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Notation

$\mathbb{F}G$	group algebra of G over the field $\mathbb{F}$	Definition 2.1.1
$\mathbb{F}_G$	trivial $\mathbb{F}G$-module	Section 2.1
$\mathrm{Irr}(G)$	set of irreducible characters of G	
$\mathrm{cd}(G)$	set of character degrees of the irreducible characters of G	
$\mathrm{Lin}(G)$	set of linear characters of G	
$\mathbb{1}_G$	trivial character of G	
$[,]$	inner product of characters	Definition 2.1.5
$\chi\big\uparrow^G/\chi\big\downarrow_G$	induction/restriction of a character χ to the group G	Section2.1
$\mathscr{C}(n)$	set of compositions of n	Section 2.2.1
$\mathscr{P}(n)$	set of partitions of n	
$[\lambda]$	the Young diagram of a partition λ	
λ'	conjugate partition of λ	
$\mathscr{H}(n)$	set of hook partitions of n	
$\mathfrak{S}_\lambda$	Young subgroup of $\mathfrak{S}_n$ associated to the partition λ	Section 2.2.2
M^λ	Young permutation module labelled by λ	
S^λ	Specht module associated to λ	
χ^λ	ordinary character afforded by S^λ	
D^λ	simple module labelled by λ	
Y^λ	Young module labelled by λ	
H_b	hook on the box b of the Young diagram	Section 2.2.3
$\mathscr{LR}(\lambda;\mu,\nu)$	Litllewood-Richardson coefficients	Theorem 2.2.6
$\mathscr{B}_n(t)$	set of partitions of n inside a $t\times t$ grid	Section 2.2.3
$\star$	operation between subset of partitions	
$\mathscr{X}(\phi;\theta)$	special character of the wreath product	
P_n	Sylow p-subgroup of $\mathfrak{S}_n$	Section 2.3.1
g_j	a generator of P_n	(2.2)
$\sigma_j^{(k)}$	a generator of the normalizer $N_{\mathfrak{S}_n}(P_n)$	Section 2.3.1
$\lambda_{(p)}$	p-core of λ	Section 2.4.1
$w_p(\lambda)$	p-weight of λ	
Z_ϕ^χ	Sylow branching coefficient of ϕ in χ	Definition 3.0.1
Ω_n^k		(4.1)
α_n	upper bound for $\mathrm{cd}(P_n)$	Section 4.1
$\mathrm{Irr}_k(G)$	irreducible characters of G of degree p^k	start of Chapter 4
$\delta_H(\theta)$	maximum degree of a costituent of $\theta\big\downarrow_H$	Definition 4.1.5
$\mathrm{Irr}_{\mathscr{H}}(\mathfrak{S}_n)$	irreducible characters of $\mathfrak{S}_n$ labelled by hook partitions	start of Chapter 5
Λ_n^k	equivalent of Ω_n^k for $p=2$ and hook partitions	Section 5.2
$\mathfrak{H}_n(t)$	equivalent of $\mathscr{B}_n(t)$ for hook partitions	
β_n	equivalent of α_n when $p=2$	Definition 5.2.2
$\diamond$	equivalent of $\star$ for hook partitions	Definition 5.2.6
A_P	set of partitions with same associated Sylow p-subgroup	Section 6.3.1
λ_P	dominant partition in A_P	Definition 6.3.6

Contents

Chapter 1

Introduction and structure of the thesis

Let G be a finite group, $\mathbb{F}$ a field and n a natural number. A *representation* of G is an homomorphism $\rho : G \to \mathrm{GL}_n(\mathbb{F})$ from G to the group $\mathrm{GL}_n(\mathbb{F})$ of invertible linear maps over an $\mathbb{F}$-vector space of dimension n. Such an homomorphism transfers the structure of G to a group of matrices, which can be investigated using linear algebra tools.

The study of the properties of a finite group can be quite challenging, but it can be simplified if we have more manageable instruments. This is the purpose of defining representations.

A representation of a finite group G defines a module for the group algebra $\mathbb{F}G$. Actually, there is a one-to-one correspondence between representations and modules, and it is equivalent studying ones rather than the others. Dealing with modules is expecially useful when the field $\mathbb{F}$ has positive characteristic. Indeed, basic instruments of ring theory are easier to handle than modular linear algebra.

In the case of $\mathbb{F}$ being the field of complex numbers $\mathbb{C}$, instead, we can associate to a representation ρ of G its *ordinary character*. This is defined as the map from G to $\mathbb{C}$ that maps an element g of G to the trace of the matrix $\rho(g)$. Hence the simplification is clear: we are dealing with maps $G \to \mathbb{C}$, so we do not even need the linear algebra.

A character is *irreducible* if it comes from a representation (or a module) that is the simplest we can build. The idea is that every character is made by combination of smaller pieces, which are the irreducible character.

We refer to Chapter 2 for a formal definition, but to be more precise we can say that every ordinary character of a finite group can be uniquely expressed as a sum of irreducible characters. This is a consequence of Maschke's theorem (Theorem 2.1.2). Hence we can restrict our investigation to the set of all irreducible characters of G, denoted by $\mathrm{Irr}(G)$.

One of the main reasons for studying characters is that many structural properties of a group can be stated in terms of its characters. For instance, we know that a finite group is abelian if and only if every irreducible character is linear (i.e. those of degree one). Another example is the Itô-Michler theorem [40, 58]. This asserts that a Sylow p-subgroup P of a finite group G is abelian and normal in G if, and only if, the character degree $\chi(1)$ is coprime to p for every irreducible character $\chi \in \mathrm{Irr}(G)$. We refer the reader to [37] for an extensive dissertation on the topic and many other examples.

We remark that sometimes properties that can be inferred from characters can be proved using classical group theory. Usually, however, a character theory proof uses clearer and more direct arguments.

We are mainly interested in this thesis in the representation theory of *symmetric groups*.

Let n be a natural number. The symmetric group, denoted $\mathfrak{S}_n$, is the group of permutations of n elements. Symmetric groups appear in many branches of mathematics and are central in finite group theory. Cayley's theorem is just one of many examples. It says that every finite group can be embedded into $\mathfrak{S}_n$ for an appropriate choice of the natural number n. Moreover, the symmetric groups provide evidence for some important conjectures, as the Alperin Weight conjecture and the McKay conjecture (see Section 3.2.1 below).

The representation theory of $\mathfrak{S}_n$ is studied through its relation with the *partitions* of n. These are decreasing sequences of natural numbers whose sum is n.

We can associate to each partition λ of n an $\mathbb{F}\mathfrak{S}_n$-module S^λ. We call this module *Specht module* and its construction is independent of the choice of the field $\mathbb{F}$.

When $\mathbb{F} = \mathbb{C}$, the Specht modules form a complete set of non-isomorphic simple $\mathbb{C}\mathfrak{S}_n$-modules. Hence we can associate to each of them an ordinary irreducible character of the symmetric group. Therefore we have a bijection from $\mathrm{Irr}(\mathfrak{S}_n)$ and the set of the partitions of n. This correspondence allows us to explicitly compute, for instance, the degree of an irreducible character or its restriction to smaller subgroups of $\mathfrak{S}_n$. In **Chapter 2**, we describe in full details some of these useful combinatorial tools.

In general, we can detect properties of a finite group using information about its 'local' subgroups, such as the Sylow p-subgroups. This interplay was one of the key ingredient to obtain the Classification of Finite Simple Groups and many other classical results. For instance, the Itô-Michler theorem mentioned above and the Brauer's Height Zero conjecture. The latter has been proved in [56] and it is an important example of the so-called local-global conjectures. We refer the reader to [54] or [62] for detailed account.

In this thesis we focus on the relationship between the ordinary characters of the symmetric groups and those of their Sylow p-subgroups.

Let G be a finite group and P be a Sylow p-subgroup of G. To link the two sets $\mathrm{Irr}(G)$ and $\mathrm{Irr}(P)$, we want to describe the restriction of an irreducible character χ of G to P. We denote this restriction by $\chi\big\downarrow_P$. The *Sylow branching coefficient* Z^χ_ϕ is the multiplicity of an irreducible character ϕ of P as a constituent of $\chi\big\downarrow_P$. Hence describing the restriction of a character to a Sylow p-subgroup means computing all the Sylow branching coefficients.

In **Chapter 3** we collect the results obtained in the study of Sylow branching coefficients, with focus on the case of G being a symmetric group.

Let P_n be a Sylow p-subgroup of the symmetric group $\mathfrak{S}_n$. In [24] and [25] the authors studied the linear constituents of the restriction $\chi\big\downarrow_{P_n}$, for $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$. Few results have been obtained instead for the description of higher-degree constituents.

In **Chapter 4** we study the set of all the irreducible characters of $\mathfrak{S}_n$ which admit a constituent of degree greater than one in their restriction to P_n. More precisely, for every natural number k we define the set

$$\Omega^k_n = \{\chi \in \mathrm{Irr}(\mathfrak{S}_n) \mid \chi\big\downarrow_{P_n} \text{ admits an irreducible constituent of degree } p^k\}.$$

The first main result of this thesis is:

We completely describe the set Ω_n^k for every choice of n, $k \in \mathbb{N}$ and an odd prime p.

We will see that Ω_n^k has a regular structure given by the combinatorics of the partitions of n.

The situation for $p = 2$ is more complicated. Hence we need to focus our attention on the irreducible characters labelled by the so-called *hook partitions*. The first result in **Chapter 5** is:

We explicitly compute some specific Sylow branching coefficients for hook partitions when $p = 2$.

After that, we study the set Λ_n^k of all the hook characters of $\mathfrak{S}_n$ which admit a constituent of degree 2^k in their restriction to a Sylow 2-subgroup. (So, Λ_n^k is the intersection of Ω_n^k and the set of all irreducible characters of $\mathfrak{S}_n$ labelled by a hook partition.)

The structure of Λ_n^k is not as nice as the one of Ω_n^k. However, another result of this chapter is:

We describe the set Λ_n^k for every choice of n, $k \in \mathbb{N}$ and $p = 2$.

In **Chapter 6** we turn to study the modular representation theory of symmetric groups. This means that $\mathbb{F}$ is a field of positive characteristic. As we already mentioned, in this setting we investigate families of $\mathbb{F}\mathfrak{S}_n$-modules. For instance, one of these family is the one of the Specht modules.

We start the chapter with a short survey on the *vertices* of indecomposable modules. These are important invariants that help to understand the structure of the modules.

Then, considering some trivial source modules, we study Scott modules and *Young modules*. The latter are the indecomposable summands of the so-called Young permutation modules. We relate these two families of modules and we obtain the last main result of this thesis:

We determine which Young modules are isomorphic to a Scott module.

Chapter 2

Preliminaries

In this chapter we summarize some results that will be useful throught this thesis.

2.1 Basics on representation theory of finite groups

We suppose that the reader is familiar with some basic ring theory. If not, we refer to the first chapters of [44] and [37] for a brief collection of the needed basics, but also for a more extended dissertation about the representation theory that we are about to present.

Some definitions and results in this chapter could have been stated in a more general setting. However the aim of this thesis is the study of finite groups, hence we specialize only in the case of a group algebra.

Definition 2.1.1. Let $\mathbb{F}$ be a field and let G be a finite group. The vector space consisting in the formal sums $\{\sum_{g \in G} a_g g \mid a_g \in \mathbb{F}\}$ with the product of the group extended linearly is an $\mathbb{F}$-algebra. This is the *group algebra* of G over the field $\mathbb{F}$, and it is denoted $\mathbb{F}G$.

Note that G is an $\mathbb{F}$-basis of $\mathbb{F}G$.

An $\mathbb{F}G$*-module* is a finite dimensional $\mathbb{F}$-vector space M for which it is uniquely defined $\mathbb{F}G \times M \to M$ that maps every $(x, m) \in \mathbb{F}G \times M$ to an element $xm \in M$. Also the followings hold for all $x, y \in \mathbb{F}G$, $m, n \in M$ and $a \in \mathbb{F}$:

$$x(m+n) = xm + xn, \quad (x+y)m = xm + ym,$$
$$(xy)m = x(ym), \quad (ax)m = a(xm) = x(am), \quad 1m = m.$$

The trivial $\mathbb{F}G$-module is denoted by $\mathbb{F}_G$ or simply $\mathbb{F}$. $\mathbb{F}G$ itself is an $\mathbb{F}G$-module under multiplication and it is called the *regular* module.

If M is an $\mathbb{F}G$-module and $N \subseteq M$ is an $\mathbb{F}G$-invariant subspace, then N is a *submodule* of M. Let M be a nonzero $\mathbb{F}G$-module, then M is *simple* if its only submodules are 0 and M. M is *indecomposable* if it cannot be written as a direct sum of two non-zero submodules.

Theorem 2.1.2 (Maschke's theorem)**.** *Let G be a finite group and $\mathbb{F}$ be a field whose characteristic does not divide the order of G. Then every $\mathbb{F}G$-module decomposes in an essentially unique way as the direct sum of its simple submodules.*

This applies in particular in characteristic zero, and it reduces representation theory to the study of simple $\mathbb{F}G$-modules. When this is not possible, we usually restrict to consider the indecomposable modules.

Let $M_n(\mathbb{F})$ be the algebra of $n \times n$ matrices over the field $\mathbb{F}$. A *representation* of $\mathbb{F}G$ is an algebra homomorphism $\mathscr{R} : \mathbb{F}G \to M_n(\mathbb{F})$. The integer n is the *degree* of $\mathscr{R}$.

Two representations $\mathscr{R}, \mathscr{S}$ of degree n are *similar* if there exists a nonsingular $n \times n$ matrix P, such that $\mathscr{R}(x) = P^{-1}\mathscr{S}(x)P$ for all $x \in \mathbb{F}G$. Similarity is an equivalence relation among representations.

Representations are a different ways of looking at modules. Indeed, there is a natural one-to-one correspondence between isomorphism classes of $\mathbb{F}G$-modules and similarity classes of representations of $\mathbb{F}G$. If $a \in \mathbb{F}G$ and M is an $\mathbb{F}G$-module, we let $a_M : M \to M$ be the M-endomorphism defined as $m \mapsto am$. Let M be an $\mathbb{F}G$-module and choose an $\mathbb{F}$-basis for M such that $R(a)$ is the matrix of a_M with respect to this basis, then R is a representation. If instead R is a representation of degree n , let V be the n-dimensional row vector space over $\mathbb{F}$. Define $av := R(a)v$ for every $v \in V, a \in \mathbb{F}G$: this gives a structure of $\mathbb{F}G$-module to V. Since a module isomorphism corresponds to a nonsingular matrix for every chosen basis, then isomorphic $\mathbb{F}G$-modules correspond to similar representations.

If M is an $\mathbb{F}G$-module and $U \subset M$ is a proper non-zero submodule of M, choose a basis b_U for U of dimension m and extend it to b_M, a basis of M. Reorder b_M so that the last m vectors are b_U. Let $\mathscr{R}$ be the representation of $\mathbb{F}G$ corresponding to M with respect to b_M and let $\mathscr{S}$ be the representation of $\mathbb{F}G$ corresponding to U. Then, for $a \in \mathbb{F}G$,

$$\mathscr{R}(a) = \begin{pmatrix} \mathscr{T}(a) & \mathscr{U}(a) \\ 0 & \mathscr{S}(a) \end{pmatrix}.$$

We say that $\mathscr{R}$ is *reducible* if it can be represented in this form, it is *irreducible* otherwise. Therefore irreducible representations correspond to simple modules.

Let $GL(n, \mathbb{F})$ be the general linear group of nonsingular $n \times n$ matrices over the field $\mathbb{F}$.

Definition 2.1.3. Let $\mathbb{F}$ be a field and G a group. Then an $\mathbb{F}$-*representation* of G is a group homomorphism $R : G \to GL(n, \mathbb{F})$ for some integer n.

A representation of $\mathbb{F}G$ determines an $\mathbb{F}$-representation of G by restriction. Conversely, an $\mathbb{F}$-representation of G determines a representation of $\mathbb{F}G$ by linear extension.

Definition 2.1.4. Let R be an $\mathbb{F}$-representation of G. Then the $\mathbb{F}$-*character* χ of G afforded by R is the function given by $\chi(g) = \mathrm{tr}R(g)$, where tr denotes the trace of a matrix.

Notice that characters are constant on the conjugacy classes of a group. Moreover, similar representations afford equal character.

We say that a character is *irreducible* if it is afforded by an irreducible representations.

We want to remark the deep connections between modules, representations and characters: isomorphism classes of $\mathbb{F}G$-modules corresponds to similarity classes of representations, and they corresponds to the same character. Also, if the starting module is simple, the representation and the afforded character are irreducible.

Ordinary representation theory

If $\mathbb{F}$ is the field of complex numbers $\mathbb{C}$, a $\mathbb{C}$-character is called an *ordinary* character. We denote by $\mathrm{Char}(G)$ the set of the ordinary characters of a finite group G, and by $\mathrm{Irr}(G)$ the set of its irreducible characters

We say that $\chi(1)$ is the *degree* of the character χ. It is actually the degree of a representation that affors χ, and also the dimension of the corresponding $\mathbb{F}G$-module. We denote by $\mathrm{cd}(G)$ the set of the character degree of the irreducible character of G: $\mathrm{cd}(G) = \{ \chi(1) \mid \chi \in \mathrm{Irr}(G) \}$.

Characters of degree one are called *linear* characters. The set of the linear characters of G is denoted by $\mathrm{Lin}(G)$. Notice that linear characters are irreducible.

The character that corresponds to the trivial module $\mathbb{C}_G$ is the function with constant value 1 on G. It is denoted $\mathbb{1}_G$ and it is called *trivial* character.

Notice that the sum of characters is a character: let R and S be two representations of G that afford the characters χ and ϕ respectively. Then the rapresentation T of G defined as

$$T(g) = \begin{pmatrix} R(g) & 0 \\ 0 & S(g) \end{pmatrix}$$

for all $g \in G$, is the one that affords $\chi + \phi$ as character. Also, if R and S correspond to the $\mathbb{C}G$-modules U and V respectively, then the representation T corresponds to the direct sum $U \oplus V$.

Since the characteristic of $\mathbb{C}$ is zero, it does not divide the order of any group. Hence by Maschke's theorem 2.1.2, every $\mathbb{C}G$-module decomposes as the direct sum of simple modules. Then characters decompose as the sum of irreducible characters. If $\chi = \sum_{\phi \in \mathrm{Irr}(G)} n_\phi \phi$ is a character of G, then those ϕ with $n_\phi > 0$ are called the *constituents* of χ.

We call *regular* character the one afforded by a representation corresponding to the regular $\mathbb{F}G$-module. This is the sum of all the irreducible characters of G taken with their multiplicities.

We will need later to describe the irreducible characters of direct products.

Let $G = H \times K$ and let ϕ and ψ be characters of H and K, respectively. Define $\chi = \phi \times \psi$ by $\chi(hk) = \phi(h)\psi(k)$ for $h \in H$ and $k \in K$.

This definition gives actually a character χ of G. Moreover,

$$\mathrm{Irr}(G) = \{ \phi \times \psi \mid \phi \in \mathrm{Irr}(H),\ \psi \in \mathrm{Irr}(K) \}.$$

This argument can be iterated for the direct products of arbitrary many finite groups.

Definition 2.1.5. Let χ, ϕ be characters of G. Then

$$[\chi, \phi] = \frac{1}{|G|} \sum_{g \in G} \chi(g)\phi(\bar{g})$$

is the *inner product* of χ and ϕ in G.

We sometimes write $[\chi, \phi]_G$ instead $[\chi, \phi]$ if we need to stress that $\chi, \phi \in \mathrm{Char}(G)$.

We collect here some basic properties of the inner product:

1. $[\chi, \phi] = [\phi, \chi]$;
2. $[c_1\chi_1 + c_2\chi_2, \phi] = c_1[\chi_1, \phi] + c_2[\chi_2, \phi]$;
3. $[\phi, \phi] > 0$ unless $\phi = 0$;
4. χ is irreducible if and only if $[\chi, \chi] = 1$.

$\mathrm{Irr}(G)$ is an orthonormal basis for $\mathrm{Char}(G)$, hence we have a method for expressing an arbitrary character in terms of the irreducible ones: let $\phi \in \mathrm{Char}(G)$, then

$$\phi = \sum_{\theta \in \mathrm{Irr}(G)} [\phi, \theta]\ \theta.$$

Let H be a subgroup of the finite group G. We would like to relate their representation theory.

$\mathbb{C}H$ is a subset of $\mathbb{C}G$. If M is a $\mathbb{C}G$-module, then M is also a $\mathbb{C}H$-module since the properties of the definitions hold for all elements in G, hence certainly hold for the elements in H. We write the corresponding $\mathbb{C}H$ module as $M{\downarrow}_H$, and call it the *restriction* of M to H. The character of the restriction is obtained from the character χ of M by evaluating it on the elements of H only. We write this character of H as $\chi{\downarrow}_H$.

If instead we have information on the representations of H and we want to say something about the representations of G, the process is more complicated than the previous one. For any subset X of $\mathbb{C}G$, we write $X(\mathbb{C}G)$ for the subspace of $\mathbb{C}G$ which is spanned by all the elements gx with $x \in X$, $g \in G$. That is,

$$X(\mathbb{C}G) = \langle gx \mid x \in X,\ g \in G \rangle.$$

$X(\mathbb{C}G)$ is then a submodule of $\mathbb{C}G$.

Now $\mathbb{C}H$ is a subset of $\mathbb{C}G$. Let U be a $\mathbb{C}H$-submodule of $\mathbb{C}H$, and let $U{\uparrow}^G$ denote the $\mathbb{C}G$-module $U(\mathbb{C}G)$. Then $U{\uparrow}^G$ is called the $\mathbb{C}G$-module *induced* from U.

If ϕ is the character that corresponds to U, then the character that corresponds to $U{\uparrow}^G$ is denoted by $\phi{\uparrow}^G$ and is called the character *induced* from ϕ.

Let ϕ be a character of H. Define the function $\dot{\phi} : G \to \mathbb{C}$ by

$$\dot{\phi}(g) = \begin{cases} \phi(g) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

The values of the induced character $\phi{\uparrow}^G$ are given by

$$(\phi{\uparrow}^G)(g) = \frac{1}{|H|} \sum_{y \in G} \dot{\phi}(y^{-1}gy)$$

for all $g \in G$.

We collect here some properties of the restriction and the induction that will be useful later.

Lemma 2.1.6. *Let χ be a character of G and ϕ be a character of $H \leq G$.*

1. *The degree $\chi\big\downarrow_H(1)$ is equal to $\chi(1)$, while $(\phi\big\uparrow^G)(1) = \dfrac{|G|}{|H|}\phi(1)$.*
2. *The restriction of an irreducible character is not necessarely irreducible. The same holds for the induction.*
3. *Let $K \leq H \leq G$ and let $\theta \in \mathrm{Char}(K)$. Then $\chi\big\downarrow_H\big\downarrow_K = \chi\big\downarrow_K$ and $\theta\big\uparrow^H\big\uparrow^G = \theta\big\uparrow^G$.*

Restriction and induction can be considered as dual operations in the sense of the Frobenius reciprocity.

Theorem 2.1.7 (The Frobenius reciprocity)**.** *Let $H \leq G$. Let χ be a character of G and let ϕ be a character of H. Then*

$$[\phi\big\uparrow^G, \chi]_G = [\phi, \chi\big\downarrow_H]_H.$$

Suppose now that N is a normal subgroup of a finite group G. Let $\theta \in \mathrm{Irr}(N)$ and $g \in G$. We define the *conjugate* character $\theta^g : N \to \mathbb{C}$ by $\theta^g(m) = \theta(g^{-1}mg)$. This definition gives a character of N. Hence by this action G permutes the irreducible characters of N, while N acts trivially.

Clifford theory relates the representation theory of a finite group and one of its normal subgroups. We state here Clifford theorem and we refer the reader to [37, Chapter 6] for the modular version of this theorem and its many important consequences.

Theorem 2.1.8 (Clifford theorem, [37, (6.2)])**.** *Let N be a normal subgroup of a finite group G and let $\chi \in \mathrm{Irr}(G)$. Let θ be an irreducible constituent of $\chi\big\downarrow_N$ and suppose $\theta = \theta_1, \theta_2, \ldots, \theta_t$ are distinct conjugates of θ in G. Then*

$$\chi\big\downarrow_N = e\sum_{i=1}^{t}\theta_i,$$

where $e = \left[\chi\big\downarrow_N, \theta\right]_N$.

Let $\mathbb{F}$ be a field. We end this section describing an important family of $\mathbb{F}G$-modules.

Let M be an $\mathbb{F}G$-module and suppose that $G \in \mathbb{F}G$ acts on M permuting an $\mathbb{F}$-basis B of M. Then M is called an *$\mathbb{F}G$-permutation module.* The value of its associated *permutation character* χ on $g \in G$ is $\chi(g) = |\{ b \in B \mid b^g = b \}|$.

Denote by G_m the stabilizer of the element $m \in M$ under the action of G. If G acts transitively on the $\mathbb{F}$-basis B of M, then the permutation character of the action is $\mathbb{1}_{G_m}\big\uparrow^G$.

2.2 The representation theory of the symmetric groups

In this section we collect some main results about representation theory of the symmetric groups. We refer the reader to [42] and [65] for more details and examples about the items that we will briefly introduce here.

2.2.1 Combinatorics of partitions

We let $\mathscr{C}(n)$ be the set of compositions of n, i.e. the set consisting of all the finite sequences $a = (a_1, a_2, \ldots, a_z)$ such that a_i is a non-negative integer for all $i \in [1, z]$ and such that $a_1 + \cdots + a_z = n$. The a_i are known as the *parts* of a.

If $\lambda \in \mathscr{C}(n)$ has strictly positive and non-increasingly ordered parts, we say that λ is a *partition* of n. We denote by $\mathscr{P}(n)$ the set of partitions of n.

We sometimes write $\lambda \vdash n$ or $|\lambda| = n$ to say $\lambda \in \mathscr{P}(n)$.

We remark that the set $\mathscr{P}(0)$ consists of only one element, that is the empty partition.

Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathscr{P}(n)$. The *length* of λ is the number of parts of λ (i.e. k), often written $l(\lambda)$. For convenience, we may use exponents to indicate multiplicities of parts; the meaning should always be clear from context. For instance, $(2, 1, 1, 1) = (2; 1^3)$, while (p^k) could denote a single part of size p^k or $(p, \ldots, p)$ where the part p appears k times, and we interpret this based on context by specifying $(p^k) \in \mathscr{P}(p^k)$ or $(p^k) \in \mathscr{P}(kp)$ respectively.

We define λ_i for $\lambda \in \mathscr{P}(n)$ and $i \in \mathbb{N}$ as follows:

$$\lambda_i := \begin{cases} \text{the } i\text{-th part of } \lambda, & \text{if } i \leq l(\lambda); \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathscr{P} = \cup_{n \in \mathbb{N}} \mathscr{P}(n)$ be the set of all partitions. Let $a \in \mathbb{N}$, $\lambda, \mu \in \mathscr{P}$. The scalar multiplication and the sum of two partitions are defined as follows:

$$a\lambda := (a\lambda_1, a\lambda_2, \ldots, a\lambda_{l(\lambda)});$$
$$\lambda + \mu := (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots, \lambda_k + \mu_k),$$

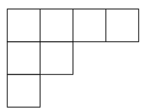
where $k = \max\{\, l(\mu), l(\lambda) \,\}$.

The *dominance ordering* on $\mathscr{P}$ is a partial order defined by

$$\mu = (\mu_1, \ldots, \mu_k) \trianglelefteq \lambda = (\lambda_1, \ldots, \lambda_k) \iff \sum_{i=1}^{m} \mu_i \leq \sum_{i=1}^{m} \lambda_i \text{ for every } m \in [1, k],$$

where again $k = \max\{\, l(\mu), l(\lambda) \,\}$. We use here the same notation as in [42, Definition 3.2].

The *Young diagram* $[\lambda]$ is an array of boxes, left aligned, having λ_j boxes in the j-th row for all $j \in [1, l(\lambda)]$. For example,



is the Young diagram of the partition $(4, 2, 1)$.

Given a partition λ, its *conjugate partition* is the partition λ' whose Young diagram is obtained by reflecting $[\lambda]$ through the main diagonal $y = -x$. In the example above, $\lambda' = (3, 2, 1, 1)$.

A λ-*tableau* t is an assignment of the numbers $\{\, 1, 2, \ldots, n \,\}$ to the boxes of $[\lambda]$ such that no number appears twice. We will denote by $t(i, j)$ the number assigned to the box of t in row i and column j. The symmetric group $\mathfrak{S}_n$ acts naturally on the set of λ-tableaux by permuting the entries within the boxes. We denote the action of $\sigma \in \mathfrak{S}_n$ on the λ-tableaux t by $\sigma(t)$.

We say that a λ-tableau is *standard* when it has the entries of each row ordered increasingly from left to right and the entries of each column ordered increasingly from top to bottom.

We conclude this section by defining a special subclass of partitions of a natural number n that we will consider extensively in the rest of the thesis, in particular in Chapter 5. Let $0 \leq k < n$ and let $\lambda = (n-k, 1, \ldots, 1) = (n-k, 1^k) \in \mathscr{P}(n)$. Then λ is called a *hook* or *hook partition*, since its Young diagram $[\lambda]$ has the shape of a hook. We denote by $\mathscr{H}(n) = \{ \lambda \in \mathscr{P}(n) \mid \lambda \text{ is a hook} \}$ the set of hook partitions of n.

2.2.2 Specht modules

We turn now to a brief summary of the theory of Specht modules, that will allow us to explicitly describe the irreducible characters of $\mathfrak{S}_n$. We refer the reader to [42, Section 4] for further details.

Let $\mathbb{F}$ be a field. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n we denote by $\mathfrak{S}_\lambda$ the subgroup of $\mathfrak{S}_n$ defined by

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k}.$$

$\mathfrak{S}_\lambda$ is the *Young subgroup* of $\mathfrak{S}_n$ associated to the partition λ.

We say that two λ-tableaux t and u are *row-equivalent* if the entries in each row of t are the same as the entries in the corresponding row of u. It is easy to see that this defines an equivalence relation on the set of λ-tableaux. We will denote by $\{t\}$ the row-equivalence class of t and we will say that $\{t\}$ is a λ-*tabloid*.

The symmetric group $\mathfrak{S}_n$ acts naturally on the set of λ-tabloids: let $\{t\}$ be a λ-tabloid and $\sigma \in \mathfrak{S}_n$, then $\sigma(\{t\}) = \{\sigma(t)\}$. Therefore we can define M^λ to be the $\mathbb{F}\mathfrak{S}_n$-permutation module generated as a vector space by the set of all λ-tabloids. The module M^λ is called *Young permutation module* and it has dimension equal to $\dfrac{n!}{\prod_{i\geq 1}(\lambda_i)!}$.

Since $\mathfrak{S}_n$ acts transitively on the set of λ-tabloids and since the stabilizer in $\mathfrak{S}_n$ of a fixed λ-tabloid $\{t\}$ is isomorphic to the Young subgroup $\mathfrak{S}_\lambda$, we have the following important isomorphism of $\mathbb{F}\mathfrak{S}_n$-modules:

$$M^\lambda \cong \mathbb{F}_{\mathfrak{S}_\lambda}\big\uparrow^{\mathfrak{S}_n}, \text{ for all } \lambda \in \mathscr{P}(n),$$

where $\mathbb{F}_{\mathfrak{S}_\lambda}$ denotes the trivial $\mathbb{F}\mathfrak{S}_\lambda$-module.

Given any λ-tableau t we denote by $C(t)$ the column stabilizer of t, namely the subgroup of $\mathfrak{S}_n$ that fixes the columns of t setwise. The λ-polytabloid corresponding to the λ-tableau t is the following element of M^λ:

$$e_t = \sum_{g \in C(t)} \operatorname{sgn}(g)\{t\}g.$$

The Specht module S^λ is the submodule of M^λ linearly generated by the polytabloids.

We will say that e_t is a *standard* λ-polytabloid if t is a standard λ-tableau. The Standard Basis Theorem proved by James in [42, Theorem 8.4] says that the standard λ-polytabloids form a basis for S^λ. In particular, the dimension of the Specht module S^λ is independent of the ground field, and equals the number of standard λ-tableaux.

When considered over a field of characteristic zero, for instance the field of the complex number $\mathbb{C}$, the family of Specht modules $\{S^\lambda \mid \lambda \in \mathscr{P}(n)\}$ is a complete set of non-isomorphic simple $\mathbb{C}\mathfrak{S}_n$-modules. For every partition λ of n we denote by χ^λ the ordinary character afforded by S^λ.

We say that a partition λ of n is *p-regular* if it has at most $p-1$ parts of any given size. The following is an important result of G.D. James:

Theorem 2.2.1. *Let $\lambda \in \mathscr{P}(n)$ and let $\mathbb{F}$ be a field with positive characteristic p. Set $D^\lambda := \dfrac{S^\lambda}{S^\lambda \cap (S^\lambda)^\perp}$. Then D^λ is non-zero if and only if λ is p-regular. Moreover, $\{D^\lambda \mid \lambda \in \mathscr{P}(n)$ p-regular$\}$ is a complete set of non-isomorphic simple $\mathbb{F}\mathfrak{S}_n$-modules.*

Here $(S^\lambda)^\perp = \left\{ x \in M^\lambda \mid \langle x, s\rangle = 0 \text{ for every } s \in S^\lambda \right\}$, where $\langle \cdot, \cdot \rangle$ is the unique bilinear form on M^λ for which $\langle \{t_1\}, \{t_2\}\rangle = 1$ if $\{t_1\} = \{t_2\}$ and $\langle \{t_1\}, \{t_2\}\rangle = 0$ if $\{t_1\} \neq \{t_2\}$.

We refer the reader to [42, Chapter 11] for more details.

Young modules

We have already constructed the Young permutation modules M^λ, for λ a partition of n. A natural question is how we can decompose these modules into indecomposable summands, the so-called *Young modules*.

A first complete parametrization can be found in the work of James in [43], Klyachko in [48] and Grabmeier in [32]. While their original description was based on Schur algebras, Erdmann in [14] described completely the Young modules using only the representation theory of the symmetric groups. (The proof of Lemma 3 of [14] contains some errors. A correction to that proof was later given by Erdmann and Schroll in [15]).

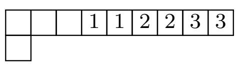
In order to state Erdmann's result we need to introduce the following definition.

Let λ be a partition of a natural number n. We say that λ is a *p-restricted* partition if the conjugate partition is p-regular. Moreover notice that if a partition λ is not p-restricted, then there exist a unique natural number r_λ and unique p-restricted partitions $\lambda(0), \lambda(1), \dots, \lambda(r_\lambda)$, such that

$$\lambda = \sum_{m=0}^{r_\lambda} \lambda(m) p^m. \tag{2.1}$$

The above expression (2.1) is called *p-adic expansion* of λ.

For instance, consider the partition $\lambda = (9,1)$ of the natural number 10 and $p = 3$. This is a 3-regular partition, but it is not 3-restricted since $\lambda' = (2, 1^8)$. We want to determine the 3-adic expansion of λ: $\lambda = \lambda(0) + 3\lambda(1) + 9\lambda(2) + \cdots$. Drawing the Young diagram of λ we can see that the only partition that can be placed three times is (2):



Therefore $\lambda(1) = (2)$, and if we remove from the Young diagram of λ three copies of (2) what remains is $\lambda(0) = (3,1)$, as we can see in the diagram above. Both (2) and $(3,1)$ are 3-restricted partitions, then $(9,1) = (3,1) + 3(2)$ is the 3-adic expansion of λ.

For each $m \in \{ 0, 1, \dots, k \}$, $\lambda(m)$ is a partition of a certain natural number which will be denoted as r_m (i.e. $\lambda(m) \vdash r_m$).

Theorem 2.2.2 ([14, Theorems 1 and 2]). *There is a set of indecomposable $\mathbb{F}\mathfrak{S}_n$-modules Y^μ one for each partition μ of n, such that the following hold for every $\lambda \vdash n$.*

(1) M^λ is isomorphic to a direct sum of Young modules Y^μ with $\mu \trianglerighteq \lambda$ and with Y^λ appearing exactly once.

(2) $Y^\lambda \cong Y^\mu$ if and only if $\lambda = \mu$.

(3) Y^λ is projective if and only if λ is p-restricted.

(4) Let λ be not p-restricted and suppose that $\lambda = \sum_{m=0}^{k} \lambda(m)p^m$ is the p-adic expansion of λ. Consider $\rho(\lambda)$ to be the partition of n which has r_m parts equal to p^m for every $m \in \{ 0, 1, \dots, k \}$. Then Y^λ has vertex a Sylow p-subgroup of $\mathfrak{S}_{\rho(\lambda)}$.

We refer to Section 2.4 for the definition of vertex of an indecomposable module.

Going back to the previous example, we have that $(9,1) \vdash 10$ is not 3-restricted and its 3-adic expansion is $(9,1) = (3,1) + 3(2)$. We are then in the case of part (4) of the above theorem. We have $r_0 = 4$ and $r_1 = 2$. Hence $\rho(\lambda)$ has 4 parts equal to 3^0 and 2 parts equal to 3^1, that is $\rho(\lambda) = (3,3,1,1,1,1) = (3^2, 1^4)$. Therefore $\mathfrak{S}_{\rho(\lambda)} = (\mathfrak{S}_3)^{\times 2} \times (\mathfrak{S}_1)^{\times 1} \cong \mathfrak{S}_3 \times \mathfrak{S}_3$. A Sylow 3-subgroup of $\mathfrak{S}_3 \times \mathfrak{S}_3$ is then $C_3 \times C_3$, and this is a vertex of $Y^{(9,1)}$.

Consider a decomposition of M^λ into a direct sum of Young modules, as in part (1) of the previous theorem. We can notice that the Young module Y^λ is the summand containing S^λ.

2.2.3 Ordinary irreducible characters of the symmetric groups

From the discussion in Section 2.2.2, the set of the ordinary irreducible characters of the symmetric group $\mathfrak{S}_n$ is naturally in bijection with $\mathscr{P}(n)$. More precisely,

$$\mathrm{Irr}(\mathfrak{S}_n) = \left\{ \chi^\lambda \mid \lambda \in \mathscr{P}(n) \right\}.$$

Here we want to collect some tools that we will use in the following chapters.

The hook length formula

The hook length formula is a closed combinatorial formula which gives the degree of an irreducible character. Notice that the degree of χ^λ, for λ a partition of n, is the dimension of the corresponding Specht module S^λ.

We need to introduce a bit of notation in order to state the related theorem. Let b be a box of the Young diagram $[\lambda]$ of λ. We denote by H_b the *hook* on b, namely the subset of boxes of $[\lambda]$ lying either to the right (namely the arm) or below b (namely the leg), including b itself. We define the *hook length* h_b to be the number of boxes in H_b.

For instance, let $\lambda = (4,2,1)$ and $b = (1,2) \in [\lambda]$. H_b is the subset of the boxes filled by a bullet in the diagram below:

	•	•	•
	•		

Hence $h_b = 4$.

Theorem 2.2.3 (Hook length formula, [18, Theorem 1])**.** *Let λ be a partition of n, then*

$$\chi^\lambda(1) = \frac{n!}{\prod_{b \in [\lambda]} h_b}.$$

For example, let $\lambda = (4,2,1)$. Filling each box b of the Young diagram $[\lambda]$ with the corresponding hook length h_b, we obtain

6	4	2	1
3	1		
1			

Hence the degree of $\chi^{(4,2,1)}$ is $\frac{7!}{6 \cdot 4 \cdot 3 \cdot 2} = 35$.

The Murnagham-Nakayama rule

The Murnagham-Nakayam rule is a recursive formula to compute a single entry in the character table of $\mathfrak{S}_n$. This rule is described and proved in [42, Chapter 21].

We call *skew-hook* a connected part of the rim of $[\lambda]$ that can be removed to leave the diagram of a partition. There is a natural one-to-one correspondence between the hooks of $[\lambda]$ and the skew-hooks of $[\lambda]$: the skew-hook that has initial box in the i-th row and final one in the j-th column corresponds to the hook H_b, with b the (i,j) box of $[\lambda]$. We call hook length (resp. leg length) of the skew-hook, the hook length (resp. leg length) of the associated hook.

For example, let $\lambda = (4,3,2)$. The skew-hook made by the boxes filled with a bullet in the first diagram below corresponds to $H_{(1,2)}$, represented in the second diagram:

		•	•
	•	•	
	•		

⟶

	•	•	•
	•		
	•		

They have hook length equal 5 and leg length equal 2.

Theorem 2.2.4 (Murnagham-Nakayama rule)**.** *Let $\lambda \in \mathscr{P}(n)$. Let $\pi\rho \in \mathfrak{S}_n$ where ρ is an r-cycle and π is a perrmutation of the remaining $n-r$ numbers.*
Let $V = \{ \nu \in \mathscr{P}(n-r) \mid [\lambda] \setminus [\nu]$ is a skew r-hook$\}$. Let $s_\lambda(\nu)$ denotes the leg length of $[\lambda] \setminus [\nu]$, $\nu \in V$. Then

$$\chi^\lambda(\pi\rho) = \sum_{\nu \in V} (-1)^{s_\lambda(\nu)} \chi^\nu(\pi).$$

If the set V is empty, the value of $\chi^\lambda(\pi\rho)$ is zero.

For instance, let $\lambda = (4,3,2)$, ρ a 5-cycle and π a 2-cycle. Then $V = \{(2,1,1)\}$, since the only 5-hook is $H_{(1,2)}$. Hence $\chi^\lambda(\rho\pi) = (-1)^2\chi^{(2,1,1)}(\pi) = \chi^{(2,1,1)}(\pi)$. Now the only 2-hook in $(2,1,1)$ is $H_{(2,1)}$. Therefore, if we apply again the Murnagham-Nakayama rule and then the hook length formula, we have

$$\chi^\lambda(\rho\pi) = \chi^{(2,1,1)}(\pi) = (-1)^1\chi^{(2)}(1) = -1.$$

The Littlewood-Richardson rule

Let $m, n \in \mathbb{N}$ with $m < n$. Given $\chi^\mu \times \chi^\nu \in \operatorname{Irr}(\mathfrak{S}_m \times \mathfrak{S}_{n-m})$, the decomposition into irreducible constituents of the induction

$$(\chi^\mu \times \chi^\nu) \uparrow^{\mathfrak{S}_n} = \sum_{\lambda \in \mathscr{P}(n)} \mathscr{LR}(\lambda; \mu, \nu) \chi^\lambda$$

is described by the Littlewood-Richardson rule (see [42, Chapter 16]). Here the natural numbers $\mathscr{LR}(\lambda; \mu, \nu)$ are called *Littlewood-Richardson coefficients.*

To be more precise, we need to introduce some notation. By a *skew shape* γ we mean a set difference of Young diagrams $[\lambda \smallsetminus \mu]$ for some partitions λ and μ with $[\mu] \subsetneq [\lambda]$, and $|\gamma| = |\lambda| - |\mu|$.

Definition 2.2.5. Let $n \in \mathbb{N}$. Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ and let $\mathscr{C} = (c_1, \ldots, c_n)$ be a sequence of positive integers. We say that $\mathscr{C}$ is of *weight* λ if $|\{ i \in [1, n] \mid c_i = j \}| = \lambda_j$ for all $j \in [1, k]$.

We say that an element c_j of $\mathscr{C}$ is *good* if $c_j = 1$ or if $j > 1$ and

$$|\{ i \in [1, j-1] \mid c_i = c_j - 1 \}| > |\{ i \in [1, j-1] \mid c_i = c_j \}| .$$

Finally, we say that the sequence $\mathscr{C}$ is *good* if c_j is good for every $j \in [1, n]$.

Theorem 2.2.6 (Littlewood-Richardson rule)**.** *Let $m, n \in \mathbb{N}$ with $m < n$. Let $\mu \vdash m$ and $\nu \vdash n - m$. Then*

$$(\chi^\mu \times \chi^\nu) \uparrow_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}}^{\mathfrak{S}_n} = \sum_{\lambda \vdash n} \mathscr{LR}(\lambda; \mu, \nu) \chi^\lambda$$

where $\mathscr{LR}(\lambda; \mu, \nu)$ equals the number of ways to replace the nodes of $[\lambda \smallsetminus \mu]$ by natural numbers such that

(i) the sequence obtained by reading the numbers from right to left, top to bottom is a good sequence of weight ν;

(ii) the numbers are non-decreasing (weakly increasing) left to right along rows; and

(iii) the numbers are strictly increasing down columns.

Remark 2.2.7. With the same notation as the above theorem, by Frobenius reciprocity we have

$$\mathscr{LR}(\lambda; \mu, \nu) = \left[(\chi^\mu \times \chi^\nu) \uparrow^{\mathfrak{S}_n}, \chi^\lambda \right]_{\mathfrak{S}_n} = \left[\chi^\mu \times \chi^\nu, \left(\chi^\lambda \right) \downarrow_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}} \right]_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}} .$$

Given $(n_1, \ldots, n_k) \in \mathscr{C}(n)$, $\lambda \in \mathscr{P}(n)$ and $\mu_j \in \mathscr{P}(n_j)$ for all $j \in [1, k]$, $\mathscr{LR}(\lambda; \mu_1, \ldots, \mu_k)$ denotes the multiplicity of χ^λ as an irreducible constituent of $(\chi^{\mu_1} \times \cdots \times \chi^{\mu_k}) \uparrow_Y^{\mathfrak{S}_n}$. Here Y denotes the Young subgroup $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \times \cdots \times \mathfrak{S}_{n_k}$ of $\mathfrak{S}_n$. This multiplicity is computed as an iterated Littlewood-Richardson coefficient.

The following lemma describes the behavior of the first parts of the partitions involved in a non-zero Littlewood-Richardson coefficient.

Lemma 2.2.8. *If $\mathscr{LR}(\lambda; \mu_1, \ldots, \mu_k) \neq 0$ then $\lambda_1 \leq \sum_{j=1}^{k} (\mu_j)_1$.*

Proof. We proceed by induction on $k \geq 2$.

Let $k = 2$. Since $\mathscr{LR}(\lambda; \mu_1, \mu_2) \neq 0$, we can fix a way to label the nodes of $[\lambda \setminus \mu_1]$ such that the conditions (i),(ii) and (iii) of Theorem 2.2.6 are satisfied. The first element of a good sequence is necessarely 1, and this has to be the entry of the last box of the first row by part (i) of the previous theorem. Hence, all the boxes in the first row of $[\lambda \setminus \mu_1]$ must be filled with the number 1, by part (ii). Since the number of boxes filled by 1 equals $(\mu_2)_1$ and there may be other 1s other than the ones in the first row, we have that $(\mu_2)_1 \geq \lambda_1 - (\mu_1)_1$.

Let now $k > 2$. One of the properties of the restriction of characters says that

$$\chi^\lambda \downarrow_{\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \times \cdots \times \mathfrak{S}_{n_k}} = \chi^\lambda \downarrow_{\mathfrak{S}_{n_1} \times \mathfrak{S}_{n-n_1}} \downarrow_{\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \times \cdots \times \mathfrak{S}_{n_k}}.$$

Suppose that ν is a partition of $n - n_1$ such that both $\mathscr{LR}(\lambda; \mu_1, \nu)$ and $\mathscr{LR}(\nu; \mu_2, \ldots, \mu_k)$ are not zero. Then we have $\mathscr{LR}(\lambda; \mu_1, \mu_2, \ldots, \mu_k) \neq 0$, and by inductive hyphotesis

$$\lambda_1 \leq (\mu_1)_1 + \nu_1 \leq (\mu_1)_1 + \sum_{j=2}^{k} (\mu_j)_1.$$

□

To conclude this section, we introduce the last combinatorial objects that we will need in the next chapters. We define $\mathscr{B}_n(t)$ as the set of those partitions of n whose Young diagram fits inside a $t \times t$ square grid, i.e. for $n, t \in \mathbb{N}$, we set

$$\mathscr{B}_n(t) := \{ \lambda \in \mathscr{P}(n) \mid \lambda_1 \leq t,\ l(\lambda) \leq t \}.$$

Notice that $\mathscr{B}_n(t)$ is closed under taking conjugate of partitions.

Let for instance $n = 4$ and $t = 3$. The Young diagram of (4) has four boxes in the first row, so it does not fits inside a 3×3 grid. Hence $(4) \notin \mathscr{B}_4(3)$. Instead, $(3, 1)$ has three boxes in the first row and only one in the second, therefore $(3, 1) \in \mathscr{B}_4(3)$

For $(n_1, \ldots, n_k) \in \mathscr{C}(n)$ and $A_j \subseteq \mathscr{P}(n_j)$ for all $j \in [1, k]$, let $A_1 \star A_2 \star \cdots \star A_k$ be the set of all the constituents of $(\chi^{\mu_1} \times \cdots \times \chi^{\mu_k}) \uparrow_{\mathfrak{S}_{|\mu_1|} \times \cdots \times \mathfrak{S}_{|\mu_k|}}^{\mathfrak{S}_{|\mu_1| + \cdots + |\mu_k|}}$ as each μ_i vary over A_i. In other words:

$$A_1 \star A_2 \star \cdots \star A_k := \{ \lambda \in \mathscr{P}(n) \mid \mathscr{LR}(\lambda; \mu_1, \ldots, \mu_k) > 0, \text{ for some } \mu_1 \in A_1, \ldots, \mu_k \in A_k \}.$$

It is easy to check that $\star$ is both commutative and associative.

The following lemma shows that boxes behave well with respect to the $\star$ operator. This is [25, Proposition 3.2].

Lemma 2.2.9. *Let $n, n', t, t' \in \mathbb{N}$ be such that $\frac{n}{2} < t \leq n$ and $\frac{n'}{2} < t' \leq n'$. Then*

$$\mathscr{B}_n(t) \star \mathscr{B}_{n'}(t') = \mathscr{B}_{n+n'}(t + t').$$

2.3 Wreath products

Here we fix the notation for characters of wreath products. For more details see [45, Chapter 4]. Let G be a finite group and let H be a subgroup of $\mathfrak{S}_n$. We denote by $G^{\times n}$ the direct product of n copies of G. The natural action of $\mathfrak{S}_n$ on the direct factors of $G^{\times n}$ induces an action of $\mathfrak{S}_n$ (and therefore of $H \leq \mathfrak{S}_n$) via automorphisms of $G^{\times n}$, giving the wreath product $G \wr H := G^{\times n} \rtimes H$.

We refer to $G^{\times n}$ as the base group of the wreath product $G \wr H$. We denote the elements of $G \wr H$ by $(g_1, \ldots, g_n; h)$ for $g_i \in G$ and $h \in H$. Let V be a $\mathbb{C}G$-module and suppose it affords the character ϕ. We let $V^{\otimes n} := V \otimes \cdots \otimes V$ (n copies) be the corresponding $\mathbb{C}G^{\times n}$-module. The left action of $G \wr H$ on $V^{\otimes n}$ defined by linearly extending

$$(g_1, \ldots, g_n; h) : v_1 \otimes \cdots \otimes v_n \mapsto g_1 v_{h^{-1}(1)} \otimes \cdots \otimes g_n v_{h^{-1}(n)},$$

turns $V^{\otimes n}$ into a $\mathbb{C}(G \wr H)$-module, which we denote by $\tilde{V}^{\otimes n}$. We denote by $\tilde{\phi}$ the character afforded by the $\mathbb{C}(G \wr H)$-module $\tilde{V}^{\otimes n}$. For any character ψ of H, we let ψ also denote its inflation to $G \wr H$ and let $\mathscr{X}(\phi;\psi) := \tilde{\phi} \cdot \psi$ be the character of $G \wr H$ obtained as the product of $\tilde{\phi}$ and ψ.

The base group $G^{\times n}$ is normal in $G \wr H$. Hence we can describe the connections between the irreducible characters of $G \wr H$ and the ones of base group using Clifford theory, in particular Theorem 2.1.8.

The irreducible characters of $G \wr H$ are of two types. The first one is the one described above: $\mathscr{X}(\phi;\psi)$ where $\phi \in \mathrm{Irr}(G)$, $\phi^{\times n} := \phi \times \cdots \times \phi$ is the corresponding irreducible character of $G^{\times n}$ and $\psi \in \mathrm{Irr}(H)$. This character has degree $\phi(1)^n \psi(1)$ and its restriction to the base group is $\psi(1)\phi^{\times n}$. Indeed, we are using Theorem 2.1.8 noticing that the action of conjugation by an element of $G \wr H$ is trivial on $\phi^{\times n}$.

The other type of irreducible character of $G \wr H$ is $\varphi = (\phi_1 \times \cdots \times \phi_n)\big\uparrow^{G \wr H}$, where $\phi_1, \ldots, \phi_n$ are irreducible characters of G not all equal. φ has degree equal to $|H|\phi_1(1)\cdots\phi_n(1)$, and using again Clifford theorem we have

$$\varphi\big\downarrow_{G^{\times n}} = \sum_{\sigma \in H} \phi_{\sigma(1)} \times \cdots \times \phi_{\sigma(n)}.$$

Indeed, $G \wr H$ acts on $\phi_1 \times \cdots \times \phi_n$ by changing their order according to the permutations that belongs to H.

If $H = C_p$ is a cyclic group of prime order p, every $\psi \in \mathrm{Irr}(G \wr C_p)$ is either of the form

(i) $\psi = \phi_1 \times \cdots \times \phi_p \big\uparrow_{G^{\times p}}^{G \wr C_p}$, where $\phi_1, \ldots \phi_p \in \mathrm{Irr}(G)$ are not all equal; or

(ii) $\psi = \mathscr{X}(\phi;\theta)$ for some $\phi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(C_p)$.

We want to record a result that will be useful later in this thesis.

Lemma 2.3.1 ([45, Lemma 4.3.9]). *Let $n \in \mathbb{N}$. Let $H \leq \mathfrak{S}_n$ and G be finite groups. Let $\phi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(H)$. Then for all $g_1, \dots, g_n \in G$ and $h \in H$,*

$$\mathscr{X}(\phi;\psi)(g_1,\dots,g_n;h) = \prod_{i=1}^{c(h)} \phi\left(g_{j_i} \cdot g_{h^{-1}(j_i)} \cdots g_{h^{-l_i+1}(j_i)}\right) \cdot \psi(h),$$

where $c(h)$ is the number of disjoint cycles in h, l_i is the length of the i^{th} cycle, and for each i, j_i is some fixed element in the i^{th} cycle.

For instance, if $n = 8$ and $h = (1,3,7,2)(5,8,6)(4)$, then

$$\mathscr{X}(\phi;\psi)(g_1,\dots,g_8;h) = \phi(g_2 g_7 g_3 g_1)\phi(g_6 g_8 g_5)\phi(g_4) \cdot \psi(h).$$

2.3.1 Sylow subgroups of $\mathfrak{S}_n$

We describe here the Sylow subgroups of a symmetric group and briefly their irreducible characters.

Let P_n denote a Sylow p-subgroup of $\mathfrak{S}_n$. Clearly P_1 is the trivial group while $P_p \cong C_p$ is cyclic of order p generated by the a p-cycle of $\mathfrak{S}_p$. If $k \geq 2$, then

$$P_{p^k} = \left(P_{p^{k-1}}\right)^{\times p} \rtimes P_p = P_{p^{k-1}} \wr P_p \cong P_p \wr \cdots \wr P_p \text{ (k-fold wreath product).}$$

We will usually write the elements of P_{p^k} as $\sigma = (\sigma_1, \dots, \sigma_p; \pi)$ where $\sigma_1, \dots, \sigma_p \in P_{p^{k-1}}$ and $\pi \in P_p$. σ is identified with the element of $\mathfrak{S}_n$ defined as follows: if $j \in [1, p^k]$ is such that $j = b + p^{k-1}(a-1)$, for some $a \in [1,p]$ and some $b \in [1, p^{k-1}]$ then $\sigma(j) := \sigma_a(b) + p^{k-1}(\pi(a)-1)$. Via this identification, a Sylow p-subgroup P_{p^k} is generated by the elements $g_1, \dots, g_p \in \mathfrak{S}_n$, where

$$g_j := \prod_{d=1}^{p^{j-1}} \left(d, d+p^{j-1}, d+2p^{j-1}, \dots, d+(p-1)p^{j-1}\right), \text{ for } j \in [1,k]. \tag{2.2}$$

Let $n = \sum_{i=1}^{r} a_i p^{k_i}$ be the p-adic expansion of n. Then $P_n \cong P_{p^{k_1}}^{\times a_1} \times P_{p^{k_2}}^{\times a_2} \times \cdots \times P_{p^{k_r}}^{\times a_r}$.

Notice that sometimes we will use the p-adic expansion $n = \sum_{i=1}^{t} p^{n_i}$, where $n_1 \geq \cdots \geq n_t \geq 0$. In this case $P_n \cong P_{p^{n_1}} \times \cdots \times P_{p^{n_t}}$. This second notation for the p-adic expansion of a natural number is useful when we do not want to consider the various exponents to be distinct.

Example 2.3.2. Suppose that $p = 3$. Then $P_3 = \langle g_1 \rangle$, $P_9 = \langle g_1, g_2 \rangle$ and $P_{27} = \langle g_1, g_2, g_3 \rangle$, where

$$\begin{aligned} g_1 &= (1,2,3), \\ g_2 &= (1,4,7)(2,5,8)(3,6,9), \\ g_3 &= (1,10,19)(2,11,20)(3,12,21)\cdots(9,18,27). \end{aligned}$$

Moreover, $P_{51} \cong P_3 \times P_3 \times P_9 \times P_9 \times P_{27}$.

We refer the reader to [45, Chapter 4] or to [64] for more details.

We denote $\{\phi_0, \phi_1, \dots, \phi_{p-1}\} = \mathrm{Irr}(P_p)$, where ϕ_0 is the trivial character of P_p. Using the facts on representations of wreath products highlighted above, it is easy to observe that linear characters of P_{p^k} are naturally labelled by elements of $[0, p-1]^{\times k}$. In fact, setting $\mathscr{X}(t) = \phi_t$ for every $t \in [0, p-1]$ and given $(j_1, \dots, j_k) \in [0, p-1]^{\times k}$, we recursively define $\mathscr{X}(j_1, \dots, j_{k-1}, j_k) \in \mathrm{Lin}(P_{p^k})$ as $\mathscr{X}(j_1, \dots, j_{k-1}, j_k) = \mathscr{X}(\mathscr{X}(j_1, \dots, j_{k-1}); \phi_{j_k})$.

For example, for any prime p and for every natural number k, the trivial character of P_{p^k} is always $\mathscr{X}(0, 0, \dots, 0)$.

Notice that an irreducible character of P_{p^k} can be only of the form $(\psi_1 \times \cdots \times \psi_p)\big\uparrow^{P_{p^k}}$ where $\psi_1, \dots, \psi_p \in \mathrm{Irr}(P_{p^{k-1}})$ are not all equal, or of the form $\mathscr{X}(\psi; \phi_i)$ with $\psi \in \mathrm{Irr}(P_{p^{k-1}})$ and $i \in [1, p-1]$.

We will need later to be able to compute the value of an irreducible character of P_{p^k} of the type $\mathscr{X}(\alpha; \beta)$ on a fixed element $(q_1, \dots, q_p; \pi) \in P_{p^k}$. To do this, we can use Lemma 2.3.1 in the case $G = P_{p^{k-1}}$ and $H = P_p \leq \mathfrak{S}_p$. Notice that $\pi \in P_p$ is either 1 or a single $p-$cycle. Hence if $\pi \neq 1$, with the same notation as the one of the lemma, $c(\pi) = 1$ and $l_1 = p$. Then we can fix $j_1 = 1$ and we obtain

$$\mathscr{X}(\alpha; \beta)(q_1, \dots, q_p; \pi) = \alpha\left(q_1 \cdot q_{\pi^{-1}(1)} \cdots q_{\pi^{-p+1}(1)}\right) \cdot \beta(\pi).$$

Otherwise, if $\pi = 1$ then $\mathscr{X}(\alpha; \beta)(q_1, \dots, q_p; 1) = \prod_{v=1}^{p} \alpha(q_v)$. Indeed, $c(1) = p$, $l_v = 1$ for every v and $\beta(1) = 1$ because it is a linear character of P_p.
We summarize this fact in the following remark.

Remark 2.3.3. Let $\mathscr{X}(\alpha; \beta) \in \mathrm{Irr}(P_{p^k})$ and $(q_1, \dots, q_p; \pi) \in P_{p^k}$. Then

$$\mathscr{X}(\alpha; \beta)(q_1, \dots, q_p; \pi) = \begin{cases} \alpha\left(q_1 \cdot q_{\pi^{-1}(1)} \cdots q_{\pi^{-p+1}(1)}\right) \cdot \beta(\pi), & \text{if } \pi \neq 1; \\ \prod_{v=1}^{p} \alpha(q_v), & \text{if } \pi = 1. \end{cases}$$

The last technical tool that we will need is a method to evaluate an irreducible character of the Sylow p-subgroup P_{p^k} on a p^k-cycle. This is proved in [28, Lemma 3.11]:

Lemma 2.3.4. *Let k be a positive integer. There exists a p^k-cycle $g \in P_{p^k}$ such that the following hold:*

(i) $\theta(g)$ is a p-th root of unity for every linear character θ of P_{p^k}.

(ii) $\chi(g) = 0$ for all $\chi \in \mathrm{Irr}(P_{p^k})$ such that p divides $\chi(1)$.

A p^k-cycle $g \in P_{p^k}$ that satisfies this lemma can be constructed recursively as follows: let $h \in P_{p^{k-1}}$ be a p^{k-1}-cycle and $\sigma \in P_p$ such that $\sigma \neq 1$, then $g = (h, 1, \dots, 1; \sigma)$. Notice that this element does not lie in the base group $(P_{p^{k-1}})^{\times p}$, hence every character properly induced from the base group vanishes on g.

For instance, let $p = 3$ and let $\sigma = (1, 2, 3) \in P_3$. Then $g = ((1, 2, 3), 1, 1; (1, 2, 3))$ is a 9-cycle in P_9, indeed $g = (1, 2, 3)(1, 4, 7)(2, 5, 8)(3, 6, 9) = (1, 4, 7, 2, 5, 8, 3, 6, 9)$.

For $k \in \mathbb{N}$, the normaliser of a Sylow p-subgroup of $\mathfrak{S}_{p^k}$ is $N_{\mathfrak{S}_{p^k}}(P_{p^k}) = P_{p^k} \rtimes H$, where $H \cong (C_{p-1})^{\times k}$. More generally, if $n = \sum_{i=1}^{r} a_i p^{k_i}$ is the p-adic expansion of n, then $N_{\mathfrak{S}_n}(P_n) = N_1 \wr \mathfrak{S}_{a_1} \times \cdots \times N_r \wr \mathfrak{S}_{a_r}$, where $N_i := N_{\mathfrak{S}_{p^{k_i}}}(P_{p^{k_i}})$ for every $i \in [1, r]$.

We now turn to an explicit presentation of the normalizer of a Sylow subgroup. In order to better understand this technical description we refer the reader to Example 2.3.5 below. Consider the specific Sylow p-subgroup P_{p^k} of $\mathfrak{S}_{p^k}$ generated by $g_1, g_2, \ldots, g_k$, defined in (2.2). Then a presentation for $N_{\mathfrak{S}_{p^k}}(P_{p^k})$ is $\langle P_{p^k}, \sigma_1^{(k)}, \sigma_2^{(k)}, \ldots, \sigma_k^{(k)} \rangle$, where the elements $\sigma_i^{(j)}$ are defined recursively as follows.

Let $P_p = \langle g_1 \rangle$ and $\sigma_1^{(1)} = (c_1, c_2, \ldots, c_{p-1}) \in \mathrm{Sym}(\{1, 2, \ldots, p\})$ be a $(p-1)$-cycle in $N_{\mathfrak{S}_p}(\langle g_1 \rangle)$ having p as a fixed point and such that $N_{\mathfrak{S}_p}(\langle g_1 \rangle) = \langle g_1 \rangle \rtimes \langle \sigma_1^{(1)} \rangle$. For any integer m, let $\tau_m \in \mathfrak{S}_{p^k}$ be the permutation $i \mapsto i + m$ with numbers modulo p^k. For $1 \leq j < k$, set

$$\sigma_j^{(k)} = \prod_{i=0}^{p-1} \tau_{ip^{k-1}} \cdot \sigma_j^{(k-1)} \cdot \tau_{-ip^{k-1}}$$

and

$$\sigma_k^{(k)} = \prod_{i=0}^{p^{k-1}-1} \tau_{-i} \cdot \left(c_1 p^{k-1}, c_2 p^{n-1}, \ldots, c_{p-1} p^{k-1} \right) \cdot \tau_i$$

with numbers modulo p^k. By construction, each $\sigma_j^{(k)}$ is a product of $(p-1)$-cycles, and the $\sigma_j^{(k)}$ commute for all j for each fixed n.

Let λ be a permutation of a certain symmetric group $\mathfrak{S}_n$ and $i \in \mathbb{N}$. The power λ^i is naturally defined as the permutation obtained by applying i times λ. By definition, we have that $\tau_0 = 1$ and for $0 < i \leq p - 1$, $\tau_{ip^{k-1}} = g_k^i = g_k^{-(p-i)}$ and $\tau_{-ip^{k-1}} = g_k^{p-i}$. Hence for $1 \leq j < k$, we can rewrite $\sigma_j^{(k)}$ as

$$\sigma_j^{(k)} = \prod_{i=0}^{p-1} \left(\sigma_j^{(k-1)} \right)^{g_k^i}.$$

By a discussion in the proof of [64, Proposition 1.5], we can consider $N_{\mathfrak{S}_{p^k}}(P_{p^k})$ as a subgroup of $N_{\mathfrak{S}_{p^{k-1}}}(P_{p^{k-1}}) \wr N_{\mathfrak{S}_p}(P_p)$. Its elements can be written as $(n_1, \ldots, n_p; \mu)$ with $n_1, \ldots, n_p \in N_{\mathfrak{S}_{p^{k-1}}}(P_{p^{k-1}})$ and $\mu \in N_{\mathfrak{S}_p}(P_p)$. We need also that $n_i \equiv n_j \pmod{P_{p^{k-1}}}$ for every i, j.

In this view, the generators $\sigma_i^{(k)}$ became $\sigma_i^{(k)} = (\sigma_i^{(k-1)}, \ldots, \sigma_i^{(k-1)}; 1)$ for $i \in [1, k-1]$, and $\sigma_k^{(k)} = (1, \ldots, 1; \sigma_1^{(1)})$.

The base group of $N_{\mathfrak{S}_{p^{k-1}}}(P_{p^{k-1}}) \wr N_{\mathfrak{S}_p}(P_p)$ can be considered to be a subgroup of $S_1 \times S_2 \times \cdots \times S_p$, where S_i is a symmetric group of p^{k-1} elements for each $i \in [1, p]$. More precisely, for every $i \in [1, p]$ we let $S_i = \mathrm{Sym}\{1 + ip^{k-1}, 2 + ip^{k-1}, \ldots, p + ip^{k-1}\}$.

To determine which is the element of $N_{\mathfrak{S}_{p^k}}(P_{p^k}) < \mathfrak{S}_{p^k}$ that corresponds to $(n_1, \ldots, n_p; \mu)$, we can then consider each n_i as a permutation in S_i, and μ as a permutation of the indices of $S_1 \times S_2 \times \cdots \times S_p$, i.e. as a permutation of the various components of this product.

To better understand this idea and the technical topics just discussed, we see an example.

Example 2.3.5. Suppose that $p = 3$. We have $N_{\mathfrak{S}_3}(P_3) = \langle P_3, \sigma_1^{(1)} \rangle$ with $\sigma_1^{(1)} = (1, 2)$.

Now $N_{\mathfrak{S}_9}(P_9) = \langle P_9, \sigma_1^{(2)}, \sigma_2^{(2)} \rangle$. Recall from Example 2.3.2 that $g_2 = (1, 4, 7)(2, 5, 8)(3, 6, 9)$. We know that $N_{\mathfrak{S}_9}(P_9) < N_{\mathfrak{S}_3}(P_3) \wr N_{\mathfrak{S}_3}(P_3)$, and the base group of the latter wreath product is $(N_{\mathfrak{S}_3}(P_3))^{\times 3}$. This can be considered as a subgroup of $\mathrm{Sym}\{1, 2, 3\} \times \mathrm{Sym}\{4, 5, 6\} \times \mathrm{Sym}\{7, 8, 9\}$.

We have that $\sigma_1^{(2)} = (\sigma_1^{(1)}, \sigma_1^{(1)}, \sigma_1^{(1)}; 1) = ((1, 2), (1, 2), (1, 2); 1)$. Then the first component $(1, 2) \in \mathrm{Sym}\{1, 2, 3\}$, the second one belongs to $\mathrm{Sym}\{4, 5, 6\}$ so it is $(1, 2)^{g_2} = (4, 5)$, and the last one belongs to $\mathrm{Sym}\{7, 8, 9\}$ so it is $(1, 2)^{g_2^2} = (7, 8)$. Therefore $\sigma_1^{(2)}$ is the product of these permutations:

$$\sigma_1^{(2)} = (1, 2)(4, 5)(7, 8).$$

The other generators, $\sigma_2^{(2)}$, is the transposition between the first and the second components, i.e. between $\mathrm{Sym}\{1, 2, 3\}$ and $\mathrm{Sym}\{4, 5, 6\}$. Hence

$$\sigma_2^{(2)} = (1, 4)(2, 5)(3, 6).$$

We can do the same thing to compute the generators of $N_{\mathfrak{S}_{27}}(P_{27}) = \langle P_{27}, \sigma_1^{(3)}, \sigma_2^{(3)}, \sigma_3^{(3)} \rangle$. We have $N_{\mathfrak{S}_{27}}(P_{27}) < N_{\mathfrak{S}_9}(P_9) \wr N_{\mathfrak{S}_3}(P_3)$ and its base group can be considered as a subgroup of $\mathrm{Sym}\{1, 2, \ldots, 9\} \times \mathrm{Sym}\{10, 11, \ldots, 18\} \times \mathrm{Sym}\{19, 20, \ldots, 27\}$. Hence $\sigma_1^{(3)}$ is the product of $\sigma_1^{(2)}$ for each of these components, and the same holds for $\sigma_2^{(3)}$:

$$\begin{aligned}
\sigma_1^{(3)} &= (1, 2)(4, 5)(7, 8) \cdot (10, 11)(13, 14)(16, 17) \cdot (19, 20)(22, 23)(25, 26), \\
\sigma_2^{(3)} &= (1, 4)(2, 5)(3, 6) \cdot (10, 13)(11, 14)(12, 15) \cdot (19, 22)(20, 23)(21, 24).
\end{aligned}$$

Finally, $\sigma_3^{(3)}$ is the transposition between $\mathrm{Sym}\{1, 2, \ldots, 9\}$ and $\mathrm{Sym}\{10, 11, \ldots, 18\}$:

$$\sigma_3^{(3)} = (1, 10)(2, 11)(3, 12)(4, 13)(5, 14)(6, 15)(7, 16)(8, 17)(9, 18).$$

Notice that $N_{\mathfrak{S}_{51}}(P_{51}) \cong N_{\mathfrak{S}_3}(P_3) \wr \mathfrak{S}_2 \times N_{\mathfrak{S}_9}(P_9) \wr \mathfrak{S}_2 \times N_{\mathfrak{S}_{27}}(P_{27})$.

We are now ready to prove the following fact, that we will need to prove Lemma 4.1.4. This will be fundamental in Chapter 4.

Lemma 2.3.6. *Let p be an odd prime, let $n \in \mathbb{N}$ and let H be a complement of P_{p^n} in $N_{\mathfrak{S}_{p^n}}(P_{p^n})$. There are no non-trivial elements of P_{p^n} that are centralized by H, i.e. H acts faithfully on P_{p^n}.*

Proof. We proceed by induction on n. If $n = 1$ then $H \cong C_{p-1}$ and $H \leq \mathrm{Aut}(P_p)$, since H acts on P_p by conjugation. Hence $H = \mathrm{Aut}(P_p)$, since $\mathrm{Aut}(P_p) \cong C_{p-1}$. If we suppose that there exists $g \in P_p$ which is centralized by H, then g has to be a fixed point for every element in $\mathrm{Aut}(P_p)$. Therefore g has to be the identity element.

If $n \geq 2$ then it is sufficient to prove the lemma for a fixed Sylow p-subgroup. In fact, if $P, Q \in \mathrm{Syl}_p(\mathfrak{S}_{p^n})$ then there exists $s \in \mathfrak{S}_{p^n}$ such that $Q = P^s$. Hence $N_{\mathfrak{S}_{p^n}}(Q) = s^{-1} N_{\mathfrak{S}_{p^n}}(P) s$. In particular, if H_P and H_Q are the complements of P and Q respectively in $N_{\mathfrak{S}_{p^n}}(P)$ and $N_{\mathfrak{S}_{p^n}}(Q)$, then $H_Q = (H_P)^s$. Therefore, suppose that $1 \neq a \in Q$ is not centralized by $h \in H_Q$. Then there exist $b \in P$ and $h' \in H_P$ such that $a = b^s$ and $h = (h')^s$. Hence

$$a = b^s \neq a^h = (b^s)^{(h')^s} = b^{h's} \text{ if and only if } b \neq b^{h'}.$$

We consider the specific Sylow p-subgroup P_{p^n} of $\mathfrak{S}_{p^n}$ generated by $g_1, g_2, \ldots, g_n$ defined in (2.2). Hence a presentation for $N_{\mathfrak{S}_{p^n}}(P_{p^n})$ is $\langle P_{p^n}, \sigma_1^{(n)}, \sigma_2^{(n)}, \ldots, \sigma_n^{(n)}\rangle$.
Let $H_n = \langle \sigma_1^{(n)}, \sigma_2^{(n)}, \ldots, \sigma_n^{(n)}\rangle$. This is a complement of P_{p^n} in $N_{\mathfrak{S}_{p^n}}(P_{p^n})$. Notice that $H_{n-1} \cong H_{n-1}^{(n)} := \langle \sigma_1^{(n)}\rangle \times \cdots \times \langle \sigma_{n-1}^{(n)}\rangle \subset H_n$.

Recall that the elements of $P_{p^n} = P_{p^{n-1}} \wr P_p$ can be written in a unique way as $x = (x_1, x_2, \ldots, x_p; y)$, where $x_i \in P_{p^{n-1}}$ for every $i \in [1, p]$, and $y \in P_p$.
Fix $1 \neq x \in P_{p^n}$. There are two cases: if $y = 1$ then there exists $i \in [1, p]$ such that $x_i \neq 1$. By the inductive hypothesis there exists $h_i \in H_{n-1}$ such that $x_i^{h_i} \neq x_i$. Let $h := \prod_{j=0}^{p-1}(h_i)^{g_n^j} \in H_{n-1}^{(n)} \subset H_n$. We have

$$x^h = (x_1^{h_i}, \ldots, x_{i-1}^{h_i}, x_i^{h_i}, x_{i+1}^{h_i}, \ldots, x_p^{h_i}; 1) \neq x.$$

If $y \neq 1$, then y cannot be centralized by $H_1 = \langle \sigma_1^{(1)}\rangle$. In particular, $y^{\sigma_1^{(1)}} \neq y$. Hence $x^{\sigma_n^{(n)}} = (x_{\sigma_1^{(1)}(1)}, \ldots, x_{\sigma_1^{(1)}(p)}; y^{\sigma_1^{(1)}}) \neq x$.

This shows that for every non-trivial element x of P_{p^n} there exists an element in H_n that does not centralize x.

□

We remark that Lemma 2.3.6 is equivalent to say that $C_{\mathfrak{S}_{p^n}}(P_{p^n}) = Z(P_{p^n})$, for any $n \in \mathbb{N}$. This Lemma does not work for $p = 2$. In that case, $N_{\mathfrak{S}_{2^n}}(P_{2^n}) = P_{2^n}$ for every $n \in \mathbb{N}$. Hence H is trivial.

The presentations of P_{p^n} and $N_{\mathfrak{S}_{p^n}}(P_{p^n})$ described in this section are the ones used in [22].

2.4 Background on modular representation theory

Let G be a finite group and $\mathbb{F}$ be a field of characteristic $p > 0$. Recall that the group algebra $\mathbb{F}G$ itself is an $\mathbb{F}G$-module under multiplication, and it is called the regular module.

An $\mathbb{F}G$-module is a *free* module if it is isomorphic to a finite direct sum of copies of the regular module $\mathbb{F}G$. An $\mathbb{F}G$-module is a *projective* module if it is a direct summand of a free module.

Let $H \leq G$; an $\mathbb{F}G$-module U is called *relatively H-projective* if it is a direct summand of $(U\downarrow_H)\uparrow^G$. This is a generalization of the projectivity, indeed a module is relatively 1-projective if and only if it is projective.

The vertex of an indecomposable $\mathbb{F}G$-module M is an important invariant of M. It was first defined by J.A. Green in [33]. It is an essentially unique subgroup of G which measures the (relative) projectivity of M. More precisely: if $Q \leq G$ is minimal with respect to the condition that M is relatively Q-projective, than Q is called a *vertex* of M. Hence the closer Q is to the identity, the nearer M is to being projective.

The vertices of an indecomposable module form a G-conjugacy class of p-subgroups of G. They are an invariant of the isomorphism class of M. Clearly M has trivial vertex if p does not divide $|G|$. For every p-subgroup Q of G, there are indecomposable $\mathbb{F}G$-modules with vertex Q.

An indecomposable Q-module S is called a *source* of M if M is a summand of $S\big\uparrow^G$. It is known that S is unique up to conjugation in $N_G(Q)$. This is [1, Theorem 4 Section 9].

These are useful results that can be found in [33]:

Theorem 2.4.1. *Let M be an indecomposable $\mathbb{F}G$-module and let $P \in \mathrm{Syl}_p(G)$ containing a vertex Q of M. Then*

$$[P : Q] \mid \dim_{\mathbb{F}}(M).$$

Obvious consequence is:

Corollary 2.4.2. *If M has dimension prime to p, then the vertices of M are the Sylow p-subgroups of G.*
In particular, the trivial module has Sylows p-vertices.

Following is a special case of Green's indecomposibility theorem:

Lemma 2.4.3. *If P is a p-group and $Q \leq P$ then the permutation module $\mathbb{F}_Q\big\uparrow^P$ is indecomposable, and has vertex Q.*

Classical costructions such as the Brauer homomorphism help us to investigate the structure of the vertices of the indecomposable modules, while others such as the Green and the Broué correspondences gives us a way to relate the global theory of a finite group to the theory of its local subgroup. We do not want to record all these constructions and results here because they are beyond what we need to understand the main contents of this thesis. We refer the reader to [1], [3] and [61] for most of the results used for the computations that we will find in Chapter 6.

We would like to mention a particular class of $\mathbb{F}G$-modules for which the results mentioned above give some more information than usual.

Definition 2.4.4. An $\mathbb{F}G$-module M is said to be a *p-permutation module* if whenever P is a p-subgroup of G, there exists an $\mathbb{F}$-basis $\mathscr{B}$ of M whose elements are permuted by P. In this case we say that $\mathscr{B}$ is a *p-permutation basis* of M, and we write $M = \langle \mathscr{B} \rangle$.

Notice that if M has a p-permutation basis with respect to a Sylow p-subgroup, then M is a p-permutation module.

The following is a characterization of the p-permutation modules.

Proposition 2.4.5 ([3, (0.4)])**.** *Let M be an indecomposable $\mathbb{F}G$-module. Then M is a p-permutation module if and only if one of the following holds:*

(i) there exists $H \leq G$ such that M is isomorphic to a summand of $\mathbb{F}_H\big\uparrow^G$;

(ii) M has a trivial source.

We are interested in this kind of modules especially for the study of the Young permutation $\mathbb{F}\mathfrak{S}_n$-modules and their indecomposable summands, the Young modules. See Section 2.2.2 for definitions.

2.4.1 Block theory

We would like to end this section with a brief recall about the theory of blocks, in particular for the case of $\mathfrak{S}_n$. For an extensive dissertation on blocks for finite groups we refer to [1, Section 13].

Each algebra has a unique decomposition into a direct sum of subalgebras, each of which is indecomposable as an algebra. Let $\mathbb{F}$ be a field of characteristic p and G be a finite group. Consider the unique decomposition of $\mathbb{F}G$ given by $\mathbb{F}G = A_1 + \cdots + A_r$. Then the subalgebras $A_1, \ldots, A_r$ are called *blocks* of $\mathbb{F}G$.

If M is an $\mathbb{F}G$-module, $A_iM = M$ and $A_jM = 0$ for all j, $j \neq i$, then we say that M *lies in the block* A_i. If M is an $\mathbb{F}G$-module then M has a unique direct sum decomposition $M = M_1 + \cdots + M_r$ where M_i lies in the block A_i.

Each simple $\mathbb{F}G$-module has to lie in a single block. Hence we can interpret blocks as a partition of the set of the irreducible characters of G.

We can now regard $\mathbb{F}G$ as a $\mathbb{F}[G \times G]$-module setting $a(g_1, g_2) = g_1 a g_2^{-1}$ for every $a \in \mathbb{F}G$, $g_1, g_2 \in G$. Therefore $\mathbb{F}G$ has a unique decomposition into the direct sum of simple $\mathbb{F}[G \times G]$-modules, the summands are the blocks of $\mathbb{F}G$.

Theorem 2.4.6 ([1, Theorem 4 Section 13]). *If B is a p-block of $\mathbb{F}G$, then B has vertex, as an $\mathbb{F}[G \times G]$-module, of the form $\delta(D) = \{ (d, d) \mid d \in D \}$, where D is a p-subgroup of G.*

The vertices of a p-block B form a conjugacy class of p-subgroup of G called the *defect groups* of G.

Blocks of $\mathfrak{S}_n$

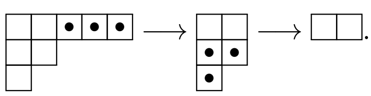
We refer the reader to [65, Section 3 and 11] for a detailed description of the p-blocks of symmetric groups and some more examples and properties.

Let now λ be a partition of n. The *p-core* $\lambda_{(p)}$ of λ is a partition which is obtained by λ, and which has no hooks length divisible by p. More precisely, if in the Young diagram $[\lambda]$ there is a hook of length a multiple of p, we remove the boxes of the corresponding skew-hook. We obtain the diagram of a new partition $[\lambda']$. We proceed in the same way until the partition obtained has no hooks of length divisible by p. This is the p-core $\lambda_{(p)}$. It does not depend on the way hooks were removed.

The *p-weight* of λ is the number of p-hooks that are removed to go from λ to $\lambda_{(p)}$. It is denoted by $w_p(\lambda)$. So we have

$$|\lambda| = |\lambda_{(p)}| + p w_p(\lambda).$$

For instance, let $p = 3$ and $\lambda = (5, 2, 1)$ be a partition of 8. We can remove first $H_{(1,3)}$ and then $H_{(2,1)}$, because they both have legth 3. What we obtain is:



This is the Young diagram of $\lambda_{(3)}$, that is (2). Since we removed two 3-hooks, $w_3(\lambda) = 2$. Indeed, $|\lambda| = 8 = |\lambda_{(3)}| + 3w_3(\lambda) = 2 + 3 \cdot 2$.

A result still referred to as the *Nakayama's Conjecture* states that the p-blocks of $\mathfrak{S}_n$ are labelled by pairs (γ, w) such that γ is a p-core partition and $|\gamma| + pw = n$. Moreover, the Specht module S^λ lies in the block $B(\gamma, w)$ if and only if λ has p-core γ and p-weight w.

We can rephrase this result looking at blocks as subsets of $\mathrm{Irr}(\mathfrak{S}_n)$. Again,eEach p-block is labelled by a pair (γ, w) such that γ is a p-core partition and $|\gamma| + pw = n$. That means: let $\lambda, \mu \in \mathscr{P}(n)$, then χ^lambda lies in the same p-block as χ^μ if and only if $\lambda_{(p)} = \mu_{(p)}$.

Proposition 2.4.7 ([65, Proposition 11.3]). *The defect group of a p-block of weight w of $\mathfrak{S}_n$ is a Sylow p-subgroup of $\mathfrak{S}_{wp}$.*

For example, consider $p = 3$ and $n = 4$. A partition of 4 can only have 3-weight zero or one. We have three 3-blocks:

$$\begin{aligned} B((1),1) &= \left\{ \chi^{(4)}, \chi^{(1^4)}, \chi^{(2,2)} \right\}, \\ B((3,1),0) &= \left\{ \chi^{(3,1)} \right\}, \\ B((2,1,1),0) &= \{ \chi(2,1,1) \}. \end{aligned}$$

Notice that these actually form a partition of $\mathrm{Irr}(\mathfrak{S}_4)$.

Moreover, we have that a defect group of $B((1),1)$ is isomorphic to a Sylow 3-subgroup of $\mathfrak{S}_3$, hence it is the cyclic group C_3. A defect group for the other two blocks has to be trivial, since it has to be a Sylow 3-subgroup of $\mathfrak{S}_0$.

Chapter 3

Sylow branching coefficients: a survey

In the introduction, we briefly mentioned the importance of the connection between the irreducible characters of a group and those of its Sylow p-subgroups. Here we make this more precise.

A famous result in the character theory of finite groups is the Itô-Michler theorem [40, 58]. This asserts that a Sylow p-subgroup P of a finite group G is abelian and normal in G if, and only if, the character degree $\chi(1)$ is coprime to p for every irreducible character $\chi \in \mathrm{Irr}(G)$.

A challenge for the last few decades has been to separate the two conditions in this theorem.The commutativity of P is characterized by Brauer's height zero conjecture, proved in [56], while the normality has been studied in [55]. In particular, one of the main theorem of this last paper says that a Sylow p-subgroup P of G is normal in G if and only if all irreducible constituents of the permutation character $\mathbb{1}_P\big\uparrow^G$ have degree coprime to p. This result has been refined recently in [27].

On the other side of the spectrum, in [63] it is shown that the character $\mathbb{1}_P\big\uparrow^G$ controls the property of a Sylow p-subgroup to be self-normalizing. More precisely, when p is an odd prime, $P = N_G(P)$ if and only if $\mathbb{1}_G$ is the only contituent of $\mathbb{1}_P\big\uparrow^G$ having degree coprime to p.

This highlights the importance of studying the permutation character $\mathbb{1}_P\big\uparrow^G$, and more generally, of the so-called Sylow branching coefficients. These have already been defined in the introduction, but we want to state here the definition formally.

Definition 3.0.1. Let p be a prime number and let P be a Sylow p-subgroup of a finite group G. Let $\chi \in \mathrm{Irr}(G)$ and $\phi \in \mathrm{Irr}(P)$ be irreducible characters of G and P, respectively. The corresponding *Sylow branching coefficient* Z^χ_ϕ is defined as the multiplicity of ϕ as an irreducible constituent of $\chi\big\downarrow_P$, the restriction of χ to P.

Recall that Frobenius reciprocity theorem tells us that $Z^\chi_\phi = [\phi, \chi\big\downarrow_P]_P = [\phi\big\uparrow^G, \chi]_G$, for any $\chi \in \mathrm{Irr}(G)$, $\phi \in \mathrm{Irr}(P)$.

We want to focus in this chapter recalling known results reached in the study of Sylow branching coefficients. Our contribution to the topic is recorded in Chapters 4 and 5.

A main observation about the restriction of a character to a Sylow can be found in [2]. Suppose that the order of a finite group G is divisible by p^n, but not by p^{n+1}. If an irreducible character $\chi \in \mathrm{Irr}(G)$ has degree divisible by p^n, it vanishes on all elements of G of order divisible by p. This implies:

Theorem 3.0.2. *Let $\chi \in \mathrm{Irr}(G)$ have degree divisible by p^n, where p^n the highest power of p that divides $|G|$. Let $P \in \mathrm{Syl}_p(G)$. Then, $\chi\big\downarrow_P$ is a multiple of the regular character of P.*

In [28] the authors focused on the irreducible characters $\chi \in \mathrm{Irr}(G)$ that have degree divisible by p. Suppose $P \in \mathrm{Syl}_p(G)$, they conjectured that if $\chi\big\downarrow_P$ has a linear constituent, then $\chi\big\downarrow_P$ has at least p different linear constituents.

In the same paper, this conjecture has been proved to hold for p-solvable groups and for the symmetric ones (Theorem 3.1.2 below). In [21] it has been proved for the alternating groups. Moreover, in [29] they studied the blockwise version of this conjecture.

Notice that for the p-solvable case, the statement is actually quite stronger.

Theorem 3.0.3 ([28, Theorem B])**.** *Let G be a p-solvable group, $P \in \mathrm{Syl}_p(G)$ and let $\chi \in \mathrm{Irr}(G)$ be a character of degree divisible by p. If $\chi\big\downarrow_P$ contains a linear constituent λ, then there exists a subgroup $U < P$ of index p such that $\chi\big\downarrow_P$ contains the character $\left(\lambda\big\downarrow_U\right)\big\uparrow^P$.*

Notice that the character $\left(\lambda\big\downarrow_U\right)\big\uparrow^P$ is not irreducible, since λ is one of its irreducible constituents. Therefore there are p different constituents, and we also know that they are the character $\left(\lambda\big\downarrow_U\right)\big\uparrow^P$.

The previous theorem has been also generalized in [67] in the case when P is a p-subgroup of G and we are not considering only linear characters.

3.1 The symmetric group case

Let us focus now on the case of the symmetric groups. We start by recording a couple of theorems proved in [28] that will be generalized in Chapter 4. In this section p is a prime.

Theorem 3.1.1 ([28, Theorem 3.1])**.** *Let $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$ and let P_n be a Sylow p-subgroup of $\mathfrak{S}_n$. Then $\chi\big\downarrow_{P_n}$ has a linear consituent.*

Notice that this theorem does not holds for a general finite group, but it is a special feature of the symmetric groups. For instance, if G is a p-group and $\chi \in \mathrm{Irr}(G)$ is not linear, then $P \in \mathrm{Syl}_p(G)$ is the whole group $P = G$ and $\chi\big\downarrow_P = \chi$ is not linear.

Theorem 3.1.2 ([28, Theorem A])**.** *Suppose that n is a positive integer and let $P_n \in \mathrm{Syl}_p(\mathfrak{S}_n)$. If $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$ has degree divisible by p, then the restriction $\chi\big\downarrow_{P_n}$ has at least p different linear constituents.*

The authors, in [28], also noticed that the number of linear constituents of the character $\chi\big\downarrow_{P_n}$ tends to be very large, but there are arbitrarly large integers n for which this number is exactly p.

The problems that the results in [28] leave open are the description of the linear constituents of which they claim the existence, and the computation of their Sylow branching coefficients.

The explicit computation of Sylow branching coefficients has been done only for few special cases. However, there are deeper results concerning their positivity in [24] and [25].

In [24], the authors determines all irreducible constituents of the permutation character $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{S}_n}$, where p is an odd prime.

Theorem 3.1.3 ([24, Theorem A]). *Let p be an odd prime, let n be a natural number and let $\lambda \in \mathscr{P}(n)$. Then χ^λ is not an irreducible constituent of $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{S}_n}$ if and only if $n = p^k$ for some $k \in \mathbb{N}$ and $\lambda \in \left\{ (p^k-1,1), (2,1^{p^k-2}) \right\}$, or $p = 3$ and λ is one of the following partitions:*

$$(2,2),\ (3,2,1),\ (5,4),(2^4,1),(4,3,2),(3^2,2,1),\ (5,5),(2^5).$$

This theorem shows that, apart from the few exceptions arising for small symmetric groups at the prime 3, given any natural number $n \in \mathbb{N}$ which is not a power of p, the restriction to P_n of any irreducible character of $\mathfrak{S}_n$ has the trivial character $\mathbb{1}_{P_n}$ as a constituent. Hence the positivity of $Z^\chi_{\mathbb{1}_{P_n}}$ is completely described for odd primes.

Notice that this does not hold if $p = 2$. For instance, the sign representation of $\mathfrak{S}_n$ restrict irreducibly and non-trivially to a Sylow 2-subgroup of $\mathfrak{S}_n$.

When p is odd, a similar description of the permutation character exists also for the alternating groups case. This is [24, Theorem C].

In [25], the work of [24] has been largely extended by considering the entire set $\mathrm{Lin}(P_n)$ of linear characters of P_n. In particular, for any linear character ϕ of P_n they studied the set $\Omega(\phi)$ consisting of all those irreducible characters χ of $\mathfrak{S}_n$ such that $Z^\chi_\phi \neq 0$.

Fix a prime $p \geq 5$, let $n \in \mathbb{N}$ and $P_n \in \mathrm{Syl}_p(\mathfrak{S}_n)$. Let ϕ be any linear character of P_n. Recalling that the set $\mathrm{Irr}(\mathfrak{S}_n)$ of ordinary irreducible characters of $\mathfrak{S}_n$ is naturally in bijection with the set $\mathscr{P}(n)$ of partitions of n, we may view $\Omega(\phi)$ as a subset of $\mathscr{P}(n)$; in other words, we set

$$\Omega(\phi) = \left\{ \lambda \in \mathscr{P}(n) \mid Z^\lambda_\phi \neq 0 \right\},$$

where, for simplicity, we used the symbol Z^λ_ϕ to denote $Z^{\chi^\lambda}_\phi$.

Let $m(\phi)$ and $M(\phi)$ be the integers defined as follows:

$$m(\phi) := \max\{\, t \in \mathbb{N} \mid \mathscr{B}_n(t) \subseteq \Omega(\phi) \,\} \text{ and } M(\phi) := \min\{\, t \in \mathbb{N} \mid \Omega(\phi) \subseteq \mathscr{B}_n(t) \,\}.$$

In [25, Theorem B], $m(\phi)$ and $M(\phi)$ are explicitly computed for any linear character ϕ of P_n and $p \geq 5$. This gives a very precise description of $\Omega(\phi)$ for all $\phi \in \mathrm{Lin}(P_n)$. In fact:

$$\mathscr{B}_n(m(\phi)) \subseteq \Omega(\phi) \subseteq \mathscr{B}_n(M(\phi)),$$

and the values $m(\phi)$ and $M(\phi)$ are closed to each other for all $\phi \in \mathrm{Lin}(P_n)$.

Notice that for the prime 3 the set $\Omega(\mathbb{1}_{P_n})$ has already been described in [24].

The paper [25] concludes with an asymptotic result. Namely, *almost all* $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$ share the following property: $Z^\chi_\phi \neq 0$ for all $\phi \in \mathrm{Lin}(P_n)$. More precisely:

Theorem 3.1.4 ([25, Theorem C]). *Let $p \geq 5$ be a prime and $n \in \mathbb{N}$. Let Ω_n be the intersection of all the sets $\Omega(\phi)$ where ϕ is free to run among the elements of $\mathrm{Lin}(P_n)$. Then*

$$\lim_{n\to\infty} \frac{|\Omega_n|}{|\mathscr{P}(n)|} = 1.$$

So far the recorded results on Sylow branching coefficients hold for p an odd prime. Usually, when $p = 2$ the situation is completely different. For instance, $\Omega(\phi)$ is no longer closed under conjugation in general, and there is not a conjecture for the structure of the sets $\Omega(\phi)$, where ϕ is a linear character of a Sylow 2-subgroup of $\mathfrak{S}_n$. Indeed, it is still an open problem for the prime 2 to determine $\Omega(\mathbb{1}_{P_n})$.

By this time, probably the reader has already noticed that there are not many results concerning constituents of the restriction $\chi\big\downarrow_{P_n}$ that are not linear. We were able to say something about this topic. We refer to Chapters 4 and 5 for our results.

3.2 The case $p = 2$ for $\mathfrak{S}_n$

From Section 3.1, it can be noticed a fracture between the knowledge accumulated on Sylow branching coefficients for the symmetric groups at odd primes and the lack of information on this topic when the prime p is equal to 2. For instance, as we already pointed out, the irreducible constituents of the permutation character $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{S}_n}$ are completely described for odd primes, but are far from being understood when $p = 2$.

We collect below some computations of Sylow branching coefficients for $p = 2$ that can be done for specific type of characters. Our contribution on the study of this case is the subject of Chapter 5.

3.2.1 On the McKay bijection

Some results concerning linear constituents of the restriction of an irreducible character to a Sylow 2-subgroup has been obtained while studying the *McKay conjecture*. This is one of the most important among the local-global conjectures mentioned in the introduction, because many other conjectures and research topics arise from it.

J. McKay [53] originally formulated his conjecture only for G a simple group and for the prime $p = 2$, We would like to state it in full generality here:

Conjecture 3.2.1 (McKay, 1972). *Let G be a finite group and p be a prime. Let P be a Sylow p-subgroup of G. Then*

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(N_G(P))|,$$

where $\mathrm{Irr}_{p'}$ denotes the subset of those ordinary irreducible characters of G of degree coprime to p, and $N_G(P)$ denotes the normaliser of P.

This conjecture holds for all finite groups of odd order at all primes p ([36]), for symmetric and general linear groups ([64]), and for every finite group for the primes 2 and 3 ([57] and [68]), where the reduction to simple groups in [38] is used.

If G is the symmetric group $\mathfrak{S}_n$, $p = 2$ and P_n is a Sylow 2-subgroup, then $N_{\mathfrak{S}_n}(P_n)$ is actually P_n. Hence the restriction of irreducible characters to Sylow 2-subgroups was studied also for its connection to the McKay conjecture. We can find some results on the topic in [39] and [20]. The achievements in the latter paper were fundamental in [23] to construct a canonical bijection between $\mathrm{Irr}_{2'}(\mathfrak{S}_n)$ and $\mathrm{Irr}_{2'}(P_n)$ for any natural number n and $P_n \in \mathrm{Syl}_2(\mathfrak{S}_n)$. Furthermore, they were used to prove the more general theorems in [28] that we already recorded in the previous section.

More precisely, in [20, Theorem 1.2] is proved that for every $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$ its restriction to a Sylow 2-subgroup P_n contains a linear constituent. Moreover, if n is a power of 2 and χ has odd degree, this linear constituent is unique.

By [20, Lemma 3.1] we know that the irreducible characters of odd degree of $\mathfrak{S}_{2^k}$ are exactly those labelled by hook partitions of 2^k. Recall that we denote by $\mathscr{H}(n)$ the set of hook partitions of the natural number n. Having said that, we would like to record here [20, Theorem 1.1].

Theorem 3.2.2. *Let $n \in \mathbb{N}$ and let $\lambda \in \mathscr{H}(2^k)$. Then $\chi^\lambda\big\downarrow_{P_{2^k}}$ admits a unique linear constituent. Such a constituent appears with multiplicity* 1.

In Theorem 3.2.3 below we identify this unique constituent by describing the associated $\{0,1\}$-sequence that characterizes linear characters of P_{2^k}. We refer to Section 2.3.1 for the description of the irreducible characters of the Sylow 2-subgroups.

The previous theorem was also made explicit in Proposition 3.2.4 and 3.2.5 of [50]. His result use the canonical basis of the Specht modules labelled by hook partitions.

From now on we will adopt the following notation. Given $z \in \mathbb{Z}$, we let $[z] \in \{0,1\}$ be such that $z \equiv [z] \bmod 2$.

Theorem 3.2.3. *Let $k \in \mathbb{N}$ and let $\lambda = (2^k - x, 1^x) \in \mathscr{H}(2^k)$ where $x = a_k 2^k + a_{k-1}2^{k-1} + \cdots + a_0 2^0$ is its binary expansion. The unique linear constituent of $\chi^\lambda\big\downarrow_{P_{2^k}}$ is*

$$\mathscr{X}\left([a_k + a_{k-1}], [a_{k-1} + a_{k-2}], \ldots, [a_1 + a_0]\right).$$

Proof. We proceed by induction on k. If $k = 1$ then $\lambda \in \{(2), (1^2)\}$. If $\lambda = (2)$ then $x = 0 = 0 \cdot 2^1 + 0 \cdot 2^0$, $[a_1 + a_0] = [0 + 0] = 0$ and $\chi^{(2)}\big\downarrow_{P_2} = \phi_0$. Similarly, $\chi^{(1^2)}\big\downarrow_{P_2} = \phi_1$. Let now $k \geq 2$ and $\lambda = \left(2^k - x, 1^x\right)$. We denote by L_λ the unique linear constituent of $\chi^\lambda\big\downarrow_{P_{2^k}}$, as prescribed by Theorem 3.2.2. By the Littlewood-Richardson rule, we have

$$\chi^\lambda\big\downarrow_{\mathfrak{S}_{2^{k-1}} \times \mathfrak{S}_{2^{k-1}}} = \left(\chi^{\left(2^{k-1}-y, 1^y\right)} \times \chi^{\left(2^{k-1}-y, 1^y\right)}\right) + \Delta,$$

where $y = \begin{cases} \frac{x}{2}, & \text{if } x \text{ is even,} \\ \frac{x-1}{2}, & \text{if } x \text{ is odd,} \end{cases}$ and Δ is the sum of irreducible constituents of the form $\phi \times \psi$, for some $\phi, \psi \in \mathrm{Irr}(\mathfrak{S}_{2^{k-1}})$ with $\phi \neq \psi$. It follows that $L_\lambda = \mathscr{X}\left(L_{\left(2^{k-1}-y, 1^y\right)}; \phi_\alpha\right)$, for some $\alpha \in \{0,1\}$. In order to find α, we consider a 2^k-cycle $g \in P_{2^k}$. Since $P_{2^k} = P_{2^{k-1}} \wr P_2$ we can choose $g = (h, 1; \gamma)$, where $h \in P_{2^{k-1}}$ is a 2^{k-1}-cycle and $\gamma \in P_2$ is a 2-cycle. Using Lemma 2.3.4 and Remark 2.3.3, we have that

$$\chi^\lambda(g) = L_\lambda(g) = L_{\left(2^{k-1}-y, 1^y\right)}(h) \cdot \phi_\alpha(\gamma) = \chi^{\left(2^{k-1}-y, 1^y\right)}(h) \cdot \chi^{(2-\alpha, \alpha)}(\gamma).$$

The Murnaghan-Nakayama rule [45, 2.4.7] implies that $(-1)^x = (-1)^y(-1)^\alpha$. It follows that

$$\alpha = \begin{cases} 0, & \text{if } x \equiv y \bmod 2, \\ 1, & \text{otherwise.} \end{cases}$$

Since $x = a_k 2^k + \cdots + a_0 2^0$ is the binary expansion of $x \in [0, 2^k - 1]$, then $y = a_k 2^{k-1} + \cdots + a_1 2^0$ is the binary expansion of $y \in [0, 2^{k-1} - 1]$. Hence, $\alpha = [a_1 + a_0]$ and we have $L_\lambda = \mathscr{X}\left(L_{(2^{k-1}-y, 1^y)}; \phi_{[a_1+a_0]}\right)$. Using the binary expansion of y and the inductive hypothesis we conclude the proof. □

This result is used in Section 5.1 to compute a large family of Sylow branching coefficients. In particular, in Theorem 5.1.4 we calculate Z^χ_ϕ for all $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$ labelled by a hook partition and all $\phi \in \mathrm{Lin}(P_n)$. This is a wide generalization of Theorem 3.2.2.

3.2.2 About the permutation character $\mathbb{1}_{P_n}\big\uparrow^{\mathfrak{S}_n}$

We end this chapter collecting some results about $Z^\chi_{\mathbb{1}_{P_n}}$ for χ an irreducible character of $\mathfrak{S}_n$.

In [51] it has been proved that, when $p = 2$, the probability that the Sylow branching coefficient $Z^\chi_{\mathbb{1}_{P_n}}$ is zero tends to zero as n tends to infinity. More precisely:

Theorem 3.2.4 ([51, Theorem B]). *For n a natural number, let P_n denote a Sylow 2-subgroup of the symmetric group $\mathfrak{S}_n$. Then almost all irreducible characters χ of $\mathfrak{S}_n$ have positive Sylow branching coefficient $Z^\chi_{\mathbb{1}_{P_n}}$. That is,*

$$\lim_{n\to\infty} \frac{\left|\left\{ \chi \in \mathrm{Irr}(\mathfrak{S}_n) \mid Z^\chi_{\mathbb{1}_{P_n}} > 0 \right\}\right|}{|\mathrm{Irr}(\mathfrak{S}_n)|} = 1.$$

Furthermore, in [51, Section 3] we can also find some explicit computations of $Z^{\chi^\lambda}_{\mathbb{1}_{P_n}}$ for partitions λ of various 'special' shapes.

To ease the notation, we denote by Z^λ the Sylow branching coefficient $Z^{\chi^\lambda}_{\mathbb{1}_{P_n}}$. In general, the strategy used in this paper is similar to the one used in our main results later: they consider the case when $\lambda \in \mathscr{P}(2^k)$ and they induct on k, before considering the case of general $n \in \mathbb{N}$. The results follow from a combination of elementary applications of the Littlewood-Richardson rule, Mackey's theorem and known results on character restrictions for symmetric groups.

Lemma 3.2.5 ([51, Lemma 3.1]). *Let λ be any partition of n. If all parts of λ are even, then $Z^\lambda > 0$.*

Lemma 3.2.6 ([51, Lemma 3.2]). *Let $n \in \mathbb{N}$ and $\lambda \in \mathscr{P}(n)$. If n is even and $l(\lambda) > \frac{n}{2}$, then $Z^\lambda = 0$. If n is odd and $l(\lambda) > \frac{n+1}{2}$, then $Z^\lambda = 0$.*

The bound on the number of parts of λ cannot be improved. For instance, from the lemma below we see that $\lambda = (2, 2, \dots, 2, \epsilon) \in \mathscr{P}(n)$ where $\epsilon \in \{0, 1\}$ satisfies $Z^\lambda = 1$ and $l(\lambda) = \frac{n}{2}$ if n is even, respectively $l(\lambda) = \frac{n+1}{2}$ if n is odd.

Lemma 3.2.7 ([51, Lemma 3.4]). *Let λ be a partition with at most two columns. Then $Z^\lambda = 0$ unless $\lambda = (2, 2, \dots, 2, \epsilon)$ where $\epsilon \in \{ 0, 1 \}$, in which case $Z^\lambda = 1$.*

Lemma 3.2.8 ([51, Lemma 3.6]). *Let $k \in \mathbb{N}_{\geq 2}$. If $\lambda = (2^k - i, 2, 1^{i-2}) \in \mathscr{P}(2^k)$ with $2 \leq i \leq 2^k - 2$, then $Z^\lambda = \binom{k-1}{k-i}$.*

3.3 Further research

In this section we collect some ideas for further research.

Let P_n be the Sylow p-subgroup of the symmetric group $\mathfrak{S}_n$. Let λ be a partition of n, and χ^λ the associated irreducible character of $\mathfrak{S}_n$.

As we have seen in the previous sections, we are far from a good understanding of the complete description of the decomposition into irreducible constituents of the restriction $\chi^\lambda{\downarrow}_{P_n}$. In Chapters 4 and 5 we have made some progress, separating the analysis of the case of an odd prime p and the case when $p = 2$.

When p is odd, we are now able to determine if there exists $\phi \in \mathrm{Irr}(P_n)$ of degree p^k among the constituents of $\chi^\lambda{\downarrow}_{P_n}$. This is the content of Theorems 4.3.1 and 4.3.3.

What is still unknown is which are the irreducible constituents of fixed degree (higher than 1) that appear in the decomposition of the restriction. Also, we do not have an explicit formula for the exact values of the Sylow branching coefficients. This problem remains open for constituents of any degree (including 1).

When $p = 2$, we are now able to understand if there exists $\phi \in \mathrm{Irr}(P_n)$ of degree 2^k among the constituents of $\chi^\lambda{\downarrow}_{P_n}$, only if λ is an hook partition. This is Theorem 5.2.11. It is natural to wonder what happens if λ is not a hook partition. This seems very difficult when $p = 2$. For instance we do not even know the constituents of $\mathbb{1}_{P_n}{\uparrow}^{\mathfrak{S}_n}$ in this case, as we already observed.

These seem to be the natural directions of the research on the Sylow branching coefficients. We believe that the combinatorial tools that led us to prove the main results in this thesis can also help to answer some of these open questions.

Let φ be an irreducible character of P_n. Frobenius reciprocity tells us that the Sylow Branching Coefficient of φ in χ^λ is $[\chi^\lambda{\downarrow}_{P_n}, \varphi] = [\chi^\lambda, \varphi{\uparrow}^{\mathfrak{S}_n}]$. Hence the problem of studying the decomposition of $\chi^\lambda{\downarrow}_{P_n}$ can be replaced by the studying of the induction $\varphi{\uparrow}^{\mathfrak{S}_n}$.

In this perspective, the main result of [26] shows that two linear characters of P_n have the same induction to $\mathfrak{S}_n$ if and only if they are $N_{\mathfrak{S}_n}(P_n)$-conjugate. We tried to generalize this result in the case of higher-degree characters. However, we discovered that this does not hold. In fact S. Law and G. Navarro found several families of counterexamples.

Thereby, there are two different directions that the research on this topic can take. The first one is to try to find subgroups $H \leq \mathfrak{S}_n$ with the property that two irreducible character of P_n of fixed degree have the same induction to $\mathfrak{S}_n$ if and only if they have the same induction to H. This is similar to the result in [26], since two irreducible characters of P_n are $N_{\mathfrak{S}_n}(P_n)$-conjugate exactly when they have the same induction to $N_{\mathfrak{S}_n}(P_n)$. Roughly speaking, the question is: if two irreducible characters of P_n have the same induction to $\mathfrak{S}_n$, did they already have the same induction to a smaller subgroup H such that $P_n \leq H \leq \mathfrak{S}_n$?

The second direction for the research is trying to understand what we usually call the *exception pairs*, i.e. the pair of irreducible characters of P_n that have the same induction to $\mathfrak{S}_n$ but are not $N_{\mathfrak{S}_n}(P_n)$-conjugate. As previously mentioned S. Law and G. Navarro found several families of exception pairs, but can we find them all? In joint work with S. Law, we have already been able to describe the action of the normalizer on the irreducible characters of P_n, but it remains to find a characterization of the characters with the same induction to the whole symmetric group $\mathfrak{S}_n$. We are trying to adapt the ideas used in [26] for linear characters, to attack the general situation of non-linear ones.

Our previous work had started from the attempt to generalize Theorem 3.1.1. However, in the same paper [28], also Theorem 3.1.2 seems to worth of some attention. This theorem tells us that if p divides the degree of an irreducible character χ of $\mathfrak{S}_n$, then $\chi\big\downarrow_{P_n}$ has p linear constituents.

Again, there are a couple of natural ways to generalize it. The first one is to trying to understand what happens when the chosen prime p does not divide the degree of the character. The second one is to focus on non-linear characters of the Sylow subgroup. Our question is: can we find a natural number l_k such that if $p^{l_k}|\chi(1)$ then there exist at least p irreducible constituents of $\chi\big\downarrow_{P_n}$ of degree p^k?

Chapter 4

Non-linear Sylow branching coefficients for $\mathfrak{S}_n$

Let $n \in \mathbb{N}$, let p be a prime and let P_n be a fixed Sylow p-subgroup of $\mathfrak{S}_n$. We would like to say something about the higher-degree constituents of the restriction of a character to P_n. As we have seen in the previous chapter, the study about the linear ones has gone quite far, while almost nothing has been said to date about the other constituents.

To be more precise, the main question studied in this chapter is the following. Given $k \in \mathbb{N}$, which and how many irreducible characters of $\mathfrak{S}_n$ admit a constituent of degree p^k in their restriction to P_n? More formally, we let $\mathrm{Irr}_k(P_n)$ denote the set consisting of all the irreducible characters of P_n of degree p^k, and we focus our attention on the subset of $\mathrm{Irr}(\mathfrak{S}_n)$ defined by

$$\Omega_n^k := \{\chi \in \mathrm{Irr}(\mathfrak{S}_n) \mid [\chi\big\downarrow_{P_n}, \phi] \neq 0, \text{ for some } \phi \in \mathrm{Irr}_k(P_n)\}. \tag{4.1}$$

As we already mentioned in Section 3.1, Theorem 3.1.1 proves that the restriction to P_n of any irreducible character of $\mathfrak{S}_n$ admits a linear constituent. In other words, $\Omega_n^0 = \mathrm{Irr}(\mathfrak{S}_n)$. This result was improved (for odd primes) in [25] where, for every linear character ϕ of P_n, the authors classify those irreducible characters χ of $\mathfrak{S}_n$ such that ϕ appears as an irreducible constituent of $\chi\big\downarrow_{P_n}$.

In this chapter we largely extend the result obtained in [28] mentioned above, since, for any odd prime number p, we are able to describe the set Ω_n^k, for any $k \in \mathbb{N}$. Surprisingly enough, these sets possess a quite regular structure.

This chapter includes substantial material taken from [30].

To describe the set Ω_n^k, we first find out which are the possible k among which we can choose, i.e. which are the possible degree for an irreducible character of the Sylow subgroup P_n. This is Lemma 4.1.3.

The study of Ω_n^k is then divided into two cases: the first one is the prime power case, that is when n is a power of the prime p, and the second one is when n is an arbitrary natural number.

Suppose that $n = p^t$ for a fixed natural number t. We know that $P_{p^t} = P_{p^{t-1}} \wr P_p$, and its base group is $\left(P_{p^{t-1}}\right)^{\times p}$. The idea is to work by induction on t. Hence, if λ is a partition of p^t, we restrict χ^λ to $\left(\mathfrak{S}_{p^{t-1}}\right)^{\times p}$ using the Littlewood-Richardson rule. Then we restrict again to the base group of the Sylow subgroup using the inductive case covering $t-1$. We have now two cases: either the character obtained is the product of distinct irreducible characters of $P_{p^{t-1}}$, or it is the product of p times the same character. Clifford theory tells us what happens when we induct the character to the Sylow subgroup P_{p^t}, as we have seen in Section 2.3.1. By Frobenius reciprocity we know that this is equivalent to restrict χ^λ directly to the Sylow subgroup. Indeed, $\left(\left(\chi^\lambda\big\downarrow_{(\mathfrak{S}_{p^{t-1}})^{\times p}}\right)\big\downarrow_{(P_{p^{t-1}})^{\times p}}\right)\big\uparrow^{P_{p^t}}$ has the same irreducible constituents of $\chi^\lambda\big\downarrow_{P_{p^t}}$. This is the main idea beyond the proof of the main results of this chapter for the prime power case. In Lemma 4.1.4 we prove that there are enough irreducible consituents of any fixed degree to be able to work by induction; Theorem 4.2.2 says that the partitions in the set $\Omega^k_{p^t}$ have Young diagram that fits in a fixed grid; finally Theorem 4.2.6 gives us the exact dimension of this grid.

We are able now to describe the case of an arbitrary natural number n, considering its p-adic expansion $n=\sum_{i=1}^t p^{n_i}$ with $n_1 \geq \cdots \geq n_t \geq 0$. Indeed, we can restrict an irreducible character of $\mathfrak{S}_n$ to $\mathfrak{S}_{p^{n_1}} \times \cdots \times \mathfrak{S}_{p^{n_t}}$ using the Littlewood-Richardson rule. Then we can restrict again to $P_n = P_{p^{n_1}} \times \cdots \times P_{p^{n_t}}$ using the knowledge obtained by the study of the prime power case. The main results are Theorem 4.3.1 and Theorem 4.3.3.

4.1 Preliminary results

In this section we start collecting some results that will be used to prove our main theorems in the second part of this chapter. From now on, unless otherwise stated, p denotes an odd prime.

Let $n \in \mathbb{N}$ and let $n=\sum_{i=1}^t p^{n_i}$ be its p-adic expansion, where $n_1 \geq n_2 \geq \cdots \geq n_t \geq 0$. Notice that n_i can be equal to n_{i+1}.

We define the integer α_n as follows. For powers of p we set

$$\alpha_1 = \alpha_p = 0 \text{ and } \alpha_{p^k} = \frac{p^{k-1}-1}{p-1}, \text{ for } k \geq 2.$$

For general $n=\sum_{i=1}^t p^{n_i}$, we set $\alpha_n = \sum_{i=1}^t \alpha_{p^{n_i}}$.

We will show in Lemma 4.1.3 that p^{α_n} is the greatest degree of an irreducible character of P_n.

Note 4.1.1. Let $\nu(m)$ be the highest power of p dividing m, $m \in \mathbb{Z}$, and let $\lfloor q \rfloor$ be the largest integer smaller than q, $q \in \mathbb{Q}$. It is interesting to note that

$$\alpha_n = \nu\left(\left\lfloor \frac{n}{p} \right\rfloor !\right).$$

The proof of this equality is quite technical. We include it for the sake of completeness.

Proof. First we can prove it for the powers of p: if $n=1$ then $\lfloor \frac{n}{p} \rfloor = 0$, and $\nu(\lfloor \frac{n}{p} \rfloor !) = \nu(1) = 0 = \alpha_1$, by definition. We claim that for $k \geq 1$, $\alpha_{p^{k+1}} = (p^k-1)/(p-1) = \nu(\lfloor p^k \rfloor !) = \nu(p^k!)$.

We have that $p^k! = p^k \cdot (p^{k-1}-1)\cdots 2 \cdot 1$, and every number in $[1, p^k-1]$ can be written as $a_{k-1}p^{k-1} + \cdots + a_1 p + a_0$ with $a_i \in [0, p-1]$ for every $i \in [0, k-1]$. Since we want to consider the highest power of p dividing these numbers, we can suppose $a_0 = 0$.

Now, the numbers that are divisible by at most p are the ones for which $a_1 \neq 0$.Hence these are $(p-1)p^{k-2}$ numbers, each of them giving 1 as contribution to the calculation of ν. The ones that are divisible at most by p^2 are the one for which $a_1 = 0$ and $a_2 \neq 0$. Hence these are $(p-1)p^{k-3}$ numbers that give 2 as contribution.

We can continue with this argument for every power of p till we reach p^{k-1}. Recall that we are counting only the numbers in $[1, p^{k-1}-1]$ and forgetting the initial p^k. Hence we have

$$\nu(p^k!) = k + \sum_{i-1}^{k-1}(p-1)p^{k-1-i} \cdot i = k + (p-1)\left[\sum_{i-1}^{k-1} ip^{k-1-i}\right].$$

We can now change the internal variable form i to $j = k-1-i$:

$$\nu(p^k!) = k + (p-1)\left[\sum_{j=0}^{k-2}(k-1-j)p^j\right] = k + (p-1)k\sum_{j=0}^{k-2}p^j - (p-1)\sum_{j=0}^{k-2}(j+1)p^j.$$

The second equality holds by splitting the summand. Now, we know that $\sum_{j=0}^{k-2} p^j = \frac{p^{k-1}-1}{p-1}$. Thus we can simplify the first part of the equation by

$$k + (p-1)k\sum_{j=0}^{k-2}p^j = k + (p-1)k\left(\frac{p^{k-1}-1}{p-1}\right) = k + k(p^{k-1}-1) = kp^{k-1}.$$

About the second summand, we can notice that $(j+1)p^j = \frac{d}{dx}(x^{j+1})|_{x=p}$, where by $\frac{d}{dx}f(x)|_{x=p}$ we denote the derivative of the function $f(x)$ in x, valued in p. Since the summand is finite, the derivative and the summand itself commute. What we have is then

$$\begin{aligned}\sum_{j=0}^{k-2}(j+1)p^j &= \sum_{j=0}^{k-2}\left(\frac{d}{dx}(x^{j+1})|_{x=p}\right) = \frac{d}{dx}\left(\sum_{j=0}^{k-2}x^{j+1}\right)|_{x=p} \\ &= \frac{d}{dx}\left(\frac{x^k-1}{x-1}\right)|_{x=p} = \frac{k(p-1)p^{k-1}-(p^k-1)}{(p-1)^2}.\end{aligned}$$

The above equation thus becames

$$\begin{aligned}\nu(p^k!) &= kp^{k-1} - (p-1)\left(\frac{k(p-1)p^{k-1}-(p^k-1)}{(p-1)^2}\right) \\ &= kp^{k-1} - kp^{k-1} + \frac{p^k-1}{p-1} = \frac{p^k-1}{p-1} = \alpha_{p^{k+1}}.\end{aligned}$$

This conclude the proof of the p-power case.

Let now n be a natural number with $n = \sum_{i=1}^t p^{n_i}$, $n_1 \geq \cdots \geq n_t \geq 0$, its p-adic expansion. By the definition of α_n and the previous computation, we have $\alpha_n = \sum_{i=1}^t \alpha_{p^{n_i}} = \sum_{i=1}^t \nu((p^{n_i-1})!)$. We claim that $\nu\left(\lfloor\frac{n}{p}\rfloor!\right) = \sum_{i=1}^t \nu((p^{n_i-1})!)$.

This can be proved by applying iteratively the following fact: for $n \geq m$,

$$\nu((p^n + p^m)!) = \nu(p^n!) + \nu(p^m!).$$

This is true because $(p^n + p^m)! = (p^n + p^m)(p^n + p^m - 1)\cdots(p^n + 1)p^n(p^n - 1)\cdots 1$, thus the contribution of the numbers in $[p^n + 1, p^n + p^m]$ to the calculation of ν is given only by what is added to p^n, since $m \leq n$. These contributions are exactly the factors of $p^m!$. About the last serie of numbers, i.e. the ones in $[1, p^n]$, these are simply the factors of $p^n!$. Hence the contribution of the two parts to ν is exactly $\nu(p^m!) + \nu(p^n!)$. $\square$

As mentioned before, $\mathrm{Irr}_k(P_n) = \{\theta \in \mathrm{Irr}(P_n) \mid \theta(1) = p^k\}$ denotes the set of irreducible character of P_n of degree p^k. In the following lemma we give a lower bound for the size of this set. The lower bound given is far from being best possible, but it will be sufficient for our purposes.

Lemma 4.1.2. *Let $k, t \in \mathbb{N}$ be such that $p^k \in \mathrm{cd}(P_{p^t})$. Then $|\mathrm{Irr}_k(P_{p^t})| \geq p$.*

Proof. We proceed by induction on t. If $t = 1$ then we know that the statement holds as $\mathrm{Irr}(P_p) = \mathrm{Irr}_0(P_p)$ has size p. As we have seen in Section 2.3.1, the elements of $\mathrm{Irr}(P_p)$ are denoted by $\phi_0, \phi_1, \ldots, \phi_{p-1}$, where we conventionally set ϕ_0 to be the trivial character. Let $t \geq 2$, and let $\psi \in \mathrm{Irr}_k(P_{p^t})$. If $\psi = \mathscr{X}(\theta; \phi_i)$ for some $\theta \in \mathrm{Irr}_k(P_{p^{t-1}})$ and $i \in [0, p-1]$, then $\mathscr{X}(\theta; \phi_j) \in \mathrm{Irr}_k(P_{p^t})$ for all $j \in [0, p-1]$. Hence $|\mathrm{Irr}_k(P_{p^t})| \geq p$. Otherwise $\psi = (\theta_1 \times \cdots \times \theta_p)\big\uparrow^{P_{p^t}}$ where $\theta_1, \ldots, \theta_p \in \mathrm{Irr}(P_{p^{t-1}})$ are not all equal. If there exists $x \in [1, p]$ such that $\theta_1(1) \neq \theta_x(1)$ then we define $\eta_1, \ldots, \eta_p \in \mathrm{Irr}_k(P_{p^t})$ as follows. For any $j \in [1, p]$ we let

$$\eta_j = (\tau_j \times \theta_2 \times \cdots \times \theta_p)\big\uparrow^{P_{p^t}},$$

where $\tau_1, \tau_2, \ldots, \tau_p$ are p distinct irreducible characters of $P_{p^{t-1}}$ of degree $\theta_1(1)$. These exist by the inductive hypothesis. On the other hand, if $\theta_1(1) = \theta_x(1)$ for all $x \in [1, p]$ then we let

$$\eta_1 = (\tau_2 \times \cdots \times \tau_2 \times \tau_1)\big\uparrow^{P_{p^t}}, \text{ and } \eta_j = (\tau_1 \times \cdots \times \tau_1 \times \tau_j)\big\uparrow^{P_{p^t}}, \text{ for all } j \in [2, p].$$

As before, here we chose $\tau_1, \tau_2, \ldots, \tau_p$ to be p distinct irreducible characters of $P_{p^{t-1}}$ of degree $\theta_1(1)$. These exist by the inductive hypothesis. In both cases $\eta_1, \ldots, \eta_p$ are p distinct elements of $\mathrm{Irr}_k(P_{p^t})$. Hence $|\mathrm{Irr}_k(P_{p^t})| \geq p$. $\square$

The next result is significant. It shows that P_n has irreducible characters of each degree $1, p, p^2, \ldots, p^{\alpha_n}$.

Lemma 4.1.3. *Let $n \in \mathbb{N}$. Then $\mathrm{cd}(P_n) = \{p^k \mid k \in [0, \alpha_n]\}$.*

Proof. Let us first suppose that $n = p^t$ is a power of p and proceed by induction on t. The case $t = 1$ is trivial, since P_p is cyclic and $\alpha_p = 0$. If $t \geq 2$, notice that $\alpha_{p^t} = 1 + p\alpha_{p^{t-1}}$. Let $k \in [0, \alpha_{p^t} - 1]$, and let $q \leq \alpha_{p^{t-1}}$ and $r \in [0, p-1]$ be such that $k = qp + r$. If $r = 0$, by the inductive hypothesis there exists $\phi \in \mathrm{Irr}(P_{p^{t-1}})$ such that $\phi(1) = p^q$. Hence for any $\psi \in \mathrm{Irr}(P_p)$, $\mathscr{X}(\phi; \psi) \in \mathrm{Irr}(P_{p^t})$ has degree p^k. If $r > 0$, then $q < \alpha_{p^{t-1}}$. By the inductive hypothesis, there exist $\phi_1, \ldots, \phi_p \in \mathrm{Irr}(P_{p^{t-1}})$ such that $\phi_i(1) = p^{q+1}$ for every $i \in [1, r]$, and $\phi_j(1) = p^q$ for every $j \in [r+1, p]$. Hence $(\phi_1 \times \cdots \times \phi_r \times \phi_{r+1} \times \cdots \times \phi_p)\big\uparrow^{P_{p^t}} \in \mathrm{Irr}(P_{p^t})$ has degree p^k. Finally, let $k = \alpha_{p^t}$. By the inductive hypothesis and by Lemma 4.1.2, there exist $\phi_1, \ldots, \phi_p \in \mathrm{Irr}(P_{p^{t-1}})$

not all equal and such that $\phi_i(1) = p^{\alpha_{p^{t-1}}}$ for all $i \in [1,p]$. Hence $(\phi_1 \times \cdots \times \phi_p)^{P_{p^t}} \in \mathrm{Irr}(P_{p^t})$ has degree p^k. This concludes the proof in the case $n = p^t$, for $t \in \mathbb{N}$.

The case where n is not a power of p follows easily. Indeed, if $n = \sum_{i=1}^{t} p^{n_i}$ is the p-adic expansion of n, where $n_1 \geq n_2 \geq \cdots \geq n_t \geq 0$, then $P_n \cong P_{p^{n_1}} \times P_{p^{n_2}} \times \cdots \times P_{p^{n_t}}$. □

Let $n, k \in \mathbb{N}$. It useful to think of Ω_n^k as a subset of $\mathscr{P}(n)$ instead of $\mathrm{Irr}(\mathfrak{S}_n)$. More precisely, for $\lambda \in \mathscr{P}(n)$, we will sometimes write $\lambda \in \Omega_n^k$ instead of $\chi^\lambda \in \Omega_n^k$.

The following is an important ingredient when proving statements by induction. For an odd prime p let χ be a non-linear character of $\mathfrak{S}_n$ and suppose that $\chi\big\downarrow_{P_n}$ has an irreducible constituent of degree p^k. Then it has at least two distinct irreducible constituents.

Lemma 4.1.4. *Let $n \in \mathbb{N}$ such that $p \leq n$ and let $\lambda \in \Omega_n^k \smallsetminus \{(n), (1^n)\}$ for some $k \in [0, \alpha_n]$. Then there are at least two distinct irreducible constituents of $(\chi^\lambda)\big\downarrow_{P_n}$ of degree p^k.*

Proof. Let us first suppose that $n = p^t$, and let us set $P = P_{p^t}$ and $N = N_{\mathfrak{S}_{p^t}}(P_{p^t})$. We observe that the only N-invariant irreducible character of P is the trivial one. To show this, we let $\mathrm{Irr}_K(P)$ denote the set of K-invariant irreducible characters of P, for any $K \leq N$. Let H be a p'-complement of P in N. Clearly $\mathrm{Irr}_N(P) = \mathrm{Irr}_H(P)$. On the other hand, by Lemma 2.3.6 we know that there are no non-trivial elements of P that are centralized by H. Hence the set $C_P(H) = \{x \in P \mid x^h = x, \text{ for all } h \in H\}$ consists of the only identity element. Using the Glauberman correspondence [37, Theorem 13.1], we get that $|\mathrm{Irr}_H(P)| = |\mathrm{Irr}(C_P(H))| = 1$. It follows that $\mathrm{Irr}_N(P) = \{\mathbb{1}_P\}$, as claimed.

Since $\lambda \notin \{(n), (1^n)\}$, by [25, Lemma 4.3] we know that $(\chi^\lambda)\big\downarrow_P$ necessarily admits a non-trivial linear constituent (direct computations show that this holds also in the case $(p, n, \lambda) = (3, 9, (3, 3, 3))$, which is not covered by the lemma). It follows that for any $k \in \mathbb{N}$ such that $\lambda \in \Omega_n^k$, we can find a non-trivial $\theta \in \mathrm{Irr}_k(P)$ such that θ is a constituent of $(\chi^\lambda)\big\downarrow_P$. Since χ^λ is N-invariant we deduce that every N-conjugate of θ is a constituent of χ^λ. The statement follows. Recalling the structure of P_n described in Section 2.3.1, we observe that the case where n is not a prime power is an easy consequence of the prime power case. □

Definition 4.1.5. Let G be a finite group and let H be a p-subgroup of G. Given a character θ of G, we let $\mathrm{cd}(\theta\big\downarrow_H)$ be the set of degrees of the irreducible constituents of $\theta\big\downarrow_H$. Moreover, we let $\partial_H(\theta)$ be the non-negative integer defined as follows:

$$\partial_H(\theta) = \max\{k \in \mathbb{N} \mid p^k \in \mathrm{cd}(\theta\big\downarrow_H)\}.$$

We refer the reader to Section 2.2.3 for the definition of the Littlewood-Richardson coefficients $\mathscr{LR}(\lambda; \nu_1, \ldots, \nu_p)$.

Proposition 4.1.6. *Let $n \in \mathbb{N}$ and let $Y = (\mathfrak{S}_{p^{n-1}})^{\times p} \leq \mathfrak{S}_{p^n}$ be such that $B \leq Y$, where $B = (P_{p^{n-1}})^{\times p}$ is the base group of P_{p^n}. Let $\lambda \in \mathscr{B}_{p^n}(p^n - 1)$. Then*

$$\partial_{P_{p^n}}(\chi^\lambda) = 1 + \max\{\partial_B(\chi^{\nu_1} \times \cdots \times \chi^{\nu_p}) \mid \chi^{\nu_1} \times \cdots \times \chi^{\nu_p} \in \mathrm{Irr}(Y) \text{ and } \mathscr{LR}(\lambda; \nu_1, \ldots, \nu_p) \neq 0\}.$$

Proof. Let $M = 1 + \max\{\partial_B(\chi^{\nu_1} \times \cdots \times \chi^{\nu_p}) \mid \chi^{\nu_1} \times \cdots \times \chi^{\nu_p} \in \mathrm{Irr}(Y) \text{ and } \mathscr{LR}(\lambda; \nu_1, \ldots, \nu_p) \neq 0\}$. Let $\mu_1, \ldots, \mu_p \in \mathscr{P}(p^{n-1})$ be such that $\mathscr{LR}(\lambda; \mu_1, \ldots, \mu_p) \neq 0$ and $M = 1 + \partial_B(\chi^{\mu_1} \times \cdots \times \chi^{\mu_p})$.

Since $\lambda \notin \{(n), (1^n)\}$, we can assume that $\mu_1, \ldots, \mu_p$ are not all in $\{(p^{n-1}), (1^{p^{n-1}})\}$. Moreover, let ϕ be an irreducible constituent of $(\chi^{\mu_1} \times \cdots \times \chi^{\mu_p})\big\downarrow_B$ such that $\phi(1) = p^{M-1}$. By Lemma 4.1.4, we can take $\phi = \phi_1 \times \cdots \times \phi_p$ with $\phi_1, \ldots, \phi_p \in \mathrm{Irr}(P_{p^{n-1}})$ not all equal. Hence $\phi\big\uparrow^{P_{p^n}} \in \mathrm{Irr}(P_{p^n})$, it has degree p^M and $\left[(\chi^\lambda)\big\downarrow_{P_{p^n}}, \phi\big\uparrow^{P_{p^n}}\right] \neq 0$. Thus $p^M \in \mathrm{cd}((\chi^\lambda)\big\downarrow_{P_{p^n}})$.

Now suppose for a contradiction that there exists an integer $N > M$ such that $p^N \in \mathrm{cd}((\chi^\lambda)\big\downarrow_{P_{p^n}})$. Then there exists $\varphi \in \mathrm{Irr}(P_{p^n})$ such that $\left[(\chi^\lambda)\big\downarrow_{P_{p^n}}, \varphi\right] \neq 0$ and $\varphi(1) = p^N$. Let $\phi_1 \times \cdots \times \phi_p$ be an irreducible constituent of $\varphi\big\downarrow_B$. Hence there exist $\mu_1, \ldots, \mu_p \in \mathscr{P}(p^{n-1})$ such that $\mathscr{LR}(\lambda; \mu_1, \ldots, \mu_p) \neq 0$ and $\left[\phi_1 \times \cdots \times \phi_p, (\chi^{\mu_1} \times \cdots \times \chi^{\mu_p})\big\downarrow_B\right] \neq 0$. We have that

$$M > \partial_B(\chi^{\mu_1} \times \cdots \times \chi^{\mu_p}) \geq N - 1,$$

since the degree of $\phi_1 \times \cdots \times \phi_p$ is either p^N or p^{N-1}. Hence $M < N < M + 1$, which is a contradiction. □

Proposition 4.1.7. *Let n be a natural number and let $n = \sum_{i=1}^t p^{n_i}$ be the p-adic expansion of n, where $n_1 \geq n_2 \geq \cdots \geq n_t \geq 0$. Let $Y = \mathfrak{S}_{p^{n_1}} \times \mathfrak{S}_{p^{n_2}} \times \cdots \times \mathfrak{S}_{p^{n_t}}$ be such that $P_n \leq Y \leq \mathfrak{S}_n$, and let λ be a partition of n. Then*

$$\partial_{P_n}(\chi^\lambda) = \max\{\partial_{P_n}(\chi^{\mu_1} \times \cdots \times \chi^{\mu_t}) \mid \chi^{\mu_1} \times \cdots \times \chi^{\mu_t} \in \mathrm{Irr}(Y) \text{ and } \mathscr{LR}(\lambda; \mu_1, \ldots, \mu_t) \neq 0\}.$$

Proof. Since $P_n = P_{p^{n_1}} \times P_{p^{n_2}} \times \cdots \times P_{p^{n_t}} \leq Y$, the statement follows. □

We refer the reader to the end of Section 2.2.3 for the definition of the $\star$ operator.

Lemma 4.1.8. *Let $n \in \mathbb{N}_{\geq 2}$, let $k \in [2, \alpha_{p^n}]$ and let $(a_1, \ldots, a_p) \in \mathscr{C}(k-1)$ be such that $a_i \in [0, \alpha_{p^{n-1}}]$, for all $i \in [1, p]$. Then*

$$\Omega^{a_1}_{p^{n-1}} \star \Omega^{a_2}_{p^{n-1}} \star \cdots \star \Omega^{a_p}_{p^{n-1}} \subseteq \Omega^k_{p^n}.$$

Proof. To ease the notation we let $q = p^{n-1}$. If $\lambda \in \Omega^{a_1}_q \star \Omega^{a_2}_q \star \cdots \star \Omega^{a_p}_q$, by definition there exists an irreducible constituent $\chi^{\mu_1} \times \cdots \times \chi^{\mu_p}$ of $(\chi^\lambda)\big\downarrow_{(\mathfrak{S}_q)^{\times p}}$ such that $\mu_i \in \Omega^{a_i}_q$ for all $i \in [1, p]$. Hence for every $i \in [1, p]$ there exists an irreducible constituent ϕ_i of $(\chi^{\mu_i})\big\downarrow_{P_q}$ such that $\phi_i(1) = p^{a_i}$. Since $k \geq 2$, there exists $j \in [1, p]$ such that $a_j \geq 1$. Hence $\mu_j \notin \{(q), (1^q)\}$. Thus, by Lemma 4.1.4 we can assume that $\phi_1, \ldots, \phi_p$ are not all equal. It follows that $(\phi_1 \times \cdots \times \phi_p)\big\uparrow^{P_{p^n}}$ is an irreducible constituent of $(\chi^\lambda)\big\downarrow_{P_{p^n}}$ of degree equal to p^k. Hence $\lambda \in \Omega^k_{p^n}$. □

Lemma 4.1.9. *Let $n \in \mathbb{N}_{\geq 2}$ and let $n = \sum_{i=1}^t p^{n_i}$ be its p-adic expansion, where $n_1 \geq n_2 \geq \cdots \geq n_t \geq 0$. Let $k \in [1, \alpha_n]$ and let $(a_1, \ldots, a_t) \in \mathscr{C}(k)$ be such that $a_i \in [0, \alpha_{p^{n_i}}]$, for all $i \in [1, t]$. Then*

$$\Omega^{a_1}_{p^{n_1}} \star \Omega^{a_2}_{p^{n_2}} \star \cdots \star \Omega^{a_t}_{p^{n_t}} \subseteq \Omega^k_{p^n}.$$

Proof. Recall that $P_n \cong P_{p^{n_1}} \times P_{p^{n_2}} \times \cdots \times P_{p^{n_t}}$ and let $\lambda \in \Omega^{a_1}_{p^{n_1}} \star \Omega^{a_2}_{p^{n_2}} \star \cdots \star \Omega^{a_t}_{p^{n_t}}$. By definition, for every $i \in [1, t]$ there exists $\phi_i \in \mathrm{Irr}(P_{p^{n_i}})$ with $\phi_i(1) = p^{a_i}$, such that $\phi_1 \times \cdots \times \phi_t$ is an irreducible constituent of $(\chi^\lambda)\big\downarrow_{P_n}$ of degree p^k. Hence $\lambda \in \Omega^k_{p^n}$. □

4.2 The prime power case

The aim of this section is to completely describe the sets $\Omega_{p^n}^k$ for all odd primes p, all natural numbers n and all $k \in [0, \alpha_{p^n}]$. We remind the reader that from Theorem 3.1.1, we know that $\Omega_{p^n}^0 = \mathscr{B}_{p^n}(p^n)$, for all $n \in \mathbb{N}$. Equivalently, every irreducible character of $\mathfrak{S}_{p^n}$ admits a linear constituent on restriction to a Sylow p-subgroup. This result will be used frequently, with no further reference.

We start by analysing the cases where $k \in \{1,2\}$. In the next lemma we show that for $j = 1, 2$, every non-linear character of $\mathfrak{S}_{p^n}$ affords an irreducible constituent of degree p^j on restriction to a Sylow p-subgroup P_{p^n}, as long as P_{p^n} has an irreducible character of degree p^j.

Lemma 4.2.1. *Let $n \in \mathbb{N}$ and $k \in [1, \alpha_{p^n}] \cap \{1,2\}$. Then $\Omega_{p^n}^k = \mathscr{B}_{p^n}(p^n - 1)$.*

Proof. Let $k = 1$. Then necessarily $n \geq 2$. By considering $\mathscr{B}_{p^n}(p^n-1)$ rather than $\mathscr{B}_{p^n}(p^n)$, we are excluding the linear characters $\chi^{(p^n)}$ and $\chi^{(1^{p^n})}$ of $\mathfrak{S}_{p^n}$. These two characters cannot have an irreducible constituent of degree p in their restriction to P_{p^n}. Hence we can observe that clearly $\Omega_{p^n}^1 \subseteq \mathscr{B}_{p^n}(p^n - 1)$.

On the other hand, if $\lambda \in \mathscr{B}_{p^n}(p^n-1)$, then there exist $\mu_1 \in \mathscr{B}_{p^{n-1}}(p^{n-1}-1)$ and $\mu_2, \ldots, \mu_p \in \mathscr{P}(p^{n-1})$ such that $\mathscr{LR}(\lambda; \mu_1, \ldots, \mu_p) \neq 0$. Using Lemma 4.1.4 we deduce that $(\chi^{\mu_1})\big\downarrow_{P_{p^{n-1}}}$ admits two distinct linear constituents. Therefore, there exists $\phi_1, \ldots, \phi_p \in \mathrm{Lin}(P_{p^{n-1}})$ not all equal and such that ϕ_i is a constituent of $(\chi^{\mu_i})\big\downarrow_{P_{p^{n-1}}}$, for all $i \in [1,p]$. It follows that $(\phi_1 \times \cdots \times \phi_p)\big\uparrow^{P_{p^n}}$ is an irreducible constituent of $(\chi^\lambda)\big\downarrow_{P_{p^n}}$ of degree p. We conclude that $\lambda \in \Omega_{p^n}^1$ and hence that $\Omega_{p^n}^1 = \mathscr{B}_{p^n}(p^n - 1)$.

Let $k = 2$. Then necessarily $n \geq 3$. It is clear that $\Omega_{p^n}^2 \subseteq \mathscr{B}_{p^n}(p^n-1)$. On the other hand, if $\lambda \in \mathscr{B}_{p^n}(p^n-1)$, then there exist $\mu_1 \in \mathscr{B}_{p^{n-1}}(p^{n-1}-1)$ and $\mu_2, \ldots, \mu_p \in \mathscr{P}(p^{n-1})$ such that $\mathscr{LR}(\lambda; \mu_1, \ldots, \mu_p) \neq 0$. We can now argue exactly as above to deduce that $(\chi^\lambda)\big\downarrow_{P_{p^n}}$ admits an irreducible constituent θ of the form $\theta = (\psi \times \phi_1 \times \cdots \times \phi_{p-1})\big\uparrow^{P_{p^n}}$, where $\psi \in \mathrm{Irr}_1(P_{p^{n-1}})$ and $\phi_1, \ldots, \phi_{p-1} \in \mathrm{Lin}(P_{p^{n-1}})$. Hence $\theta(1) = p^2$, $\lambda \in \Omega_{p^n}^2$ and therefore we have that $\Omega_{p^n}^2 = \mathscr{B}_{p^n}(p^n - 1)$. □

Lemma 4.2.1 is a special case of the following more general result.

Theorem 4.2.2. *Let $n \in \mathbb{N}$ and let $k \in [0, \alpha_{p^n}]$. Then there exists $t_n^k \in [\frac{p^n+1}{2}, p^n]$ such that $\Omega_{p^n}^k = \mathscr{B}_{p^n}(t_n^k)$. Moreover, if $k \in [0, \alpha_{p^n} - 1]$, then $t_n^{k+1} \in \{t_n^k - 1, t_n^k\}$.*

Proof. We proceed by induction on n. If $n = 1$, then $\alpha_p = 0$ and $\Omega_p^0 = \mathscr{P}(p)$. If $n \geq 2$, we assume that the statement holds for $n-1$. If $k = 0$ then $\Omega_{p^n}^0 = \mathscr{B}_{p^n}(p^n)$ and $t_n^0 = p^n$. Moreover, by Lemma 4.2.1 we know that $t_n^1 = p^n - 1 = t_n^0 - 1$, as required. The case $k = 1$ is completely treated by Lemma 4.2.1. In fact, we know that $\Omega_{p^n}^1 = \mathscr{B}_{p^n}(p^n - 1)$ and that $t_n^2 = p^n - 1 = t_n^1$, as required. We can now suppose that $k \geq 2$. We define

$$\mathscr{L}(k-1) = \{(j_1, \ldots, j_p) \in \mathscr{C}(k-1) \mid j_i \in [0, \alpha_{p^{n-1}}] \text{ for all } i \in [1,p]\}.$$

Moreover, we set

$$M = \max\left\{ t_{n-1}^{j_1} + \cdots + t_{n-1}^{j_p} \;\middle|\; (j_1, \ldots, j_p) \in \mathscr{L}(k-1) \right\}.$$

Notice that for any $j \in [0, \alpha_{p^{n-1}}]$, the value t_{n-1}^{j} is well-defined by induction as the integer such that $\Omega_{p^{n-1}}^{j} = \mathscr{B}_{p^{n-1}}(t_{n-1}^{j})$. We claim that $M = t_n^k$. In other words, we want to prove that $\Omega_{p^n}^k = \mathscr{B}_{p^n}(M)$. Let $(j_1, \dots, j_p) \in \mathscr{L}(k-1)$ be such that $M = t_{n-1}^{j_1} + \dots + t_{n-1}^{j_p}$. By the inductive hypothesis and by Lemmas 2.2.9 and 4.1.8, we have that

$$\mathscr{B}_{p^n}(M) = \mathscr{B}_{p^{n-1}}(t_{n-1}^{j_1}) \star \dots \star \mathscr{B}_{p^{n-1}}(t_{n-1}^{j_p}) = \Omega_{p^{n-1}}^{j_1} \star \dots \star \Omega_{p^{n-1}}^{j_p} \subseteq \Omega_{p^n}^k.$$

For the opposite inclusion, suppose for a contradiction that $\lambda \in \Omega_{p^n}^k \smallsetminus \mathscr{B}_{p^n}(M)$. Since p is odd, we have that $\Omega_{p^n}^k$ is closed under conjugation of partitions. Hence, we can assume that $\lambda_1 \geq M+1$. Since $\lambda \in \Omega_{p^n}^k$, there exists an irreducible constituent θ of $(\chi^\lambda)\big\downarrow_{P_{p^n}}$ with $\theta(1) = p^k$.
• If $\theta = (\phi_1 \times \dots \times \phi_p)\big\uparrow^{P_{p^n}}$ with $\phi_1, \dots, \phi_p \in \mathrm{Irr}(P_{p^{n-1}})$ not all equal, then there exists $(j_1, \dots, j_p) \in \mathscr{L}(k-1)$ such that $\phi_i(1) = p^{j_i}$ for all $i \in [1,p]$. Then, for every $i \in [1,p]$ there exists an irreducible constituent χ^{μ_i} of $(\phi_i)\big\uparrow^{\mathfrak{S}_{p^{n-1}}}$ such that $\left[\chi^{\mu_1} \times \dots \times \chi^{\mu_p}, (\chi^\lambda)\big\downarrow_{(\mathfrak{S}_{p^{n-1}})^{\times p}}\right] \neq 0$. Hence using the inductive hypothesis, we have that $\mu_i \in \Omega_{p^{n-1}}^{j_i} = \mathscr{B}_{p^{n-1}}(t_{n-1}^{j_i})$, for all $i \in [1,p]$. Hence

$$M \geq t_{n-1}^{j_1} + \dots + t_{n-1}^{j_p} \geq \lambda_1 \geq M+1,$$

where the first inequality holds by definition of M and the second one by Lemma 2.2.8. This is a contradiction.
• On the other hand, if $\theta = \mathscr{X}(\phi; \psi)$ for some $\phi \in \mathrm{Irr}(P_{p^{n-1}})$ and $\psi \in \mathrm{Irr}(P_p)$, then, $\phi(1) = p^{\frac{k}{p}}$ and there exist $\mu_1, \dots, \mu_p \in \Omega_{p^{n-1}}^{\frac{k}{p}}$ such that $\mathscr{LR}(\lambda; \mu_1, \dots, \mu_p) \neq 0$. Hence, using the inductive hypothesis we have that

$$\lambda \in \left(\Omega_{p^{n-1}}^{\frac{k}{p}}\right)^{\star p} = \left(\mathscr{B}_{p^{n-1}}(t_{n-1}^{\frac{k}{p}})\right)^{\star p}.$$

Here we denoted by $A^{\star p}$ the p-fold $\star$-product $A \star \dots \star A$. By the inductive hypothesis we also know that $t_{n-1}^{\frac{k}{p}} \in \left\{ t_{n-1}^{\frac{k}{p}-1} - 1, t_{n-1}^{\frac{k}{p}-1} \right\}$. Using Lemma 2.2.8 we obtain that

$$M + 1 \leq \lambda_1 \leq p t_{n-1}^{\frac{k}{p}} \leq (p-1) t_{n-1}^{\frac{k}{p}} + t_{n-1}^{\frac{k}{p}-1} \leq M.$$

This is a contradiction. Notice that the last inequality above follows from the definition of M, as $(\frac{k}{p}, \dots, \frac{k}{p}, \frac{k}{p} - 1) \in \mathscr{L}(k-1)$.

For $k \in [2, \alpha_{p^n} - 1]$, what we have proved so far is summarised here.

$$\Omega_{p^n}^k = \mathscr{B}_{p^n}(T), \text{ with } T = \max\left\{ t_{n-1}^{j_1} + \dots + t_{n-1}^{j_p} \;\middle|\; (j_1, \dots, j_p) \in \mathscr{L}(k-1) \right\}$$
$$\Omega_{p^n}^{k+1} = \mathscr{B}_{p^n}(V), \text{ with } V = \max\left\{ t_{n-1}^{h_1} + \dots + t_{n-1}^{h_p} \;\middle|\; (h_1, \dots, h_p) \in \mathscr{L}(k) \right\}.$$

Let $(j_1, \dots, j_p) \in \mathscr{L}(k-1)$ be such that $T = t_{n-1}^{j_1} + \dots + t_{n-1}^{j_p}$. Without loss of generality, we can assume that $j_1 < \alpha_{p^{n-1}}$. Then $(j_1+1, j_2, \dots, j_p) \in \mathscr{L}(k)$. By the inductive hypothesis we know that $t_{n-1}^{j_1+1} \in \left\{ t_{n-1}^{j_1} - 1, t_{n-1}^{j_1} \right\}$. Hence

$$V \geq t_{n-1}^{j_1+1} + t_{n-1}^{j_2} + \dots + t_{n-1}^{j_p} \in \{ T-1, T \}. \tag{4.2}$$

On the other hand, let $(h_1, \dots, h_p) \in \mathscr{L}(k)$ be such that $V = t_{n-1}^{h_1} + \dots + t_{n-1}^{h_p}$. Since $k \geq 2$, without loss of generality we can assume that $h_1 > 0$. Then $(h_1 - 1, h_2, \dots, h_p) \in \mathscr{L}(k-1)$. Thus, as above:

$$V = t_{n-1}^{h_1} + \dots + t_{n-1}^{h_p} \leq T, \tag{4.3}$$

since $t_{n-1}^{h_1} \in \left\{ t_{n-1}^{h_1 - 1} - 1, t_{n-1}^{h_1 - 1} \right\}$. Inequalities (4.2) and (4.3) imply that $V \in \{ T-1, T \}$. □

We refer the reader to the second part of Example 4.2.8 for a description of the key steps of the proof of Theorem 4.2.2 in a small concrete instance.

The following definitions may seem artificial but are crucial for determining the exact value of t_n^k for all $n, k \in \mathbb{N}$.

Definition 4.2.3. Let $n \in \mathbb{N}_{\geq 2}$ and let $x \in [1, p^{n-2}]$. We define the integers m_x and $\ell(n, x)$ as follows:

$$m_x = \min\{m \mid x \leq p^{m-2}\}, \text{ and } \ell(n, x) = n - m_x + 1.$$

We prove that $\sum_{x=1}^{p^{n-2}} \ell(n, x) = \alpha_{p^n}$ in Lemma 4.2.4 below. For $x \in [1, p^{n-2}]$ we let

$$A_x := \left[\sum_{j=1}^{x-1} \ell(n, j) + 1, \sum_{j=1}^{x} \ell(n, j) \right].$$

We observe that $\{A_1, A_2, \dots, A_{p^{n-2}}\}$ is a partition of $[1, \alpha_{p^n}]$ and that $|A_x| = \ell(n, x)$ for all $x \in [1, p^{n-2}]$. We refer the reader to Example 4.2.8 for a description of these objects in a specific setting.

For the convenience of the reader we give a more informal explanation of Definition 4.2.3 above. For fixed $n \geq 2$, we define an increasing sequence $0 = a_0 < a_1 < a_2 < \dots < a_{p^{n-2}} = \alpha_{p^n}$ as follows. First $a_1 = n - 1$. Then $a_i - a_{i-1} = n - 2$, for $i = 2, \dots, p$. Next $a_i - a_{i-1} = n - 3$, for $i = p+1, p+2, \dots, p^2$. Continue in this manner, we find that $a_i - a_{i-1} = 1$, for $i = p^{n-3} + 1, \dots, p^{n-2}$. Now set $A_i := (a_{i-1}, a_i]$, for $i = 1, \dots, p^{n-2}$. Then $\{A_1, A_2, \dots, A_{p^{n-2}}\}$ is clearly a partition of $[1, p^{n-2}]$.

Lemma 4.2.4. *With the notation in Definition 4.2.3, we have that* $\sum_{x=1}^{p^{n-2}} \ell(n, x) = \alpha_{p^n}$.

Proof. If $n = 2$, then $\ell(2, 1) = 1 = \alpha_{p^2}$. Let $n \geq 3$ and $i \in [0, n-3]$, then for every $x \in [p^i + 1, p^{i+1}]$, $m_x = i + 3$ and $\ell(n, x) = n - i - 2$. Hence

$$\begin{aligned}
\sum_{x=1}^{p^{n-2}} \ell(n, x) &= \ell(n, 1) + \sum_{i=0}^{n-3} \sum_{x=p^i+1}^{p^{i+1}} \ell(n, x) = (n-1) + \sum_{i=0}^{n-3} p^i (p-1)(n-i-2) \\
&= (n-1) - (n-2) + \left(\sum_{i=1}^{n-3} p^i [(n-i-1) - (n-i-2)] \right) + p^{n-2}[n - (n-3) - 2] \\
&= 1 + \left(\sum_{i=1}^{n-3} p^i \right) + p^{n-2} = \alpha_{p^n}.
\end{aligned}$$

□

The following technical lemma will be useful to prove Theorem 4.2.6.

Lemma 4.2.5. *Let $n \in \mathbb{N} \geq 2$. If $x = pa + r$, for some $r \in [0, p-1]$ and $a \in \mathbb{N}$, then*

$$p \cdot \sum_{j=1}^{a} \ell(n-1, j) + r \cdot \ell(n-1, a+1) = \sum_{j=1}^{x} \ell(n, j) - 1.$$

Proof. Notice that $\ell(n,1) = n-1$ and if $y \in [2,p]$, $\ell(n,y) = n-2$. Thus

$$\sum_{y=1}^{p} \ell(n,y) = p\ell(n-1,1) + 1.$$

Moreover, for $j \in \mathbb{N}$ we have that

$$\sum_{y=jp+1}^{jp+p} \ell(n,y) = p\ell(n-1, j+1).$$

This follows by observing that $\ell(n,y) = \ell(n-1, j+1)$, for all $y \in [jp+1, jp+p]$.

Using these facts, we deduce that

$$\begin{aligned}\sum_{j=1}^{x} \ell(n,j) &= \sum_{j=1}^{p} \ell(n,j) + \sum_{j=1}^{a-1} \sum_{y=jp+1}^{jp+p} \ell(n,y) + \sum_{i=1}^{r} \ell(n, ap+i) \\ &= 1 + p\ell(n-1,1) + p\sum_{j=1}^{a-1} \ell(n-1, j+1) + r\ell(n-1, a+1).\end{aligned}$$

□

Following is one of our key results.

Theorem 4.2.6. *Let $n \geq 2$, $k \in [1, \alpha_{p^n}]$ and let $x \in [1, p^{n-2}]$ be such that $k \in A_x$. Then $\Omega^k_{p^n} = \mathscr{B}_{p^n}(p^n - x)$.*

Proof. We proceed by induction on n: if $n = 2$ then $\alpha_{p^2} = 1$ and necessarily $k = 1$ as $A_1 = \{1\}$. By Lemma 4.2.1, we have that $\Omega^1_{p^2} = \mathscr{B}_{p^2}(p^2 - 1)$, as required. If $n \geq 3$, we proceed by induction on the parameter $x \in [1, p^{n-2}]$. For $x = 1$, we want to show that for every $k \in A_1 = [1, \ell(n,1)]$ we have that $\Omega^k_{p^n} = \mathscr{B}_{p^n}(p^n - 1)$. Using Theorem 4.2.2 and Lemma 4.2.1, we know that

$$\Omega^{\ell(n,1)}_{p^n} \subseteq \Omega^k_{p^n} \subseteq \Omega^1_{p^n} = \mathscr{B}_{p^n}(p^n - 1).$$

Hence, it is enough to show that $\Omega^{\ell(n,1)}_{p^n} = \mathscr{B}_{p^n}(p^n - 1)$. Since $\ell(n,1) = \ell(n-1,1) + 1$, we use Lemma 4.1.8, the inductive hypothesis on n and Theorem 3.1.1, to deduce that

$$\Omega^{\ell(n,1)}_{p^n} \supseteq \Omega^{\ell(n-1,1)}_{p^n} \star \left(\Omega^0_{p^n}\right)^{\star p-1} = \mathscr{B}_{p^n}(p^n - 1) \star \left(\mathscr{B}_{p^n}(p^n)\right)^{\star p-1}.$$

Using Lemma 2.2.9 we conclude that $\mathscr{B}_{p^n}(p^n-1)\subseteq\Omega_{p^n}^{\ell(n,1)}$ and therefore that $\mathscr{B}_{p^n}(p^n-1)=\Omega_{p^n}^{\ell(n,1)}$.

Let us now suppose that $x\geq 2$ and that $k\in A_x$. To ease the notation, for any $y\in[1,p^{n-2}]$ we let $f_n(y)=\sum_{j=1}^{y}\ell(n,j)$. With this notation we have that $A_x=[f_n(x-1)+1,f_n(x)]$. Using Theorem 4.2.2 and arguing exactly as above, we observe that in order to show that $\Omega_{p^n}^{k}=\mathscr{B}_{p^n}(p^n-x)$, it is enough to prove that

$$(1)\ \Omega_{p^n}^{f_n(x-1)+1}=\mathscr{B}_{p^n}(p^n-x)\ \text{and that}\ (2)\ \Omega_{p^n}^{f_n(x)}=\mathscr{B}_{p^n}(p^n-x).$$

To prove (1), we start by observing that by the inductive hypothesis we know that the statement holds for any $j\in A_{x-1}$. In particular we have that $\Omega_{p^n}^{f_n(x-1)}=\mathscr{B}_{p^n}(p^n-(x-1))$. By Theorem 4.2.2 it follows that $\Omega_{p^n}^{f_n(x-1)+1}=\mathscr{B}_{p^n}(T)$, for some $T\in\{p^n-x,p^n-(x-1)\}$. It is therefore enough to show that $\lambda=(p^n-(x-1),x-1)\notin\Omega_{p^n}^{f_n(x-1)+1}$. Let $\mu_1,\dots,\mu_p\in\mathscr{P}(p^{n-1})$ be such that $\mathscr{LR}(\lambda;\mu_1,\dots,\mu_p)\neq 0$. By Lemma 2.2.8 for every $i\in[1,p]$, there exists $a_i\in\mathbb{N}$ such that $(\mu_i)_1=p^{n-1}-a_i$ and such that $\sum_{j=1}^{p}a_j\leq x-1$. In particular, for every $i\in[1,p]$ we have that

$$\mu_i\in\mathscr{B}_{p^{n-1}}(p^{n-1}-a_i)\smallsetminus\mathscr{B}_{p^{n-1}}(p^{n-1}-(a_i+1))=\Omega_{p^{n-1}}^{f_{n-1}(a_i)}\smallsetminus\Omega_{p^{n-1}}^{f_{n-1}(a_i)+1},$$

where the equality is guaranteed by the inductive hypothesis on n.

Let $B=\big(P_{p^{n-1}}\big)^{\times p}$ be the base group of P_{p^n} and let $Y=(\mathfrak{S}_{p^{n-1}})^{\times p}\leq\mathfrak{S}_{p^n}$ be such that $B\leq Y$. Let $\eta=\chi^{\mu_1}\times\cdots\times\chi^{\mu_p}\in\mathrm{Irr}(Y)$ and let $x-1=ap+r$, for some $a\in\mathbb{N}$ and $r\in[0,p-1]$. We observe that

$$\begin{aligned}\partial_B(\eta)&=\sum_{j=1}^{p}f_{n-1}(a_j)=\sum_{j=1}^{p}\sum_{i=1}^{a_j}\ell(n-1,i)\\&\leq p\cdot\Big(\sum_{j=1}^{a}\ell(n-1,j)\Big)+r\cdot\ell(n-1,a+1)\\&=\sum_{j=1}^{x-1}\ell(n,j)-1=f_n(x-1)-1.\end{aligned}$$

Here, the inequality follows immediately by observing that $\ell(n-1,s)\geq\ell(n-1,s+1)$ for all $s\in\mathbb{N}$. On the other hand, the third equality holds by Lemma 4.2.5. Using Proposition 4.1.6, we deduce that $\partial_{P_{p^n}}(\chi^\lambda)\leq f_n(x-1)$. It follows that $\lambda\notin\Omega_{p^n}^{f_n(x-1)+1}$, as desired.

To prove (2), we recall that by (1) above we have that $\Omega_{p^n}^{f_n(x-1)+1}=\mathscr{B}_{p^n}(p^n-x)$. Hence, Theorem 4.2.2 implies that $\Omega_{p^n}^{f_n(x)}\subseteq\mathscr{B}_{p^n}(p^n-x)$. On the other hand, writing $x=ap+r$ for some $a\in\mathbb{N}$ and $r\in[0,p-1]$, and using Lemma 4.2.5, we have that:

$$\begin{aligned}\Omega_{p^n}^{f_n(x)}&=\Omega_{p^n}^{1+p\cdot\big(\sum_{j=1}^{a}\ell(n-1,j)\big)+r\cdot\ell(n-1,a+1)}\\&\supseteq\big(\Omega_{p^{n-1}}^{f_{n-1}(a+1)}\big)^{\star r}\star\big(\Omega_{p^{n-1}}^{f_{n-1}(a)}\big)^{\star p-r}\end{aligned}$$

$$= \left(\mathscr{B}_{p^{n-1}}(p^{n-1}-(a+1))\right)^{\star r} \star \left(\mathscr{B}_{p^{n-1}}(p^{n-1}-a)\right)^{\star p-r}$$
$$= \mathscr{B}_{p^n}(p^n - x).$$

Here the first inclusion follows from Lemma 4.1.8. The second equality holds by the inductive hypothesis. Finally, the last equality is given by Lemma 2.2.9. The proof is complete. □

In the following corollary we collect a number of facts useful to have a better understanding of the structure of the sets $\Omega^k_{p^n}$ for all $n \in \mathbb{N}$ and all $k \in [0, \alpha_{p^n}]$.

Corollary 4.2.7. *Let $n \in \mathbb{N}$ and let $1 \leq k < t \leq \alpha_{p^n}$. The following hold.*

(i) $\mathscr{B}_{p^n}(p^n - p^{n-2}) = \Omega^{\alpha_{p^n}}_{p^n} \subseteq \Omega^t_{p^n} \subseteq \Omega^k_{p^n}$.

(ii) $\Omega^k_{p^n} = \Omega^t_{p^n}$ *if, and only if, there exists* $x \in [1, p^{n-2}]$ *such that* $k, t \in A_x$.

(iii) Given $x \in [1, p^{n-2}]$ *we have that* $|\{k \in [1, \alpha_{p^n}] \mid \Omega^k_{p^n} = \mathscr{B}(p^n - x)\}| = \ell(n, x)$.

Proof. Recalling that $|A_x| = \ell(n, x)$ for every $x \in [1, p^{n-2}]$, (i), (ii) and (iii) follow immediately by Theorem 4.2.6. □

Table 4.1: Let $p = 3$. According to Theorem 4.2.6, the structure of $\Omega^k_{p^n}$ is recorded in the entry corresponding to row k and column n.

$\Omega^k_{p^n}$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$k = 0$	$\mathscr{B}_3(3)$	$\mathscr{B}_{3^2}(3^2)$	$\mathscr{B}_{3^3}(3^3)$	$\mathscr{B}_{3^4}(3^4)$
$k = 1$	$\emptyset$	$\mathscr{B}_{3^2}(3^2-1)$	$\mathscr{B}_{3^3}(3^3-1)$	$\mathscr{B}_{3^4}(3^4-1)$
$k = 2$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^3}(3^3-1)$	$\mathscr{B}_{3^4}(3^4-1)$
$k = 3$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^3}(3^3-2)$	$\mathscr{B}_{3^4}(3^4-1)$
$k = 4$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^3}(3^3-3)$	$\mathscr{B}_{3^4}(3^4-2)$
$k = 5$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4-2)$
$k = 6$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4-3)$
$k = 7$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4-3)$
$k = 8$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4-4)$
$k = 9$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4-5)$
$k = 10$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4-6)$
$k = 11$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4-7)$
$k = 12$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4-8)$
$k = 13$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_{3^4}(3^4-9)$
$k = 14$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Example 4.2.8. Let $p = 3$ and fix $n = 4$. Following the notation introduced in Definition 4.2.3, we have $3^{4-2} = 9$ and $\ell(4,1) = 3$, $\ell(4,2) = \ell(4,3) = 2$, $\ell(4,4) = \cdots = \ell(4,9) = 1$. Hence

$$A_1 = \{1,2,3\},\ A_2 = \{4,5\},\ A_3 = \{6,7\},\ A_4 = \{8\},\ A_5 = \{9\}, \ldots, A_9 = \{13\}.$$

Observe that $\{ A_1, \ldots, A_9 \}$ is a partition of $[1, \alpha_{3^4}] = [1, 13]$, as required. Using Theorem 4.2.6, we have a complete description of $\Omega_{3^4}^k$, for all $k \in [1, 13]$. In particular, we have

$$\Omega_{3^4}^1 = \Omega_{3^4}^2 = \Omega_{3^4}^3 = \mathscr{B}_{3^4}(3^4 - 1),\ \Omega_{3^4}^4 = \Omega_{3^4}^5 = \mathscr{B}_{3^4}(3^4 - 2),\ \Omega_{3^4}^6 = \Omega_{3^4}^7 = \mathscr{B}_{3^4}(3^4 - 3),$$

$$\Omega_{3^4}^8 = \mathscr{B}_{3^4}(3^4 - 4),\ \Omega_{3^4}^9 = \mathscr{B}_{3^4}(3^4 - 5), \ldots, \Omega_{3^4}^{13} = \mathscr{B}_{3^4}(3^4 - 9).$$

These sets are recorded in the fourth column of Table 4.1.

We use the second part of this example to illustrate a key step of the proof of Theorem 4.2.2. Let $n = k = 4$. Theorem 4.2.2 tells us that $\Omega_{3^4}^4 = \mathscr{B}_{3^4}(t_4^4)$, therefore we wish to compute t_4^4. Following the notation introduced in the proof of Theorem 4.2.2 we have that

$$\mathscr{L}(3) = \{(j_1, j_2, j_3) \in \mathscr{C}(3) \mid j_i \in [0, \alpha_{3^3}] = [0, 4],\ \text{for all } i \in [1, 3]\} = \{(3, 0, 0), (2, 1, 0), (1, 1, 1)\}.$$

Working by induction we can assume that we know the values t_3^j for every $j \in [0, 4]$. This can be comfortably read off the third column of Table 4.1. We set

$$M = \max \{ t_3^3 + t_3^0 + t_3^0, t_3^2 + t_3^1 + t_3^0, t_3^1 + t_3^1 + t_3^1 \} = \max \{ 3^4 - 2, 3^4 - 2, 3^4 - 3 \} = 3^4 - 2.$$

We conclude that $t_4^4 = M = 3^4 - 2$.

4.3 Arbitrary natural numbers

The aim of this section is to complete our investigation by extending Theorem 4.2.6 to any arbitrary natural number. In order to do this, we first extend Theorem 4.2.2. We recall that p is a fixed odd prime.

Theorem 4.3.1. *Let $n \in \mathbb{N}$ and let $k \in [0, \alpha_n]$. There exists $T_n^k \in [1, n]$ such that $\Omega_n^k = \mathscr{B}_n(T_n^k)$. Moreover, $T_n^{k+1} \in \{T_n^k - 1, T_n^k\}$, for all $k \in [0, \alpha_n - 1]$.*

Proof. We proceed by induction on $n \in \mathbb{N}$. If $n = 1$, then necessarily $k = 0$ and $\Omega_1^0 = \mathscr{B}_1(1)$. If $n \geq 2$, let $n = \sum_{i=1}^t p^{n_i}$ be the p-adic expansion of n, with $n_1 \geq \cdots \geq n_t \geq 0$. By Theorem 4.2.2, for every $i \in [1, t]$ and every $d_i \in [0, \alpha_{p^{n_i}}]$, there exists $t_{n_i}^{d_i} \in [\frac{p^{n_i}}{2} + 1, p^{n_i}]$ such that $\Omega_{p^{n_i}}^{d_i} = \mathscr{B}_{p^{n_i}}\left(t_{n_i}^{d_i}\right)$. Similarly to the procedure used to prove Theorem 4.2.2, we define

$$\mathscr{J}(k) = \{(j_1, \ldots, j_t) \in \mathscr{C}(k) \mid j_i \in [0, \alpha_{p^{n_i}}] \text{ for all } i \in [1, t]\}.$$

Moreover, we set

$$M = \max \left\{ \sum_{i=1}^t t_{n_i}^{d_i} \,\middle|\, (d_1, \ldots, d_t) \in \mathscr{J}(k) \right\}.$$

We claim that $\Omega_n^k = \mathscr{B}_n(M)$.

Let $(d_1, \ldots, d_t) \in \mathscr{J}(k)$ be such that $M = \sum_{i=1}^t t_{n_i}^{d_i}$. Then using Lemma 2.2.9, Theorem 4.2.2 and Lemma 4.1.8 we have that

$$\mathscr{B}_n(M) = \mathscr{B}_{p^{n_1}}\left(t_{n_1}^{d_1}\right) \star \cdots \star \mathscr{B}_{p^{n_t}}\left(t_{n_t}^{d_t}\right) = \Omega_{p^{n_1}}^{d_1} \star \cdots \star \Omega_{p^{n_t}}^{d_t} \subseteq \Omega_n^k.$$

Suppose now for a contradiction that $\lambda \in \Omega_n^k \smallsetminus \mathscr{B}_n(M)$. Without loss of generality we can assume that $\lambda_1 \geq M+1$. Let $\phi = \phi_1 \times \cdots \times \phi_t$ be an irreducible constituent of $(\chi^\lambda)\big\downarrow_{P_n}$ with $\phi_i(1) = p^{d_i}$ for every $i \in [1,t]$ and $\sum_{i=1}^t d_i = k$. Observe that $(d_1,\ldots,d_t) \in \mathscr{J}(k)$. For every $i \in [1,t]$, let $\mu_i \in \mathscr{P}(p^{n_i})$ be such that $[(\chi^{\mu_i})\big\downarrow_{P_{p^{n_i}}}, \phi_i] \neq 0$ and such that $\chi^{\mu_1} \times \cdots \times \chi^{\mu_t}$ is an irreducible constituent of $(\chi^\lambda)\big\downarrow_Y$. Here $Y = \mathfrak{S}_{p^{n_1}} \times \cdots \times \mathfrak{S}_{p^{n_t}} \leq \mathfrak{S}_n$ is chosen so that $P_n \leq Y$. Thus by Theorem 4.2.6, $\mu_i \in \Omega_{p^{n_i}}^{d_i} = \mathscr{B}_{p^{n_i}}\left(t_{n_i}^{d_i}\right)$ for every $i \in [1,t]$. Hence,

$$\lambda \in \mathscr{B}_{p^{n_1}}\left(t_{n_1}^{d_1}\right) \star \cdots \star \mathscr{B}_{p^{n_t}}\left(t_{n_t}^{d_t}\right) = \mathscr{B}_n\left(\sum_{i=1}^t t_{n_i}^{d_i}\right).$$

By Lemma 2.2.8 and our assumptions, we have that

$$M+1 \leq \lambda_1 \leq \sum_{i=1}^t t_{n_i}^{d_i} \leq M,$$

which is a contradiction.

In summary, for $k \in [0, \alpha_n - 1]$ the following holds:

$$\Omega_n^k = \mathscr{B}_n(M), \text{ where } M = \max\left\{ \sum_{i=1}^t t_{n_i}^{d_i} \;\middle|\; (d_1,\ldots,d_t) \in \mathscr{J}(k) \right\}, \text{ and}$$

$$\Omega_n^{k+1} = \mathscr{B}_n(T), \text{ where } T = \max\left\{ \sum_{i=1}^t t_{n_i}^{f_i} \;\middle|\; (f_1,\ldots,f_t) \in \mathscr{J}(k+1) \right\}.$$

Let $(d_1,\ldots,d_t) \in \mathscr{J}(k)$ be such that $M = \sum_{i=1}^t t_{n_i}^{d_i}$. Since $k \leq \alpha_n - 1$, there exists $i \in [1,t]$ such that $d_i \leq \alpha_{p^{n_i}} - 1$. Hence $(d_1,\ldots,d_{i-1},d_i+1,d_{i+1},\ldots,d_t) \in \mathscr{J}(k+1)$ and $t_{n_i}^{d_i+1} \in \left\{ t_{n_i}^{d_i} - 1, t_{n_i}^{d_i} \right\}$, by Theorem 4.2.2. Thus,

$$M - 1 = -1 + \sum_{i=1}^t t_{n_i}^{d_i} \leq t_{n_1}^{d_1} + \cdots + t_{n_{i-1}}^{d_{i-1}} + t_{n_i}^{d_i+1} + t_{n_{i+1}}^{d_{i+1}} + \cdots + t_{n_t}^{d_t} \leq T.$$

On the other hand, let $(f_1,\ldots,f_t) \in \mathscr{J}(k+1)$ be such that $T = \sum_{i=1}^t t_{n_i}^{f_i}$. Without loss of generality we can assume that $f_1 \geq 1$. Then $(f_1 - 1, f_2, \ldots, f_t) \in \mathscr{J}(k)$ and by Theorem 4.2.2, $t_{n_1}^{f_1} \in \left\{ t_{n_1}^{f_1-1} - 1, t_{n_1}^{f_1-1} \right\}$. Hence

$$T = \sum_{i=1}^t t_{n_i}^{f_i} \leq t_{n_1}^{f_1-1} + t_{n_2}^{f_2} + \cdots + t_{n_t}^{f_t} \leq M.$$

It follows that $T = M$ or $T = M-1$. This concludes the proof. □

Theorem 4.3.1 shows that for every $n \in \mathbb{N}$ and $k \in [0, \alpha_n]$ there exists an integer, denoted by T_n^k, such that $\Omega_n^k = \mathscr{B}_n(T_n^k)$. In order to prove our main result, i.e. to precisely compute the value T_n^k for all $n \in \mathbb{N}$ and $k \in [0, \alpha_n]$, we start by fixing some notation that will be kept throughout this section. We remark that for $n < p^2$ we have that P_n is abelian and that $\Omega_n^0 = \mathscr{P}(n)$. For this reason we focus on the case $n \geq p^2$.

Notation 4.3.2. Let $n \geq p^2$ be a natural number and let $n = \sum_{i=1}^{t} p^{n_i}$ be the p-adic expansion of n, where $n_1 \geq n_2 \geq \cdots \geq n_t \geq 0$. Let $\mathscr{R} := \{(i, y) \mid i \in [1, t], \text{ and } y \in [1, p^{n_i - 2}]\}$. Using Definition 4.2.3, we can associate to every $(i, y) \in \mathscr{R}$ the integer $\ell(n_i, y)$.

We define a total order $\triangleright$ on $\mathscr{R}$ as follows. Given (i, y) and (j, z) in $\mathscr{R}$ we say that $(i, y) \triangleright (j, z)$ if and only if one of the following hold:

(i) $\ell(n_i, y) > \ell(n_j, z)$, or

(ii) $\ell(n_i, y) = \ell(n_j, z)$ and $i < j$, or

(iii) $\ell(n_i, y) = \ell(n_j, z)$ and $i = j$ and $y < z$.

Let $N := \left\lfloor \frac{n}{p^2} \right\rfloor$ and notice that $N = |\mathscr{R}|$. Let $\phi : \mathscr{R} \longrightarrow [1, N]$ be the bijection mapping $(i, y) \mapsto x$ if and only if the pair (i, y) is the x-th greatest element in the totally ordered set $(\mathscr{R}, \triangleright)$. We use this bijection to relabel the integers $\ell(n_i, y)$, for all $(i, y) \in \mathscr{R}$. In particular, we let $\ell(x) := \ell(n_i, y)$ if $\phi((i, y)) = x$. Recalling Definition 4.2.3, we observe that the definition of $\triangleright$ implies that $\ell(1) \geq \ell(2) \geq \cdots \geq \ell(N)$.

Finally, for any $\alpha \in [1, N]$ we let $F_n(\alpha) = \sum_{a=1}^{\alpha} \ell(a)$ and $A_\alpha = [\{F_n(\alpha - 1) + 1, F_n(\alpha)\}]$. We observe that $\{A_1, A_2, \ldots, A_N\}$ is a partition of $[1, \alpha_n]$ (this follows easily from Lemma 4.2.4). We refer the reader to Example 4.3.6 for an explicit description of these objects in a concrete case.

Theorem 4.3.3. *Let $n \in \mathbb{N}_{\geq p^2}$ and $k \in [1, \alpha_n]$. Let $x \in [1, N]$ be such that $k \in A_x$. Then $\Omega_n^k = \mathscr{B}_n(n - x)$.*

Proof. As in Notation 4.3.2, let $n = \sum_{i=1}^{t} p^{n_i}$ be the p-adic expansion of n, where $n_1 \geq n_2 \geq \cdots \geq n_t \geq 0$. We proceed by induction on x. If $x = 1$ then $k \in A_1 = [1, \ell(1)] = [1, \ell(n_1, 1)]$, because $\phi((n_1, 1)) = 1$. By Theorem 4.2.6 we know that $\Omega_{p^{n_1}}^k = \mathscr{B}_{p^{n_1}}(p^{n_1} - 1)$. Moreover, $\Omega_{p^m}^0 = \mathscr{B}_{p^m}(p^m)$ for all $m \in \mathbb{N}$. Thus, using first Lemma 4.1.9 and then Lemma 2.2.9, we deduce that

$$\Omega_n^k \supseteq \Omega_{p^{n_1}}^k \star \Omega_{p^{n_2}}^0 \star \cdots \star \Omega_{p^{n_t}}^0 = \mathscr{B}_{p^{n_1}}(p^{n_1} - 1) \star \mathscr{B}_{p^{n_2}}(p^{n_2}) \star \cdots \star \mathscr{B}_{p^{n_t}}(p^{n_t}) = \mathscr{B}_n(n - 1).$$

Since $(n) \notin \Omega_n^k$, we conclude that $\Omega_n^k = \mathscr{B}_n(n - 1)$, as desired. Let us now set $x \geq 2$ and assume that the statement holds for any $s \in A_{x-1} = [F_n(x - 1) + 1, F_n(x)]$. From Theorem 4.3.1 we know that

$$\Omega_n^{F_n(x-1)+1} \subseteq \Omega_n^k \subseteq \Omega_n^{F_n(x)},$$

hence it is enough to show that:

$$(1)\ \Omega_n^{F_n(x-1)+1} = \mathscr{B}_n(n - x), \text{ and that } (2)\ \Omega_n^{F_n(x)} = \mathscr{B}_n(n - x).$$

Here $F_n(y) = \sum_{j=1}^{y} \ell(j)$, exactly as explained in Notation 4.3.2.

To prove (1), we first notice that $\Omega_n^{F_n(x-1)} = \mathscr{B}_n(n - (x - 1))$ by the inductive hypothesis. Hence, Theorem 4.3.1 implies that $\Omega_n^{F_n(x-1)+1} = \mathscr{B}_n(T)$, for some $T \in \{n - x, n - (x - 1)\}$.

Therefore it suffices to prove that $\lambda = (n-(x-1), x-1) \notin \Omega_n^{F_n(x-1)+1}$. Let $\{G_1, G_2, \dots, G_t\}$ be the partition of $[1, x-1]$ defined by

$$G_i = \{y \in [1, x-1] \mid \phi^{-1}(y) = (i, z), \text{ for some } z \in [1, p^{n_i-2}]\}, \text{ for all } i \in [1,t].$$

To ease the notation we let $g_i = |G_i|$ for all $i \in [1,t]$, and we remark that $g_1+g_2+\cdots+g_t = x-1$.

Let $Y = \mathfrak{S}_{p^{n_1}} \times \mathfrak{S}_{p^{n_2}} \times \cdots \times \mathfrak{S}_{p^{n_t}}$ be a Young subgroup of $\mathfrak{S}_n$ containing P_n. For every $i \in [1,t]$ let $\mu^i \in \mathscr{P}(p^{n_i})$ be such that $\mathscr{LR}(\lambda; \mu^1, \dots, \mu^t) \neq 0$. Then Lemma 2.2.8 implies that there exist $a_1, a_2, \dots, a_t \in \mathbb{Z}$ such that

$$(\mu^i)_1 = p^{n_i} - (g_i + a_i) \text{ for all } i \in [1,t], \text{ and such that } \sum_{i=1}^t a_i \leq 0.$$

In particular, using Theorem 4.2.6 we have that for every $i \in [1,t]$,

$$\mu^i \in \mathscr{B}_{p^{n_i}}(p^{n_i} - (g_i+a_i)) \smallsetminus \mathscr{B}_{p^{n_i}}(p^{n_i} - (g_i+a_i+1)) = \Omega_{p^{n_i}}^{f_{n_i}(g_i+a_i)} \smallsetminus \Omega_{p^{n_i}}^{f_{n_i}(g_i+a_i)+1}.$$

Recycling the notation used in the proof of Theorem 4.2.6, here $f_m(a) := \sum_{j=1}^a \ell(m,j)$. It follows that

$$\partial_{P_{p^{n_i}}}(\chi^{\mu^i}) = \sum_{j=1}^{g_i+a_i} \ell(n_i, j) = \begin{cases} \sum_{y \in G_i} \ell(y) + \sum_{j=g_i+1}^{g_i+a_i} \ell(n_i,j) & \text{if } a_i \geq 0, \\ \sum_{y \in G_i} \ell(y) - \sum_{j=g_i+a_i}^{g_i} \ell(n_i,j) & \text{if } a_i < 0. \end{cases}$$

Hence, letting $\chi = \chi^{\mu^1} \times \chi^{\mu^2} \times \cdots \times \chi^{\mu^t}$, we have that

$$\partial_{P_n}(\chi) = \sum_{i=1}^t \sum_{y \in G_i} \ell(y) + E - F, \text{ where } E = \sum_{\substack{i=1 \\ a_i > 0}}^t \sum_{j=g_i+1}^{g_i+a_i} \ell(n_i,j), \text{ and } F = \sum_{\substack{i=1 \\ a_i<0}}^t \sum_{j=g_i+a_i}^{g_i} \ell(n_i,j).$$

We claim that $E - F \leq 0$. To see this, we notice that the definition of the set G_i implies that $\phi((i,y)) > x-1$ for all $y \geq g_i+1$. On the other hand, for the same reasons, we have that $\phi((j,z)) \leq x-1$ for all $z \leq g_j$. Therefore every summand $\ell(n_i, y)$ appearing in E is smaller than or equal to any summand $\ell(n_j, z)$ appearing in F. Since $\sum_{i=1}^t a_i \leq 0$ we have that $E - F \leq 0$, as desired. Using Proposition 4.1.7 we conclude that

$$\partial_{P_n}(\chi^\lambda) \leq \sum_{i=1}^t \sum_{y \in G_i} \ell(y) = \sum_{y=1}^{x-1} \ell(y) = F_n(x-1) < F_n(x-1)+1.$$

Hence $\lambda \notin \Omega_n^{F_n(x-1)+1}$ and therefore $\Omega_n^{F_n(x-1)+1} = \mathscr{B}_n(n-x)$ as required.

To prove (2) we observe that the equality (1) shown above implies that $\Omega_n^{F_n(x)} \subseteq \mathscr{B}_n(n-x)$, by Theorem 4.3.1. To show that the opposite inclusion holds we use an idea that is similar to the one used to prove (1). In particular, we let $\{H_1, H_2, \dots, H_t\}$ be the partition of $[1,x]$ defined by

$$H_i = \{y \in [1,x] \mid \phi^{-1}(y) = (i,z), \text{ for some } z \in [1, p^{n_i-2}]\}, \text{ for all } i \in [1,t].$$

To ease the notation we let $h_i = |H_i|$ for all $i \in [1,t]$, and we remark that $h_1 + h_2 + \cdots + h_t = x$. We also introduce the following notation. For each $i \in [1,t]$, we let

$$\Gamma_i := \sum_{y \in H_i} \ell(y) = \sum_{j=1}^{h_i} \ell(n_i, j) = f_{n_i}(h_i).$$

We observe that $(\Gamma_1, \Gamma_2, \ldots, \Gamma_t) \in \mathscr{C}(F_n(x))$ and that $\Gamma_i \in [0, \alpha_{p^{n_i}}]$, for all $i \in [1,t]$. We can now use Lemma 4.1.9, Theorem 4.2.6 and Lemma 2.2.9 (in this order) to deduce that

$$\Omega_n^{F_n(x)} \supseteq \Omega_{p^{n_1}}^{\Gamma_1} \star \Omega_{p^{n_2}}^{\Gamma_2} \star \cdots \star \Omega_{p^{n_t}}^{\Gamma_t} = \mathscr{B}_{p^{n_1}}(p^{n_1} - h_1) \star \mathscr{B}_{p^{n_2}}(p^{n_2} - h_2) \star \cdots \star \mathscr{B}_{p^{n_t}}(p^{n_t} - h_t) = \mathscr{B}_n(n - x).$$

We obtain that $\Omega_n^{F_n(x)} = \mathscr{B}_n(n-x)$, and the proof is concluded. □

As we have done for the prime power case in Corollary 4.2.7, we record some facts to understand better the set Ω_n^k for every $n \in \mathbb{N}$ and $k \in [0, \alpha_n]$. Keeping the notation introduced in 4.3.2, we recall that $N = \left\lfloor \frac{n}{p^2} \right\rfloor$.

Corollary 4.3.4. *Let $n \in \mathbb{N}$ and $n = \sum_{i=1}^{t} p^{n_i}$ its p-adic expansion, where $n_1 \geq n_2 \geq \cdots \geq n_t \geq 0$. Let $1 \leq k < t \leq \alpha_n$. The following hold.*

(i) $\mathscr{B}_n(n-N) = \Omega_n^{\alpha_n} \subseteq \Omega_n^t \subseteq \Omega_n^k$.

(ii) $\Omega_n^k = \Omega_n^t$ *if, and only if, there exists* $x \in [1,N]$ *such that* $k, t \in A_x$.

(iii) Given $x \in [1,N]$ *we have that* $|\{ k \in [1, \alpha_n] \mid \Omega_n^k = \mathscr{B}_n(n-x) \}| = \ell(x)$.

Proof. Since $|A_x| = \ell(x)$ for every $x \in [1,N]$, (i), (ii) and (iii) hold by Theorem 4.3.3. □

A second consequence of Theorem 4.3.3 is the following asymptotic result.

Corollary 4.3.5. *Let $\Omega_n = \bigcap_k \Omega_n^k$, where k runs over $[0, \alpha_n]$. Then*

$$\lim_{n \to \infty} \frac{|\Omega_n|}{|\mathscr{P}(n)|} = 1.$$

Proof. There are various combinatorial methods to prove this corollary. We use instead a result of Erdős and Lehner [16, (1.4)] which guarantees that, given a function $f(n)$ that diverges as n tends to infinity, then for all but $o(|\mathscr{P}(n)|)$ partitions λ of n, the quantities λ_1 and $l(\lambda)$ lie between $\sqrt{n} \cdot (\frac{\log n}{d} \pm f(n))$ where d is a constant. By Theorem 4.3.3, we observe that $\Omega_n = \Omega_n^{\alpha_n} = \mathscr{B}(n-N)$, where $N = \left\lfloor \frac{n}{p^2} \right\rfloor$. Since $n - N \geq n/2$, the statement follows. □

Example 4.3.6. Let $p = 3$ and $n = 3^3 + 3^2 + 3$. Following Notation 4.3.2, we have $n_1 = 3$, $n_2 = 2$ and $n_3 = 1$. Hence $\mathscr{R} = \{ (1,1), (1,2), (1,3), (2,1) \}$, since $[1, 3^{n_3 - 2}] = \emptyset$. Observe that $|\mathscr{R}| = 4 = \left\lfloor \frac{n}{3^2} \right\rfloor$. Using Definition 4.2.3, we can see that $\ell(3,1) = 2$, $\ell(3,2) = \ell(3,3) = 1$ and $\ell(2,1) = 1$. Hence, the definition of the total order $\rhd$ on $\mathscr{R}$ implies that $(1,1) \rhd (1,2) \rhd (1,3) \rhd (2,1)$. Thus $\ell(1) = 2$, $\ell(2) = \ell(3) = \ell(4) = 1$ and

$$A_1 = \{ 1, 2 \}, \ A_2 = \{ 3 \}, \ A_3 = \{ 4 \}, \ A_4 = \{ 5 \}.$$

Notice that $\{ A_1, \dots, A_4 \}$ is a partition of $[1, \alpha_n] = [1, 5]$, as required. Moreover by Theorem 4.3.3 we have $\Omega_n^1 = \Omega_n^2 = \mathscr{B}_n(n-1)$, $\Omega_n^3 = \mathscr{B}_n(n-2)$, $\Omega_n^4 = \mathscr{B}_n(n-3)$, $\Omega_n^5 = \mathscr{B}_n(n-4)$.

Using the notation of Theorem 4.3.1, the above computation gives that $T_n^2 = n-1$. Following the proof of Theorem 4.3.1, we can compute T_n^2 in a different way. We have

$$\mathscr{J}(2) = \{ (j_1, j_2, j_3) \in \mathscr{C}(2) \mid j_1 \in [0,4],\ j_2 \in [0,1],\ j_3 \in \{ 0 \} \} = \{ (2,0,0), (1,1,0) \}.$$

Hence $M = \max \{ t_3^2 + t_2^0 + t_1^0, t_3^1 + t_2^1 + t_1^0 \} = \max \{ n-1, n-2 \} = n-1$. Thus $T_n^2 = M = n-1$, as expected. Notice that $n_3 = 1$ does not contribute at all to the computations. In fact in $\mathscr{R}$ there are no elements of the form $(3, y)$, $y \in \mathbb{N}$. Furthermore, by looking at the third column of Table 4.2, we can see that $T_n^k = T_{n-3}^k - 3$ for every $k \in \mathbb{N}$. A second example of this fact can be found by observing that the first two columns of Table 4.2 are equal.

Table 4.2: Let $p = 3$. According to Theorem 4.3.3, the structure of Ω_n^k is recorded in the entry corresponding to row k and column n.

Ω_n^k	$n = 3 + 3^3$	$n = 2 \cdot 3 + 3^3$	$n = 3^2 + 3^3$	$n = 3^3 + 3^3$	$n = 3^3 + 3^4$
$k = 0$	$\mathscr{B}_n(n)$	$\mathscr{B}_n(n)$	$\mathscr{B}_n(n)$	$\mathscr{B}_n(n)$	$\mathscr{B}_n(n)$
$k = 1$	$\mathscr{B}_n(n-1)$	$\mathscr{B}_n(n-1)$	$\mathscr{B}_n(n-1)$	$\mathscr{B}_n(n-1)$	$\mathscr{B}_n(n-1)$
$k = 2$	$\mathscr{B}_n(n-1)$	$\mathscr{B}_n(n-1)$	$\mathscr{B}_n(n-1)$	$\mathscr{B}_n(n-1)$	$\mathscr{B}_n(n-1)$
$k = 3$	$\mathscr{B}_n(n-2)$	$\mathscr{B}_n(n-2)$	$\mathscr{B}_n(n-2)$	$\mathscr{B}_n(n-2)$	$\mathscr{B}_n(n-1)$
$k = 4$	$\mathscr{B}_n(n-3)$	$\mathscr{B}_n(n-3)$	$\mathscr{B}_n(n-3)$	$\mathscr{B}_n(n-2)$	$\mathscr{B}_n(n-2)$
$k = 5$	$\emptyset$	$\emptyset$	$\mathscr{B}_n(n-4)$	$\mathscr{B}_n(n-3)$	$\mathscr{B}_n(n-2)$
$k = 6$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_n(n-4)$	$\mathscr{B}_n(n-3)$
$k = 7$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_n(n-5)$	$\mathscr{B}_n(n-3)$
$k = 8$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_n(n-6)$	$\mathscr{B}_n(n-4)$
$k = 9$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_n(n-4)$
$k = 10$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_n(n-5)$
$k = 11$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_n(n-6)$
$k = 12$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_n(n-7)$
$k = 13$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_n(n-8)$
$k = 14$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_n(n-9)$
$k = 15$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_n(n-10)$
$k = 16$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_n(n-11)$
$k = 17$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathscr{B}_n(n-12)$
$k = 18$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Remark 4.3.7. This chapter treats the case of odd primes. When $p = 2$, linear constituents of the restriction to Sylow 2-subgroups of odd degree characters of $\mathfrak{S}_n$ were studied in [20], as we mentioned in Section 3.2.1. Despite this, the object of our study seems to be particularly difficult when $p = 2$. For instance, we immediately notice in this case that the set $\Omega_4^1 = \{(3,1), (2,1,1)\}$ and therefore is not of the form $\mathscr{B}_4(T)$, for any $T \in \{1, 2, 3, 4\}$. This shows that the main

theorems of the present chapter do not hold for the prime 2. Even if this irregularity might disappear for larger natural numbers, more serious obstacles arise in this setting. For example, Lemma 4.1.4 above asserts that the restriction to P_n of every non-linear irreducible character of $\mathfrak{S}_n$ cannot admit a unique irreducible constituent of a certain degree. This is a crucial ingredient in the proofs of our main results. Unfortunately, this is plainly false when the prime is 2. For instance, if $\lambda = (2^n - 1, 1)$ then $(\chi^\lambda)\big\downarrow_{P_{2^n}}$ admits a unique constituent of degree 2^k, for all $k \in \{0, 1, \dots, n-1\}$.

The study of $p = 2$ case will be the subject of next chapter.

4.4 Relation to the alternating groups

We conclude this chapter by describing the analogue of the set Ω_n^k for the alternating groups. The standard reference for the representation theory of alternating groups is [45, Chapter 2.5]. Since $\mathfrak{A}_n$ is a normal subgroup of the symmetric group $\mathfrak{S}_n$, the description of $\mathrm{Irr}(\mathfrak{A})$ is given by the application of Clifford theory.

Let $n \in \mathbb{N}$. Let λ be a partition of n and λ' its conjugate partition. Recall that we denote by χ^λ the irreducible character of $\mathfrak{S}_n$ labelled by λ. If $\lambda \neq \lambda'$ then the restriction $(\chi^\lambda)\big\downarrow_{\mathfrak{A}_n}$ is an irreducible character of the alternating group $\mathfrak{A}_n$. Otherwise, if $\lambda = \lambda'$ then $(\chi^\lambda)\big\downarrow_{\mathfrak{A}_n}$ splits into two irreducible constituents that we will denote by φ_λ^+ and φ_λ^-. If we are not considering the trivial case $n < p$, by [45, Theorem 2.5.7] it follows in particular that $(\varphi_\lambda^+)^g = \varphi_\lambda^-$, where $g \in N_{\mathfrak{S}_n}(P_n) \setminus \mathfrak{A}_n$. All the irreducible characters of $\mathfrak{A}_n$ are of one of these two forms.

If p is an odd prime then $P_n \leq \mathfrak{A}_n$, and so $P_n \in \mathrm{Syl}_p(\mathfrak{A}_n) = \mathrm{Syl}_p(\mathfrak{S}_n)$. We would like to study the set of the irreducible characters of $\mathfrak{A}_n$ that have an irreducible constituent of fixed degree in the restriction to P_n. Let $k \in \mathbb{N}$ be such that $p^k \in \mathrm{cd}(P_n)$. Denote

$$\Gamma_n^k = \left\{ \chi \in \mathrm{Irr}(\mathfrak{A}_n) \mid [\chi\big\downarrow_{P_n}, \phi] \neq 0, \text{ for some } \phi \in \mathrm{Irr}_k(P_n) \right\}.$$

If $\lambda \neq \lambda'$, $((\chi^\lambda)\big\downarrow_{\mathfrak{A}_n})\big\downarrow_{P_n} = (\chi^\lambda)\big\downarrow_{P_n}$. Hence, the irreducible character $(\chi^\lambda)\big\downarrow_{\mathfrak{A}_n}$ has the same constituents of $(\chi^\lambda)\big\downarrow_{P_n}$ when restricted to P_n. That means: if $\lambda \neq \lambda'$ and $\lambda \in \Omega_n^k$ then $(\chi^\lambda)\big\downarrow_{\mathfrak{A}_n} \in \Gamma_n^k$.

Suppose now $\lambda = \lambda' \in \Omega_n^k$. Let θ be an irreducible constituent of $(\chi^\lambda)\big\downarrow_{P_n}$ of degree p^k. We have $(\chi^\lambda)\big\downarrow_{P_n} = ((\chi^\lambda)\big\downarrow_{\mathfrak{A}_n})\big\downarrow_{P_n} = (\varphi_\lambda^+)\big\downarrow_{P_n} + (\varphi_\lambda^-)\big\downarrow_{P_n}$. Therefore, without loss of generality we can suppose that $\left[(\varphi_\lambda^+)\big\downarrow_{P_n}, \theta\right] \neq 0$. Hence

$$\begin{aligned} 0 \neq \left[(\varphi_\lambda^+)\big\downarrow_{P_n}, \theta\right] &= \left[((\varphi_\lambda^+)\big\downarrow_{P_n})^g, \theta^g\right] \\ &= \left[((\varphi_\lambda^+)^g\big\downarrow_{P_n}), \theta^g\right] = \left[((\varphi_\lambda^-)\big\downarrow_{P_n}), \theta^g\right]. \end{aligned}$$

Thus φ_λ^+, $\varphi_\lambda^- \in \Gamma_n^k$, since θ^g is an irreducible character of P_n of the same degree as θ.

The argument above leads us to state this theorem.

Theorem 4.4.1. *Let p be an odd prime, $n \in \mathbb{N}$ and $p^k \in \mathrm{cd}(P_n)$. Then*

$$\Gamma_n^k = \left\{ (\chi^\lambda)\big\downarrow_{\mathfrak{A}_n} \,\middle|\, \lambda \neq \lambda' \in \Omega_n^k \right\} \cup \left\{ \varphi_\lambda^+, \varphi_\lambda^- \mid \lambda = \lambda' \in \Omega_n^k \right\}.$$

Chapter 5

Sylow branching coefficients for $p = 2$

As we already discussed in Chapter 3, there is gap of knowledge between Sylow branching coefficients at odd primes and when the prime p is equal to 2.

The aim of this chapter is to advance in the study of Sylow branching coefficients at the prime 2. Our results are taken from [31].

From now on we fix $p = 2$ and we let P_n be a Sylow 2-subgroup of $\mathfrak{S}_n$. In particular, here we focus on irreducible characters labelled by *hook partitions* $\mathscr{H}(n)$. We denote this subset of $\mathrm{Irr}(\mathfrak{S}_n)$ by $\mathrm{Irr}_{\mathscr{H}}(\mathfrak{S}_n)$.

In this setting we are able to compute the Sylow branching coefficients Z^{χ}_{ϕ} for all $\chi \in \mathrm{Irr}_{\mathscr{H}}(\mathfrak{S}_n)$ and all $\phi \in \mathrm{Lin}(P_n)$. This is Theorem 5.1.4, and its proof requires some technical lemmas that use extensively the Littlewood-Richardson rule.

We then focus on higher degree constituents in Section 5.2. Here we study again the set Ω^k_n, but restricting it to the hook partitions in the set. More precisely for any $k \in \mathbb{N}$ such that $2^k \in \mathrm{cd}(P_n)$, we will study $\Lambda^k_n = \Omega^k_n \cap \mathscr{H}(n)$. We proceed in a similar way to what has been done in Chapter 4: we first explicit which are the possible degrees for an irreducible character of P_n in Proposition 5.2.3, then we divide the study of Λ^k_n into the case when n is a power of 2 and the case when n is an arbitrary natural number.

The substancial difference from the case of an odd prime is that we need to prove that the sets Λ^k_n are one inside the other (Theorem 5.2.5). This is no more a consequence of the description of the sets. Moreover, we cannot prove separately from the main theorems that we have enough irreducible constituents of the same degree to go on with the induction. In the odd prime case this was Lemma 4.1.4, while in this section this result is included as part (2) of both the Theorems 5.2.10 and 5.2.11.

The combinatorial arguments used in this chapter are similar to the ones used before.

5.1 Computing linear Sylow branching coefficients

The goal of this section is to explicitly compute the Sylow branching coefficients Z^{χ}_{ϕ} for all $\chi \in \mathrm{Irr}_{\mathscr{H}}(\mathfrak{S}_n)$ and all $\phi \in \mathrm{Lin}(P_n)$. This generalizes Theorem 3.2.2 to any arbitrary natural number.

The following lemma says that if we restrict an hook character of $\mathfrak{S}_n$ to a Young subgroup $\mathfrak{S}_\alpha$, $\alpha = (n_1, \ldots, n_t) \in \mathscr{C}(n)$, we obtain a sum of products of hook characters of $\mathfrak{S}_{n_i}$.

Lemma 5.1.1. *Let $\alpha = (n_1, \ldots, n_t) \in \mathscr{C}(n)$ and $h \in \mathscr{H}(n)$. Let $\lambda_i \in \mathscr{P}(n_i)$ for all $i \in [1, t]$. If $\mathscr{LR}(h; \lambda_1, \ldots, \lambda_t) \neq 0$ then $\lambda_i \in \mathscr{H}(n_i)$ for all $i \in [1, t]$.*

Proof. This is a direct consequence of the Littlewood-Richardson rule. □

In lemma below we explicitly compute the Littlewood-Richardson coefficient of the restriction of an hook character.

Lemma 5.1.2. *Let $\alpha = (n_1, \ldots, n_t) \in \mathscr{C}(n)$ and $h = (n - x, 1^x) \in \mathscr{H}(n)$. For each $i \in [1, t]$ let $h_i = (n_i - x_i, 1^{x_i}) \in \mathscr{H}(n_i)$, for some $x_i \in [0, n_i - 1]$. Finally, let $y = x - \sum_{i=1}^t x_i$. Then*

$$\mathscr{LR}(h; h_1, \ldots, h_t) = \binom{t-1}{y}.$$

Proof. We proceed by induction on t. If $t = 2$, by Littlewood-Richardson rule we have that $\mathscr{LR}(h; h_1, h_2) = 0$, unless $x_1 + x_2 \in \{x-1, x\} \cap \mathbb{N}$. In such cases we have that $\mathscr{LR}(h; h_1, h_2) = 1$. These facts agree with the desired statement as $t - 1 = 1$. Let us now suppose that $t \geq 3$ and let $K_1 = (n_2 + \cdots + n_t - (x - x_1), 1^{x - x_1})$, $K_2 = (n_2 + \cdots + n_t - (x - x_1 - 1), 1^{x - x_1 - 1}) \in \mathscr{H}(n_2 + \cdots + n_t)$. Using the Littlewood-Richardson rule we observe that $\mathscr{LR}(h; h_1, \theta) = 0$, unless $\theta \in \{K_1, K_2\}$. Moreover, $\mathscr{LR}(h; h_1, K_1) = \mathscr{LR}(h; h_1, K_2) = 1$. By the inductive hypothesis we know that $\mathscr{LR}(K_1; h_2, \ldots, h_t) = \binom{t-2}{y}$ and $\mathscr{LR}(K_2; h_2, \ldots, h_t) = \binom{t-2}{y-1}$. Then, we can conclude that

$$\mathscr{LR}(h; h_1, \ldots, h_t) = \binom{t-2}{y} + \binom{t-2}{y-1} = \binom{t-1}{y}.$$

□

We give an example of the computation of the Littlewood-Richardson coefficient $\mathscr{LR}(\lambda; \mu, \nu)$ in the case $\lambda \in \mathscr{H}(2^n)$ and $\mu, \nu \in \mathscr{H}(2^{n-1})$. This case will be repeatedly used in the following section.

Example 5.1.3. Let $n > 1$ and $\lambda = (2^n - x, 1^x) \in \mathscr{H}(2^n)$, where $x \in [1, 2^n]$. We would like to restrict χ^λ to $Y := \mathfrak{S}_{2^{n-1}} \times \mathfrak{S}_{2^{n-1}}$. Using the Littlewood-Richardson rule we find the following decomposition:

$$\chi^\lambda\big\downarrow_Y = \chi^{(2^{n-1}-x, 1^x)} \times \chi^{(2^{n-1})} + \sum_{y=0}^{x-1} \chi^{(2^{n-1}-y, 1^y)} \times \left(\chi^{(2^{n-1}-(x-y), 1^{x-y})} + \chi^{(2^{n-1}-(x-y-1), 1^{x-y-1})} \right).$$

Notice the symmetry of this decomposition: if $\chi^{(2^{n-1}-a, 1^a)} \times \chi^{(2^{n-1}-b, 1^b)}$ appears as an irreducible constituent, then also $\chi^{(2^{n-1}-b, 1^b)} \times \chi^{(2^{n-1}-a, 1^a)}$ appears, and with the same multiplicity.

Let $\mu = (2^{n-1} - t, 1^t)$ and $\nu = (2^{n-1} - z, 1^z)$. Define

$$V := \{\, (x, 0) \,\} \cup \{\, (y, x-y), (y, x-y-1) \mid y \in [0, x-1] \,\}.$$

We find

$$\mathscr{LR}(\lambda; \mu, \nu) = \begin{cases} 1, & \text{if } (t, z) \in V; \\ 0, & \text{otherwise.} \end{cases}$$

Now we are able to explicitly compute the Sylow branching coefficients Z_ϕ^χ for $\chi \in \mathrm{Irr}_{\mathscr{H}}(\mathfrak{S}_n)$ and $\phi \in \mathrm{Lin}(P_n)$. Let $n \in \mathbb{N}$ and let $n = 2^{k_1}+\cdots+2^{k_t}$ be its binary expansion. Given $\phi \in \mathrm{Lin}(P_n)$, there exist unique $(h_1,\dots,h_t) \in \mathscr{H}(k_1)\times\mathscr{H}(k_2)\times\cdots\times\mathscr{H}(k_t)$ such that $\phi = L_{h_1}\times\cdots\times L_{h_t}$. Here for all $i \in [1,t]$, L_{h_i} denotes the only linear constituent of $\chi^{h_i}\big\downarrow_{P_{2^{k_i}}}$, as described by Theorems 3.2.2 and 3.2.3. In this case, we denote ϕ by $\phi(h_1,\dots,h_t)$.

Theorem 5.1.4. *Let $n \in \mathbb{N}$ and let $k_1 > \cdots > k_t \geq 0$ be such that $n = 2^{k_1}+\cdots+2^{k_t}$. Let $x \in [0,n-1]$ and let $h = (n-x,1^x) \in \mathscr{H}(n)$. For each $i \in [1,t]$ let $x_i \in [0,2^{k_i}-1]$ and set $h_i = (2^{k_i}-x_i,1^{x_i})$. Finally let $y = x - \sum_{i=1}^t x_i$. Then*

$$[\chi^h\big\downarrow_{P_n}, \phi(h_1,\dots,h_t)] = \binom{t-1}{y}.$$

Proof. For each $i \in [1,t]$ we let $n_i = 2^{k_i}$. Set $\alpha = (n_1,\dots,n_t) \in \mathscr{P}(n)$ and let $Y = \mathfrak{S}_\alpha$ be such that $P_n \leq Y$. Let $\mathscr{H} = \mathscr{H}(n_1)\times\mathscr{H}(n_2)\times\cdots\times\mathscr{H}(n_t)$ and $\phi = \phi(h_1,\dots,h_t)$. From Lemma 5.1.1 we know that

$$(\chi^h)\big\downarrow_Y = \sum_{(\lambda_1,\dots,\lambda_t)\in\mathscr{H}} \mathscr{LR}(h;\lambda_1,\dots,\lambda_t)(\chi^{\lambda_1}\times\cdots\times\chi^{\lambda_t}).$$

By Theorem 3.2.2, $[(\chi^{\lambda_1}\times\cdots\times\chi^{\lambda_t})\big\downarrow_{P_n},\phi] = 0$ unless $(\lambda_1,\dots,\lambda_t) = (h_1,\dots,h_t)$. In the latter case we have $[(\chi^{h_1}\times\cdots\times\chi^{h_t})\big\downarrow_{P_n},\phi] = 1$. These observations, together with Lemma 5.1.2 imply that

$$[(\chi^h)\big\downarrow_{P_n},\phi] = [((\chi^h)\big\downarrow_Y)\big\downarrow_{P_n},\phi] = \mathscr{LR}(h;h_1,\dots,h_t) = \binom{t-1}{y}.$$

□

This theorem gives in particular [28, Theorem 3.1], for every natural number n and every hook partition $h \in \mathscr{H}(n)$, the restriction $\chi^h\big\downarrow_{P_n}$ has a linear constituent.

We conclude this section with a result that will prove to be useful in the second part of the chapter. Note that $\chi^{(2^n-1,1)}$ has degree $2^n - 1 = \sum_{k=0}^{n-1} 2^k$. There is a corresponding decomposition of $\chi^{(2^n-1,1)}\big\downarrow_{P_n}$.

Lemma 5.1.5. *Let $n \in \mathbb{N}$, $\lambda = (2^n-1,1)$. Then $\chi^\lambda\big\downarrow_{P_n} = \sum_{k=0}^{n-1}\theta_k$, where $\theta_k \in \mathrm{Irr}(P_{2^n})$ has degree 2^k for $k \in [0,n-1]$.*

Proof. We proceed by induction on n. If $n = 1$ then $\lambda = (1^2)$ and $\chi^\lambda\big\downarrow_{P_2} = \phi_1$, as desired. If $n \geq 2$, by the Littlewood-Richardson rule we have

$$\chi^\lambda\big\downarrow_{\mathfrak{S}_{2^{n-1}}\times\mathfrak{S}_{2^{n-1}}} = \chi^{(2^{n-1})}\times\chi^{(2^{n-1})} + \chi^{(2^{n-1})}\times\chi^{(2^{n-1}-1,1)} + \chi^{(2^{n-1}-1,1)}\times\chi^{(2^{n-1})}.$$

However, $\chi^{(2^{n-1})}\big\downarrow_{P_{2^{n-1}}} = \mathbb{1}_{P_{2^{n-1}}}$ and by the inductive hypothesis, $\chi^{(2^{n-1}-1,1)}\big\downarrow_{P_{2^{n-1}}} = \sum_{i=0}^{n-2}\psi_i$, where $\psi_i \in \mathrm{Irr}(P_{2^{n-1}})$ and $\psi_i(1) = 2^i$. Notice also that $\psi_0 \neq L_{(2^{n-1})} = \mathbb{1}_{P_{2^{n-1}}}$. Hence

$$\chi^\lambda\big\downarrow_{P_{2^{n-1}}\times P_{2^{n-1}}} = \left(\mathbb{1}_{P_{2^{n-1}}}\times\mathbb{1}_{P_{2^{n-1}}}\right) + \sum_{i=0}^{n-2}\left(\mathbb{1}_{P_{2^{n-1}}}\times\psi_i + \psi_i\times\mathbb{1}_{P_{2^{n-1}}}\right).$$

Therefore $\chi^\lambda\big\downarrow_{P_{2^n}} = L_\lambda + \sum_{i=0}^{n-2}\left(\mathbb{1}_{P_{2^{n-1}}} \times \psi_i\right)\big\uparrow^{P_{2^n}}$. The proof is then concluded by observing that $L_\lambda(1) = 2^0$ and that $\left(\mathbb{1}_{P_{2^{n-1}}} \times \psi_i\right)\big\uparrow^{P_{2^n}}(1) = 2^{i+1}$ for every $i \in [0, n-2]$. □

Remark 5.1.6. The characters $\theta_k \in \mathrm{Irr}(P_{2^n})$, for every $k \in [0, n-1]$, have been explicitly determined in the proof of the lemma above. We have indeed a recurcive description: $\theta_0 = \mathscr{X}(0, \ldots, 0, 1)$ and $\theta_k = \left(\mathbb{1}_{P_{2^{n-1}}} \times \psi_{k-1}\right)\big\uparrow^{P_{2^n}}$, where $\chi^{(2^{n-1}-1,1)}\big\downarrow_{P_{2^{n-1}}} = \sum_{i=0}^{n-2}\psi_i$.

5.2 On non-linear Sylow branching coefficients

In Sections 3.2.1 and 5.1 we completely described the linear constituents of the Sylow restriction of irreducible characters labelled by hook partitions. The aim of this section is to continue our investigation by focusing on irreducible constituents of higher degree, mimicking what have been done in Chapter 4. More precisely for any $k \in \mathbb{N}$ such that $2^k \in \mathrm{cd}(P_n)$, we will study the structure of the set Λ_n^k defined as follows:

$$\begin{aligned}\Lambda_n^k &= \Omega_n^k \cap \mathscr{H}(n)\\ &= \left\{ \lambda \in \mathscr{H}(n) \;\middle|\; \chi^\lambda\big\downarrow_{P_n} \text{ has an irreducible constituent of degree } 2^k \right\}.\end{aligned}$$

Recall from Section 2.2.3 the definition of the set $\mathscr{B}_n(t) = \{ \lambda \in \mathscr{P}(n) \mid \lambda_1 \leq t,\ l(\lambda) \leq t \}$. In order to ease the notation, from now on given $n, t \in \mathbb{N}$ such that $t \leq n$ we will let $\mathfrak{H}_n(t)$ be the subset of hook partitions $\mathscr{H}(n)$ defined by

$$\mathfrak{H}_n(t) := \mathscr{B}_n(t) \cap \mathscr{H}(n),$$

i.e. all hook partitions $(n-l, 1^l)$ of n with $n - t < l < t$.
It is an important and easy observation that the sets Λ_n^k and $\mathfrak{H}_n(t)$ are closed under conjugation of partitions.

We start the section with a short example. On one hand this will help the reader understand the behavior of the sets Λ_n^k for small values of n. On the other hand this will serve as base case of some of the later induction arguments.

Example 5.2.1. Here we compute Λ_n^k whenever n is a small power of 2. More precisely we will restrict our attention to the cases $n \in \{2, 4, 8\}$.

If $n = 2$ then there is not much to say as $\mathfrak{S}_2 = P_2$. We just recall the notation introduced in Section 2.3 and write $\mathrm{Irr}(P_2) = \{ \phi_0, \phi_1 \}$ where ϕ_0 is the trivial character of P_2.

Let now $n = 4$. Then $P_4 \cong C_2 \wr C_2$ admits four linear characters $\mathscr{X}(i, j)$, for $i, j \in \{0, 1\}$ and a unique irreducible character φ of degree 2. In particular, $\varphi = (\phi_0 \times \phi_1)\big\uparrow^{P_4}$. It follows that $\mathrm{cd}(P_4) = \{1, 2\}$, hence we will be interested in computing the sets Λ_4^0 and Λ_4^1. In order to do this we are going to study the restriction to P_4 of those irreducible characters of $\mathfrak{S}_4$ that are labelled by partitions contained in the set $\mathscr{H}(4) = \{(4), (3, 1), (2, 1^2), (1^4)\}$. It is not difficult to see that:

$$\chi^{(4)}\big\downarrow_{P_4} = \mathbb{1}_{P_4} = \mathscr{X}(0, 0) \text{ and } \chi^{(3,1)}\big\downarrow_{P_4} = \mathscr{X}(0, 1) + \varphi.$$

Since the sets Λ_4^0 and Λ_4^1 are closed under conjugation of partitions, we conclude that

$$\Lambda_4^0 = \mathscr{H}(4) = \mathfrak{H}_4(4), \text{ and } \Lambda_4^1 = \{(3, 1), (2, 1^2)\} = \mathfrak{H}_4(3) = \mathscr{H}(4) \setminus \{(4), (1^4)\}.$$

Finally, let us consider the case where $n = 8$. Here $P_8 = (P_4 \times P_4) \rtimes P_2 \cong P_4 \wr P_2$. Since the base group $P_4 \times P_4$ is naturally a subgroup of an appropriately chosen Young subgroup $Y \cong \mathfrak{S}_4 \times \mathfrak{S}_4$ of $\mathfrak{S}_8$, our strategy is to restrict irreducible characters of $\mathfrak{S}_8$ to Y, to inductively deduce information on their restriction to $P_4 \times P_4$ and finally to obtain results on their restriction to P_8. Consider for instance $\lambda = (6, 1^2)$ and let $\chi = \chi^\lambda$. Using the Littlewood-Richardson rule we know that $\chi^{(4)} \times \chi^{(2,1^2)}$ is an irreducible constituent of $\chi\big\downarrow_Y$. Using this, together with the calculations we have done for the case $n = 4$ (inductive step) we deduce that $\mathscr{X}(0,0) \times \varphi$ is an irreducible constituent of $\chi\big\downarrow_{P_4 \times P_4}$. We conclude that $(\mathscr{X}(0,0) \times \varphi)\big\uparrow^{P_8}$ is an irreducible constituent (of degree 4) of $\chi\big\downarrow_{P_8}$. This shows that $(6, 1^2) \in \Lambda_8^2$. With completely similar arguments we obtain that

$$\Lambda_8^0 = \mathfrak{H}_8(8) \text{ and that } \Lambda_8^1 = \Lambda_8^2 = \mathfrak{H}_8(7) = \mathscr{H}(8) \setminus \{(8), (1^8)\}.$$

This is all we need to compute since $\mathrm{cd}(P_8) = \{1, 2, 4\}$.

From now on we will denote by β_n the maximal integer k such that 2^k is the degree of an irreducible character of P_n. This is formally defined and explained in the following Definition 5.2.2 and Proposition 5.2.3.

Definition 5.2.2. For any natural number t we define the integer β_{2^t} as follows. We set

$$\beta_1 = \beta_2 = 0, \quad \beta_4 = 1 \text{ and } \beta_{2^t} = 2^{t-2} + 2^{t-3} - 1 \text{ for every } t \geq 3.$$

Notice in particular that $\beta_{2^t} = 2\beta_{2^{t-1}} + 1$, for any $t \geq 4$. Let now $n \in \mathbb{N}$, and $n = \sum_{i=1}^r 2^{n_i}$ be its binary expansion. We define $\beta_n := \sum_{i=1}^r \beta_{2^{n_i}}$.

Proposition 5.2.3. *Given $n \in \mathbb{N}$, we have that $\mathrm{cd}(P_n) = \left\{ 2^j \mid j \in [0, \beta_n] \right\}$. Moreover, if $n \geq 8$, then $|\{\theta \in \mathrm{Irr}(P_n) \mid \theta(1) = 2^{\beta_n}\}| \geq 3$.*

Proof. Suppose first of all that $n = 2^t$ is a power of 2. We proceed by induction on t. If $t \in \{1, 2, 3\}$, the proposition holds as we can see in Example 5.2.1. Let us now fix $t \geq 4$ and recall that $\beta_{2^t} = 2\beta_{2^{t-1}} + 1$. Let $k \in [0, \beta_{2^t} - 1]$ and let $q \in [0, \alpha_{2^{t-1}}]$, $r \in \{0, 1\}$ be such that $k = 2q + r$. Suppose first that $r = 0$, by the inductive hypothesis there exists $\phi \in \mathrm{Irr}(P_{2^{t-1}})$ such that $\phi(1) = 2^q$. Hence for every $\psi \in \mathrm{Irr}(P_2)$, $\mathscr{X}(\phi; \psi) \in \mathrm{Irr}(P_{2^t})$ has degree 2^k. If instead $r = 1$, then $q < \beta_{2^{t-1}}$. If $q = 0$ then $k = 1$, and $|\mathrm{Lin}(P_{2^t})| = 2^t > 2$. Hence we can choose $L, L' \in \mathrm{Lin}(P_{2^t})$, $L \neq L'$ and $(L \times L')\big\uparrow^{P_{2^t}}$ is an irreducible character of degree $2 = 2^k$. If $1 \leq q < \beta_{2^{t-1}}$, by the inductive hypothesis there exist $\phi, \psi \in \mathrm{Irr}(P_{2^{t-1}})$ such that $\phi(1) = 2^{q-1}$, $\psi(1) = 2^{q+1}$. Hence $(\phi \times \psi)\big\uparrow^{P_{2^t}} \in \mathrm{Irr}(P_{2^t})$ and it has degree 2^k. If $k = \beta_{2^t}$, by the inductive hypothesis there exist ψ_1, ψ_2 and ψ_3 distinct irreducible characters of $P_{2^{t-1}}$ of degree $2^{\beta_{2^{t-1}}}$. Since $\beta_{2^t} = 2\beta_{2^{t-1}} + 1$ we obtain that $(\psi_1 \times \psi_2)\big\uparrow^{P_{2^t}}, (\psi_2 \times \psi_3)\big\uparrow^{P_{2^t}}, (\psi_1 \times \psi_3)\big\uparrow^{P_{2^t}} \in \mathrm{Irr}(P_{2^t})$ are three distinct irreducible characters of degree $2^{\beta_{2^t}}$.

Suppose now that $n \in \mathbb{N}$ is arbitrary and let $n = \sum_{i=1}^r 2^{n_i}$ with $n_1 \geq \cdots \geq n_r \geq 0$, be its binary expansion. From Section 2.3 we know that $P_n = P_{2^{n_1}} \times \cdots \times P_{2^{n_r}}$ and therefore that $\mathrm{Irr}(P_n) = \{ \phi_1 \times \cdots \times \phi_r \mid \phi_i \in \mathrm{Irr}(P_{2^{n_i}}),\ i = 1, \ldots, r \}$. Using this together with the information obtained above in the 2-power case, we easily obtain that $\mathrm{cd}(P_n) = \left\{ 2^j \mid j \in [0, \beta_n] \right\}$ and that $|\{\theta \in \mathrm{Irr}(P_n) \mid \theta(1) = 2^{\beta_n}\}| \geq 3$, for any $n \geq 8$. $\square$

Recall that in Section 3.1 we denoted the Sylow branching coefficient corresponding to the characters $\chi^\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$ and $\phi \in \mathrm{Irr}(P_n)$ as Z^λ_ϕ (instead of $Z^{\chi^\lambda}_\phi$).

We begin by giving some precise information concerning Sylow branching coefficients Z^λ_ϕ, where ϕ is a degree 2 irreducible character of P_n. In order to do this, we introduce the following notation. For $\lambda = (2^n - x, 1^x) \in \mathscr{H}(2^n)$ with $x \in [0, 2^n - 1]$, we define $a^x_n \in \mathbb{N}$ as follows:

$$a^x_n := \left| \left\{ \phi \in \mathrm{Irr}(P_{2^n}) \;\middle|\; \phi(1) = 2 \text{ and } Z^\lambda_\phi \neq 0 \right\} \right|.$$

Our first result will be used in the proof of the main theorems of this section.

Lemma 5.2.4. *Let $n \geq 2$ and $\lambda = (2^n - x, 1^x) \in \mathscr{H}(2^n)$. Then $a^x_n = \min\{x, 2^n - 1 - x\}$.*

Proof. Recall that $P_{2^n} = B \rtimes P_2$, for some $B \leq \mathfrak{S}_{2^n}$ such that $B \cong P_{2^{n-1}} \times P_{2^{n-1}}$. Let $Y = \mathfrak{S}_{2^{n-1}} \times \mathfrak{S}_{2^{n-1}} \leq \mathfrak{S}_{2^n}$ be chosen such that $B \leq Y$. Let $\phi \in \mathrm{Irr}(P_{2^n})$ be such that $\phi(1) = 2$. Then $\phi = (L_1 \times L_2)\big\uparrow_B^{P_{2^n}}$, for some $L_1, L_2 \in \mathrm{Lin}(P_{2^{n-1}})$ such that $L_1 \neq L_2$. In particular, we have that $Z^\lambda_\phi \neq 0$ if and only if $\left[\chi^\lambda\big\downarrow_B, L_1 \times L_2\right] \neq 0$. Using this observation together with the Littlewood-Richardson rule and Theorem 3.2.2, we deduce that $Z^\lambda_\phi \neq 0$ if and only if $\left[\chi^\lambda\big\downarrow_Y, \chi_1 \times \chi_2\right] \neq 0$ where χ_i is the unique character in $\mathrm{Irr}_{\mathscr{H}}(\mathfrak{S}_{2^{n-1}})$ such that $\left[\chi_i\big\downarrow_{P_{2^{n-1}}}, L_i\right] \neq 0$.

Let us first suppose that $x \leq 2^n - 1 - x$, in other words $x \in \left[0, 2^{n-1} - 1\right]$. If $x = 0$, then $\lambda = (2^n)$ and $a^0_n = 0$. Otherwise using the Littlewood-Richardson rule we have

$$\begin{aligned}\chi^\lambda\big\downarrow_Y =& \sum_{y=0}^{m} \chi^{\left(2^{n-1}-y, 1^y\right)} \times \left[\chi^{\left(2^{n-1}-(x-y), 1^{x-y}\right)} + \chi^{\left(2^{n-1}-(x-y-1), 1^{x-y-1}\right)}\right] \\ &+ \sum_{y=0}^{m} \left[\chi^{\left(2^{n-1}-(x-y), 1^{x-y}\right)} + \chi^{\left(2^{n-1}-(x-y-1), 1^{x-y-1}\right)}\right] \times \chi^{\left(2^{n-1}-y, 1^y\right)},\end{aligned} \tag{5.1}$$

where $m = \lfloor x/2 \rfloor$. From Equation (5.1) we can see that there are exactly x distinct unordered pairs $(\mu_1, \mu_2) \in \mathscr{H}(2^{n-1}) \times \mathscr{H}(2^{n-1})$ such that $\mu_1 \neq \mu_2$ and such that $\left[\chi^\lambda\big\downarrow_Y, \chi^{\mu_1} \times \chi^{\mu_2}\right] \neq 0$. Using the observations discussed at the start of the proof we deduce that $a^x_n = x$.

If $x \geq 2^n - 1 - x$ then we are in the case $x \in \left[2^{n-1}, 2^n - 1\right]$ and we consider λ', the conjugate partition of λ. We observe that $\lambda' = \left(x + 1, 1^{2^n - x - 1}\right)$ and $0 \leq 2^n - x - 1 \leq 2^{n-1} - 1$. Since $\chi^{\lambda'}\big\downarrow_{P_{2^n}} = \chi^\lambda\big\downarrow_{P_{2^n}} \cdot \chi^{\left(1^{2^n}\right)}\big\downarrow_{P_{2^n}}$, we deduce that $a^x_n = a^{2^n - x - 1}_n = 2^n - x - 1$, as desired. □

Using this lemma we can notice that $a^x_n \neq 0$ if and only if $x \notin \{0, 2^n - 1\}$, that is $\lambda \notin \{(2^n), (1^{2^n})\}$. We deduce that $\mathscr{H}^1_{2^n} = \mathfrak{H}_{2^n}(2^n - 1)$ for every natural number $n \geq 2$.

We are now ready to state and prove one of the main results of this section. This theorem tells us that if an hook character has an irreducible constituent of degree 2^k in its restriction to P_n, then it has also an irreducible constituent of degree 2^ℓ, for every $\ell \leq k$.

Theorem 5.2.5. *Let $n \in \mathbb{N}$. For every $l, k \in [0, \beta_n]$ such that $\ell \leq k$, we have*

$$\Lambda^k_n \subseteq \Lambda^\ell_n.$$

Proof. Suppose first that $n = 2^t$, for some $t \in \mathbb{N}$. Clearly, it is enough to show that

$$\Lambda_{2^t}^k \subseteq \Lambda_{2^t}^{k-1}, \text{ for every } k \in [1, \beta_{2^t}].$$

We proceed by induction on $t \geq 2$. If $t = 2$, then the statement holds by direct computation (see Example 5.2.1). Let $t \geq 3$ and let $\lambda \in \Lambda_{2^t}^k$. By definition there exists an irreducible constituent θ of $\chi^\lambda\big\downarrow_{P_{2^t}}$ of degree 2^k. If $\theta = \mathscr{X}(\psi;\alpha)$ with $\psi \in \mathrm{Irr}(P_{2^{t-1}})$, $\psi(1) = 2^{\frac{k}{2}}$ and $\alpha \in \mathrm{Lin}(P_2)$, then there exist $\mu, \nu \in \mathscr{H}(2^{t-1})$ such that $\mathscr{LR}(\lambda;\mu,\nu) \neq 0$ and ψ is a constituent of both $\chi^\mu\big\downarrow_{P_{2^{t-1}}}$ and $\chi^\nu\big\downarrow_{P_{2^{t-1}}}$. In particular $\mu, \nu \in \Lambda_{2^{t-1}}^{\frac{k}{2}}$. If $\frac{k}{2} > 1$, using the inductive hypothesis for $\frac{k}{2}$ and then for $\frac{k}{2}-1$, we have that $\mu \in \Lambda_{2^{t-1}}^{\frac{k}{2}-2}$. Therefore there exists an irreducible constituent ψ' of $\chi^\mu\big\downarrow_{P_{2^{t-1}}}$ of degree $2^{\frac{k}{2}-2}$. Hence we have that $(\psi' \times \psi)\big\uparrow^{P_{2^t}}$ is an irreducible constituent of $\chi^\lambda\big\downarrow_{P_{2^t}}$ of degree 2^{k-1} and we conclude that $\lambda \in \Lambda_{2^t}^{k-1}$. Otherwise, if $\frac{k}{2} = 1$ then $k = 2$ and $\lambda \notin \{(2^t), (1^{2^t})\}$. Hence $\lambda \in \mathfrak{H}_{2^t}(2^t - 1)$. By Lemma 5.2.4 we know that $\mathfrak{H}_{2^t}(2^t - 1) = \Lambda_{2^t}^1$. It follows that $\lambda \in \Lambda_{2^t}^{k-1} = \Lambda_{2^t}^1$.

Suppose now that $\theta = (\theta_1 \times \theta_2)\big\uparrow^{P_{2^t}}$ with $\theta_1, \theta_2 \in \mathrm{Irr}(P_{2^{t-1}})$, $\theta_1 \neq \theta_2$ and $\theta_1(1) = 2^{h_1}$, $\theta_2(1) = 2^{h_2}$ where $h_1 + h_2 = k - 1$. Then there exist $\mu_1, \mu_2 \in \mathscr{H}(2^{t-1})$ such that $\mathscr{LR}(\lambda;\mu_1,\mu_2) \neq 0$ and θ_1 (respectively θ_2) is an irreducible constituent of $\chi^{\mu_1}\big\downarrow_{P_{2^{t-1}}}$ (respectively $\chi^{\mu_2}\big\downarrow_{P_{2^{t-1}}}$). In particular, $\mu_i \in \Lambda_{2^{t-1}}^{h_i}$ for $i = 1, 2$. Suppose $h_1 = h_2 = \frac{k-1}{2} := h$. If $h = 0$ then $k = 1$ and we know that $\lambda \in \Lambda_{2^t}^0 = \mathscr{H}_{2^t}^{k-1}$ by Theorem 5.1.4. If $h > 0$ then by induction we have that $\mu_1 \in \Lambda_{2^{t-1}}^h \subseteq \Lambda_{2^{t-1}}^{h-1}$. Hence there exists an irreducible constituent ψ of $\chi^{\mu_1}\big\downarrow_{P_{2^{t-1}}}$ of degree 2^{h-1}. Therefore $(\psi \times \theta_2)\big\uparrow^{P_{2^t}}$ is an irreducible constituent of $\chi^\lambda\big\downarrow_{P_{2^t}}$ of degree 2^{k-1}, and so $\lambda \in \Lambda_{2^t}^{k-1}$.

If instead $h_1 \neq h_2$, we can suppose without loss of generality that $h_1 > h_2$. In particular $h_1 \geq 1$, so we can use the inductive hypothesis and we have $\mu_1 \in \Lambda_{2^{t-1}}^{h_1-1}$. Hence there exists an irreducible constituent ψ of $\chi^{\mu_1}\big\downarrow_{P_{2^{t-1}}}$ of degree 2^{h_1-1}. If $\psi \neq \theta_2$ then considering $(\psi \times \theta_2)\big\uparrow^{P_{2^t}}$ we deduce that $\lambda \in \Lambda_{2^t}^{k-1}$, since $\theta_2(1) = 2^{h_2}$ and $h_1 + h_2 = k-1$.. If instead $\psi = \theta_2$ then in particular $h_2 = h_1 - 1$. Suppose $h_2 > 0$ and use the inductive hypothesis. We have $\mu_2 \in \Lambda_{2^{t-1}}^{h_2} \subseteq \Lambda_{2^{t-1}}^{h_2-1}$. Hence there exists $\psi' \in \mathrm{Irr}(P_{2^{t-1}})$ such that $\left[\psi', \chi^{\mu_2}\big\downarrow_{P_{2^{t-1}}}\right] \neq 0$ and $\psi'(1) = 2^{h_2-1}$. We conclude that $\lambda \in \Lambda_{2^t}^{k-1}$ since $(\theta_1 \times \psi')\big\uparrow^{P_{2^t}}$ is an irreducible constituent of $\chi^\lambda\big\downarrow_{P_{2^t}}$ of degree 2^{k-1}. If $h_2 = 0$ then $k = 2$ and arguing as before, using Lemma 5.2.4, we deduce that $\lambda \in \Lambda_{2^t}^1 = \Lambda_{2^t}^{k-1}$.

Let now $n \in \mathbb{N}$ and let $n = \sum_{i=1}^r 2^{n_i}$ be its binary expansion. Let $\lambda \in \Lambda_n^k$, then there exists $\phi \in \mathrm{Irr}(P_n)$ such that $\left[\phi, \chi^\lambda\big\downarrow_{P_n}\right] \neq 0$ and $\phi(1) = 2^k$. By the structure of the irreducible characters of P_n in section 2.3, $\phi = \phi_1 \times \cdots \times \phi_r$, with $\phi_i \in \mathrm{Irr}(P_{2^{n_i}})$ and $\phi_i(1) = 2^{j_i}$ for every $i = 1, \ldots, r$, where $j_1 + \cdots + j_r = k$. Therefore for every $i = 1, \ldots, r$ there exists $\mu_i \in \mathscr{H}(2^{n_i})$ such that ϕ_i is an irreducible constituent of $\chi^{\mu_i}\big\downarrow_{P_{2^{n_i}}}$ and $\mathscr{LR}(\lambda;\mu_1,\ldots,\mu_r) \neq 0$. Let $\ell \leq k$ and for every $i = 1, \ldots, r$ let $d_i \in [0, \alpha_{2^{n_i}}]$, $d_i \leq j_i$ such that $d_1 + \cdots + d_r = \ell$. By previous case, for every $i = 1, \ldots, r$, $\mu_i \in \Lambda_{2^{n_i}}^{j_i} \subseteq \Lambda_{2^{n_i}}^{d_i}$. Hence there exists an irreducible constituent ψ_i of $\chi^{\mu_i}\big\downarrow_{P_{2^{n_i}}}$ of degree 2^{d_i}. Therefore $\psi_1 \times \cdots \times \psi_r$ is an irreducible constituent of $\chi^\lambda\big\downarrow_{P_n}$ of degree 2^ℓ, and so $\lambda \in \Lambda_n^\ell$. $\square$

We introduce here a combinatorial operation between hook partitions that is very similar to the $\star$ operation described before Lemma 2.2.9.

Definition 5.2.6. Let $n, m \in \mathbb{N}$ and $A \subseteq \mathscr{P}(n)$, $B \subseteq \mathscr{P}(m)$.

$$A \diamond B := (A \star B) \cap \mathscr{H}(n+m).$$

The following three lemmas are the analogue of Lemma 2.2.9, Lemma 4.1.8 and Lemma 4.1.9, respectively.

Lemma 5.2.7. *Let $n, n', t, t' \in \mathbb{N}$ be such that $\frac{n}{2} < t \leq n$ and $\frac{n'}{2} < t' \leq n'$. Then*

$$\mathfrak{H}_n(t) \diamond \mathfrak{H}_{n'}(t') = \mathfrak{H}_{n+n'}(t+t').$$

Proof. By definition, we have $\mathfrak{H}_n(t) \subseteq \mathscr{B}_n(t)$ and $\mathfrak{H}_{n'}(t') \subseteq \mathscr{B}_{n'}(t')$. Hence

$$\begin{aligned} \mathfrak{H}_n(t) \diamond \mathfrak{H}_{n'}(t') &\subseteq \mathscr{B}_n(t) \diamond \mathscr{B}_{n'}(t') = \big(\mathscr{B}_n(t) \star \mathscr{B}_{n'}(t')\big) \cap \mathscr{H}(n+n') \\ &= \mathscr{B}_{n+n'}(t+t') \cap \mathscr{H}(n+n') = \mathfrak{H}_{n+n'}(t+t'), \end{aligned}$$

where the second equality holds by Lemma 2.2.9. To prove the other inclusion it is enough to show that $\mathscr{B}_n(t) \diamond \mathscr{B}_{n'}(t') \subseteq \mathfrak{H}_n(t) \diamond \mathfrak{H}_{n'}(t')$. Let $\lambda \in \mathscr{B}_n(t) \diamond \mathscr{B}_{n'}(t')$, then $\lambda \in \mathscr{H}(n+n')$ and there exist $\mu \in \mathscr{B}_n(t)$, $\mu' \in \mathscr{B}_{n'}(t')$ such that $\mathscr{LR}(\lambda; \mu, \mu') \neq 0$. However by Lemma 5.1.1, μ and μ' have to be hook partitions. Therefore $\mu \in \mathfrak{H}_n(t)$, $\mu' \in \mathfrak{H}_{n'}(t')$ and $\lambda \in \mathfrak{H}_n(t) \diamond \mathfrak{H}_{n'}(t')$. □

Lemma 5.2.8. *Let $n \geq 2$ and $i, j \in [0, \beta_{2^{n-1}}]$ be such that $i \neq j$. Then*

$$\Lambda^i_{2^{n-1}} \diamond \Lambda^j_{2^{n-1}} \subseteq \Lambda^{i+j+1}_{2^n}.$$

Proof. Let $\lambda \in \Lambda^i_{2^{n-1}} \diamond \Lambda^j_{2^{n-1}}$. By definition $\lambda \in \mathscr{H}(2^n)$ and there exist $\mu \in \Lambda^i_{2^{n-1}}$, $\nu \in \Lambda^j_{2^{n-1}}$ such that $\mathscr{LR}(\lambda; \mu, \nu) \neq 0$. Hence there exist $\phi, \psi \in \mathrm{Irr}(P_{2^{n-1}})$ such that $\left[\phi, \chi^\mu\big\downarrow_{P_{2^{n-1}}}\right] \neq 0$, $\left[\psi, \chi^\nu\big\downarrow_{P_{2^{n-1}}}\right] \neq 0$ and $\phi(1) = 2^i$, $\psi(1) = 2^j$. Since $i \neq j$, we have that $\theta = (\phi \times \psi)\big\uparrow^{P_{2^n}} \in \mathrm{Irr}(P_{2^n})$. Moreover, $\theta(1) = 2^{i+j+1}$ and $\left[\theta, \chi^\lambda\big\downarrow_{P_{2^n}}\right] \neq 0$. It follows that $\lambda \in \Lambda^{i+j+1}_{2^n}$. □

Lemma 5.2.9. *Let $n \in \mathbb{N}$ and let $n = \sum_{i=1}^r 2^{n_i}$ be its binary expansion. Suppose that for every $i = 1, \dots, r$, $j_i \in [0, \beta_{2^{n_i}}]$ is such that $j_1 + \cdots + j_r = k \in [0, \beta_n]$. Then*

$$\Lambda^{j_1}_{2^{n_1}} \diamond \cdots \diamond \Lambda^{j_r}_{2^{n_r}} \subseteq \Lambda^k_n.$$

Proof. Let $\lambda \in \Lambda^{j_1}_{2^{n_1}} \diamond \cdots \diamond \Lambda^{j_r}_{2^{n_r}}$. By definition, for every $i = 1, \dots, r$ there exists $\mu_i \in \Lambda^{j_i}_{2^{n_i}}$ such that $\mathscr{LR}(\lambda; \mu_1, \dots, \mu_r) \neq 0$. Hence there exists an irreducible constituent ϕ_i of $\chi^{\mu_i}\big\downarrow_{P_{2^{n_i}}}$ of degree 2^{j_i} for every $i = 1, \dots, r$. Since $P_n = P_{2^{n_1}} \times \cdots \times P_{2^{n_r}} \leq \mathfrak{S}_{2^{n_1}} \times \cdots \times \mathfrak{S}_{2^{n_r}} \leq \mathfrak{S}_n$, it follows that $\phi_1 \times \cdots \times \phi_r$ is an irreducible constituent of $\chi^\lambda\big\downarrow_{P_n}$ of degree 2^k. Hence $\lambda \in \Lambda^k_n$. □

We are now ready to state the second main result of the section. In particular, we are able to show that the sets Λ^k_n have a very regular structure. In Theorem 5.2.10 we first deal with the case where n is a power of 2. Then in Theorem 5.2.11 we show that for any $n \in \mathbb{N}$ and any $k \in [0, \beta_n]$ there exists $T^k_n \in \mathbb{N}$ such that $\Lambda^k_n = \mathfrak{H}_n(T^k_n)$.

Notice that we are extending to $p = 2$ the notation for t^k_n and T^k_n, defined in Theorems 4.2.2 and 4.3.1 respectively.

Theorem 5.2.10. *Let $n \in \mathbb{N}$ and $k \in [0, \beta_{2^n}]$. Then:*

(1) there exists $t_n^k \in \left[2^{n-1}+1, 2^n\right]$ such that $\Lambda_{2^n}^k = \mathfrak{H}_{2^n}(t_n^k)$;

(2) if $k > 1$, then for every $\lambda \in \mathfrak{H}_{2^n}(t_n^k - 1)$, $\chi^\lambda\downarrow_{P_{2^n}}$ has at least three distinct irreducible constituents of degree 2^k.

Proof. We proceed by induction on n. If $n \in \{1,2\}$ then the theorem holds (see Example 5.2.1). For $n \geq 3$ we proceed by induction on k. By Theorem 3.2.2 we know that $\Lambda_{2^n}^0 = \mathfrak{H}_{2^n}(2^n)$ and hence that $t_n^0 = 2^n$. If instead $k = 1$, by Lemma 5.2.4 we have that $\Lambda_{2^n}^1 = \mathfrak{H}_{2^n}(2^n - 1)$ and hence that $t_n^1 = 2^n - 1$. Suppose now that $k = 2$. We want to show that $\Lambda_{2^n}^2 = \mathfrak{H}_{2^n}(2^n - 1)$. We have that

$$\mathfrak{H}_{2^n}(2^n-1) = \mathfrak{H}_{2^{n-1}}(2^{n-1}) \diamond \mathfrak{H}_{2^{n-1}}(2^{n-1}-1) = \Lambda_{2^{n-1}}^0 \diamond \Lambda_{2^{n-1}}^1 \subseteq \Lambda_{2^n}^2,$$

where the first equality holds by Lemma 5.2.7 and where the last inclusion holds by Lemma 5.2.8. On the other hand, we clearly have that $\Lambda_{2^n}^2 \subseteq \mathfrak{H}_{2^n}(2^n-1)$, because $(2^n), (1^{2^n}) \notin \Lambda_{2^n}^2$. To conclude, we need to show that for every $\lambda \in \mathfrak{H}_{2^n}(t_n^2 - 1) = \mathfrak{H}_{2^n}(2^n - 2)$, $\chi^\lambda\downarrow_{P_{2^n}}$ has three distinct irreducible constituents of degree 2^2. If $\lambda_1 < 2^n - 2$ then $\lambda \in \mathfrak{H}_{2^n}(2^n - 3) = \mathfrak{H}_{2^{n-1}}(2^{n-1}-3) \diamond \mathfrak{H}_{2^{n-1}}(2^{n-1})$, by Lemma 5.2.7. Hence there exist $\mu \in \mathfrak{H}_{2^{n-1}}(2^{n-1}-3)$ and $\nu \in \mathfrak{H}_{2^{n-1}}(2^{n-1})$ such that $\mathscr{LR}(\lambda;\mu,\nu) \neq 0$. In particular, if $\mu_1 = 2^{n-1} - x \leq 2^{n-1} - 3$ then using Lemma 5.2.4, we deduce that there exist three distinct irreducible constituents θ_1, θ_2 and θ_3 of $\chi^\mu\downarrow_{P_{2^{n-1}}}$ of degree 2. On the other hand, $\nu \in \mathfrak{H}_{2^{n-1}}(2^{n-1}) = \Lambda_{2^{n-1}}^0$. Hence there exists a linear constituent L of $\chi^\nu\downarrow_{P_{2^{n-1}}}$. Therefore $(\theta_1 \times L)\uparrow^{P_{2^n}}, (\theta_2 \times L)\uparrow^{P_{2^n}}$ and $(\theta_3 \times L)\uparrow^{P_{2^n}}$ are three distinct irreducible contituents of $\chi^\lambda\downarrow_{P_{2^n}}$ of degree 2^2. Consider now the case $\lambda_1 = 2^n - 2$, then $\mathscr{LR}(\lambda; (2^{n-1}-1, 1), (2^{n-1}-1,1)) \neq 0 \neq \mathscr{LR}(\lambda; (2^{n-1}-2, 1^2), (2^{n-1}))$. By Lemma 5.1.5, we know that $\chi^{(2^{n-1}-1,1)}\downarrow_{P_{2^{n-1}}}$ has an irreducible consituent θ of degree 2. Hence there exists $\alpha \in \mathrm{Irr}(P_2)$ such that $\mathscr{X}(\theta;\alpha)$ is an irreducible constituent of $\chi^\lambda\downarrow_{P_{2^n}}$ of degree 4. Moreover, by Lemma 5.2.4 we know that $\chi^{(2^{n-1}-2,1^2)}\downarrow_{P_{2^{n-1}}}$ admits two distinct irreducible constituents ψ_1 and ψ_2 of degree 2. We conclude that $\left(\psi_1 \times \mathbb{1}_{P_{2^{n-1}}}\right)\uparrow^{P_{2^n}}$, $\left(\psi_2 \times \mathbb{1}_{P_{2^{n-1}}}\right)\uparrow^{P_{2^n}}$ and $\mathscr{X}(\theta;\alpha)$ are three distinct irreducible constituents of $\chi^\lambda\downarrow_{P_{2^n}}$ of degree 2^2.

Let us now suppose that $k \geq 3$. From now on we will denote $w := \frac{k-1}{2}$. We define $M \in \mathbb{N}$ as follows.

$$M := \max\left\{ t_{n-1}^i + t_{n-1}^j, 2t_{n-1}^w + \delta_{n-1}^w \;\middle|\; i,j,w \in [0, \beta_{2^{n-1}}],\ i+j = k-1 \text{ and } i \neq j \right\},$$

where for $h \in [0, \beta_{2^{n-1}}]$,

t_{n-1}^h is defined inductively such that $\Lambda_{2^{n-1}}^h = \mathfrak{H}_{2^{n-1}}(t_{n-1}^h)$;

$\lambda_{n-1}^h := \left(t_{n-1}^h, 1, \ldots, 1\right) \in \mathfrak{H}_{2^{n-1}}(t_{n-1}^h)$; and

$$\delta_{n-1}^h := \begin{cases} 0, & \text{if } \chi^{\lambda_{n-1}^h}\downarrow_{P_{2^{n-1}}} \text{ has two distinct irreducible constituents of degree } 2^h; \\ -1, & \text{if } \chi^{\lambda_{n-1}^h}\downarrow_{P_{2^{n-1}}} \text{ has a unique irreducible constituent of degree } 2^h. \end{cases}$$

We will now show that $M = t_n^k$, or equivalently that $\Lambda_{2^n}^k = \mathfrak{H}_{2^n}(M)$.

To show that $\Lambda_{2^n}^k \supseteq \mathfrak{H}_{2^n}(M)$, we need to split our discussion into three cases, depending on the value M.

1. First, let us suppose that $M = t_{n-1}^i + t_{n-1}^j$, for some $i, j \in [0, \beta_{2^{n-1}}]$, $i + j = k - 1$ and $i \neq j$. We have

$$\mathfrak{H}_{2^n}(M) = \mathfrak{H}_{2^{n-1}}(t_{n-1}^i) \diamond \mathfrak{H}_{2^{n-1}}(t_{n-1}^j) = \Lambda_{2^{n-1}}^i \diamond \Lambda_{2^{n-1}}^j \subseteq \Lambda_{2^n}^k,$$

respectively by Lemma 5.2.7, inductive hypothesis and Lemma 5.2.8. Moreover, since $k \geq 3$, without loss of generality we can assume that $i > 1$. Hence $\mathfrak{H}_{2^n}(M-1) = \mathfrak{H}_{2^{n-1}}(t_{n-1}^i - 1) \diamond \mathfrak{H}_{2^{n-1}}(t_{n-1}^j)$ by Lemma 5.2.7. If $\lambda \in \mathfrak{H}_{2^n}(M-1)$, then there exist $\mu \in \mathfrak{H}_{2^{n-1}}(t_{n-1}^i - 1)$ and $\nu \in \mathfrak{H}_{2^{n-1}}(t_{n-1}^j)$ such that $\mathscr{LR}(\lambda; \mu, \nu) \neq 0$. By the inductive hypothesis there exist three distinct irreducible constituents θ_1, θ_2 and θ_3 of $\chi^\mu \big\downarrow_{P_{2^{n-1}}}$ of degree 2^i, and by definition there exists an irreducible constituent ψ of $\chi^\nu \big\downarrow_{P_{2^{n-1}}}$ of degree 2^j. It follows that $(\theta_1 \times \psi)\big\uparrow^{P_{2^n}}$, $(\theta_2 \times \psi)\big\uparrow^{P_{2^n}}$ and $(\theta_3 \times \psi)\big\uparrow^{P_{2^n}}$ are three distinct irreducible constituents of $\chi^\lambda \big\downarrow_{P_{2^n}}$ of degree 2^k.

2. For the second case we assume that $\delta_{n-1}^w = 0$ and $M = 2t_{n-1}^w$. In this setting we observe that k must be strictly greater than 3. In fact, if $k = 3$ then Example 5.2.1 and Lemma 5.2.4 show that $M = 2t_{n-1}^1 = 2(2^{n-1} - 1) = 2^n - 2$. On the other hand, by definition of M we know that $M = \max\left\{ t_{n-1}^0 + t_{n-1}^2, 2^n - 2 \right\} = t_{n-1}^0 + t_{n-1}^2 = 2^{n-1} + 2^{n-1} - 1 = 2^n - 1$ and this is a contradiction. Hence we can assume that $k > 3$. Let $\lambda \in \mathfrak{H}_{2^n}(M)$. If $\lambda_1 = M$, then $\mathscr{LR}\left(\lambda; \lambda_{n-1}^w, \lambda_{n-1}^w\right) \neq 0$. Recall that $\delta_{n-1}^w = 0$ means that $\chi^{\lambda_{n-1}^w} \big\downarrow_{P_{2^{n-1}}}$ has two distinct irreducible constituents θ_1 and θ_2 of degree 2^w. Hence $(\theta_1 \times \theta_2)\big\uparrow^{P_{2^n}}$ is an irreducible constituent of $\chi^\lambda \big\downarrow_{P_{2^n}}$ of degree 2^k and therefore $\lambda \in \Lambda_{2^n}^k$.

   If $t_{n-1}^w \leq \lambda_1 < M$, then $\mathscr{LR}\left(\lambda; \lambda_{n-1}^w, \mu\right) \neq 0$ for some $\mu \in \mathscr{H}(2^{n-1})$, with $\mu_1 = \lambda_1 - t_{n-1}^w < M - t_{n-1}^w = t_{n-1}^w$. In particular, $\mu \in \mathfrak{H}_{2^{n-1}}\left(t_{n-1}^w - 1\right)$. Since $w = \frac{k-1}{2} > 1$, by the inductive hypothesis there exist three distinct irreducible constituents ψ_1, ψ_2 and ψ_3 of $\chi^\mu \big\downarrow_{P_{2^{n-1}}}$ of degree 2^w. Let θ_1 and θ_2 be as in the previous case, and suppose without loss of generality that $\theta_1 \notin \{ \psi_2, \psi_3 \}$ and that $\theta_2 \neq \psi_1$. Then $(\theta_1 \times \psi_2)\big\uparrow^{P_{2^n}}, (\theta_1 \times \psi_3)\big\uparrow^{P_{2^n}}$ and $(\theta_2 \times \psi_1)\big\uparrow^{P_{2^n}}$ are three distinct irreducible constituents of $\chi^\lambda \big\downarrow_{P_{2^n}}$ of degree 2^k. In particular, $\lambda \in \Lambda_{2^n}^k$.

   Suppose now that $1 \leq \lambda_1 < t_{n-1}^w$, then we have

$$\begin{aligned}(\lambda')_1 &= 2^n + 1 - \lambda_1 > 2^n + 1 - t_{n-1}^w \geq 2^n + 1 - 2^{n-1} \\ &= 2^{n-1} + 1 \geq t_{n-1}^{\frac{k+1}{2}} + 1 > t_{n-1}^{\frac{k+1}{2}}.\end{aligned}$$

   Since $\lambda' \in \mathfrak{H}_{2^n}(M)$, from the previous case we deduce that $\lambda' \in \Lambda_{2^n}^k$ and we conclude that $\lambda \in \Lambda_{2^n}^k$, as $\Lambda_{2^n}^k$ is closed under conjugation of partitions.

3. Finally consider the case where $\delta_{n-1}^w = -1$ and $M = 2t_{n-1}^w - 1$. Arguing exactly as above we observe that $k > 3$ and hence that $w = \frac{k-1}{2} > 1$. Moreover, in this case we

have that $\chi^{\lambda^w_{n-1}}\downarrow_{P_{2^{n-1}}}$ has a unique irreducible constituent ψ of degree 2^w. Let us fix $\lambda \in \mathfrak{H}_{2^n}(M)$. If $\lambda_1 = M$, then $\mathscr{LR}\left(\lambda; \lambda^w_{n-1}, \mu\right) \neq 0$, for some $\mu \in \mathfrak{H}_{2^{n-1}}\left(t^w_{n-1} - 1\right)$. Indeed $\mathfrak{H}_{2^n}(M) = \mathfrak{H}_{2^{n-1}}\left(t^w_{n-1}\right) \diamond \mathfrak{H}_{2^{n-1}}\left(t^w_{n-1} - 1\right)$, by Lemma 5.2.7. Using the inductive hypothesis on μ, we have that there exist three distinct irreducible constituents θ_1, θ_2 and θ_3 of $\chi^\mu\downarrow_{P_{2^{n-1}}}$ of degree 2^w. Without loss of generality we can suppose that $\psi \neq \theta_1$, hence $(\psi \times \theta_1)\uparrow^{P_{2^n}}$ is an irreducible constituent of $\chi^\lambda\downarrow_{P_{2^n}}$ of degree 2^k. Therefore $\lambda \in \Lambda^k_{2^n}$.
If instead $\lambda_1 < M$, by Lemma 5.2.7 we have

$$\lambda \in \mathfrak{H}_{2^n}(M-1) = \mathfrak{H}_{2^{n-1}}\left(t^w_{n-1} - 1\right) \diamond \mathfrak{H}_{2^{n-1}}\left(t^w_{n-1} - 1\right).$$

Hence there exist $\mu, \nu \in \mathfrak{H}_{2^{n-1}}\left(t^w_{n-1} - 1\right)$ such that $\mathscr{LR}(\lambda; \mu, \nu) \neq 0$. By induction there exist three distinct irreducible constituents θ_1, θ_2 and θ_3 of $\chi^\mu\downarrow_{P_{2^{n-1}}}$ and three distinct ones σ_1, σ_2 and σ_3 of $\chi^\nu\downarrow_{P_{2^{n-1}}}$, all of them of degree 2^w. Without loss of generality we can suppose that $\theta_1 \notin \{\sigma_2, \sigma_3\}$ and $\theta_2 \neq \sigma_3$. Hence $(\theta_1 \times \sigma_2)\uparrow^{P_{2^n}}, (\theta_1 \times \sigma_3)\uparrow^{P_{2^n}}$ and $(\theta_2 \times \sigma_3)\uparrow^{P_{2^n}}$ are three distinct irreducible constituents of $\chi^\lambda\downarrow_{P_{2^n}}$ of degree 2^k. In particular, $\lambda \in \Lambda^k_{2^n}$.

We claim next that $\Lambda^k_{2^n} \subseteq \mathfrak{H}_{2^n}(M)$. To do this, suppose by contradiction that there exists $\lambda \in \Lambda^k_{2^n} \smallsetminus \mathfrak{H}_{2^n}(M)$ and without loss of generality suppose also that $\lambda_1 \geq M+1$. By definition there exists $\phi \in \mathrm{Irr}(P_{2^n})$ such that $\left[\chi^\lambda\downarrow_{P_{2^n}}, \phi\right] \neq 0$ and $\phi(1) = 2^k$.

To prove the claim there are three cases to consider

Case 1. Suppose first that $\phi = \mathscr{X}(\psi, \alpha)$ with $\psi \in \mathrm{Irr}(P_{2^{n-1}})$, $\psi(1) = 2^{\frac{k}{2}}$ and $\alpha \in \mathrm{Lin}(P_2)$. Hence there exist $\mu, \nu \in \mathscr{H}(2^{n-1})$ such that $\mathscr{LR}(\lambda; \mu, \nu) \neq 0$ and ψ is both an irreducible constituent of $\chi^\mu\downarrow_{P_{2^{n-1}}}$ and of $\chi^\nu\downarrow_{P_{2^{n-1}}}$. Therefore $\mu, \nu \in \Lambda^{\frac{k}{2}}_{2^{n-1}}$. By the inductive hypothesis and by Theorem 5.2.5, we have

$$\mathfrak{H}_{2^{n-1}}\left(t^{\frac{k}{2}}_{n-1}\right) = \Lambda^{\frac{k}{2}}_{2^{n-1}} \subseteq \Lambda^{\frac{k}{2}-1}_{2^{n-1}} = \mathfrak{H}_{2^{n-1}}\left(t^{\frac{k}{2}-1}_{n-1}\right).$$

In particular, $t^{\frac{k}{2}}_{n-1} \leq t^{\frac{k}{2}-1}_{n-1}$ and this implies that

$$2t^{\frac{k}{2}}_{n-1} \leq t^{\frac{k}{2}}_{n-1} + t^{\frac{k}{2}-1}_{n-1} \leq M.$$

This inequality gives that $\mathfrak{H}_{2^n}\left(2t^{\frac{k}{2}}_{n-1}\right) \subseteq \mathfrak{H}_{2^n}(M)$. Since $\mu, \nu \in \Lambda^{\frac{k}{2}}_{2^{n-1}} = \mathfrak{H}_{2^{n-1}}\left(t^{\frac{k}{2}}_{n-1}\right)$, we have $\lambda \in \mathfrak{H}_{2^{n-1}}\left(t^{\frac{k}{2}}_{n-1}\right) \diamond \mathfrak{H}_{2^{n-1}}\left(t^{\frac{k}{2}}_{n-1}\right)$. Using Lemma 5.2.7 we conclude that

$$\lambda \in \mathfrak{H}_{2^{n-1}}\left(t^{\frac{k}{2}}_{n-1}\right) \diamond \mathfrak{H}_{2^{n-1}}\left(t^{\frac{k}{2}}_{n-1}\right) = \mathfrak{H}_{2^n}\left(2t^{\frac{k}{2}}_{n-1}\right) \subseteq \mathfrak{H}_{2^n}(M).$$

This is a contradiction as $\lambda_1 \geq M+1$.

Case 2. Suppose now that $\phi = (\phi_1 \times \phi_2)\big\uparrow^{P_{2^n}}$, where $\phi_1, \phi_2 \in \mathrm{Irr}(P_{2^{n-1}})$ and $\phi_1 \neq \phi_2$. Then there exist $\mu_1, \mu_2 \in \mathscr{H}(2^{n-1})$ such that $\mathscr{LR}(\lambda; \mu_1, \mu_2) \neq 0$ and ϕ_1 (respectively ϕ_2) is an irreducible constituent of $\mu_1\big\downarrow_{P_{2^{n-1}}}$ (respectively $\mu_2\big\downarrow_{P_{2^{n-1}}}$). We need to distinguish two cases: the first one holds when $\phi_1(1) = 2^i$ and $\phi_2(1) = 2^j$, with $i, j \in [0, \beta_{2^{n-1}}]$, $i \neq j$ and $i + j = k - 1$. In this case $\mu_1 \in \Lambda^i_{2^{n-1}}$ and $\mu_2 \in \Lambda^j_{2^{n-1}}$. Hence, by the inductive hypothesis, Lemma 5.2.7 and the definition of M we get that

$$\lambda \in \Lambda^i_{2^{n-1}} \diamond \Lambda^j_{2^{n-1}} = \mathfrak{H}_{2^{n-1}}\left(t^i_{n-1}\right) \diamond \mathfrak{H}_{2^{n-1}}\left(t^j_{n-1}\right) = \mathfrak{H}_{2^{n-1}}\left(t^i_{n-1} + t^j_{n-1}\right) \subseteq \mathfrak{H}_{2^n}(M).$$

This is a contradiction as $\lambda_1 \geq M + 1$.

Case 3. Finally consider when $\phi_1 \neq \phi_2$ but $\phi_1(1) = \phi_2(1) = 2^w$. In this setting we have that $\mu_1, \mu_2 \in \Lambda^w_{2^{n-1}}$. By the inductive hypothesis $\Lambda^w_{2^{n-1}} = \mathfrak{H}_{2^{n-1}}\left(t^w_{n-1}\right)$.
In particular, $(\mu_1)_1, (\mu_2)_1 \leq t^w_{n-1}$. Hence we have

$$2t^w_{n-1} \leq M + 1 \leq \lambda_1 \leq (\mu_1)_1 + (\mu_2)_2 \leq 2t^w_{n-1}, \tag{5.2}$$

where the first relation holds by definition of M, the second one by assumption and the third one by Lemma 2.2.8. Therefore (5.2) is a chain of equalities and in particular, $(\mu_1)_1 = (\mu_2)_1 = t^w_{n-1}$. Hence $\mu_1 = \mu_2 = \lambda^w_{n-1}$. Then $\delta^w_{n-1} = 0$ since by assumption ϕ_1 and ϕ_2 are two distinct irreducible constituents of $\lambda^w_{n-1}\big\downarrow_{P_{2^{n-1}}}$ of degree 2^w. By definition,

$$M = \max\left\{ t^i_{n-1} + t^j_{n-1}, 2t^w_{n-1} \;\middle|\; i, j, w \in [0, \beta_{2^{n-1}}],\ i + j = k - 1 \text{ and } i \neq j \right\}.$$

Since (5.2) is a chain of equalities we get $M+1 = \lambda_1 = 2t^w_{n-1} \leq M$, which is a contradiction.

This proves the claim. □

We can now prove our main result in this chapter:

Theorem 5.2.11. *Let $n \in \mathbb{N}$ and $k \in [0, \beta_n]$. Then:*

(1) there exists $T^k_n \in [1, n]$ such that $\Lambda^k_n = \mathfrak{H}_n(T^k_n)$;

(2) if $k > 1$, for every $\lambda \in \mathfrak{H}_n(T^k_n - 1)$, $\chi^\lambda\big\downarrow_{P_n}$ has three distinct irreducible constituents of degree 2^k.

Proof. We proceed by induction on n: if $n = 1$ then $k = 0$ and the statement is obvious. Let $n \geq 2$ and let $n = \sum_{i=1}^r 2^{n_i}$ be its binary expansion. By Theorem 5.2.10, for every $i \in [1, r]$ and every $d_i \in [0, \beta_{2^{n_i}}]$ there exists $t^{d_i}_{n_i} \in \left[2^{n_i-1} + 1, 2^{n_i}\right]$ such that $\Lambda^{d_i}_{2^{n_i}} = \mathfrak{H}_{2^{n_i}}\left(t^{d_i}_{n_i}\right)$. Define

$$M := \max\left\{ t^{j_1}_{n_1} + \cdots + t^{j_r}_{n_r} \;\middle|\; j_i \in [0, \beta_{2^{n_i}}] \text{ for every } i \in [1, r] \text{ and } j_1 + \cdots + j_r = k \right\}.$$

We want to prove that $\Lambda^k_n = \mathfrak{H}_n(M)$. Let $j_1 \in [0, \beta_{2^{n_1}}], \ldots, j_r \in [0, \beta_{2^{n_r}}]$ be such that $M = t^{j_1}_{n_1} + \cdots + t^{j_r}_{n_r}$. We have

$$\mathfrak{H}_n(M) = \mathfrak{H}_{2^{n_1}}(t^{j_1}_{n_1}) \diamond \cdots \diamond \mathfrak{H}_{2^{n_r}}(t^{j_r}_{n_r}) = \Lambda^{j_1}_{2^{n_1}} \diamond \cdots \diamond \Lambda^{j_r}_{2^{n_r}} \subseteq \Lambda^k_n,$$

where the first equality holds by Lemma 5.2.7, the second by Theorem 5.2.10 and the inclusion by Lemma 5.2.9.

To prove part (1) of the theorem it remains to show that $\Lambda_n^k \subseteq \mathfrak{H}_n(M)$. Let $\lambda \in \Lambda_n^k$. Then there exists an irreducible constituent ϕ of $\chi^\lambda\big\downarrow_{P_n}$ of degree 2^k. The structure of P_n discussed in section 2.3 implies that $\phi = \phi_1 \times \cdots \times \phi_r$, for some $\phi_i \in \mathrm{Irr}(P_{2^{n_i}})$ for every $i \in [1, r]$ such that $\phi_i(1) = 2^{j_i}$ and such that $j_1 + \cdots + j_r = k$. Hence there exists $\mu_i \in \mathscr{H}(2^{n_i})$ with ϕ_i as an irreducible constituent of the restriction $\chi^{\mu_i}\big\downarrow_{P_{2^{n_i}}}$ for every $i \in [1, r]$, such that $\mathscr{L}\mathscr{R}(\lambda; \mu_1, \dots, \mu_r) \neq 0$. In particular, by Theorem 5.2.10 there exists $t_{n_i}^{j_i} \in \left[2^{n_i-1}+1, 2^{n_i}\right]$ such that $\mu_i \in \Lambda_{2^{n_i}}^{j_i} = \mathfrak{H}_{2^{n_i}}(t_{n_i}^{j_i})$. Therefore by Lemma 5.2.7,

$$\lambda \in \mathfrak{H}_{2^{n_1}}\left(t_{n_1}^{j_1}\right) \diamond \cdots \diamond \mathfrak{H}_{2^{n_r}}\left(t_{n_r}^{j_r}\right) = \mathfrak{H}_n\left(t_{n_1}^{j_1} + \cdots + t_{n_r}^{j_r}\right) \subseteq \mathfrak{H}_n\left(M\right),$$

where the last inclusion follows from the definition of M.

In order to prove statement (2), let us fix $k > 1$. From the discussion above, we know that $\Lambda_n^k = \mathfrak{H}_n(T_n^k)$, where $T_n^k = t_{n_1}^{j_1} + \cdots + t_{n_r}^{j_r}$ for suitable $j_1 \in [0, \beta_{2^{n_1}}], \dots, j_r \in [0, \beta_{2^{n_r}}]$ such that $j_1 + \cdots + j_r = k$. Since $k > 1$, we can suppose without loss of generality that $j_1 > 1$. Let $\lambda \in \mathfrak{H}_n(T_n^k - 1)$. By Lemma 5.2.7, $\mathfrak{H}_n(T_n^k - 1) = \mathfrak{H}_{2^{n_1}}\left(t_{n_1}^{j_1} - 1\right) \diamond \cdots \diamond \mathfrak{H}_{2^{n_r}}\left(t_{n_r}^{j_r}\right)$. Hence there exist $\mu_1 \in \mathfrak{H}_{2^{n_1}}\left(t_{n_1}^{j_1} - 1\right)$ and $\mu_i \in \mathfrak{H}_{2^{n_i}}\left(t_{n_i}^{j_i}\right)$ for every $i \in [2, r]$ such that $\mathscr{L}\mathscr{R}(\lambda; \mu_1, \mu_2, \dots, \mu_r) \neq 0$. By definition there exists an irreducible constituent ψ_i of $\chi^{\mu_i}\big\downarrow_{P_{2^{n_i}}}$ of degree 2^{j_i} for every $i \in [2, r]$. By Theorem 5.2.10, there exist three distinct irreducible constituents ϕ_1, ϕ_2 and ϕ_3 of $\chi^{\mu_1}\big\downarrow_{P_{2^{n_1}}}$ of degree 2^{j_1}. Therefore $\phi_1 \times \psi_2 \times \cdots \times \psi_r$, $\phi_2 \times \psi_2 \times \cdots \times \psi_r$ and $\phi_3 \times \psi_2 \times \cdots \times \psi_r$ are three distinct irreducible constituents of $\chi^\lambda\big\downarrow_{P_n}$ of degree 2^k. □

We conclude this chapter by explicitly computing $T_n^{\alpha_n}$. By doing this, we manage to identify those characters $\chi \in \mathrm{Irr}_{\mathscr{H}}(\mathfrak{S}_n)$ such that $\chi\big\downarrow_{P_n}$ admits irreducible constituents of every possible degree.

Definition 5.2.12. For $n \in \mathbb{N}$, define $\tau_n \in [2^{n-1}+1, 2^n]$ by

$$\tau_1 = 2,\ \tau_2 = 3, \tau_3 = 7,\ \tau_4 = 13,\ \tau_5 = 26,\ \text{and}$$
$$\tau_n = 2^{n-1} + 2^{n-2} + 2^{n-5} + 2^{n-6} \text{ for } n \geq 6.$$

Notice that $\tau_n = 2\tau_{n-1}$ for any $n \geq 7$.

As usual, we start by studying the case where n is a power of 2.

Proposition 5.2.13. *For every $n \in \mathbb{N}$, $T_{2^n}^{\beta_{2^n}} = \tau_n$. Moreover, if $n \geq 6$ and $\lambda \in \Lambda_{2^n}^{\beta_{2^n}}$ then there exist at least three distinct irreducible constituents of $\chi^\lambda\big\downarrow_{P_{2^n}}$ of degree $2^{\beta_{2^n}}$.*

Proof. We proceed by induction on n. If $n \leq 6$ then the statement holds by direct computation. The cases $n \in \{1, 2, 3\}$ has already been computed in Example 5.2.1. If $n = 4$, $\beta_{2^4} = 5$ and an irreducible character of degree 2^5 is of the form $(\gamma \times \delta)\big\uparrow^{P_{2^4}}$ with $\gamma(1) = \delta(1) = 2^2$. Hence $\lambda \in \Lambda_{2^4}^5$ if $(\chi^\lambda)\big\downarrow_{\mathfrak{S}_8 \times \mathfrak{S}_8}$ has a constituent $\mu_1 \times \mu_2$ such that $\mu_1, \mu_2 \in \Lambda_8^2 = \mathfrak{H}_8(7)$. Then by Littlewood-Richardson rule, we can see that this happens if $\lambda_1 \in [4, 13]$. Thus $\Lambda_{16}^8 = \mathfrak{H}_{16}(13)$. Notice that $(13, 1^3)$ has only two distinct irreducible constituents of degree 2^5 in its restriction to P_{2^4}, say θ_1 and θ_2. We can argue in the same way for the case $n = 5$. Notice that $\Lambda_{2^5}^{11} = \mathfrak{H}_{2^5}(26)$, but if $\lambda = (26, 1^6)$ then its restriction to the Sylow subgroup P_{2^5} has a unique irreducible constituent of maximum degree 2^{11}, achieved by inducting $\theta_1 \times \theta_2$ to P_{2^5}. Finally, let $n = 6$ and $\lambda \in \mathscr{H}(2^6)$. If $\lambda = (52, 1^{12})$ then there is $\chi^{(26,1^6)} \times \chi^{(26,1^6)}$ in the restriction $(\chi^\lambda)\big\downarrow_{\mathfrak{S}_{2^5} \times \mathfrak{S}_{2^5}}$, but this not gives an irreducible character of degree $2^{\beta}_{2^6} = 2^{23}$. Indeed, as we have seen in the previous case, $(\chi^{(26,1^6)})\big\downarrow_{P_{2^5}}$ has a unique irreducible constituent of degree 2^{11}. Hence, $\Lambda_{2^6}^{23} = \mathfrak{H}_{2^6}(51)$ and $51 = 2^5 + 2^4 + 2^1 + 2^0$, as we wanted. Notice also that since $(\chi^\mu)\big\downarrow_{P_{2^5}}$ has at least three distinct irreducible contituents of degree 2^{11} for $\mu \in \Lambda_{2^5}^{11} \setminus \{ (26, 1^6), (7, 1^{25}) \}$, then every $\lambda \in \Lambda_{2^6}^{23}$ has at least three distinct irreducible constituents in the restriction to P_{2^6}.

Let now $n \geq 7$. We want to show that $\Lambda_{2^n}^{\beta_{2^n}} = \mathfrak{H}_{2^n}(\tau_n)$. Given $\lambda \in \Lambda_{2^n}^{\beta_{2^n}}$, there exists an irreducible constituent ϕ of $\chi^\lambda\big\downarrow_{P_n}$ of degree $2^{\beta_{2^n}}$. As we have seen in the proof of Proposition 5.2.3, $\phi = (\phi_1 \times \phi_2)\big\uparrow^{P_{2^n}}$ with $\phi_1, \phi_2 \in \mathrm{Irr}(P_{2^{n-1}})$, $\phi_1 \neq \phi_2$ and $\phi_1(1) = \phi_2(1) = 2^{\beta_{2^{n-1}}}$. Hence, there exists $\mu_1, \mu_2 \in \mathscr{H}(2^{n-1})$ such that $\mathscr{LR}(\lambda; \mu_1, \mu_2) \neq 0$ and such that $\left[\chi^{\mu_i}\big\downarrow_{P_{2^{n-1}}}, \phi_i\right] \neq 0$ for all $i \in \{1, 2\}$. In particular, $\mu_1, \mu_2 \in \Lambda_{2^{n-1}}^{\beta_{2^{n-1}}}$ and by the inductive hypothesis we know that $\Lambda_{2^{n-1}}^{\beta_{2^{n-1}}} = \mathfrak{H}_{2^{n-1}}(\tau_{n-1})$. Using these observations together with Lemma 5.2.7 we conclude that

$$\lambda \in \mathfrak{H}_{2^{n-1}}(\tau_{n-1}) \diamond \mathfrak{H}_{2^{n-1}}(\tau_{n-1}) = \mathfrak{H}_{2^n}(2\tau_{n-1}) = \mathfrak{H}_{2^n}(\tau_n).$$

In order to prove the other inclusion, we consider $\lambda \in \mathfrak{H}_{2^n}(\tau_n)$. From Lemma 5.2.7 and the inductive hypothesis, we know that

$$\mathfrak{H}_{2^n}(\tau_n) = \mathfrak{H}_{2^{n-1}}(\tau_{n-1}) \diamond \mathfrak{H}_{2^{n-1}}(\tau_{n-1}) = \Lambda_{2^{n-1}}^{\beta_{2^{n-1}}} \diamond \Lambda_{2^{n-1}}^{\beta_{2^{n-1}}}.$$

Therefore there exist $\mu,\nu \in \Lambda_{2^{n-1}}^{\beta_{2^{n-1}}}$ such that $\chi^\mu \times \chi^\nu \mid (\chi^\lambda)\big\downarrow_{\mathfrak{S}_{2^{n-1}}\times\mathfrak{S}_{2^{n-1}}}$. The inductive hypothesis implies that both $\chi^\mu\big\downarrow_{P_{2^{n-1}}}$ and $\chi^\nu\big\downarrow_{P_{2^{n-1}}}$ admit three distinct irreducible constituents of degree $2^{\beta_{2^{n-1}}}$. Denote by ϕ_1,ϕ_2,ϕ_3 those constituents of $\chi^\mu\big\downarrow_{P_{2^{n-1}}}$ and by ψ_1,ψ_2,ψ_3 those constituents of $\chi^\nu\big\downarrow_{P_{2^{n-1}}}$. If $\psi_j \notin \{\phi_1,\phi_2,\phi_3\}$ for some $j\in[1,3]$, then $(\phi_1\times\psi_j)\big\uparrow^{P_{2^n}}, (\phi_2\times\psi_j)\big\uparrow^{P_{2^n}}$ and $(\phi_3\times\psi_j)\big\uparrow^{P_{2^n}}$ are three distinct irreducible constituents of $\chi^\lambda\big\downarrow_{P_{2^n}}$ of degree $2^{\beta_{2^n}}$. On the other hand, if $\psi_j \in \{\phi_1,\phi_2,\phi_3\}$ for all $j\in[1,3]$, then we can assume without loss of generality that $\phi_i=\psi_i$ for all $i\in[1,3]$. In this case we have that $(\phi_1\times\psi_2)\big\uparrow^{P_{2^n}}, (\phi_2\times\psi_3)\big\uparrow^{P_{2^n}}$ and $(\phi_3\times\psi_1)\big\uparrow^{P_{2^n}}$ are three distinct irreducible constituents of $\chi^\lambda\big\downarrow_{P_{2^n}}$ of degree $2^{\beta_{2^n}}$. Moreover, $\lambda\in\Lambda_{2^n}^{\beta_{2^n}}$ as desired. □

Proposition 5.2.14. *If $n\in\mathbb{N}$ and $n=\sum_{i=1}^r 2^{n_i}$ is its binary expansion, then $T_n^{\beta_n}=\sum_{i=1}^r \tau_{n_i}$.*

Proof. Arguing exactly as in the proof of Theorem 5.2.11, we have that

$$T_n^{\beta_n} = \max\{T_{n_1}^{j_1}+\cdots+T_{n_r}^{j_r} \mid j_i\in[0,\beta_{2^{n_i}}] \text{ for } i\in[1,r], \text{ and } \sum_{i=1}^r j_i=\beta_n\}.$$

Since $\beta_n=\sum_{i=1}^r \beta_{2^{n_i}}$, we deduce that $T_n^{\beta_n}=T_{n_1}^{\beta_{2^{n_1}}}+\cdots+T_{n_r}^{\beta_{2^{n_r}}}$. The statement now follows from Proposition 5.2.13. □

Remark 5.2.15. By Theorems 5.2.5 and 5.2.11 we know that for every $k\in[0,\beta_n]$ we have $\mathfrak{H}_n(T_n^{\beta_n})=\Lambda_n^{\beta_n}\subseteq\Lambda_n^k$. From Proposition 5.2.14 we observe that the majority of the elements of $\mathscr{H}(n)$ are contained in $\mathfrak{H}_n(T_n^{\beta_n})$. This shows that the restriction to P_n of most of the irreducible characters labelled by hook partitions admit irreducible constituents of every possible degree.

Chapter 6

Vertices of $\mathbb{F}\mathfrak{S}_n$-modules

Let $\mathbb{F}$ be a field and let $\mathfrak{S}_n$ be the symmetric group on n elements. Let $\mathbb{F}\mathfrak{S}_n$ be the associated group algebra. It is an open problem to determine the vertices of some important families of $\mathbb{F}\mathfrak{S}_n$-modules, for instance the Specht modules.

In this chapter we would like to collect some results on this topic. Notice that a previous survey on vertices of simple modules has been made by S. Danz and B. Külshammer in [8].

We dedicate the first section of this chapter to the vertices of Specht modules. First we present a characterization of simple Specht modules: Theorem 6.1.1 for $p = 2$ and Theorem 6.1.2 for p odd. Then we completely describe the vertices of simple Specht modules in Theorems 6.1.3 and 6.1.4. To do this we define pairs of partitions and signed Young modules.

After that we show a characterization of indecomposable Specht module (Theorem 6.1.5). In this case we do not have a generic description of the vertices. However we describe a lower bound in Theorem 6.1.8 and we explicitely compute few special cases.

We then turn to Section 6.2 in which we collect the results about vertices of simple modules D^λ. There are some special cases for which the computation has been done, in particular for a simple module labelled by a hook partition.

We include some results concerning trivial source Specht modules in the last section, right before turning to the definition of a Scott module. Finally we are able to say something about the decomposition of a permutation module M^λ into Young modules. Indeed, only one of these indecomposable summand is isomorphic to a Scott module. It is the the main result of this chapter to determine which one. This is Theorem 6.3.7.

Recall that $\mathbb{F}_G$ denotes the trivial $\mathbb{F}G$-module when G is a finite group.

6.1 Vertices of Specht modules

The first family of $\mathbb{F}\mathfrak{S}_n$-modules that we consider are the Specht modules. Recall that if λ is a partition of n, S^λ is the associated Specht module.

The knowledge that we obtain about the vertices of a Specht module is deeper if the module is simple. Therefore we split this section into two parts: the first one focuses on simple Specht modules, and the second one on indecomposable Specht modules.

In Section 2.2.2 we have seen that if $\mathbb{F}$ has characteristic zero, $\{S^\lambda \mid \lambda \in \mathscr{P}(n)\}$ is a set of representatives for the isomorphism classes of simple $\mathbb{F}\mathfrak{S}_n$-modules. Hence from now on, we will consider $\mathbb{F}$ to be a field of positive characteristic p.

6.1.1 Simple Specht modules

A characterization of simple Specht modules in the case $p = 2$ is given by G.D. James and A. Mathas in [46].

Theorem 6.1.1. *The Specht module S^λ is simple in characteristic 2 if and only if*

(i) $\lambda_i - \lambda_{i+1} \equiv -1 \mod 2^{m(\lambda_{i+1}-\lambda_{i+2})}$ *for all $i \geq 1$; or*

(ii) $\lambda'_i - \lambda'_{i+1} \equiv -1 \mod 2^{m(\lambda'_{i+1}-\lambda'_{i+2})}$ *for all $i \geq 1$; or*

(iii) $\lambda = (2, 2)$.

Here, for an integer k, $m(k)$ denote the least non-negative integer such that $k < 2^{m(k)}$.

In the same paper, they conjectured a characterization of simple Specht modules for $p \geq 3$ that was subsequently proved by M. Fayers in [17] using also results by S. Lyle [52].

Theorem 6.1.2. *Let $p \geq 3$. Then S^λ is simple if and only if in the Young diagram $[\lambda]$ there are no nodes (a, b), (a, y) and (x, b) such that*

$$\nu_p(h_{(a,b)}) > 0, \text{ and}$$
$$\nu_p(h_{(x,b)}) \neq \nu_p(h_{(a,b)}) \neq \nu_p(h_{(a,y)}).$$

Here $\nu_p(h)$ denotes the highest power of p dividing a natural number h, while $h_{(a,b)}$ is the hoook-length of the node (a, b).

The partitions labelling the simple Specht modules of $\mathbb{F}\mathfrak{S}_n$ are usually called JM-*partitions*. We denote by $\mathrm{JM}(n)_p$ the set of the JM-partitions of n, where p is the characteristic of the base field $\mathbb{F}$.

The vertices of simple Specht modules have been completely described, since every simple Specht module is a signed Young module, by [34], and there is a complete characterization of the vertices of the signed Young modules in [11]. The latter are a natural generalization of the usual Young module that we defined in Section 2.2.2. We describe them next.

Let $\mathscr{P}^2(n)$ be the set of all pairs $(\lambda \mid \mu)$ of partitions λ, μ such that $|\lambda| + |\mu| = n$. We allow λ or μ to be the empty partition $\emptyset$. For $(\lambda \mid \mu) \in \mathscr{P}^2(n)$, the *signed Young permutation $F\mathfrak{S}_n$-module* is

$$M(\lambda \mid \mu) := \left(\mathbb{F}_{\mathfrak{S}_\lambda} \boxtimes \mathrm{sgn}_{\mathfrak{S}_\mu}\right)\big\uparrow_{\mathfrak{S}_\lambda \times \mathfrak{S}_\mu}^{\mathfrak{S}_n}.$$

In the case when $\mu = \emptyset$, we obtain the usual Young permutation module, that is $M^\lambda = M(\lambda \mid \emptyset)$.

Let $(\lambda \mid \mu), (\alpha \mid \beta) \in \mathscr{P}^2(n)$. We say that $(\lambda \mid \mu)$ *dominates* $(\alpha \mid \beta)$ (and we write $(\lambda \mid \mu) \trianglerighteq (\alpha \mid \beta)$) if for every $k \geq 1$,

(i)

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \alpha_i, \text{ and}$$

(ii)

$$|\lambda| + \sum_{i=1}^{k} \mu_i \geq |\alpha| + \sum_{i=1}^{k} \beta_i.$$

This gives a partial order $\trianglerighteq$ on $\mathscr{P}^2(n)$, which is also called the *dominance order.*

The indecomposable *signed Young modules* have been defined in [11] inductively: for any $(\lambda \mid p\mu) \in \mathscr{P}^2(n)$, $M(\lambda \mid p\mu)$ has an indecomposable direct summand $Y(\lambda \mid p\mu)$ appearing with multiplicity 1, and the remaining indecomposable summands are isomorphic to indecomposable summands of other signed permutation modules $M(\alpha \mid p\beta)$ such that $(\alpha \mid p\beta) \triangleright (\lambda \mid p\mu)$.

More generally, for an arbitrary $(\alpha \mid \beta) \in \mathscr{P}^2(n)$, the signed Young permutation module $M(\alpha \mid \beta)$ is isomorphic to a direct sum of some signed Young $\mathbb{F}\mathfrak{S}_n$-modules $Y(\lambda \mid p\mu)$ such that $(\lambda \mid p\mu) \trianglerighteq (\alpha \mid \beta)$. Notice that if $\mu = \emptyset$, we recover the usual Young module $Y^\lambda \cong Y(\lambda \mid \emptyset)$.

Let $(\lambda \mid p\mu) \in \mathscr{P}^2(n)$. Using the notation of the p-adic expansion from (2.1), set $r = \max\{ r_\lambda, r_\mu + 1 \}$ and define the following composition of n:

$$\mathfrak{V}(\lambda \mid p\mu) := \left(1^{|\lambda(0)|}, p^{|\lambda(1)|+|\mu(0)|}, (p^2)^{|\lambda(2)|+|\mu(1)|}, \ldots, (p^r)^{|\lambda(r)|+|\mu(r-1)|}\right).$$

Theorem 6.1.3 ([10, Theorem 3.8 (b)])**.** *Let $(\lambda \mid p\mu) \in \mathscr{P}^2(n)$ and $\rho = \mathfrak{V}(\lambda \mid p\mu)$. Each Sylow p-subgroup of $\mathfrak{S}_\rho$ is a vertex of the indecomposable signed Young module $Y(\lambda \mid p\mu)$. Moreover, $Y(\lambda \mid p\mu)$ has trivial source.*

Finally, by D.J. Hemmer [34] we know that every simple Specht module is isomorphic to a signed Young module. More precisely:

Theorem 6.1.4 ([10, Theorem 5.1])**.** *Let $\lambda \in \mathrm{JM}(n)$. The simple Specht $\mathbb{F}\mathfrak{S}_n$-module S^λ is isomorphic to $Y(\phi(\lambda))$, where the map $\phi : \mathscr{P}(n) \to \mathscr{P}^2(n)$ is defined by*

$$\phi(\lambda) = (\alpha \mid p\beta),$$

where $\alpha = (\lambda'(0))'$ and $p\beta = \lambda' - \lambda'(0)$.

6.1.2 Indecomposable Specht modules

G.D. James, in [42, Corollary 13.18] showed that almost all Specht modules are indecomposable.

Theorem 6.1.5. *Let $\mathbb{F}$ be a field of prime characteristic p and let λ be a partition of n. If $p > 2$, or if $p = 2$ and the parts of λ are distinct, then S^λ is indecomposable.*

It remains to understand what happens when $p = 2$ and the parts of λ are not distinct. In [41, Example 1] it is shown that $S^{(5,1^2)}$ is decomposable over a field of characteristic 2. Moreover, in [60], Gwendolen Murphy gave necessary and sufficient conditions for the Specht module $S^{(2m+1-r,1^r)}$ to be decomposable if $p = 2$. Hence she found infinitely many Specht modules that are decomposable.

In [12] the authors describe a set of partitions that label Specht modules which are isomorphic to Young modules. So these Specht modules are indecomposable, for every characteristic. Similar techniques were used in [13] to give decompositions of several types of Specht modules in characteristic 2 as direct sums of indecomposable modules.

However, it is still an open question if there are other decomposable Specht modules yet to be discovered.

The vertices of indecomposable Specht modules have only been found in a few special cases. However, there is a general lower bound. More precisely M. Wildon proved in [70] that:

Theorem 6.1.6. *Let λ be a partition of n, let t be a λ-tableau and denote by $H(t)$ the group of those permutations which stabilize the rows of t and also permutes the columns of equal length in t.If the Specht module S^λ, defined over a field of characteristic p, is indecomposable, then it has a vertex containing a subgroup isomorphic to a Sylow p-subgroup of $H(t)$.*

Corollary 6.1.7. *If the Specht module S^λ, defined over a field of characteristic p, is indecomposable, then it has a vertex containing a Sylow p-subgroup of $S_{\lambda_1-\lambda_2}$.*

In [19], E. Giannelli generalizes and improves the lower bound on the vertices given in Theorem 6.1.6:

Theorem 6.1.8. *Let n be a natural number and let $\mathbb{F}$ be a field of prime characteristic p. Let λ be a partition of n and let t be a λ-tableau. If the Specht module S^λ defined over $\mathbb{F}$ is indecomposable, then each of its vertices contains a subgroup conjugate to a Sylow p-subgroup of $H(t) \times H(t')$, where t' denotes the transposed tableau of t.*

In the case of hook partitions of a natural number n that is not divisible by p, this theorem is enough to describe the vertices of the corresponding Specht modules.

Let $\lambda = (n-k, 1^k)$, $p \nmid n$. The lower bound on the vertices of S^λ is attained. In fact, in this case we have $H(t) \times H(t') \cong \mathfrak{S}_k \times \mathfrak{S}_{n-k-1}$, hence by [69] we have that the vertex of S^λ is conjugate to a Sylow p-subgroup of $\mathfrak{S}_k \times \mathfrak{S}_{n-k-1}$.

In [19], E. Giannelli also determined the vertices of a certain family of Specht modules. Recall the definitions of p-core and p-weight of a partition from Section 2.4.1..

Theorem 6.1.9. *Let n be a natural number and let $\lambda = (m, x_1, x_2, \ldots, x_k)$ be a partition of n such that the partition $(x_1, x_2, \ldots, x_k)$ is a p-core partition of $n-m$. Denote by ρ and ω the p-core and the p-weight of λ respectively. Then the vertex of S^λ is equal to defect group of the corresponding block $B(\rho, \omega)$.*

6.2 All simple modules of $\mathbb{F}\mathfrak{S}_n$

The vertices of a simple module D^λ, λ a p-regular partition of n, have been completely determined for $n \leq 14$ in [9]. Also, the vertices of all simple $\mathbb{F}\mathfrak{S}_n$-modules (n arbitrary) of dimension $d \leq 1000$ have been established in [5].

There is no general description of the vertices of the simple $\mathbb{F}\mathfrak{S}_n$-modules, not even conjecturally. However, there are some results for special situations, which are now outline.

Two-part partitions

S. Danz in [6] describes the vertices of $D^{(n-m,m)}$ in the case where $p > 2$ and $m < \frac{p(p+1)}{2}$. They are either defect groups of the corresponding block or Sylow p-subgroups of the Young subgroup $\mathfrak{S}_{n-2m} \times \mathfrak{S}_m \times \mathfrak{S}_m$ of $\mathfrak{S}_n$. Moreover, one can say precisely when the second alternative occurs. The same result is true for arbitrary two-part partitions whenever $p > 2$ and $n < 2p^2$. However, not much is known about the vertices of $D^{(n-m,m)}$ in the case when $p = 2$.

Completely splittable modules

D^λ is called *completely splittable* if the restriction of D^λ to $\mathfrak{S}_m$ is semisimple for $m = 1, \ldots, n-1$. A. Kleschev in [49] has given a combinatorial characterization of such modules: the simple $F\mathfrak{S}_n$-module D^λ is completely splittable if and only if $\lambda_1 - \lambda_k + k \leq p$. It was shown in [6] that the vertices of the completely splittable $F\mathfrak{S}_n$-modules are always defect groups of the corresponding blocks. Such defect groups are known. See Proposition 2.4.7.

Hook partitions

If $\lambda = (n-r, 1^r)$, for some $r \in \{0, \ldots, p-1\}$, then λ is called a *p-regular hook partition* of n. Let $\mathbb{F}$ be a field of characteristic $p > 0$, and let $n \in \mathbb{N}$. The vertices of the simple $\mathbb{F}\mathfrak{S}_n$-module $D^{(n-r,1^r)}$ are known. Their exhaustive description is due to the collective work of several authors: M.H. Peel in [66], M. Wildon in [69], J. Müller and R. Zimmermann in [59], and S. Danz in [4] and [7]. We summarize these results in the following theorem:

Theorem 6.2.1. *Let $\mathbb{F}$ be a field of characteristic $p > 0$, and let $n \in \mathbb{N}$. Let further $r \in \{0, 1, \ldots, p-1\}$, and let Q be a vertex of the simple $\mathbb{F}\mathfrak{S}_n$-module $D^{(n-r,1^r)}$.*

(a) If $p \nmid n$ then Q is $\mathfrak{S}_n$-conjugate to a Sylow p-subgroup of $\mathfrak{S}_{n-r-1} \times \mathfrak{S}_r$.

(b) If $p = 2$, $p \mid n$ and $(n, r) \neq (4, 1)$ then Q is a Sylow 2-subgroup of $\mathfrak{S}_n$.

(c) If $p = 2$, $n = 4$ and $r = 1$ then Q is the unique Sylow 2-subgroup of $\mathfrak{A}_4$.

(d) If $p > 2$ and $p \mid n$ then Q is a Sylow p-subgroup of $\mathfrak{S}_n$.

6.3 Trivial source modules

As we have briefly seen in Section 2.4, we can say more about the vertices of p-permutation module, i.e. the $\mathbb{F}G$-modules that have trivial source.

We know that simple Specht modules have trivial source by Theorem 6.1.3. In addition, if a Specht module is isomorphic to a Young module it also has trivial source. This holds by the definition of the Young modules in Theorem 2.2.2.

If $p \geq 3$ and $n < 3p$, the vertex of a $\mathbb{F}\mathfrak{S}_n$-module could be the trivial subgroup, C_p, or $C_p \times C_p$. From the discussion in the proof of Theorem 1 in [69], these cases can be determined by considering the p-weight of the corresponding partition. First of all a Specht module corresponding to a weight zero partition is projective, and therefore has a trivial vertex. If a partition has weight one then the labelled Specht module has vertex C_p, and if a partition has weight two, the corresponding Specht module has vertex $C_p \times C_p$.

In [35] there is a characterization of the Specht modules of trivial source corresponding to partitions of weight one.

Theorem 6.3.1. *Let $p > 3$ and let B be a weight one block of $\mathfrak{S}_n$. Order the partitions corresponding to Specht modules in this block by the lexicographic order, $\lambda(p) \lhd \lambda(p-1) \lhd \cdots \lhd \lambda(1)$. The Specht modules of this block with trivial source are exactly those corresponding to the partitions $\{\, \lambda(2k+1) \mid k \geq 0 \,\}$.*

In [47], Y. Jiang completely classified the trivial source Specht modules labelled by hook partitions. When $p > 2$, he also classified the trivial source Specht modules labelled by two-part partitions. Moreover, for $p = 2$ he proved a conjecture stated in [35].

More precisely, the following are the main results in [47]:

Theorem 6.3.2 ([47, Theorem A])**.** *Let n and r be integers such that $0 \leq r < n$. Then $S^{(n-r,1^r)}$ is a trivial source module if and only if the partition $(n-r, 1^r)$ falls into precisely one of the following cases:*

(i) $(n-r, 1^r) \in \mathrm{JM}(n)_p$;

(ii) $p > 3$, $n = p$, $0 < r < p-1$ *and* $2 \mid r$;

(iii) $p = 2$, $2 \nmid n$, $r > 1$. $r \neq n-2$, $n-1$ *and* $n \equiv 2r+1 \pmod{2^L}$ *where the integer* L *satisfies* $2^{L-1} \leq r < 2^L$.

Theorem 6.3.3 ([47, Theorem B])**.** *Let $p > 2$ and n, r be integer such that $n > 0$ and $0 \leq 2r \leq n$. Then $S^{(n-r,r)}$ is a trivial source module if and only if the partition $(n-r, r) \in \mathrm{JM}(n)_p$.*

Theorem 6.3.4 ([47, Theorem C])**.** *Let $p = 2$ and n be an integer such that $n \geq 4$. Let λ be a partition of n with 2-weight two and $k_2(\lambda)$ be its 2-core. Then S^λ is a trivial source module if and only if λ falls into exactly one of the following cases:*

(i) $\lambda \in \mathrm{JM}(n)_2$;

(ii) $\lambda \notin \mathrm{JM}(n)_2$, $S^\lambda \cong Y^\mu$ *and* $\mu = k_2(\lambda) + (2,2)$.

6.3.1 Other families of modules

In this chapter we consider some more families of trivial source $\mathbb{F}\mathfrak{S}_n$-modules. We already talked about Young modules in Section 2.2.2, but there is another family that we would like to define: the so-called Scott modules.

Let $\mathbb{F}$ be a field and G be a finite group. Given a subgroup H of G there exists a unique indecomposable summand U of the permutation module $\mathbb{F}_H\uparrow^G$ such that the trivial $\mathbb{F}G$-module is a submodule of U. This holds by Frobenius reciprocity theorem 2.1.7. We say that U is the *Scott module* of G associated to H and we denote it by $\mathrm{Sc}(G, H)$.

The following summarizes the main properties of Scott modules (see [3, Theorems (2.1) and (3.2)]):

Theorem 6.3.5. *Let G be a finite group, H a subgroup of G and $P \in \mathrm{Syl}_p(H)$. Then the Scott module $\mathrm{Sc}(G, P)$ is isomorphic to $\mathrm{Sc}(G, H)$ and is uniquely determined up to isomorphism among the summands of $\mathbb{F}_H\uparrow^G$ by either of the following properties:*

(i) $\mathbb{F}_G$ is isomorphic to a submodule of $\mathrm{Sc}(G, P)$;

(ii) $\mathbb{F}_G$ is isomorphic to a quotient of $\mathrm{Sc}(G, P)$.

Moreover, $\mathrm{Sc}(G, P)$ has vertex P.

Notice that, by definition, Scott modules have trivial source.

When we consider symmetric groups, the most interesting permutation modules are the Young permutation modules $M^\lambda \cong \mathbb{F}_{\mathfrak{S}_\lambda}\uparrow^{\mathfrak{S}_n}$, for λ a partition of n. Hence we know, by the properties listed above, that one of the indecomposable summand of M^λ is the Scott module $\mathrm{Sc}(\mathfrak{S}_n, \mathfrak{S}_\lambda)$. Since the indecomposable summands of a Young permutation module are the Young modules (see Theorem 2.2.2), a natural question arises: which Young module Y^μ, $\mu \trianglerighteq \lambda$, is isomorphic to the Scott module $\mathrm{Sc}(\mathfrak{S}_n, \mathfrak{S}_\lambda)$? This Young module Y^μ has to appear with multiplicity one. The first to give an answer to this was J. Grabmeier in [32]. Here we give our answer to this question.

Let λ be a partition of n. We denote by P_λ a Sylow p-subgroup of the Young subgroup $\mathfrak{S}_\lambda$. Notice that, if λ and μ are partitions of n such that $P_\lambda \cong P_\mu$, then by the Theorem 6.3.5 we have:

$$\mathrm{Sc}(\mathfrak{S}_n, \mathfrak{S}_\lambda) \cong \mathrm{Sc}(\mathfrak{S}_n, P_\lambda) \cong \mathrm{Sc}(\mathfrak{S}_n, P_\mu) \cong \mathrm{Sc}(\mathfrak{S}_n, \mathfrak{S}_\mu).$$

Therefore, for P a p-subgroup of $\mathfrak{S}_n$ we can define

$$A_P := \{\, \lambda \in \mathscr{P}(n) \mid P_\lambda \cong P \,\}.$$

Then for every $\lambda \in A_P$, $\mathrm{Sc}(\mathfrak{S}_n, \mathfrak{S}_\lambda) \cong \mathrm{Sc}(\mathfrak{S}_n, P)$.

Denote by λ_P the partition of n such that $\mathrm{Sc}(\mathfrak{S}_n, P) \cong Y^{\lambda_P}$.

By the above argument, we necessarely have that $\lambda_P \trianglerighteq \lambda$ for every $\lambda \in A_P$, and Y^{λ_P} appears with coefficient 1 in the decomposition of M^λ for every $\lambda \in A_P$. Notice also that Y^{λ_P} cannot be one of the indecomposable summand of M^λ if $\lambda \notin A_P$. For instance, for every natural number n, $\mathrm{Sc}(\mathfrak{S}_n, P_n) \cong Y^{(n)} \cong \mathbb{F}_{\mathfrak{S}_n}$ and $Y^{(n)}$ is always a Scott module.

We would like to remark, before going on with our discussion, that if $n < p$ then for every $\lambda \in \mathscr{P}(n)$ we have $P_\lambda \cong 1$. Hence, $A_1 = \mathscr{P}(n)$ and the partition that dominates all the others is (n). Then $\mathrm{Sc}(\mathfrak{S}_n, \mathfrak{S}_\lambda) \cong Y^{(n)} \cong \mathbb{F}_{\mathfrak{S}_n}$, for every $\lambda \in \mathscr{P}(n)$, is the only Young module that is isomorphic to a trivial Scott module.

Suppose that $n = \sum_{i=0}^{r} b_i p^i$, for some $r \geq 0$ and arbitrary non-negative integers $b_0, \ldots, b_r$. Let $P \cong \times_{i=0}^{r} (P_{p^i})^{b_i}$. A partition $\lambda \vdash n$ belongs to A_P if for every $i \in [0, p-1]$, p^i occur in the p-adic expansion of the parts of λ exactly b_i times.

Definition 6.3.6. Let d_i be the residue of b_i modulo $p-1$ and set $c_i = \dfrac{b_i - d_i}{p-1}$. Define λ_P by letting p^i occur in the p-adic expansion of the first c_i parts with digit $p-1$, and in part $c_i + 1$ with digit d_i. More formally, for $j \geq 1$

$$(\lambda_P)_j := \sum_{i=0}^{r} \alpha_{i,j} p^i \text{ with } \alpha_{i,j} = \begin{cases} p-1, & \text{if } j \leq c_i \\ d_i, & \text{if } j = c_i + 1 \\ 0, & \text{otherwise.} \end{cases}$$

This definition gives a partition $\lambda_P \in A_P$. Indeed, for every $j \geq 1$, $P_{(\lambda_P)_j} \cong \times_{i=0}^{r} (P_{p^i})^{\alpha_{i,j}}$, since $\alpha_{i,j} \leq p-1$ by definition. Hence, $P_{\lambda_P} \cong P_{(\lambda_P)_1} \times P_{(\lambda_P)_2} \times \cdots \cong \times_{i=0}^{r} (P_{p^i})^{\sum_{j\geq 1} \alpha_{i,j}}$, reordering the various factors. However, $\sum_{j \geq 1} \alpha_{i,j} = c_i(p-1) + d_i = b_i$. Therefore $P_{\lambda_P} \cong P$ and so $\lambda_P \in A_P$.

Theorem 6.3.7. *λ_P dominates every partition in A_P. In particular* $\mathrm{Sc}(\mathfrak{S}_n, \mathfrak{S}_\mu) \cong Y^{\lambda_P}$ *appears with multiplicity* 1 *in the decomposition of M^μ, for every $\mu \in A_p$.*

Proof. Let $\mu \in A_P$. We have $P_\mu \cong P \cong \times_{i=0}^{r} (P_{p^i})^{b_i}$ with $b_i \in \mathbb{N}$ and $\sum_{i=0}^{r} b_i p^i = n$. Hence $\mu_j = \sum_{i=0}^{r} \beta_{i,j} p^i$ for every $j \geq 1$, for some $0 \leq \beta_{i,j} \leq p-1$ such that $\sum_{j \geq 1} \beta_{i,j} = b_i$ for every $i \in [0, r]$.

Now, $\lambda_P \trianglerighteq \mu$ if and only if for every $x \geq 1$,

$$\sum_{j=0}^{x} (\lambda_P)_j \geq \sum_{j=1}^{x} \mu_j = \sum_{j=1}^{x} \sum_{i=0}^{r} \beta_{i,j} p^i = \sum_{i=0}^{r} \left(\sum_{j=1}^{x} \beta_{i,j} \right) p^i.$$

However $\beta_{i,j} \leq p-1$ for every possible choice of i and j. Hence $\sum_{j=1}^{x} \leq x(p-1)$. Moreover, $\sum_{j \geq 1} \beta_{i,j} = b_i$ for every $i \in [0, r]$, and $\sum_{j=1}^{x} \beta_{i,j} \leq \sum_{j \geq 1} \beta_{i,j} = b_i$. Therefore $\sum_{j=1}^{x} \mu_j \leq \sum_{i=0}^{r} \min\{b_i, x(p-1)\} p^i$.

By definition of $\alpha_{i,j}$, we have that $\sum_{j=1}^{x} \alpha_{i,j} = \min\{b_i, x(p-1)\}$ for every $i \in [0, r]$. Indeed,

$$\sum_{j=1}^{x} \alpha_{i,j} = \begin{cases} d_i = b_i, & \text{if } c_i = 0; \\ p - 1 + d_i = b_i, & \text{if } c_i = 1 \ (x \geq 2); \\ 2(p-1) + d_i = b_i, & \text{if } c_i = 2 \ (x \geq 3); \\ \ldots & \\ x(p-1), & \text{if } c_i \geq x. \end{cases}$$

Finally:

$$\sum_{j=1}^{x}\mu_j \le \sum_{i=0}^{r}\left(\sum_{j=1}^{x}\alpha_{i,j}\right)p^i = \sum_{j=1}^{x}\sum_{i=0}^{r}\alpha_{i,j}p^i = \sum_{j=1}^{x}(\lambda_P)_j.$$

□

Example 6.3.8. Consider the case $n = p = 5$. We have

$$(5) \trianglerighteq (4,1) \trianglerighteq (3,2) \trianglerighteq (3,1,1) \trianglerighteq (2,2,1) \trianglerighteq (2,1,1,1) \trianglerighteq (1,1,1,1,1).$$

Then $P_\lambda \cong P_5$, if $\lambda = (5)$, and $P_\lambda \cong 1$, otherwise. Hence $A_{P_5} = \{(5)\}$ and $A_1 = \mathscr{P}(5)\setminus\{(5)\}$. Therefore we can have only two Young modules that are isomorphic to Scott modules: $Y^{(5)} \cong \mathrm{Sc}(\mathfrak{S}_5, P_5)$ and $Y^{(4,1)} \cong \mathrm{Sc}(\mathfrak{S}_5, 1)$, since $(4,1)$ dominates all partitions except (5).

These information can help us to compute the decomposition of a Young permutation module M^λ for some small cases.

If $\lambda = (5)$, we already know that $M^\lambda = Y^{(5)} \cong \mathbb{F}_{\mathfrak{S}_5}$.

If $\lambda = (4,1)$, by Theorem 2.2.2 we have that $M^\lambda = cY^{(5)} \oplus Y^{(4,1)}$ for some $c \ge 0$. However, by the above computation we know that both $Y^{(5)}$ and $Y^{(4,1)}$ are Scott modules. If they both appear with positive coefficient as summands of M^λ, then we will have two indecomposable summand with the trivial $\mathbb{F}\mathfrak{S}_5$-module as a submodule. By the argument at the beginning of this section, this is impossible. Hence we have $M^{(4,1)} = Y^{(4,1)}$.

If $\lambda = (3,2)$, the Young permutation module M^λ has dimension 10. Both the Young modules $\mathrm{Sc}(\mathfrak{S}_5, P_\lambda) \cong Y^{(4,1)}$ and $Y^{(3,2)}$ have dimension at least 5, since $M^{(4,1)} = Y^{(4,1)}$ and since $S^{(3,2)}$ (that has dimension 5) is contained in $Y^{(3,2)}$. They both have to appear once in the decomposition of M^λ. Hence necessarely $Y^{(3,2)}$ has dimension exactly 5 and $M^{(\lambda)} = Y^{(4,1)} \oplus Y^{(3,2)}$.

Notice that the knowledge of these decompositions gives us also a basis for some Young modules. Since M^λ is generated by the λ-tabloids, a basis for $Y^{(4,1)}$ is formed by the $(4,1)$-tabloids. Also, since $S^{(3,2)} = Y^{(3,2)}$, we know that it is generated by the standard basis for the Specht module $S^{(3,2)}$.

We know that both Scott and Young modules have trivial source. Moreover, for a partition μ of n, the vertex of $\mathrm{Sc}(\mathfrak{S}_n, \mathfrak{S}_\mu)$ is P_μ, while the vertex of Y^μ is given by $P_{\rho(\mu)}$ where $\rho(\mu)$ is defined in Theorem 2.2.2. If $\mu \in A_P$ we have just seen that $\mathrm{Sc}(\mathfrak{S}_n, \mathfrak{S}_\mu) \cong \mathrm{Sc}(\mathfrak{S}_n, P) \cong Y^{\lambda_P}$. Hence we have $P_\mu \cong P \cong P_{\rho(\lambda_P)}$, so $\rho(\lambda_P) \in A_P$.

The following lemmas show that there is only a possible choice for $\rho(\lambda_P) \in A_P$.

Lemma 6.3.9. *For every $P \cong \times_{i=0}^{r}(P_{p^i})^{b_i}$, where $b_i \ge 0$ and $\sum_{i=0}^{r} b_i p^i = n$, there exists a unique $\rho \in A_P$ whose parts are powers of p. This is the least dominant partition in A_P.*

Proof. If $\mu \in A_P$, then $\mu_j = \sum_{i=0}^{r}\beta_{i,j}p^i$ for every $j \ge 1$, with $\beta_{i,j} \le p-1$ for every i,j and $\sum_{j\ge 1}\beta_{i,j} = b_i$ for every $i \in [0,r]$. Hence ρ has to be the partition of n defined by $\rho_j = p^{r-s}$ where $j \in \left[\sum_{k=0}^{s-1} b_{r-k}+1, \sum_{k=0}^{s} b_{r-k}\right]$, and $\rho_j = p^r$ if $j \le b_r$.

Let now $x = \sum_{i=r-s}^{r} b_i + y$ with $y \le b_{r-s-1}$, $s \in [0, r-1]$. By definition, $\sum_{j=1}^{x}\rho_j = \sum_{i=r-s}^{r} b_i p^i + y p^{r-s-1}$. Notice that $\mu_j \ge \mu_{j+1}$ for every $j \ge 1$, since μ is a partition. In particular,

$\beta_{r,1} \geq \beta_{r,2} \geq \cdots$ and $\sum_{j\geq 1} \beta_{r,j} = b_r$. Then necessarely $\beta_{r,1} \geq 1$, and if $\beta_{r,j} = 0$ then $\beta_{r,j+1} = 0$. Therefore, $\sum_{j=1}^{b_r} \mu_j \geq b_r p^r$. To realize $b_{r-1}p^{r-1}$ we need at most other b_{r-1} "positions", so $\sum_{j=1}^{b_r+b_{r-1}} \mu_j \geq b_r p^r + b_{r-1}p^{r-1}$. And so on: $\sum_{j=1}^{x} \mu_j \geq \sum_{i=r-s}^{r} b_i p^i + y p^{r-s-1} = \sum_{j=1}^{x} \rho_j$. Therefore a generic partition $\mu \in A_P$ dominates ρ. □

Lemma 6.3.10. *Let $\rho \in A_P$ as in the previous lemma. Then $\rho = \rho(\lambda_P)$.*

Proof. We need to show that the definition of ρ in the previous lemma and the definition of $\rho(\lambda_P)$ in Theorem 2.2.2 coincide.

Recall that $(\lambda_P)_j = \sum_{i=0}^{r} \alpha_{i,j} p^i$ for every $j \geq 1$. Define $\lambda(i)_j = \alpha_{i,j}$ for every possible choice of i, j. Then $\lambda(i) = (\alpha_{i,1}, \alpha_{i,2}, \alpha_{i,3}, \dots) = ((p-1)^{c_i}, d_i)$ for every $i \in [0, r]$. We claim that $\lambda_P = \sum_{i=0}^{r} \lambda(i) p^i$ is the p-adic expansion of λ. If this is true, we have $|\lambda(i)| = c_i(p-1) + d_i = b_i$, by definition of $\alpha_{i,j}$. Hence $\rho(\lambda_P)$ is exactly ρ.

To prove the claim we need to check that $\lambda_P = \sum_{i=0}^{r} \lambda(i) p^i$, but this is true. Hence it remains only to check that for every $i \in [0, r]$, $\lambda(i)$ is p-restricted. Notice that $\lambda(i)' = ((c_i+1)^{d_i}, c_i^{p-1-d_i})$. By assumption, $0 \leq d_i < p-1$, hence $p - 1 - d_i \leq p - 1 < p$. Therefore $\lambda(i)'$ is p-regular, and $\lambda(i)$ is p-restricted. □

Let us recap what we have found so far: fix a set of partitions A_P and order its elements with the dominance order. The maximal partition λ_P is the one such that its corresponding Young module Y^{λ_P} is a Scott module, and the minimal partition $\rho(\lambda_P)$ is the one that describes the vertex of this module, that is $P \cong P_{\rho(\lambda_P)}$. We refer to Example 6.3.12 for an explanatory computation.

We conclude pointing out that every Young module labelled by a partition in A_P has vertex P (not only the one that is isomorphic to a Scott module).

Proposition 6.3.11. *For every $\mu \in A_P$, Y^μ has vertex P.*

Proof. Let $\mu \in A_P$, then $\mu_j = \sum_{i=0}^{r} \beta_{i,j} p^i$ with $\beta_{i,j} \leq p-1$ and $\sum_{j\geq 1} \beta_{i,j} = b_i$ for every possible choice of i, j. Let $\mu(i)_j = \beta_{i,j}$ for every i, j. Then $\mu = \sum_{i=0} \mu(i) p^i$ is the p-adic expansion of μ. Indeed: for $i \in [0, r]$, $l(\mu(i)) = k$. Then $\mu(i) = (\beta_{i,1}, \dots, \beta_{i,k})$. If $k_1 = \max\{j \geq 1 \mid \beta_{i,j} > \beta_{i,k}\}$ and $k_h = \max\{j \geq 1 \mid \beta_{i,j} > \beta_{i,k_{h-1}}\}$ for every h, then $\mu(i)' = (k^{\beta_{i,k}}, k_1^{\beta_{i,k_1} - \beta_{i,k}}, k_2^{\beta_{i,k_2} - \beta_{i,k_1}}, \dots)$. However $1 \leq \beta_{i,j} \leq p-1$ for every i, j, and $\beta_{i,1} \geq \cdots \geq \beta_{i,k}$ for every i, imply that $\beta_{i,j} - \beta_{i,j'} \leq (p-1) - 1 = p - 2$ if $j' > j$. That means that $\mu(i)'$ is p-regular, and $\mu(i)$ is p-restricted.

Now, let $\rho(\mu)$ be the partition of n which has $|\mu(i)|$ parts equal to p^i. Since $|\mu(i)| = \sum_{j\geq 1} \beta_{i,j} = b_i$ for every $i \in [0, r]$, we have that $\rho(\mu)$ coincide with $\rho(\lambda_P)$. Hence Y^μ has vertex $P_{\rho(\mu)} \cong P_{\rho(\lambda_P)} \cong P$. □

Example 6.3.12. Let $p = 3$ and $n = 9$. We partition the set $\mathscr{P}(9)$ into five sets: A_{P_9}, $A_{C_3\times C_3\times C_3}$, $A_{C_3\times C_3}$, A_{C_3} and A_1.

Let us focus for instance on C_3. We have

$$A_{C_3} = \{(5,2,2), (5,2,1^2), (5,1^4), (4,2,2,1), (4,2,1^3), (4,1^5), (3,2,2,2), (3,2,2,1^2), (3,2,1^4), (3,1^6)\}.$$

Every partition in this set labels a Young module that has vertex C_3, by Proposition 6.3.11.

The partition that dominates all the others in A_{C_3} is $(5,2,2)$, therefore $\lambda_{C_3} = (5,2,2)$.
Indeed in the notation of Definition 6.3.6, $C_3 \cong (P_{p^0})^6 \times P_{p^1}$ where $6 = b_0$ and $1 = b_1$. Hence, $c_0 = 3$, $d_0 = 0$ and $c_1 = 0$, $d_1 = 1$. Then we can compute the parts of λ_{C_3}:

if $j = 1$, $\alpha_{0,1} = 2$ and $\alpha_{1,1} = 1$, so $(\lambda_{C_3})_1 = 5$;

if $j = 2$, $\alpha_{0,2} = 2$ and $\alpha_{1,2} = 0$, so $(\lambda_{C_3})_2 = 2$;

if $j = 3$, $\alpha_{0,3} = 2$ and $\alpha_{1,3} = 0$, so $(\lambda_{C_3})_3 = 2$ and the other parts are zero.

Thus we have that $Y^{(5,2,2)}$ is a Scott module.

As we said in Lemma 6.3.9, there is a unique partition in A_{C_3} whose parts are power of 3 and it is dominated by every other partition in the same set: $\rho = (3,1^6)$.
In the proof of this lemma we described a way to detect ρ: as in the above computation, $r = 1$ and $b_0 = 6$, $b_1 = 1$. Hence $\rho_1 = 3$ and $\rho_j = 1$ for $j \in [2,7]$, as we expected.
This construction coincide with the one of $\rho(\lambda_{C_3})$ that we have in Theorem 2.2.2, indeed the 3-adic expansion of $(5,2,2)$ is $(1) \cdot 3^1 + (2,2,2) \cdot 3^0$, and $(1) \in \mathscr{P}(1)$, $(2,2,2) \in \mathscr{P}(6)$.
The vertex of $(5,2,2)$ can be recovered also by $\rho((5,2,2)) = (3,1^6)$, because it is isomorphic to a Sylow 3-subgroup of $\mathfrak{S}_{(3,1^6)} \cong \mathfrak{S}_3$, hence to $P_3 = C_3$.

There are exactly five Young modules that are Scott modules when $p = 3$ and $n = 9$, and these are the ones labelled by $\lambda_{P_9} = (9)$, $\lambda_{C_3 \times C_3 \times C_3} = (6,3)$, $\lambda_{C_3 \times C_3} = (8,1)$, $\lambda_{C_3} = (5,2,2)$ and $\lambda_1 = (2,2,2,2,1)$.

Notice that $A_{P_9} = \{(9)\}$, hence $\lambda_{P_9} = \rho(\lambda_{P_9}) = (9)$ and the only Young module with vertex P_9 is the trivial one. This happens whenever n is a power of p.

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