality. Always considering a stochastic mortality rate, Delong [43] address the pricing and hedging problem for a group of life insurance liabilities in the presence of systematic mortality risks in a market model driven by a Levy process. In Eichler et al. [48], the authors analyze the valuation of catastrophe derivatives, while in Liang and Lu [71] they investigate the pricing problem for life insurance policies with equity-indexed life contingent payments, in a financial market charcaterized by shot-noise effects in the stock prices. Finally, results on the valuation of pure endowment policies under partial information via backward stochastic differential equations can be found in Ceci et al. [27]. It is worth noting that the indifference pricing approch is widely used also in non-life insurance, for instance to evaluate insurance-linked securities, see e.g. Liu et al. [72].

Here, according to [37], we investigate the indifference pricing problem of pure endowment contracts for an insurance company in a financial market where the risky asset price dynamics exhibits jumps and is affected by regime changes, when the hazard rate governing the population mortality is stochastic. A pure endowment is a life insurance policy which pays a fixed amount to the policyholder at maturity if and only if she/he survives the term. To the best of our knowledge, indifference pricing of life-insurance liabilities in a Markov-modulated framework accounting for a market behavior affected by long-term macroeconomic conditions and possible jumps in the risky asset price dynamics and stochastic hazard rate, is taken up for the first time.

## 3.2 Setting

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ endowed with a filtration $\mathbb{G} = \{\mathcal{G}_t,\ t \in [0, T]\}$, satisfying the usual conditions of completeness and right continuity, where $T > 0$ is a fixed, finite time horizon. Specifically, the filtration $\mathbb{G}$ is given by

$$\mathbb{G} = \mathbb{F} \vee \mathbb{F}^I,$$

where the filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \in [0, T]\}$ models the information flow in the financial market and $\mathbb{F}^I = \{\mathcal{F}^I_t,\ t \in [0, T]\}$ contains information about the lifetime of the individual insured. We assume that the subfiltrations $\mathbb{F}$ and $\mathbb{F}^I$ are independent.

To describe some possible structural changes in economic conditions, we introduce an irreducible and continuous time Markov chain $Y = \{Y_t,\ t \in [0, T]\}$ with finite state space $\mathcal{E} = \{1, 2, \ldots, K\}$, whose transition probabilities satisfy

$$\mathbf{P}(Y_{t+\delta t} = j | Y_t = i) = q_{ij}\delta t + o(\delta t),\ i \neq j;\ \ \mathbf{P}(Y_{t+\delta t} = i | Y_t = i) = 1 + q_{ii}\delta t + o(\delta t),$$

when $\delta t \longrightarrow 0$, where for each $i \in \mathcal{E}$ we have

$$q_{ij} \geq 0 \ \text{ for each } i \neq j \quad \text{and} \quad q_{ii} = -\sum_{j=1}^{K} q_{ij}.$$

Here, $Y_t$ represents the regime of the economy at time $t$, and $K$ the number of regimes. Let $Q = (q_{ij})_{i,j=1,\ldots,K}$ denote the generating $Q$-matrix of the Markov chain $Y$. It is convenient to represent $Y$ as a stochastic integral with respect to a Poisson random measure. Following the description of Basak et al. [10], for $i, j \in \mathcal{E}$, with $i \neq j$, we denote by $\Delta_{ij}$ the consecutive (with respect to the lexicographic ordering on $\mathcal{E} \times \mathcal{E}$) left-closed right-open intervals of the real line, each having length $q_{ij}$ and define a function $h : \mathcal{E} \times \mathbb{R} \longrightarrow \mathbb{R}^K$ by embedding $\{1, 2, \ldots, K\}$ into $\mathbb{R}^K$ (identifying $i$ with $e_i \in \mathbb{R}^K$), as follows

$$h(i, z) = \begin{cases} j - i, & \text{if } z \in \Delta_{ij} \\ 0, & \text{otherwise.} \end{cases}$$

Then, we get

$$Y_t = Y_0 + \int_0^t \int_{\mathbb{R}} h(Y_{v-}, z)\mathcal{P}(\mathrm{d}v, \mathrm{d}z), \quad t \in [0, T], \tag{3.1}$$

where the integration is over the interval $(0, t]$ and $\mathcal{P}(\mathrm{d}t, \mathrm{d}z,)$ is a Poisson random measure with intensity $m(\mathrm{d}z)\mathrm{d}t$, with $m(\mathrm{d}z)$ being the Lebesgue measure on $\mathbb{R}$. Let $\hat{\mathcal{P}}(\mathrm{d}t, \mathrm{d}z)$ be the compensated Poisson random measure, i.e. $\hat{\mathcal{P}}(\mathrm{d}t, \mathrm{d}z) = \mathcal{P}(\mathrm{d}t, \mathrm{d}z) - m(\mathrm{d}z)\mathrm{d}t$.

In this setting, we consider a financial market consisting of a locally risk-free money market account and one stock as a risky asset. The price process $S^0 = \{S_t^0,\ t \in [0, T]\}$ of the locally risk-free asset is described by

$$\mathrm{d}S_t^0 = rS_t^0\mathrm{d}t, \quad S_0^0 = 1,$$

where $r$ is a positive constant denoting the risk-less interest rate. The risky asset price process $S = \{S_t,\ t \in [0, T]\}$ evolves over time according to the following Markov-modulated dynamics

$$\mathrm{d}S_t = S_{t-}\left\{\mu(t, Y_t)\mathrm{d}t + \sigma(t, Y_t)\mathrm{d}W_t^S + K_1(t, Y_{t-})\mathrm{d}N_t^1 - K_2(t, Y_{t-})\mathrm{d}N_t^2\right\}, \quad S_0 = s > 0. \tag{3.2}$$

Here, $W^S = \{W_t^S,\ t \in [0, T]\}$ is a standard Brownian motion independent of $Y$ and $N^1 = \{N_t^1,\ t \in [0, T]\}$ and $N^2 = \{N_t^2,\ t \in [0, T]\}$ are independent Poisson processes defined on $(\Omega, \mathcal{F}, \mathbf{P}; \mathbb{F})$. Furthermore, we suppose that $N^1$, $N^2$ are independent of $W^S$ and $Y$ and that the $\mathbb{F}$-intensities of $N^1$ and $N^2$ are positive deterministic functions $\Theta_1 : [0, T] \longrightarrow (0, +\infty)$ and $\Theta_2 : [0, T] \longrightarrow (0, +\infty)$, respectively. The coefficients $\mu : [0, T] \times \mathcal{E} \longrightarrow (0, +\infty)$ and $\sigma : [0, T] \times \mathcal{E} \longrightarrow (0, +\infty)$ are measurable functions which model the appreciation rate and the volatility of the stock, respectively, such that $\mu(t, i) > r$, for all $(t, i) \in [0, T] \times \mathcal{E}$ and

$$\int_0^T \left(\mu(t, Y_t) + \sigma^2(t, Y_t)\right) \mathrm{d}t < \infty \quad \mathbf{P}\text{-a.s.}. \tag{3.3}$$

Moreover, $K_1 : [0, T] \times \mathcal{E} \longrightarrow (0, +\infty)$ and $K_2 : [0, T] \times \mathcal{E} \longrightarrow (0, +\infty)$ are measurable functions such that $K_l(t, i) > 0$, $l = 1, 2$, and $K_2(t, i) < 1$, for every $(t, i) \in [0, T] \times \mathcal{E}$. From (3.1) and (3.2) it is clear that the pair $(S, Y)$ is an $(\mathbb{F}, \mathbf{P})$-Markov process. We observe that the stock price $S$ is described by a jump diffusion process where the appreciation rate and the volatility depend on a Markov chain representing the regimes of the economy. Taking a mixture of continuous and jump processes for the stock price dates back to Merton [77] and it can also be found in more recent papers, see e.g. Ceci and Gerardi [23] and Xiao and Zhao [100]. This financial market model is actually reasonable, since recent research provides strong empirical evidence of jumps in stock prices, see e.g. Jawadi et al. [68]. Moreover, the risky asset behavior could be also affected by long-term macroeconomic conditions that should be included in the framework and represented by another stochastic process. Therefore, the presence of an exogenous term affecting the stock price makes the model even more realistic. This stochastic factor may represent some environmental conditions, social circumstances, economic crisis or natural phenomena, that can have a considerable impact on financial returns. The economic effects of catastrophic events, climate changes and pandemics, as for instance the COVID-19, on the financial market are recently analyzed, see, e.g., Baek et al. [7], Just and Echaust [69], Tesselaar et al. [93], Wang et al. [99]. Here, we address this modeling issue by assuming that all these exogenous events are aggregated to create different regimes, as e.g. in Sotomayor and Cadenillas [92], Altay et al. [3], Cretarola and Figà-Talamanca [36].

**Remark 3.1.** *By the Doléans-Dade exponential formula, condition $K_2(t,i) < 1$ allows us to write*

$$S_t = s e^{L_t}, \quad t \in [0,T],$$

*where the logreturn process $L = \{L_t,\ t \in [0,T]\}$ is given by*

$$\mathrm{d}L_t = \big(\mu(t,Y_t) - \frac{1}{2}\sigma^2(t,Y_t)\big)\mathrm{d}t + \sigma(t,Y_t)\mathrm{d}W_t^S + \ln(1+K_1(t,Y_{t-}))\mathrm{d}N_t^1 + \ln(1-K_2(t,Y_{t-}))\mathrm{d}N_t^2,$$

*with $L_0 = 0$.*

**Proposition 3.1.** *If we assume that*

$$\int_0^T \big(K_1^2(t,Y_{t-})\Theta_1(t) + K_2^2(t,Y_{t-})\Theta_2(t)\big)\,\mathrm{d}t < \infty \quad \mathbf{P}\text{-}a.s., \tag{3.4}$$

*then the process $S$ is an $\mathbb{F}$-semimartingale with decomposition*

$$S_t = s + A_t^S + M_t^S,$$

*where $A^S = \{A_t^S,\ t \in [0,T]\}$ defined as*

$$A_t^S = \int_0^t S_{v-}\left(\mu(v,Y_{v-}) + K_1(v,Y_{v-})\Theta_1(v) + K_2(v,Y_{v-})\Theta_2(v)\right)\mathrm{d}v,$$

*is an $\mathbb{R}$-valued process with finite variation paths and $A_0^S = 0$, while $M^S = \{M_t^S,\ t \in [0,T]\}$ given by*

$$M_t^S = \int_0^t S_v \sigma(v, Y_v) \mathrm{d}W_v^S + \int_0^t S_{v-} K_1(v, Y_{v-}) \{\mathrm{d}N_v^1 - \Theta_1(v)\mathrm{d}v\} - \int_0^t S_{v-} K_2(v, Y_{v-}) \{\mathrm{d}N_v^2 - \Theta_2(v)\mathrm{d}v\}$$

*is an $\mathbb{F}$-local martingale with $M_0^S = 0$.*

*Proof.* Conditions (3.3) and (3.4) imply that the process $R^S = \{R_t^S,\ t \in [0,T]\}$ defined as

$$R_t^S = \int_0^t \left(\mu(v, Y_v)\mathrm{d}v + \sigma(v, Y_v)\mathrm{d}W_v^S + K_1(v, Y_{v-})\mathrm{d}N_v^1 - K_2(v, Y_{v-})\mathrm{d}N_v^2\right)$$

is an $\mathbb{F}$-semimartingale. Noting that

$$\mathrm{d}S_t = S_{t-}\mathrm{d}R_t^S,$$

we can conclude the proof. □

Now, we consider an individual to be insured and a stochastic model for the mortality of the equivalent age cohort of the population. We assume that the hazard rate (or force of mortality) is governed by a diffusion process, i.e. we describe the mortality intensity as a stochastic process $\Lambda = \{\lambda_t,\ t \in [0,T]\}$ that is given by the following SDE

$$\mathrm{d}\lambda_t = \zeta_1(t, \lambda_t)\lambda_t \mathrm{d}t + \zeta_2(t, \lambda_t)\lambda_t \mathrm{d}W_t^\Lambda, \quad \lambda_0 = \lambda > 0. \tag{3.5}$$

Here, $W^\Lambda = \{W_t^\Lambda,\ t \in [0,T]\}$ is an additional standard Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P}; \mathbb{F}^I)$.

Moreover, $\zeta_1 : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\zeta_2 : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ are two measurable functions such that a unique positive strong solution to (3.5) exists and the following conditions hold

$$\mathbb{E}\left[\int_0^T |\zeta_1(t, \lambda_t)\lambda_t| \mathrm{d}t + \int_0^T \zeta_2(t, \lambda_t)^2 \lambda_t^2 \mathrm{d}t\right] < \infty, \tag{3.6}$$

$$\sup_{t \in [0,T]} \mathbb{E}\left[\lambda_t^2\right] < \infty. \tag{3.7}$$

These conditions are satisfied if, for instance, the coefficients of the SDE (3.5) fulfill the classical Lipschitz and sublinear growth conditions, see e.g. Gihman and Skorohod [54]. We observe that, the mortality rate of the insured is generally different from that of its age cohort. However, to keep the framework tractable we consider individuals subjected to the same stochastic hazard rate, as e.g. in Ludkovski and Young [76]. We point out that we are not the first to consider stochastic mortality rates, see e.g. Milevsky and Promislow [78], Dahl [38], Dahl and Møller [39], Biffis [15], Ludkovski and Young [76]. Indeed, empirical evidence suggests that wars, medical breakthroughs, developments in healthcare and improved lifestyles combine to affect human mortality in a fluctuating and unpredictable manner. The uncertainty given by minuscule and continuous movements of the mortality intensity is usually represented by a Brownian motion, see [20] for an overview. As a consequence, it seems reasonable to require that in our setting the exogenous stochastic factor, representing long-term environmental changes, does not affect the mortality intensity; therefore the insurance market remains independent of the financial market.

Let $\tau$ be a non negative random variable on $(\Omega, \mathcal{F}, \mathbf{P})$ which represents the remaining lifetime of the given individual of the reference population with mortality rate $\Lambda$. Denote by $D = \{D_t,\ t \in [0,T]\}$ the death indicator process associated to $\tau$ by setting $D_t := \mathbf{1}_{\{\tau \le t\}}$, for every $t \in [0,T]$. We assume that $D$ is an $\mathbb{F}^I$-adapted process independent of $\Lambda$.

## 3.3 The indifference pricing problem formulation

Now, we assume that the insurance company writes a unit-linked life insurance policy, which is a long term insurance contract whose payoff depends on the insured remaining lifetime and on the underlying stock. In particular, we consider a pure endowment contract with maturity of $T$ years, which yields a fixed sum of money if the policyholder is still alive at that time. Then, the associated payoff is given by the random variable

$$G_T := \tilde{K}\mathbf{1}_{\{\tau > T\}} = \tilde{K}(1 - D_T), \tag{3.8}$$

where $\tilde{K}$ is a positive constant. The goal is to evaluate the pure endowment policy with payoff given by (3.8) in the Markov-modulated model outlined above. Since the financial market consists of two primary securities and several sources of random shocks due to mortality events and structural changes in economic conditions, it turns out to be incomplete. Therefore, we apply the indifference pricing approach assuming that the insurance company preferences towards the risk are described by an exponential utility function of the form

$$u(x) = -e^{-\gamma x}, \quad x \in \mathbb{R},$$

where $\gamma$ is a positive parameter which measures the absolute risk aversion. In the underlying financial market, the insurance company starts out with an initial wealth $x_0$, and then proceeds to trade dynamically among the locally risk-free asset and the risky asset, following a self-financing strategy. Let $\Pi = \{\Pi_t,\ t \in [0,T]\}$ be the total amount of wealth invested in the stock, with the remainder of wealth in the money market account. Thereby, $W_t - \Pi_t$ will be the capital invested in the risk-free asset at time $t$. The insurance company is also allowed to short-sell and to borrow/lend any infinitesimal amount, so that $\Pi_t \in \mathbb{R}$, for each $t \in [0,T]$. Precisely, given an initial wealth $x_0 \geq 0$, the insurance company wealth process $\{X_t^\Pi,\ t \in [0,T]\}$ associated to a given strategy $\Pi$ evolves over time as

$$\begin{aligned} \mathrm{d}X_t^\Pi &= \Pi_t \frac{\mathrm{d}S_t}{S_{t-}} + (X_t^\Pi - \Pi_t)\frac{\mathrm{d}S_t^0}{S_t^0} \\ &= \left(rX_t^\Pi + \Pi_t\left(\mu(t,Y_t) - r\right)\right)\mathrm{d}t + \Pi_t\sigma(t,Y_t)\mathrm{d}W_t^S + \Pi_t\Big(K_1(t,Y_{t-})\mathrm{d}N_t^1 - K_2(t,Y_{t-})\mathrm{d}N_t^2\Big), \end{aligned} \tag{3.9}$$

with $X_0^\Pi = x_0 \geq 0$.

**Remark 3.2.** *It can be checked that the solution to the SDE* (3.9) *is given by*

$$\begin{aligned} X_t^\Pi = X_0^\Pi e^{rt} &+ \int_0^t e^{r(t-s)}\Pi_s(\mu(s,Y_s) - r)\mathrm{d}s + \int_0^t e^{r(t-s)}\Pi_s\sigma(s,Y_s)\mathrm{d}W_s^S \\ &+ \int_0^t e^{r(t-s)}\Pi_s\Big(K_1(s,Y_{s-})\mathrm{d}N_s^1 - K_2(s,Y_{s-})\mathrm{d}N_s^2\Big), \end{aligned} \tag{3.10}$$

*with* $X_0^\Pi = x_0 \geq 0$.

In order to characterize the indifference price of the pure endowment, we introduce two optimal investment problems: one related to an insurance company that does not issue the mortality-contingent claim and the other one to a company who sells the policy. We start by defining the class of admissible strategies.

**Definition 3.1.** *An* admissible *strategy is a self-financing portfolio identified by an* $\mathbb{R}$*-valued* $\mathbb{G}$*-predictable process* $\Pi = \{\Pi_t,\ t \in [0,T]\}$ *such that*

$$\mathbb{E}\left[\int_0^T |\Pi_t|(\mu(t,Y_t) - r)\mathrm{d}t\right] < \infty,$$

$$\mathbb{E}\left[\int_0^T \Pi_t^2\sigma^2(t,Y_t)\mathrm{d}t\right] < \infty,$$

$$\mathbb{E}\left[\int_0^T |\Pi_t|\Big(K_1(t,Y_{t-})\Theta_1(t)+K_2(t,Y_{t-})\Theta_2(t)\Big)\mathrm{d}t\right] < \infty. \tag{3.11}$$

*We denote by $\mathcal{A}$ the set of $\mathbb{G}$-admissible strategies. Whenever the controls are restricted to the time interval $[t,T]$, we will use the notation $\mathcal{A}_t$.*

Now, we assume that the following assumptions are in force throughout the chapter.

**Assumption 3.1.**

*(i) There exist three positive constants $M_1$, $M_2$ and $M_3$ such that*

$$\Theta_1(t) \leq M_1, \quad \Theta_2(t) \leq M_2, \quad K_1(t,i) \leq M_3, \quad \textit{for every } (t,i) \in [0,T]\times\mathcal{E}.$$

*(ii) There is a constant $M_4 > 0$ such that $\frac{\mu(t,i)-r}{\sigma(t,i)} \leq M_4$, for every $(t,i) \in [0,T]\times\mathcal{E}$.*

In this way, we consider securities not too risky, in every market regime. Indeed, the random jumps in the stock have restrictions in terms of intensities and coefficients, just as the Sharpe ratio is bounded, in order to avoid extreme peaks in price dynamics.

In particular, Assumption 3.1(i) provides a sufficient condition for a strategy $\Pi$ to be admissible, as shown in the next result.

**Proposition 3.2.** *Let $\Pi = \{\Pi_t,\ t \in [0,T]\}$ be a $\mathbb{G}$-predictable strategy with values in $\mathbb{R}$. Assume there exists a square-integrable function $\eta : [0,T]\times\mathcal{E} \to (0,+\infty)$ such that*

$$|\Pi_t| \leq \eta(t,Y_t), \quad t \in [0,T], \quad \mathbf{P}-a.s. \tag{3.12}$$

*and*

$$\int_0^T \eta(s,i)\left((\mu(s,i)-r)+\eta(s,i)\sigma^2(s,i)\right)\mathrm{d}s < \infty, \quad \forall i \in \mathcal{E}. \tag{3.13}$$

*Then, $\Pi$ is an admissible strategy, i.e. $\Pi \in \mathcal{A}$.*

*Proof.* We note that by (3.13), we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T |\Pi_s|\left((\mu(s,Y_s)-r)+\Pi_s\sigma^2(s,Y_s)\right)\mathrm{d}s\right] \\
&\quad\leq \mathbb{E}\left[\int_0^T \eta(s,Y_s)\left((\mu(s,Y_s)-r)+\eta(s,Y_s)\sigma^2(s,Y_s)\right)\mathrm{d}s\right] \\
&\quad\leq \max_{i=1,\dots,K}\int_0^T \eta(s,i)\left((\mu(s,i)-r)+\eta(s,i)\sigma^2(s,i)\right)\mathrm{d}s < \infty.
\end{aligned}$$

Finally, in view of Assumption 3.1(i), condition (3.11) is satisfied and this concludes the proof. □

We consider the case where the insurance company simply invests its wealth in the financial market, without writing the insurance derivative. Then, the goal is the following.

**Problem 3.1.** *To maximize the expected utility of its terminal wealth, i.e. to solve*

$$\sup_{\Pi \in \mathcal{A}} \mathbb{E}\Big[ -e^{-\gamma X_T^\Pi} \Big].$$

Let $(t, x, i) \in [0, T] \times \mathbb{R} \times \mathcal{E}$. In a dynamic framework, we define the corresponding value function $\bar{V}$ by

$$\bar{V}(t, x, i) := \sup_{\Pi \in \mathcal{A}_t} \mathbb{E}_{t,x,i}\Big[ -e^{-\gamma X_T^\Pi(t,x)} \Big], \tag{3.14}$$

where $\mathbb{E}_{t,x,i}$ denotes the conditional expectation given $X_t^\Pi = x$ and $Y_t = i$, and $\{X_s^\Pi(t, x),\ s \in [t, T]\}$ stands for the solution to equation (3.9) with initial condition $X_t^\Pi = x$. Note that, since the coefficients $\mu$, $\sigma$, $K_1$ and $K_2$ only depend on $t$ and $i$, it is possible to absorb the stock price in the wealth and therefore to remove the variable corresponding to $S$.

Now, we suppose that the insurance company invests its wealth in the market, also issuing a pure endowment contract with payoff given in (3.8). In this case, the goal of the insurance company is the following.

**Problem 3.2.** *To maximize the expected utility of its terminal wealth, i.e. to solve*

$$\sup_{\Pi \in \mathcal{A}} \mathbb{E}\Big[ -e^{-\gamma (X_T^\Pi - G_T)} \Big],$$

*where $G_T$ is defined in* (3.8)*.*

Let $(t, x, \lambda, i) \in [0, T] \times \mathbb{R} \times (0, +\infty) \times \mathcal{E}$. We define the corresponding value function $V$ as

$$V(t, x, \lambda, i) := \sup_{\Pi \in \mathcal{A}_t} \mathbb{E}_{t,x,\lambda,i}\Big[ -e^{-\gamma (X_T^\Pi(t,x) - G_T)} \Big], \tag{3.15}$$

where $\mathbb{E}_{t,x,\lambda,i}$ denotes the conditional expectation given $X_t^\Pi = x$, $\lambda_t = \lambda$ and $Y_t = i$ and we implicitly condition on $G_t = \tilde{K}$.

**Remark 3.3.** *We note that the control $\Pi = 0$ is admissible and such that*

$$\mathbb{E}_{t,x,i}\left[ e^{-\gamma X_T^0(t,x)} \right] < \infty,$$

*for each $(t, x, i) \in [0, T] \times \mathbb{R} \times \mathcal{E}$.*

$$\mathbb{E}_{t,x,\lambda,i}\left[ e^{-\gamma (X_T^0(t,x) - G_T)} \right] < \infty,$$

*for each* $(t,x,\lambda,i)\in[0,T]\times\mathbb{R}\times(0,+\infty)\times\mathcal{E}$*. This implies that*

$$\operatorname*{ess\,sup}_{\Pi\in\mathcal{A}_t}\mathbb{E}\left[-e^{-\gamma X_T^\Pi}\right]>-\infty,\quad \operatorname*{ess\,sup}_{\Pi\in\mathcal{A}_t}\mathbb{E}\left[-e^{-\gamma(X_T^\Pi-G_T)}\right]>-\infty,\quad \mathbf{P}-a.s.,\ t\in[0,T],$$

*and as a consequence that*

$$\sup_{\Pi\in\mathcal{A}}\mathbb{E}\left[-e^{-\gamma X_T^\Pi}\right]>-\infty,\quad \sup_{\Pi\in\mathcal{A}}\mathbb{E}\left[-e^{-\gamma(X_T^\Pi-G_T)}\right]>-\infty.$$

## 3.4 The optimal investment problems

In this section, we solve the two optimization problems introduced in Section 3.3. Specifically, applying the classical stochastic control approach based on the HJB equation, we characterize the optimal investment strategies and provide verification results for the value functions $\bar{V}$ and $V$ given in (3.14) and (3.15), respectively.

### 3.4.1 The pure investment problem

Here, we consider the case where the insurance company simply invests in the underlying financial market, without underwriting any claims. Thus the corresponding value function $\bar{V}$ is given by (3.14).

Firstly, we see that the financial model has a Markovian structure, i.e. the couple $(X^\Pi,Y)$ is a $(\mathbb{G},\mathbf{P})$ Markov process. After introducing a suitable class of functions, we compute its infinitesimal generator. Let $\bar{\mathcal{L}}_i^\Pi$ denote the Markov generator of $(X^\Pi,Y)$, associated with a constant control $\Pi\in\mathbb{R}$, and let $\mathcal{D}(\bar{\mathcal{L}}_i^\Pi)$ denote the domain of this generator, for each $i\in\mathcal{E}$.

**Definition 3.2.** *The set* $\mathcal{D}(\bar{\mathcal{L}}_i^\Pi)$ *denotes the class of functions* $f(\cdot,\cdot,i)\in C^1([0,T])\times C^2(\mathbb{R})$*, for each* $i\in\mathcal{E}$*, such that for every constant* $\Pi\in\mathbb{R}$*, we have*

$$\mathbb{E}\Big[\int_0^T\Big(\sigma(v,Y_v)\Pi\frac{\partial f}{\partial x}(v,X_v^\Pi,Y_v)\Big)^2\mathrm{d}v\Big]<\infty, \tag{3.16}$$

*and*

$$\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}\left|f\big(v,X_v^\Pi,Y_{v-}+h(Y_{v-},z)\big)-f(v,X_v^\Pi,Y_{v-})\right|m(\mathrm{d}z)\mathrm{d}v\right]<\infty, \tag{3.17}$$

$$\mathbb{E}\left[\int_0^T\left|f\big(v,X_{v-}^\Pi+\Pi K_1(v,Y_{v-}),Y_{v-}\big)-f(v,X_{v-}^\Pi,Y_{v-})\right|\Theta_1(v)\mathrm{d}v\right]<\infty,$$

$$\mathbb{E}\left[\int_0^T\left|f\big(v,X_{v-}^\Pi-\Pi K_2(v,Y_{v-}),Y_{v-}\big)-f(v,X_{v-}^\Pi,Y_{v-})\right|\Theta_2(v)\mathrm{d}v\right]<\infty.$$

**Lemma 3.1.** *Let $f \in \mathcal{D}(\bar{\mathcal{L}}_i^{\Pi})$. For any constant strategy $\Pi \in \mathbb{R}$, the stochastic process $(X^{\Pi}, Y)$ is a Markov couple with infinitesimal generator given by*

$$\begin{aligned}
&\bar{\mathcal{L}}_i^{\Pi} f(t,x,i) \\
&\quad= \frac{\partial f}{\partial t}(t,x,i) + \big[rw + (\mu(t,i) - r)\Pi\big]\frac{\partial f}{\partial x}(t,x,i) + \frac{1}{2}\sigma^2(t,i)\Pi^2\frac{\partial^2 f}{\partial x^2}(t,x,i) + \sum_{j\in\mathcal{E}} q_{ij} f(t,x,j) \\
&\quad+ \Theta_1(t)\big\{\bar{V}(t, x + \Pi K_1(t,i), i) - \bar{V}(t,x,i)\big\} + \Theta_2(t)\big\{\bar{V}(t, x - \Pi K_2(t,i), i) - \bar{V}(t,x,i)\big\}.
\end{aligned} \tag{3.18}$$

*Proof.* In view of (3.9) and (3.1), by applying Itô's formula to the stochastic process $f(t, X_t^{\Pi}, Y_t)$, we have

$$f(t, X_t^{\Pi}, Y_t) = f(0, X_0^{\Pi}, Y_0) + \int_0^t \bar{\mathcal{L}}^{\Pi} f(u, X_u^{\Pi}, Y_u)\mathrm{d}u + m_t,$$

where

$$\begin{aligned}
m_t = m_0 &+ \int_0^t \Pi_v \sigma(v, X_v)\frac{\partial f}{\partial x}(v, X_v^{\Pi}, Y_v)\mathrm{d}W_v^S + \\
&+ \int_0^t \int_{\mathbb{R}} \big\{f\big(v, X_v^{\Pi}, Y_{v-} + h(Y_{v-}, z)\big) - f(v, X_v^{\Pi}, Y_{v-})\big\}\hat{\mathcal{P}}(\mathrm{d}v, \mathrm{d}z) \\
&+ \int_0^t \big\{f\big(v, X_{v-}^{\Pi} + \Pi_v K_1(v, Y_{v-}), Y_{v-})\big) - f(v, X_{v-}^{\Pi}, Y_{v-})\big\}\{\mathrm{d}N_v^1 - \Theta_1(v)\mathrm{d}v\} \\
&+ \int_0^t \big\{f\big(v, X_{v-}^{\Pi} - \Pi_v K_2(v, Y_{v-}), Y_{v-})\big) - f(v, X_{v-}^{\Pi}, Y_{v-})\big\}\{\mathrm{d}N_v^2 - \Theta_2(v)\mathrm{d}v\}.
\end{aligned} \tag{3.19}$$

We only need to prove that the process $m = \{m_t,\ t \in [0,T]\}$ is a $(\mathbb{G}, \mathbf{P})$-martingale. According to the Itô integral theory, by (3.16), the first integral in (3.19) is well-defined and turns out to be a $(\mathbb{G}, \mathbf{P})$-martingale. Furthermore, due to (3.17), we have that also the jump terms in (3.19) are $(\mathbb{G}, \mathbf{P})$-martingales, (see e.g. [40, Theorem 26.12(2)] and [19, Lemma L3, Ch.II] for further details about the martingale property related to a Poisson random measure and a Poisson process, respectively). □

Next, let us consider the HJB equation with final condition that the value function $\bar{V}$ is expected to solve, if sufficiently smooth:

$$\begin{cases} \sup_{\Pi\in\mathbb{R}} \bar{\mathcal{L}}_i^{\Pi}\bar{V}(t,x,i) = 0, & \forall (t,x,i) \in [0,T) \times \mathbb{R} \times \mathcal{E}, \\ \bar{V}(T,x,i) = -e^{-\gamma x}, & \forall (x,i) \in \mathbb{R} \times \mathcal{E}, \end{cases} \tag{3.20}$$

where $\bar{\mathcal{L}}_i^{\Pi}$ denotes the Markov generator of $(X^{\Pi}, Y)$ associated with a constant control $\Pi \in \mathbb{R}$, given by (3.18).

**Remark 3.4.** *Since the pair* $(X^{\Pi}, Y)$ *is a Markov process, any Markovian control is of the form* $\Pi_t = \Pi(t, X_t^{\Pi}, Y_t)$. *The generator* $\bar{\mathcal{L}}_i^{\Pi} f(t,x,i)$ *associated to a general Markovian strategy can be easily obtained by replacing* $\Pi$ *with* $\Pi(t,x,i)$ *in* (3.18).

Now, we conjecture a solution to equation (3.20). Due to the exponential form of the boundary condition, it is natural to guess that the solution of the above HJB equation also has an exponential structure. Therefore, we consider the ansatz $\bar{V}(t,x,i) = -e^{-\gamma x e^{r(T-t)}}\varphi(t,i)$, with $(t,x,i) \in [0,T] \times \mathbb{R} \times \mathcal{E}$, for a suitable function $\varphi$, which is motivated by the following result.

**Proposition 3.3.** *Assume that there exists a unique function* $\varphi(\cdot, i)$, *for each* $i \in \mathcal{E}$, *solution to the following Cauchy problem:*

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}(t,i) = H(t, \varphi(t,i)), \qquad t \in [0,T), \\ \varphi(T,i) = 1, \end{cases} \tag{3.21}$$

*where*

$$H(t, \varphi(t,i)) = -\sum_{j \in \mathcal{E}} \varphi(t,j) q_{ij} - \varphi(t,i) \inf_{\Pi \in \mathbb{R}} \bar{\Psi}^{\Pi}(t,i), \tag{3.22}$$

*with the function* $\bar{\Psi}^{\Pi} : [0,T] \times \mathcal{E} \to \mathbb{R}$ *defined by*

$$\begin{aligned} \bar{\Psi}^{\Pi}(t,i) = &-\gamma e^{r(T-t)}(\mu(t,i) - r)\Pi + \frac{1}{2}\gamma^2 e^{2r(T-t)}\sigma^2(t,i)\Pi^2 + \Theta_1(t)\big(e^{-\gamma \Pi K_1(t,i) e^{r(T-t)}} - 1\big) \\ &+ \Theta_2(t)\left(e^{\gamma \Pi K_2(t,i) e^{r(T-t)}} - 1\right). \end{aligned} \tag{3.23}$$

*Then, the function*

$$\bar{V}(t,x,i) = -e^{-\gamma x e^{r(T-t)}}\varphi(t,i), \tag{3.24}$$

*solves the HJB problem given in* (3.20).

*Proof.* From the expression (3.24), we can easily verify that the original HJB problem given in (3.20) reads as follows

$$\begin{aligned} &\frac{\partial \varphi}{\partial t}(t,i) + \sum_{j \in \mathcal{E}} \varphi(t,j) q_{ij} + \inf_{\Pi \in \mathbb{R}} \Big\{ -\gamma e^{r(T-t)} \varphi(t,i)(\mu(t,i) - r)\Pi + \frac{1}{2}\gamma^2 e^{2r(T-t)} \varphi(t,i) \sigma^2(t,i) \Pi^2 \\ &\quad + \varphi(t,i)\Theta_1(t)\big(e^{-\gamma \Pi K_1(t,i) e^{r(T-t)}} - 1\big) + \varphi(t,i)\Theta_2(t)\big(e^{\gamma \Pi K_2(t,i) e^{r(T-t)}} - 1\big)\Big\} = 0, \quad t \in [0,T), \end{aligned} \tag{3.25}$$

with final condition $\varphi(T,i)=1$, for all $i\in\mathcal{E}$. Thus, if we define the function $\bar{\Psi}^{\Pi}$ by means of expression (3.23), equation (3.25) can be written as

$$\frac{\partial\varphi}{\partial t}(t,i)+\sum_{j\in\mathcal{E}}\varphi(t,j)q_{ij}+\varphi(t,i)\inf_{\Pi\in\mathbb{R}}\bar{\Psi}^{\Pi}(t,i)=0.$$

Moreover, the terminal condition in (3.20) implies that $\varphi(T,i)=1$, for each $i\in\mathcal{E}$. Hence, we find out the problem (3.21). □

The previous result suggests to focus on the minimization of the function (3.23), that is the aim of the next subsection.

**Optimal investment strategy without the insurance derivative**

In order to characterize the optimal portfolio for a company that does not write the life insurance derivative, we study the following minimization problem

$$\inf_{\Pi\in\mathbb{R}}\bar{\Psi}^{\Pi}(t,i), \tag{3.26}$$

where the function $\bar{\Psi}^{\Pi}$ is introduced in (3.23).

**Proposition 3.4.** *The following equation*

$$\begin{aligned}&\sigma^2(t,i)\gamma e^{r(T-t)}\Pi-(\mu(t,i)-r)\\&\quad=K_1(t,i)\Theta_1(t)e^{-\alpha\Pi K_1(t,i)e^{r(T-t)}}-K_2(t,i)\Theta_2(t)e^{\gamma\Pi K_2(t,i)e^{r(T-t)}}.\end{aligned} \tag{3.27}$$

*admits at least a solution $\widehat{\Pi}(t,i)$ in $\mathbb{R}$ for any $(t,i)\in[0,T]\times\mathcal{E}$ and the minimization problem* (3.26) *has a unique solution $\Pi^*(t,i)=\widehat{\Pi}(t,i)$, for all $(t,i)\in[0,T]\times\mathcal{E}$.*

*Proof.* Firstly, we observe that $\bar{\Psi}^{\Pi}(t,i)$ is continuous with respect to $\Pi\in\mathbb{R}$, for every $(t,i)\in[0,T]\times\mathcal{E}$ and has continuous first and second order derivatives with respect to $\Pi\in\mathbb{R}$, which are respectively given by

$$\begin{aligned}\frac{\partial\bar{\Psi}^{\Pi}}{\partial\Pi}(t,i)&=-\alpha e^{r(T-t)}(\mu(t,i)-r)+\sigma^2(t,i)\gamma^2e^{2r(T-t)}\Pi-\gamma e^{r(T-t)}K_1(t,i)\Theta_1(t)e^{-\gamma\Pi K_1(t,i)e^{r(T-t)}}\\&\quad+\alpha e^{r(T-t)}K_2(t,i)\Theta_2(t)e^{\gamma\Pi K_2(t,i)e^{r(T-t)}},\\\frac{\partial^2\bar{\Psi}^{\Pi}}{\partial\Pi^2}(t,i)&=\gamma^2e^{2r(T-t)}\sigma^2(t,i)+\gamma^2e^{2r(T-t)}K_1^2(t,i)\Theta_1(t)e^{-\gamma\Pi K_1(t,i)e^{r(T-t)}}\\&\quad+\gamma^2e^{2r(T-t)}K_2^2(t,i)\Theta_2(t)e^{\gamma\Pi K_2(t,i)e^{r(T-t)}}.\end{aligned}$$

Note that these derivatives are well defined and the second order derivative is strictly positive, i.e. $\dfrac{\partial^2 \bar{\Psi}^{\Pi}}{\partial \Pi^2}(t,i) > 0$, for every $(t,i) \in [0,T] \times \mathcal{E}$; therefore, the function $\bar{\Psi}^{\Pi}(t,i)$ is strictly convex in $\Pi \in \mathbb{R}$. Moreover, it is easy to check that, for any $(t,i) \in [0,T] \times \mathcal{E}$, we have

$$\lim_{\Pi \longrightarrow +\infty} \frac{\partial \bar{\Psi}^{\Pi}}{\partial \Pi}(t,i) \longrightarrow +\infty,$$

while

$$\lim_{\Pi \longrightarrow -\infty} \frac{\partial \bar{\Psi}^{\Pi}}{\partial \Pi}(t,i) \longrightarrow -\infty.$$

As a consequence, being $\dfrac{\partial \bar{\Psi}^{\Pi}}{\partial \Pi}(t,i)$ a continuous function in $\Pi \in \mathbb{R}$, there exists $\widehat{\Pi}(t,i) \in \mathbb{R}$ such that $\dfrac{\partial \bar{\Psi}^{\Pi}}{\partial \Pi}(t,i) = 0$, for every $(t,i) \in [0,T] \times \mathcal{E}$, that is, (3.27) is satisfied. Since the function $\bar{\Psi}^{\Pi}(t,i)$ is strictly convex, the stationary point $\widehat{\Pi}(t,i) \in \mathbb{R}$ is unique and provides the unique minimizer $\Pi^*(t,i) = \widehat{\Pi}(t,i)$ on $\mathbb{R}$. □

**Remark 3.5.** *We point out that the optimal portfolio strategy* $\Pi^*$ *evolves over time and changes according to the different economic regimes. This is due to the fact that* $\Pi^*$ *solves equation* (3.27) *and thus it depends on time and on the Markov chain. Moreover, observing* (3.27)*, we also note that* $\Pi^*$ *does not depend on wealth, as usually happens when the investor's preferences are described by a utility function of exponential type.*

In the next result, we pick out the range in which the optimal investment strategy varies, even though we do not know it explicitly.

**Proposition 3.5** (Properties of $\Pi^*$)**.** *The following condition is satisfied*

$$\min\left\{0, \frac{\ln\left(\frac{\mu(t,i)-r}{M_2}\right)}{\gamma e^{r(T-t)}}\right\} \leq \Pi^*(t,i) \leq \frac{\mu(t,i) - r + M_3 M_1}{\sigma^2(t,i)\gamma e^{r(T-t)}},$$

*for all* $(t,i) \in [0,T] \times \mathcal{E}$*, where* $M_1, M_2, M_3 > 0$ *are the constants limiting the functions* $\Theta_1, \Theta_2, K_1$*, respectively introduced in Assumption 3.1.*

*Proof.* By Proposition 3.4 (we omit the dependence in $\Pi^*$ on $(t,i)$), we get the upper limit and the lower limit for $\Pi^*$. If $\Pi^*$ is non-negative, we have

$$\begin{aligned}
0 &= \sigma^2(t,i)\gamma e^{r(T-t)}\Pi^* - (\mu(t,i) - r) - K_1(t,i)\Theta_1(t)e^{-\gamma\Pi^* K_1(t,i)e^{r(T-t)}} \\
&\quad + K_2(t,i)\Theta_2(t)e^{\gamma\Pi^* K_2(t,i)e^{r(T-t)}} \\
&> \sigma^2(t,i)\gamma e^{r(T-t)}\Pi^* - (\mu(t,i) - r) - K_1(t,i)\Theta_1(t)e^{-\gamma\Pi^* K_1(t,i)e^{r(T-t)}} \\
&\geq \sigma^2(t,i)\gamma e^{r(T-t)}\Pi^* - (\mu(t,i) - r) - M_3 M_1 e^{-\gamma\Pi^* K_1(t,i)e^{r(T-t)}} \\
&\geq \sigma^2(t,i)\gamma e^{r(T-t)}\Pi^* - (\mu(t,i) - r) - M_3 M_1,
\end{aligned}$$

which implies

$$\Pi^*(t,i) \leq \frac{\mu(t,i) - r + M_3 M_1}{\sigma^2(t,i)\gamma e^{r(T-t)}},$$

for all $(t,i) \in [0,T] \times \mathcal{E}$. Otherwise, if $\Pi^*$ is non-positive, we get

$$\begin{aligned}
0 &= \sigma^2(t,i)\gamma e^{r(T-t)}\Pi^* - (\mu(t,i) - r) - K_1(t,i)\Theta_1(t)e^{-\gamma\Pi^* K_1(t,i)e^{r(T-t)}} \\
&\quad + K_2(t,i)\Theta_2(t)e^{\gamma\Pi^* K_2(t,i)e^{r(T-t)}} \\
&< -(\mu(t,i) - r) + K_2(t,i)\Theta_2(t)e^{\gamma\Pi^* K_2(t,i)e^{r(T-t)}} \\
&\leq -(\mu(t,i) - r) - M_2 e^{\gamma\Pi^* e^{r(T-t)}},
\end{aligned}$$

that leads to

$$\Pi^*(t,i) \geq \frac{\ln\left(\frac{\mu(t,i)-r}{M_2}\right)}{\gamma e^{r(T-t)}},$$

for all $(t,i) \in [0,T] \times \mathcal{E}$. □

**Verification Theorem**

Now, we are ready to state a verification result which ensures that the value function $\bar{V}$ is the unique solution of the HJB problem (3.20).

**Theorem 3.1** (Verification Theorem)**.** *Suppose that the Cauchy problem* (3.21) *admits a classical solution* $\varphi(\cdot,i) \in C^1\big((0,T[\big) \cap C\big([0,T]\big)$*, for each* $i \in \mathcal{E}$*. Then, the function* $\bar{V}$ : $[0,T] \times \mathbb{R} \times \mathcal{E} \longrightarrow \mathbb{R}$ *defined by*

$$\bar{V}(t,x,i) = -e^{-\gamma x e^{r(T-t)}}\varphi(t,i)$$

*is the value function in* (3.14)*. Consequently, the strategy* $\Pi^*_t = \Pi^*(t,Y_t)$ *described in Proposition 3.4 is an optimal control.*

*Proof.* The proof uses similar arguments as in that of Theorem 3.2 below for the problem with the insurance derivative. Note that Problem 3.1 corresponds to a special case of Problem 3.2, choosing $G_T = 0$. Nevertheless, for the sake of clarity we trace the fundamental steps of the proof. By Proposition 3.3, the function $\bar{V}(t,x,i)$ defined in equation (3.24) solves the HJB problem (3.20). Hence, for any $(t,x,i) \in [0,T] \times \mathbb{R} \times \mathcal{E}$, we have

$$\bar{\mathcal{L}}_i^{\Pi} \bar{V}(s, X_s^{\Pi}(t,x), Y_s(t,i)) \leq 0, \quad \forall s \in [t,T],\ \Pi \in \mathcal{A}_t, \tag{3.28}$$

where we recall that $\{X_s^{\Pi}(t,x),\ s \in [t,T]\}$ and $\{Y_s(t,i),\ s \in [t,T]\}$ denote the solutions to equations (3.9) and (3.1) at time $s \in [t,T]$, starting from $(t,x) \in [0,T] \times \mathbb{R}$ and $(t,i) \in [0,T] \times \mathcal{E}$, respectively. Clearly, $\bar{V}(\cdot,\cdot,i) \in C^{1,2}([0,T] \times \mathbb{R})$, for each $i \in \mathcal{E}$. In view of (3.9), by applying Itô's formula, we have

$$\bar{V}(T, X_T^{\Pi}(t,x), Y_T(t,i)) = \bar{V}(t,x,i) + \int_t^T \bar{\mathcal{L}}_i^{\Pi} \bar{V}(v, X_v^{\Pi}(t,x), Y_v(t,i)) \mathrm{d}v + M_T, \tag{3.29}$$

where $M = \{M_r,\ r \in [t,T]\}$ is the stochastic process given by

$$\begin{aligned} M_r = &\int_t^r \Pi_v \sigma(v, Y_v) \frac{\partial \bar{V}}{\partial x}(v, X_v^{\Pi}, Y_v) \mathrm{d}W_v^S \\ &+ \int_t^r \int_{\mathbb{R}} \left\{ \bar{V}\big(v, X_v^{\Pi}, Y_{v-} + h(Y_{v-}, z)\big) - \bar{V}(v, X_v^{\Pi}, Y_{v-}) \right\} \hat{\mathcal{P}}(\mathrm{d}v, \mathrm{d}z) \\ &+ \int_t^r \left\{ \bar{V}\big(v, X_{v-}^{\Pi} + \Pi_v K_1(v, Y_{v-}), Y_{v-}\big) - \bar{V}(v, X_{v-}^{\Pi}, Y_{v-}) \right\} \{\mathrm{d}N_v^1 - \Theta_1(v)\mathrm{d}v\} \\ &+ \int_t^r \left\{ \bar{V}\big(v, X_{v-}^{\Pi} - \Pi_v K_2(v, Y_{v-}), Y_{v-}\big) - \bar{V}(v, X_{v-}^{\Pi}, Y_{v-}) \right\} \{\mathrm{d}N_v^2 - \Theta_2(v)\mathrm{d}v\}. \end{aligned}$$

In order to prove that $M$ is a $(\mathbb{G}, \mathbf{P})$-local martingale, we use a localization argument, taking

$$\tau_n := \inf\{s \in [t,T] \mid X_s^{\Pi} < -n\}, \quad n \in \mathbb{N},$$

which defines a non-decreasing sequence of stopping times $\{\tau_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \longrightarrow +\infty} \tau_n = +\infty$.

Therefore, taking the conditional expectation with respect to $X_t^{\Pi} = x$ and $Y_t = i$ on both sides of (3.29), with $T$ replaced by $T \wedge \tau_n$, by (3.28) we obtain that

$$\mathbb{E}_{t,x,i}\big[\bar{V}(T \wedge \tau_n, X_{T \wedge \tau_n}^{\Pi}(t,x), Y_{T \wedge \tau_n}(t,i))\big] \leq \bar{V}(t,x,i),$$

for every $\Pi \in \mathcal{A}_t$, $t \in [\![0, T \wedge \tau_n]\!]$, $n \in \mathbb{N}$. Now, we note that

$$\mathbb{E}\big[\big(\bar{V}(T \wedge \tau_n, X_{T \wedge \tau_n}^{\Pi}(t,x), Y_{T \wedge \tau_n}(t,i))\big)^2\big] = \mathbb{E}\big[e^{-2\gamma X_{T \wedge \tau_n}^{\Pi} e^{r(T \wedge \tau_n - t)}} \varphi(T \wedge \tau_n, i)^2\big] < \infty.$$

Consequently, $\{\bar{V}(T \wedge \tau_n, X^{\Pi}_{T\wedge\tau_n}(t,x), Y_{T\wedge\tau_n}(t,i))\}_{n\in\mathbb{N}}$ is a family of uniformly integrable random variables. Hence, it converges almost surely. Since $\{\tau_n\}_{n\in\mathbb{N}}$ is a bounded and non-decreasing sequence of random times and $\mathbf{P}(|X^{\Pi}_t| < +\infty) = 1$, see (3.10), we can apply the dominated convergence theorem and, taking the limit for $n \longrightarrow +\infty$, and we get

$$\begin{aligned}\mathbb{E}_{t,x,i}\big[\bar{V}(T, X^{\Pi}_T(t,x), Y_T(t,i))\big] &= \lim_{n\longrightarrow+\infty} \mathbb{E}_{t,x,i}\big[\bar{V}(T\wedge\tau_n, X^{\Pi}_{T\wedge\tau_n}(t,x), Y_{T\wedge\tau_n}(t,i))\big]\\ &\leq \bar{V}(t,x,i),\end{aligned}$$

for every $\Pi \in \mathcal{A}_t$, $t \in [0,T]$. As a byproduct, since $\Pi^*(t,i)$ given in Proposition 3.4 realizes the infimum in (3.26), we have that $\bar{\mathcal{L}}^{\Pi^*}_i \bar{V}(t,x,i) = 0$ and, performing the computations above, we get the equality

$$\mathbb{E}_{t,x,i}\Big[-e^{-\gamma X^{\Pi^*}_T(t,x)}\Big] = \sup_{\Pi\in\mathcal{A}_t} \mathbb{E}_{t,x,i}\Big[-e^{-\gamma X^{\Pi}_T(t,x)}\Big] = \bar{V}(t,x,i),$$

that is, $\Pi^*_t = \Pi^*_t(t, Y_t)$ is an optimal control. □

**Remark 3.6.** *We outline that it all boils down to solve ODEs. Indeed, thanks to Theorem 3.1, the value function given by* (3.14) *can be characterized as a transformation of the solution* $\varphi$ *to a certain system of ODEs with a particular terminal condition. As regards existence and uniqueness of a solution to this specific Cauchy problem* (3.21)*, we refer to Walter [97, Theorem VII, Chapter II:6] or to Baran et al. [9, Section* 6*]. According to [97], if* $H$ *given in* (3.22) *is a locally Lipschitz function with respect to the second variable, uniformly in* $t$*, we get that there exists a unique solution* $\varphi(t,i)$*, for every* $t \in [0,T]$*, for all* $i \in \mathcal{E}$*. Requiring that* $\mu$*,* $\sigma$*,* $K_1$ *and* $K_2$ *are continuous functions is a sufficient condition for the regularity of function* $H$ *and, as a consequence, the smoothness of* $\varphi$*. Otherwise,* (3.21) *can be seen as a trivial case of the Cauchy problem faced by [9]. Supposing that* $\mu(\cdot,i)$ *and* $\sigma(\cdot,i)$ *are continuous functions in* $t \in [0,T]$*, for all* $i \in \mathcal{E}$*, ensures that* $\inf_{\Pi\in\mathbb{R}} \bar{\Psi}(t,i)$ *is bounded with respect to the first variable and thus all required hypotheses are satisfied.*

The next result provides the optimal investment portfolio strategy corresponding to Problem 3.1.

**Proposition 3.6.** *Assume existence and uniqueness of a classical solution to the the HJB equation with final condition* (3.20)*. Moreover, suppose that for all* $(t,i) \in [0,T] \times \mathcal{E}$*,*

$$\sigma(t,i) > \sigma > 0. \tag{3.30}$$

*Then, the process* $\{\Pi^*(t,i),\ t \in [0,T]\}$ *characterized in Proposition 3.4 provides the optimal investment strategy for Problem 3.1.*

*Proof.* Let

$$\eta(t,i) = \max\left\{ \frac{\left|\ln\left(\frac{\mu(t,i)-r}{M_2}\right)\right|}{\gamma e^{r(T-t)}}, \frac{\mu(t,i)-r+M_3M_1}{\sigma^2(t,i)\gamma e^{r(T-t)}} \right\}, \quad (t,i) \in [0,T]\times\mathcal{E}.$$

We show that conditions (3.12) and (3.13) in Proposition 3.2 are satisfied. By Proposition 3.5, we immediately have $\Pi^*(t,Y_t) \le \eta(t,Y_t)$ and $\Pi^*(t,Y_t) \ge -\eta(t,Y_t)$, for every $t \in [0,T]$. Moreover, by (3.30) and Assumption 3.1, we get condition (3.13). Then, the process $\{\Pi^*(t,i),\ t\in[0,T]\}$ is an admissible investment strategy and the statement follows by applying the Verification Theorem 3.1 and Proposition 3.4. □

### 3.4.2 The investment problem with the insurance derivative

Now, we suppose that the insurance company, in addition to investing in financial securities, can write a pure endowment contract, whose payoff is given in (3.8).

The following result guarantees that the financial-insurance model outlined in Section 3.2 has a Markovian structure, i.e. the vector process $(X^\Pi,\Lambda,Y)$ is a $(\mathbb{G},\mathbf{P})$-Markov-process. Let $\mathcal{L}_i^\Pi$ denote the Markov generator of $(X^\Pi,\Lambda,Y)$ associated with a constant control $\Pi\in\mathbb{R}$ and let $\mathcal{D}(\mathcal{L}_i^\Pi)$ denote its domain.

**Definition 3.3.** *The set* $\mathcal{D}(\mathcal{L}_i^\Pi)$ *denotes the class of functions* $f(\cdot,\cdot,\cdot,i)\in C^1([0,T])\times C^2(\mathbb{R}\times(0,+\infty))$*, for each* $i\in\mathcal{E}$*, such that for every constant* $\Pi\in\mathbb{R}$*, we have*

$$\mathbb{E}\left[\int_0^T \left(\sigma(v,Y_v)\Pi\frac{\partial f}{\partial w}(v,X_v^\Pi,\lambda_v,Y_v)\right)^2 \mathrm{d}v\right] < \infty, \tag{3.31}$$
$$\mathbb{E}\left[\int_0^T \left(\zeta_2(v,\lambda_v)\lambda_v\frac{\partial f}{\partial \lambda}(v,X_v^\Pi,\lambda_v,Y_v)\right)^2 \mathrm{d}v\right] < \infty,$$

*and*

$$\mathbb{E}\left[\int_0^T\int_{\mathbb{R}} \left|f\left(v,X_v^\Pi,\lambda_v,Y_{v-}+h(Y_{v-},z)\right)-f(v,X_v^\Pi,\lambda_v,Y_{v-})\right| m(\mathrm{d}z)\mathrm{d}v\right] < \infty, \tag{3.32}$$
$$\mathbb{E}\left[\int_0^T \left|f\left(v,X_{v-}^\Pi+\Pi K_1(v,Y_{v-}),\lambda_v,Y_{v-}\right)\right)-f(v,X_{v-}^\Pi,\lambda_v,Y_{v-})\right| \Theta_1(v)\mathrm{d}v\right] < \infty,$$
$$\mathbb{E}\left[\int_0^T \left|f\left(v,X_{v-}^\Pi-\Pi K_2(v,Y_{v-}),\lambda_v,Y_{v-}\right)\right)-f(v,X_{v-}^\Pi,\lambda_v,Y_{v-})\right| \Theta_2(v)\mathrm{d}v\right] < \infty.$$

**Lemma 3.2.** *The stochastic process* $(X^{\Pi}, \Lambda, Y)$ *is a Markov process on* $(\Omega, \mathcal{F}, \mathbf{P}; \mathbb{G})$*, with infinitesimal generator* $\mathcal{L}_i^{\Pi}$ *for all constant strategies* $\Pi \in \mathbb{R}$ *given by*

$$\begin{aligned}\mathcal{L}_i^{\Pi} f(t,x,\lambda,i) =& \frac{\partial f}{\partial t}(t,x,\lambda,i) + \big[rx + (\mu(t,i) - r)\Pi\big]\frac{\partial f}{\partial x}(t,x,\lambda,i) + \zeta_1(t,\lambda)\lambda\frac{\partial f}{\partial \lambda}(t,x,\lambda,i) \\ &+ \frac{1}{2}\sigma^2(t,i)\Pi^2\frac{\partial^2 f}{\partial x^2}(t,x,\lambda,i) + \frac{1}{2}\zeta_2^2(t,\lambda)\lambda^2\frac{\partial^2 f}{\partial \lambda^2}(t,x,\lambda,i) + \sum_{j\in\mathcal{E}} q_{ij} f(t,x,\lambda,j) \\ &+ \Theta_1(t)\big\{V(t, x + \Pi K_1(t,i), \lambda, i) - V(t,x,\lambda,i)\big\} \\ &+ \Theta_2(t)\big\{V(t, x - \Pi K_2(t,i), \lambda, i) - V(t,x,\lambda,i)\big\},\end{aligned}$$

*for every* $i \in \mathcal{E}$*. The domain of the generator* $\mathcal{L}_i^{\Pi}$ *is* $\mathcal{D}(\mathcal{L}_i^{\Pi})$*, for each* $i \in \mathcal{E}$*.*

*Proof.* In view of (3.2), (3.5) and (3.9), by applying Itô's formula to the stochastic process $f(t, X_t^{\Pi}, \lambda_t, Y_t)$, we have

$$f(t, X_t^{\Pi}, \lambda_t, Y_t) = f(0, X_0^{\Pi}, \lambda_0, Y_0) + \int_0^t \mathcal{L}^{\Pi} f(u, X_u^{\Pi}, \lambda_u, Y_u)\mathrm{d}u + m_t,$$

where

$$\begin{aligned} m_t = m_0 &+ \int_0^t \Pi_v \sigma(v, Y_v)\frac{\partial f}{\partial x}(v, X_v^{\Pi}, \lambda_v, Y_v)\mathrm{d}W_v^S + \int_0^t \zeta_2(v,\lambda_v)\lambda_v\frac{\partial f}{\partial \lambda}(v, X_v^{\Pi}, \lambda_v, Y_v)\mathrm{d}W_v^{\Lambda} \\ &+ \int_0^t\int_{\mathbb{R}} \big\{f\big(v, X_v^{\Pi}, \lambda_v, Y_{v-} + h(Y_{v-}, z)\big) - f(v, X_v^{\Pi}, \lambda_v, Y_{v-})\big\}\hat{\mathcal{P}}(\mathrm{d}v, \mathrm{d}z) \\ &+ \int_0^t \big\{f\big(v, X_{v-}^{\Pi} + \Pi_v K_1(v, Y_{v-}), \lambda_v, Y_{v-})\big) - f(v, X_{v-}^{\Pi}, \lambda_v, Y_{v-})\big\}\{\mathrm{d}N_v^1 - \Theta_1(v)\mathrm{d}v\} \\ &+ \int_0^t \big\{f\big(v, X_{v-}^{\Pi} - \Pi_v K_2(v, Y_{v-}), \lambda_v, X_{v-})\big) - f(v, X_{v-}^{\Pi}, \lambda_v, Y_{v-})\big\}\{\mathrm{d}N_v^2 - \Theta_2(v)\mathrm{d}v\}. \end{aligned} \tag{3.33}$$

We only need to prove that the process $m = \{m_t,\ t \in [0,T]\}$ is a $(\mathbb{G}, \mathbf{P})$-martingale. By (3.31), the first two integrals in (3.33) are well-defined and turn out to be $(\mathbb{G}, \mathbf{P})$-martingales. Furthermore, due to (3.32), we have that also the jump terms in (3.33) are $(\mathbb{G}, \mathbf{P})$-martingales, (see e.g. [40, Theorem 26.12(2)] and [19, Lemma L3, Ch.II] for further details about the martingale property related to a Poisson random measure and a Poisson process, respectively). ☐

Let us consider the HJB equation that the value function $V$ is expected to solve, if sufficiently smooth:

$$\sup_{\Pi\in\mathbb{R}} \mathcal{L}_i^{\Pi} V(t,x,\lambda,i) + \lambda\big(\bar{V}(t,x,i) - V(t,x,\lambda,i)\big) = 0, \tag{3.34}$$

for all $(t, x, \lambda, i) \in [0, T) \times \mathbb{R} \times (0, +\infty) \times \mathcal{E}$, with the final condition

$$V(T, x, \lambda, i) = -e^{-\gamma(x-\tilde{K})}, \tag{3.35}$$

for all $(x, \lambda, i) \in \mathbb{R} \times (0, +\infty) \times \mathcal{E}$.

For the sake of clarity, we show how to obtain a formal derivation of the HJB equation (3.34) associated to the problem with the insurance derivative. A formal derivation of equation (3.34) can be obtained by following Björk [16, Section 19.3] and Fleming and Soner [51, Section $III.7$]. To this aim, we apply the Bellman's dynamic programming principle that, in this context, it is formulated as follows.

**Proposition 3.7** (Bellman optimality principle). *Let $(t, x, \lambda, i) \in [0, T] \times \mathbb{R} \times (0, +\infty) \times \mathcal{E}$. Then, for $t \le t + h \le T$ and $\Pi \in \mathcal{A}_t$, we have*

$$V(t, x, \lambda, i) \ge \mathbb{E}_{t,x,\lambda,i}\left[V(t+h, X^{\Pi}_{t+h}, \lambda_{t+h}, Y_{t+h})\right], \tag{3.36}$$

*where $V$ is the value function introduced in* (3.15)*. Moreover, equality holds in* (3.36) *if, and only if, the arbitrary control $\Pi$ on the interval $[t, t+h]$ is optimal.*

Firstly, we employ the Dynamic Programming Principle of Bellman. The idea is that if the company follows the optimal strategy on $[t, T]$, her/his expected utility is at least as great as if she/he invests arbitrarily on $[t, t+h[$ and then optimally on $[t+h, T]$, for $h$ sufficiently small such that $t + h < T$. In the application of the dynamic programming principle, we must consider whether the policyholder survives from time $t$ until time $t + h$, as in Young and Zariphopoulou [102], Moore and Young [80], Ludkovski and Young [76] and Young [101]. Consider an individual aged $l$, who is seeking to buy a pure endowment policy. For the rest of this section, we write $(l)$ to refer to this individual. For each $h$ such that $t + h < T$, if the individual $(l+t)$ survives for another $h$ years until time $t+h$, which happens with probability ${}_hp_{l+t}$, the insurance company still faces the endowment risk on the time interval $[t+h, T]$. In this case, by (3.15), the maximum expected utility derived by investing optimally on $[t+h, T]$ is $V(t+h, X^{\Pi}_{t+h}, \lambda_{t+h}, Y_{t+h})$. However, if the individual $(l+t)$ dies in $[t, t+h]$, an event that happens with probability ${}_hq_{l+t}$, then the company is not longer at risk for the endowment payout. Hence, by (3.14), the maximum expected utility derived by investing optimally on

$[t+h,T]$ is $\bar{V}(t+h, X^{\Pi}_{t+h}, Y_{t+h})$.

From (3.36), we have

$$V(t,x,\lambda,i) \geq {}_hp_{l+t}\mathbb{E}_{t,x,\lambda,i}\Big[V(t+h, X^{\Pi}_{t+h}, \lambda_{t+h}, Y_{t+h})\Big] + {}_hq_{l+t}\mathbb{E}_{t,x,i}\Big[\bar{V}(t+h, X^{\Pi}_{t+h}, Y_{t+h})\Big].$$

If we assume enough regularity conditions and appropriate integrability on the value functions and their derivatives, by applying Itô's formula and conditioning on $X^{\Pi}_t = x$, $\lambda_t = \lambda$ and $Y_t = i$, we get

$$\begin{aligned}
V(t,x,\lambda,i) \geq\ & {}_hp_{l+t}V(t,x,\lambda,i) +{}_h q_{l+t}\bar{V}(t,x,i) \\
&+{}_h p_{l+t}\mathbb{E}_{t,x,\lambda,i}\Bigg[\int_t^{t+h}\Bigg\{\frac{\partial V}{\partial t} + \Big[rX^{\Pi}_v + \big(\mu(v,Y_v)-r\big)\Pi_v\Big]\frac{\partial V}{\partial x}\mathrm{d}v\Bigg\}\Bigg] \\
&+{}_h p_{l+t}\mathbb{E}_{t,x,\lambda,i}\Bigg[\int_t^{t+h}\Bigg\{\zeta_1(v,\lambda_v)\lambda_v\frac{\partial V}{\partial \lambda} + \frac{1}{2}\sigma^2(v,Y_v)\Pi_v^2\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\zeta_2^2(v,\lambda_v)\lambda_v^2\frac{\partial^2 V}{\partial \lambda^2}\Bigg\}\mathrm{d}v\Bigg] \\
&+{}_h p_{l+t}\mathbb{E}_{t,x,\lambda,i}\Bigg[\int_t^{t+h}\Bigg\{\sum_{j\in\mathcal{E}}V(v,X^{\Pi}_v,\lambda_v,j)q_{v,j}\Bigg\}\mathrm{d}v\Bigg] \\
&+{}_h p_{l+t}\mathbb{E}_{t,x,\lambda,i}\Bigg[\int_t^{t+h}\Theta_1(v)\big\{V(v,X^{\Pi}_v+\Pi_vK_1(v,i),\lambda_v,Y_v) - V(v,X^{\Pi}_v,\lambda_v,Y_v)\big\}\mathrm{d}v\Bigg] \\
&+{}_h p_{l+t}\mathbb{E}_{t,x,\lambda,i}\Bigg[\int_t^{t+h}\Theta_2(v)\big\{V(v,X^{\Pi}_v-\Pi_vK_2(v,i),\lambda_v,i) - V(v,X^{\Pi}_v,\lambda_v,Y_v)\big\}\mathrm{d}v\Bigg] \\
&+{}_h q_{l+t}\mathbb{E}_{t,x,i}\Bigg[\int_t^{t+h}\Bigg\{\frac{\partial \bar{V}}{\partial t} + \Big[rX^{\Pi}_v + \big(\mu(v,Y_v)-r\big)\Pi_v\Big]\frac{\partial \bar{V}}{\partial x}\Bigg\}\mathrm{d}v\Bigg] \\
&+{}_h q_{l+t}\mathbb{E}_{t,x,i}\Bigg[\int_t^{t+h}\Bigg\{\frac{1}{2}\sigma(v,Y_v)^2\Pi_v^2\frac{\partial^2 V}{\partial x^2} + \sum_{j\in\mathcal{E}}\bar{V}(v,X^{\Pi}_v,j)q_{v,j}\Bigg\}\mathrm{d}v\Bigg] \\
&+{}_h q_{l+t}\mathbb{E}_{t,x,i}\Bigg[\int_t^{t+h}\Theta_1(v)\big\{\bar{V}(v,X^{\Pi}_v+\Pi_vK_1(v,i),Y_v) - \bar{V}(v,X^{\Pi}_v,Y_v)\big\}\mathrm{d}v\Bigg] \\
&+{}_h q_{l+t}\mathbb{E}_{t,x,i}\Bigg[\int_t^{t+h}\Theta_2(v)\big\{\bar{V}(v,X^{\Pi}_v-\Pi_vK_2(v,i),i) - \bar{V}(v,X^{\Pi}_v,Y_v)\big\}\mathrm{d}v\Bigg].
\end{aligned}$$

To keep the formulas readable, in the integrals above we have suppressed the independent variables $(v, X^{\Pi}_v, \lambda_v, Y_v)$ and $(v, X^{\Pi}_v, Y_v)$ of the partial derivatives of $V$ and $\bar{V}$, respectively. By subtracting ${}_hp_{l+t}V(t,x,\lambda,i)$ from both sides of inequality and dividing both sides by $h$, we obtain

$$\begin{aligned}
\frac{{}_hq_{l+t}}{h}V(t,x,\lambda,i) \geq\ & \frac{{}_hq_{l+t}}{h}\bar{V}(t,x,i) \\
&+{}_h p_{l+t}\mathbb{E}_{t,x,\lambda,i}\Bigg[\int_t^{t+h}\frac{1}{h}\Bigg\{\frac{\partial V}{\partial t} + \Big[rX^{\Pi}_v + \big(\mu(v,Y_v)-r\big)\Pi_v\Big]\frac{\partial V}{\partial x}\mathrm{d}v\Bigg\}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+_h\, p_{l+t}\mathbb{E}_{t,x,\lambda,i}\left[\int_t^{t+h}\frac{1}{h}\left\{\zeta_1(v,\lambda_v)\lambda_v\frac{\partial V}{\partial\lambda}+\frac{1}{2}\sigma^2(v,Y_v)\Pi_v^2\frac{\partial^2 V}{\partial x^2}+\frac{1}{2}\zeta_2^2(v,\lambda_v)\lambda_v^2\frac{\partial^2 V}{\partial\lambda^2}\right\}\mathrm{d}v\right]\\
&+_h\, p_{l+t}\mathbb{E}_{t,x,\lambda,i}\left[\int_t^{t+h}\frac{1}{h}\left\{\sum_{j\in\mathcal{E}}V(v,X_v^\Pi,\lambda_v,j)q_{v,j}\right\}\mathrm{d}v\right]\\
&+_h\, p_{l+t}\mathbb{E}_{t,x,\lambda,i}\left[\int_t^{t+h}\frac{1}{h}\left\{\Theta_1(v)\big\{V(v,X_v^\Pi+\Pi_vK_1(v,i),\lambda_v,Y_v)-V(v,X_v^\Pi,\lambda_v,Y_v)\big\}\right\}\mathrm{d}v\right]\\
&+_h\, p_{l+t}\mathbb{E}_{t,x,\lambda,i}\left[\int_t^{t+h}\frac{1}{h}\left\{\Theta_2(v)\big\{V(v,X_v^\Pi-\Pi_vK_2(v,i),\lambda_v,i)-V(v,X_v^\Pi,\lambda_v,Y_v)\big\}\right\}\mathrm{d}v\right]\\
&+_h\, q_{l+t}\mathbb{E}_{t,x,i}\left[\int_t^{t+h}\frac{1}{h}\left\{\frac{\partial\bar{V}}{\partial t}+\Big[rX_v^\Pi+\big(\mu(v,Y_v)-r\big)\Pi_v\Big]\frac{\partial\bar{V}}{\partial x}\right\}\mathrm{d}v\right]\\
&+_h\, q_{l+t}\mathbb{E}_{t,x,i}\left[\int_t^{t+h}\frac{1}{h}\left\{\frac{1}{2}\sigma(v,Y_v)^2\Pi_v^2\frac{\partial^2 V}{\partial x^2}+\sum_{j\in\mathcal{E}}\bar{V}(v,X_v^\Pi,j)q_{v,j}\right\}\mathrm{d}v\right]\\
&+_h\, q_{l+t}\mathbb{E}_{t,x,i}\left[\int_t^{t+h}\frac{1}{h}\left\{\Theta_1(v)\big\{\bar{V}(v,X_v^\Pi+\Pi_vK_1(v,i),Y_v)-\bar{V}(v,X_v^\Pi,Y_v)\big\}\right\}\mathrm{d}v\right]\\
&+_h\, q_{l+t}\mathbb{E}_{t,x,i}\left[\int_t^{t+h}\frac{1}{h}\left\{\Theta_2(v)\big\{\bar{V}(v,X_v^\Pi-\Pi_vK_2(v,i),i)-\bar{V}(v,X_v^\Pi,Y_v)\big\}\right\}\mathrm{d}v\right].
\end{aligned}
$$

We observe that as $h\longrightarrow 0^+$, we have

$$
{}_hp_{l+t}\longrightarrow 1,\quad {}_hq_{l+t}\longrightarrow 0\quad\text{and}\quad\frac{{}_hq_{l+t}}{h}\longrightarrow\lambda_t,
$$

for each $t\in[0,T]$. Consequently, taking the limit as $h\longrightarrow 0^+$ yields

$$
\begin{aligned}
0\geq&\lambda\big(\bar{V}(t,x,i)-V(t,x,\lambda,i)\big)+\frac{\partial V}{\partial t}+\big[rx+\big(\mu(t,i)-r\big)\Pi\big]\frac{\partial V}{\partial x}+\zeta_1(t,\lambda)\lambda\frac{\partial V}{\partial\lambda}\\
&+\frac{1}{2}\Pi^2\sigma^2(t,i)\frac{\partial^2 V}{\partial x^2}+\frac{1}{2}\zeta_2^2(t,\lambda)\lambda^2\frac{\partial^2 V}{\partial\lambda^2}+\sum_{j\in\mathcal{E}}V(t,x,\lambda,j)q_{ij}\\
&+\Theta_1(t)\big\{V(t,x+\Pi K_1(t,i),\lambda,i)-V(t,x,\lambda,i)\big\}\\
&+\Theta_2(t)\big\{V(t,x-\Pi K_2(t,i),\lambda,i)-V(t,x,\lambda,i)\big\}.
\end{aligned}
$$

By the arbitrariness of the investment strategy $\Pi$, the previous inequality holds for every $\Pi\in\mathcal{A}_t$. Finally, we note that along the optimum, we have

$$
\begin{aligned}
0=&\lambda\left(\bar{V}(t,x,i)-V(t,x,\lambda,i)\right)+\frac{\partial V}{\partial t}+rx\frac{\partial V}{\partial x}+\zeta_1(t,\lambda)\lambda\frac{\partial V}{\partial\lambda}\\
&+\frac{1}{2}\zeta_2^2(t,\lambda)\lambda^2\frac{\partial^2 V}{\partial\lambda^2}+\sum_{j\in\mathcal{E}}V(t,x,\lambda,j)q_{ij}\\
&+\sup_{\Pi\in\mathbb{R}}\Bigg[\big(\mu(t,i)-r\big)\Pi\frac{\partial V}{\partial x}+\frac{1}{2}\sigma^2(t,i)\Pi^2\frac{\partial^2 V}{\partial x^2}+\Theta_1(t)\big\{V(t,x+\Pi K_1(t,i),\lambda,i)-V(t,x,\lambda,i)\big\}\\
&\qquad+\Theta_2(t)\big\{V(t,x-\Pi K_2(t,i),\lambda,i)-V(t,x,\lambda,i)\big\}\Bigg],
\end{aligned}
$$

for every $(t, x, \lambda, i) \in [0, T) \times \mathbb{R} \times (0, +\infty) \times \mathcal{E}$ and $V(T, x, \lambda, i) = -e^{-\gamma(x-\tilde{K})}$, for each $(x, \lambda, i) \in \mathbb{R} \times (0, +\infty) \times \mathcal{E}$, which coincides with (3.34).

Now, based on the nature of exponential utility, we guess that $V$ can be traced back to $\bar{V}$, thanks to a function which does not depend on wealth; thus we introduce the following ansatz

$$V(t, x, \lambda, i) = -e^{-\gamma x e^{r(T-t)}} \varphi(t, i) \phi(t, \lambda), \tag{3.37}$$

with $(t, x, \lambda, i) \in [0, T] \times \mathbb{R} \times (0, +\infty) \times \mathcal{E}$, where $\varphi$ solves (3.21), while the function $\phi$ is non-negative and does not depend on $x$.

From (3.37), replacing all the derivatives and performing some computations, the final value problem (3.34)-(3.35) reduces to

$$\frac{\partial \phi}{\partial t}(t, \lambda) + \zeta_1(t, \lambda)\lambda \frac{\partial \phi}{\partial \lambda}(t, \lambda) + \frac{1}{2}\zeta_2^2(t, \lambda)\lambda^2 \frac{\partial^2 \phi}{\partial \lambda^2}(t, \lambda) - \lambda(\phi(t, \lambda) - 1) = 0, \tag{3.38}$$

for all $(t, \lambda) \in [0, T) \times (0, +\infty)$, and

$$\phi(T, \lambda) = e^{\gamma \tilde{K}}, \tag{3.39}$$

for every $\lambda > 0$.

We observe that the PDE in (3.38) is linear and a solution exists under suitable conditions on model coefficients; see, e.g. Pham [86, Theorem5.3] or Colaneri and Frey [31, Theorem 1].

Clearly, if the function $\phi$ is a classical solution of the Cauchy problem (3.38), then $V(\cdot, \cdot, \cdot, i) \in C^{1,2,2}([0, T] \times \mathbb{R} \times (0, +\infty))$, for each $i \in \mathcal{E}$ and we have that $V(t, x, \lambda, i) = -e^{-\gamma x e^{r(T-t)}} \varphi(t, i)\phi(t, \lambda)$ solves the original HJB equation given in (3.34).

Now, we can state the verification result, which can be used to verify that the candidate solution is indeed the value function in (3.15).

**Theorem 3.2** (Verification Theorem). *Let* $\varphi(\cdot, i) \in C^1\big((0, T)\big) \cap C\big([0, T]\big)$ *and* $\phi(\cdot, \cdot) \in C^1\big((0, T) \times (0, +\infty)\big) \cap C\big([0, T] \times \mathbb{R}^+\big)$, *for each* $i \in \mathcal{E}$, *be classical solutions of the Cauchy problems* (3.21) *and* (3.38), *respectively. Then, the function* $V : [0, T] \times \mathbb{R} \times (0, +\infty) \times \mathcal{E} \longrightarrow \mathbb{R}$ *defined by* (3.37) *is the value function in* (3.15). *Consequently, the strategy* $\Pi_t^* = \Pi^*(t, X_t)$ *described in Proposition 3.4 is an optimal control.*

*Proof.* Let $\varphi : [0, T] \times \mathcal{E} \longrightarrow \mathbb{R}$ be a function such that $\varphi(\cdot, i) \in C^1\big((0, T)\big) \cap C\big([0, T]\big)$, for each $i \in \mathcal{E}$, and suppose that it is a solution of the problem (3.21). Moreover, let $\phi$ :

$[0,T]\times(0,+\infty)\longrightarrow(0,+\infty)$ be a function such that $\phi(\cdot,\cdot)\in C^1\big((0,T)\times(0,+\infty)\big)\cap C\big([0,T]\times(0,+\infty)\big)$, and suppose that it solves the problem (3.38). Now, taking $V$ defined in (3.37), we have that $V$ is a solution of the problem (3.34). This implies that, for every $(t,x,\lambda,i)\in[0,T]\times\mathbb{R}\times(0,+\infty)\times\mathcal{E}$

$$\begin{aligned}&\mathcal{L}_i^{\Pi}V(r,X_r^{\Pi}(t,x),\lambda_r(t,\lambda),Y_r(t,i))\\&\quad+\lambda_r(t,\lambda)\left(\bar{V}(r,X_r^{\Pi}(t,x),y_r(t,i))-V(r,x_r^{\Pi}(t,x),\lambda_r(t,\lambda),Y_r(t,i))\right)\leq 0,\quad r\in[t,T],\end{aligned}\tag{3.40}$$

for all $\Pi\in\mathcal{A}_t$, where $\{\lambda_r(t,\lambda),\ r\in[t,T]\}$ denotes the solution to equation (3.5) with initial condition $\lambda_t=\lambda$ and $\bar{V}$ is the value function of the pure investment problem given in (3.14). In view of (3.9), by applying Itô's formula, we have

$$\begin{aligned}V(T,X_T^{\Pi}(t,x),\lambda_T(t,\lambda),Y_T(t,i))&=V(t,\lambda,i)+\int_t^T\mathcal{L}_i^{\Pi}V(v,X_v^{\Pi}(t,x),\lambda_v(t,\lambda),Y_v(t,i))\mathrm{d}v\\&+\int_t^T\lambda_v(t,\lambda)\left(\bar{V}(v,X_v^{\Pi}(t,x),Y_v(t,i))-V(v,X_v^{\Pi}(t,x),\lambda_v(t,\lambda),Y_v(t,i))\right)\mathrm{d}v+M_T,\end{aligned}\tag{3.41}$$

where $M=\{M_r,\ r\in[t,T]\}$ is the stochastic process given by

$$\begin{aligned}M_r=&\int_t^r\Pi_v\sigma(v,Y_v)\frac{\partial V}{\partial x}(v,X_v^{\Pi},\lambda_v,Y_v)\mathrm{d}W_v^S+\int_t^r\zeta_2(v,\lambda_v)\lambda_v\frac{\partial V}{\partial\lambda}(v,X_v^{\Pi},\lambda_v,Y_v)\mathrm{d}W_v^{\Lambda}\\&+\int_t^r\int_{\mathbb{R}}\left\{V\big(v,X_v^{\Pi},\lambda_v,Y_{v-}+h(Y_{v-},z)\big)-V(v,X_v^{\Pi},\lambda_v,Y_{v-})\right\}\hat{\mathcal{P}}(\mathrm{d}v,\mathrm{d}z)\\&+\int_t^r\left\{V\big(v,X_{v-}^{\Pi}+\Pi_vK_1(v,Y_{v-}),\lambda_v,Y_{v-})\big)-V(v,X_{v-}^{\Pi},\lambda_v,Y_{v-})\right\}\{\mathrm{d}N_v^1-\Theta_1(v)\mathrm{d}v\}\\&+\int_t^r\left\{V\big(v,X_{v-}^{\Pi}-\Pi_vK_2(v,Y_{v-}),\lambda_v,Y_{v-})\big)-V(v,X_{v-}^{\Pi},\lambda_v,Y_{v-})\right\}\{\mathrm{d}N_v^2-\Theta_2(v)\mathrm{d}v\}.\end{aligned}$$

Now, we prove that $M$ is a $(\mathbb{G},\mathbf{P})$-local martingale. Precisely, we need to show that

$$\mathbb{E}\left[\int_t^{T\wedge\tau_n}\left(\sigma(v,Y_v)\Pi_v\frac{\partial V}{\partial w}(v,X_v^{\Pi},\lambda_v,Y_v)\right)^2\mathrm{d}v\right]<\infty,$$
$$\mathbb{E}\left[\int_t^{T\wedge\tau_n}\left(\zeta_2(v,\lambda_v)\lambda_v\frac{\partial V}{\partial\lambda}(v,X_v^{\Pi},\lambda_v,Y_v)\right)^2\mathrm{d}v\right]<\infty,$$

for a suitable, non-decreasing sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\longrightarrow+\infty}\tau_n=+\infty$. Taking expression (3.37) into account, we note that

$$\begin{aligned}\frac{\partial V}{\partial x}(t,x,\lambda,i)&=\gamma\phi(t,\lambda)\varphi(t,i)e^{r(T-t)-\gamma xe^{r(T-t)}},\\\frac{\partial V}{\partial\lambda}(t,x,\lambda,i)&=-\frac{\partial\phi}{\partial\lambda}(t,\lambda)\varphi(t,i)e^{-\gamma xe^{r(T-t)}}.\end{aligned}$$

Let us define a sequence of random times $\{\tau_n\}_{n\in\mathbb{N}}$ by setting

$$\tau_n := \inf\{s \in [t,T] \mid X_s^{\Pi} < -n, \lambda_s > n, \phi(s,\lambda_s) > n, \frac{\partial \phi}{\partial \lambda}(s,\lambda_s) > n\}, \quad n \in \mathbb{N}.$$

Throughout the proof, we denote by $C_n$ any constant depending on $n \in \mathbb{N}$. Consequently, we get that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^{T\wedge\tau_n} \Big(\sigma(v,Y_v)\Pi_v \frac{\partial V}{\partial x}(v, X_v^{\Pi}, \lambda_v, Y_v)\Big)^2 \mathrm{d}v\Big] \\
&\qquad = \mathbb{E}\Big[\int_0^{T\wedge\tau_n} \sigma^2(v,Y_v)\Pi_v^2\Big(\gamma\phi(v,\lambda_v)\varphi(v,Y_v)e^{r(T-v)-\gamma X_v^{\Pi} e^{r(T-v)}}\Big)^2 dv\Big] \\
&\qquad \le C_n \mathbb{E}\Big[\int_0^T \sigma^2(v,Y_v)\Pi_v^2 dv\Big] < \infty, \quad \forall n \in \mathbb{N},
\end{aligned}$$

since $\Pi$ is admissible. Further, by (3.6) we have that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^{T\wedge\tau_n} \Big(\zeta_2(v,\lambda_v)\lambda_v \frac{\partial V}{\partial \lambda}(v, X_v^{\Pi}, \lambda_v, Y_v)\Big)^2 \mathrm{d}v\Big] \\
&\qquad = \mathbb{E}\Big[\int_0^{T\wedge\tau_n} \Big(\zeta_2(v,\lambda_v)\lambda_v \frac{\partial \phi}{\partial \lambda}(v,\lambda_v)\varphi(v,Y_v)e^{-\gamma X_v^{\Pi} e^{r(T-t)}}\Big)^2 \mathrm{d}v\Big] \\
&\qquad \le C_n \mathbb{E}\Big[\int_0^T \zeta_2(v,\lambda_v)^2\lambda_v^2 \mathrm{d}v\Big] < \infty, \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Furthermore, due to the boundedness of function $V$ until time $\tau_n$, we have that the stopped process

$$\Big\{\int_t^{r\wedge\tau_n}\int_{\mathbb{R}} \Big\{V\big(v, X_v^{\Pi}, \lambda_v, Y_{v-} + h(Y_{v-},z)\big) - V\big(v, X_v^{\Pi}, \lambda_v, Y_{v-}\big)\Big\}\hat{\mathcal{P}}(\mathrm{d}v,\mathrm{d}z),\ r \in [t,T]\Big\}$$

is a $(\mathbb{G},\mathbf{P})$-martingale (see e.g. [40, Theorem 26.12(2)]), for every $n \in \mathbb{N}$. Finally, even the stopped processes

$$\Big\{\int_t^{r\wedge\tau_n} \Big\{V\big(v, X_{v-}^{\Pi} + \Pi_v K_1(v,Y_{v-}), \lambda_v, Y_{v-}\big) - V\big(v, X_{v-}^{\Pi}, \lambda_v, Y_v\big)\Big\}\{\mathrm{d}N_v^1 - \Theta_1(v)\mathrm{d}v\},\ r \in [t,T]\Big\}$$

and

$$\Big\{\int_t^{r\wedge\tau_n} \Big\{V\big(v, X_{v-}^{\Pi} - \Pi_v K_2(v,Y_{v-}), \lambda_v, Y_{v-}\big) - V\big(v, X_{v-}^{\Pi}, \lambda_v, Y_v\big)\Big\}\{\mathrm{d}N_v^2 - \Theta_2(v)\mathrm{d}v\},\ r \in [t,T]\Big\}$$

are $(\mathbb{G},\mathbf{P})$-martingales, (see e.g. [19, Lemma L3, Ch.II]). Thus, the process $\{M_r,\ r \in [t,T]\}$ turns out to be a $(\mathbb{G},\mathbf{P})$-local martingale and $\{\tau_n\}_{n\in\mathbb{N}}$ is a localizing sequence for $\{M_r,\ r \in$

$[t, T]\}$. Therefore, taking the conditional expectation of both sides of (3.41) with respect to $X_t^\Pi = x$, $\lambda_t = \lambda$ and $Y_t = i$ with $T$ replaced by $T \wedge \tau_n$, by (3.40) we obtain that

$$\mathbb{E}_{t,x,\lambda,i}\big[V(T \wedge \tau_n, X^\Pi_{T\wedge\tau_n}(t,x), \lambda_{T\wedge\tau_n}, Y_{T\wedge\tau_n}(t,i))\big] \le V(t,x,\lambda,i),$$

for every $\Pi \in \mathcal{A}_t$, $t \in [\![0, T \wedge \tau_n]\!]$, $n \in \mathbb{N}$. Now, we note that

$$\begin{aligned}&\mathbb{E}\left[\big(V(T \wedge \tau_n, X^\Pi_{T\wedge\tau_n}(t,x), \lambda_{T\wedge\tau_n}(t,\lambda), Y_{T\wedge\tau_n}(t,i))\big)^2\right]\\ &\quad = \mathbb{E}\left[e^{-2\gamma X^\Pi_{T\wedge\tau_n} e^{r(T\wedge\tau_n - t)}} \varphi(T\wedge\tau_n, Y_{T\wedge\tau_n})^2 \phi(T\wedge\tau_n, \lambda_{T\wedge\tau_n})^2\right] \le \widetilde{C},\end{aligned}$$

for a positive constant $\widetilde{C}$. This means that $\{V(T \wedge \tau_n, X^\Pi_{T\wedge\tau_n}(t,x), \lambda_{T\wedge\tau_n}, Y_{T\wedge\tau_n}(t,i))\}_{n\in\mathbb{N}}$ is a family of uniformly integrable random variables. Hence, it converges almost surely. Since $\{\tau_n\}_{n\in\mathbb{N}}$ is a bounded and non-decreasing sequence of random times and $\mathbf{P}(|X_t^\Pi| < +\infty) = 1$, see (3.10), in view of (3.7), we can apply the dominated convergence theorem and, taking the limit for $n \longrightarrow +\infty$, we get

$$\begin{aligned}&\mathbb{E}_{t,x,\lambda,i}\big[V(T, X^\Pi_T(t,x), \lambda_T(t,\lambda), Y_T(t,i))\big]\\ &\quad = \lim_{n\longrightarrow+\infty} \mathbb{E}_{t,x,\lambda,i}\big[V(T\wedge\tau_n, X^\Pi_{T\wedge\tau_n}(t,x), \lambda_{T\wedge\tau_n}(t,\lambda), Y_{T\wedge\tau_n}(t,i))\big]\\ &\quad \le V(t,x,\lambda,i),\end{aligned}$$

for every $\Pi \in \mathcal{A}_t$, $t \in [0, T]$. By the terminal condition (3.35) and the previous inequality, we get

$$\mathbb{E}_{t,x,\lambda,i}\big[-e^{-\gamma(X^\Pi_T(t,x) - \tilde{K})}\big] \le V(t,x,\lambda,i),$$

for every $\Pi \in \mathcal{A}_t$, $t \in [0, T]$. Finally, since the insurance payment does not depend on the risky asset price, we have that $\Pi^*(t,x,\lambda,i) = \Pi^*(t,i)$ given in Proposition 3.4 yields that $\mathcal{L}_i^{\Pi^*} V(t,x,\lambda,i) + \lambda\big(\bar{V}(t,x,i) - V(t,x,\lambda,i)\big) = 0$; then, if we apply the above arguments to $\Pi^*$ and replacing $\mathcal{L}_i^\Pi$ with $\mathcal{L}_i^{\Pi^*}$, we find the equality

$$\sup_{\Pi\in\mathcal{A}_t} \mathbb{E}_{t,x,\lambda,i}\big[-e^{-\gamma(X^\Pi_T(t,x) - \tilde{K})}\big] = V(t,x,\lambda,i),$$

which implies that the process $\Pi^*(t, Y_t)$ is an optimal Markovian control. □

**Remark 3.7.** *We observe that the optimal investment strategy $\Pi^*(t, Y_t)$ turns out to be the same as the pure investment problem. This result relies on the fact that the payoff of a pure endowment treaty does not depend on the stock price process. In other words, the optimal* *portfolio for the investment problem with the insurance policy equals the strategy without insurance risks when the insurance payment is independent of the risky asset price process. This statement is the same as that provided in Delong [43] and Liang and Lu [71] when the risky asset price dynamics is driven by a Lévy process and a shot-noise process, respectively.*

## 3.5 The indifference price of the pure endowment

Now, we compute explicitly the indifference price for the pure endowment contract whose payoff is given by (3.8) in the market model outlined in Section 3.2.

Firstly, we provide the formal definition of the indifference price charged by an insurance company which writes a pure endowment. Recall that $\bar{V}$ and $V$ are the value functions introduced in (3.14) and (3.15), respectively.

**Definition 3.4.** *Given $X_t = x$, $\lambda_t = \lambda$ and $Y_t = i$, the* indifference price *process or* reservation price *process $P = \{P_t,\ t \in [0,T]\}$ of the insurance company related to the pure endowment contract is defined at any time $t \in [0,T]$ as the $\mathbb{G}$-adapted process implicit solution to the equation*

$$\bar{V}(t,x,i) = V(t, w + x + P_t, \lambda, i). \tag{3.42}$$

*In other words, $P$ is the price that makes the company indifferent, in terms of expected utility, between not selling and selling the insurance policy for the price $P$ now and paying the benefits at maturity (provided the insured person be still alive).*

In our framework, we obtain the following explicit characterization of the indifference price process.

**Proposition 3.8.** *Under the same hypotheses of Theorem 3.2, for every $t \in [0,T]$, the indifference price of the insurance company related to the pure endowment with maturity $T$ is given by*

$$P_t = P(t,\lambda;T) = \frac{\ln\big(\phi(t,\lambda)\big)}{\gamma e^{r(T-t)}}, \tag{3.43}$$

*for all $(t,\lambda) \in [0,T] \times (0,\infty)$, where the function $\phi$ solves the Cauchy problem (3.38).*

*Proof.* By Theorem 3.1 and Theorem 3.2, equation (3.42) reads as

$$e^{-\gamma x e^{r(T-t)}} \varphi(t,i) = e^{-(x+P_t)\gamma e^{r(T-t)}} \varphi(t,i)\phi(t,\lambda),$$

and then

$$e^{P_t \gamma e^{r(T-t)}} = \phi(t, \lambda),$$

from which, computing the logarithm of both members, we get (3.43). □

**Remark 3.8.** *We briefly comment the expression achieved for the indifference price in* (3.43)*, underlining the similarities and differences with results in the existing actuarial literature. It is well known that an exponential utility function like* $u(w) = -e^{-\alpha w}$*, for* $w \in \mathbb{R}$*, implies that the indifference price of a pure endowment depends on the risk aversion coefficient, the interest rate and the logarithm of the function that links the two value functions and is independent of wealth (see e.g. Young and Zariphopoulou [102], Young [101], Moore and Young [80], Ludkovski and Young [76]). Here, the indifference price shares the same features. Clearly, since we deal with a pure endowment policy for individuals subjected to a stochastic hazard rate, in our framework the function* $\phi$ *that binds the investment problem with claim to the pure investment problem depends on the mortality intensity, rather than the stock price as in the case of equity-indexed policies. Therefore, the randomness effect introduced by the stochastic hazard rate has a significant impact on the price, as in Ludkovski and Young [76]. Moreover, we notice that the current state of the market does not influence the reservation price. This means that in our model, the regime of the economy does not affect directly the indifference price of such type of insurance contracts but only the amount invested in the financial assets.*

Next, we show that under the indifference pricing principle, the premium solves a terminal value problem.

**Corollary 3.8.1.** *For every* $(t, \lambda) \in [0, T] \times (0, +\infty)$ *the indifference premium* $P(t, \lambda; T)$ *satisfies the following PDE*

$$\begin{aligned} rP(t,\lambda;T) &= \frac{\partial P}{\partial t}(t,\lambda;T) + \zeta_1(t,\lambda)\lambda \frac{\partial P}{\partial \lambda}(t,\lambda;T) \\ &+ \frac{1}{2}\zeta_2^2(t,\lambda)\lambda^2 \left\{ \frac{\partial^2 P}{\partial \lambda^2}(t,\lambda;T) + \gamma e^{r(T-t)} \left( \frac{\partial P}{\partial \lambda}(t,\lambda;T) \right)^2 \right\} + \frac{\lambda}{\gamma e^{r(T-t)}} \left( e^{-P(t,\lambda;T)\gamma e^{r(T-t)}} - 1 \right), \end{aligned}$$

*with boundary condition* $P(T, \lambda; T) = \tilde{K}$*, for each* $\lambda \in (0, \infty)$.

*Proof.* It follows from a straightforward combination of (3.38) and (3.43). □

Furthermore, we provide a probabilistic representation for the indifference price process $P$, by a generalization of the Feynman-Kac formula. Indeed, if the function $\phi$ solves the Cauchy problem (3.38), we can represent $\phi$ as an expectation via an extension of the Feynman-Kac formula. More precisely, using the linear PDE for $\phi - 1$, it is easy to see that

$$\phi(t,\lambda) - 1 = \mathbb{E}_{t,\lambda}\big[e^{-\int_t^T \lambda_v \mathrm{d}v}\big(e^{\gamma\tilde{K}} - 1\big)\big],$$

for every $(t,\lambda) \in [0,T] \times (0,+\infty)$. Thus, as a consequence, we have that

$$\phi(t,\lambda) = 1 + \big(e^{\gamma\tilde{K}} - 1\big)\mathbb{E}_{t,\lambda}\big[e^{-\int_t^T \lambda_v \mathrm{d}v}\big],$$

where $\mathbb{E}_{t,\lambda}$ denotes the conditional expectation given $\lambda_t = \lambda$, for every $(t,\lambda) \in [0,T] \times (0,+\infty)$. We outline that $\mathbb{E}_{t,\lambda}\Big[e^{-\int_t^T \lambda_v \mathrm{d}v}\Big]$ is the conditional probability that an individual will survive until time $T$ given that she/he is alive at time $t$. Hence, representing the function $\phi$ as

$$\phi(t,\lambda) = e^{\gamma\tilde{K}}\mathbb{E}_{t,\lambda}\big[e^{-\int_t^T \lambda_v \mathrm{d}v}\big] + \Big(1 - \mathbb{E}_{t,\lambda}\big[e^{-\int_t^T \lambda_v \mathrm{d}v}\big]\Big) = \mathbb{E}_{t,\lambda}\big[e^{\gamma G_T}\big],$$

for every $(t,\lambda) \in [0,T] \times (0,+\infty)$, the indifference price of the insurance company related to a pure endowment contract can be written as

$$P_t = P(t,\lambda;T) = \frac{\ln\Big(\mathbb{E}_{t,\lambda}\big[e^{\gamma G_T}\big]\Big)}{\gamma e^{r(T-t)}},$$

for every $(t,\lambda) \in [0,T] \times (0,+\infty)$.

### 3.5.1 Indifference price for a portfolio of pure endowments

In this subsection, we evaluate a panel of insurance policies, extending the previous results. Put another way, we no longer consider a single life insurance policy, we deal with a portfolio consisting of pure endowments issued to a group of $n \in \mathbb{N}$ individuals, who are all the same age with indipendent and identically distributed times until death. We suppose that the loss payable at the maturity $T$ equals the amount $\tilde{K} > 0$, for each of the policyholders who have not died yet. Thus, the value function (3.15) is replaced by

$$V^{(n)}(t,x,\lambda,i) := \sup_{\Pi \in \mathcal{A}_t(\mathbb{G})} \mathbb{E}_{t,x,\lambda,i}\Big[ - e^{-\gamma(X_T^{\Pi} - G_T^{(n)})}\Big],$$

where $G_T^{(n)} = m\tilde{K}$, with $\tilde{K} > 0$ constant, if there are exactly $m$ individuals alive at time $T$ out of the group of $n$ insured individuals alive at time $t$. Analogously to (3.34)-(3.35), $V^{(n)}$ solves a final value problem, specifically it solves the following HJB equation

$$\sup_{\Pi \in \mathbb{R}} \mathcal{L}_i^{\Pi} V^{(n)}(t,x,\lambda,i) + n\lambda\big(V^{(n-1)}(t,x,i) - V^{(n)}(t,x,\lambda,i)\big) = 0,$$

for all $(t,x,\lambda,i) \in [0,T) \times \mathbb{R} \times (0,+\infty) \times \mathcal{E}$, with the final condition

$$V^{(n)}(T,x,\lambda,i) = -e^{-\gamma(x-n\tilde{K})},$$

for all $(x,\lambda,i) \in \mathbb{R} \times (0,+\infty) \times \mathcal{E}$, in which $V^{(0)} = \bar{V}$. Note that $V^{(1)} = V$ in (3.34). One can easily shows that

$$V^{(n)}(t,x,\lambda,i) = \bar{V}(t,x,i)\phi^{(n)}(t,\lambda),$$

where $\phi^{(n)} : [0,T] \times (0,+\infty) \longrightarrow (0,+\infty)$ solves the linear PDE

$$\frac{\partial \phi^{(n)}}{\partial t}(t,\lambda) + \zeta_1(t,\lambda)\lambda \frac{\partial \phi^{(n)}}{\partial \lambda}(t,\lambda) + \frac{1}{2}\zeta_2(t,\lambda)^2\lambda^2 \frac{\partial^2 \phi^{(n)}}{\partial \lambda^2}(t,\lambda) - n\lambda\left(\phi^{(n)}(t,\lambda) - \phi^{(n-1)}(t,\lambda)\right) = 0, \tag{3.44}$$

for all $(t,\lambda) \in [0,T] \times (0,+\infty)$, with final condition

$$\phi^{(n)}(T,\lambda) = e^{\gamma n \tilde{K}}, \tag{3.45}$$

for every $\lambda > 0$, where $\phi^{(0)} \equiv 1$. Thus, the indifference price of $n$ pure endowments $P^{(n)}$ is an implicit solution of the following equation

$$\bar{V}(t,x,i) = V^{(n)}(t, x + P_t^{(n)}, \lambda, i),$$

for every $(t,x,\lambda,i) \in [0,T] \times \mathbb{R} \times (0,+\infty) \times \mathcal{E}$. Similarly to (3.43), the reservation price of the insurance company related to $n$ pure endowment contracts is given by

$$P_t^{(n)} = P^{(n)}(t,\lambda,i;T) = \frac{\ln\left(\phi^{(n)}(t,\lambda)\right)}{\gamma e^{r(T-t)}},$$

for all $(t,\lambda,i) \in [0,T] \times (0,+\infty) \times \mathcal{E}$, where the function $\phi^{(n)}$ solves the Cauchy problem (3.44)-(3.45).

### 3.5.2 Indifference price for a term life insurance

Finally, we analyze the indifference price of another type of a mortality-contingent claim, the so-called term life insurance that can be defined as follows.

**Definition 3.5.** *A term life insurance contract with maturity $T$ is a life insurance policy where the amount is paid at time $T$ if the policyholder dies before time $T$. The associated payoff is given by the random variable*

$$G_T := \tilde{K}\mathbf{1}_{\{\tau \leq T\}}, \tag{3.46}$$

*where $\tilde{K}$ is a positive constant.*

We determine the indifference price of a term life insurance policy whose payoff is given by (3.46) in the Markov-modulated model outlined in Section 3.2.

The goal of the insurance company remains to maximize the expected utility of her/his terminal wealth. Then, we consider the problem with the new kind of insurance derivative

$$\sup_{\Pi \in \mathcal{A}} \mathbb{E}\Big[ - e^{-\gamma(X_T^{\Pi} - \tilde{K})}\Big].$$

Thus, the corresponding value function is given by

$$V(t, x, \lambda, i) := \sup_{\Pi \in \mathcal{A}_t} \mathbb{E}_{t,x,\lambda,i}\Big[ - e^{-\gamma(X_T^{\Pi} - \tilde{K})}\Big],$$

for every $(t, x, \lambda, i) \in [0, T] \times \mathbb{R} \times (0, +\infty) \times \mathcal{E}$. Note that it is exactly the same defined in (3.15), namely it equals the value function of the problem with the pure endowment. Thus, proceeding as above, the HJB problem for $V$ is given by

$$\sup_{\Pi \in \mathbb{R}} \mathcal{L}_i^{\Pi} V(t, x, \lambda, i) + \lambda\big(\bar{V}(t, x - \tilde{K}e^{-r(T-t)}, i) - V(t, x, \lambda, i)\big) = 0,$$

for all $(t, x, \lambda, i) \in [0, T) \times \mathbb{R} \times (0, +\infty) \times \mathcal{X}$, with final condition

$$V(T, x, \lambda, i) = -e^{-\gamma(x - \tilde{K})},$$

for all $(x, \lambda, i) \in \mathbb{R} \times (0, +\infty) \times \mathcal{E}$, where $\bar{V}$ is introduced in (3.14). Note that the HJB equation corresponds to (3.34) with $\bar{V}(t, x, i)$ replaced by $\bar{V}(t, x - \tilde{K}e^{-r(T-t)}, i)$, since the insurance company has to pay the amount $\tilde{K}$ at time $T$ for the death of the policyholder and so she/he needs to charge $\tilde{K}e^{-r(T-t)}$ at time $t$ in order to cover this payout. One can easily shows that

$$V(t,x,\lambda,i) = \bar{V}(t,x,i)\xi(t,\lambda),$$

where the function $\xi : [0,T] \times (0,+\infty) \longrightarrow (0,+\infty)$ solves the linear PDE

$$\frac{\partial \xi}{\partial t}(t,\lambda) + \zeta_1(t,\lambda)\lambda\frac{\partial \xi}{\partial \lambda}(t,\lambda) + \frac{1}{2}\zeta_2(t,\lambda)^2\lambda^2\frac{\partial^2 \xi}{\partial \lambda^2}(t,\lambda) - \lambda(e^{\gamma\tilde{K}} - \xi(t,\lambda)) = 0, \tag{3.47}$$

for all $(t,\lambda) \in [0,T] \times (0,+\infty)$, with final condition

$$\phi(T,\lambda) = e^{\gamma\tilde{K}}, \tag{3.48}$$

for every $\lambda > 0$. We note that this final value problem (3.47)-(3.48) is very similar to (3.38)-(3.39). Hence, the reservation price of the insurance company related to a term life contract is given by

$$P_t = P(t,\lambda,i;T) = \frac{\ln\Big(\xi(t,\lambda)\Big)}{\gamma e^{r(T-t)}},$$

for all $(t,\lambda,i) \in [0,T] \times (0,+\infty) \times \mathcal{E}$, where the function $\xi$ solves the Cauchy problem (3.47)-(3.39).

## 3.6 Numerical experiment

In order to illustrate certain qualitative features of the model that are difficult to verify analytically, we present some numerical results based on the theoretical framework developed previously. In particular, we aim to investigate how the regime-switching and the stochastic hazard rate affect the decisions of the insurance company, for both optimization problems, with or without liabilities. Precisely, to analyze such dependency, we compute the optimal investment strategy, the value functions and the indifference price for a pure endowment policy, numerically.

To simplify the analysis, we provide a toy example with only two economic regimes: we suppose that the Markov chain $Y$ has only two states that can be interpreted as the 'good' and 'bad' economic regimes, respectively. For instance, the good regime could represent a market in economic boom whereas the bad regime could be a market in economic recession in which security prices are expected to fall. We also call these two regimes of the market 'bull' market and 'bear' market, respectively.

Let us make some assumptions and fix some parameter values. We start with the infinitesimal generator of the 2-state Markov chain that describes the rate our Markov chain moves between states: specifically $q_{ij}$ indicates the average of number of switches in an unit time, from state $i$ to $j$. Since empirical observations of the market suggest that it is more likely to pass from a good economic state to a bad one than the opposite, we choose $q_{12} > q_{21}$; in particular, we take $q_{12} = 0.2$ and $q_{21} = 0.1$.

For the sake of simplicity, we consider that functions $\mu$, $\sigma$, $K_1$ and $K_2$ depend only on the Markov chain. Thus, by (1.5), the risky asset price dynamics is given by

$$\mathrm{d}S_t = S_{t-}\{\mu_i \mathrm{d}t + \sigma_i \mathrm{d}W_t^S + K_{1,i}\mathrm{d}N_t^1 - K_{2,i}\mathrm{d}N_t^2\}, \quad S_0 > 0,\ i = 1, 2,$$

where $\mu_i$, $\sigma_i$, $K_{1,i}$ and $K_{2,i}$ denote the expected rate of return, the volatility and the jump coefficients in the $i$-th regime, for $i = 1, 2$. By way of example, we set the initial value of the stock price to be $S_0 = 1$ and the short-term interest rate to be $r = 5\%$. As shown by French et al. [52], the appreciation rate of the underlying risky asset is higher in a growing economy, so we assume that $\mu_1 > \mu_2$. Moreover, in each economic regime, the return of the risky asset should be higher than that of the risk-free rate, as required also in our modeling framework. Further, we suppose that volatility is lower in a good economy, i.e. $\sigma_1 < \sigma_2$, because Hamilton and Gang [57] find that economic recessions represent the main factor that drives fluctuations in the volatility of stock returns. Furthermore, let us assume that $\frac{\mu_1 - r}{\sigma_1^2} > \frac{\mu_2 - r}{\sigma_2^2}$. Indeed, according to French et al. [52], even though the expected market risk premium (defined as the expected return on the stock minus the risk-free interest rate) is usually higher during a 'bear' market than during a 'bull' market, the volatility of the stock offsets the effect of this quantity and, as a consequence, the ratio 'expected excess return/return variance' is greater when the economic conditions are good. Regarding the jump terms, we consider two homogeneous Poisson processes $N^1$ and $N^2$ with constant intensities $\Theta_1 = 0.3$ and $\Theta_2 = 0.4$. We observe, simulating trajectories of the stock price $S$, that the higher are the values of function $K_1$, the higher is the price. On the other hand, any increase in the coefficient $K_2$ leads to smaller prices for the risky stock. Moreover, we note that large values of $K_2$ cause dizzying upward or downward peaks in the stock behavior over time, even though the intensity $\Theta_2$ is tiny. Therefore, since in a market with good economic conditions stock prices are rising or are expected to rise, we suppose that $K_{1,1} > K_{1,2}$ and

| **Regime** | $\mu$ | $\sigma$ | $K_1$ | $K_2$ |
|---|---|---|---|---|
| $e_1$ (good) | 0.15 | 0.15 | 0.15 | 0.3 |
| $e_2$ (bad) | 0.12 | 0.25 | 0.1 | 0.35 |

Table 3.1: Simulation market parameters.

$K_{2,1} < K_{2,2}$.

On the basis of all these considerations, we fix the parameter values as summarized in Table 3.1.

Since the underlying market is a continuous time model, we need to discretize it by Monte Carlo simulation. The time horizon is taken to be $T = 10$ years and we discretize time with a total of 1000 time steps (that means that we take into account about two updates of $S$ every workweek), each of width $\Delta t = \frac{1}{100}$).

In order to get an idea of our model, we plot three trajectories of the risky asset $S$ in Figure 3.1. We notice that the stock price exhibits jumps at switching times of the Markov chain. Moreover, the risky asset price is greater during a 'bull market' rather than during a 'bear' market and it, as it is reasonable.

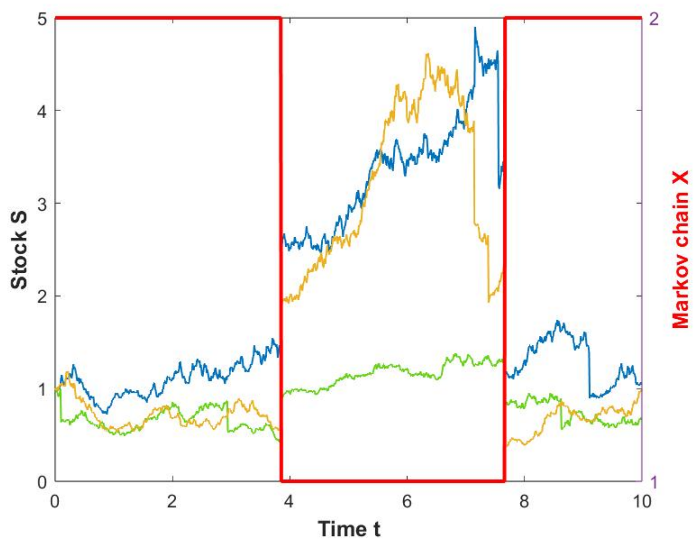



Figure 3.1: The effect of the regime-switching on the stock price $S$.

Next, we compute the optimal investment strategy based on Proposition 3.4. The aim is to investigate how it is sensitive to economic regimes during the trading period. In Figure 3.2 we plot the optimal dynamic portfolio given by (3.27), as a function of time. We clearly note that

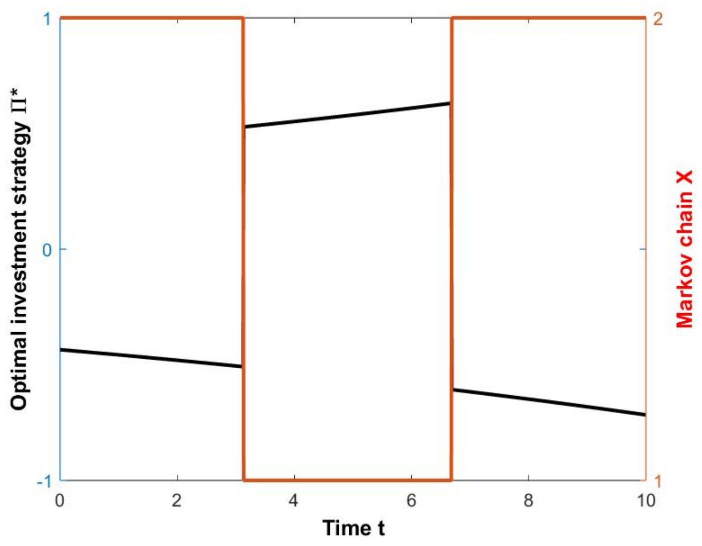



Figure 3.2: The effect of regime-switching on the optimal strategy $\Pi^*$.

when the market state changes, the company opts for a different investment portfolio, since a regime switch leads to a sudden change in the optimal strategy. Further, we note that in a good economy (namely when the stock price presents a high rate of return, small fluctuations and a few peaks) the amount invested in the risky asset is always positive and increasing with respect to time. Thus, when the good regime is in force, the amount invested in risky asset grows up. Instead, if the market scenario is bad, the strategy is negative, meaning that when the economic conditions are bad, the insurance company prefers to short-sell the risky asset.

After that, we take into account a pure endowment policy and we investigate its indifference price, studying the effect of the hazard rate over the years.

In this numerical example of our proposed model, we assume that the hazard rate follows a mean-reverting Brownian Gompertz model, similar to the one proposed in Milevsky and Promislow [78], i.e.

$$\lambda_t = \lambda_0 e^{c_1 t + c_2 Z_t}, \quad c_1, c_2, \lambda_0 > 0,$$
$$\mathrm{d}Z_t = -mZ_t \mathrm{d}t + \mathrm{d}W_t^Z, \quad Z_0 = 0, \quad m \geq 0,$$

with $c_1 = 0.083$, $c_2 = 0.1$, $\lambda_0 = 0.01$ and $m = 0.5$. Let us observe that this choice corresponds to (3.5), considering $\zeta_1(t,\lambda) = c_1 + m\ln(\lambda_0) + \frac{1}{2}c_2^2 - m\ln(\lambda) + mc_1 t$ and $\zeta_2(t,\lambda) = c_2\lambda$, for all $(t,\lambda)$. This model ensures that the hazard rate is kept positive and does not explode on $[0,T]$, since it is an exponential function that depends on a stochastic factor $Y$ with a mean reversion behavior.

In this context, based on the results obtained above, we compute the indifference price of an insurance company related to a pure endowment contract that pays $\tilde{K} = 1$ (without loss of generality) if the policyholder is still alive after 10 years from purchaising the policy. Thus, the payoff is easily given by the random variable

$$G_T := \mathbf{1}_{\{\tau > T\}},$$

recalling that $\tau$ represents the remaining lifetime of the insured.

Now, we briefly analyze the value function $\bar{V}$ related to the company that simply invests her/his wealth in the market and the value function $V$ related to the company who also issues a pure endowment contract.

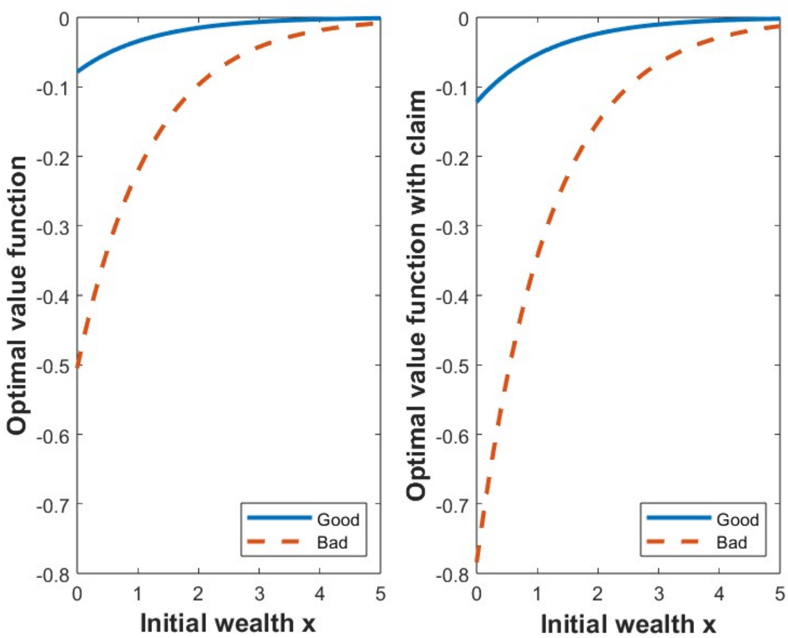



Figure 3.3: Optimal value at time 0 as a function of wealth when the economic regime is $i = 1$ (solid line) or $i = 2$ (dashed line). Left panel: the pure investment problem. Right panel: the investment problem with the insurance contract.

Figure 3.3 illustrates the value functions $\bar{V}$ (left panel) and $V$ (right panel) at time $t = 0$, with respect to the initial wealth $x$, associated to the optimal strategy computed above, when the market state is good (solid line) or bad (dashed line). The two panels are different but exhibit a similar behavior: at the initial time, the optimal value functions are increasing with respect to wealth, in both regimes. It is worth noting that values reached by functions $\bar{V}$ and $V$ are always higher in a 'bull' market, as it is reasonable. Furthermore, we can outline that regime-switching influences our market model and its effect is tangibile also for the optimal value functions: different economic conditions imply different value functions. We highlight that this gap becomes greater when the company, beyond investing in financial assets (risky and not), also writes a life insurance contract.

We conclude this section, investigating the indifference price of a pure endowment policy, in order to highlight the dependence of a life insurance contract on mortality force and time to expiration. In view of the probabilistic representation provided above, the indifference price charged by the insurance company is determined as

$$P_t = P(t, \lambda; T) = \frac{\ln\left(1 + (e^{\gamma} - 1)\mathbb{E}_{t,\lambda}\left[e^{-\int_t^T \lambda_v \mathrm{d}v}\right]\right)}{\gamma e^{r(T-t)}}, \tag{3.49}$$

for every $(t, \lambda) \in [0, T] \times (0, +\infty)$. Using the standard Monte Carlo technique (with parameter $M = 5000$) to evaluate expectations with respect to the probability measure $\mathbf{P}$, we employ this formula.

From expression (3.49), we note that the price is a function of time and hazard rate. It is easy to see that economic regimes do not affect the price which instead strongly depends on the risk aversion coefficient and the risk-free interest rate. In particular, it is easy to see that the indifference price increases as risk aversion increases and, at the same time, it decreases as long as the interest rate increases. Instead, the dependence on the mortality intensity $\lambda$ is not explicit in the indifference price: thus we perform a sensitivity analysis in order to analyze numerically the impact of the hazard rate on the price. First of all, we study the effect of changing initial mortality rate on the indifference price charged at the beginning of the trading period.

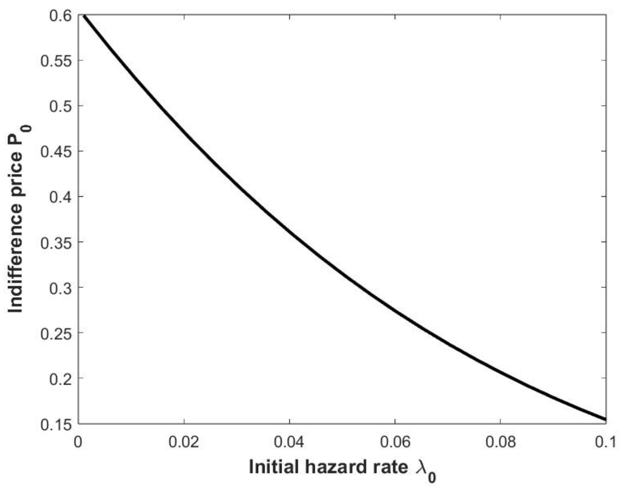



Figure 3.4: The effect of the hazard rate on the indifference price at time $t = 0$.

Figure 3.4 shows the behavior of the indifference price $P_0$ charged by the company at the initial time $t = 0$, with respect to the initial hazard rate $\lambda_0$. We observe that when the mortality intensity is low its premium is greater. In other words, larger force of mortality decreases the indifference price, as it is reasonable to expect for such type of life insurance treaties. This is consistent with common intuition as an endowment payout is less likely, under an higher mortality rate.

Finally, we investigate the evolution of the indifference price over the time. For the sake of simplicity, we assume a constant mortality rate (such as in some numerical experiments of Moore and Young [80]). In this framework, we calculate the indifference premium related to a pure endowment policy for our insurance company endowed with exponential utilities preferences.

In Figure 3.5, we plot the indifference price as a function of time to maturity, for three different constant hazard rates: $\lambda = 0.01$ (solid line), $\lambda = 0.05$ (dashed line) and $\lambda = 0.1$ (dot line).

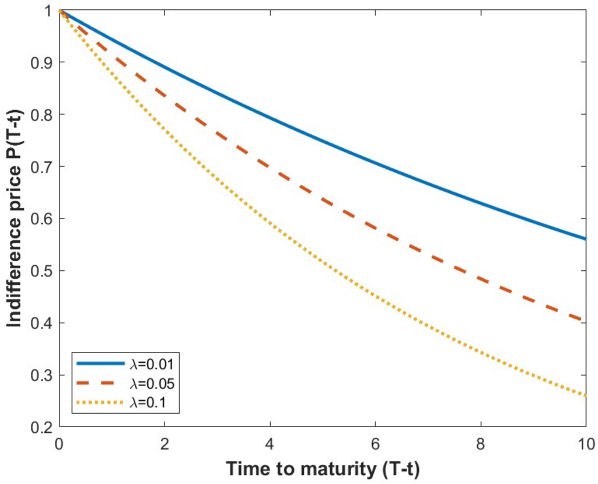



Figure 3.5: The effect of the hazard rate on the indifference price for several different deferral periods.

It is evident that the higher is the hazard rate, the lower is the indifference price for a pure endowment policy, whether the market is in a good regime or not; in other terms, the price is more sensitive to variations of deferral periods, when the population mortality intensity is more pronounced. Moreover, it is worth emphasizing that the indifference price is a decreasing function of time of maturity, namely the premium is bigger for shorter deferral periods, as usually happens.

# Conclusions

This thesis addresses a few optimization problems related to insurance, analyzing three different situations often faced by an insurance company.

Firstly, we use the forward performance criteria in the well known optimal investment and reinsurance problem. Considering forward utilities to describe the behavior of an insurance company allows for a significant flexibility in incorporating changing market opportunities and agents' attitudes in a dynamically consistent manner. Our setting allows for mutual dependence between the insurance and the financial markets. We construct a class of forward dynamic exponential utilities which are obtained by penalizing the standard utility function with a stochastic process that accounts for the riskiness related to insurance claims which are not covered by reinsurance, and the financial market, e.g. via the Sharpe ratio. Consequently, we solve the corresponding utility maximization problem using the Bellman optimality principle. We further characterize the corresponding optimal portfolio strategy and the optimal level of proportional reinsurance. This approach allows us to obtain the value function in closed form and consequently, to characterize explicitly the optimal portfolio strategy and the reinsurance. Although the approach of forward utilities is, nowadays, well known in the literature in the purely financial framework, the major contribution of this part of the thesis is to extend the analysis to the reinsurance problem, which is not a trivial extension of the existing literature, in particular in presence of a common factor process. Finally we discuss a dynamic version of the certainty equivalence for forward performance criteria and provide a comparison with the standard backward setting, along the optimal investment-reinsurance strategies. This study of the optimal investment and reinsurance problem in the forward setting can be completed by including stochastic risk tolerance as, e.g., in Žitković [67], and by considering for example the pricing of the reinsurance contract under the *indifference pricing approach*. This is left to future work.

In our second problem, we consider other two standard optimization settings in actuarial science, under the constraint that the terminal surplus at a deterministic and finite time $T$ follows a given distribution. This approach has the important advantage to being able to compute risk measures which are typically based on the distribution of the surplus at a future date, and hence to compute the capital required by, e.g. Solvency regulations. We consider first the case where the reinsurance is allowed to pay dividends. Here the optimal strategy depends on the objective function, that is either the value of expected discounted dividends (to be maximized) or the ruin probability (to be minimized). In both cases the main result states that the optimal strategy is decided at the initial time. We prove that the optimal strategy in both cases should be decided at time zero and that the strategy leading to the maximal discounted dividend value starts with high payments in the very beginning and decreases approaching the time horizon, whereas the strategy minimizing the ruin probability behaves in an opposite way. Since the dividend strategy acts solely on the drift of the wealth, we can compare different strategies path by path and thus we can consider updates at discrete points in time and continuously in time. Next, we analyze the ruin minimization problem for a company that purchases a reinsurance contract for a pool of insured or a branch of business and aims to achieve a target terminal distribution. Since reinsurance controls affect both the drift and the volatility of the wealth process, a pathwise comparison is not possible anymore and an optimal strategy may not even exists under certain scenarios. In the two-period case, we are still able to obtain an explicit solution with probabilistic methods. However, the problem becomes immediately more complicated when we increase the number of periods, even if we restrict to deterministic strategies. We plan to work on the $n$-period model for the reinsurance setting and we will also generalize to the continuous time model, although the two period case is the most realistic from a practical viewpoint, since reinsurance contracts are usually difficult to be updated before maturity.

The third project concerns the indifference pricing of mortality contingent claims in a stochastic factor model accounting for a market behavior affected by long-term macroeconomic conditions described by a continuous time Markov chain, possible jumps in the risky asset price dynamics and stochastic hazard rate. We prove verification results for the value functions of the problems with and without the insurance liability, via classical solutions to a linear PDE and a system of ODEs by applying the classical stochastic control approach based on the HJB equation. In addition, we provide some characterizations of the indifference price of a pure endowment: via a classical solution to a linear PDE, as the solution of a final value problem and in terms of its probabilistic representation by means of an extension of the Feynman-Kac formula. We also generalize these results, characterizing the indifference price for a group of insurance contracts and for a term life insurance. Finally, we perform a sensitivity analysis in case of a two-state Markov chain to highlight some interesting features of the indifference price, e.g. under higher mortality, the indifference premium of such type of contract is lower, as it is reasonable since an endowment payout is less likely in that circumstance. The evaluation of more complex insurance products, such as equity-linked policies, possibly under different utility preferences, is left for future research.

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